

TCR

git diff solution (Jens Heuseveldt, Ludo Pulles, Pim Spelier)

CONTENTS

0.1.	De winnende aanpak	2
0.2.	Wrong Answer	2
0.3.	Covering problems	2
0.4.	Game theory	2
1.	math	2
1.1.	Primitive Root	3
1.2.	Tonelli-Shanks algorithm	3
1.3.	Numeric Integration	3
1.4.	Fast Hadamard Transform	3
1.5.	Tridiagonal Matrix Algorithm	3
1.6.	Mertens Function	3
1.7.	Summatory Phi	3
1.8.	Josephus problem	3
1.9.	Number of Integer Points under Line	3
1.10.	Numbers and Sequences	4
2.	Datastructures	4
2.1.	Segment tree $\mathcal{O}(\log n)$	4
2.2.	Binary Indexed Tree $\mathcal{O}(\log n)$	4
2.3.	Disjoint-Set / Union-Find $\mathcal{O}(\alpha(n))$	4
2.4.	AVL Tree Balanced Binary Search Tree $\mathcal{O}(\log n)/\mathcal{O}(\log n)$	5
2.5.	Cartesian tree	5
2.6.	Heap	6
2.7.	Dancing Links	6
2.8.	Misof Tree	6
2.9.	k-d Tree	6
2.10.	Sqrt Decomposition	7
2.11.	Monotonic Queue	7
2.12.	Convex Hull Trick	7
2.13.	Sparse Table	8
3.	Graph Algorithms	8
3.1.	Maximum matching $\mathcal{O}(nm)$	8
3.2.	Hopcroft-Karp bipartite matching $\mathcal{O}(E\sqrt{V})$	8
3.3.	Depth first searches	8
3.4.	Cycle Detection $\mathcal{O}(V + E)$	9
3.5.	Shortest path	9
3.6.	Maximum Flow Algorithms	10
3.7.	Minimal Spanning Tree	10
3.8.	Topological Sort	10
3.9.	Euler Path	11
3.10.	Heavy-Light Decomposition	11
3.11.	Centroid Decomposition	11
3.12.	Least Common Ancestors, Binary Jumping	11
3.13.	Tarjan's Off-line Lowest Common Ancestors Algorithm	12
3.14.	Minimum Mean Weight Cycle	12
3.15.	Minimum Arborescence	12
3.16.	Blossom algorithm	12
3.17.	Maximum Density Subgraph	13
3.18.	Maximum-Weight Closure	13

3.19.	Maximum Weighted Independent Set in a Bipartite Graph
3.20.	Synchronizing word problem
4.	String algorithms
4.1.	Trie
4.2.	Z-algorithm $\mathcal{O}(n)$
4.3.	Suffix array $\mathcal{O}(n \log^2 n)$
4.4.	Longest Common Subsequence $\mathcal{O}(n^2)$
4.5.	Levenshtein Distance $\mathcal{O}(n^2)$
4.6.	Knuth-Morris-Pratt algorithm $\mathcal{O}(N + M)$
4.7.	Aho-Corasick Algorithm $\mathcal{O}(N + \sum_{i=1}^m S_i)$
4.8.	eerTree
4.9.	Suffix Automaton
4.10.	Hashing
5.	Geometry
5.1.	Convex Hull $\mathcal{O}(n \log n)$
5.2.	Rotating Calipers $\mathcal{O}(n)$
5.3.	Closest points $\mathcal{O}(n \log n)$
5.4.	Great-Circle Distance
5.5.	3D Primitives
5.6.	Polygon Centroid
5.7.	Rectilinear Minimum Spanning Tree
5.8.	Formulas
6.	Miscellaneous
6.1.	Binary search $\mathcal{O}(\log(hi - lo))$
6.2.	Fast Fourier Transform $\mathcal{O}(n \log n)$
6.3.	Minimum Assignment (Hungarian Algorithm) $\mathcal{O}(n^3)$
6.4.	Partial linear equation solver $\mathcal{O}(N^3)$
6.5.	Cycle-Finding
6.6.	Longest Increasing Subsequence
6.7.	Dates
6.8.	Simplex
7.	Geometry (CP3)
7.1.	Points and lines
7.2.	Polygon
7.3.	Triangle
7.4.	Circle
8.	Combinatorics
8.1.	The Twelfefold Way
9.	Useful Information
10.	Misc
10.1.	Debugging Tips
10.2.	Solution Ideas
11.	Formulas
11.1.	Physics
11.2.	Markov Chains
11.3.	Burnside's Lemma
11.4.	Bézout's identity
11.5.	Misc
	Practice Contest Checklist

At the start of a contest, create the following files in the home-dir:

.vimrc:

```

1  set nu sw=4 ts=4 sts=4 noet ai hls shcf=-ic
12 sy on | colo slate
13
13 .bashrc:
13
13 alias gsubmit='g++ -Wall -Wshadow -std=c++14'
13 alias gll='gsubmit -DLOCAL -g'
13 gsettings set
13   ⇨ org.compiz.core:/org/compiz/profiles/unity/plugins/core/
13   ⇨ vsize 3
14 gsettings set
14   ⇨ org.compiz.core:/org/compiz/profiles/unity/plugins/core/
14   ⇨ hsize 3
14
14 Test script (usage: ./test.sh A/B/..)
15
15 g++ -std=c++11 -O2 $1.cpp
15 for i in $(ls *.in)
15 do
16     j="$(echo $i | sed 's/\.in$/.ans$')"
16     ./a.out < $i > output
16     diff output $j || exit
16     echo "$i is correct"
17 done
17 echo "You may submit!"
17
17                                     template.cpp
17
17 #include <bits/stdc++.h>
17 using namespace std;
17
17 // BBST + order statistics (if supported by judge)
17 // iterator find_by_order(int r) (zero based)
17 // int order_of_key(TK v)
17 #include <ext/pb_ds/assoc_container.hpp>
17 #include <ext/pb_ds/tree_policy.hpp>
17 using namespace __gnu_pbds;
17 template<class TK, class TM> using order_tree = tree<TK, TM,
20   ⇨ less<TK>, rb_tree_tag,
21   ⇨ tree_order_statistics_node_update>;
21 template<class TV> using order_set = order_tree<TV,
22   ⇨ null_type>;
22
23 typedef long long ll;
23 typedef long double ld;
23 typedef pair<int, int> ii;
23 typedef vector<int> vi;
24 typedef vector<vi> vvi;
24 typedef vector<ii> vii;
24
24 #define x first
24 #define y second
24 #define pb push_back
24 #define eb emplace_back
24 #define rep(i,a,b) for(auto i=(a);i!=(b); ++i)
24 #define all(v) (v).begin(), (v).end()
24 #define rs resize
24 #define DBG(x) cerr << __LINE__ << ": " << #x << " = " << (x)
25   ⇨ << endl

```

```
template<class T> using min_queue = priority_queue<T,
↳ vector<T>, greater<T>>;
template<class T> int size(const T &x) { return x.size(); }
↳ // copy the ampersand(&)!

const int INF = 2147483647;
const ll LLINF = ~(1LL<<63); // = 9.223.372.036.854.775.807
const ld PI = acos(-1.0);

void run() {

}

signed main() {
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    (cout << fixed).precision(18);
    run();
    return 0;
}
```

0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Ludo moet **ALLE** opgaves **goed** lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zware) test cases.
- Gebruik ll indien wellicht nodig.

0.2. Wrong Answer.

- (1) Print de oplossing om te debuggen!
- (2) Kijk naar wellicht makkelijkere problemen.
- (3) Bedenk zelf test cases met **randgevallen**!
- (4) Controleer de **precisie**.
- (5) Controleer op **overflow** (gebruik **OVERAL** ll, ld).
Kijk naar overflows in tussenantwoorden bij modulo.
- (6) Controleer op **typo's**.
- (7) Loop de voorbeeld test case accuraat langs.
- (8) Controleer op off-by-one-errors (in indices of lus-grenzen)?

Detecting overflow This GNU builtin checks for over- and under-flow. Result is in res if successful:

```
bool isOverflown = __builtin_[add|mul|sub]-overflow(a, b,
↳ &res);
```

0.3. Covering problems.

Minimum edge cover \iff Maximum independent set

Matching: A set of edges without common vertices (*Maximum is the **largest** such set, maximal is a set which you cannot add more edges to without breaking the property*).

Minimum Vertex Cover: A set vertices (cover) such that each edge in the graph is incident to at least one vertex of the set.

Minimum Edge Cover: A set of edges (cover) such that every vertex is incident to at least one edge of the set.

Maximum Independent Set: A set of vertices in a graph such that no two of them are adjacent.

König's theorem: In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover

0.4. **Game theory.** A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

Nim: Let $X = \bigoplus_{i=1}^n x_i$, then $(x_i)_{i=1}^n$ is a winning position iff $X \neq 0$. Find a move by picking k such that $x_k > x_k \oplus X$.

Misère Nim: Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles.

Staricase Nim: Stones are moved down a staircase and only removed from the last pile. $(x_i)_{i=1}^n$ is an L -position if $(x_{2i-1})_{i=1}^{n/2}$ is (i.e. only look at odd-numbered piles).

Moore's Nim_k: The player may remove from at most k piles (Nim = Nim₁). Expand the piles in base 2, do a carry-less addition in base $k + 1$ (i.e. the number of ones in each column should be divisible by $k + 1$).

Dim⁺: The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is $k + 1$ where 2^k is the largest power of 2 dividing the pile size.

Aliquot game: Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just k .

Nim (at most half): Write $n + 1 = 2^m y$ with m maximal, then the Sprague-Grundy function of n is $(y - 1)/2$.

Lasker's Nim: Players may alternatively split a pile into two new non-empty piles. $g(4k + 1) = 4k + 1$, $g(4k + 2) = 4k + 2$, $g(4k + 3) = 4k + 4$, $g(4k + 4) = 4k + 3$ ($k \geq 0$).

Hackenbush on trees: A tree with stalks $(x_i)_{i=1}^n$ may be replaced with a single stalk with length $\bigoplus_{i=1}^n x_i$.

A useful identity: $\bigoplus_{x=0}^{a-1} x = \{0, a - 1, 1, a\}[a \bmod 4]$.

1. MATH

```
int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }
```

```
// greatest common divisor
ll gcd(ll a, ll b) { while (b) a %= b, swap(a, b); return a;
↳ };
// least common multiple
```

```
ll lcm(ll a, ll b) { return a / gcd(a, b) * b; }
ll mod(ll a, ll b) { return (a %= b) < 0 ? a + b : a; }
```

```
// safe multiplication (ab % m) for m <= 4e18 in O(log b)
ll mod_mul(ll a, ll b, ll m) {
    ll r = 0;
    while (b) {
        if (b & 1) r = (r + a) % m; a = (a + a) % m; b >>= 1;
    }
    return r;
}
```

```
// safe exponentation (a^b % m) for m <= 2e9 in O(log b)
ll mod_pow(ll a, ll b, ll m) {
    ll r = 1;
    while (b) {
        if (b & 1) r = (r * a) % m; // r = mod_mul(r, a, m);
        a = (a * a) % m; // a = mod_mul(a, a, m);
```

```
        b >>= 1;
    }
    return r;
}

// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
    ll xx = y = 0, yy = x = 1;
    while (b) {
        x -= a / b * xx; swap(x, xx);
        y -= a / b * yy; swap(y, yy);
        a %= b; swap(a, b);
    }
    return a;
}

// Chinese remainder theorem: returns (u, v) s.t.: x = u (mod
↳ v) <=> x = a (mod n) and x = b (mod m) for n,m <= 1e9
pair<ll, ll> crt(ll a, ll n, ll b, ll m) {
    ll s, t, d = egcd(n, m, s, t);
    if (mod(a - b, d)) return { 0, -1 };
    return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}

// phi[i] = #{ 0 < j <= i | gcd(i, j) = 1 }
vi totient(int N) {
    vi phi(N);
    for (int i = 0; i < N; i++) phi[i] = i;
    for (int i = 2; i < N; i++)
        if (phi[i] == i)
            for (int j = i; j < N; j += i) phi[j] -= phi[j] / i;
    return phi;
}

// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
    ll ans = 1;
    while (n) {
        ll np = n % p, kp = k % p;
        if (np < kp) return 0;
        ans = mod(ans * binom(np, kp), p); // (np C kp)
        n /= p; k /= p;
    }
    return ans;
}

// returns if n is prime for n < 3e24 ( > 2^64)
// but use mul_mod for n > 2e9!!!
bool millerRabin(ll n){
    if (n < 2 || n % 2 == 0) return n == 2;
    ll d = n - 1, ad, s = 0, r;
    for (; d % 2 == 0; d /= 2) s++;
    for (int a : { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
↳ 41 }) {
```

```

    if (n == a) return true;
    if ((ad = mod_pow(a, d, n)) == 1) continue;
    for (r = 0; r < s && ad + 1 != n; r++)
        ad = (ad * ad) % n;
    if (r == s) return false;
}
return true;
}

```

1.1. Primitive Root.

```

ll primitive_root(ll m) {
    vector<ll> div;
    for (ll i = 1; i*i < m; i++) {
        if ((m-1) % i == 0) {
            if (i < m) div.pb(i);
            if (m/i < m) div.pb(m/i); } }
    rep(x,2,m) {
        bool ok = true;
        for (ll d : div)
            if (mod_pow(x, d, m) == 1) {
                ok = false; break; }
        if (ok) return x; }
    return -1; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.2. **Tonelli-Shanks algorithm.** Given prime p and integer $1 \leq n < p$, returns the square root r of n modulo p . There is also another solution given by $-r$ modulo p .

```

ll legendre(ll a, ll p) {
    if (a % p == 0) return 0;
    if (p == 2) return 1;
    return mod_pow(a, (p-1)/2, p) == 1 ? 1 : -1; }
ll tonelli_shanks(ll n, ll p) {
    assert(legendre(n,p) == 1);
    if (p == 2) return 1;
    ll s = 0, q = p-1, z = 2;
    while (~q & 1) s++, q >= 1;
    if (s == 1) return mod_pow(n, (p+1)/4, p);
    while (legendre(z,p) != -1) z++;
    ll c = mod_pow(z, q, p),
        r = mod_pow(n, (q+1)/2, p),
        t = mod_pow(n, q, p),
        m = s;
    while (t != 1) {
        ll i = 1, ts = (ll)t*t % p;
        while (ts != 1) i++, ts = ((ll)ts * ts) % p;
        ll b = mod_pow(c, 1LL<<(m-i-1), p);
        r = ((ll)r * b % p;
        t = ((ll)t * b % p * b % p;
        c = ((ll)b * b % p;
        m = i; }
    return r; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.3. **Numeric Integration.** Numeric integration using Simpson's rule.

```

double integrate(double (*)(double), double a, double b,
    double delta = 1e-6) {

```

```

    if (abs(a - b) < delta)
        return (b-a)/8 *
            (f(a) + 3*f((2*a+b)/3) + 3*f((a+2*b)/3) + f(b));
    return integrate(f, a,
        (a+b)/2, delta) + integrate(f, (a+b)/2, b, delta); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.4. **Fast Hadamard Transform.** Computes the Hadamard transform of the given array. Can be used to compute the XOR-convolution of arrays, exactly like with FFT. For AND-convolution, use $(x+y, y)$ and $(x-y, y)$. For OR-convolution, use $(x, x+y)$ and $(x, -x+y)$. **Note:** Size of array must be a power of 2.

```

void fht(vi &arr, bool inv=false, int l=0, int r=-1) {
    if (r == -1) { fht(arr, inv, 0, size(arr)); return; }
    if (l+1 == r) return;
    int k = (r-l)/2;
    if (!inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k, r);
    rep(i,l,l+k) { int x = arr[i], y = arr[i+k];
        if (!inv) arr[i] = x-y, arr[i+k] = x+y;
        else arr[i] = (x+y)/2, arr[i+k] = (-x+y)/2; }
    if (inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k, r); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.5. **Tridiagonal Matrix Algorithm.** Solves a tridiagonal system of linear equations $a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$ where $a_1 = c_n = 0$. Beware of numerical instability.

```

#define MAXN 5000
long double A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
    C[0] /= B[0]; D[0] /= B[0];
    rep(i,1,n-1) C[i] /= B[i] - A[i]*C[i-1];
    rep(i,1,n)
        D[i] = (D[i] - A[i] * D[i-1]) / (B[i] - A[i] * C[i-1]);
    X[n-1] = D[n-1];
    for (int i = n-2; i>=0; i--)
        X[i] = D[i] - C[i] * X[i+1]; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.6. **Mertens Function.** Mertens function is $M(n) = \sum_{i=1}^n \mu(i)$. Let $L \approx (n \log \log n)^{2/3}$ and the algorithm runs in $O(n^{2/3})$.

```

#define L 9000000
int mob[L], mer[L];
unordered_map<ll,ll> mem;
ll M(ll n) {
    if (n < L) return mer[n];
    if (mem.find(n) != mem.end()) return mem[n];
    ll ans = 0, done = 1;
    for (ll i = 2; i*i <= n; i++) ans += M(n/i), done = i;
    for (ll i = 1; i*i <= n; i++)
        ans += mer[i] * (n/i - max(done, n/(i+1)));
    return mem[n] = 1 - ans; }
void sieve() {
    for (int i = 1; i < L; i++) mer[i] = mob[i] = 1;
    for (int i = 2; i < L; i++) {
        if (mer[i]) {
            mob[i] = -1;
            for (int j = i+i; j < L; j += i)

```

```

        mer[j] = 0, mob[j] = (j/i)%i == 0 ? 0 : -mob[j/i]; }
    mer[i] = mob[i] + mer[i-1]; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.7. **Summatory Phi.** The summatory phi function $\Phi(n) = \sum_{i=1}^n \phi(i)$. Let $L \approx (n \log \log n)^{2/3}$ and the algorithm runs in $O(n^{2/3})$.

```

#define N 10000000
ll sp[N];
unordered_map<ll,ll> mem;
ll sumphi(ll n) {
    if (n < N) return sp[n];
    if (mem.find(n) != mem.end()) return mem[n];
    ll ans = 0, done = 1;
    for (ll i = 2; i*i <= n; i++) ans += sumphi(n/i), done = i;
    for (ll i = 1; i*i <= n; i++)
        ans += sp[i] * (n/i - max(done, n/(i+1)));
    return mem[n] = n*(n+1)/2 - ans; }
void sieve() {
    for (int i = 1; i < N; i++) sp[i] = i;
    for (int i = 2; i < N; i++) {
        if (sp[i] == i) {
            sp[i] = i-1;
            for (int j = i+i; j < N; j += i) sp[j] -= sp[j] / i; }
        sp[i] += sp[i-1]; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.8. **Josephus problem.** Last man standing out of n if every k th is killed. Zero-based, and does not kill 0 on first pass.

```

int J(int n, int k) {
    if (n == 1) return 0;
    if (k == 1) return n-1;
    if (n < k) return (J(n-1,k)+k)%n;
    int np = n - n/k;
    return k*((J(np,k)+np-n*k%np)%np) / (k-1); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.9. **Number of Integer Points under Line.** Count the number of integer solutions to $Ax + By \leq C$, $0 \leq x \leq n$, $0 \leq y$. In other words, evaluate the sum $\sum_{x=0}^n \left\lfloor \frac{C-Ax}{B} + 1 \right\rfloor$. To count all solutions, let $n = \lfloor \frac{C}{A} \rfloor$. In any case, it must hold that $C - nA \geq 0$. Be very careful about overflows.

```

ll floor_sum(ll n, ll a, ll b, ll c) {
    if (c == 0) return 1;
    if (c < 0) return 0;
    if (a % b == 0) return (n+1)*(c/b+1)-n*(n+1)/2*a/b;
    if (a >= b) return floor_sum(n,a%b,b,c)-a/b*n*(n+1)/2;
    ll t = (c-a*n+b)/b;
    return floor_sum((c-b*t)/b,b,a,c-b*t)+t*(n+1); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.10. **Numbers and Sequences.** Some random prime numbers: 1031, 32771, 1048583, 8125344, 33554467, 9982451653 1073741827, 34359738421, 1099511627791, 35184372088891, 1125899906842679, 36028797018963971.

More random prime numbers: $10^3 + \{-9, -3, 9, 13\}$, $10^6 + \{-17, 3, 33\}$, $10^9 + \{7, 9, 21, 33, 87\}$.

	840	32
	720 720	240
Some maximal divisor counts:	735 134 400	1344
	963 761 198 400	6 720
	866 421 317 361 600	26 880
	897 612 484 786 617 600	103 680

2. DATASTRUCTURES

2.1. Segment tree $\mathcal{O}(\log n)$. Standard segment tree

```
typedef /* Tree element */ S;
const int n = 1 << 20; S t[2 * n];
```

```
// required axiom: associativity
S combine(S l, S r) { return l + r; } // sum segment tree
S combine(S l, S r) { return max(l, r); } // max segment tree
```

```
void build() { for (int i = n; --i; ) t[i] = combine(t[2 * i - 1], t[2 * i]); }
```

```
// set value v on position i
void update(int i, S v) { for (t[i += n] = v; i /= 2; ) t[i] = combine(t[2 * i - 1], t[2 * i]); }
```

```
// sum on interval [l, r]
S query(int l, int r) {
    S resL, resR;
    for (l += n, r += n; l < r; l /= 2, r /= 2) {
        if (l & 1) resL = combine(resL, t[l++]);
        if (r & 1) resR = combine(t[--r], resR);
    }
    return combine(resL, resR);
}
```

Lazy segment tree

```
struct node {
    int l, r, x, lazy;
    node() {}
    node(int _l, int _r) : l(_l), r(_r), x(INF), lazy(0) {}
    node(int _l, int _r, int _x) : node(_l, _r) { x = _x; }
    node(node a, node b) : node(a.l, b.r) { x = min(a.x, b.x); }
    void update(int v) { x = v; }
    void range_update(int v) { lazy = v; }
    void apply() { x += lazy; lazy = 0; }
    void push(node &u) { u.lazy += lazy; }
};
```

```
struct segment_tree {
    int n;
    vector<node> arr;
    segment_tree() {}
```

```
segment_tree(const vector<ll> &a) : n(size(a)), arr(4*n) {
    mk(a, 0, 0, n-1); }
node mk(const vector<ll> &a, int l, int r) {
    int m = (l+r)/2;
    return arr[l] = l > r ? node(l, r) :
        l == r ? node(l, r, a[l]) :
        node(mk(a, 2*i+1, l, m), mk(a, 2*i+2, m+1, r)); }
node update(int at, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (at < hl || hr < at) return arr[i];
    if (hl == at && at == hr) {
        arr[i].update(v); return arr[i]; }
    return arr[i] =
        node(update(at, v, 2*i+1), update(at, v, 2*i+2)); }
node query(int l, int r, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return node(hl, hr);
    if (l <= hl && hr <= r) return arr[i];
    return node(query(l, r, 2*i+1), query(l, r, 2*i+2)); }
node range_update(int l, int r, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return arr[i];
    if (l <= hl && hr <= r)
        return arr[i].range_update(v), propagate(i), arr[i];
    return arr[i] = node(range_update(l, r, v, 2*i+1),
        range_update(l, r, v, 2*i+2)); }
void propagate(int i) {
    if (arr[i].l < arr[i].r)
        arr[i].push(arr[2*i+1]), arr[i].push(arr[2*i+2]);
    arr[i].apply(); } }
```

Persistent segment tree

```
int segcnt = 0;
struct segment {
    int l, r, lid, rid, sum;
} segs[2000000];
int build(int l, int r) {
    if (l > r) return -1;
    int id = segcnt++;
    segs[id].l = l;
    segs[id].r = r;
    if (l == r) segs[id].lid = -1, segs[id].rid = -1;
    else {
        int m = (l + r) / 2;
        segs[id].lid = build(l, m);
        segs[id].rid = build(m + 1, r); }
    segs[id].sum = 0;
    return id; }
int update(int idx, int v, int id) {
    if (id == -1) return -1;
    if (idx < segs[id].l || idx > segs[id].r) return id;
    int nid = segcnt++;
    segs[nid].l = segs[id].l;
    segs[nid].r = segs[id].r;
```

```
segs[nid].lid = update(idx, v, segs[id].lid);
segs[nid].rid = update(idx, v, segs[id].rid);
segs[nid].sum = segs[id].sum + v;
return nid; }
int query(int id, int l, int r) {
    if (r < segs[id].l || segs[id].r < l) return 0;
    if (l <= segs[id].l && segs[id].r <= r) return
        segs[id].sum;
    return query(segs[id].lid, l, r)
        + query(segs[id].rid, l, r); }
```

2.2. Binary Indexed Tree $\mathcal{O}(\log n)$. Use one-based indices ($i > 0$)!

```
int bit[MAXN + 1];

// arr[i] += v
void update(int i, int v) {
    while (i <= MAXN) bit[i] += v, i += i & -i;
}

// returns sum of arr[i], where i: [1, i]
int query(int i) {
    int v = 0; while (i) v += bit[i], i -= i & -i; return v;
}
```

Use this if you add things, which depend on i :

```
struct fenwick_tree {
    int n; vi data;
    fenwick_tree(int _n) : n(_n), data(vi(n)) {}
    void update(int at, int by) {
        while (at < n) data[at] += by, at |= at + 1; }
    int query(int at) {
        int res = 0;
        while (at >= 0) res += data[at], at = (at & (at + 1)) - 1;
        return res; }
    int rsq(int a, int b) { return query(b) - query(a - 1); }
};
struct fenwick_tree_sq {
    int n; fenwick_tree x1, x0;
    fenwick_tree_sq(int _n) : n(_n), x1(fenwick_tree(n)),
        x0(fenwick_tree(n)) {}
    // insert f(y) = my + c if x <= y
    void update(int x, int m, int c) {
        x1.update(x, m); x0.update(x, c); }
    int query(int x) { return x*x1.query(x) + x0.query(x); }
};
void range_update(fenwick_tree_sq &s, int a, int b, int k) {
    s.update(a, k, k * (1 - a)); s.update(b+1, -k, k * b); }
int range_query(fenwick_tree_sq &s, int a, int b) {
    return s.query(b) - s.query(a-1); }
```

2.3. Disjoint-Set / Union-Find $\mathcal{O}(\alpha(n))$.

```
struct dsu {
    vi par, rnk;
    dsu(int n) : par(n, 0), rnk(n, -1) {}
    int find(int i) { return par[i] < 0 ? i : par[i] =
        find(par[i]); }
```

```

void unite(int a, int b) {
    if ((a = find(a)) == (b = find(b))) return;
    if (rnk[a] < rnk[b]) swap(a, b);
    if (rnk[a] == rnk[b]) rnk[a]++;
    par[a] += par[b]; par[b] = a;
}
};

```

2.4. AVL Tree Balanced Binary Search Tree $\mathcal{O}(\log n)/\mathcal{O}(\log n)$.

```

#define AVL_MULTISSET 0
template <class T> struct avl_tree {
    struct node {
        T item; node *p, *l, *r;
        int size, height;
        node(const T &item, node *_p = NULL) : item(item),
        p(_p),
        l(NULL), r(NULL), size(1), height(0) { } };

    node *root;
    avl_tree() : root(NULL) { }
    inline int sz(node *n) const { return n ? n->size : 0; }
    inline int height(node *n) const {
        return n ? n->height : -1; }
    inline bool left_heavy(node *n) const {
        return n && height(n->l) > height(n->r); }
    inline bool right_heavy(node *n) const {
        return n && height(n->r) > height(n->l); }
    inline bool too_heavy(node *n) const {
        return n && abs(height(n->l) - height(n->r)) > 1; }

    void delete_tree(node *n) { if (n) {
        delete_tree(n->l), delete_tree(n->r); delete n; } }
    node*& parent_leg(node *n) {
        if (!n->p) return root;
        if (n->p->l == n) return n->p->l;
        if (n->p->r == n) return n->p->r;
        assert(false); }
    void augment(node *n) {
        if (!n) return;
        n->size = 1 + sz(n->l) + sz(n->r);
        n->height = 1 + max(height(n->l), height(n->r)); }
    #define rotate(l, r) {
        node *l = n->l;
        l->p = n->p;
        parent_leg(n) = l;
        n->l = l->r;
        if (l->r) l->r->p = n;
        l->r = n, n->p = l;
        augment(n), augment(l)
    }
    void left_rotate(node *n) { rotate(r, l); }
    void right_rotate(node *n) { rotate(l, r); }
    void fix(node *n) {
        while (n) { augment(n);
            if (too_heavy(n)) {
                if (left_heavy(n) && right_heavy(n->l))
                    left_rotate(n->l);

```

```

                else if (right_heavy(n) && left_heavy(n->r))
                    right_rotate(n->r);
                if (left_heavy(n)) right_rotate(n);
                else left_rotate(n);
                n = n->p; }
        n = n->p; } }
    inline int size() const { return sz(root); }
    node* find(const T &item) const {
        node *cur = root;
        while (cur) {
            if (cur->item < item) cur = cur->r;
            else if (item < cur->item) cur = cur->l;
            else break; }
        return cur; }
    node* insert(const T &item) {
        node *prev = NULL, **cur = &root;
        while (*cur) {
            prev = *cur;
            if ((*cur)->item < item) cur = &(*cur)->r;
            #if AVL_MULTISSET
            else cur = &(*cur)->l;
            #else
            else if (item < (*cur)->item) cur = &(*cur)->l;
            else return *cur;
            #endif
        }
        node *n = new node(item, prev);
        *cur = n, fix(n); return n; }
    void erase(const T &item) { erase(find(item)); }
    void erase(node *n, bool free = true) {
        if (!n) return;
        if (!n->l && n->r) parent_leg(n) = n->r, n->r->p = n->p;
        else if (n->l && !n->r)
            parent_leg(n) = n->l, n->l->p = n->p;
        else if (n->l && n->r) {
            node *s = successor(n);
            erase(s, false);
            s->p = n->p, s->l = n->l, s->r = n->r;
            if (n->l) n->l->p = s;
            if (n->r) n->r->p = s;
            parent_leg(n) = s, fix(s);
            return;
        } else parent_leg(n) = NULL;
        fix(n->p), n->p = n->l = n->r = NULL;
        if (free) delete n; }
    node* successor(node *n) const {
        if (!n) return NULL;
        if (n->r) return nth(0, n->r);
        node *p = n->p;
        while (p && p->r == n) n = p, p = p->p;
        return p; }
    node* predecessor(node *n) const {
        if (!n) return NULL;
        if (n->l) return nth(n->l->size-1, n->l);
        node *p = n->p;
        while (p && p->l == n) n = p, p = p->p;

```

```

        return p; }
    node* nth(int n, node *cur = NULL) const {
        if (!cur) cur = root;
        while (cur) {
            if (n < sz(cur->l)) cur = cur->l;
            else if (n > sz(cur->l))
                n -= sz(cur->l) + 1, cur = cur->r;
            else break;
        } return cur; }
    int count_less(node *cur) {
        int sum = sz(cur->l);
        while (cur) {
            if (cur->p && cur->p->r == cur) sum += 1 +
            sz(cur->p->l);
            cur = cur->p;
        } return sum; }
    void clear() { delete_tree(root), root = NULL; } };

```

Use this easy implementation for a map:

```

template <class K, class V> struct avl_map {
    struct node {
        K key; V value;
        node(K k, V v) : key(k), value(v) { }
        bool operator <(const node &other) const {
            return key < other.key; } };
    avl_tree<node> tree;
    V& operator [] (K key) {
        typename avl_tree<node>::node *n =
            tree.find(node(key, V(0)));
        if (!n) n = tree.insert(node(key, V(0)));
        return n->item.value; } };

```

2.5. Cartesian tree.

```

struct node {
    int x, y, sz;
    node *l, *r;
    node(int _x, int _y)
        : x(_x), y(_y), sz(1), l(NULL), r(NULL) { } };
int tsize(node* t) { return t ? t->sz : 0; }
void augment(node *t) {
    t->sz = 1 + tsize(t->l) + tsize(t->r); }
pair<node*, node*> split(node *t, int x) {
    if (!t) return make_pair((node*)NULL, (node*)NULL);
    if (t->x < x) {
        pair<node*, node*> res = split(t->r, x);
        t->r = res.first; augment(t);
        return make_pair(t, res.second); }
    pair<node*, node*> res = split(t->l, x);
    t->l = res.second; augment(t);
    return make_pair(res.first, t); }
node* merge(node *l, node *r) {
    if (!l) return r; if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r); augment(l); return l; }
    r->l = merge(l, r->l); augment(r); return r; }
node* find(node *t, int x) {

```



```

while (t) {
    if (x < t->x) t = t->l;
    else if (t->x < x) t = t->r;
    else return t; }
return NULL; }

node* insert(node *t, int x, int y) {
    if (find(t, x) != NULL) return t;
    pair<node*, node*> res = split(t, x);
    return merge(res.first,
        merge(new node(x, y), res.second)); }

node* erase(node *t, int x) {
    if (!t) return NULL;
    if (t->x < x) t->r = erase(t->r, x);
    else if (x < t->x) t->l = erase(t->l, x);
    else { node *old = t; t = merge(t->l, t->r); delete old; }
    if (t) augment(t); return t; }

int kth(node *t, int k) {
    if (k < tsize(t->l)) return kth(t->l, k);
    else if (k == tsize(t->l)) return t->x;
    else return kth(t->r, k - tsize(t->l) - 1); }

```

2.6. **Heap.** An implementation of a binary heap.

```

#define RESIZE
#define SWP(x,y) tmp = x, x = y, y = tmp
struct default_int_cmp {
    default_int_cmp() {}
    bool operator()(const int &a, const int &b) {
        return a < b; } };

template <class Compare = default_int_cmp> struct heap {
    int len, count, *q, *loc, tmp;
    Compare _cmp;
    inline bool cmp(int i, int j) { return _cmp(q[i], q[j]); }
    inline void swp(int i, int j) {
        SWP(q[i], q[j]), SWP(loc[q[i]], loc[q[j]]); }
    void swim(int i) {
        while (i > 0) {
            int p = (i - 1) / 2;
            if (!cmp(i, p)) break;
            swp(i, p), i = p; } }
    void sink(int i) {
        while (true) {
            int l = 2*i + 1, r = l + 1;
            if (l >= count) break;
            int m = r >= count || cmp(l, r) ? l : r;
            if (!cmp(m, i)) break;
            swp(m, i), i = m; } }

    heap(int init_len = 128)
        : count(0), len(init_len), _cmp(Compare()) {
        q = new int[len], loc = new int[len];
        memset(loc, 255, len << 2); }
    ~heap() { delete[] q; delete[] loc; }
    void push(int n, bool fix = true) {
        if (len == count || n >= len) {
#ifdef RESIZE
            int newlen = 2 * len;
            while (n >= newlen) newlen *= 2;

```

```

            int *newq = new int[newlen], *newloc = new int[newlen];
            rep(i, 0, len) newq[i] = q[i], newloc[i] = loc[i];
            memset(newloc + len, 255, (newlen - len) << 2);
            delete[] q, delete[] loc;
            loc = newloc, q = newq, len = newlen;
        } else
            assert(false);
    }
    #endif
    assert(loc[n] == -1);
    loc[n] = count, q[count++] = n;
    if (fix) swim(count-1); }
    void pop(bool fix = true) {
        assert(count > 0);
        loc[q[0]] = -1, q[0] = q[--count], loc[q[0]] = 0;
        if (fix) sink(0);
    }
    int top() { assert(count > 0); return q[0]; }
    void heapify() { for (int i = count - 1; i > 0; i--)
        if (cmp(i, (i - 1) / 2)) swp(i, (i - 1) / 2); }
    void update_key(int n) {
        assert(loc[n] != -1), swim(loc[n]), sink(loc[n]); }
    bool empty() { return count == 0; }
    int size() { return count; }
    void clear() { count = 0, memset(loc, 255, len << 2); }; }

```

2.7. **Dancing Links.** An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```

template <class T>
struct dancing_links {
    struct node {
        T item;
        node *l, *r;
        node(const T &_item, node *_l = NULL, node *_r = NULL)
            : item(_item), l(_l), r(_r) {
            if (l) l->r = this;
            if (r) r->l = this; } };
    node *front, *back;
    dancing_links() { front = back = NULL; }
    node *push_back(const T &item) {
        back = new node(item, back, NULL);
        if (!front) front = back;
        return back; }
    node *push_front(const T &item) {
        front = new node(item, NULL, front);
        if (!back) back = front;
        return front; }
    void erase(node *n) {
        if (!n->l) front = n->r; else n->l->r = n->r;
        if (!n->r) back = n->l; else n->r->l = n->l; }
    void restore(node *n) {
        if (!n->l) front = n; else n->l->r = n;
        if (!n->r) back = n; else n->r->l = n; } };

```

2.8. **Misof Tree.** A simple tree data structure for inserting, erasing, and querying the n th largest element.

```

#define BITS 15
struct misof_tree {
    int cnt[BITS][1<<BITS];
    misof_tree() { memset(cnt, 0, sizeof(cnt)); }
    void insert(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]++, x >>= 1); }
    void erase(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]--, x >>= 1); }
    int nth(int n) {
        int res = 0;
        for (int i = BITS-1; i >= 0; i--)
            if (cnt[i][res <= 1] <= n) n -= cnt[i][res], res |= 1;
        return res; } };

```

2.9. **k-d Tree.** A k -dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```

#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
    struct pt {
        double coord[K];
        pt() {}
        pt(double c[K]) { rep(i, 0, K) coord[i] = c[i]; }
        double dist(const pt &other) const {
            double sum = 0.0;
            rep(i, 0, K) sum += pow(coord[i] - other.coord[i], 2.0);
            return sqrt(sum); } };
    struct cmp {
        int c;
        cmp(int _c) : c(_c) {}
        bool operator()(const pt &a, const pt &b) {
            for (int i = 0, cc; i <= K; i++) {
                cc = i == 0 ? c : i - 1;
                if (abs(a.coord[cc] - b.coord[cc]) > EPS)
                    return a.coord[cc] < b.coord[cc];
            }
            return false; } };
    struct bb {
        pt from, to;
        bb(pt _from, pt _to) : from(_from), to(_to) {}
        double dist(const pt &p) {
            double sum = 0.0;
            rep(i, 0, K) {
                if (p.coord[i] < from.coord[i])
                    sum += pow(from.coord[i] - p.coord[i], 2.0);
                else if (p.coord[i] > to.coord[i])
                    sum += pow(p.coord[i] - to.coord[i], 2.0);
            }
            return sqrt(sum); }
        bb bound(double l, int c, bool left) {
            pt nf(from.coord), nt(to.coord);
            if (left) nt.coord[c] = min(nt.coord[c], l);
            else nf.coord[c] = max(nf.coord[c], l);
            return bb(nf, nt); } };
    struct node {

```

```

    pt p; node *l, *r;
    node(pt _p, node *_l, node *_r)
        : p(_p), l(_l), r(_r) { } };
node *root;
// kd_tree() : root(NULL) { }
kd_tree(vector<pt> pts) {
    root = construct(pts, 0, size(pts) - 1, 0); }
node* construct(vector<pt> &pts, int from, int to, int c) {
    if (from > to) return NULL;
    int mid = from + (to - from) / 2;
    nth_element(pts.begin() + from, pts.begin() + mid,
        pts.begin() + to + 1, cmp(c));
    return new node(pts[mid],
        construct(pts, from, mid - 1, INC(c)),
        construct(pts, mid + 1, to, INC(c))); }
bool contains(const pt &p) { return _con(p, root, 0); }
bool _con(const pt &p, node *n, int c) {
    if (!n) return false;
    if (cmp(c)(p, n->p)) return _con(p, n->l, INC(c));
    if (cmp(c)(n->p, p)) return _con(p, n->r, INC(c));
    return true; }
void insert(const pt &p) { _ins(p, root, 0); }
void _ins(const pt &p, node *n, int c) {
    if (!n) n = new node(p, NULL, NULL);
    else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
    else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c)); }
void clear() { _clr(root); root = NULL; }
void _clr(node *n) {
    if (n) _clr(n->l), _clr(n->r), delete n; }
pt nearest_neighbour(const pt &p, bool allow_same=true) {
    assert(root);
    double mn = INFINITY, cs[K];
    rep(i,0,K) cs[i] = -INFINITY;
    pt from(cs);
    rep(i,0,K) cs[i] = INFINITY;
    pt to(cs);
    return _nn(p, root, bb(from, to), mn, 0,
        allow_same).first;
}
pair<pt, bool> _nn(const pt &p, node *n, bb b,
    double &mn, int c, bool same) {
    if (!n || b.dist(p) > mn) return make_pair(pt(), false);
    bool found = same || p.dist(n->p) > EPS,
        l1 = true, l2 = false;
    pt resp = n->p;
    if (found) mn = min(mn, p.dist(resp));
    node *n1 = n->l, *n2 = n->r;
    rep(i,0,2) {
        if (i == 1 || cmp(c)(n->p, p))
            swap(n1, n2), swap(l1, l2);
        pair<pt, bool> res = _nn(p, n1,
            b.bound(n->p.coord[c], c, l1), mn, INC(c), same);
        if (res.second &&
            (!found || p.dist(res.first) < p.dist(resp)))
            resp = res.first, found = true;
    }
}

```

```

    }
    return make_pair(resp, found); } };

2.10. Sqrt Decomposition. Design principle that supports many operations in amortized  $\sqrt{n}$  per operation.
struct segment {
    vi arr;
    segment(vi _arr) : arr(_arr) { } };
vector<segment> T;
int K;
void rebuild() {
    int cnt = 0;
    rep(i,0,size(T))
        cnt += size(T[i].arr);
    K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
    vi arr(cnt);
    for (int i = 0, at = 0; i < size(T); i++)
        rep(j,0,size(T[i].arr))
            arr[at++] = T[i].arr[j];
    T.clear();
    for (int i = 0; i < cnt; i += K)
        T.push_back(segment(vi(arr.begin()+i,
            arr.begin()+min(i+K, cnt)))); }
int split(int at) {
    int i = 0;
    while (i < size(T) && at >= size(T[i].arr))
        at -= size(T[i].arr), i++;
    if (i >= size(T)) return size(T);
    if (at == 0) return i;
    T.insert(T.begin() + i + 1,
        segment(vi(T[i].arr.begin() + at, T[i].arr.end())));
    T[i] = segment(vi(T[i].arr.begin(), T[i].arr.begin() +
        at));
    return i + 1; }
void insert(int at, int v) {
    vi arr; arr.push_back(v);
    T.insert(T.begin() + split(at), segment(arr)); }
void erase(int at) {
    int i = split(at); split(at + 1);
    T.erase(T.begin() + i); }
// vim: cc=60 ts=2 sts=2 sw=2:

2.11. Monotonic Queue. A queue that supports querying for the minimum element. Useful for sliding window algorithms.
struct min_stack {
    stack<int> S, M;
    void push(int x) {
        S.push(x);
        M.push(M.empty() ? x : min(M.top(), x)); }
    int top() { return S.top(); }
    int mn() { return M.top(); }
    void pop() { S.pop(); M.pop(); }
    bool empty() { return S.empty(); } };
struct min_queue {
    min_stack inp, outp;
    void push(int x) { inp.push(x); }
}

```

```

void fix() {
    if (outp.empty()) while (!inp.empty())
        outp.push(inp.top(), inp.pop()); }
int top() { fix(); return outp.top(); }
int mn() {
    if (inp.empty()) return outp.mn();
    if (outp.empty()) return inp.mn();
    return min(inp.mn(), outp.mn()); }
void pop() { fix(); outp.pop(); }
bool empty() { return inp.empty() && outp.empty(); } };

```

2.12. **Convex Hull Trick.** If converting to integers, look out for division by 0 and $\pm\infty$.

```

struct convex_hull_trick {
    vector<pair<double,double> > h;
    double intersect(int i) {
        return (h[i+1].second-h[i].second) /
            (h[i].first-h[i+1].first); }
    void add(double m, double b) {
        h.push_back(make_pair(m,b));
        while (size(h) >= 3) {
            int n = size(h);
            if (intersect(n-3) < intersect(n-2)) break;
            swap(h[n-2], h[n-1]);
            h.pop_back(); } }
    double get_min(double x) {
        int lo = 0, hi = size(h) - 2, res = -1;
        while (lo <= hi) {
            int mid = lo + (hi - lo) / 2;
            if (intersect(mid) <= x) res = mid, lo = mid + 1;
            else hi = mid - 1; }
        return h[res+1].first * x + h[res+1].second; } };

```

And dynamic variant:

```

const ll is_query = -(1LL<<62);
struct Line {
    ll m, b;
    mutable function<const Line*> succ;
    bool operator<(const Line& rhs) const {
        if (rhs.b != is_query) return m < rhs.m;
        const Line* s = succ();
        if (!s) return 0;
        ll x = rhs.m;
        return b - s->b < (s->m - m) * x; } };
// will maintain upper hull for maximum
struct HullDynamic : public multiset<Line> {
    bool bad(iterator y) {
        auto z = next(y);
        if (y == begin()) {
            if (z == end()) return 0;
            return y->m == z->m && y->b <= z->b; }
        auto x = prev(y);
        if (z == end()) return y->m == x->m && y->b <= x->b;
        return (x->b - y->b)*(z->m - y->m) >=
            (y->b - z->b)*(y->m - x->m); }
    void insert_line(ll m, ll b) {
        auto y = insert({ m, b }); }
}

```

```

y->succ = [=] { return next(y) == end() ? 0 : &*next(y);
↪ };
if (bad(y)) { erase(y); return; }
while (next(y) != end() && bad(next(y))) erase(next(y));
while (y != begin() && bad(prev(y))) erase(prev(y)); }
ll eval(ll x) {
    auto l = *lower_bound((Line) { x, is_query });
    return l.m * x + l.b; } };

```

2.13. Sparse Table.

```

struct sparse_table { vvi m;
    sparse_table(vi arr) {
        m.push_back(arr);
        for (int k = 0; (1<<(++k)) <= size(arr); ) {
            m.push_back(vi(size(arr)-(1<<k)+1));
            rep(i,0,size(arr)-(1<<k)+1)
                m[k][i] = min(m[k-1][i], m[k-1][i+(1<<(k-1))]); } }
    int query(int l, int r) {
        int k = 0; while (1<<(k+1) <= r-l+1) k++;
        return min(m[k][l], m[k][r-(1<<k)+1]); } };

```

3. GRAPH ALGORITHMS

3.1. Maximum matching $\mathcal{O}(nm)$.

```
const int sizeL = 1e4, sizeR = 1e4;
```

```

bool vis[sizeR];
int par[sizeR]; // par : R -> L
vi adj[sizeL]; // adj : L -> (N -> R)

```

```

bool match(int u) {
    for (int v : adj[u]) {
        if (vis[v] continue; vis[v] = true;
        if (par[v] == -1 || match(par[v])) {
            par[v] = u;
            return true;
        }
    }
    return false;
}

```

// perfect matching iff ret == sizeL == sizeR

```

int maxmatch() {
    fill_n(par, sizeR, -1); int ret = 0;
    for (int i = 0; i < sizeL; i++) {
        fill_n(vis, sizeR, false);
        ret += match(i);
    }
    return ret;
}

```

3.2. Hopcroft-Karp bipartite matching $\mathcal{O}(E\sqrt{V})$.

```

#define MAXN 5000
int dist[MAXN+1], q[MAXN+1];
#define dist(v) dist[v == -1 ? MAXN : v]
struct bipartite_graph {
    int N, M, *L, *R; vi *adj;

```

```

bipartite_graph(int _N, int _M) : N(_N), M(_M),
    L(new int[N]), R(new int[M]), adj(new vi[N]) {}
~bipartite_graph() { delete[] adj; delete[] L; delete[] R;
↪ }
bool bfs() {
    int l = 0, r = 0;
    rep(v,0,N) if(L[v] == -1) dist(v) = 0, q[r++] = v;
    else dist(v) = INF;
    dist(-1) = INF;
    while(l < r) {
        int v = q[l++];
        if(dist(v) < dist(-1)) {
            iter(u, adj[v]) if(dist(R[*u]) == INF)
                dist(R[*u]) = dist(v) + 1, q[r++] = R[*u]; } }
    return dist(-1) != INF; }
bool dfs(int v) {
    if(v != -1) {
        iter(u, adj[v])
            if(dist(R[*u]) == dist(v) + 1)
                if(dfs(R[*u])) {
                    R[*u] = v, L[v] = *u;
                    return true; }
        dist(v) = INF;
        return false; }
    return true; }
void add_edge(int i, int j) { adj[i].push_back(j); }
int maximum_matching() {
    int matching = 0;
    memset(L, -1, sizeof(int) * N);
    memset(R, -1, sizeof(int) * M);
    while(bfs()) rep(i,0,N)
        matching += L[i] == -1 && dfs(i);
    return matching; } };
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.2.1. Minimum Vertex Cover in Bipartite Graphs.

```

#include "hopcroft_karp.cpp"
vector<bool> alt;
void dfs(bipartite_graph &g, int at) {
    alt[at] = true;
    iter(it,g.adj[at]) {
        alt[*it + g.N] = true;
        if (g.R[*it] != -1 && !alt[g.R[*it]]) dfs(g, g.R[*it]); }
↪ }
vi mvc_bipartite(bipartite_graph &g) {
    vi res; g.maximum_matching();
    alt.assign(g.N + g.M, false);
    rep(i,0,g.N) if (g.L[i] == -1) dfs(g, i);
    rep(i,0,g.N) if (!alt[i]) res.push_back(i);
    rep(i,0,g.M) if (alt[g.N + i]) res.push_back(g.N + i);
    return res; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.3. Depth first searches.

3.3.1. Cut Points and Bridges.

```

const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;

void dfs(const vvi &adj, vi &cp, vii &bri, int u, int p) {
    low[u] = num[u] = curnum++;
    int cnt = 0; bool found = false;
    rep(i,0,size(adj[u])) {
        int v = adj[u][i];
        if (num[v] == -1) {
            dfs(adj, cp, bri, v, u);
            low[u] = min(low[u], low[v]);
            cnt++;
            found = found || low[v] >= num[u];
            if (low[v] > num[u]) bri.push_back(ii(u, v));
        } else if (p != v) low[u] = min(low[u], num[v]); }
    if (found && (p != -1 || cnt > 1)) cp.push_back(u); }

pair<vi,vii> cut_points_and_bridges(const vvi &adj) {
    int n = size(adj);
    vi cp; vii bri;
    memset(num, -1, n << 2);
    curnum = 0;
    rep(i,0,n) if (num[i] == -1) dfs(adj, cp, bri, i, -1);
    return make_pair(cp, bri); }

```

3.3.2. Strongly Connected Components $\mathcal{O}(V + E)$.

```

vvi adj, comps;
vi tidx, lnk, cnr, st;
vector<bool> vis;
int age, ncomps;

void tarjan(int v) {
    tidx[v] = lnk[v] = ++age; vis[v] = true; st.pb(v);
    for (int w : adj[v]) {
        if (!tidx[w]) tarjan(w), lnk[v] = min(lnk[v], lnk[w]);
        else if (vis[w]) lnk[v] = min(lnk[v], tidx[w]);
    }
    if (lnk[v] != tidx[v]) return;
    comps.pb(vi());
    int w;
    do {
        vis[w = st.back()] = false; cnr[w] = ncomps;
↪     comps.back().pb(w);
        st.pop_back();
    } while (w != v);
    ncomps++;
}

void findSCC(int n) {
    age = ncomps = 0; vis.assign(n, false); tidx.assign(n, 0);
    lnk.resize(n); cnr.resize(n); comps.clear();
    for (int i = 0; i < n; i++)
        if (tidx[i] == 0) tarjan(i);
}

```


3.3.3. Dominator graph.

```
const int N = 1234567;
```

```
vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], parent[N], sdom[N], p[N], best[N],
↳ idom[N], link[N];
```

```
void dfs(int v) {
    pos[v] = cnt;
    order[cnt++] = v;
    for (int u : g[v]) {
        if (pos[u] == -1) {
            parent[u] = v;
            dfs(u);
        }
    }
}
```

```
int find_best(int x) {
    if (p[x] == x) return best[x];
    int u = find_best(p[x]);
    if (pos[sdom[u]] < pos[sdom[best[x]]])
        best[x] = u;
    p[x] = p[p[x]];
    return best[x];
}
```

```
void dominators(int n, int root) {
    fill_n(pos, n, -1);
    cnt = 0;
    dfs(root);
    for (int i = 0; i < n; i++)
        for (int u : g[i]) g_rev[u].push_back(i);
    for (int i = 0; i < n; i++)
        p[i] = best[i] = sdom[i] = i;
    for (int it = cnt - 1; it >= 1; it--) {
        int w = order[it];
        for (int u : g_rev[w]) {
            int t = find_best(u);
            if (pos[sdom[t]] < pos[sdom[w]])
                sdom[w] = sdom[t];
        }
        bucket[sdom[w]].push_back(w);
        idom[w] = sdom[w];
        for (int u : bucket[parent[w]])
            link[u] = find_best(u);
        bucket[parent[w]].clear();
        p[w] = parent[w];
    }
    for (int it = 1; it < cnt; it++) {
        int w = order[it];
        idom[w] = idom[link[w]];
    }
}
```

3.3.4. 2-SAT $\mathcal{O}(V + E)$. Include findSCC.

```
void init2sat(int n) { adj.assign(2 * n, vi()); }

// vl, vr = true -> variable l, variable r should be negated.
void imply(int xl, bool vl, int xr, bool vr) {
    adj[2 * xl + vl].pb(2 * xr + vr); adj[2 * xr + !vr].pb(2 *
↳ xl + !vl); }

void satOr(int xl, bool vl, int xr, bool vr) { imply(xl, !vl,
↳ xr, vr); }

void satConst(int x, bool v) { imply(x, !v, x, v); }
void satIff(int xl, bool vl, int xr, bool vr) {
    imply(xl, vl, xr, vr); imply(xr, vr, xl, vl);}

bool solve2sat(int n, vector<bool> &sol) {
    findSCC(2 * n);
    for (int i = 0; i < n; i++)
        if (cncr[2 * i] == cncr[2 * i + 1]) return false;
    vector<bool> seen(n, false); sol.assign(n, false);
    for (vi &comp : comps) {
        for (int v : comp) {
            if (seen[v / 2]) continue;
            seen[v / 2] = true; sol[v / 2] = v & 1;
        }
    }
    return true;
}
```

3.4. Cycle Detection $\mathcal{O}(V + E)$.

vvi adj; // assumes bidirected graph, adjust accordingly

```
bool cycle_detection() {
    stack<int> s; vector<bool> vis(MAXN, false); vi par(MAXN,
↳ -1); s.push(0);
    vis[0] = true;
    while(!s.empty()) {
        int cur = s.top(); s.pop();
        for(int i : adj[cur]) {
            if(vis[i] && par[cur] != i) return true;
            s.push(i); par[i] = cur; vis[i] = true;
        }
    }
    return false;}

}
```

3.5. Shortest path.

3.5.1. Dijkstra $\mathcal{O}(|E| \log |V|)$.

```
int *dist, *dad;
struct cmp {
    bool operator()(int a, int b) {
        return dist[a] != dist[b] ? dist[a] < dist[b] : a < b; }
};
pair<int*, int*> dijkstra(int n, int s, vii *adj) {
    dist = new int[n];
    dad = new int[n];
    rep(i,0,n) dist[i] = INF, dad[i] = -1;
    set<int, cmp> pq;
    dist[s] = 0, pq.insert(s);
```

```
while (!pq.empty()) {
    int cur = *pq.begin(); pq.erase(pq.begin());
    rep(i,0,size(adj[cur])) {
        int nxt = adj[cur][i].first;
        ndist = dist[cur] + adj[cur][i].second;
        if (ndist < dist[nxt]) pq.erase(nxt),
            dist[nxt] = ndist, dad[nxt] = cur, pq.insert(nxt);
    }
}
return pair<int*, int*>(dist, dad); }
```

3.5.2. Floyd-Warshall $\mathcal{O}(V^3)$.

```
int n = 100; ll d[MAXN][MAXN];
for (int i = 0; i < n; i++) fill_n(d[i], n, 1e18);
// set direct distances from i to j in d[i][j] (and d[j][i])
for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
        for (int k = 0; k < n; k++)
            d[j][k] = min(d[j][k], d[j][i] + d[i][k]);
```

3.5.3. Bellman Ford $\mathcal{O}(VE)$. This is only useful if there are edges with weight $w_{ij} < 0$ in the graph.

```
vector< pair<pii, ll> > edges; // ((from, to), weight)
vector<ll> dist;
```

// when undirected, add back edges

```
bool bellman_ford(int V, int source) {
    dist.assign(V, 1e18); dist[source] = 0;
```

```
bool updated = true; int loops = 0;
while (updated && loops < n) {
    updated = false;
    for (auto e : edges) {
        int alt = dist[e.x.x] + e.y;
        if (alt < dist[e.x.y]) {
            dist[e.x.y] = alt; updated = true;
        }
    }
}
return loops < n; // loops >= n: negative cycles
}
```

3.5.4. IDA* algorithm.

```
int n, cur[100], pos;
int calch() {
    int h = 0;
    rep(i,0,n) if (cur[i] != 0) h += abs(i - cur[i]);
    return h; }
int dfs(int d, int g, int prev) {
    int h = calch();
    if (g + h > d) return g + h;
    if (h == 0) return 0;
    int mn = INF;
    rep(di,-2,3) {
        if (di == 0) continue;
        int nxt = pos + di;
```

```

    if (nxt == prev) continue;
    if (0 <= nxt && nxt < n) {
        swap(cur[pos], cur[nxt]);
        swap(pos,nxt);
        mn = min(mn, dfs(d, g+1, nxt));
        swap(pos,nxt);
        swap(cur[pos], cur[nxt]); }
    if (mn == 0) break; }
return mn; }
int idastar() {
    rep(i,0,n) if (cur[i] == 0) pos = i;
    int d = calch();
    while (true) {
        int nd = dfs(d, 0, -1);
        if (nd == 0 || nd == INF) return d;
        d = nd; } }

```

3.6. Maximum Flow Algorithms.

3.6.1. Dinic's Algorithm $\mathcal{O}(V^2E)$.

```

struct edge { ll t, r, c, f; };
int S, T, h[MAXN]; vector<edge> g[MAXN];

```

```

void addEdge(int u, int v, ll c) {
    g[u].pb({v, (ll)g[v].size(), c, 0});
    g[v].pb({u, (ll)g[u].size()-1,0,0});
}
void dinicBfs() {
    fill_n(h, MAXN, 0); h[S] = 1;
    queue<int> q; q.push(S);
    while (!q.empty()) {
        int v = q.front(); q.pop();
        for (edge e : g[v]) if (!h[e.t] && e.f < e.c)
            h[e.t] = h[v]+1, q.push(e.t);
    }
}

```

```

ll dinicDFS(int v, ll mf) {
    if (v == T) return mf;
    ll f = 0; bool sat = true;
    for (edge &e : g[v]) {
        if (h[e.t] != h[v] + 1 || e.f >= e.c) continue;
        ll df = dinicDFS(e.t, min(mf - f, e.c - e.f));
        f += df, e.f += df, g[e.t][e.r].f -= df;
        sat &= e.f == e.c; if (mf == f) break;
    }
    if (sat) h[v] = 0;
    return f;
}

```

```

ll dinicMaxFlow(ll f = 0) {
    while (dinicBfs(), h[T]) f += dinicDFS(S, LLINF);
    return f;
}

```

3.6.2. *Min-cost max-flow*. Find the cheapest possible way of sending a certain amount of flow through a flow network.

```

const int maxn = 300;

struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
    g.pb({u, v, 0, c, w});
    g.pb({v, u, 0, 0, -w});
}

void sp(int s, int t) {
    fill_n(D, V, LLINF); D[s] = 0;
    for (int ng = g.size(), _ = V; --_ > 0; ) {
        bool ok = false;
        for (int i = 0; i < ng; i++)
            if (D[g[i].x] != LLINF && g[i].f < g[i].c && D[g[i].x]
                + g[i].w < D[g[i].y]) {
                    D[g[i].y] = D[g[i].x] + g[i].w;
                    par[g[i].y] = i; ok = true;
                }
        if (!ok) break;
    }
}

```

```

void minCostMaxFlow(int s, int t, ll &c, ll &f) {
    for (c = f = 0; sp(s, t), D[t] < LLINF; ) {
        ll df = LLINF, dc = 0;
        for (int v = t, e; e = par[v], v != s; v = g[e].x) df =
            min(df, g[e].c - g[e].f);
        for (int v = t, e; e = par[v], v != s; v = g[e].x) g[e].f
            += df, g[e].f -= df, dc += g[e].w;
        f += df; c += dc * df;
    }
}

```

3.6.3. *Gomory-Hu Tree - All Pairs Maximum Flow*. An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in $\mathcal{O}(|V|^2)$ plus $|V| - 1$ times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is $\mathcal{O}(|V|^3|E|)$. NOTE: Not sure if it works correctly with disconnected graphs.

```

#include "dinic.cpp"
bool same[MAXV];
pair<vii, vvi> construct_gh_tree(flow_network &g) {
    int n = g.n, v;
    vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
    rep(s,1,n) {
        int l = 0, r = 0;
        par[s].second = g.max_flow(s, par[s].first, false);
        memset(d, 0, n * sizeof(int));
        memset(same, 0, n * sizeof(bool));
        d[q[r++] = s] = 1;
        while (l < r) {
            same[v = q[l++]] = true;
            for (int i = g.head[v]; i != -1; i = g.e[i].nxt)
                if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
                    d[q[r++] = g.e[i].v] = 1; }
    }
}

```

```

    rep(i,s+1,n)
        if (par[i].first == par[s].first && same[i])
            par[i].first = s;
    g.reset(); }
rep(i,0,n) {
    int mn = INF, cur = i;
    while (true) {
        cap[cur][i] = mn;
        if (cur == 0) break;
        mn = min(mn, par[cur].second), cur = par[cur].first; }
    }
return make_pair(par, cap); }
int compute_max_flow(int s, int t, const pair<vii, vvi> &gh)
    {
    int cur = INF, at = s;
    while (gh.second[at][t] == -1)
        cur = min(cur, gh.first[at].second),
        at = gh.first[at].first;
    return min(cur, gh.second[at][t]); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.7. Minimal Spanning Tree.

3.7.1. Kruskal $\mathcal{O}(E \log V)$.

```

struct edge { int x, y, w; };
vector<edge> edges;

```

```

ll kruskal(int n) { // n: #vertices
    uf_init(n);
    sort(all(edges), [] (edge a, edge b) -> bool { return a.w <
        b.w; });
    ll ret = 0;
    for (edge e : edges)
        if (uf_find(e.x) != uf_find(e.y))
            ret += e.w, uf_union(e.x, e.y);
    return ret;
}

```

3.8. Topological Sort.

3.8.1. Modified Depth-First Search.

```

void tsort_dfs(int cur, char* color, const vvi& adj,
    stack<int>& res, bool& cyc) {
    color[cur] = 1;
    rep(i,0,size(adj[cur])) {
        int nxt = adj[cur][i];
        if (color[nxt] == 0)
            tsort_dfs(nxt, color, adj, res, cyc);
        else if (color[nxt] == 1)
            cyc = true;
        if (cyc) return; }
    color[cur] = 2;
    res.push(cur); }
vvi tsort(int n, vvi adj, bool& cyc) {
    cyc = false;
    stack<int> S;
}

```

```

vi res;
char* color = new char[n];
memset(color, 0, n);
rep(i,0,n) {
    if (!color[i]) {
        tsort_dfs(i, color, adj, S, cyc);
        if (cyc) return res; } }
while (!S.empty()) res.push_back(S.top()), S.pop();
return res; }

```

3.9. **Euler Path.** Finds an euler path (or circuit) in a directed graph, or reports that none exist.

```

#define MAXV 1000
#define MAXE 5000
vi adj[MAXV];
int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
    int start = -1, end = -1, any = 0, c = 0;
    rep(i,0,n) {
        if (outdeg[i] > 0) any = i;
        if (indeg[i] + 1 == outdeg[i]) start = i, c++;
        else if (indeg[i] == outdeg[i] + 1) end = i, c++;
        else if (indeg[i] != outdeg[i]) return ii(-1,-1); }
    if ((start == -1) != (end == -1) || (c != 2 && c != 0))
        return ii(-1,-1);
    if (start == -1) start = end = any;
    return ii(start, end); }
bool euler_path() {
    ii se = start_end();
    int cur = se.first, at = m + 1;
    if (cur == -1) return false;
    stack<int> s;
    while (true) {
        if (outdeg[cur] == 0) {
            res[--at] = cur;
            if (s.empty()) break;
            cur = s.top(); s.pop();
        } else s.push(cur), cur = adj[cur][--outdeg[cur]]; }
    return at == 0; }

```

And an undirected version, which finds a cycle.

```

multiset<int> adj[1010];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
    if (at == to) return it;
    L.insert(it, at), --it;
    while (!adj[at].empty()) {
        int nxt = *adj[at].begin();
        adj[at].erase(adj[at].find(nxt));
        adj[nxt].erase(adj[nxt].find(at));
        if (to == -1) {
            it = euler(nxt, at, it);
            L.insert(it, at);
            --it;
        } else {
            it = euler(nxt, to, it);

```

```

        to = -1; } }
        return it; }
// euler(0,-1,L.begin())

3.10. Heavy-Light Decomposition.
#include "../data-structures/segment_tree.cpp"
const int ID = 0;
int f(int a, int b) { return a + b; }
struct HLD {
    int n, curhead, curloc;
    vi sz, head, parent, loc;
    vvi adj; segment_tree values;
    HLD(int _n) : n(_n), sz(n, 1), head(n),
        parent(n, -1), loc(n), adj(n) {
        vector<ll> tmp(n, ID); values = segment_tree(tmp); }
    void add_edge(int u, int v) {
        adj[u].push_back(v); adj[v].push_back(u); }
    void update_cost(int u, int v, int c) {
        if (parent[v] == u) swap(u, v); assert(parent[u] == v);
        values.update(loc[u], c); }
    int csz(int u) {
        rep(i,0,size(adj[u])) if (adj[u][i] != parent[u])
            sz[u] += csz(adj[parent[adj[u][i]]] = u[i]);
        return sz[u]; }
    void part(int u) {
        head[u] = curhead; loc[u] = curloc++;
        int best = -1;
        rep(i,0,size(adj[u]))
            if (adj[u][i] != parent[u] &&
                (best == -1 || sz[adj[u][i]] > sz[best]))
                best = adj[u][i];
        if (best != -1) part(best);
        rep(i,0,size(adj[u]))
            if (adj[u][i] != parent[u] && adj[u][i] != best)
                part(curhead = adj[u][i]); }
    void build(int r = 0) {
        curloc = 0, csz(curhead = r), part(r); }
    int lca(int u, int v) {
        vi uat, vat; int res = -1;
        while (u != -1) uat.push_back(u), u = parent[head[u]];
        while (v != -1) vat.push_back(v), v = parent[head[v]];
        u = size(uat) - 1, v = size(vat) - 1;
        while (u >= 0 && v >= 0 && head[uat[u]] == head[vat[v]])
            res = (loc[uat[u]] < loc[vat[v]] ? uat[u] : vat[v]),
            u--, v--;
        return res; }
    int query_upto(int u, int v) { int res = ID;
        while (head[u] != head[v])
            res = f(res, values.query(loc[head[u]], loc[u]).x),
            u = parent[head[u]];
        return f(res, values.query(loc[v] + 1, loc[u]).x); }
    int query(int u, int v) { int l = lca(u, v);
        return f(query_upto(u, l), query_upto(v, l)); } };
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.11. Centroid Decomposition.

```

#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
    path[MAXV][LGMAXV],
    sz[MAXV], seph[MAXV],
    shortest[MAXV];
struct centroid_decomposition {
    int n; vvi adj;
    centroid_decomposition(int _n) : n(_n), adj(n) { }
    void add_edge(int a, int b) {
        adj[a].push_back(b); adj[b].push_back(a); }
    int dfs(int u, int p) {
        sz[u] = 1;
        rep(i,0,size(adj[u]))
            rep(i,0,size(adj[u]))
                if (adj[u][i] != p) sz[u] += dfs(adj[u][i], u);
        return sz[u]; }
    void makepaths(int sep, int u, int p, int len) {
        jmp[u][seph[sep]] = sep, path[u][seph[sep]] = len;
        int bad = -1;
        rep(i,0,size(adj[u])) {
            if (adj[u][i] == p) bad = i;
            else makepaths(sep, adj[u][i], u, len + 1);
        }
        if (p == sep)
            swap(adj[u][bad], adj[u].back()), adj[u].pop_back(); }
    void separate(int h=0, int u=0) {
        dfs(u,-1); int sep = u;
        down: iter(nxt,adj[sep])
            if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2) {
                sep = *nxt; goto down; }
        seph[sep] = h, makepaths(sep, sep, -1, 0);
        rep(i,0,size(adj[sep])) separate(h+1, adj[sep][i]); }
    void paint(int u) {
        rep(h,0,seph[u]+1)
            shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
                path[u][h]); }

    int closest(int u) {
        int mn = INF/2;
        rep(h,0,seph[u]+1)
            mn = min(mn, path[u][h] + shortest[jmp[u][h]]);
        return mn; } };
// vim: cc=60 ts=2 sts=2 sw=2:

3.12. Least Common Ancestors, Binary Jumping.
const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
    rep(i, 0, SZ) BP[i][0] = P[i];
    rep(j, 1, LOGSZ) rep(i, 0, SZ)
        BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
    if (H[a] > H[b]) swap(a, b);
    int dh = H[b] - H[a], j = 0;

```

```

rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
while (BP[a][j] != BP[b][j]) j++;
while (--j >= 0) if (BP[a][j] != BP[b][j])
    a = BP[a][j], b = BP[b][j];
return a == b ? a : P[a];
}

```

3.13. Tarjan's Off-line Lowest Common Ancestors Algorithm.

```

#include "../data-structures/union_find.cpp"
struct tarjan_olca {
    int *ancestor;
    vi *adj, answers;
    vii *queries;
    bool *colored;
    union_find uf;
    tarjan_olca(int n, vi *_adj) : adj(_adj), uf(n) {
        colored = new bool[n];
        ancestor = new int[n];
        queries = new vii[n];
        memset(colored, 0, n);
    }
    void query(int x, int y) {
        queries[x].push_back(ii(y, size(answers)));
        queries[y].push_back(ii(x, size(answers)));
        answers.push_back(-1);
    }
    void process(int u) {
        ancestor[u] = u;
        rep(i,0,size(adj[u])) {
            int v = adj[u][i];
            process(v);
            uf.unite(u,v);
            ancestor[uf.find(u)] = u;
        }
        colored[u] = true;
        rep(i,0,size(queries[u])) {
            int v = queries[u][i].first;
            if (colored[v]) {
                answers[queries[u][i].second] = ancestor[uf.find(v)];
            }
        }
    }
};
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.14. **Minimum Mean Weight Cycle.** Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```

double min_mean_cycle(vector<vector<pair<int,double>>> adj){
    int n = size(adj); double mn = INFINITY;
    vector<vector<double>> arr(n+1, vector<double>(n, mn));
    arr[0][0] = 0;
    rep(k,1,n+1) rep(j,0,n) iter(it,adj[j])
        arr[k][it->first] = min(arr[k][it->first],
                                it->second + arr[k-1][j]);

    rep(k,0,n) {
        double mx = -INFINITY;
        rep(i,0,n) mx = max(mx, (arr[n][i]-arr[k][i])/(n-k));
        mn = min(mn, mx);
    }
    return mn;
}
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.15. **Minimum Arborescence.** Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root r to each vertex. Returns a vector of size n , where the i th element is the edge for the i th vertex. The answer for the root is undefined!

```

#include "../data-structures/union_find.cpp"
struct arborescence {
    int n; union_find uf;
    vector<vector<pair<ii,int>>> adj;
    arborescence(int _n) : n(_n), uf(n), adj(n) {}
    void add_edge(int a, int b, int c) {
        adj[b].push_back(make_pair(ii(a,b),c));
    }
    vii find_min(int r) {
        vi vis(n,-1), mn(n,INF); vii par(n);
        rep(i,0,n) {
            if (uf.find(i) != i) continue;
            int at = i;
            while (at != r && vis[at] == -1) {
                vis[at] = i;
                iter(it,adj[at]) if (it->second < mn[at] &&
                    uf.find(it->first.first) != at)
                    mn[at] = it->second, par[at] = it->first;
                if (par[at] == ii(0,0)) return vii();
                at = uf.find(par[at].first);
            }
            if (at == r || vis[at] != i) continue;
            union_find tmp = uf; vi seq;
            do { seq.push_back(at); at = uf.find(par[at].first); } while (at != seq.front());
            iter(it,seq) uf.unite(*it,seq[0]);
            int c = uf.find(seq[0]);
            vector<pair<ii,int>> nw;
            iter(it,seq) iter(jt,adj[*it])
                nw.push_back(make_pair(jt->first,
                    jt->second - mn[*it]));
            adj[c] = nw;
            vii rest = find_min(r);
            if (size(rest) == 0) return rest;
            ii use = rest[c];
            rest[at = tmp.find(use.second)] = use;
            iter(it,seq) if (*it != at)
                rest[*it] = par[*it];
            return rest;
        }
    }
};
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.16. **Blossom algorithm.** Finds a maximum matching in an arbitrary graph in $O(|V|^4)$ time. Be wary of loop edges.

```

#define MAXV 300
bool marked[MAXV], emarked[MAXV][MAXV];
int S[MAXV];
vi find_augmenting_path(const vector<vi> &adj, const vi &m){
    int n = size(adj), s = 0;
    vi par(n,-1), height(n), root(n,-1), q, a, b;
    memset(marked,0,sizeof(marked));
    memset(emarked,0,sizeof(emarked));
    rep(i,0,n) if (m[i] >= 0) emarked[i][m[i]] = true;

```

```

        else root[i] = i, S[s++] = i;
    while (s) {
        int v = S[--s];
        iter(wt,adj[v]) {
            int w = *wt;
            if (emarked[v][w]) continue;
            if (root[w] == -1) {
                int x = S[s++] = m[w];
                par[w]=v, root[w]=root[v], height[w]=height[v]+1;
                par[x]=w, root[x]=root[w], height[x]=height[w]+1;
            } else if (height[w] % 2 == 0) {
                if (root[v] != root[w]) {
                    while (v != -1) q.push_back(v), v = par[v];
                    reverse(q.begin(), q.end());
                    while (w != -1) q.push_back(w), w = par[w];
                    return q;
                } else {
                    int c = v;
                    while (c != -1) a.push_back(c), c = par[c];
                    c = w;
                    while (c != -1) b.push_back(c), c = par[c];
                    while (!a.empty() && !b.empty() && a.back() == b.back())
                        c = a.back(), a.pop_back(), b.pop_back();
                    memset(marked,0,sizeof(marked));
                    fill(par.begin(), par.end(), 0);
                    iter(it,a) par[*it] = 1; iter(it,b) par[*it] = 1;
                    par[c] = s = 1;
                    rep(i,0,n) root[par[i] = par[i] ? 0 : s++] = i;
                    vector<vi> adj2(s);
                    rep(i,0,n) iter(it,adj[i]) {
                        if (par[*it] == 0) continue;
                        if (par[i] == 0) {
                            if (!marked[par[*it]]) {
                                adj2[par[i]].push_back(par[*it]);
                                adj2[par[*it]].push_back(par[i]);
                                marked[par[*it]] = true;
                            }
                            else adj2[par[i]].push_back(par[*it]);
                        }
                    }
                    vi m2(s, -1);
                    if (m[c] != -1) m2[m2[par[m[c]]] = 0] = par[m[c]];
                    rep(i,0,n) if (par[i] != 0 && m[i] != -1 && par[m[i]] != 0)
                        m2[par[i]] = par[m[i]];
                    vi p = find_augmenting_path(adj2, m2);
                    int t = 0;
                    while (t < size(p) && p[t]) t++;
                    if (t == size(p)) {
                        rep(i,0,size(p)) p[i] = root[p[i]];
                        return p;
                    }
                    if (!p[0] || (m[c] != -1 && p[t+1] != par[m[c]]))
                        reverse(p.begin(), p.end()), t = size(p)-t-1;
                    rep(i,0,t) q.push_back(root[p[i]]);
                    iter(it,adj[root[p[t-1]]) {
                        if (par[*it] != (s = 0)) continue;
                        a.push_back(c), reverse(a.begin(), a.end());
                        iter(jt,b) a.push_back(*jt);
                        while (a[s] != *it) s++;

```

```

    if ((height[*it] & 1) ^ (s < size(a) - size(b)))
        reverse(a.begin(), a.end()), s = size(a)-s-1;
    while(a[s] != c) q.push_back(a[s]), s = (s+1)%size(a);
    q.push_back(c);
    rep(i,t+1,size(p)) q.push_back(root[p[i]]);
    return q; } } }
    emarked[v][w] = emarked[w][v] = true; }
    marked[v] = true; } return q; }
vii max_matching(const vector<vi> &adj) {
    vi m(size(adj), -1), ap; vii res, es;
    rep(i,0,size(adj)) iter(it,adj[i]) es.emplace_back(i,*it);
    random_shuffle(es.begin(), es.end());
    iter(it,es) if (m[it->first] == -1 && m[it->second] == -1)
        m[it->first] = it->second, m[it->second] = it->first;
    do { ap = find_augmenting_path(adj, m);
        rep(i,0,size(ap)) m[m[ap[i^1]]] = ap[i] = ap[i^1];
    } while (!ap.empty());
    rep(i,0,size(m)) if (i < m[i]) res.emplace_back(i, m[i]);
    return res; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.17. **Maximum Density Subgraph.** Given (weighted) undirected graph G . Binary search density. If g is current density, construct flow network: (S, u, m) , $(u, T, m + 2g - d_u)$, $(u, v, 1)$, where m is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty S -component, then maximum density is smaller than g , otherwise it's larger. Distance between valid densities is at least $1/(n(n-1))$. Edge case when density is 0. This also works for weighted graphs by replacing d_u by the weighted degree, and doing more iterations (if weights are not integers).

3.18. **Maximum-Weight Closure.** Given a vertex-weighted directed graph G . Turn the graph into a flow network, adding weight ∞ to each edge. Add vertices S, T . For each vertex v of weight w , add edge (S, v, w) if $w \geq 0$, or edge $(v, T, -w)$ if $w < 0$. Sum of positive weights minus minimum $S - T$ cut is the answer. Vertices reachable from S are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

3.19. **Maximum Weighted Independent Set in a Bipartite Graph.** This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges $(S, u, w(u))$ for $u \in L$, $(v, T, w(v))$ for $v \in R$ and (u, v, ∞) for $(u, v) \in E$. The minimum S, T -cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

3.20. **Synchronizing word problem.** A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

4. STRING ALGORITHMS

4.1. Trie.

```
const int SIGMA = 26;
```

```
struct trie {
    bool word; trie **adj;
```

```

    trie() : word(false), adj(new trie*[SIGMA]) {
        for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
    }

```

```

void addWord(const string &str) {
    trie *cur = this;
    for (char ch : str) {
        int i = ch - 'a';
        if (!cur->adj[i]) cur->adj[i] = new trie();
        cur = cur->adj[i];
    }
    cur->word = true;
}

```

```

bool isWord(const string &str) {
    trie *cur = this;
    for (char ch : str) {
        int i = ch - 'a';
        if (!cur->adj[i]) return false;
        cur = cur->adj[i];
    }
    return cur->word;
}
};

```

4.2. Z-algorithm $\mathcal{O}(n)$.

// $z[i]$ = length of longest substring starting from $s[i]$
 \hookrightarrow which is also a prefix of s .

```

vi z_function(const string &s) {
    int n = (int) s.length();
    vi z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]]) ++z[i];
        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    }
    return z;
}

```

4.3. **Suffix array** $\mathcal{O}(n \log^2 n)$. This creates an array $P[0], P[1], \dots, P[n-1]$ such that the suffix $S[i \dots n]$ is the $P[i]^{th}$ suffix of S when lexicographically sorted.

```
typedef pair<pii, int> tii;
```

```
const int maxlogn = 17, int maxn = 1 << maxlogn;
```

```
tii make_triple(int a, int b, int c) { return tii(pii(a, b),
    c); }
```

```
int p[maxlogn + 1][maxn]; tii L[maxn];
```

```

int suffixArray(string S) {
    int N = S.size(), stp = 1, cnt = 1;
    for (int i = 0; i < N; i++) p[0][i] = S[i];
    for (; cnt < N; stp++, cnt <= 1) {

```

```

        for (int i = 0; i < N; i++)
            L[i] = tii(pii(p[stp-1][i], i + cnt < N ? p[stp-1][i +
    cnt] : -1), i);
        sort(L, L + N);
        for (int i = 0; i < N; i++)
            p[stp][L[i].y] = i > 0 && L[i].x == L[i-1].x ?
    p[stp][L[i-1].y] : i;
    }
    return stp - 1; // result is in p[stp - 1][0 .. (N - 1)]
}

```

4.4. **Longest Common Subsequence** $\mathcal{O}(n^2)$. SUBSTRING: consecutive characters !!!

```
int dp[STR_SIZE][STR_SIZE]; // DP problem
```

```

int lcs(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i < n1; i++) {
        for (int j = 0; j < n2; j++) {
            if (i == 0 || j == 0) dp[i][j] = 0;
            else if (w1[i - 1] == w2[j - 1]) dp[i][j] = dp[i - 1][j
    - 1] + 1;
            else dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);
        }
    }
    return dp[n1][n2];
}

```

// backtrace

```

string getLCS(const string &w1, const string &w2) {
    int i = w1.size(), j = w2.size(); string ret = "";
    while (i > 0 && j > 0) {
        if (w1[i - 1] == w2[j - 1]) ret += w1[--i], j--;
        else if (dp[i][j - 1] > dp[i - 1][j]) j--;
        else i--;
    }
    reverse(ret.begin(), ret.end());
    return ret;
}

```

4.5. **Levenshtein Distance** $\mathcal{O}(n^2)$. Also known as the 'Edit distance'.

```
int dp[MAX_SIZE][MAX_SIZE]; // DP problem
```

```

int levDist(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i <= n1; i++) dp[i][0] = i; // removal
    for (int j = 0; j <= n2; j++) dp[0][j] = j; // insertion
    for (int i = 1; i <= n1; i++)
        for (int j = 1; j <= n2; j++)
            dp[i][j] = min(
                1 + min(dp[i - 1][j], dp[i][j - 1]),
                dp[i - 1][j - 1] + (w1[i - 1] != w2[j - 1])
            );
    return dp[n1][n2];
}

```


4.6. Knuth-Morris-Pratt algorithm $\mathcal{O}(N + M)$.

```
int kmp_search(const string &word, const string &text) {
    int n = word.size();
    vi T(n + 1, 0);
    for (int i = 1, j = 0; i < n; ) {
        if (word[i] == word[j]) T[++i] = ++j; // match
        else if (j > 0) j = T[j]; // fallback
        else i++; // no match, keep zero
    }
    int matches = 0;
    for (int i = 0, j = 0; i < text.size(); ) {
        if (text[i] == word[j]) {
            i++;
            if (++j == n) { // match at interval [i - n, i)
                matches++; j = T[j];
            }
        } else if (j > 0) j = T[j];
        else i++;
    }
    return matches;
}
```

4.7. Aho-Corasick Algorithm $\mathcal{O}(N + \sum_{i=1}^m |S_i|)$. All given P must be unique!

```
const int MAXP = 100, MAXLEN = 200, SIGMA = 26, MAXTRIE =
    MAXP * MAXLEN;

int nP;
string P[MAXP], S;

int pnr[MAXTRIE], to[MAXTRIE][SIGMA], sLink[MAXTRIE],
    dLink[MAXTRIE], nnodes;

void ahoCorasick() {
    fill_n(pnr, MAXTRIE, -1);
    for (int i = 0; i < MAXTRIE; i++) fill_n(to[i], SIGMA, 0);
    fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE, 0);
    nnodes = 1;
    // STEP 1: MAKE A TREE
    for (int i = 0; i < nP; i++) {
        int cur = 0;
        for (char c : P[i]) {
            int i = c - 'a';
            if (to[cur][i] == 0) to[cur][i] = nnodes++;
            cur = to[cur][i];
        }
        pnr[cur] = i;
    }
    // STEP 2: CREATE SUFFIX_LINKS AND DICT_LINKS
    queue<int> q; q.push(0);
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (int c = 0; c < SIGMA; c++) {
            if (to[cur][c]) {
                int sl = sLink[to[cur][c]] = cur == 0 ? 0 :
                    to[sLink[cur]][c];
            }
        }
    }
}
```

```
// if all strings have equal length, remove this:
dLink[to[cur][c]] = pnr[sl] >= 0 ? sl : dLink[sl];
q.push(to[cur][c]);
} else to[cur][c] = to[sLink[cur]][c];
}
}
// STEP 3: TRAVERSE S
for (int cur = 0, i = 0, n = S.size(); i < n; i++) {
    cur = to[cur][S[i] - 'a'];
    for (int hit = pnr[cur] >= 0 ? cur : dLink[cur]; hit; hit
        = dLink[hit]) {
        cerr << P[pnr[hit]] << " found at [" << (i + 1 -
            P[pnr[hit]].size()) << ", " << i << "]" << endl;
    }
}
}
```

4.8. **eerTree**. Constructs an eerTree in $\mathcal{O}(n)$, one character at a time.

```
#define MAXN 100100
#define SIGMA 26
#define BASE 'a'
char *s = new char[MAXN];
struct state {
    int len, link, to[SIGMA];
} *st = new state[MAXN+2];
struct eertree {
    int last, sz, n;
    eertree() : last(1), sz(2), n(0) {
        st[0].len = st[0].link = -1;
        st[1].len = st[1].link = 0;
    }
    int extend() {
        char c = s[n++]; int p = last;
        while (n - st[p].len - 2 < 0 || c != s[n - st[p].len -
            2])
            p = st[p].link;
        if (!st[p].to[c-BASE]) {
            int q = last = sz++;
            st[p].to[c-BASE] = q;
            st[q].len = st[p].len + 2;
            do { p = st[p].link;
            } while (p != -1 && (n < st[p].len + 2 ||
                c != s[n - st[p].len - 2]));
            if (p == -1) st[q].link = 1;
            else st[q].link = st[p].to[c-BASE];
            return 1;
        }
        last = st[p].to[c-BASE];
        return 0;
    }
};
// vim: cc=60 ts=2 sts=2 sw=2:
```

4.9. **Suffix Automaton**. Minimum automata that accepts all suffixes of a string with $\mathcal{O}(n)$ construction. The automata itself is a DAG therefore suitable for DP, examples are counting unique substrings, occurrences of substrings and suffix.

```
// TODO: Add longest common subsring
const int MAXL = 100000;
struct suffix_automaton {
```

```
vi len, link, occur, cnt;
vector<map<char,int>> next;
vector<bool> isclone;
ll *occuratleast;
int sz, last;
string s;
suffix_automaton() : len(MAXL*2), link(MAXL*2),
    occur(MAXL*2), next(MAXL*2), isclone(MAXL*2) { clear(); }
void clear() { sz = 1; last = len[0] = 0; link[0] = -1;
    next[0].clear(); isclone[0] = false; }
bool issubstr(string other){
    for(int i = 0, cur = 0; i < size(other); ++i){
        if(cur == -1) return false; cur = next[cur][other[i]];
    }
    return true;
}
void extend(char c){ int cur = sz++; len[cur] =
    len[last]+1;
    next[cur].clear(); isclone[cur] = false; int p = last;
    for(; p != -1 && !next[p].count(c); p = link[p])
        next[p][c] = cur;
    if(p == -1){ link[cur] = 0; }
    else{ int q = next[p][c];
        if(len[p] + 1 == len[q]){ link[cur] = q; }
        else { int clone = sz++; isclone[clone] = true;
            len[clone] = len[p] + 1;
            link[clone] = link[q]; next[clone] = next[q];
            for(; p != -1 && next[p].count(c) && next[p][c] == q;
                p = link[p]){
                next[p][c] = clone;
                link[q] = link[cur] = clone;
            } } last = cur; }
    void count(){
        cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
        map<char,int>::iterator i;
        while(!S.empty()){
            ii cur = S.top(); S.pop();
            if(cur.second){
                for(i = next[cur.first].begin();
                    i != next[cur.first].end(); ++i){
                    cnt[cur.first] += cnt[(*i).second]; } }
            else if(cnt[cur.first] == -1){
                cnt[cur.first] = 1; S.push(ii(cur.first, 1));
                for(i = next[cur.first].begin();
                    i != next[cur.first].end(); ++i){
                    S.push(ii((*i).second, 0)); } } } }
        string lexicok(ll k){
            int st = 0; string res; map<char,int>::iterator i;
            while(k){
                for(i = next[st].begin(); i != next[st].end(); ++i){
                    if(k <= cnt[(*i).second]){ st = (*i).second;
                        res.push_back((*i).first); k--; break;
                    } else { k -= cnt[(*i).second]; } } }
            return res;
        }
        void countoccur(){
            for(int i = 0; i < sz; ++i){ occur[i] = 1 - isclone[i]; }
            vii states(sz);
```

```

    for(int i = 0; i < sz; ++i){ states[i] = ii(len[i],i); }
    sort(states.begin(), states.end());
    for(int i = size(states)-1; i >= 0; --i){
        int v = states[i].second;
        if(link[v] != -1) { occur[link[v]] += occur[v]; } } } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

4.10. **Hashing.** Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```

struct hasher { int b = 311, m; vi h, p;
    hasher(string s, int _m)
        : m(_m), h(size(s)+1), p(size(s)+1) {
        p[0] = 1; h[0] = 0;
        rep(i,0,size(s)) p[i+1] = (ll)p[i] * b % m;
        rep(i,0,size(s)) h[i+1] = ((ll)h[i] * b + s[i]) % m; }
    int hash(int l, int r) {
        return (h[r+1] + m - (ll)h[l] * p[r-l+1] % m) % m; } } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

5. GEOMETRY

```
const double EPS = 1e-7, PI = acos(-1.0);
```

```

typedef long long NUM; // EITHER double OR long long
typedef pair<NUM, NUM> pt;
typedef pt frac;
typedef pair<double, double> vec;
struct line { NUM a, b, c; } // ax + by + c = 0

```

```

vec getvec(pt p, pt dp, frac t) {
    return vec(p.x + 1. * dp.x * t.x / t.y, p.y + 1. * dp.y *
    ↪ t.x / t.y);
}

```

```

pt operator+(pt p, pt q) { return pt(p.x + q.x, p.y + q.y); }
pt operator-(pt p, pt q) { return pt(p.x - q.x, p.y - q.y); }

```

```

pt& operator+=(pt &p, pt q) { return p = p + q; }
pt& operator-=(pt &p, pt q) { return p = p - q; }

```

```

pt operator*(pt p, NUM l) { return pt(p.x * l, p.y * l); }
pt operator/(pt p, NUM l) { return pt(p.x / l, p.y / l); }

```

```

NUM operator*(pt p, pt q) { return p.x * q.x + p.y * q.y; }
NUM operator^(pt p, pt q) { return p.x * q.y - p.y * q.x; }

```

```

istream& operator>>(istream &in, pt &p) { return in >> p.x >>
    ↪ p.y; }
ostream& operator<<(ostream &out, pt p) { return out << '('
    ↪ << p.x << ", " << p.y << ')'; }

```

```

NUM lenSq(pt p) { return p * p; }
NUM lenSq(pt p, pt q) { return lenSq(p - q); }
double len(pt p) { return sqrt(p * p); }
double len(pt p, pt q) { return len(p - q); }

```

```
double angle(pt a, pt o, pt b) {
```

```

    return acos(a - o) * (b - o) / len(a - o) / len(b - o));
}

```

```

pt rotate(pt p, double rad) {
    return pt(p.x * cos(rad) - p.y * sin(rad),
    ↪ p.x * sin(rad) + p.y * cos(rad));
}

```

```

// ----- LINES -----
// square distance from pt a to line bc
frac distPtLineSq(pt a, pt b, pt c) {
    a -= b, c -= b;
    return frac((a ^ c) * (a ^ c), c * c);
}

```

```

// square distance from pt a to line segment bc
frac distPtSegmentSq(pt a, pt b, pt c) {
    a -= b; c -= b;
    NUM dot = a * c, len = c * c;
    if (dot <= 0) return frac(a * a, 1);
    if (dot >= len) return frac((a - c) * (a - c), 1);
    return frac(a * a * len - dot * dot, len);
}

```

```

// projects pt a onto line segment bc
frac proj(pt a, pt b, pt c) { return frac((a - b) * (c - b),
    ↪ (c - b) * (c - b)); }
vec projv(pt a, pt b, pt c) { return getvec(b, c - b, proj(a,
    ↪ b, c)); }

```

```

bool collinear(pt a, pt b, pt c) { return ((a - b) ^ (a - c))
    ↪ == 0; }

```

```

bool pointOnSegment(pt a, pt b, pt c) {
    NUM dot = (a - b) * (c - b), len = (c - b) * (c - b);
    return collinear(a, b, c) && 0 <= dot && dot <= len;
}

```

```

// true => 1 intersection, false => parallel, so 0 or \infty
    ↪ solutions
bool linesIntersect(pt a, pt b, pt c, pt d) { return ((a - b)
    ↪ ^ (c - d)) != 0; }
vec lineLineIntersection(pt a, pt b, pt c, pt d) {
    double det = (a - b) ^ (c - d); pt ret = (c - d) * (a ^ b)
    ↪ - (a - b) * (c ^ d);
    return vec(ret.x / det, ret.y / det);
}

```

```

// dp, dq are directions from p, q
// intersection at p + t_i dp, for 0 <= i < return value
int segmentIntersection(pt p, pt dp, pt q, pt dq, frac &t0,
    ↪ frac &t1){
    if (dp * dp == 0) swap(p, q), swap(dp, dq); // dq = 0
    if (dp * dp == 0) { t0 = t1 = frac(0, 1); return p == q; }
    ↪ // dp = dq = 0

```

```

    pt dpq = (q - p); NUM c = dp ^ dq, c0 = dpq ^ dp, c1 = dpq
    ↪ ^ dq;
    if (c == 0) { // parallel, dp > 0, dq >= 0
        if (c0 != 0) return 0; // not collinear
        NUM v0 = dpq * dp, v1 = v0 + dq * dp, dp2 = dp * dp;
        if (v1 < v0) swap(v0, v1);
        t0 = frac(v0 = max(v0, (NUM) 0), dp2);
        t1 = frac(v1 = min(v1, dp2), dp2);
        return (v0 <= v1) + (v0 < v1);
    } else if (c < 0) c = -c, c0 = -c0, c1 = -c1;
    t0 = t1 = frac(c1, c);
    return 0 <= min(c0, c1) && max(c0, c1) <= c;
}

```

```

// Returns TWICE the area of a polygon to keep it an integer
NUM polygonTwiceArea(const vector<pt> &pts) {
    NUM area = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j = i++)
        area += pts[i] ^ pts[j];
    return abs(area); // area < 0 <=> pts ccw
}

```

```

bool pointInPolygon(pt p, const vector<pt> &pts) {
    double sum = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j = i++)
    ↪ {
        if (pointOnSegment(p, pts[i], pts[j])) return true; //
    ↪ boundary
        double angle = acos((pts[i] - p) * (pts[j] - p) /
    ↪ len(pts[i], p) / len(pts[j], p));
        sum += ((pts[i] - p) ^ (pts[j] - p)) < 0 ? angle :
    ↪ -angle;
        return abs(abs(sum) - 2 * PI) < EPS;
    }
}

```

5.1. Convex Hull $\mathcal{O}(n \log n)$.

```

// the convex hull consists of: { pts[ret[0]], pts[ret[1]],
    ↪ ... pts[ret.back()] }
vi convexHull(const vector<pt> &pts) {
    if (pts.empty()) return vi();
    vi ret, ord;
    int n = pts.size(), st = min_element(all(pts)) -
    ↪ pts.begin();
    rep(i, 0, n)
        if (pts[i] != pts[st]) ord.pb(i);
    sort(all(ord), [&pts,&st] (int a, int b) {
        pt p = pts[a] - pts[st], q = pts[b] - pts[st];
        return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) < lenSq(q);
    });
    ret.pb(st);
    for (int i : ord) {
        // use '>' to include ALL points on the hull-line
        for (int s = ret.size() - 1; s > 0 && ((pts[ret[s-1]] -
    ↪ pts[ret[s]]) ^ (pts[i] - pts[ret[s]])) >= 0; s--)

```

```

    ret.pop_back();
    ret.pb(i);
}
return ret;
}

5.2. Rotating Calipers  $\mathcal{O}(n)$ . Finds the longest distance between two
points in a convex hull.

NUM rotatingCalipers(vector<pt> &hull) {
    int n = hull.size(), a = 0, b = 1;
    if (n <= 1) return 0.0;
    while (((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
    hull[b])) > 0) b++;
    NUM ret = 0.0;
    while (a < n) {
        ret = max(ret, lenSq(hull[a], hull[b]));
        if (((hull[(a + 1) % n] - hull[a % n]) ^ (hull[(b + 1) %
    n] - hull[b])) <= 0) a++;
        else if (++b == n) b = 0;
    }
    return ret;
}

5.3. Closest points  $\mathcal{O}(n \log n)$ .

int n; pt pts[maxn];

struct byY {
    bool operator()(int a, int b) const { return pts[a].y <
    pts[b].y; }
};

inline NUM dist(pii p) {
    return hypot(pts[p.x].x - pts[p.y].x, pts[p.x].y -
    pts[p.y].y);
}

pii minpt(pii p1, pii p2) { return (dist(p1) < dist(p2)) ? p1
    : p2; }

// closest pts (by index) inside pts[l ... r], with sorted y
// values in ys
pii closest(int l, int r, vi &ys) {
    if (r - l == 2) { // don't assume 1 here.
        ys = { l, l + 1 };
        return pii(l, l + 1);
    } else if (r - l == 3) { // brute-force
        ys = { l, l + 1, l + 2 };
        sort(ys.begin(), ys.end(), byY());
        return minpt(pii(l, l + 1), minpt(pii(l, l + 2), pii(l +
    1, l + 2)));
    }
    int m = (l + r) / 2; vi yl, yr;
    pii delta = minpt(closest(l, m, yl), closest(m, r, yr));
    NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
    pts[m].x);

```

```

    merge(yl.begin(), yl.end(), yr.begin(), yr.end(),
    back_inserter(ys), byY());
    deque<int> q;
    for (int i : ys) {
        if (abs(pts[i].x - xm) <= ddelta) {
            for (int j : q) delta = minpt(delta, pii(i, j));
            q.pb(i);
            if (q.size() > 8) q.pop_front(); // magic from
    Introduction to Algorithms.
        }
    }
    return delta;
}

5.4. Great-Circle Distance. Computes the distance between two
points (given as latitude/longitude coordinates) on a sphere of radius
r.

double gc_distance(double pLat, double pLong,
    double qLat, double qLong, double r) {
    pLat *= pi / 180; pLong *= pi / 180;
    qLat *= pi / 180; qLong *= pi / 180;
    return r * acos(cos(pLat) * cos(qLat) * cos(pLong - qLong)
    +
        sin(pLat) * sin(qLat)); }
// vim: cc=60 ts=2 sts=2 sw=2:

5.5. 3D Primitives.

#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)
#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
    double x, y, z;
    point3d() : x(0), y(0), z(0) {}
    point3d(double _x, double _y, double _z)
        : x(_x), y(_y), z(_z) {}
    point3d operator+(P(p)) const {
        return point3d(x + p.x, y + p.y, z + p.z); }
    point3d operator-(P(p)) const {
        return point3d(x - p.x, y - p.y, z - p.z); }
    point3d operator-() const {
        return point3d(-x, -y, -z); }
    point3d operator*(double k) const {
        return point3d(x * k, y * k, z * k); }
    point3d operator/(double k) const {
        return point3d(x / k, y / k, z / k); }
    double operator%(P(p)) const {
        return x * p.x + y * p.y + z * p.z; }
    point3d operator*(P(p)) const {
        return point3d(y*p.z - z*p.y,
            z*p.x - x*p.z, x*p.y - y*p.x); }
    double length() const {
        return sqrt(*this % *this); }
    double distTo(P(p)) const {
        return (*this - p).length(); }
    double distTo(P(A), P(B)) const {
        // A and B must be two different points

```

```

        return ((*this - A) * (*this - B)).length() /
    A.distTo(B); }
    point3d normalize(double k = 1) const {
        // length() must not return 0
        return (*this) * (k / length()); }
    point3d getProjection(P(A), P(B)) const {
        point3d v = B - A;
        return A + v.normalize((v % (*this - A)) / v.length()); }
    point3d rotate(P(normal)) const {
        // normal must have length 1 and be orthogonal to the
    vector
        return (*this) * normal; }
    point3d rotate(double alpha, P(normal)) const {
        return (*this) * cos(alpha) + rotate(normal) *
    sin(alpha); }
    point3d rotatePoint(P(O), P(axe), double alpha) const {
        point3d Z = axe.normalize(axe % (*this - O));
        return O + Z + (*this - O - Z).rotate(alpha, O); }
    bool isZero() const {
        return abs(x) < EPS && abs(y) < EPS && abs(z) < EPS; }
    bool isOnLine(L(A, B)) const {
        return ((A - *this) * (B - *this)).isZero(); }
    bool isInSegment(L(A, B)) const {
        return isOnLine(A, B) && ((A - *this) % (B -
    *this)) < EPS; }
    bool isInSegmentStrictly(L(A, B)) const {
        return isOnLine(A, B) && ((A - *this) % (B -
    *this)) < -EPS; }
    double getAngle() const {
        return atan2(y, x); }
    double getAngle(P(u)) const {
        return atan2((*this * u).length(), *this % u); }
    bool isOnPlane(PL(A, B, C)) const {
        return
            abs((A - *this) * (B - *this) % (C - *this)) < EPS; }
    };
    int line_line_intersect(L(A, B), L(C, D), point3d &O) {
        if (abs((B - A) * (C - A) % (D - A)) > EPS) return 0;
        if (((A - B) * (C - D)).length() < EPS)
            return A.isOnLine(C, D) ? 2 : 0;
        point3d normal = ((A - B) * (C - B)).normalize();
        double s1 = (C - A) * (D - A) % normal;
        O = A + ((B - A) / (s1 + ((D - B) * (C - B) % normal))) *
    s1;
        return 1; }
    int line_plane_intersect(L(A, B), PL(C, D, E), point3d &O) {
        double V1 = (C - A) * (D - A) % (E - A);
        double V2 = (D - B) * (C - B) % (E - B);
        if (abs(V1 + V2) < EPS)
            return A.isOnPlane(C, D, E) ? 2 : 0;
        O = A + ((B - A) / (V1 + V2)) * V1;
        return 1; }
    bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
        point3d &P, point3d &Q) {
        point3d n = nA * nB;

```

```

    if (n.isZero()) return false;
    point3d v = n * nA;
    P = A + (n * nA) * ((B - A) % nB / (v % nB));
    Q = P + n;
    return true; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

5.6. Polygon Centroid.

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

5.7. Rectilinear Minimum Spanning Tree. Given a set of n points in the plane, and the aim is to find a minimum spanning tree connecting these n points, assuming the Manhattan distance is used. The function `candidates` returns at most $4n$ edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```
#define MAXN 100100
```

```

struct RMST {
    struct point {
        int i; ll x, y;
        point() : i(-1) { }
        ll d1() { return x + y; }
        ll d2() { return x - y; }
        ll dist(point other) {
            return abs(x - other.x) + abs(y - other.y); }
        bool operator <(const point &other) const {
            return y == other.y ? x > other.x : y < other.y; }
    } best[MAXN], arr[MAXN], tmp[MAXN];
    int n;
    RMST() : n(0) { }
    void add_point(int x, int y) {
        arr[arr[n].i = n].x = x, arr[n++].y = y; }
    void rec(int l, int r) {
        if (l >= r) return;
        int m = (l+r)/2;
        rec(l,m), rec(m+1,r);
        point bst;
        for (int i = l, j = m+1, k = l; i <= m || j <= r; k++) {
            if (j > r || (i <= m && arr[i].d1() < arr[j].d1())) {
                tmp[k] = arr[i++];
                if (bst.i != -1 && (best[tmp[k].i].i == -1
                    || best[tmp[k].i].d2() < bst.d2()))
                    best[tmp[k].i] = bst;
            } else {
                tmp[k] = arr[j++];
                if (bst.i == -1 || bst.d2() < tmp[k].d2())
                    bst = tmp[k]; }
            rep(i,l,r+1) arr[i] = tmp[i]; }
    vector<pair<ll,ii>> candidates() {
        vector<pair<ll, ii>> es;

```

```

        rep(p,0,2) {
            rep(q,0,2) {
                sort(arr, arr+n);
                rep(i,0,n) best[i].i = -1;
                rec(0,n-1);
                rep(i,0,n) {
                    if(best[arr[i].i].i != -1)
                        es.push_back({arr[i].dist(best[arr[i].i]),
                            {arr[i].i, best[arr[i].i].i}});
                    swap(arr[i].x, arr[i].y);
                    arr[i].x *= -1, arr[i].y *= -1; } }
                rep(i,0,n) arr[i].x *= -1; }
            return es; } } };
// vim: cc=60 ts=2 sts=2 sw=2:

```

5.8. Formulas. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two-dimensional vectors.

- $a \cdot b = |a||b| \cos \theta$, where θ is the angle between a and b .
- $a \times b = |a||b| \sin \theta$, where θ is the signed angle between a and b .
- $a \times b$ is equal to the area of the parallelogram with two of its sides formed by a and b . Half of that is the area of the triangle formed by a and b .
- **Euler's formula:** $V - E + F = 2$
- Side lengths a, b, c can form a triangle iff. $a + b > c$, $b + c > a$ and $a + c > b$.
- Sum of internal angles of a regular convex n -gon is $(n - 2)\pi$.
- **Law of sines:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:** $b^2 = a^2 + c^2 - 2ac \cos B$
- Internal tangents of circles $(c_1, r_1), (c_2, r_2)$ intersect at $(c_1 r_2 + c_2 r_1) / (r_1 + r_2)$, external intersect at $(c_1 r_2 - c_2 r_1) / (r_1 + r_2)$.

6. MISCELLANEOUS

6.1. Binary search $\mathcal{O}(\log(hi - lo))$.

```

bool test(int n);

int search(int lo, int hi) {
    // assert(test(lo) && !test(hi));
    while (hi - lo > 1) {
        int m = (lo + hi) / 2;
        (test(m) ? lo : hi) = m;
    }
    // assert(test(lo) && !test(hi));
    return lo;
}

```

6.2. Fast Fourier Transform $\mathcal{O}(n \log n)$. Given two polynomials $A(x) = a_0 + a_1x + \dots + a_{n/2}x^{n/2}$ and $B(x) = b_0 + b_1x + \dots + b_{n/2}x^{n/2}$, FFT calculates all coefficients of $C(x) = A(x) \cdot B(x) = c_0 + c_1x + \dots + c_nx^n$, with $c_i = \sum_{j=0}^i a_j b_{i-j}$.

```

typedef complex<double> cpx;
const int logmaxn = 20, maxn = 1 << logmaxn;

cpx a[maxn] = {}, b[maxn] = {}, c[maxn];

void fft(cpx *src, cpx *dest) {
    for (int i = 0, rep = 0; i < maxn; i++, rep = 0) {

```

```

        for (int j = i, k = logmaxn; k-- > 1) rep = (rep <<
            ↪ 1) | (j & 1);
        dest[rep] = src[i];
    }
    for (int s = 1, m = 1; m <= maxn; s++, m *= 2) {
        cpx r = exp(cpx(0, 2.0 * PI / m));
        for (int k = 0; k < maxn; k += m) {
            cpx cr(1.0, 0.0);
            for (int j = 0; j < m / 2; j++) {
                cpx t = cr * dest[k + j + m / 2]; dest[k + j + m / 2]
            ↪ = dest[k + j] - t;
                dest[k + j] += t; cr *= r;
            }
        }
    }
}

```

```

void multiply() {
    fft(a, c); fft(b, a);
    for (int i = 0; i < maxn; i++) b[i] = conj(a[i] * c[i]);
    fft(b, c);
    for (int i = 0; i < maxn; i++) c[i] = conj(c[i]) / (1.0 *
        ↪ maxn);
}

```

6.3. Minimum Assignment (Hungarian Algorithm) $\mathcal{O}(n^3)$.

```
int a[MAXN + 1][MAXN + 1]; // matrix, 1-based
```

```
int minimum_assignment(int n, int m) { // n rows, m columns
    vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);

```

```

    for (int i = 1; i <= n; i++) {
        p[0] = i;
        int j0 = 0;
        vi minv(m + 1, INF);
        vector<char> used(m + 1, false);
        do {
            used[j0] = true;
            int i0 = p[j0], delta = INF, j1;
            for (int j = 1; j <= m; j++)
                if (!used[j]) {
                    int cur = a[i0][j] - u[i0] - v[j];
                    if (cur < minv[j]) minv[j] = cur, way[j] = j0;
                    if (minv[j] < delta) delta = minv[j], j1 = j;
                }
            for (int j = 0; j <= m; j++) {
                if (used[j]) u[p[j]] += delta, v[j] -= delta;
                else minv[j] -= delta;
            }
            j0 = j1;
        } while (p[j0] != 0);
        do {
            int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
        } while (j0);
    }
}

```

```
// column j is assigned to row p[j]
// for (int j = 1; j <= m; ++ j) ans[p[j]] = j;
return -v[0];
}
```

6.4. Partial linear equation solver $\mathcal{O}(N^3)$.

```
typedef double NUM;
const int MAXROWS = 200, MAXCOLS = 200;
const NUM EPS = 1e-5;

// F2: bitset<MAXCOLS+1> mat[MAXROWS]; bitset<MAXROWS> vals;
NUM mat[MAXROWS][MAXCOLS + 1], vals[MAXCOLS]; bool
↪ hasval[MAXCOLS];
bool is0(NUM a) { return -EPS < a && a < EPS; }
```

```
// finds x such that Ax = b
// A_ij is mat[i][j], b_i is mat[i][m]
int solvemat(int n, int m) {
    // F2: vals.reset();
    int pr = 0, pc = 0;
    while (pc < m) {
        int r = pr, c;
        while (r < n && is0(mat[r][pc])) r++;
        if (r == n) { pc++; continue; }

        // F2: mat[pr] ^= mat[r]; mat[r] ^= mat[pr]; mat[pr] ^=
↪ mat[r];
        for (c = 0; c <= m; c++) swap(mat[pr][c], mat[r][c]);

        r = pr++; c = pc++;
        // F2: vals.set(pc, mat[pr][m]);
        NUM div = mat[r][c];
        for (int col = c; col <= m; col++) mat[r][col] /= div;
        for (int row = 0; row < n; row++) {
            if (row == r) continue;
            // F2: if (mat[row].test(c)) mat[row] ^= mat[r];
            NUM times = -mat[row][c];
            for (int col = c; col <= m; col++) mat[row][col] +=
↪ times * mat[r][col];
        }
    } // now mat is in RREF
```

```
for (int r = pr; r < n; r++)
    if (!is0(mat[r][n])) return 0;
// F2: return 1;
fill_n(hasval, n, false);
for (int col = 0, row; col < n; col++) {
    hasval[col] = !is0(mat[row][col]);
    if (!hasval[col]) continue;
    for (int c = col + 1; c < n; c++) {
        if (!is0(mat[row][c])) hasval[col] = false;
    }
    if (hasval[col]) vals[col] = mat[row][n];
    row++;
}
```

```
for (int i = 0; i < n; i++)
    if (!hasval[i]) return 2;
return 1;
}
```

6.5. Cycle-Finding.

```
ii find_cycle(int x0, int (*f)(int)) {
    int t = f(x0), h = f(t), mu = 0, lam = 1;
    while (t != h) t = f(t), h = f(f(h));
    h = x0;
    while (t != h) t = f(t), h = f(h), mu++;
    h = f(t);
    while (t != h) h = f(h), lam++;
    return ii(mu, lam); }
// vim: cc=60 ts=2 sts=2 sw=2:
```

6.6. Longest Increasing Subsequence.

```
vi lis(vi arr) {
    vi seq, back(size(arr)), ans;
    rep(i, 0, size(arr)) {
        int res = 0, lo = 1, hi = size(seq);
        while (lo <= hi) {
            int mid = (lo+hi)/2;
            if (arr[seq[mid-1]] < arr[i]) res = mid, lo = mid + 1;
            else hi = mid - 1; }
        if (res < size(seq)) seq[res] = i;
        else seq.push_back(i);
        back[i] = res == 0 ? -1 : seq[res-1]; }
    int at = seq.back();
    while (at != -1) ans.push_back(at), at = back[at];
    reverse(ans.begin(), ans.end());
    return ans; }
// vim: cc=60 ts=2 sts=2 sw=2:
```

6.7. Dates.

```
int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
    return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
        367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
        3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
        d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
    int x, n, i, j;
    x = jd + 68569;
    n = 4 * x / 146097;
    x -= (146097 * n + 3) / 4;
    i = (4000 * (x + 1)) / 1461001;
    x -= 1461 * i / 4 - 31;
    j = 80 * x / 2447;
    d = x - 2447 * j / 80;
    x = j / 11;
    m = j + 2 - 12 * x;
    y = 100 * (n - 49) + i + x; }
// vim: cc=60 ts=2 sts=2 sw=2:
```

6.8. Simplex.

```
typedef long double DOUBLE;
typedef vector<DOUBLE> VD;
typedef vector<VD> VVD;
typedef vector<int> VI;
const DOUBLE EPS = 1e-9;
struct LPSolver {
    int m, n;
    VI B, N;
    VVD D;
    LPSolver(const VVD &A, const VD &b, const VD &c) :
        m(b.size()), n(c.size()),
        N(n + 1), B(m), D(m + 2, VD(n + 2)) {
        for (int i = 0; i < m; i++) for (int j = 0; j < n; j++)
            D[i][j] = A[i][j];
        for (int i = 0; i < m; i++) { B[i] = n + i; D[i][n] = -1;
            D[i][n + 1] = b[i]; }
        for (int j = 0; j < n; j++) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m + 1][n] = 1; }
    void Pivot(int r, int s) {
        double inv = 1.0 / D[r][s];
        for (int i = 0; i < m + 2; i++) if (i != r)
            for (int j = 0; j < n + 2; j++) if (j != s)
                D[i][j] -= D[r][j] * D[i][s] * inv;
        for (int j = 0; j < n + 2; j++) if (j != s) D[r][j] *= inv;
        for (int i = 0; i < m + 2; i++) if (i != r) D[i][s] *=
↪ -inv;
        D[r][s] = inv;
        swap(B[r], N[s]); }
    bool Simplex(int phase) {
        int x = phase == 1 ? m + 1 : m;
        while (true) {
            int s = -1;
            for (int j = 0; j <= n; j++) {
                if (phase == 2 && N[j] == -1) continue;
                if (s == -1 || D[x][j] < D[x][s] ||
                    D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
            if (D[x][s] > -EPS) return true;
            int r = -1;
            for (int i = 0; i < m; i++) {
                if (D[i][s] < EPS) continue;
                if (r == -1 || D[i][n + 1] / D[i][s] < D[r][n + 1] /
                    D[r][s] || (D[i][n + 1] / D[i][s]) == (D[r][n + 1] /
                    D[r][s]) && B[i] < B[r]) r = i; }
            if (r == -1) return false;
            Pivot(r, s); } }
    DOUBLE Solve(VD &x) {
        int r = 0;
        for (int i = 1; i < m; i++) if (D[i][n + 1] < D[r][n + 1])
            r = i;
        if (D[r][n + 1] < -EPS) {
            Pivot(r, n);
            if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
                return -numeric_limits<DOUBLE>::infinity();
        }
        for (int i = 0; i < m; i++) if (B[i] == -1) {
            int s = -1;

```



```

    for (int j = 0; j <= n; j++)
        if (s == -1 || D[i][j] < D[i][s] ||
            D[i][j] == D[i][s] && N[j] < N[s])
            s = j;
    Pivot(i, s); } }
if (!Simplex(2)) return numeric_limits<DOUBLE>::infinity();
x = VD(n);
for (int i = 0; i < m; i++) if (B[i] < n)
    x[B[i]] = D[i][n + 1];
return D[m][n + 1]; } }

// Two-phase simplex algorithm for solving linear programs
// of the form
//      maximize      c^T x
//      subject to    Ax <= b
//                  x >= 0
// INPUT: A -- an m x n matrix
//        b -- an m-dimensional vector
//        c -- an n-dimensional vector
//        x -- a vector where the optimal solution will be
//              stored
// OUTPUT: value of the optimal solution (infinity if
//         unbounded above, nan if infeasible)
// To use this code, create an LPSolver object with A, b,
// and c as arguments. Then, call Solve(x).
// #include <iostream>
// #include <iomanip>
// #include <vector>
// #include <cmath>
// #include <limits>
// using namespace std;
// int main() {
//     const int m = 4;
//     const int n = 3;
//     DOUBLE _A[m][n] = {
//         { 6, -1, 0 },
//         { -1, -5, 0 },
//         { 1, 5, 1 },
//         { -1, -5, -1 }
//     };
//     DOUBLE _b[m] = { 10, -4, 5, -5 };
//     DOUBLE _c[n] = { 1, -1, 0 };
//     VVD A(m);
//     VD b(_b, _b + m);
//     VD c(_c, _c + n);
//     for (int i = 0; i < m; i++) A[i] = VD(_A[i], _A[i] + n);
//     LPSolver solver(A, b, c);
//     VD x;
//     DOUBLE value = solver.Solve(x);
//     cerr << "VALUE: " << value << endl; // VALUE: 1.29032
//     cerr << "SOLUTION:"; // SOLUTION: 1.74194 0.451613 1
//     for (size_t i = 0; i < x.size(); i++) cerr << " " <<
//     x[i];
//     cerr << endl;
//     return 0;
// }
// vim: cc=60 ts=2 sts=2 sw=2:

```

```

7. GEOMETRY (CP3)

7.1. Points and lines.
#define INF 1e9
#define EPS 1e-9
#define PI acos(-1.0) // important constant; alternative
    ↪ #define PI (2.0 * acos(0.0))

double DEG_to_RAD(double d) { return d * PI / 180.0; }

double RAD_to_DEG(double r) { return r * 180.0 / PI; }

struct point { double x, y; // only used if more precision
    ↪ is needed
    point() { x = y = 0.0; } // default
    ↪ constructor
    point(double _x, double _y) : x(_x), y(_y) {} //
    ↪ user-defined
    bool operator < (point other) const { // override less than
    ↪ operator
        if (fabs(x - other.x) > EPS) // useful
    ↪ for sorting
            return x < other.x; // first criteria , by
    ↪ x-coordinate
            return y < other.y; } // second criteria, by
    ↪ y-coordinate
        // use EPS (1e-9) when testing equality of two floating
    ↪ points
        bool operator == (point other) const {
            return (fabs(x - other.x) < EPS && (fabs(y - other.y) <
    ↪ EPS)); } };

double dist(point p1, point p2) { // Euclidean
    ↪ distance
        // hypot(dx, dy) returns sqrt(dx * dx +
    ↪ dy * dy)
        return hypot(p1.x - p2.x, p1.y - p2.y); } //
    ↪ return double

// rotate p by theta degrees CCW w.r.t origin (0, 0)
point rotate(point p, double theta) {
    double rad = DEG_to_RAD(theta); // multiply theta with
    ↪ PI / 180.0
    return point(p.x * cos(rad) - p.y * sin(rad),
        p.x * sin(rad) + p.y * cos(rad)); }

struct line { double a, b, c; }; // ax + by + c = 0

// the answer is stored in the third parameter (pass by
    ↪ reference)
void pointsToLine(point p1, point p2, line &l) {
    if (fabs(p1.x - p2.x) < EPS) { // vertical
    ↪ line is fine
        l.a = 1.0; l.b = 0.0; l.c = -p1.x; //
    ↪ default values

```

```

    } else {
        l.a = -(double)(p1.y - p2.y) / (p1.x - p2.x);
        l.b = 1.0; // IMPORTANT: we fix the value of
    ↪ b to 1.0
        l.c = -(double)(l.a * p1.x) - p1.y;
    } }

bool areParallel(line l1, line l2) { // check
    ↪ coefficients a & b
    return (fabs(l1.a-l2.a) < EPS) && (fabs(l1.b-l2.b) < EPS);
    ↪ }

bool areSame(line l1, line l2) { // also check
    ↪ coefficient c
    return areParallel(l1 ,l2) && (fabs(l1.c - l2.c) < EPS); }

// returns true (+ intersection point) if two lines are
    ↪ intersect
bool areIntersect(line l1, line l2, point &p) {
    if (areParallel(l1, l2)) return false; // no
    ↪ intersection
    // solve system of 2 linear algebraic equations with 2
    ↪ unknowns
    p.x = (l2.b * l1.c - l1.b * l2.c) / (l2.a * l1.b - l1.a *
    ↪ l2.b);
    // special case: test for vertical line to avoid division
    ↪ by zero
    if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
    else p.y = -(l2.a * p.x + l2.c);
    return true; }

struct vec { double x, y; // name: 'vec' is different from
    ↪ STL vector
    vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) { // convert 2 points to
    ↪ vector a->b
    return vec(b.x - a.x, b.y - a.y); }

vec scale(vec v, double s) { // nonnegative s = [<1 ..
    ↪ 1 .. >1]
    return vec(v.x * s, v.y * s); } //
    ↪ shorter.same.longer

point translate(point p, vec v) { // translate p
    ↪ according to v
    return point(p.x + v.x , p.y + v.y); }

// convert point and gradient/slope to line
void pointSlopeToLine(point p, double m, line &l) {
    l.a = -m; //
    ↪ always -m

```

```

    l.b = 1; //
    ↪ always 1
    l.c = -((l.a * p.x) + (l.b * p.y)); //
    ↪ compute this

void closestPoint(line l, point p, point &ans) {
    line perpendicular; // perpendicular to l and pass
    ↪ through p
    if (fabs(l.b) < EPS) { // special case 1:
    ↪ vertical line
        ans.x = -(l.c); ans.y = p.y; return; }

    if (fabs(l.a) < EPS) { // special case 2:
    ↪ horizontal line
        ans.x = p.x; ans.y = -(l.c); return; }

    pointSlopeToLine(p, 1 / l.a, perpendicular); //
    ↪ normal line
    // intersect line l with this perpendicular line
    // the intersection point is the closest point
    areIntersect(l, perpendicular, ans); }

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
    point b;
    closestPoint(l, p, b); // similar to
    ↪ distToLine
    vec v = toVec(p, b); // create
    ↪ a vector
    ans = translate(translate(p, v), v); // translate
    ↪ p twice

double dot(vec a, vec b) { return (a.x * b.x + a.y * b.y); }

double norm_sq(vec v) { return v.x * v.x + v.y * v.y; }

// returns the distance from p to the line defined by
// two points a and b (a and b must be different)
// the closest point is stored in the 4th parameter (byref)
double distToLine(point p, point a, point b, point &c) {
    // formula: c = a + u * ab
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    c = translate(a, scale(ab, u)); //
    ↪ translate a to c
    return dist(p, c); // Euclidean distance
    ↪ between p and c

// returns the distance from p to the line segment ab defined
    ↪ by
// two points a and b (still OK if a == b)
// the closest point is stored in the 4th parameter (byref)
double distToLineSegment(point p, point a, point b, point &c)
    ↪ {
    vec ap = toVec(a, p), ab = toVec(a, b);

```

```

    double u = dot(ap, ab) / norm_sq(ab);
    if (u < 0.0) { c = point(a.x, a.y); //
    ↪ closer to a
        return dist(p, a); // Euclidean distance
    ↪ between p and a
    if (u > 1.0) { c = point(b.x, b.y); //
    ↪ closer to b
        return dist(p, b); // Euclidean distance
    ↪ between p and b
        return distToLine(p, a, b, c); // run distToLine
    ↪ as above

double angle(point a, point o, point b) { // returns angle
    ↪ aob in rad
    vec oa = toVec(o, a), ob = toVec(o, b);
    return acos(dot(oa, ob) / sqrt(norm_sq(oa) * norm_sq(ob)));
    ↪ }

double cross(vec a, vec b) { return a.x * b.y - a.y * b.x; }

// note: to accept collinear points, we have to change the '>'
    ↪ '0'
// returns true if point r is on the left side of line pq
bool ccw(point p, point q, point r) {
    return cross(toVec(p, q), toVec(p, r)) > 0; }

// returns true if point r is on the same line as the line pq
bool collinear(point p, point q, point r) {
    return fabs(cross(toVec(p, q), toVec(p, r))) < EPS; }

7.2. Polygon.
// returns the perimeter, which is the sum of Euclidian
    ↪ distances
// of consecutive line segments (polygon edges)
double perimeter(const vector<point> &P) {
    double result = 0.0;
    for (int i = 0; i < (int)P.size()-1; i++) // remember that
    ↪ P[0] = P[n-1]
        result += dist(P[i], P[i+1]);
    return result; }

// returns the area, which is half the determinant
double area(const vector<point> &P) {
    double result = 0.0, x1, y1, x2, y2;
    for (int i = 0; i < (int)P.size()-1; i++) {
        x1 = P[i].x; x2 = P[i+1].x;
        y1 = P[i].y; y2 = P[i+1].y;
        result += (x1 * y2 - x2 * y1);
    }
    return fabs(result) / 2.0; }

// returns true if we always make the same turn while
    ↪ examining
// all the edges of the polygon one by one
bool isConvex(const vector<point> &P) {

```

```

    int sz = (int)P.size();
    if (sz <= 3) return false; // a point/sz=2 or a line/sz=3
    ↪ is not convex
    bool isLeft = ccw(P[0], P[1], P[2]); //
    ↪ remember one result
    for (int i = 1; i < sz-1; i++) // then compare
    ↪ with the others
        if (ccw(P[i], P[i+1], P[(i+2) == sz ? 1 : i+2])) !=
    ↪ isLeft)
            return false; // different sign -> this
    ↪ polygon is concave
    return true; // this
    ↪ polygon is convex

// returns true if point p is in either convex/concave
    ↪ polygon P
bool inPolygon(point pt, const vector<point> &P) {
    if ((int)P.size() == 0) return false;
    double sum = 0; // assume the first vertex is equal to
    ↪ the last vertex
    for (int i = 0; i < (int)P.size()-1; i++) {
        if (ccw(pt, P[i], P[i+1]))
            sum += angle(P[i], pt, P[i+1]); //
    ↪ left turn/ccw
        else sum -= angle(P[i], pt, P[i+1]); //
    ↪ right turn/cw
    return fabs(fabs(sum) - 2*PI) < EPS; }

// line segment p-q intersect with line A-B.
point lineIntersectSeg(point p, point q, point A, point B) {
    double a = B.y - A.y;
    double b = A.x - B.x;
    double c = B.x * A.y - A.x * B.y;
    double u = fabs(a * p.x + b * p.y + c);
    double v = fabs(a * q.x + b * q.y + c);
    return point((p.x * v + q.x * u) / (u+v), (p.y * v + q.y *
    ↪ u) / (u+v)); }

// cuts polygon Q along the line formed by point a -> point b
// (note: the last point must be the same as the first point)
vector<point> cutPolygon(point a, point b, const
    ↪ vector<point> &Q) {
    vector<point> P;
    for (int i = 0; i < (int)Q.size(); i++) {
        double left1 = cross(toVec(a, b), toVec(a, Q[i])), left2
    ↪ = 0;
        if (i != (int)Q.size()-1) left2 = cross(toVec(a, b),
    ↪ toVec(a, Q[i+1]));
        if (left1 > -EPS) P.push_back(Q[i]); // Q[i] is on
    ↪ the left of ab
        if (left1 * left2 < -EPS) // edge (Q[i], Q[i+1])
    ↪ crosses line ab
            P.push_back(lineIntersectSeg(Q[i], Q[i+1], a, b));

```

```

    }
    if (!P.empty() && !(P.back() == P.front()))
        P.push_back(P.front()); // make P's first point =
    ↪ P's last point
    return P; }

point pivot;
bool angleCmp(point a, point b) { //
    ↪ angle-sorting function
    if (collinear(pivot, a, b))
    ↪ // special case
        return dist(pivot, a) < dist(pivot, b); // check which
    ↪ one is closer
    double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
    double d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0; } //
    ↪ compare two angles

vector<point> CH(vector<point> P) { // the content of P may
    ↪ be reshuffled
    int i, j, n = (int)P.size();
    if (n <= 3) {
        if (!(P[0] == P[n-1])) P.push_back(P[0]); // safeguard
    ↪ from corner case
        return P; // special case, the
    ↪ CH is P itself
    }

    // first, find P0 = point with lowest Y and if tie:
    ↪ rightmost X
    int P0 = 0;
    for (i = 1; i < n; i++)
        if (P[i].y < P[P0].y || (P[i].y == P[P0].y && P[i].x >
    ↪ P[P0].x))
        P0 = i;

    point temp = P[0]; P[0] = P[P0]; P[P0] = temp; // swap
    ↪ P[P0] with P[0]

    // second, sort points by angle w.r.t. pivot P0
    pivot = P[0]; // use this global
    ↪ variable as reference
    sort(++P.begin(), P.end(), angleCmp); // we do
    ↪ not sort P[0]

    // third, the ccw tests
    vector<point> S;
    S.push_back(P[n-1]); S.push_back(P[0]); S.push_back(P[1]);
    ↪ // initial S
    i = 2; // then, we
    ↪ check the rest
    while (i < n) { // note: N must be >= 3 for this
    ↪ method to work
        j = (int)S.size()-1;

```

```

        if (ccw(S[j-1], S[j], P[i])) S.push_back(P[i++]); //
    ↪ left turn, accept
        else S.pop_back(); } // or pop the top of S until we
    ↪ have a left turn
        return S; } //
    ↪ return the result

7.3. Triangle.
double perimeter(double ab, double bc, double ca) {
    return ab + bc + ca; }

double perimeter(point a, point b, point c) {
    return dist(a, b) + dist(b, c) + dist(c, a); }

double area(double ab, double bc, double ca) {
    // Heron's formula, split sqrt(a * b) into sqrt(a) *
    ↪ sqrt(b); in implementation
    double s = 0.5 * perimeter(ab, bc, ca);
    return sqrt(s) * sqrt(s - ab) * sqrt(s - bc) * sqrt(s -
    ↪ ca); }

double area(point a, point b, point c) {
    return area(dist(a, b), dist(b, c), dist(c, a)); }

double rInCircle(double ab, double bc, double ca) {
    return area(ab, bc, ca) / (0.5 * perimeter(ab, bc, ca)); }

double rInCircle(point a, point b, point c) {
    return rInCircle(dist(a, b), dist(b, c), dist(c, a)); }

// assumption: the required points/lines functions have been
    ↪ written
// returns 1 if there is an inCircle center, returns 0
    ↪ otherwise
// if this function returns 1, ctr will be the inCircle
    ↪ center
// and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point &ctr, double
    ↪ &r) {
    r = rInCircle(p1, p2, p3);
    if (fabs(r) < EPS) return 0; // no
    ↪ inCircle center

    line l1, l2; // compute these two angle
    ↪ bisectors
    double ratio = dist(p1, p2) / dist(p1, p3);
    point p = translate(p2, scale(toVec(p2, p3), ratio / (1 +
    ↪ ratio)));
    pointsToLine(p1, p, l1);

    ratio = dist(p2, p1) / dist(p2, p3);
    p = translate(p1, scale(toVec(p1, p3), ratio / (1 +
    ↪ ratio)));
    pointsToLine(p2, p, l2);

```

```

    areIntersect(l1, l2, ctr); // get their
    ↪ intersection point
    return 1; }

double rCircumCircle(double ab, double bc, double ca) {
    return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

double rCircumCircle(point a, point b, point c) {
    return rCircumCircle(dist(a, b), dist(b, c), dist(c, a)); }

// assumption: the required points/lines functions have been
    ↪ written
// returns 1 if there is a circumCenter center, returns 0
    ↪ otherwise
// if this function returns 1, ctr will be the circumCircle
    ↪ center
// and r is the same as rCircumCircle
int rCircumCircle(point p1, point p2, point p3, point &ctr,
    ↪ double &r) {
    double a = p2.x - p1.x, b = p2.y - p1.y;
    double c = p3.x - p1.x, d = p3.y - p1.y;
    double e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
    double f = c * (p1.x + p3.x) + d * (p1.y + p3.y);
    double g = 2.0 * (a * (p3.y - p2.y) - b * (p3.x - p2.x));
    if (fabs(g) < EPS) return 0;

    ctr.x = (d*e - b*f) / g;
    ctr.y = (a*f - c*e) / g;
    r = dist(p1, ctr); // r = distance from center to 1 of the
    ↪ 3 points
    return 1; }

// returns true if point d is inside the circumCircle defined
    ↪ by a,b,c
int inCircumCircle(point a, point b, point c, point d) {
    return (a.x - d.x) * (b.y - d.y) * ((c.x - d.x) * (c.x -
    ↪ d.x) + (c.y - d.y) * (c.y - d.y)) +
        (a.y - d.y) * ((b.x - d.x) * (b.x - d.x) + (b.y -
    ↪ d.y) * (b.y - d.y)) * (c.x - d.x) +
        ((a.x - d.x) * (a.x - d.x) + (a.y - d.y) * (a.y -
    ↪ d.y)) * (b.x - d.x) * (c.y - d.y) -
        ((a.x - d.x) * (a.x - d.x) + (a.y - d.y) * (a.y -
    ↪ d.y)) * (b.y - d.y) * (c.x - d.x) -
        (a.y - d.y) * (b.x - d.x) * ((c.x - d.x) * (c.x -
    ↪ d.x) + (c.y - d.y) * (c.y - d.y)) -
        (a.x - d.x) * ((b.x - d.x) * (b.x - d.x) + (b.y -
    ↪ d.y) * (b.y - d.y)) * (c.y - d.y) > 0 ? 1 : 0;
}

bool canFormTriangle(double a, double b, double c) {
    return (a + b > c) && (a + c > b) && (b + c > a); }

```

7.4. Circle.

```
int insideCircle(point_i p, point_i c, int r) { // all
↪ integer version
    int dx = p.x - c.x, dy = p.y - c.y;
    int Euc = dx * dx + dy * dy, rSq = r * r;           //
↪ all integer
    return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
↪ //inside/border/outside

bool circle2PtsRad(point p1, point p2, double r, point &c) {
    double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
                (p1.y - p2.y) * (p1.y - p2.y);
    double det = r * r / d2 - 0.25;
    if (det < 0.0) return false;
    double h = sqrt(det);
    c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
    c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
    return true; }           // to get the other center, reverse
↪ p1 and p2
```

8. COMBINATORICS

Catalan	$C_0 = 1, C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \frac{4n-2}{n+1} C_{n-1}$	
Stirling 1st kind	$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0, \begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$	#perms of n objs with exactly k cycles
Stirling 2nd kind	$\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1, \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$	#ways to partition n objs into k nonempty sets
Euler	$\left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\rangle = 1, \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (n-k) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle$	#perms of n objs with exactly k ascents
Euler 2nd Order	$\left\langle\!\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle\!\right\rangle = (k+1) \left\langle\!\left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle\!\right\rangle + (2n-k-1) \left\langle\!\left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle\!\right\rangle$	#perms of $1, 1, 2, 2, \dots, n, n$ with exactly k ascents
Bell	$B_1 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	#partitions of $1..n$ (Stirling 2nd, no limit on k)

#labeled rooted trees	n^{n-1}				
#labeled unrooted trees	n^{n-2}				
#forests of k rooted trees	$\frac{k}{n} \binom{n}{k} n^{n-k}$				
$\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$	$\sum_{i=1}^n i^3 = n^2(n+1)^2/4$				
$!n = n \times !(n-1) + (-1)^n$	$!n = (n-1)(!(n-1) + !(n-2))$				
$\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}$	$\sum_i \binom{n-i}{i} = F_{n+1}$				
$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$	$x^k = \sum_{i=0}^k i! \left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\} \binom{x}{i} = \sum_{i=0}^k \left\langle \begin{smallmatrix} k \\ i \end{smallmatrix} \right\rangle \binom{x+i}{k}$				
$a \equiv b \pmod{x, y} \Rightarrow a \equiv b \pmod{\text{lcm}(x, y)}$	$\sum_{d n} \phi(d) = n$				
$ac \equiv bc \pmod{m} \Rightarrow a \equiv b \pmod{\frac{m}{\text{gcd}(c, m)}}$	$(\sum_{d n} \sigma_0(d))^2 = \sum_{d n} \sigma_0(d)^3$				
p prime $\Leftrightarrow (p-1)! \equiv -1 \pmod{p}$	$\text{gcd}(n^a - 1, n^b - 1) = n^{\text{gcd}(a, b)} - 1$				
$\sigma_x(n) = \prod_{i=0}^r \frac{p_i^{(a_i+1)x-1}}{p_i^x - 1}$	$\sigma_0(n) = \prod_{i=0}^r (a_i + 1)$				
$\sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$					
$2^{\omega(n)} = O(\sqrt{n})$	$\sum_{i=1}^n 2^{\omega(i)} = O(n \log n)$				
$d = v_i t + \frac{1}{2} a t^2$	$v_f^2 = v_i^2 + 2 a d$				
$v_f = v_i + a t$	$d = \frac{v_i + v_f}{2} t$				
8.1. The Twelfefold Way. Putting n balls into k boxes.					
Balls	same	distinct	same	distinct	
Boxes	same	same	distinct	distinct	Remarks
-	$p_k(n)$	$\sum_{i=0}^k \left\{ \begin{smallmatrix} n \\ i \end{smallmatrix} \right\}$	$\binom{n+k-1}{k-1}$	k^n	$p_k(n)$: #partitions of n into $\leq k$ positive parts
size ≥ 1	$p(n, k)$	$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	$\binom{n-1}{k-1}$	$k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	$p(n, k)$: #partitions of n into k positive parts
size ≤ 1	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n! \binom{k}{n}$	$[cond]$: 1 if $cond = true$, else 0

9. USEFUL INFORMATION

10. Misc

10.1. Debugging Tips.

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
 - Getting **NaN**? Make sure **acos** etc. are not getting values out of their range (perhaps **1+eps**).
 - Rounding negative numbers?
 - Outputting in scientific notation?
- Wrong Answer?
 - Read the problem statement again!
 - Are multiple test cases being handled correctly? Try repeating the same test case many times.
 - Integer overflow?
 - Think very carefully about boundaries of all input parameters
 - Try out possible edge cases:
 - * $n = 0, n = -1, n = 1, n = 2^{31} - 1$ or $n = -2^{31}$
 - * List is empty, or contains a single element
 - * n is even, n is odd
 - * Graph is empty, or contains a single vertex
 - * Graph is a multigraph (loops or multiple edges)
 - * Polygon is concave or non-simple
 - Is initial condition wrong for small cases?
 - Are you sure the algorithm is correct?
 - Explain your solution to someone.
 - Are you using any functions that you don't completely understand? Maybe STL functions?
 - Maybe you (or someone else) should rewrite the solution?
 - Can the input line be empty?
- Run-Time Error?
 - Is it actually Memory Limit Exceeded?

10.2. Solution Ideas.

- Dynamic Programming
 - Parsing CFGs: CYK Algorithm
 - Drop a parameter, recover from others
 - Swap answer and a parameter
 - When grouping: try splitting in two
 - 2^k trick
 - When optimizing
 - * Convex hull optimization
 - $dp[i] = \min_{j < i} \{dp[j] + b[j] \times a[i]\}$
 - $b[j] \geq b[j + 1]$
 - optionally $a[i] \leq a[i + 1]$
 - $O(n^2)$ to $O(n)$
 - * Divide and conquer optimization
 - $dp[i][j] = \min_{k < j} \{dp[i - 1][k] + C[k][j]\}$
 - $A[i][j] \leq A[i][j + 1]$
 - $O(kn^2)$ to $O(kn \log n)$
 - sufficient: $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$, $a \leq b \leq c \leq d$ (QI)
 - * Knuth optimization
 - $dp[i][j] = \min_{i < k < j} \{dp[i][k] + dp[k][j] + C[i][j]\}$
 - $A[i][j - 1] \leq A[i][j] \leq A[i + 1][j]$
 - $O(n^3)$ to $O(n^2)$

• sufficient: QI and $C[b][c] \leq C[a][d]$, $a \leq b \leq c \leq d$

- Greedy
- Randomized
- Optimizations
 - Use bitset (/64)
 - Switch order of loops (cache locality)
- Process queries offline
 - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
 - Mo's algorithm
 - Sqrt decomposition
 - Store 2^k jump pointers
- Data structure techniques
 - Sqrt buckets
 - Store 2^k jump pointers
 - 2^k merging trick
- Counting
 - Inclusion-exclusion principle
 - Generating functions
- Graphs
 - Can we model the problem as a graph?
 - Can we use any properties of the graph?
 - Strongly connected components
 - Cycles (or odd cycles)
 - Bipartite (no odd cycles)
 - * Bipartite matching
 - * Hall's marriage theorem
 - * Stable Marriage
 - Cut vertex/bridge
 - Biconnected components
 - Degrees of vertices (odd/even)
 - Trees
 - * Heavy-light decomposition
 - * Centroid decomposition
 - * Least common ancestor
 - * Centers of the tree
 - Eulerian path/circuit
 - Chinese postman problem
 - Topological sort
 - (Min-Cost) Max Flow
 - Min Cut
 - * Maximum Density Subgraph
 - Huffman Coding
 - Min-Cost Arborescence
 - Steiner Tree
 - Kirchhoff's matrix tree theorem
 - Prüfer sequences
 - Lovász Toggle
 - Look at the DFS tree (which has no cross-edges)
 - Is the graph a DFA or NFA?
 - * Is it the Synchronizing word problem?
- math
 - Is the function multiplicative?
 - Look for a pattern

- Permutations
 - * Consider the cycles of the permutation
- Functions
 - * Sum of piecewise-linear functions is a piecewise-linear function
 - * Sum of convex (concave) functions is convex (concave)
- Modular arithmetic
 - * Chinese Remainder Theorem
 - * Linear Congruence
- Sieve
- System of linear equations
- Values too big to represent?
 - * Compute using the logarithm
 - * Divide everything by some large value
- Linear programming
 - * Is the dual problem easier to solve?
- Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
 - 2-SAT
 - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ($\log(n)$)
- Strings
 - Trie (maybe over something weird, like bits)
 - Suffix array
 - Suffix automaton (+DP?)
 - Aho-Corasick
 - eertree
 - Work with $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
 - Lazy propagation
 - Persistent
 - Implicit
 - Segment tree of X
- Geometry
 - Minkowski sum (of convex sets)
 - Rotating calipers
 - Sweep line (horizontally or vertically?)
 - Sweep angle
 - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem

- Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)
- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

11. FORMULAS

- **Legendre symbol:** $\left(\frac{a}{b}\right) = a^{(b-1)/2} \pmod{b}$, b odd prime.
- **Heron’s formula:** A triangle with side lengths a, b, c has area $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.
- **Pick’s theorem:** A polygon on an integer grid strictly containing i lattice points and having b lattice points on the boundary has area $i + \frac{b}{2} - 1$. (Nothing similar in higher dimensions)
- **Euler’s totient:** The number of integers less than n that are coprime to n are $n \prod_{p|n} \left(1 - \frac{1}{p}\right)$ where each p is a distinct prime factor of n .
- **König’s theorem:** In any bipartite graph $G = (L \cup R, E)$, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover. Let U be the set of unmatched vertices in L , and Z be the set of vertices that are either in U or are connected to U by an alternating path. Then $K = (L \setminus Z) \cup (R \cap Z)$ is the minimum vertex cover.
- A minumum Steiner tree for n vertices requires at most $n - 2$ additional Steiner vertices.
- The number of vertices of a graph is equal to its minimum vertex cover number plus the size of a maximum independent set.
- **Lagrange polynomial** through points $(x_0, y_0), \dots, (x_k, y_k)$ is $L(x) = \sum_{j=0}^k y_j \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$
- **Hook length formula:** If λ is a Young diagram and $h_\lambda(i, j)$ is the hook-length of cell (i, j) , then then the number of Young tableaux $d_\lambda = n! / \prod h_\lambda(i, j)$.
- **Möbius inversion formula:** If $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(d) f(n/d)$. If $f(n) = \sum_{m=1}^n g(\lfloor n/m \rfloor)$, then $g(n) = \sum_{m=1}^n \mu(m) f(\lfloor \frac{n}{m} \rfloor)$.
- #primitive pythagorean triples with hypotenuse $< n$ approx $n/(2\pi)$.
- **Frobenius Number:** largest number which can’t be expressed as a linear combination of numbers a_1, \dots, a_n with non-negative coefficients. $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$, $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d - 1)$. An integer $x > (\max_i a_i)^2$ can be expressed in such a way iff. $x \mid \gcd(a_1, \dots, a_n)$.

11.1. Physics.

- **Snell’s law:** $\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$

11.2. **Markov Chains.** A Markov Chain can be represented as a weighted directed graph of states, where the weight of an edge represents the probability of transitioning over that edge in one timestep. Let $P^{(m)} = (p_{ij}^{(m)})$ be the probability matrix of transitioning from state i to state j in m timesteps, and note that $P^{(1)}$ is the adjacency matrix of the graph. **Chapman-Kolmogorov:** $p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$. It follows that $P^{(m+n)} = P^{(m)} P^{(n)}$ and $P^{(m)} = P^m$. If $p^{(0)}$ is the initial probability distribution (a vector), then $p^{(0)} P^{(m)}$ is the probability distribution after m timesteps.

The return times of a state i is $R_i = \{m \mid p_{ii}^{(m)} > 0\}$, and i is *aperiodic* if $\gcd(R_i) = 1$. A MC is aperiodic if any of its vertices is aperiodic. A MC is *irreducible* if the corresponding graph is strongly connected. A distribution π is stationary if $\pi P = \pi$. If MC is irreducible then $\pi_i = 1/\mathbb{E}[T_i]$, where T_i is the expected time between two visits at i . π_j/π_i is the expected number of visits at j in between two consecutive visits at i . A MC is *ergodic* if $\lim_{m \rightarrow \infty} p^{(0)} P^m = \pi$. A MC is ergodic iff. it is irreducible and aperiodic.

A MC for a random walk in an undirected weighted graph (un-weighted graph can be made weighted by adding 1-weights) has $p_{uv} = w_{uv} / \sum_x w_{ux}$. If the graph is connected, then $\pi_u = \sum_x w_{ux} / \sum_v \sum_x w_{vx}$. Such a random walk is aperiodic iff. the graph is not bipartite.

An *absorbing* MC is of the form $P = \begin{pmatrix} Q & R \\ 0 & I_r \end{pmatrix}$. Let $N = \sum_{m=0}^\infty Q^m = (I_t - Q)^{-1}$. Then, if starting in state i , the expected number of steps till absorption is the i -th entry in $N1$. If starting in state i , the probability of being absorbed in state j is the (i, j) -th entry of NR .

Many problems on MC can be formulated in terms of a system of recurrence relations, and then solved using Gaussian elimination.

11.3. **Burnside’s Lemma.** Let G be a finite group that acts on a set X . For each g in G let X^g denote the set of elements in X that are fixed by g . Then the number of orbits

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

$$Z(S_n) = \frac{1}{n} \sum_{l=1}^n a_l Z(S_{n-l})$$

11.4. **Bézout’s identity.** If (x, y) is any solution to $ax + by = d$ (e.g. found by the Extended Euclidean Algorithm), then all solutions are given by

$$\left(x + k \frac{b}{\gcd(a, b)}, y - k \frac{a}{\gcd(a, b)}\right)$$

11.5. Misc.

11.5.1. *Binomial transform.* If $a_n = \sum_{k=0}^n \binom{n}{k} b_k$, then $b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$.

11.5.2. *Generating functions.* Ordinary (o.g.f.): $A(x) := \sum_{n=0}^\infty a_n x^n$. Exponential (e.g.f.): $A(x) := \sum_{n=0}^\infty a_n x^n / n!$. o.g.f. convolution: $c_n = \sum_{k=0}^n a_k b_{n-k}$ (use FFT for $\mathcal{O}(n \log n)$). e.g.f. convolution: $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$ (use FFT for $\mathcal{O}(n \log n)$).

11.5.3. *General linear recurrences.* If $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$, then $A(x) = \frac{a_0}{1-B(x)}$. We also can compute all a_n with Divide-and-Conquer algorithm in $\mathcal{O}(n \log^2 nn)$.

11.5.4. *Inverse polynomial modulo x^l .* Given $A(x)$, find $B(x)$ such that $A(x)B(x) = 1 + x^l Q(x)$ for some $Q(x)$. Step 1: Start with $B_0(x) = 1/a_0$ Step 2: $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \pmod{x^{2^{k+1}}}$.

11.5.5. *Fast subset convolution.* Given array a_i of size 2^k calculate $b_i = \sum_{j \& i = i} a_j$.

```
for (int b = 1; b < (1 << k); b <= 1)
    for (int i = 0; i < (1 << k); i++)
        if (!(i & b)) a[i | b] += a[i];
// inv: if (!(i & b)) a[i | b] -= a[i];
```

11.5.6. *Determinants and PM.*

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}$$

$$\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i, \sigma(i)}$$

$$\begin{aligned} pf(A) &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)} \\ &= \sum_{M \in \text{PM}(n)} \text{sgn}(M) \prod_{(i,j) \in M} a_{i,j} \end{aligned}$$

11.5.7. *BEST Theorem.* Count directed Eulerian cycles. Number of OST given by Kirchoff’s Theorem (remove r/c with root) $\# \text{OST}(G, r) \cdot \prod_v (d_v - 1)!$

11.5.8. *Primitive Roots.* Only exists when n is $2, 4, p^k, 2p^k$, where p odd prime. Assume n prime. Number of primitive roots $\phi(\phi(n))$ Let g be primitive root. All primitive roots are of the form g^k where $k, \phi(p)$ are coprime. k -roots: $g^{i \cdot \phi(n)/k}$ for $0 \leq i < k$

11.5.9. *Sum of primes.* For any multiplicative f : $S(n, p) = S(n, p - 1) - f(p) \cdot (S(n/p, p - 1) - S(p - 1, p - 1))$

11.5.10. *Floor.*

$$\begin{aligned} \lfloor \lfloor x/y \rfloor / z \rfloor &= \lfloor x/(yz) \rfloor \\ x \% y = x - y \lfloor x/y \rfloor \end{aligned}$$

PRACTICE CONTEST CHECKLIST

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use printf/scanf with long long/long double?
- Are `__int128` and `__float128` available?
- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: `i?python[23]`, `factor`.
- Try submitting with `assert(false)` and `assert(true)`.
- Return-value from `main`.
- Look for directory with sample test cases.
- Make sure printing works.
- Remove this page from the notebook (optionally).