

TCR

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```

.bashrc
alias gg='g++ -std=c++17 -Wall -Wshadow'
alias g='gg -DDEBUG -g -fsanitize=address,undefined'

.vimrc

set nu rnu sw=4 ts=4 sts=4 noet ai hls shcf=-ic
sy on | colo slate

Test script (usage: ./test.sh A/B/..)
g++ -g -Wall -fsanitize=address,undefined
-Wfatal-errors -std=c++17 $1.cc || exit
for i in $(ls *.in)
do
    j="$(echo $i | sed 's/.in/.ans/')"
    ./a.out < $i > output
    diff output $j || echo "!!WA on $i!!"
done

template.cc

#include <bits/stdc++.h>
using namespace std;

typedef long long ll;
typedef long double ld;
typedef pair<ll, ll> ii;
typedef vector<ll> vi;
typedef vector<vi> vvi;
typedef vector<ii> vii;

#define x first
#define y second
#define pb push_back
#define eb emplace_back
#define rep(i,a,b) for(auto i=(a); i<(b); ++i)
#define REP(i,n) rep(i,0,n)
#define all(v) begin(v), end(v)
#define sz(v) ((int) (v).size())
#define rs resize
#define DBG(x) cerr << __LINE__ << ": " \
<< #x << " = " << (x) << endl

template<class T> ostream& operator<<(ostream &os,
const vector<T> &v) {
    os << "\n[";
    for(const T &x : v) os << x << ', ';
    return os << "]\n";
}

namespace std { template<class T1, class T2>
struct hash<pair<T1,T2>> { public:
    size_t operator() (const pair<T1,T2> &p) const {
        size_t x = hash<T1>()(p.x), y = hash<T2>()(p.y);
        return x ^ (y + 0x9e3779b9 + (x<<6) + (x>>2));
    }
}; }

void run() {

```

```
signed main() {
    // DON'T MIX "scanf" and "cin"!
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    cout << fixed << setprecision(20);
    run();
    return 0;
}
```

template.py

```
# reading input:
from sys import *
n,m = [ int(x) for x in
    ↪ stdin.readline().rstrip().split() ]
stdout.write( str(n*m)+"\n" )
# set operations:
from itertools import *
for (x,y) in product(range(3),repeat=2):
    stdout.write( str(3*x+y)+" " )
print()
for L in combinations(range(4),2):
    stdout.write( str(L)+" " )
print()
# fancy lambdas:
from functools import *
y = reduce( lambda x,y: x+y, map( lambda x: x*x,
    ↪ range(4) ), -3 )
print(y)
# formatting:
from math import *
stdout.write( "{0:.2f}\n".format(pi) )
```

0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Iedereen (incl. Ludo) moet **ALLE** opgaves goed lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zware) test cases.
- Gebruik ll.

0.2. Wrong Answer.

- Print de oplossing om te debuggen!
- Kijk naar wellicht makkelijkere problemen.
- Bedenk zelf test cases met **randgevallen**!
- Controleer de **precisie**.
- Controleer op **overflow** (gebruik **OVERAL** ll, ld).
- *Kijk naar overflows in tussenantwoorden bij modulo.*
- Controleer op **typo's**.
- Loop de voorbeeld test case accuraat langs.
- Controleer op off-by-one-errors (in indices of lus-grenzen)?

Detecting overflow: This GNU builtin checks for over- and underflow. Result is in res if successful:

```
bool isOverflown =
    ↪ __builtin_[add|mul|sub]_overflow(a, b, &res);
```

1. MATHEMATICS

XOR sum: $\bigoplus_{x=0}^{a-1} x = \{0, a-1, 1, a\}[a \bmod 4]$.

```
int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }
```

```
// greatest common divisor
ll gcd(ll a, ll b) { while(b) a%=b, swap(a,b); return a; }
// least common multiple
ll lcm(ll a, ll b) { return a/gcd(a, b)*b; }
ll mod(ll a, ll b) { return (a%=b) < 0 ? a+b : a; }
```

```
// ab % m for m <= 4e18 in O(log b)
ll mod_mul(ll a, ll b, ll m) {
    ll r = 0;
    while(b) {
        if (b & 1) r = mod(r+a,m);
        a = mod(a+a,m); b >>= 1;
    }
    return r;
}
```

```
// a^b % m for m <= 2e9 in O(log b)
ll mod_pow(ll a, ll b, ll m) {
    ll r = 1;
    while(b) {
        if (b & 1) r = (r * a) % m; // mod_mul
        a = (a * a) % m; // mod_mul
        b >>= 1;
    }
    return r;
}
```

```
// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
    ll xx = y = 0, yy = x = 1;
    while (b) {
        x -= a / b * xx; swap(x, xx);
        y -= a / b * yy; swap(y, yy);
        a %= b; swap(a, b);
    }
    return a;
}
```

```
// Chinese Remainder Theorem: returns (u, v) s.t.
// x=u (mod v) <=> x=a (mod n) and x=b (mod m)
pair<ll, ll> crt(ll a, ll n, ll b, ll m) {
    ll s, t, d = egcd(n, m, s, t); //n,m<=1e9
    if (mod(a - b, d)) return { 0, -1 };
    return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}
```

```
// phi[i] = #{ 0 < j <= i | gcd(i, j) = 1 } sieve
vi totient(int N) {
    vi phi(N);
    for (int i = 0; i < N; i++) phi[i] = i;
    for (int i = 2; i < N; i++) if (phi[i] == i)
        for (int j = i; j < N; j+=i) phi[j] -= phi[j]/i;
    return phi;
}
```

```
// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
    ll ans = 1;
    while (n) {
        ll np = n % p, kp = k % p;
        if (np < kp) return 0;
        ans = mod(ans * binom(np, kp), p); // (np C kp)
        n /= p; k /= p;
    }
    return ans;
}
```

```
// returns if n is prime for n < 3e24 (>2^64)
// but use mul_mod for n > 2e9.
bool millerRabin(ll n) {
    if (n < 2 || n % 2 == 0) return n == 2;
    ll d = n - 1, ad, s = 0, r;
    for (; d % 2 == 0; d /= 2) s++;
    for (int a : { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 }) {
        if (n == a) return true;
        if ((ad = mod_pow(a, d, n)) == 1) continue;
        for (r = 0; r < s && ad + 1 != n; r++)
            ad = (ad * ad) % n;
        if (r == s) return false;
    }
    return true;
}
```

1.1. Primitive Root $O(\sqrt{m})$. Returns a generator of \mathbb{F}_m^* . If m not prime, replace $m-1$ by totient of m .

```
ll primitive_root(ll m) {
    vector<ll> div;
    for (ll i = 1; i*i < m; i++)
        if ((m-1) % i == 0) {
            if (i < m-1) div.pb(i);
            if ((m-1)/i < m) div.pb((m-1)/i);
        }
    rep(x,2,m) {
        bool ok = true;
        for (ll d : div) if (mod_pow(x, d, m) == 1)
            { ok = false; break; }
        if (ok) return x;
    }
    return -1;
}
```

1.2. Tonelli-Shanks algorithm. Given prime p and integer $1 \leq n < p$, returns the square root r of n modulo p . There is also another solution given by $-r$ modulo p .

```
11 legendre(11 a, 11 p) {
    if (a % p == 0) return 0;
    return p == 2 || mod_pow(a, (p-1)/2, p) == 1 ? 1 :
        -1;
}
11 tonelli_shanks(11 n, 11 p) {
    assert(legendre(n,p) == 1);
    if (p == 2) return 1;
    11 s = 0, q = p-1, z = 2;
    while (~q & 1) s++, q >= 1;
    if (s == 1) return mod_pow(n, (p+1)/4, p);
    while (legendre(z,p) != -1) z++;
    11 c = mod_pow(z, q, p),
        r = mod_pow(n, (q+1)/2, p),
        t = mod_pow(n, q, p),
        m = s;
    while (t != 1) {
        11 i = 1, ts = (11)t*t % p;
        while (ts != 1) i++, ts = ((11)ts * ts) % p;
        11 b = mod_pow(c, 1LL<<(m-i-1), p);
        r = (11)r * b % p;
        t = (11)t * b % p * b % p;
        c = (11)b * b % p;
        m = i;
    }
    return r;
}
```

1.3. Numeric Integration. Numeric integration using Simpson's rule.

```
1d numint(1d (*f)(1d), 1d a, 1d b, 1d EPS = 1e-6) {
    1d ba = b - a, m=(a+b)/2;
    return abs(ba) < EPS
        ? ba/8*(f(a)+f(b)+f(a+ba/3)*3+f(b-ba/3)*3)
        : numint(f,a,m,EPS) + numint(f,m,b,EPS);
}
```

1.4. Fast Hadamard Transform. Computes XOR-convolutions in $O(k2^k)$ on k bits.

For AND-convolution, use $(x+y, y)$, $(x-y, y)$.

For OR-convolution, use $(x, x+y)$, $(x, -x+y)$.

Note: The array size must be a power of 2.

```
void fht(vi &A, bool inv=false, int l, int r) {
    if (l+1 == r) return;
    int k = (r-l)/2;
    if (!inv) fht(A, inv, l, l+k), fht(A, inv, l+k,
        -> r);
    rep(i, l, l+k) {
        11 x = A[i], y = A[i+k];
        if (!inv) A[i] = x-y, A[i+k] = x+y;
        else A[i] = (x+y)/2, A[i+k] = (-x+y)/2;
    }
}
```

```
if (inv) fht(A, inv, l, l+k), fht(A, inv, l+k, r);
}
```

1.5. Tridiagonal Matrix Algorithm. Solves a tridiagonal system of linear equations

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$$

where $a_1 = c_n = 0$. Beware of numerical instability.

```
#define MAXN 5000
1d A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
    C[0] /= B[0]; D[0] /= B[0];
    rep(i, 1, n-1) C[i] /= B[i] - A[i]*C[i-1];
    rep(i, 1, n) D[i] =
        (D[i] - A[i]*D[i-1]) / (B[i] - A[i]*C[i-1]);
    X[n-1] = D[n-1];
    for (int i = n-1; i--;) X[i] = D[i] - C[i]*X[i+1];
}
```

1.6. Number of Integer Points under Line. Count the number of integer solutions to $Ax + By \leq C$, $0 \leq x \leq n$, $0 \leq y$. In other words, evaluate the sum $\sum_{x=0}^n \lfloor \frac{C-Ax}{B} + 1 \rfloor$. To count all solutions, let $n = \lfloor \frac{C}{A} \rfloor$. In any case, it must hold that $C - nA \geq 0$. Be very careful about overflows.

```
11 floor_sum(11 n, 11 a, 11 b, 11 c) {
    if (c == 0) return 1;
    if (c < 0) return 0;
    if (a % b == 0) return
        -> (n+1)*(c/b+1)-n*(n+1)/2*a/b;
    if (a >= b) return
        -> floor_sum(n, a%b, b, c)-a/b*n*(n+1)/2;
    11 t = (c-a*n+b)/b;
    return floor_sum((c-b*t)/b, b, a, c-b*t)+t*(n+1);
}
```

1.7. Solving linear recurrences. Given some brute-forced sequence $s[0], s[1], \dots, s[2n-1]$, Berlekamp-Massey finds the shortest possible recurrence relation in $\mathcal{O}(n^2)$. After that, `lin_rec` finds $s[k]$ in $\mathcal{O}(n^2 \log k)$.

```
// Given a sequence s[0], ..., s[2n-1] finds the
-> smallest linear recurrence
// of size <= n compatible with s.
1d BerlekampMassey(const vl &s, 11 mod) {
    int n = sz(s), L = 0, m = 0;
    vl C(n), B(n), T;
    C[0] = B[0] = 1;
    11 b = 1;
    REP(i, n) {
        ++m;
        11 d = s[i] % mod;
        rep(j, 1, L+1) d = (d + C[j] * s[i-j]) % mod;
        if (!d) continue;
        T = C;
        11 coef = d * modpow(b, mod-2, mod) % mod;
        rep(j, m, n) C[j] = (C[j] - coef * B[j-m]) % mod;
        if (2 * L > i) continue;
    }
}
```

```
L = i + 1 - L;
B = T; b = d; m = 0;
}
C.resize(L + 1);
C.erase(C.begin());
for (auto &x : C) x = (mod - x) % mod;
return C;
}
```

```
// Input: A[0,...,n-1], C[0,...,n-1] satisfying
// A[i] = \sum_{j=1}^n C[j-1] A[i-j],
// Outputs A[k]
11 lin_rec(const vl &A, const vl &C, 11 k, 11 mod){
    int n = sz(A);
    auto combine = [&](vl a, vl b) {
        vl res(sz(a) + sz(b) - 1, 0);
        REP(i, sz(a)) REP(j, sz(b))
            res[i+j] = (res[i+j] + a[i]*b[j]) % mod;
        for (int i = 2*n; i > n; --i) REP(j, n)
            res[i-1-j] = (res[i-1-j] + res[i]*C[j]) % mod;
        res.resize(n + 1);
        return res;
    };
    vl pol(n + 1), e(pol);
    pol[0] = e[1] = 1;
    for (++k; k; k /= 2) {
        if (k % 2) pol = combine(pol, e);
        e = combine(e, e);
    }
    11 res = 0;
    REP(i, n) res = (res + pol[i + 1] * A[i]) % mod;
    return res;
}
```

1.8. Misc.

1.8.1. Josephus problem. Last man standing out of n if every k th is killed. Zero-based, and does not kill 0 on first pass.

```
int J(int n, int k) {
    if (n == 1 || k == 1) return n-1;
    if (n < k) return (J(n-1, k)+k)%n;
    int np = n - n/k;
    return k*((J(np, k)+np-n%k*np)%np) / (k-1);
}
```

• **Prime numbers:**

1031, 32771, 1048583, 8125344, 33554467, 9982451653, 1073741827, 34359738421, 1099511627791, 35184372088891, 1125899906842679, 36028797018963971.

$10^3 + \{-9, -3, 9, 13\}$, $10^6 + \{-17, 3, 33\}$, $10^9 + \{7, 9, 21, 33, 87\}$.

• **Generating functions:** Ordinary (ogf): $A(x) := \sum_{n=0}^{\infty} a_n x^n$.

Calculate product $c_n = \sum_{k=0}^n a_k b_{n-k}$ with FFT.

Exponential (e.g.f.): $A(x) := \sum_{n=0}^{\infty} a_n x^n / n!$,

$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$ (use FFT).

- **General linear recurrences:** If $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$, then $A(x) = \frac{a_0}{1-B(x)}$.
- **Inverse polynomial modulo x^l :** Given $A(x)$, find $B(x)$ such that $A(x)B(x) = 1 + x^l Q(x)$ for some $Q(x)$.
Step 1: Start with $B_0(x) = 1/a_0$
Step 2: $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \bmod x^{2^{k+1}}$.
- **Fast subset convolution:** Given array a_i of size 2^k calculate $b_i = \sum_{j \& i = i} a_j$.
for (int b = 1; b < (1 << k); b <= 1)
 for (int i = 0; i < (1 << k); i++)
 if (!(i & b)) a[i | b] += a[i];
// inv: if (!(i & b)) a[i | b] -= a[i];
- **Primitive Roots:** It only exists when n is $2, 4, p^k, 2p^k$, where p odd prime. If g is a primitive root, all primitive roots are of the form g^k where $k, \phi(p)$ are coprime (hence there are $\phi(\phi(p))$ primitive roots).
- **Maximum number of divisors:**

$\leq N$	10^3	10^6	10^9	10^{12}	10^{18}
m	840	720720	735134400	963761198400	
$\sigma_0(m)$	32	240	1344	6270	103680

For $n = 10^{18}, m = 897612484786617600$.

2. DATASTRUCTURES

2.1. Order tree.

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
template<class TK, class TM> using order_tree =
    tree<TK, TM, greater<TK>, rb_tree_tag,
    tree_order_statistics_node_update>;
template<class TK> using order_set =
    order_tree<TK, null_type>;
```

```
vi s;
order_set<ii> t;
void update( ll k, ll v ) {
    t.erase( ii{ s[k], k } );
    s[k] = v;
    t.insert( ii{ s[k], k } );
}
```

```
signed main() {
    ll n = 4;
    s.resize(n, 0);
    rep(i, 0, n) t.insert(ii{0, i});
    update( 2, 3 );
    cout << t.find_by_order( 2 )->y << endl;
    cout << t.order_of_key( ii{s[3], 3} ) << endl;
}
```

2.2. Segment tree $\mathcal{O}(\log n)$.

2.2.1. Standard segment tree.

```
typedef int S; // or define your own object
const int n = 1 << 20;
S t[2 * n];

// combine must be an associative function!
S combine(S l, S r) { return l+r; } //or max(l,r) etc

void build() {
    for (int i = n; --i; )
        t[i] = combine(t[2 * i], t[2 * i + 1]);
}

// set value v on position i
void update(int i, S v) {
    for (t[i+=n] = v; i /= 2; )
        t[i] = combine(t[2 * i], t[2 * i + 1]);
}

// sum on interval [l, r)
S query(int l, int r) {
    S resL = 0, resR = 0;
    for (l += n, r += n; l < r; l /= 2, r /= 2) {
        if (l & 1) resL = combine(resL, t[l++]);
        if (r & 1) resR = combine(t[--r], resR);
    }
    return combine(resL, resR);
}
```

2.2.2. Lazy segment tree.

Be careful: all intervals are right-closed $[\ell, r]$.

```
struct node {
    int l, r, x, lazy;
    node() {}
    node(int _l, int _r) : l(_l), r(_r), x(INT_MAX),
        lazy(0) {}
    node(int _l, int _r, int _x) : node(_l, _r) { x = _x; }
    node(node a, node b) : node(a.l, b.r) { x = min(a.x, b.x); }
    void update(int v) { x = v; }
    void range_update(int v) { lazy = v; }
    void apply() { x += lazy; lazy = 0; }
    void push(node &u) { u.lazy += lazy; }
};
```

```
struct segment_tree {
    int n;
    vector<node> arr;
    segment_tree() {}
    segment_tree(const vi &a) : n(sz(a)), arr(4*n) {
        mk(a, 0, 0, n-1); }
    node mk(const vi &a, int i, int l, int r) {
        int m = (l+r)/2;
        return arr[i] = l > r ? node(l, r) :
            l == r ? node(l, r, a[l]) :
            node(mk(a, 2*i+1, l, m), mk(a, 2*i+2, m+1, r));
    }
    node update(int at, ll v, int i=0) {
```

```
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (at < hl || hr < at) return arr[i];
    if (hl == at && at == hr) {
        arr[i].update(v); return arr[i]; }
    return arr[i] =
        node(update(at, v, 2*i+1), update(at, v, 2*i+2));
}
node query(int l, int r, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return node(hl, hr);
    if (l <= hl && hr <= r) return arr[i];
    return node(query(l, r, 2*i+1), query(l, r, 2*i+2));
}
node range_update(int l, int r, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return arr[i];
    if (l <= hl && hr <= r) {
        arr[i].range_update(v);
        propagate(i);
        return arr[i];
    }
    return arr[i] = node(range_update(l, r, v, 2*i+1),
        range_update(l, r, v, 2*i+2));
}
void propagate(int i) {
    if (arr[i].l < arr[i].r) {
        arr[i].push(arr[2*i+1]);
        arr[i].push(arr[2*i+2]);
    }
    arr[i].apply();
}
};
```

2.2.3. Persistent segment tree.

Be careful: all intervals are right-closed $[\ell, r]$, including build.

```
int segcnt = 0;
struct segment {
    int l, r, lid, rid, sum;
} S[2000000];

int build(int l, int r) {
    if (l > r) return -1;
    int id = segcnt++;
    S[id].l = l;
    S[id].r = r;
    if (l == r) S[id].lid = -1, S[id].rid = -1;
    else {
        int m = (l + r) / 2;
        S[id].lid = build(l, m);
        S[id].rid = build(m + 1, r);
    }
    S[id].sum = 0;
    return id;
}
```

```

}
int update(int idx, int v, int id) {
    if (id == -1) return -1;
    if (idx < S[id].l || idx > S[id].r) return id;
    int nid = segcnt++;
    S[nid].l = S[id].l;
    S[nid].r = S[id].r;
    S[nid].lid = update(idx, v, S[id].lid);
    S[nid].rid = update(idx, v, S[id].rid);
    S[nid].sum = S[id].sum + v;
    return nid;
}
int query(int id, int l, int r) {
    if (r < S[id].l || S[id].r < l) return 0;
    if (l <= S[id].l && S[id].r <= r) return S[id].sum;
    return query(S[id].lid, l, r) + query(S[id].rid, l, r);
}

```

2.3. Binary Indexed Tree $\mathcal{O}(\log n)$. Use one-based indices ($i > 0$)!

```

struct BIT {
    int n; vi A;
    BIT(int _n) : n(_n), A(_n+1, 0) {}
    BIT(vi &v) : n(sz(v)), A(1) {
        for (auto x:v) A.pb(x);
        for (int i=1, j; j=i&-i, i<=n; i++)
            if (i+j <= n) A[i+j] += A[i];
    }
    void update(int i, ll v) { // a[i] += v
        while (i <= n) A[i] += v, i += i&-i;
    }
    ll query(int i) { // sum_{j<=i} a[j]
        ll v = 0;
        while (i) v += A[i], i -= i&-i;
        return v;
    }
};

struct rangeBIT {
    int n; BIT b1, b2;
    rangeBIT(int _n) : n(_n), b1(_n), b2(_n+1) {}
    rangeBIT(vi &v) : n(sz(v)), b1(v), b2(sz(v)+1) {}
    void pupdate(int i, ll v) { b1.update(i, v); }
    void rupdate(int i, int j, ll v) { // a[i,...,j] += v
        b2.update(i, v);
        b2.update(j+1, -v);
        b1.update(j+1, v*j);
        b1.update(i, (1-i)*v);
    }
    ll query(int i) { return b1.query(i) + b2.query(i)*i; }
};

```

2.4. Disjoint-Set / Union-Find $\mathcal{O}(\alpha(n))$.

```

struct dsu { vi p; dsu(int n) : p(n, -1) {}
    int find(int i) {
        return p[i] < 0 ? i : p[i] = find(p[i]);
    }
    void unite(int a, int b) {

```

```

        if ((a = find(a)) == (b = find(b))) return;
        if (p[a] > p[b]) swap(a, b);
        p[a] += p[b]; p[b] = a;
    }
};

```

2.5. Cartesian tree.

```

struct node {
    int x, y, sz;
    node *l, *r;
    node(int _x, int _y)
        : x(_x), y(_y), sz(1), l(NULL), r(NULL) {}
};
int tsize(node* t) { return t ? t->sz : 0; }
void augment(node* t) {
    t->sz = 1 + tsize(t->l) + tsize(t->r);
}
pair<node*, node*> split(node* t, int x) {
    if (!t) return make_pair((node*)NULL, (node*)NULL);
    if (t->x < x) {
        pair<node*, node*> res = split(t->r, x);
        t->r = res.x; augment(t);
        return make_pair(t, res.y);
    }
    pair<node*, node*> res = split(t->l, x);
    t->l = res.y; augment(t);
    return make_pair(res.x, t);
}
node* merge(node* l, node* r) {
    if (!l) return r; if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r); augment(l); return l;
    }
    r->l = merge(l, r->l); augment(r); return r;
}
node* find(node* t, int x) {
    while (t) {
        if (x < t->x) t = t->l;
        else if (t->x < x) t = t->r;
        else return t;
    }
    return NULL;
}
node* insert(node* t, int x, int y) {
    if (find(t, x) != NULL) return t;
    pair<node*, node*> res = split(t, x);
    return merge(res.x, merge(new node(x, y), res.y));
}
node* erase(node* t, int x) {
    if (!t) return NULL;
    if (t->x < x) t->r = erase(t->r, x);
    else if (x < t->x) t->l = erase(t->l, x);
    else { node* old = t; t = merge(t->l, t->r); delete old; }
    if (t) augment(t); return t;
}
int kth(node* t, int k) {
    if (k < tsize(t->l)) return kth(t->l, k);
    else if (k == tsize(t->l)) return t->x;
    else return kth(t->r, k - tsize(t->l) - 1);
}

```

2.6. Heap. An implementation of a binary heap.

```

template <class Comp = less<int>> struct heap {
    vi q, loc; Comp op;
    heap() : op(Comp()) {}
    bool cmp(int i, int j) { return op(q[i], q[j]); }

```

```

    void swp(int i, int j) {
        swap(q[i], q[j]), swap(loc[q[i]], loc[q[j]]);
    }
    void swim(int i) {
        for (int p; i; swp(i, p), i = p)
            if (!cmp(i, p=(i-1)/2)) break;
    }
    void sink(int i) {
        for (int j; (j=2*i+1) < sz(q); swp(j, i), i=j) {
            if (j+1 < sz(q) && cmp(j+1, j)) ++j;
            if (!cmp(j, i)) break;
        }
    }
    void push(int n) {
        while (n >= sz(loc)) loc.pb(-1);
        assert(loc[n] == -1);
        loc[n] = sz(q), q.pb(n);
        swim(sz(q) - 1);
    }
    int top() { assert(!empty()); return q[0]; }
    int pop() {
        int res = top();
        q[0] = q.back(), q.pop_back();
        loc[q[0]] = 0, loc[res] = -1;
        sink(0); return res;
    }
    void heapify() {
        for (int i=sz(q); --i; )
            if (cmp(i, (i-1)/2)) swp(i, (i-1)/2);
    }
    void update_key(int n) {
        assert(loc[n] != -1);
        swim(loc[n]), sink(loc[n]);
    }
    int size() { return sz(q); }
    bool empty() { return !size(); }
    void clear() { q.clear(), loc.clear(); }
};

```

2.7. Dancing Links. An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```

template <class T>
struct dancing_links {
    struct node {
        T item;
        node *l, *r;
        node(const T &_item, node *_l=NULL, node *_r=NULL)
            : item(_item), l(_l), r(_r) {
            if (!l) l->r = this;
            if (!r) r->l = this;
        }
        node *front, *back;
        dancing_links() { front = back = NULL; }
        node *push_back(const T &item) {
            back = new node(item, back, NULL);
            if (!front) front = back;
            return back;
        }

```



```

node *push_front(const T &item) {
    front = new node(item, NULL, front);
    if (!back) back = front;
    return front; }
void erase(node *n) {
    if (!n->l) front = n->r; else n->l->r = n->r;
    if (!n->r) back = n->l; else n->r->l = n->l; }
void restore(node *n) {
    if (!n->l) front = n; else n->l->r = n;
    if (!n->r) back = n; else n->r->l = n; } }

```

2.8. **Misof Tree.** A simple tree data structure for inserting, erasing, and querying the n th largest element.

```

const int BITS = 15;
struct misof_tree {
    int cnt[BITS][1<<BITS];
    misof_tree() { memset(cnt,0,sizeof(cnt)); }
    void insert(int x) {
        for (int i=0; i<BITS; cnt[i++][x]++, x >>= 1); }
    void erase(int x) {
        for (int i=0; i<BITS; cnt[i++][x]--, x >>= 1); }
    int nth(int n) {
        int res = 0;
        for (int i = BITS-1; i >= 0; i--)
            if (cnt[i][res <= 1] <= n)
                n -= cnt[i][res], res |= 1;
        return res;
    }
};

```

2.9. **k-d Tree.** A k -dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```

#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
    struct pt {
        double coord[K];
        pt() {}
        pt(double c[K]) { REP(i,K) coord[i] = c[i]; }
        double dist(const pt &other) const {
            double sum = 0.0;
            REP(i,K) sum +=
                pow(coord[i]-other.coord[i],2);
            return sqrt(sum); } };
    struct cmp {
        int c;
        cmp(int _c) : c(_c) {}
        bool operator () (const pt &a, const pt &b) {
            for (int i = 0, cc; i <= K; i++) {
                cc = i == 0 ? c : i - 1;
                if (abs(a.coord[cc] - b.coord[cc]) > EPS)
                    return a.coord[cc] < b.coord[cc];
            }
            return false; } };
    struct bb {
        pt from, to;
        bb(pt _from, pt _to) : from(_from), to(_to) {}
    };
};

```

```

double dist(const pt &p) {
    double sum = 0.0;
    REP(i,K) {
        if (p.coord[i] < from.coord[i])
            sum += pow(from.coord[i] - p.coord[i],
                2.0);
        else if (p.coord[i] > to.coord[i])
            sum += pow(p.coord[i] - to.coord[i], 2.0);
    }
    return sqrt(sum); }
bb bound(double l, int c, bool left) {
    pt nf(from.coord), nt(to.coord);
    if (left) nt.coord[c] = min(nt.coord[c], l);
    else nf.coord[c] = max(nf.coord[c], l);
    return bb(nf, nt); } };
struct node {
    pt p; node *l, *r;
    node(pt _p, node *_l, node *_r)
        : p(_p), l(_l), r(_r) {} };
node *root;

// kd_tree() : root(NULL) {}
kd_tree(vector<pt> pts) {
    root = construct(pts, 0, size(pts) - 1, 0); }
node* construct(vector<pt> &pts, int fr, int to,
    int c) {
    if (fr > to) return NULL;
    int mid = fr + (to-fr) / 2;
    nth_element(pts.begin() + fr, pts.begin() + mid,
        pts.begin() + to + 1, cmp(c));
    return new node(pts[mid],
        construct(pts, fr, mid - 1, INC(c)),
        construct(pts, mid + 1, to, INC(c))); }
bool contains(const pt &p) { return
    _con(p,root,0); }
bool _con(const pt &p, node *n, int c) {
    if (!n) return false;
    if (cmp(c)(p, n->p)) return _con(p,n->l,INC(c));
    if (cmp(c)(n->p, p)) return _con(p,n->r,INC(c));
    return true; }
void insert(const pt &p) { _ins(p, root, 0); }
void _ins(const pt &p, node* &n, int c) {
    if (!n) n = new node(p, NULL, NULL);
    else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
    else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c));
}
void clear() { _clr(root); root = NULL; }
void _clr(node *n) {
    if (n) _clr(n->l), _clr(n->r), delete n; }
pt nearest_neighbour(const pt &p, bool same=true)
    {
    assert(root);
    double mn = INFINITY, cs[K];
    REP(i,K) cs[i] = -INFINITY;
    pt from(cs);
    REP(i,K) cs[i] = INFINITY;
    pt to(cs);
}

```

```

return _nn(p, root, bb(from, to), mn, 0,
    same.x);
}
pair<pt, bool> _nn(const pt &p, node *n, bb b,
    double &mn, int c, bool same) {
    if (!n || b.dist(p) > mn)
        return make_pair(pt(), false);
    bool found = same || p.dist(n->p) > EPS,
        l1 = true, l2 = false;
    pt resp = n->p;
    if (found) mn = min(mn, p.dist(resp));
    node *n1 = n->l, *n2 = n->r;
    REP(i,2) {
        if (i == 1 || cmp(c)(n->p, p))
            swap(n1, n2), swap(l1, l2);
        auto res = _nn(p, n1, b.bound(n->p.coord[c],
            c, l1), mn, INC(c), same);
        if (res.y && (!found || p.dist(res.x) <
            p.dist(resp)))
            resp = res.x, found = true;
    }
    return make_pair(resp, found); } };

```

2.10. **Sqrt Decomposition.** Design principle that supports many operations in amortized \sqrt{n} per operation.

```

struct segment {
    vi arr;
    segment(vi _arr) : arr(_arr) {} };
vector<segment> T;
int K;
void rebuild() {
    int cnt = 0;
    rep(i,0,size(T))
        cnt += size(T[i].arr);
    K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
    vi arr(cnt);
    for (int i = 0, at = 0; i < size(T); i++)
        rep(j,0,size(T[i].arr))
            arr[at++] = T[i].arr[j];
    T.clear();
    for (int i = 0; i < cnt; i += K)
        T.push_back(segment(vi(arr.begin()+i,
            arr.begin()+min(i+K,
                cnt)))); }
int split(int at) {
    int i = 0;
    while (i < size(T) && at >= size(T[i].arr))
        at -= size(T[i].arr), i++;
    if (i >= size(T)) return size(T);
    if (at == 0) return i;
    T.insert(T.begin() + i + 1,
        segment(vi(T[i].arr.begin() + at,
            T[i].arr.end())));
    T[i] = segment(vi(T[i].arr.begin(),
        T[i].arr.begin() + at));
    return i + 1; }

```

```

void insert(int at, int v) {
    vi arr; arr.push_back(v);
    T.insert(T.begin() + split(at), segment(arr)); }
void erase(int at) {
    int i = split(at); split(at + 1);
    T.erase(T.begin() + i); }

```

2.11. **Monotonic Queue.** A queue that supports querying for the minimum element. Useful for sliding window algorithms.

```

struct min_stack {
    stack<int> S, M;
    void push(int x) {
        S.push(x);
        M.push(M.empty() ? x : min(M.top(), x)); }
    int top() { return S.top(); }
    int mn() { return M.top(); }
    void pop() { S.pop(); M.pop(); }
    bool empty() { return S.empty(); } };

struct min_queue {
    min_stack inp, outp;
    void push(int x) { inp.push(x); }
    void fix() {
        if (outp.empty()) while (!inp.empty())
            outp.push(inp.top()), inp.pop(); }
    int top() { fix(); return outp.top(); }
    int mn() {
        if (inp.empty()) return outp.mn();
        if (outp.empty()) return inp.mn();
        return min(inp.mn(), outp.mn()); }
    void pop() { fix(); outp.pop(); }
    bool empty() { return inp.empty() && outp.empty(); } };

```

2.12. **Line container à la ‘Convex Hull Trick’** $\mathcal{O}(n \log n)$. Container where you can add lines of the form $y_i(x) = k_i x + m_i$ and query $\max_i y_i(x)$.

```

bool Q;
struct Line {
    mutable ll k, m, p;
    bool operator<(const Line& o) const {
        return Q ? p < o.p : k < o.k;
    }
};

struct LineContainer : multiset<Line> {
    // (for doubles, use inf = 1/.0, div(a,b) = a/b)
    const ll inf = LLONG_MAX;
    ll div(ll a, ll b) { // floored division
        return a / b - ((a ^ b) < 0 && a % b); }
    bool isect(iterator x, iterator y) {
        if (y == end()) { x->p = inf; return false; }
        if (x->k == y->k)
            x->p = x->m > y->m ? inf : -inf;
        else
            x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    }
    void add(ll k, ll m) {

```

```

        auto z = insert({k, m, 0}), y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y))
            isect(x, y = erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
    }
    ll query(ll x) {
        assert(!empty());
        Q=1; auto l = *lower_bound({0,0,x}); Q=0;
        return l.k * x + l.m;
    }
};

```

2.13. **Sparse Table** $\mathcal{O}(\log n)$ per query.

```

struct sparse_table {
    vvi m;
    sparse_table(vi arr) {
        m.pb(arr);
        for (int k=0; (1<<(+k)) <= sz(arr); ) {
            int w = (1<<k), hw = w/2;
            m.pb(vi(sz(arr) - w + 1);
            for (int i = 0; i+w <= sz(arr); i++) {
                m[k][i] = min(m[k-1][i], m[k-1][i+hw]);
            }
        }
        int query(int l, int r) { // query min in [l,r]
            int k = 31 - __builtin_clz(r-l); // k = 0;
            // while (1<<(k+1) <= r-l+1) k++;
            return min(m[k][l], m[k][r-(1<<k)+1]);
        }
    }
};

```

3. GRAPH ALGORITHMS

3.1. Shortest path.

3.1.1. **Dijkstra** $\mathcal{O}(|E| \log |V|)$.

```

// (dist, prev)
pair<vi,vi> dijkstra(const vector<vii> &G, int s) {
    vi d(sz(G), LLONG_MAX), p(sz(G), -1);
    set<ii> Q{ ii{ d[s] = 0, s } }; // (dist[v], v)
    while (!Q.empty()) {
        int v = Q.begin()->y;
        Q.erase(Q.begin());
        for(ii e : G[v]) if (d[v] + e.y < d[e.x]) {
            Q.erase(ii(d[e.x], e.x));
            Q.emplace(d[e.x] = d[v] + e.y, e.x);
            p[e.x] = v;
        }
    }
    return {d, p};
}

```

3.1.2. **Floyd-Warshall** $\mathcal{O}(V^3)$. Be careful with negative edges! Note: $|d[i][j]|$ can grow exponentially, and INFTY + negative < INFTY.

```

const ll INF = 1LL << 61;
void floyd_warshall(vvi& d) {
    ll n = d.size();
    REP(i,n) REP(j,n) REP(k,n)
        if(d[j][i] < INF && d[i][k] < INF) // neg edges!
            d[j][k] = max(-INF,
                min(d[j][k], d[j][i] + d[i][k]));
}

vvi d(n,vi(n,INF));
REP(i,n) d[i][i] = 0;

```

3.1.3. **Bellman Ford** $\mathcal{O}(VE)$. This is only useful if there are edges with weight $w_{ij} < 0$ in the graph.

```

const ll INF = 1LL << 61;
// G[u] = { (v,w) | edge u->v, cost w }
vi bellman_ford(vector<vii> G, ll s) {
    ll n = G.size();
    vi d(n, INF); d[s] = 0;
    REP(loops, n) REP(u, n) if(d[u] != INF)
        for(ii e : G[u]) if(d[u] + e.y < d[e.x])
            d[e.x] = d[u] + e.y;
    // detect paths of -INF length
    for( ll change = 1; change--> 0; )
        REP(u, n) if(d[u] != INF)
            for(ii e : G[u]) if(d[u] + e.y < d[e.x])
                d[e.x] = -INF, change = 1;
    return d;
}

```

3.1.4. **IDA* algorithm.**

```

int n, cur[100], pos;
int calch() {
    int h = 0;
    rep(i,0,n) if (cur[i] != 0) h += abs(i - cur[i]);
    return h; }
int dfs(int d, int g, int prev) {
    int h = calch();
    if (g + h > d) return g + h;
    if (h == 0) return 0;
    int mn = INT_MAX;
    rep(di,-2,3) {
        if (di == 0) continue;
        int nxt = pos + di;
        if (nxt == prev) continue;
        if (0 <= nxt && nxt < n) {
            swap(cur[pos], cur[nxt]);
            swap(pos,nxt);
            mn = min(mn, dfs(d, g+1, nxt));
            swap(pos,nxt);
            swap(cur[pos], cur[nxt]);
        }
        if (mn == 0) break;
    }
    return mn; }
int idastar() {
    rep(i,0,n) if (cur[i] == 0) pos = i;
    int d = calch();
}

```

```

while (true) {
    int nd = dfs(d, 0, -1);
    if (nd == 0 || nd == INT_MAX) return d;
    d = nd; } }

```

3.2. Maximum Matching.

Matching: A set of edges without common vertices (*Maximum is the largest such set, maximal is a set which you cannot add more edges to without breaking the property*).

Minimum Vertex Cover: A set of vertices such that each edge in the graph is incident to at least one vertex of the set.

Minimum Edge Cover: A set of edges such that every vertex is incident to at least one edge of the set.

Maximum Independent Set: A set of vertices in a graph such that no two of them are adjacent.

Minimum edge cover \iff Maximum independent set.

König's theorem: In any bipartite graph $G = (L \cup R, E)$, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover. Let U be the set of unmatched vertices in L , and Z be the set of vertices that are either in U or are connected to U by an alternating path. Then $K = (L \setminus Z) \cup (R \cap Z)$ is the minimum vertex cover.

In any bipartite graph,

$$\text{maxmatch} = \text{MVC} = V - \text{MIS}.$$

See 3.2.2.

3.2.1. Standard bipartite matching $\mathcal{O}(|L| \cdot |R|)$.

vector<bool> vis; vi par; vvi adj; // L->{R,...}

```

bool match(int u) {
    for (int v : adj[u]) {
        if (vis[v]) continue;
        vis[v] = true;
        if (par[v] == -1 || match(par[v])) {
            par[v] = u; return true; }
    }
    return false;
}

```

// perfect matching iff ret == L == R

```

int maxmatch(int L, int R) {
    par.assign(R, -1);
    int ret = 0;
    REP(i, L) vis.assign(R, false), ret += match(i);
    return ret;
}

```

3.2.2. Hopcroft-Karp bipartite matching $\mathcal{O}(E\sqrt{V})$.

```

struct bi_graph {
    int n, m, s; vvi G; vi L, R, d;
    bi_graph(int _n, int _m) : n(_n), m(_m), s(0),
        G(n), L(n, -1), R(m, n), d(n+1) {}
    void add_edge(int a, int b) { G[a].pb(b); }
    bool bfs() {
        queue<int> q; d[n] = LLONG_MAX;
        REP(v, n)
            if (L[v] < 0) d[v] = 0, q.push(v);
            else d[v] = LLONG_MAX;
        while (!q.empty()) {
            int v = q.front(); q.pop();
            if (d[v] >= d[n]) continue;
            for (int u : G[v]) if (d[R[u]] == LLONG_MAX)
                d[R[u]] = d[v]+1, q.push(R[u]);
        }
        return d[n] != LLONG_MAX;
    }
    bool dfs(int v) {
        if (v == n) return true;
        for (int u : G[v])
            if (d[R[u]] == d[v]+1 && dfs(R[u])) {
                R[u] = v; L[v] = u; return true;
            }
        d[v] = LLONG_MAX; return false;
    }
    int max_match() {
        while (bfs()) REP(i, n) s += L[i]<0 && dfs(i);
        return s;
    }

    void dfs2(int v, vector<bool> &alt) {
        alt[v] = true;
        for (int u : G[v]) {
            alt[u+n] = true;
            if (R[u] != n && !alt[R[u]]) dfs2(R[u], alt);
        }
    }

    vi min_vertex_cover() {
        vector<bool> alt(n+m, false); vi res;
        max_match();
        REP(i, n) if (L[i] < 0) dfs2(i, alt);
        // !alt[i] (i<n) OR alt[i] (i >= n)
        REP(i, n+m) if (alt[i] != (i<n)) res.pb(i);
        return res;
    }
};

```

3.2.3. Stable marriage. With n men, $m \geq n$ women, n preference lists of women for each men, and for every woman j a preference of men defined by $\text{pref}[][j]$ (lower is better) find for every man a woman such that no pair of a men and a woman want to run off together.

// n = aantal mannen, m = aantal vrouwen
// voor een man i, is order[i] de prefere

```

vi stable(int n, int m, vvi order, vvi pref) {
    queue<int> q;
    REP(i, n) q.push(i);
    vi mas(m, -1), mak(n, -1), p(n, 0);
    while (!q.empty()) {
        int k = q.front();
        q.pop();
        int s = order[k][p[k]], k2 = mas[s];
        if (mas[s] == -1) {
            mas[s] = k;
            mak[k] = s;
        } else if (pref[k][s] < pref[k2][s]) {
            mas[s] = k;
            mak[k] = s;
            mak[k2] = -1;
            q.push(k2);
        } else {
            q.push(k);
        }
        p[k]++;
    }
    return mak;
}

```

3.3. Depth first searches.

3.3.1. Topological Sort $\mathcal{O}(V + E)$.

```

vi topo(vvi &adj) { // requires C++14
    int n=sz(adj); vector<bool> vis(n, 0); vi ans;
    auto dfs = [&](int v, const auto& f)->void {
        vis[v] = true;
        for (int w : adj[v]) if (!vis[w]) f(w, f);
        ans.pb(v);
    };
    REP(i, n) if (!vis[i]) dfs(i, dfs);
    reverse(all(ans));
    return ans;
}

```

3.4. Cycle Detection $\mathcal{O}(V + E)$.

```

// returns cycle in a connected undirected graph,
vi find_cycle(const vvi &G, int v0) { // if exists
    vi p(sz(G), -1), h(sz(G), 0), s{v0}; h[v0] = 1;
    while (!s.empty()) {
        int v = s.back(); s.pop_back();
        for (int w : G[v])
            if (!h[w]) s.pb(w), p[w] = v, h[w] = h[v]+1;
            else if (w != p[v]) {
                deque<int> cyc{v};
                while (v != w)
                    if (h[v] > h[w]) cyc.pb(v = p[v]);
                    else cyc.push_front(w, w = p[w]);
                return vi(all(cyc));
            }
    }
    return {};
}

```


3.4.1. Cut Points and Bridges $O(V + E)$.

```

const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;

void dfs(vvi &adj, vi &cp, vii &bs, int u, int p) {
    low[u] = num[u] = curnum++;
    int cnt = 0; bool found = false;
    REP(i, sz(adj[u])) {
        int v = adj[u][i];
        if (num[v] == -1) {
            dfs(adj, cp, bs, v, u);
            low[u] = min(low[u], low[v]);
            cnt++;
            found = found || low[v] >= num[u];
            if (low[v] > num[u]) bs.pb(u, v);
        } else if (p != v) low[u] = min(low[u], num[v]);
    }
    if (found && (p != -1 || cnt > 1)) cp.pb(u);
}

pair<vi, vii> cut_points_and_bridges(vvi &adj) {
    int n = size(adj);
    vi cp; vii bs;
    memset(num, -1, n << 2);
    curnum = 0;
    REP(i, n) if (num[i] < 0) dfs(adj, cp, bs, i, -1);
    return make_pair(cp, bs);
}

```

3.4.2. Strongly Connected Components $O(V + E)$.

```

struct SCC {
    int n, age=0, ncomps=0; vvi adj, comps;
    vi tidx, lnk, cnr, st; vector<bool> vis;
    SCC(vvi &adj) : n(sz(_adj)), adj(_adj),
        tidx(n, 0), lnk(n), cnr(n), vis(n, false) {
        REP(i, n) if (!tidx[i]) dfs(i);
    }

    void dfs(int v) {
        tidx[v] = lnk[v] = ++age;
        vis[v] = true; st.pb(v);
        for (int w : adj[v]) {
            if (!tidx[w])
                dfs(w, lnk[v] = min(lnk[v], lnk[w]));
            else if (vis[w]) lnk[v] = min(lnk[v],
                tidx[w]);
        }
        if (lnk[v] != tidx[v]) return;
        comps.pb(vi());
        int w;
        do {
            vis[w = st.back()] = false; cnr[w] = ncomps;
            comps.back().pb(w);
            st.pop_back();
        } while (w != v);
        ncomps++;
    }
}

```

```

}
};

```

3.4.3. 2-SAT $O(V + E)$. Uses SCC.

```

struct TwoSat {
    int n; SCC *scc = nullptr; vvi adj;
    TwoSat(int _n) : n(_n), adj(_n*2, vi()) {}
    ~TwoSat() { delete scc; }

    // l => r, i.e. r is true or ~l
    void imply(int l, int r) {
        adj[n+l].pb(n+r); adj[n+~r].pb(n+~l); }
    void OR(int a, int b) { imply(~a, b); }
    void CONST(int a) { OR(a, a); }
    void IFF(int a, int b) { imply(a, b); imply(b, a); }

    bool solve(vector<bool> &sol) {
        delete scc; scc = new SCC(adj);
        REP(i, n) if (scc->cnr[n+i] == scc->cnr[n+~i])
            return false;
        vector<bool> seen(n, false);
        sol.assign(n, false);
        for (vi &cc : scc->comps) for (int v : cc) {
            int i = v < n ? n + (~v) : v - n;
            if (!seen[i]) seen[i] = true, sol[i] = v >= n;
        }
        return true;
    }
};

```

3.4.4. Dominator graph.

- A node d dominates a node n if every path from the entry node to n must go through d .
- The immediate dominator (idom) of a node n is the unique node that strictly dominates n but does not strictly dominate any other node that strictly dominates n .

```

const int N = 1e6;
vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], par[N], sdом[N];
int p[N], best[N], idом[N], link[N];

void dfs(int v) {
    pos[v] = cnt;
    order[cnt++] = v;
    for (int u : g[v])
        if (pos[u] < 0) par[u] = v, dfs(u);
}

int find_best(int x) {
    if (p[x] == x) return best[x];
    int u = find_best(p[x]);
    if (pos[sdom[u]] < pos[sdom[best[x]]])
        best[x] = u;
    p[x] = p[p[x]];
    return best[x];
}

```

```

void dominators(int n, int root) {
    fill_n(pos, n, -1);
    cnt = 0;
    dfs(root);
    REP(i, n) for (int u : g[i]) g_rev[u].pb(i);
    REP(i, n) p[i] = best[i] = sdom[i] = i;

    for (int it = cnt - 1; it >= 1; it--) {
        int w = order[it];
        for (int u : g_rev[w]) {
            int t = find_best(u);
            if (pos[sdom[t]] < pos[sdom[w]])
                sdom[w] = sdom[t];
        }
        bucket[sdom[w]].pb(w);
        idом[w] = sdom[w];
        for (int u : bucket[par[w]])
            link[u] = find_best(u);
        bucket[par[w]].clear();
        p[w] = par[w];
    }

    for (int it = 1; it < cnt; it++) {
        int w = order[it];
        idом[w] = idом[link[w]];
    }
}

```

3.5. Min Cut / Max Flow.

3.5.1. Dinic's Algorithm $O(V^2E)$.

```

struct Edge { int t; ll c, f; };
struct Dinic {
    vi H, P; vvi E;
    vector<Edge> G;
    Dinic(int n) : H(n), P(n), E(n) {}

    void addEdge(int u, int v, ll c) {
        E[u].pb(G.size()); G.pb({v, c, 0LL});
        E[v].pb(G.size()); G.pb({u, 0LL, 0LL});
    }

    ll dfs(int t, int v, ll f) {
        if (v == t || !f) return f;
        for (; P[v] < (int) E[v].size(); P[v]++) {
            int e = E[v][P[v]], w = G[e].t;
            if (H[w] != H[v] + 1) continue;
            ll df = dfs(t, w, min(f, G[e].c - G[e].f));
            if (df > 0) {
                G[e].f += df, G[e ^ 1].f -= df;
                return df;
            }
        }
        return 0;
    }

    ll maxflow(int s, int t, ll f = 0) {
        while (1) {
            fill(all(H), 0); H[s] = 1;
            queue<int> q; q.push(s);

```

```

while (!q.empty()) {
    int v = q.front(); q.pop();
    for (int w : E[v])
        if (G[w].f < G[w].c && !H[G[w].t])
            H[G[w].t] = H[v] + 1, q.push(G[w].t);
}
if (!H[t]) return f;
fill(all(P), 0);
while (ll df = dfs(t, s, LLONG_MAX)) f += df;
}
};

```

3.5.2. *Min-cost max-flow* $O(n^2m^2)$. Find the cheapest possible way of sending a certain amount of flow through a flow network.

```

const int maxn = 300;

struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
    g.pb({u, v, 0, c, w});
    g.pb({v, u, 0, 0, -w});
}

void sp(int s, int t) {
    fill_n(D, V, LLONG_MAX); D[s] = 0;
    for (int ng = g.size(), _ = V; _--;) {
        bool ok = false;
        for (int i = 0; i < ng; i++)
            if (D[g[i].x] != LLONG_MAX && g[i].f < g[i].c
                && D[g[i].x] + g[i].w < D[g[i].y]) {
                D[g[i].y] = D[g[i].x] + g[i].w;
                par[g[i].y] = i; ok = true;
            }
        if (!ok) break;
    }
}

void minCostMaxFlow(int s, int t, ll &c, ll &f) {
    for (c = f = 0; sp(s, t), D[t] < LLONG_MAX; ) {
        ll df = LLONG_MAX, dc = 0;
        for (int v = t, e; e = par[v], v != s; v =
            ↪ g[e].x) df = min(df, g[e].c - g[e].f);
        for (int v = t, e; e = par[v], v != s; v =
            ↪ g[e].x) g[e].f += df, g[e^1].f -= df, dc +=
            ↪ g[e].w;
        f += df; c += dc * df;
    }
}

```

3.5.3. *Gomory-Hu Tree - All Pairs Maximum Flow*. An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in $O(|V|^2)$ plus $|V|-1$ times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is

$O(|V|^3|E|)$. NOTE: Not sure if it works correctly with disconnected graphs.

```

#include "dinic.cpp"
bool same[MAXV];
pair<vii, vvi> construct_gh_tree(flow_network &g) {
    int n = g.n, v;
    vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
    rep(s, 1, n) {
        int l = 0, r = 0;
        par[s].second = g.max_flow(s, par[s].first,
            ↪ false);
        memset(d, 0, n * sizeof(int));
        memset(same, 0, n * sizeof(bool));
        d[q[r++]] = s; l = 1;
        while (l < r) {
            same[v = q[l++]] = true;
            for (int i = g.head[v]; i != -1; i =
                ↪ g.e[i].nxt)
                if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
                    d[q[r++]] = g.e[i].v; l = 1;
        }
        rep(i, s+1, n)
            if (par[i].first == par[s].first && same[i])
                par[i].first = s;
        g.reset();
    }
    rep(i, 0, n) {
        int mn = INT_MAX, cur = i;
        while (true) {
            cap[cur][i] = mn;
            if (cur == 0) break;
            mn = min(mn, par[cur].second), cur =
                ↪ par[cur].first;
        }
        return make_pair(par, cap);
    }
}

int compute_max_flow(int s, int t, const pair<vii,
    ↪ vvi> &gh) {
    int cur = INT_MAX, at = s;
    while (gh.second[at][t] == -1)
        cur = min(cur, gh.first[at].second),
        at = gh.first[at].first;
    return min(cur, gh.second[at][t]);
}

```

3.6. Minimal Spanning Tree $O(E \log V)$.

```

struct edge { int x, y; ll w; };
ll kruskal(int n, vector<edge> edges) {
    dsu D(n);
    sort(all(edges), [] (edge a, edge b) -> bool {
        return a.w < b.w; });
    ll ret = 0;
    for (edge e : edges)
        if (D.find(e.x) != D.find(e.y))
            ret += e.w, D.unite(e.x, e.y);
    return ret;
}

```

3.7. *Euler Path* $O(V + E)$ hopefully. Finds an Euler Path (or circuit) in a *directed* graph iff one exists.

```

const int MAXV = 1000, MAXE = 5000;
vi adj[MAXV];

```

```

int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
    int start = -1, end = -1, any = 0, c = 0;
    REP(i, n) {
        if (outdeg[i] > 0) any = i;
        if (indeg[i] + 1 == outdeg[i]) start = i, c++;
        else if (indeg[i] == outdeg[i] + 1) end = i, c++;
        else if (indeg[i] != outdeg[i]) return ii(-1, -1);
    }
    if ((start == -1) != (end == -1) || (c != 2 && c))
        return ii(-1, -1);
    if (start == -1) start = end = any;
    return ii(start, end);
}

bool euler_path() {
    ii se = start_end();
    int cur = se.first, at = m + 1;
    if (cur == -1) return false;
    stack<int> s;
    while (true) {
        if (outdeg[cur] == 0) {
            res[--at] = cur;
            if (s.empty()) break;
            cur = s.top(); s.pop();
        } else s.push(cur), cur =
            ↪ adj[cur][--outdeg[cur]];
    }
    return at == 0;
}

```

Finds an Euler *cycle* in a *undirected* graph:

```

const int MAXV = 1000;
multiset<int> adj[MAXV];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
    if (at == to) return it;
    L.insert(it, at), --it;
    while (!adj[at].empty()) {
        int nxt = *adj[at].begin();
        adj[at].erase(adj[at].find(nxt));
        adj[nxt].erase(adj[nxt].find(at));
        if (to == -1) {
            it = euler(nxt, at, it);
            L.insert(it, at);
            --it;
        } else {
            it = euler(nxt, to, it);
            to = -1;
        }
    }
    return it;
}
// usage: euler(0, -1, L.begin());

```

3.8. Heavy-Light Decomposition.

```

struct HLD {
    vvi adj; int cur_pos = 0;
    vi par, dep, hvy, head, pos;

    HLD(int n, const vvi &A) : adj(all(A)), par(n),
        dep(n), hvy(n, -1), head(n), pos(n) {

```

```

    cur_pos = 0; dfs(0); decomp(0, 0);
}

int dfs(int v) { // determine parent/depth/sizes
    int wei = 1, mw = 0;
    for (int c : adj[v]) if (c != par[v]) {
        par[c] = v, dep[c] = dep[v]+1;
        int w = dfs(c);
        wei += w;
        if (w > mw) mw = w, hvy[v] = c;
    }
    return wei;
}

// pos: index in SegmentTree, head: root of path
void decomp(int v, int h) {
    head[v] = h, pos[v] = cur_pos++;
    if (hvy[v] != -1) decomp(hvy[v], h);
    for (int c : adj[v])
        if (c != par[v] && c != hvy[v]) decomp(c, c);
}

// requires queryST(a, b) = max{A[i] | a ≤ i < b}.
int query(int a, int b) {
    int res = 0;
    for (; head[a] != head[b]; b = par[head[b]]) {
        if (dep[head[a]] > dep[head[b]]) swap(a, b);
        res = max(res, queryST(pos[head[b]], pos[b]+1));
    }
    if (dep[a] > dep[b]) swap(a, b);
    return max(res, queryST(pos[a], pos[b]+1));
}
};

```

3.9. Centroid Decomposition.

```

#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
    path[MAXV][LGMAXV],
    sz[MAXV], seph[MAXV],
    shortest[MAXV];
struct centroid_decomposition {
    int n; vvi adj;
    centroid_decomposition(int _n : n(_n), adj(n) {}
    void add_edge(int a, int b) {
        adj[a].push_back(b); adj[b].push_back(a);
    }
    int dfs(int u, int p) {
        sz[u] = 1;
        rep(i, 0, size(adj[u]))
            if (adj[u][i] != p) sz[u] += dfs(adj[u][i],
                ↳ u);
        return sz[u];
    }
    void makepaths(int sep, int u, int p, int len) {
        jmp[u][seph[sep]] = sep, path[u][seph[sep]] =
            ↳ len;
        int bad = -1;
        rep(i, 0, size(adj[u])) {
            if (adj[u][i] == p) bad = i;

```

```

        else makepaths(sep, adj[u][i], u, len + 1);
    }
    if (p == sep)
        swap(adj[u][bad], adj[u].back()),
        ↳ adj[u].pop_back();
    void separate(int h=0, int u=0) {
        dfs(u, -1); int sep = u;
        down: iter(nxt, adj[sep])
            if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2)
                ↳ {
                    sep = *nxt; goto down;
                }
        seph[sep] = h, makepaths(sep, sep, -1, 0);
        rep(i, 0, size(adj[sep])) separate(h+1,
            ↳ adj[sep][i]);
    }
    void paint(int u) {
        rep(h, 0, seph[u]+1)
            shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
                path[u][h]);
    }

    int closest(int u) {
        int mn = INT_MAX/2;
        rep(h, 0, seph[u]+1)
            mn = min(mn, path[u][h] +
                ↳ shortest[jmp[u][h]]);
        return mn;
    }
};

```

3.10. Least Common Ancestors, Binary Jumping.

```

const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
    rep(i, 0, SZ) BP[i][0] = P[i];
    rep(j, 1, LOGSZ) rep(i, 0, SZ)
        BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
    if (H[a] > H[b]) swap(a, b);
    int dh = H[b] - H[a], j = 0;
    rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
    while (BP[a][j] != BP[b][j]) j++;
    while (--j >= 0) if (BP[a][j] != BP[b][j])
        a = BP[a][j], b = BP[b][j];
    return a == b ? a : P[a];
}

```

3.11. Miscellaneous.

3.11.1. *Misra-Gries D+1-edge coloring.* Finds a max, $\deg(i) + 1$ -edge coloring where there all incident edges have distinct colors. Finding a D -edge coloring is NP-hard.

```

struct Edge { int to, col, rev; };

```

```

struct MisraGries {
    int N, K=0; vvi F;
    vector<vector<Edge>> G;

    MisraGries(int n) : N(n), G(n) {}

```

```

// add an undirected edge, NO DUPLICATES ALLOWED
void addEdge(int u, int v) {
    G[u].pb({v, -1, (int) G[v].size()});
    G[v].pb({u, -1, (int) G[u].size()-1});
}

void color(int v, int i) {
    vi fan = { i };
    vector<bool> used(G[v].size());
    used[i] = true;
    for (int j = 0; j < (int) G[v].size(); j++)
        if (!used[j] && G[v][j].col >= 0 &&
            ↳ F[G[v][fan.back()]].to[G[v][j].col] < 0)
            used[j] = true, fan.pb(j), j = -1;
    int c = 0; while (F[v][c] >= 0) c++;
    int d = 0; while (F[G[v][fan.back()]].to[d] >=
        ↳ 0) d++;
    int w = v, a = d, k = 0, ccol;
    while (true) {
        swap(F[w][c], F[w][d]);
        if (F[w][c] >= 0) G[w][F[w][c]].col = c;
        if (F[w][d] >= 0) G[w][F[w][d]].col = d;
        if (F[w][a^c^d] < 0) break;
        w = G[w][F[w][a]].to;
    }
    do {
        Edge &e = G[v][fan[k]];
        ccol = F[e.to][d] < 0 ? d :
            ↳ G[v][fan[k+1]].col;
        if (e.col >= 0) F[e.to][e.col] = -1;
        F[e.to][ccol] = e.rev;
        F[v][ccol] = fan[k];
        e.col = G[e.to][e.rev].col = ccol;
        k++;
    } while (ccol != d);
}

// finds a K-edge-coloring
void color() {
    REP(v, N) K = max(K, (int) G[v].size() + 1);
    F = vvi(N, vi(K, -1));
    REP(v, N) for (int i = G[v].size(); i--;)
        if (G[v][i].col < 0) color(v, i);
}
};

```

3.11.2. *Minimum Mean Weight Cycle.* Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```

double
↳ min_mean_cycle(vector<vector<pair<int, double>>>
↳ adj) {
    int n = size(adj); double mn = INFINITY;
    vector<vector<double>> arr(n+1, vector<double>(n,
        ↳ mn));

```

```

arr[0][0] = 0;
rep(k,1,n+1) rep(j,0,n) iter(it,adj[j])
    arr[k][it->first] = min(arr[k][it->first],
        it->second +
        ↪ arr[k-1][j]);

rep(k,0,n) {
    double mx = -INFINITY;
    rep(i,0,n) mx = max(mx,
        ↪ (arr[n][i]-arr[k][i])/(n-k));
    mn = min(mn, mx); }
return mn; }

```

3.11.3. *Minimum Arborescence*. Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root r to each vertex. Returns a vector of size n , where the i th element is the edge for the i th vertex. The answer for the root is undefined!

$\mathcal{O}(EV)$ runtime and $\mathcal{O}(E)$ memory:

```

#include "../datastructures/union_find.cpp"
struct arborescence {
    int n; union_find uf;
    vector<vector<pair<ii,int>>> adj;
    arborescence(int _n) : n(_n), uf(n), adj(n) { }
    void add_edge(int a, int b, int c) {
        adj[b].eb(ii(a,b),c); }
    vii find_min(int r) {
        vi vis(n,-1), mn(n,INT_MAX); vii par(n);
        REP(i, n) {
            if (uf.find(i) != i) continue;
            int at = i;
            while (at != r && vis[at] == -1) {
                vis[at] = i;
                for (auto it : adj[at])
                    if (it.y < mn[at] && uf.find(it.x.x) !=
                        ↪ at)
                        mn[at] = it.y, par[at] = it.x;
            if (par[at] == ii(0,0)) return vii();
            at = uf.find(par[at].x);
        }
        if (at == r || vis[at] != i) continue;
        union_find tmp = uf;
        vi seq;
        do seq.pb(at), at = uf.find(par[at].x);
        while (at != seq.front());
        int c = uf.find(seq[0]);
        for (auto it : seq) uf.unite(it, c);
        for (auto &jt : adj[c]) jt.y -= mn[c];
        for (auto it : seq) {
            if (it == c) continue;
            for (auto jt : adj[it])
                adj[c].eb(jt.x, jt.y - mn[it]);
            adj[it].clear();
        }
        vii rest = find_min(r);
        if (rest.empty()) return rest;
        ii use = rest[c];

```

```

        rest[at = tmp.find(use.y)] = use;
        for (int it : seq) if (it != at)
            rest[it] = par[it];
        return rest;
    }
    return par; } };

 $\mathcal{O}(V^2 \log V)$  runtime and  $\mathcal{O}(E)$  memory:
const int oo = 0x3f3f3f3f, MAXN = 4024;

// N = #V, R = root
int N, R;
// for each node a list of pairs (predecessor,
↪ cost):
vector<pii> g[MAXN];
int pred[MAXN], label[MAXN], node[MAXN],
↪ helper[MAXN];

int get_node(int n) {
    return node[n] == n ? n :
        (node[n] = get_node(node[n]));
}

int update_node(int n) {
    int m = oo;
    for (auto ed : g[n]) m = min(m, ed.y);
    REP(j, sz(g[n])) {
        g[n][j].y -= m;
        if (g[n][j].y == 0)
            pred[n] = g[n][j].x;
    }
    return m;
}

ll cycle(vi &active, int n, int &cend) {
    n = get_node(n);
    if (label[n] == 1) return false;
    if (label[n] == 2) { cend = n; return 0; }

    active.pb(n);
    label[n] = 2;
    auto res = cycle(active, pred[n], cend);
    if (cend == n) {
        int F = find(all(active), n)-active.begin();
        vi todo(active.begin() + F, active.end());
        active.resize(F);
        vii> newg;
        for (auto i: todo) node[i] = n;
        for (auto i: todo) for (auto &ed : g[i])
            helper[ed.x = get_node(ed.x)] = ed.y;
        for (auto i: todo) for (auto ed : g[i])
            helper[ed.x] = min(ed.y, helper[ed.x]);
        for (auto i: todo) for (auto ed: g[i]) {
            if (helper[ed.x] != oo && ed.x != n) {
                newg.eb(ed.x, helper[ed.x]);
                helper[ed.x] = oo;
            }
        }
    }
}

```

```

g[n] = newg;
res += update_node(n);
label[n] = 0;
cend = -1;
return cycle(active, n, cend) + res;
}
if (cend == -1) {
    active.pop_back();
    label[n] = 1;
}
return res;
}

// Calculates value of minimal arborescence from R,
// assuming it exists.
// NOTE: N, R must be initialized at this point!!!
// Algo changes g!!
ll min_arbor() {
    ll res = 0;
    REP(i, N) {
        node[i] = i;
        if (i != R) res += update_node(i);
    }
    REP(i, N) label[i] = (i==R);
    REP(i, N) {
        if (label[i] == 1 || get_node(i) != i)
            continue;
        vi active;
        int cend = -1;
        res += cycle(active, i, cend);
    }
    return res;
}

```

3.11.4. *Maximum Density Subgraph*. Given (weighted) undirected graph G . Binary search density. If g is current density, construct flow network: (S, u, m) , $(u, T, m + 2g - d_u)$, $(u, v, 1)$, where m is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty S -component, then maximum density is smaller than g , otherwise it's larger. Distance between valid densities is at least $1/(n(n-1))$. Edge case when density is 0. This also works for weighted graphs by replacing d_u by the weighted degree, and doing more iterations (if weights are not integers).

3.11.5. *Maximum-Weight Closure*. Given a vertex-weighted directed graph G . Turn the graph into a flow network, adding weight ∞ to each edge. Add vertices S, T . For each vertex v of weight w , add edge (S, v, w) if $w \geq 0$, or edge $(v, T, -w)$ if $w < 0$. Sum of positive weights minus minimum $S - T$ cut is the answer. Vertices reachable from S are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

3.11.6. *Maximum Weighted Independent Set in a Bipartite Graph.* This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges $(S, u, w(u))$ for $u \in L$, $(v, T, w(v))$ for $v \in R$ and (u, v, ∞) for $(u, v) \in E$. The minimum S, T -cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

3.11.7. *Synchronizing word problem.* A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

4. STRING ALGORITHMS

4.1. Trie.

```
const int SIGMA = 26;

struct trie {
    bool word; trie **adj;

    trie() : word(false), adj(new trie*[SIGMA]) {
        for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
    }

    void addWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) cur->adj[i] = new trie();
            cur = cur->adj[i];
        }
        cur->word = true;
    }

    bool isWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) return false;
            cur = cur->adj[i];
        }
        return cur->word;
    }
};
```

4.2. Z-algorithm $\mathcal{O}(n)$.

// $z[i]$ = length of longest substring starting from $s[i]$ which is also a prefix of s .

```
vi z_function(const string &s) {
    int n = (int) s.length();
    vi z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]])
            ++z[i];
    }
```

```
        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    }
    return z;
}
```

4.3. **Suffix array** $\mathcal{O}(n \log n)$. Lexicographically sorts the cyclic shifts of S where $p[0]$ is the index of the smallest string, etc.

```
vi sort_cyclic_shifts(const string &s) {
    const int alphabet = 256, n = sz(s);

    vi p(n), c(n), cnt(max(alphabet, n), 0);
    REP(i, n) cnt[s[i]]++;
    partial_sum(all(cnt), cnt.begin());
    REP(i, n) p[--cnt[s[i]]] = i;
    c[p[0]] = 0;
    int cl = 1;
    rep(i, 1, n) {
        if (s[p[i]] != s[p[i-1]]) cl++;
        c[p[i]] = cl - 1;
    }

    vi pn(n), cn(n);
    for (int h = 0, l = 1; l < n; l*=2, ++h) {
        REP(i, n) {
            pn[i] = p[i] - (l << h);
            if (pn[i] < 0) pn[i] += n;
        }
        fill(cnt.begin(), cnt.begin() + cl, 0);
        REP(i, n) cnt[c[pn[i]]]++;
        rep(i, 1, cl) cnt[i] += cnt[i-1];
        for (int i = n-1; i >= 0; i--)
            p[--cnt[c[pn[i]]]] = pn[i];
        cn[p[0]] = 0;
        cl = 1;
        rep(i, 1, n) {
            if (c[p[i]] != c[p[i-1]] || c[(p[i]+l)%n]
                != c[(p[i-1]+l)%n]) cl++;
            cn[p[i]] = cl - 1;
        }
        c.swap(cn);
    }
    return p;
}
```

```
vi suffix_array(string s) {
    s += '\0';
    vi v = sort_cyclic_shifts(s);
    v.erase(v.begin());
    return v;
}
```

4.4. **Longest Common Subsequence** $\mathcal{O}(n^2)$. SUBSTRING: consecutive characters !!!

```
int dp[STR_SIZE][STR_SIZE]; // DP problem

int lcs(const string &w1, const string &w2) {
```

```
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i < n1; i++) {
        for (int j = 0; j < n2; j++) {
            if (i == 0 || j == 0) dp[i][j] = 0;
            else if (w1[i-1] == w2[j-1])
                dp[i][j] = dp[i-1][j-1] + 1;
            else dp[i][j] = max(dp[i-1][j], dp[i][j-1]);
        }
    }
    return dp[n1][n2];
}

// backtrace
string getLCS(const string &w1, const string &w2) {
    int i = w1.size(), j = w2.size(); string ret = "";
    while (i > 0 && j > 0) {
        if (w1[i-1] == w2[j-1]) ret += w1[--i], j--;
        else if (dp[i][j-1] > dp[i-1][j]) j--;
        else i--;
    }
    reverse(ret.begin(), ret.end());
    return ret;
}
```

4.5. **Levenshtein Distance** $\mathcal{O}(n^2)$. Minimal number of insertions, removals and edits required to transform one string in the other.

```
int dp[MAX_SIZE][MAX_SIZE]; // DP problem

int levDist(const string &w1, const string &w2) {
    int n1 = sz(w1)+1, n2 = sz(w2)+1;
    REP(i, n1) dp[i][0] = i; // removal
    REP(j, n2) dp[0][j] = j; // insertion
    rep(i, 1, n1) rep(j, 1, n2)
        dp[i][j] = min(
            1 + min(dp[i-1][j], dp[i][j-1]),
            dp[i-1][j-1] + (w1[i-1] != w2[j-1])
        );
    return dp[n1][n2];
}
```

4.6. **Knuth-Morris-Pratt algorithm** $\mathcal{O}(N + M)$.

```
int kmp(const string &word, const string &text) {
    int n = word.size();
    vi T(n + 1, 0);
    for (int i = 1, j = 0; i < n; ) {
        if (word[i] == word[j]) T[++i] = ++j; // match
        else if (j > 0) j = T[j]; // fallback
        else i++; // no match, keep zero
    }
    int matches = 0;
    for (int i = 0, j = 0; i < text.size(); ) {
        if (text[i] == word[j]) {
            i++;
            if (++j == n) // match at interval [i - n, i)
                matches++, j = T[j];
        } else if (j > 0) j = T[j];
    }
```



```

    else i++;
}
return matches;
}

```

4.7. **Aho-Corasick Algorithm** $\mathcal{O}(N + \sum_{i=1}^m |S_i|)$. Dictionary substring matching as automaton. All given P must be unique!

```

const int MAXP = 100, MAXLEN = 200, SIGMA = 26,
↳ MAXTRIE = MAXP * MAXLEN;

int nP;
string P[MAXP], S;

int pnr[MAXTRIE], to[MAXTRIE][SIGMA],
↳ sLink[MAXTRIE], dLink[MAXTRIE], nnodes;

void ahoCorasick() {
    fill_n(pnr, MAXTRIE, -1);
    for (int i = 0; i < MAXTRIE; i++) fill_n(to[i],
↳ SIGMA, 0);
    fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE,
↳ 0);
    nnodes = 1;
    // STEP 1: MAKE A TREE
    for (int i = 0; i < nP; i++) {
        int cur = 0;
        for (char c : P[i]) {
            int i = c - 'a';
            if (to[cur][i] == 0) to[cur][i] = nnodes++;
            cur = to[cur][i];
        }
        pnr[cur] = i;
    }
    // STEP 2: CREATE SUFFIX_LINKS AND DICT_LINKS
    queue<int> q; q.push(0);
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (int c = 0; c < SIGMA; c++) {
            if (to[cur][c]) {
                int sl = sLink[to[cur][c]] = cur == 0 ? 0 :
↳ to[sLink[cur]][c];
                // if all strings have equal length, remove
↳ this:
                dLink[to[cur][c]] = pnr[sl] >= 0 ? sl :
↳ dLink[sl];
                q.push(to[cur][c]);
            } else to[cur][c] = to[sLink[cur]][c];
        }
    }
    // STEP 3: TRAVERSE S
    for (int cur = 0, i = 0, n = S.size(); i < n; i++)
↳ {
        cur = to[cur][S[i] - 'a'];
        for (int hit = pnr[cur] >= 0 ? cur : dLink[cur];
↳ hit; hit = dLink[hit]) {

```

```

        cerr << P[pnr[hit]] << " found at [" << (i + 1
↳ - P[pnr[hit]].size()) << ", " << i << "]"
↳ << endl;
    }
}

4.8. eerTree. Constructs an eerTree in  $\mathcal{O}(n)$ , one character at
a time.

#define MAXN 100100
#define SIGMA 26
#define BASE 'a'
char *s = new char[MAXN];
struct state {
    int len, link, to[SIGMA];
} *st = new state[MAXN+2];
struct eertree {
    int last, sz, n;
    eertree() : last(1), sz(2), n(0) {
        st[0].len = st[0].link = -1;
        st[1].len = st[1].link = 0; }
    int extend() {
        char c = s[n++]; int p = last;
        while (n - st[p].len - 2 < 0 || c != s[n -
↳ st[p].len - 2])
            p = st[p].link;
        if (!st[p].to[c-BASE]) {
            int q = last = sz++;
            st[p].to[c-BASE] = q;
            st[q].len = st[p].len + 2;
            do { p = st[p].link;
                } while (p != -1 && (n < st[p].len + 2 ||
                c != s[n - st[p].len - 2]));
            if (p == -1) st[q].link = 1;
            else st[q].link = st[p].to[c-BASE];
            return 1; }
        last = st[p].to[c-BASE];
        return 0; } };

```

4.9. **Suffix Tree**. Compressed suffix trie with $\leq 2n$ vertices. Works with characters in ASCII range [64, 128].

```

const int N = 200000; // p: parent, s: suffix link
string a; // edge p[v] -> v, contains
↳ a[l[v]..r[v]-1]
int t[N][64], l[N], r[N], p[N], s[N], tv, tp, ts,
↳ la;

void ukkadd(int c) {
    if (r[tp] <= ts) {
        if (t[tp][c] == -1) {
            t[tp][c] = ts; l[ts] = la; p[ts++] = tp;
            tv = s[tp]; tp = r[tp]; ukkadd(c); return;
        }
        tv = t[tp][c]; tp = l[tp];
    }
    if (tp == -1 || c == a[tp]-64) { tp++; return; }
}

```

```

l[ts+1] = la; p[ts+1] = ts;
l[ts] = l[tp]; r[ts] = tp; p[ts] = p[tp];
t[ts][c] = ts+1; t[ts][a[tp]-64] = tv;
l[tp] = tp; p[tp] = ts; t[p[ts]][a[l[ts]]-64] = ts;
tv = s[p[ts]]; tp = l[ts];
while (tp < r[ts])
    tv = t[tp][a[tp]-64], tp += r[tp] - l[tp];
if (tp == r[ts]) s[ts] = tv;
else s[ts] = ts+2;
tp = r[tp] - (tp - r[ts]); ts += 2; ukkadd(c);
}

void build() {
    memset(t, -1, sizeof t);
    fill_n(t[1], 64, 0); fill_n(r, N, sz(a));
    l[0] = l[1] = -1; la = tv = tp = r[0] = r[1] = 0; s[0] = 1; ts = 2;
    for (; la < sz(a); la++) ukkadd(a[la] - 64);
}

bool has_substr(const string &S) { //  $\mathcal{O}(|S|)$ 
    int v = 0, it = 0, n = sz(S);
    while (it < n) {
        int c = S[it++] - 64;
        if ((v = t[v][c]) < 0) return 0;
        for (int i = l[v]; it < n && ++i < r[v]; )
            if (S[it++] != a[i]) return 0;
    }
    return 1;
}

4.10. Suffix Automaton. Minimum automata that accepts
all suffixes of a string with  $\mathcal{O}(n)$  construction. The automata
itself is a DAG therefore suitable for DP, examples are counting
unique substrings, occurrences of substrings and suffix.

// TODO: Add longest common substring
const int MAXL = 100000;
struct suffix_automaton {
    vi len, link, occur, cnt;
    vector<map<char, int>> next;
    vector<bool> isclone;
    ll *occuratleast;
    int sz, last;
    string s;
    suffix_automaton() : len(MAXL*2), link(MAXL*2),
        occur(MAXL*2), next(MAXL*2), isclone(MAXL*2) {
        clear(); }
    void clear() { sz = 1; last = len[0] = 0; link[0]
↳ = -1;
        next[0].clear(); isclone[0] =
        false; }
    bool issustr(string other) {
        for (int i = 0, cur = 0; i < size(other); ++i) {
            if (cur == -1) return false; cur =
↳ next[cur][other[i]]; }
        return true; }
}

```

```

void extend(char c){ int cur = sz++; len[cur] =
↳ len[last]+1;
next[cur].clear(); isclone[cur] = false; int p =
↳ last;
for(; p != -1 && !next[p].count(c); p = link[p])
next[p][c] = cur;
if(p == -1){ link[cur] = 0; }
else{ int q = next[p][c];
if(len[p] + 1 == len[q]){ link[cur] = q; }
else{ int clone = sz++; isclone[clone] =
↳ true;
len[clone] = len[p] + 1;
link[clone] = link[q]; next[clone] =
↳ next[q];
for(; p != -1 && next[p].count(c) &&
↳ next[p][c] == q;
p = link[p]){
next[p][c] = clone; }
link[q] = link[cur] = clone;
} } last = cur; }

void count(){
cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
map<char,int>::iterator i;
while(!S.empty()){
ii cur = S.top(); S.pop();
if(cur.y){
for(i = next[cur.x].begin();
i != next[cur.x].end();++i){
cnt[cur.x] += cnt[(i).y]; } }
else if(cnt[cur.x] == -1){
cnt[cur.x] = 1; S.push(ii(cur.x, 1));
for(i = next[cur.x].begin();
i != next[cur.x].end();++i){
S.push(ii((i).y, 0)); } } } }

string lexicok(ll k){
int st=0; string res; map<char,int>::iterator i;
while(k){
for(i = next[st].begin(); i != next[st].end();
↳ ++i){
if(k <= cnt[(i).y]){ st = (i).y;
res.push_back((i).x); k--; break;
} else { k -= cnt[(i).y]; } } }
return res; }

void countoccur(){
REP(i, sz) occur[i] = 1 - isclone[i];
vii states(sz);
REP(i, sz) states[i] = ii(len[i],i);
sort(states.begin(), states.end());
for(int i = size(states)-1; i >= 0; --i){
int v = states[i].y;
if (link[v] != -1)
occur[link[v]] += occur[v]; } } }

```

4.11. **Hashing.** Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```

struct hasher {
int b = 311, m; vi h, p;
hasher(string s, int _m) :
m(_m), h(sz(s)+1), p(sz(s)+1) {
p[0] = 1; h[0] = 0;
REP(i, sz(s)) p[i+1] = (ll)p[i] * b % m;
REP(i, sz(s)) h[i+1] = ((ll)h[i] * b + s[i]) % m;
}
int hash(int l, int r) {
return (h[r+1] + m - (ll)h[l]*p[r-l+1] % m) % m;
}
};

```

5. GEOMETRY

```

const ld EPS = 1e-7, PI = acos(-1.0);
typedef ld NUM; // EITHER ld OR ll
typedef pair<NUM, NUM> pt;

pt operator+(pt p, pt q){ return {p.x+q.x, p.y+q.y}; }
pt operator-(pt p, pt q){ return {p.x-q.x, p.y-q.y}; }
pt operator*(pt p, NUM n){ return {p.x*n, p.y*n}; }

pt& operator+=(pt &p, pt q){ return p = p+q; }
pt& operator-=(pt &p, pt q){ return p = p-q; }

NUM operator*(pt p, pt q){ return p.x*q.x+p.y*q.y; }
NUM operator^(pt p, pt q){ return p.x*q.y-p.y*q.x; }

// square distance from p to line ab
ld distPtLineSq(pt p, pt a, pt b) {
p -= a; b -= a;
return ld(p^b) * (p^b) / (b*b);
}

// square distance from p to linesegment ab
ld distPtSegmentSq(pt p, pt a, pt b) {
p -= a; b -= a;
NUM dot = p*b, len = b*b;
if (dot <= 0) return p*p;
if (dot >= len) return (p-b)*(p-b);
return p*p - ld(dot)*dot/len;
}

// Test if p is on line segment ab
bool segmentHasPoint(pt p, pt a, pt b) {
pt u = p-a, v = p-b;
return abs(u^v) < EPS && u*v <= 0;
}

// projects p onto the line ab
pair<ld,ld> proj(pt p, pt a, pt b) {
p -= a; b -= a;
return a + b*(ld(b*p) / (b*b));
}

bool col(pt a, pt b, pt c) {
return abs((a-b) ^ (a-c)) < EPS;
}

```

```

// true => 1 intersection, false => parallel or same
bool linesIntersect(pt a, pt b, pt c, pt d) {
return abs((a-b) ^ (c-d)) > EPS;
}

pair<ld,ld> lineLineIntersection(pt a, pt b, pt c,
↳ pt d) {
ld det = (a-b) ^ (c-d);
assert(abs(det) > EPS);
return ((c-d)*(a^b) - (a-b)*(c^d)) *
↳ (ld(1.0)/det);
}

// dp, dq are directions from p, q
// intersection at p + t_i dp, for 0 <= i < return
↳ value
int segmentIntersection(pt p, pt dp, pt q, pt dq,
pt &A, pt &B) {
if (abs(dp * dp)<EPS)
swap(p,q), swap(dp,dq); // dq=0
if (abs(dp * dp)<EPS) {
A = p; // dp = dq = 0
return p == q;
}

pt dpq = q-p;
NUM c = dp^dq, c0 = dpq^dp, c1 = dpq^dq;
if (abs(c) < EPS) { // parallel, dp > 0, dq >= 0
if (abs(c0) > EPS) return 0; // not collinear
NUM v0 = dpq*dp, v1 = v0 + dq*dp, dp2 = dp*dp;
if (v1 < v0) swap(v0, v1);

v0 = max(v0, NUM(0));
v1 = min(v1, dp2);

A = p + dp * (ld(v0) / dp2);
B = p + dp * (ld(v1) / dp2);

return (v0 <= v1) + (v0 < v1);
}

if (c < 0) {
c = -c; c0 = -c0; c1 = -c1;
}

A = p + dp * (ld(c1)/c);
return 0 <= min(c0,c1) && max(c0,c1) <= c;
}

// Returns TWICE the area of a polygon (for
↳ integers)
NUM polygonTwiceArea(const vector<pt> &p) {
NUM area = 0;
for (int n = sz(p), i=0, j=n-1; i<n; j = i++)
area += p[i] ^ p[j];
return abs(area); // area < 0 ==> p ccw
}

```

```

bool insidePolygon(const vector<pt> &pts, pt p, bool
↳ strict = true) {
    int n = 0;
    for (int N = sz(pts), i = 0, j = N - 1; i < N; j =
↳ i++) {
        // if p is on edge of polygon
        if (segmentHasPoint(p, pts[i], pts[j])) return
↳ !strict;
        // or: if(distPtSegmentSq(p, pts[i], pts[j]) <=
↳ EPS) return !strict;

        // increment n if segment intersects line from p
        n += (max(pts[i].y, pts[j].y) > p.y &&
↳ min(pts[i].y, pts[j].y) <= p.y &&
        ((pts[j] - pts[i])^(p-pts[i])) > 0) ==
↳ (pts[i].y <= p.y));
    }
    return n & 1; // inside if odd number of
↳ intersections
}

```

5.1. Convex Hull $\mathcal{O}(n \log n)$.

```

// the convex hull consists of: { pts[ret[0]],
↳ pts[ret[1]], ... pts[ret.back()] }
vi convexHull(const vector<pt> &pts) {
    if (pts.empty()) return vi();
    vi ret, ord;
    int n = pts.size(), st = min_element(all(pts)) -
↳ pts.begin();
    rep(i, 0, n)
        if (pts[i] != pts[st]) ord.pb(i);
    sort(all(ord), [&pts, &st] (int a, int b) {
        pt p = pts[a] - pts[st], q = pts[b] - pts[st];
        return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) <
↳ lenSq(q);
    });
    ord.pb(st); ret.pb(st);
    for (int i : ord) {
        // use '>' to include ALL points on the
↳ hull-line
        for (int s = ret.size() - 1; s > 0 &&
↳ ((pts[ret[s-1]] - pts[ret[s]]) ^ (pts[i] -
        pts[ret[s]])) >= 0; s--)
            ret.pop_back();
        ret.pb(i);
    }
    ret.pop_back();
    return ret;
}

```

5.2. Rotating Calipers $\mathcal{O}(n)$. Finds the longest distance between two points in a convex hull.

```

NUM rotatingCalipers(vector<pt> &hull) {
    int n = hull.size(), a = 0, b = 1;
    if (n <= 1) return 0.0;

```

```

    while ((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
↳ hull[b])) > 0) b++;
    NUM ret = 0.0;
    while (a < n) {
        ret = max(ret, lenSq(hull[a], hull[b]));
        if (((hull[(a + 1) % n] - hull[a % n]) ^
↳ (hull[(b + 1) % n] - hull[b])) <= 0) a++;
        else if (++b == n) b = 0;
    }
    return ret;
}

```

5.3. Closest points $\mathcal{O}(n \log n)$.

```

int n; pt pts[maxn];

struct byY {
    bool operator()(int a, int b) const { return
↳ pts[a].y < pts[b].y; }
};

inline NUM dist(ii p) { return hypot(pts[p.x].x -
↳ pts[p.y].x, pts[p.x].y - pts[p.y].y); }

ii minpt(ii p1, ii p2) { return dist(p1) < dist(p2)
↳ ? p1 : p2; }

// closest pts (by index) inside pts[l ... r], with
↳ sorted y values in ys
ii closest(int l, int r, vi &ys) {
    if (r - l == 2) { // don't assume 1 here.
        ys = { l, l + 1 };
        return ii(l, l + 1);
    } else if (r - l == 3) { // brute-force
        ys = { l, l + 1, l + 2 };
        sort(all(ys), byY());
        return minpt(ii(l, l + 1), minpt(ii(l, l + 2),
↳ ii(l + 1, l + 2)));
    }
    int m = (l + r) / 2; vi yl, yr;
    ii delta = minpt(closest(l, m, yl), closest(m, r,
↳ yr));
    NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
↳ pts[m].x);
    merge(all(yl), all(yr), back_inserter(ys), byY());
    deque<int> q;
    for (int i : ys) if (abs(pts[i].x - xm) <= ddelta)
        {
            for (int j : q) delta = minpt(delta, ii(i, j));
            q.pb(i);
            if (q.size() > 8) q.pop_front(); // magic from
↳ Introduction to Algorithms.
        }
    return delta;
}

```

5.4. Great-Circle Distance. Computes the distance between two points (given as latitude/longitude coordinates) on a sphere of radius r .

```

ld gc_distance(ld pLat, ld pLong, ld qLat, ld qLong,
↳ ld r) {
    pLat *= pi / 180; pLong *= pi / 180;
    qLat *= pi / 180; qLong *= pi / 180;
    return r * acos(cos(pLat)*cos(qLat)*cos(pLong -
↳ qLong) + sin(pLat)*sin(qLat)); }

```

5.5. Delaunay triangulation.

<https://cp-algorithms.com/geometry/delaunay.html>
typedef long long ll;

```

bool ge(const ll& a, const ll& b) { return a >= b; }
bool le(const ll& a, const ll& b) { return a <= b; }
bool eq(const ll& a, const ll& b) { return a == b; }
bool gt(const ll& a, const ll& b) { return a > b; }
bool lt(const ll& a, const ll& b) { return a < b; }
int sgn(const ll& a) { return (a>0) - (a<0); }

```

```

struct pt {
    ll x, y;
    pt() { }
    pt(ll _x, ll _y) : x(_x), y(_y) { }
    pt operator-(const pt& p) const {
        return pt(x - p.x, y - p.y); }
    ll cross(const pt& p) const {
        return x*p.y - y*p.x; }
    ll cross(const pt& a, const pt& b) const {
        return (a - *this).cross(b - *this); }
    ll dot(const pt& p) const {
        return x*p.x + y*p.y; }
    ll dot(const pt& a, const pt& b) const {
        return (a - *this).dot(b - *this); }
    ll lenSq() const { return this->dot(*this); }
    bool operator==(const pt& p) const {
        return eq(x, p.x) && eq(y, p.y); }
};

```

```
const pt inf_pt = pt(1e18, 1e18);
```

```

struct Quad { // `QuadEdge` originally
    pt O; // origin
    Quad *rot = nullptr, *onext = nullptr;
    bool used = false;
    Quad* rev() const { return rot->rot; }
    Quad* lnext() const {
        return rot->rev()->onext->rot; }
    Quad* oprev() const {
        return rot->onext->rot; }
    pt dest() const { return rev()->O; }
};

```

```

Quad* make_edge(pt from, pt to) {
    Quad* e1 = new Quad, e2 = new Quad;
    Quad* e3 = new Quad, e4 = new Quad;

```

```

e1->O = from; e2->O = to;
e3->O = e4->O = inf_pt;
e1->rot = e3; e2->rot = e4;
e3->rot = e2; e4->rot = e1;
e1->onext = e1; e2->onext = e2;
e3->onext = e4; e4->onext = e3;
return e1;
}

void splice(Quad* a, Quad* b) {
    swap(a->onext->rot->onext, b->onext->rot->onext);
    swap(a->onext, b->onext);
}

void delete_edge(Quad* e) {
    splice(e, e->oprev());
    splice(e->rev(), e->rev()->oprev());
    delete e->rot; delete e->rev()->rot;
    delete e; delete e->rev();
}

Quad* connect(Quad* a, Quad* b) {
    Quad* e = make_edge(a->dest(), b->O);
    splice(e, a->lnext());
    splice(e->rev(), b);
    return e;
}

bool left_of(pt p, Quad* e) {
    return gt(p.cross(e->O, e->dest()), 0);
}
bool right_of(pt p, Quad* e) {
    return lt(p.cross(e->O, e->dest()), 0);
}

template <class T> T det3(T a1, T a2, T a3,
    T b1, T b2, T b3, T c1, T c2, T c3) {
    return a1*(b2*c3 - c2*b3) - a2*(b1*c3 - c1*b3)
        + a3*(b1*c2 - c1*b2);
}

// Calculate directly with __int128, or with angles
bool in_circle(pt a, pt b, pt c, pt d) {
    #if defined(__LP64__) || defined(_WIN64)
        __int128 det = 0;
        det -= det3<__int128>(b.x,b.y,b.lenSq(),
            c.x,c.y,c.lenSq(), d.x,d.y,d.lenSq());
        det += det3<__int128>(a.x,a.y,a.lenSq(),
            c.x,c.y,c.lenSq(), d.x,d.y,d.lenSq());
        det -= det3<__int128>(a.x,a.y,a.lenSq(),
            b.x,b.y,b.lenSq(), d.x,d.y,d.lenSq());
        det += det3<__int128>(a.x,a.y,a.lenSq(),
            b.x,b.y,b.lenSq(), c.x,c.y,c.lenSq());
        return det > 0;
    #else
        auto ang = [](pt l, pt mid, pt r) {
            ll x = mid.dot(l, r), y = mid.cross(l, r);
            return atan2((ld) x, (ld) y);
        };
        return (ang(a,b,c) + ang(c,d,a)

```

```

        - ang(b,c,d) - ang(d,a,b)) > 1e-8;
    #endif
}

pair<Quad*, Quad*> build_tr(int l, int r,
    vector<pt>& p) {
    if (r - l + 1 == 2) {
        Quad* res = make_edge(p[l], p[r]);
        return make_pair(res, res->rev());
    }
    if (r - l + 1 == 3) {
        Quad *a = make_edge(p[l], p[l+1]);
        Quad *b = make_edge(p[l+1], p[r]);
        splice(a->rev(), b);
        int sg = sgn(p[l].cross(p[l + 1], p[r]));
        if (sg == 0) return make_pair(a, b->rev());
        Quad* c = connect(b, a);
        if (sg == 1) return make_pair(a, b->rev());
        return make_pair(c->rev(), c);
    }
    int mid = (l + r) / 2;
    Quad *ldo, *ldi, *rdo, *rdi;
    tie(ldo, ldi) = build_tr(l, mid, p);
    tie(rdi, rdo) = build_tr(mid + 1, r, p);
    while (true) {
        if (left_of(rdi->O, ldi)) {
            ldi = ldi->lnext(); continue;
        }
        if (right_of(ldi->O, rdi)) {
            rdi = rdi->rev()->onext; continue;
        }
        break;
    }
    Quad* B = connect(rdi->rev(), ldi);
    auto valid = [&B](Quad* e) {
        return right_of(e->dest(), B);
    };

    if (ldi->O == ldo->O) ldo = B->rev();
    if (rdi->O == rdo->O) rdo = B;
    while (true) {
        Quad* lc = B->rev()->onext; // left candidate
        if (valid(lc)) {
            while (in_circle(B->dest(), B->O,
                lc->dest(), lc->onext->dest())) {
                Quad* t = lc->onext;
                delete_edge(lc);
                lc = t;
            }
        }
        Quad* rc = B->oprev(); // right candidate
        if (valid(rc)) {
            while (in_circle(B->dest(), B->O,
                rc->dest(), rc->oprev()->dest())) {
                Quad* t = rc->oprev();
                delete_edge(rc);
                rc = t;
            }
        }
        if (!valid(lc) && !valid(rc)) break;
        if (!valid(lc) || (valid(rc) && in_circle(

```

```

        lc->dest(), lc->O, rc->O, rc->dest())) {
            B = connect(rc, B->rev());
            else B = connect(B->rev(), lc->rev());
        }
        return make_pair(ldo, rdo);
    }

    vector<tuple<pt, pt, pt>> delaunay(vector<pt> p) {
        sort(all(p), [](const pt& a, const pt& b) {
            return lt(a.x, b.x) ||
                (eq(a.x, b.x) && lt(a.y, b.y));
        });
        auto res = build_tr(0, sz(p) - 1, p);
        Quad* e = res.first;
        vector<Quad*> edges = {e};
        while (lt(e->onext->dest().cross(e->dest(), e->O), 0))
            e = e->onext;
        auto add = [&p, &e, &edges]() {
            Quad* cur = e;
            do {
                cur->used = true;
                p.pb(cur->O);
                edges.pb(cur->rev());
                cur = cur->lnext();
            } while (cur != e);
        };
        add(); p.clear();

        int kek = 0;
        while (kek < sz(edges))
            if (!(e = edges[kek++]->used) add());
        vector<tuple<pt, pt, pt>> ans;
        for (int i = 0; i < sz(p); i += 3)
            ans.pb(make_tuple(p[i], p[i + 1], p[i + 2]));
        return ans;
    }
}

```

5.6. 3D Primitives.

```

#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)
#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
    double x, y, z;
    point3d() : x(0), y(0), z(0) {}
    point3d(double _x, double _y, double _z)
        : x(_x), y(_y), z(_z) {}
    point3d operator+(P(p)) const {
        return point3d(x + p.x, y + p.y, z + p.z);
    }
    point3d operator-(P(p)) const {
        return point3d(x - p.x, y - p.y, z - p.z);
    }
    point3d operator-() const {
        return point3d(-x, -y, -z);
    }
    point3d operator*(double k) const {
        return point3d(x * k, y * k, z * k);
    }
    point3d operator/(double k) const {
        return point3d(x / k, y / k, z / k);
    }
    double operator%(P(p)) const {

```

```

    return x * p.x + y * p.y + z * p.z; }
point3d operator*(P(p)) const {
    return point3d(y*p.z - z*p.y,
        z*p.x - x*p.z, x*p.y - y*p.x); }
double length() const {
    return sqrt(*this % *this); }
double distTo(P(p)) const {
    return (*this - p).length(); }
double distTo(P(A), P(B)) const {
    // A and B must be two different points
    return ((*this - A) * (*this - B)).length() /
        ⇨ A.distTo(B); }
point3d normalize(double k = 1) const {
    // length() must not return 0
    return (*this) * (k / length()); }
point3d getProjection(P(A), P(B)) const {
    point3d v = B - A;
    return A + v.normalize((v % (*this - A)) /
        ⇨ v.length()); }
point3d rotate(P(normal)) const {
    //normal must have length 1 and be orthogonal to
    ⇨ the vector
    return (*this) * normal; }
point3d rotate(double alpha, P(normal)) const {
    return (*this) * cos(alpha) + rotate(normal) *
        ⇨ sin(alpha); }
point3d rotatePoint(P(O), P(axe), double alpha)
    ⇨ const {
    point3d Z = axe.normalize(axe % (*this - O));
    return O + Z + (*this - O - Z).rotate(alpha, O);
    ⇨ }
bool isZero() const {
    return abs(x) < EPS && abs(y) < EPS && abs(z) <
        ⇨ EPS; }
bool isOnLine(L(A, B)) const {
    return ((A - *this) * (B - *this)).isZero(); }
bool isInSegment(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
        ⇨ *this)) < EPS; }
bool isInSegmentStrictly(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
        ⇨ *this)) < EPS; }
double getAngle() const {
    return atan2(y, x); }
double getAngle(P(u)) const {
    return atan2((*this * u).length(), *this % u); }
bool isOnPlane(PL(A, B, C)) const {
    return
        abs((A - *this) * (B - *this) % (C - *this)) <
        ⇨ EPS; } }
int line_line_intersect(L(A, B), L(C, D), point3d
    ⇨ &O) {
    if (abs((B - A) * (C - A) % (D - A)) > EPS) return
        ⇨ 0;
    if (((A - B) * (C - D)).length() < EPS)
        return A.isOnLine(C, D) ? 2 : 0;
    point3d normal = (A - B) * (C - B).normalize();

```

```

    double s1 = (C - A) * (D - A) % normal;
    O = A + ((B - A) / (s1 + ((D - B) * (C - B) %
        ⇨ normal))) * s1;
    return 1; }
int line_plane_intersect(L(A, B), PL(C, D, E),
    ⇨ point3d &O) {
    double V1 = (C - A) * (D - A) % (E - A);
    double V2 = (D - B) * (C - B) % (E - B);
    if (abs(V1 + V2) < EPS)
        return A.isOnPlane(C, D, E) ? 2 : 0;
    O = A + ((B - A) / (V1 + V2)) * V1;
    return 1; }
bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
    point3d &P, point3d &Q) {
    point3d n = nA * nB;
    if (n.isZero()) return false;
    point3d v = n * nA;
    P = A + (n * nA) * ((B - A) % nB / (v % nB));
    Q = P + n;
    return true; }

```

5.7. Polygons Centroid.

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

5.8. Rectilinear Minimum Spanning Tree. Given a set of n points in the plane, and the aim is to find a minimum spanning tree connecting these n points, assuming the Manhattan distance is used. The function candidates returns at most $4n$ edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```

#define MAXN 100100
struct RMST {
    struct point {
        int i; ll x, y;
        point() : i(-1) { }
        ll d1() { return x + y; }
        ll d2() { return x - y; }
        ll dist(point other) {
            return abs(x - other.x) + abs(y - other.y); }
        bool operator < (const point &other) const {
            return y == other.y ? x > other.x : y < other.y; }
    } best[MAXN], A[MAXN], tmp[MAXN];
    int n;
    RMST() : n(0) { }
    void add_point(int x, int y) {
        A[A[n].i = n].x = x, A[n++].y = y; }

```

```

void rec(int l, int r) {
    if (l >= r) return;
    int m = (l+r)/2;
    rec(l,m), rec(m+1,r);
    point bst;
    for(int i=l, j=m+1, k=1; i <= m || j <= r; k++){
        if(j>r || (i <= m && A[i].d1() < A[j].d1())){
            tmp[k] = A[i++];
            if (bst.i != -1 && (best[tmp[k].i].i == -1
                || best[tmp[k].i].d2() < bst.d2()))
                best[tmp[k].i] = bst;
        } else {
            tmp[k] = A[j++];
            if (bst.i == -1 || bst.d2() < tmp[k].d2())
                bst = tmp[k]; } }
    rep(i,l,r+1) A[i] = tmp[i]; }
vector<pair<ll,ii>> candidates() {
    vector<pair<ll, ii>> es;
    REP(p, 2) {
        REP(q, 2) {
            sort(A, A+n);
            REP(i, n) best[i].i = -1;
            rec(0, n-1);
            REP(i, n) {
                if(best[A[i].i].i != -1)
                    es.pb({A[i].dist(best[A[i].i]),
                        {A[i].i, best[A[i].i].i}});
                swap(A[i].x, A[i].y);
                A[i].x *= -1, A[i].y *= -1; } }
            REP(i, n) A[i].x *= -1; }
        return es; } }

```

5.9. Points and lines (CP3).

```
const ld EPS = 1e-9;
```

```
ld DEG_to_RAD(ld d) { return d*PI/180.0; }
ld RAD_to_DEG(ld r) { return r*180.0/PI; }

```

```

struct point { ld x, y;
    point() { x = y = 0.0; }
    point(ld _x, ld _y) : x(_x), y(_y) { }
    // useful for sorting
    bool operator < (point other) const {
        if (fabs(x - other.x) > EPS)
            return x < other.x;
        return y < other.y; }
    // use EPS (1e-9) when testing for equality
    bool operator == (point other) const {
        ⇨ fabs(y-other.y) < EPS; }
    }
};

ld dist(point p1, point p2) {
    // hypot(dx, dy) returns sqrt(dx * dx + dy * dy)
    return hypot(p1.x - p2.x, p1.y - p2.y);
}

```



```
// rotate p by rad RADIANS CCW w.r.t origin (0, 0)
point rotate(point p, ld rad) {
    return point(p.x*cos(rad) - p.y*sin(rad),
                p.x*sin(rad) + p.y*cos(rad));
}

// lines are (x,y) s.t. ax + by = c. AND b=0,1.
struct line { ld a, b, c; };

// gives line through p1, p2
line pointsToLine(point p1, point p2) {
    if (fabs(p1.x - p2.x) < EPS) // vertical line
        return { 1.0, 0.0, -p1.x };
    else return {
        -(ld)(p1.y - p2.y) / (p1.x - p2.x),
        1.0,
        -(ld)(1.a * p1.x) - p1.y;
    };
}

bool areParallel(line l1, line l2) {
    return fabs(l1.a-l2.a)<EPS && fabs(l1.b-l2.b)<EPS;
}

bool areSame(line l1, line l2) {
    return areParallel(l1,l2) && fabs(l1.c-l2.c)<EPS;
}

// returns true (+ intersection) if l1,l2 intersect
bool areIntersect(line l1, line l2, point &p) {
    if (areParallel(l1, l2)) return false; // 0 or inf
    // solve two equations:
    p.x = (l2.b * l1.c - l1.b * l2.c) / (l2.a * l1.b - l1.a * l2.b);
    // special case: test for vertical line:
    if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
    else p.y = -(l2.a * p.x + l2.c);
    return true;
}

// name: `vec` is different from STL vector
struct vec { ld x, y;
    vec(ld _x, ld _y) : x(_x), y(_y) {} };
// convert 2 points to vector a->b
vec toVec(point a, point b) {
    return vec(b.x - a.x, b.y - a.y);
}
vec scale(vec v, ld s) { return vec(v.x*s, v.y*s); }
// translate p according to v
point translate(point p, vec v) {
    return point(p.x + v.x, p.y + v.y);
}

// convert point and gradient/slope to line
void pointSlopeToLine(point p, ld m, line &l) {
    l.a = -m; // always -m
    l.b = 1; // always 1
    l.c = -(l.a * p.x) + (l.b * p.y);
}

void closestPoint(line l, point p, point &ans) {
```

```
if (fabs(l.b) < EPS) { // case 1: vertical line
    ans.x = -(l.c); ans.y = p.y; return; }

if (fabs(l.a) < EPS) { // case 2: horizontal line
    ans.x = p.x; ans.y = -(l.c); return; }
// normal line:
line perpendicular;
pointSlopeToLine(p, 1 / l.a, perpendicular);
// intersect line l with this perpendicular line
// the intersection point is the closest point
areIntersect(l, perpendicular, ans);

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
    point b;
    closestPoint(l, p, b); // similar to distToLine
    return point(2*b.x - p.x, 2*b.y - p.y);

ld dot(vec a, vec b) { return a.x*b.x + a.y*b.y; }
ld cross(vec a,vec b){ return a.x*b.y - a.y*b.x; }
ld norm_sq(vec v) { return v.x*v.x + v.y*v.y; }

// returns the distance from p to the line defined
// by points a and b (a != b), closest point in c.
ld distToLine(point p, point a, point b, point &c) {
    // formula: c = a + u * ab
    vec ap = toVec(a, p), ab = toVec(a, b);
    ld u = dot(ap, ab) / norm_sq(ab);
    c = translate(a, scale(ab, u));
    return dist(p, c);
}

// returns the distance from p to the line segment
// ab defined by points a and b (still OK if a == b)
// the closest point is stored in c byref.
ld distToLineSegment(point p, point a, point b,
    & point &c) {
    vec ap = toVec(a, p), ab = toVec(a, b);
    ld u = dot(ap, ab) / norm_sq(ab);
    if (u < 0.0) { c = point(a.x, a.y);
        return dist(p, a); } // closer to a
    if (u > 1.0) { c = point(b.x, b.y);
        return dist(p, b); } // closer to b
    // otherwise closest is perp to line:
    return distToLine(p, a, b, c);
}

// returns angle aob in rad
ld angle(point a, point o, point b) {
    vec oa = toVec(o, a), ob = toVec(o, b);
    return acos(dot(oa, ob) / sqrt(norm_sq(oa) * norm_sq(ob)));
}

// note: to accept collinear points, change '> 0'
// returns true if r is on the left side of line pq
bool ccw(point p, point q, point r) {
    return cross(toVec(p, q), toVec(p, r)) > 0;
}

// returns true if r is on the same line as line pq
```

```
bool collinear(point p, point q, point r) {
    return fabs(cross(toVec(p,q), toVec(p,r))) < EPS;
}

5.10. Polygon (CP3). Polygons have  $P_0 = P_{n-1}$  here.

typedef vector<point> poly;

// returns the perimeter: sum of Euclidean distances
// of consecutive line segments (polygon edges)
ld perimeter(const poly &P) {
    ld result = 0.0;
    REP(i, sz(P)-1) // remember that P[0] = P[n-1]
        result += dist(P[i], P[i+1]);
    return result;
}

// returns the area, which is half the determinant
ld area(const poly &P) {
    ld result = 0.0;
    REP(i, sz(P)-1)
        result += P[i].x*P[i+1].y - P[i+1].x*P[i].y;
    return result;
}

// returns true if we always make the same turn
// throughout the polygon
bool isConvex(const poly &P) {
    int n = sz(P);
    if (n <= 3) return false; // point=2; line=3
    bool isLeft = ccw(P[0], P[1], P[2]);
    rep(i, n-2) if (ccw(P[i], P[i+1],
        P[(i+2) == n ? 1 : i+2]) != isLeft)
        return false; // different sign -> concave
    return true;
} // convex

// returns true if pt is in polygon P
bool inPolygon(point pt, const poly &P) {
    if (sz(P) == 0) return false;
    ld sum = 0; // Assume P[0] == P[n-1]
    REP(i, sz(P)-1) {
        if (ccw(pt, P[i], P[i+1]))
            sum += angle(P[i], pt, P[i+1]);
        else sum -= angle(P[i], pt, P[i+1]);
    }
    return fabs(fabs(sum) - 2*PI) < EPS;
}

// line segment p-q intersect with line A-B.
point lineIntersectSeg(point p, point q,
    point A, point B) {
    ld a = B.y - A.y;
    ld b = A.x - B.x;
    ld c = B.x * A.y - A.x * B.y;
    ld u = fabs(a * p.x + b * p.y + c);
    ld v = fabs(a * q.x + b * q.y + c);
    return point((p.x*v + q.x*u) / (u+v),
        (p.y*v + q.y*u) / (u+v));
}
```

```
// cuts polygon Q along the line formed by a -> b
// (note: Q[0] == Q[n-1] is assumed)
poly cutPolygon(point a, point b, const poly &Q) {
    poly P;
    REP(i, sz(Q)) {
        ld left1 = cross(toVec(a,b), toVec(a,Q[i]));
        ld left2 = 0;
        if (i != sz(Q)-1)
            left2 = cross(toVec(a, b), toVec(a, Q[i+1]));
        if (left1 > -EPS)
            P.pb(Q[i]); // Q[i] is left of ab
        if (left1 * left2 < -EPS)
            // edge Q[i]--Q[i+1] crosses line ab
            P.pb(lineIntersectSeg(Q[i], Q[i+1], a, b));
    }
    if (!P.empty() && !(P.back() == P.front()))
        P.pb(P.front()); // make P[0] == P[n-1]
    return P; }

point pivot; // sorts points by angle around pivot
bool angleCmp(point a, point b) {
    if (collinear(pivot, a, b)) // special case
        return dist(pivot, a) < dist(pivot, b);
    ld d1x = a.x - pivot.x, d1y = a.y - pivot.y;
    ld d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0;
}

poly CH(poly P) { // no order of P assumed!
    int i, j, n = sz(P)
    if (n <= 3) {
        // safeguard from corner case
        if (!(P[0] == P[n-1])) P.pb(P[0]);
        return P; // special case, the CH is P itself
    }

    // P0 = point with lowest Y (if tie rightmost X)
    int P0 = 0;
    rep(i, 1, n) if (P[i].y < P[P0].y
        || (P[i].y == P[P0].y && P[i].x > P[P0].x))
        P0 = i;
    // swap P[P0] with P[0]:
    point temp = P[0]; P[0] = P[P0]; P[P0] = temp;

    // second, sort points by angle w.r.t. pivot P0
    pivot = P[0];
    sort(++P.begin(), P.end(), angleCmp); // keep P[0]

    // third, the ccw tests
    poly S = { P[n-1], P[0], P[1] }; // initial S
    i = 2; // then, we check the rest
    while (i < n) { // required: N must be >= 3
        j = sz(S) - 1;
        if (ccw(S[j-1], S[j], P[i]))
            S.pb(P[i++]); // left turn, accept
        else // pop top of S when right turn
            S.pop_back();
    }
}
```

```
return S;
}

5.11. Triangle (CP3).
ld perimeter(point a, point b, point c) {
    return dist(a, b) + dist(b, c) + dist(c, a); }

ld area(ld ab, ld bc, ld ca) {
    // Heron's formula
    ld s = 0.5 * (ab+bc+ca);
    return sqrt(s)*sqrt(s-ab)*sqrt(s-bc)*sqrt(s-ca);
}

ld area(point a, point b, point c) {
    return area(dist(a, b), dist(b, c), dist(c, a));
}

ld rInCircle(ld ab, ld bc, ld ca) {
    return area(ab, bc, ca) * 2.0 / (ab+bc+ca);
}

ld rInCircle(point a, point b, point c) {
    return rInCircle(dist(a,b), dist(b,c), dist(c,a));
}

// assumption: the required points/lines functions
// have been written.
// Returns if there is an inCircle center
// if it returns TRUE, ctr will be the inCircle
// center and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point
    &ctr, ld &r) {
    r = rInCircle(p1, p2, p3);
    if (fabs(r) < EPS) return false;

    line l1, l2; // compute these two angle bisectors
    ld ratio = dist(p1, p2) / dist(p1, p3);
    point p = translate(p2,
        scale(toVec(p2, p3), ratio / (1 + ratio)));
    pointsToLine(p1, p, l1);

    ratio = dist(p2, p1) / dist(p2, p3);
    p = translate(p1,
        scale(toVec(p1, p3), ratio / (1 + ratio)));
    pointsToLine(p2, p, l2);
    // get their intersection point:
    areIntersect(l1, l2, ctr);
    return true;
}

ld rCircumCircle(ld ab, ld bc, ld ca) {
    return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

ld rCircumCircle(point a, point b, point c) {
    return rCircumCircle(
        dist(a,b), dist(b,c), dist(c,a));
}

// assumption: the required points/lines functions
// have been written.
// Returns 1 iff there is a circumCenter center
```

```
// if this function returns 1, ctr will be the
// circumCircle center and r = rCircumCircle
bool circumCircle(point p1, point p2, point p3,
    &ctr, ld &r) {
    ld a = p2.x - p1.x, b = p2.y - p1.y;
    ld c = p3.x - p1.x, d = p3.y - p1.y;
    ld e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
    ld f = c * (p1.x + p3.x) + d * (p1.y + p3.y);
    ld g = 2.0 * (a * (p3.y - p2.y) - b * (p3.x - p2.x));
    if (fabs(g) < EPS) return false;

    ctr.x = (d*e - b*f) / g;
    ctr.y = (a*f - c*e) / g;
    r = dist(p1, ctr); // r = dist(center, p_i)
    return true;
}

// returns if pt d is inside the circumCircle
// defined by a,b,c
bool inCircumCircle(point a, point b,
    point c, point d) {
    vec va=toVec(a,d), vb=toVec(b,d), vc=toVec(c,d);
    return 0 <
        va.x * vb.y * (vc.x*vc.x + vc.y*vc.y) +
        va.y * (vb.x*vb.x + vb.y*vb.y) * vc.x +
        (va.x*va.x + va.y*va.y) * vb.x * vc.y -
        (va.x*va.x + va.y*va.y) * vb.y * vc.x -
        va.y * vb.x * (vc.x*vc.x + vc.y*vc.y) -
        va.x * (vb.x*vb.x + vb.y*vb.y) * vc.y;
}

bool canFormTriangle(ld a, ld b, ld c) {
    return a+b > c && a+c > b && b+c > a; }

5.12. Circle (CP3).

int insideCircle(point_i p, point_i c, int r) { //
    &ctr, ld &r) {
    // all integer version
    int dx = p.x - c.x, dy = p.y - c.y;
    int Euc = dx * dx + dy * dy, rSq = r * r;
    // all integer
    return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
    //inside/border/outside

bool circle2PtsRad(point p1, point p2, double r,
    &ctr) {
    double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
        (p1.y - p2.y) * (p1.y - p2.y);
    double det = r * r / d2 - 0.25;
    if (det < 0.0) return false;
    double h = sqrt(det);
    c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
    c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
    return true; } // to get the other center,
    &ctr, ld &r) {
    // reverse p1 and p2
```

5.13. **Formulas.** Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two-dimensional vectors.

- $a \cdot b = |a||b| \cos \theta$, where θ is the angle between a and b .
- $a \times b = |a||b| \sin \theta$, where θ is the signed angle between a and b .
- $a \times b$ is equal to the area of the parallelogram with two of its sides formed by a and b . Half of that is the area of the triangle formed by a and b .
- **Euler's formula:** $V - E + F = 2$
- Side lengths a, b, c can form a triangle iff. $a + b > c$, $b + c > a$ and $a + c > b$.
- Sum of internal angles of a regular convex n -gon is $(n - 2)\pi$.
- **Law of sines:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:** $b^2 = a^2 + c^2 - 2ac \cos B$
- Internal tangents of circles $(c_1, r_1), (c_2, r_2)$ intersect at $(c_1 r_2 + c_2 r_1) / (r_1 + r_2)$, external intersect at $(c_1 r_2 - c_2 r_1) / (r_1 + r_2)$.

6. MISCELLANEOUS

6.1. **Binary search** $\mathcal{O}(\log(hi - lo))$.

```
bool test(int n);

int search(int lo, int hi) {
    assert(test(lo) && !test(hi)); // BE CERTAIN
    while (hi - lo > 1) {
        int m = (lo + hi) / 2;
        (test(m) ? lo : hi) = m;
    }
    // assert(test(lo) && !test(hi));
    return lo;
}
```

6.2. **Fast Fourier Transform** $\mathcal{O}(n \log n)$. Given two polynomials $A(x) = a_0 + a_1x + \dots + a_n/2x^{n/2}$ and $B(x) = b_0 + b_1x + \dots + b_n/2x^{n/2}$, FFT calculates all coefficients of $C(x) = A(x) \cdot B(x) = c_0 + c_1x + \dots + c_nx^n$, with $c_i = \sum_{j=0}^i a_j b_{i-j}$.

```
typedef complex<double> cpx;
const int LOGN = 19, MAXN = 1 << LOGN;
```

```
int rev[MAXN];
cpx rt[MAXN], a[MAXN] = {}, b[MAXN] = {};
```

```
void fft(cpx *A) {
    REP(i, MAXN) if (i < rev[i]) swap(A[i],
    ↪ A[rev[i]]);
    for (int k = 1; k < MAXN; k *= 2)
        for (int i = 0; i < MAXN; i += 2*k) REP(j, k) {
            cpx t = rt[j + k] * A[i + j + k];
            A[i + j + k] = A[i + j] - t;
            A[i + j] += t;
        }
}
```

```
void multiply() { // a = convolution of a * b
    const ld PI = acos(-1.0);
    rev[0] = 0; rt[1] = cpx(1, 0);
    REP(i, MAXN) rev[i] = (rev[i/2] | (i&1)<<LOGN)/2;
    for (int k = 2; k < MAXN; k *= 2) {
        cpx z(cos(PI/k), sin(PI/k));
        rep(i, k/2, k) rt[2*i] = rt[i], rt[2*i+1] = rt[i]*z;
    }
    fft(a); fft(b);
    REP(i, MAXN) a[i] *= b[i] / (double)MAXN;
    reverse(a+1, a+MAXN); fft(a);
}
```

6.3. **Minimum Assignment (Hungarian Algorithm)** $\mathcal{O}(n^3)$.

```
int a[MAXN + 1][MAXM + 1]; // matrix, 1-based
int minimum_assignment(int n, int m) { // n rows, m
    ↪ columns
    vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);
    for (int i = 1; i <= n; i++) {
        p[0] = i;
        int j0 = 0;
        vi mv(m + 1, INT_MAX);
        vector<char> used(m + 1, false);
        do {
            used[j0] = true;
            int i0 = p[j0], delta = INT_MAX, j1;
            for (int j = 1; j <= m; j++)
                if (!used[j]) {
                    int cur = a[i0][j] - u[i0] - v[j];
                    if (cur < mv[j]) mv[j] = cur, way[j] = j0;
                    if (mv[j] < delta) delta = mv[j], j1 = j;
                }
            for (int j = 0; j <= m; j++) {
                if (used[j]) u[p[j]] += delta, v[j] -= delta;
                else mv[j] -= delta;
            }
            j0 = j1;
        } while (p[j0] != 0);
        do {
            int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
        } while (j0);
        // column j is assigned to row p[j]
        return -v[0];
    }
}
```

6.4. **Partial linear equation solver** $\mathcal{O}(N^3)$.

```
typedef double NUM;
const int ROWS = 200, COLS = 200;
const NUM EPS = 1e-5;

// F2: bitset<COLS+1> M[ROWS]; bitset<ROWS> vals;
NUM M[ROWS][COLS + 1], vals[COLS];
bool hasval[COLS];

bool is0(NUM a) { return -EPS < a && a < EPS; }
```

```
// finds x such that Ax = b
// A_ij is M[i][j], b_i is M[i][m]
int solveM(int n, int m) {
    // F2: vals.reset();
    int pr = 0, pc = 0;
    while (pc < m) {
        int r = pr, c;
        while (r < n && is0(M[r][pc])) r++;
        if (r == n) { pc++; continue; }

        // F2: M[pr]^=M[r]; M[r]^=M[pr]; M[pr]^=M[r];
        for (c = 0; c <= m; c++)
            swap(M[pr][c], M[r][c]);

        r = pr++; c = pc++;
        // F2: vals.set(pc, M[pr][m]);
        NUM div = M[r][c];
        for (int col = c; col <= m; col++)
            M[r][col] /= div;
        REP(row, n) {
            if (row == r) continue;
            // F2: if (M[row].test(c)) M[row] ^= M[r];
            NUM times = -M[row][c];
            for (int col = c; col <= m; col++)
                M[row][col] += times * M[r][col];
        }
        // now M is in RREF

        for (int r = pr; r < n; r++)
            if (!is0(M[r][m])) return 0;
        // F2: return 1;
        fill_n(hasval, n, false);
        for (int col = 0, row; col < m; col++) {
            hasval[col] = !is0(M[row][col]);
            if (!hasval[col]) continue;
            for (int c = col + 1; c < m; c++) {
                if (!is0(M[row][c])) hasval[c] = false;
            }
            if (hasval[col]) vals[col] = M[row][m];
            row++;
        }
        REP(i, n) if (!hasval[i]) return 2;
        return 1;
    }
}
```

6.5. **Cycle-Finding.**

```
ii find_cycle(int x0, int (*f)(int)) {
    int t = f(x0), h = f(t), mu = 0, lam = 1;
    while (t != h) t = f(t), h = f(f(h));
    h = x0;
    while (t != h) t = f(t), h = f(h), mu++;
    h = f(t);
    while (t != h) h = f(h), lam++;
    return ii(mu, lam);
}
```

6.6. Longest Increasing Subsequence.

```

vi lis(vi arr) {
    vi seq, back(sz(arr)), ans;
    REP(i, sz(arr)) {
        int res = 0, lo = 1, hi = sz(seq);
        while (lo <= hi) {
            int mid = (lo+hi)/2;
            if (arr[seq[mid-1]] < arr[i]) res = mid, lo =
                mid + 1;
            else hi = mid - 1;
        }
        if (res < sz(seq)) seq[res] = i;
        else seq.pb(i);
        back[i] = res == 0 ? -1 : seq[res-1];
    }
    int at = seq.back();
    while (at != -1) ans.pb(at), at = back[at];
    reverse(all(ans));
    return ans;
}

```

6.7. Dates.

```

int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
    return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
        367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
        3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
        d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
    int x, n, i, j;
    x = jd + 68569;
    n = 4 * x / 146097;
    x -= (146097 * n + 3) / 4;
    i = (4000 * (x + 1)) / 1461001;
    x -= 1461 * i / 4 - 31;
    j = 80 * x / 2447;
    d = x - 2447 * j / 80;
    x = j / 11;
    m = j + 2 - 12 * x;
    y = 100 * (n - 49) + i + x; }

```

6.8. Simplex.

```

typedef vector<ld> VD;
typedef vector<VVD> VVD;
const ld EPS = 1e-9;
struct LPSolver {
    int m, n; vi B, N; VVD D;
    LPSolver(const VVD &A, const VD &b, const VD &c) :
        m(b.size()), n(c.size()),
        N(n + 1), B(m), D(m + 2, VD(n + 2)) {
        REP(i, m) REP(j, n) D[i][j] = A[i][j];
        REP(i, m) { B[i] = n + i; D[i][n] = -1;
            D[i][n + 1] = b[i]; }
        REP(j, n) N[j] = j, D[m][j] = -c[j];
        N[n] = -1; D[m + 1][n] = 1;
    }
    void Pivot(int r, int s) {

```

```

        double inv = 1.0 / D[r][s];
        REP(i, m+2) if (i != r) REP(j, n+2) if (j != s)
            D[i][j] -= D[r][j] * D[i][s] * inv;
        REP(j, n+2) if (j != s) D[r][j] *= inv;
        REP(i, m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]); }
    bool Simplex(int phase) {
        int x = phase == 1 ? m + 1 : m;
        while (true) {
            int s = -1;
            for (int j = 0; j <= n; j++) {
                if (phase == 2 && N[j] == -1) continue;
                if (s == -1 || D[x][j] < D[x][s] ||
                    D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
            if (D[x][s] > -EPS) return true;
            int r = -1;
            REP(i, m) {
                if (D[i][s] < EPS) continue;
                if (r == -1 || D[i][n + 1] / D[i][s] < D[r][n +
                    1] /
                    D[r][s] || (D[i][n + 1] / D[i][s]) ==
                    D[r][n + 1] / D[r][s] && B[i] < B[r]) r = i; }
            if (r == -1) return false;
            Pivot(r, s); } }
    ld Solve(VD &x) {
        int r = 0;
        rep(i, 1, m) if (D[i][n+1] < D[r][n+1]) r = i;
        if (D[r][n + 1] < -EPS) {
            Pivot(r, n);
            if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
                return -numeric_limits<ld>::infinity();
            REP(i, m) if (B[i] == -1) {
                int s = -1;
                for (int j = 0; j <= n; j++)
                    if (s == -1 || D[i][j] < D[i][s] ||
                        D[i][j] == D[i][s] && N[j] < N[s])
                        s = j;
                Pivot(i, s); }
        }
        if (!Simplex(2)) return
            numeric_limits<ld>::infinity();
        x = VD(n);
        for (int i = 0; i < m; i++) if (B[i] < n)
            x[B[i]] = D[i][n + 1];
        return D[m][n + 1]; } }
// 2-phase simplex solves linear system:
//      maximize      c^T x
//      subject to    Ax <= b, x >= 0
// INPUT: A -- an m x n matrix
//      b -- an m-dimensional vector
//      c -- an n-dimensional vector
//      x -- optimal solution (by reference)
// OUTPUT: c^T x (inf. if unbounded above, nan if
//      infeasible)
// *** Example ***

```

```

// const int m = 4, n = 3;
// ld _A[m][n] = {{6,-1,0}, {-1,-5,0},
//      {1,5,1}, {-1,-5,-1}};
// ld _b[m] = {10,-4,5,-5}, _c[n] = {1,-1,0};
// VVD A(m);
// VD b(_b, _b + m), c(_c, _c + n), x;
// REP(i, m) A[i] = VD(_A[i], _A[i] + n);
// LPSolver solver(A, b, c);
// ld value = solver.Solve(x);
// cerr << "VALUE: " << value << endl; // 1.29032
// cerr << "SOLUTION: "; // 1.74194 0.451613 1
// REP(i, sz(x)) cerr << " " << x[i];
// cerr << endl;

```

7. COMBINATORICS

- Catalan numbers (valid bracket seq's of length $2n$):
 $C_0 = 1, C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1}$.
- Stirling 1th kind ($\# \pi \in \mathfrak{S}_n$ with exactly k cycles):
 $[0] = [0] = \delta_{0n}, [n] = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.
- Stirling 2nd kind (k -partitions of $[n]$):
 $\begin{Bmatrix} n \\ 1 \end{Bmatrix} = \begin{Bmatrix} n \\ n \end{Bmatrix} = 1, \begin{Bmatrix} n \\ k \end{Bmatrix} = k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}$.
- Bell numbers (partitions of $[n]$):
 $B_0 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^n \begin{Bmatrix} n \\ k \end{Bmatrix}$.
- Euler ($\# \pi \in \mathfrak{S}_n$ with exactly k ascents):
 $\langle 0 \rangle = \langle n-1 \rangle = 1, \langle n \rangle = (k+1) \langle n-1 \rangle + (n-k) \langle n-1 \rangle_{k-1}$.
- Euler 2nd order (nr perms of $1, 1, 2, 2, \dots, n, n$ with exactly k ascents):
 $\langle\langle n \rangle\rangle = (k+1) \langle\langle n-1 \rangle\rangle + (2n-k-1) \langle\langle n-1 \rangle\rangle_{k-1}$.
- Rooted trees: n^{n-1} , unrooted: n^{n-2} .
- Forests of k rooted trees: $\binom{n}{k} k \cdot n^{n-k-1}$.
- $1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}, 1^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

- $\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}, \sum_i \binom{n-i}{i} = F_{n+1}$
- $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, x^k = \sum_{i=0}^k i! \begin{Bmatrix} k \\ i \end{Bmatrix} \langle x \rangle_i = \sum_{i=0}^k \langle x \rangle_i \begin{Bmatrix} k \\ i \end{Bmatrix}$
- $a \equiv b \pmod{x, y} \Leftrightarrow a \equiv b \pmod{\text{lcm}(x, y)}$.
- $ac \equiv bc \pmod{m} \Leftrightarrow a \equiv b \pmod{m/\text{gcd}(c, m)}$.
- $\text{gcd}(n^a - 1, n^b - 1) = \text{gcd}(a, b) - 1$.
- Möbius inversion formula:** If $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(d) f(n/d)$. If $f(n) = \sum_{m=1}^n g(\lfloor n/m \rfloor)$, then $g(n) = \sum_{m=1}^n \mu(m) f(\lfloor \frac{n}{m} \rfloor)$.
- Inclusion-Exclusion:** If $g(T) = \sum_{S \subseteq T} f(S)$, then

$$f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} g(T).$$

Corollary: $b_n = \sum_{k=0}^n \binom{n}{k} a_k \Leftrightarrow a_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} b_k$.

- **The Twelfold Way:** Putting n balls into k boxes. $p(n, k)$ is # partitions of n in k parts, each > 0 . $p_k(n) = \sum_{i=0}^k p(n, k)$.

Balls	same	distinct	same	distinct
Boxes	same	same	distinct	distinct
-	$p_k(n)$	$\sum_{i=0}^k \binom{n}{i}$	$\binom{n+k-1}{k-1}$	k^n
size ≥ 1	$p(n, k)$	$\binom{n}{k}$	$\binom{n-1}{k-1}$	$k! \binom{n}{k}$
size ≤ 1	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n! \binom{k}{n}$

8. FORMULAS

- **Legendre symbol:** $\left(\frac{a}{b}\right) = a^{(b-1)/2} \pmod{b}$, b odd prime.
- **Heron's formula:** A triangle with side lengths a, b, c has area $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.
- **Shoelace formula:** $A = \frac{1}{2} |\sum_{i=0}^{n-1} x_i y_{i+1} - x_{i+1} y_i|$.
- **Pick's theorem:** A polygon on an integer grid strictly containing i lattice points and having b lattice points on the boundary has area $i + \frac{b}{2} - 1$. (Nothing similar in higher dimensions)
- **Absorption probabilities** A random walk on $[0, n]$ with probability p to increase and q to decrease, starting at k has at n absorption probability $\frac{(q/p)^k - 1}{(q/p)^n - 1}$ if $q \neq p$, and k/n if $q = p$.
- A minimum Steiner tree for n vertices requires at most $n - 2$ additional Steiner vertices.
- **Lagrange polynomial** through points $(x_0, y_0), \dots, (x_k, y_k)$ is

$$L(x) = \sum_{j=0}^k y_j \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}.$$

- **Hook length formula:** If λ is a Young diagram and $h_\lambda(i, j)$ is the hook-length of cell (i, j) , then the number of Young tableaux $d_\lambda = n! / \prod h_\lambda(i, j)$.
- #primitive pythagorean triples with hypotenuse $< n$ approx $n/(2\pi)$.
- **Frobenius Number:** largest number which can't be expressed as a linear combination of numbers a_1, \dots, a_n with non-negative coefficients. $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$, $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d - 1)$. An integer $x > (\max_i a_i)^2$ can be expressed in such a way iff. $x \mid \gcd(a_1, \dots, a_n)$.
- **Snell's law:** $v_2 \sin \theta_1 = v_1 \sin \theta_2$ gives the shortest path between two media.
- **BEST theorem:** The number of Eulerian cycles in a directed graph G is:

$$t_w(G) \prod_{v \in G} (\deg v - 1)!,$$

where $t_w(G)$ is the number of arborescences ("directed spanning" tree) rooted at w : $t_w(G) = \det(q_{ij})_{i,j \neq w}$, with $q_{ij} = [i = j] \text{indeg}(i) - \#\{(i, j) \in E\}$.

- **Burnside's Lemma:** Let a finite group G act on a set X . Denote $X^g = \{x \in X \mid gx = x\}$. For each g in G let X^g denote the set of elements in X that are fixed by g . Then the number of orbits is:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

- **Bézout's identity:** If (x, y) is a solution to $ax + by = d$ (x, y can be found with EGCD), then all solutions are given by

$$(x + k \cdot \text{lcm}(a, b)/a, y - k \cdot \text{lcm}(a, b)/b), \quad k \in \mathbb{Z}$$

9. GAME THEORY

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

- **Nim:** Let $X = \bigoplus_{i=1}^n x_i$, then $(x_i)_{i=1}^n$ is a winning position iff $X \neq 0$. Find a move by picking k such that $x_k > x_k \oplus X$.
- **Misère Nim:** Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles. The second player wins (a_1, \dots, a_n) if 1) there is a pile $a_i > 1$ and $\bigoplus_{i=1}^n a_i = 0$ or 2) all $a_i \leq 1$ and $\bigoplus_{i=1}^n a_i = 1$.
- **Staircase Nim:** Stones are moved down a staircase and only removed from the last pile. $(x_i)_{i=1}^n$ is an L -position if $(x_{2i-1})_{i=1}^{n/2}$ is (i.e. only look at odd-numbered piles).
- **Moore's Nim $_k$:** The player may remove from at most k piles (Nim = Nim $_1$). Expand the piles in base 2, do a carry-less addition in base $k+1$ (i.e. the number of ones in each column should be divisible by $k+1$).
- **Dim $^+$:** The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is $k+1$ where 2^k is the largest power of 2 dividing the pile size.
- **Aliquot game:** Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just k .
- **Nim (at most half):** Write $n+1 = 2^m y$ with m maximal, then the Sprague-Grundy function of n is $(y-1)/2$.
- **Lasker's Nim:** Players may alternatively split a pile into two new non-empty piles. $g(4k+1) = 4k+1$, $g(4k+2) = 4k+2$, $g(4k+3) = 4k+4$, $g(4k+4) = 4k+3$ ($k \geq 0$).
- **Hackenbush on trees:** A tree with stalks $(x_i)_{i=1}^n$ may be replaced with a single stalk with length $\bigoplus_{i=1}^n x_i$.

10. SCHEDULING THEORY

Let p_j be the time task j takes on a machine, d_j the deadline, C_j the time it is completed, $L_j = C_j - d_j$ the lateness, $T_j = \max(L_j, 0)$ the tardiness, $U_j = 1$ iff $T_j > 0$ and else 0.

- One machine, minimise L_{\max} : do the tasks in increasing deadline
- One machine, minimise $\sum_j w_j C_j$: do the task increasing in p_j/w_j
- One machine, minimise $\sum_{j=1}^n C_j$ under the condition that all tasks can be done on time:
 - (1) Initialise $k = n, \tau = \sum_j p_j, J = [n]$
 - (2) Take $i_k \in J$ with $d_{i_k} \geq \tau$ and $p_{i_k} \geq p_\ell$ for $\ell \in J$ with $d_\ell \geq \tau$
 - (3) $\tau := \tau - p_{i_k}, k := k - 1, J := J - \{i_k\}$. If $k \neq 0$, go to step 2.
 - (4) The optimale schedule is i_1, \dots, i_n .
- One machine, minimise $\sum_j U_j$. Add all tasks in order of increasing deadline; if adding a task makes it contrary with its deadline, remove the processed task with the highest processing time.
- Two machines (all tasks have to be done on both machines, in any order), minimise C_{\max} : a greedy algorithm, when a machine is free it picks a task that hasn't been done yet on either machine and has longest processing time on the other machine.
- Two machines (all tasks have to be done first on machine 1, then machine 2), minimise C_{\max} . There is an optimal schedule with on both machines the same order of tasks. Take $X = \{j : p_{1j} \leq p_{2j}\}$ and Y the complement. Sort X increasing in p_{1j} and Y decreasing in p_{2j} . Then X, Y is an optimal schedule.
- Two machines (all tasks have to be done first on machine 1, then on 2, or vice versa), minimise C_{\max} : let J_{12} be the tasks that have to be done first on machine 1, then on 2 and similar J_{21} . Use the above algorithm to find S_{12}, S_{21} optimal for J_{12}, J_{21} . Then optimal is S_{12}, S_{21} for M1 and S_{21}, S_{12} for M2. (If there are tasks that have to be done on only one machine, do them in the middle.)

11. DEBUGGING TIPS

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
 - Getting NaN? Make sure `acos` etc. are not getting values out of their range (perhaps $1+\epsilon$).
 - Rounding negative numbers?
 - Outputting in scientific notation?
- Wrong Answer?
 - Read the problem statement again!
 - Are multiple test cases being handled correctly? Try repeating the same test case many times.
 - Integer overflow?
 - Think very carefully about boundaries of all input parameters
 - Try out possible edge cases:
 - * $n = 0, n = -1, n = 1, n = 2^{31} - 1$ or $n = -2^{31}$
 - * List is empty, or contains a single element
 - * n is even, n is odd
 - * Graph is empty, or contains a single vertex
 - * Graph is a multigraph (loops or multiple edges)
 - * Polygon is concave or non-simple
 - Is initial condition wrong for small cases?
 - Are you sure the algorithm is correct?
 - Explain your solution to someone.
 - Are you using any functions that you don't completely understand? Maybe STL functions?
 - Maybe you (or someone else) should rewrite the solution?
 - Can the input line be empty?
- Run-Time Error?
 - Is it actually Memory Limit Exceeded?

11.1. Dynamic programming optimizations.

- Convex Hull
 - $dp[i] = \min_{j < i} \{dp[j] + b[j] \times a[i]\}$
 - $b[j] \geq b[j+1]$
 - optionally $a[i] \leq a[i+1]$
 - $O(n^2)$ to $O(n)$ (see 2.12).
 - Divide & Conquer
 - $dp[i][j] = \min_{k < j} \{dp[i-1][k] + C[k][j]\}$
 - $A[i][j] \leq A[i][j+1]$
 - sufficient:

$$C[a][c] + C[b][d] \leq C[a][d] + C[b][c], (a \leq b \leq c \leq d) \quad (\text{QI})$$
 - $O(kn^2)$ to $O(kn \log n)$
- `vvi A; // A[i][j] is voor [i,j]`

```
void divco(ll ls, ll rs, ll lt, ll rt, vi &t, vi
↳ &s) {
    // berekent t[_][lt,rt))
    if(lt >= rt) return;
    ll ms = ls, mt = (lt + rt)/2;
    t[mt] = -INF;
    rep(i,ls,rs) {
        if (i >= mt) break;
        if (s[i] + A[i][mt] > t[mt]) {
            t[mt] = s[i] + A[i][mt];
            ms = i;
        }
    }
    divco(ls,ms+1,lt,mt,t,s);
    divco(ms,rs,mt+1,rt,t,s);
}
```

• Knuth

- $dp[i][j] = \min_{i < k < j} \{dp[i][k] + dp[k][j] + C[i][j]\}$
- $A[i][j-1] \leq A[i][j] \leq A[i+1][j]$
- $O(n^3)$ to $O(n^2)$
- sufficient: QI and $C[b][c] \leq C[a][d], a \leq b \leq c \leq d$

11.2. Solution Ideas.

- Dynamic Programming
 - Parsing CFGs: CYK Algorithm
 - Drop a parameter, recover from others
 - Swap answer and a parameter
 - When grouping: try splitting in two
 - 2^k trick
- Greedy
- Randomized
- Optimizations
 - Use bitset (/64)
 - Switch order of loops (cache locality)
- Process queries offline
 - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
 - Mo's algorithm
 - Sqrt decomposition
 - Store 2^k jump pointers
- Data structure techniques
 - Sqrt buckets
 - Store 2^k jump pointers
 - 2^k merging trick
- Counting
 - Inclusion-exclusion principle
 - Generating functions
- Graphs

- Can we model the problem as a graph?
- Can we use any properties of the graph?
- Strongly connected components
- Cycles (or odd cycles)
- Bipartite (no odd cycles)
 - * Bipartite matching
 - * Hall's marriage theorem
 - * Stable Marriage
- Cut vertex/bridge
- Biconnected components
- Degrees of vertices (odd/even)
- Trees
 - * Heavy-light decomposition
 - * Centroid decomposition
 - * Least common ancestor
 - * Centers of the tree
- Eulerian path/circuit
- Chinese postman problem
- Topological sort
- (Min-Cost) Max Flow
- Min Cut
 - * Maximum Density Subgraph
- Huffman Coding
- Min-Cost Arborescence
- Steiner Tree
- Kirchoff's matrix tree theorem
- Prüfer sequences
- Lovász Toggle
- Look at the DFS tree (which has no cross-edges)
- Is the graph a DFA or NFA?
 - * Is it the Synchronizing word problem?
- math
 - Is the function multiplicative?
 - Look for a pattern
 - Permutations
 - * Consider the cycles of the permutation
 - Functions
 - * Sum of piecewise-linear functions is a piecewise-linear function
 - * Sum of convex (concave) functions is convex (concave)
 - Modular arithmetic
 - * Chinese Remainder Theorem
 - * Linear Congruence
 - Sieve
 - System of linear equations
 - Values too big to represent?
 - * Compute using the logarithm

- * Divide everything by some large value
- Linear programming
- * Is the dual problem easier to solve?
- Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
 - 2-SAT
 - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ($\log(n)$)
- Strings
 - Trie (maybe over something weird, like bits)
 - Suffix array
 - Suffix automaton (+DP?)
 - Aho-Corasick
 - `exerTree`
 - Work with $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
 - Lazy propagation
 - Persistent
 - Implicit
 - Segment tree of X
- Geometry
 - Minkowski sum (of convex sets)
 - Rotating calipers
 - Sweep line (horizontally or vertically?)
 - Sweep angle
 - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem
 - Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)

- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

PRACTICE CONTEST CHECKLIST

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use `printf/scanf` with long long/long double?
- Are `__int128` and `__float128` available?
- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: `i?python[23]`, `factor`.
- Try submitting with `assert(false)` and `assert(true)`.
- Omitting `return 0;` still works?
- Look for directory with sample test cases.
- Make sure printing works.