

TCR

git diff solution (Jens Heuseveldt, Ludo Pulles, Pim Spelier)

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	Practice Contest Checklist

At the start of a contest, create the following files in the home-dir:
.vimrc:

```

1  set nu sw=4 ts=4 sts=4 noet ai hls shcf=-ic
12 sy on | colo slate
13
13 .bashrc:
13
13 alias gsubmit='g++ -Wall -Wshadow -std=c++14'
13 alias gll='gsubmit -DLOCAL -g'
13 gsettings set
13   ⇨ org.compiz.core:/org/compiz/profiles/unity/plugins/core/
13   ⇨ vsize 3
14 gsettings set
14   ⇨ org.compiz.core:/org/compiz/profiles/unity/plugins/core/
14   ⇨ hsize 3
14
14 Test script (usage: ./test.sh A/B/..)
15
15 g++ -std=c++11 -O2 $1.cpp
15 for i in $(ls *.in)
15 do
16     j="$(echo $i | sed 's/\.in$/.ans$')"
16     ./a.out < $i > output
16     diff output $j || exit
16     echo "$i is correct"
17 done
17 echo "You may submit!"
17
17                                     template.cpp
17
17 #include <bits/stdc++.h>
17 using namespace std;
17
17 // BBST + order statistics (if supported by judge)
17 // iterator find_by_order(int r) (zero based)
17 // int order_of_key(TK v)
17 #include <ext/pb_ds/assoc_container.hpp>
17 #include <ext/pb_ds/tree_policy.hpp>
17 using namespace __gnu_pbds;
17 template<class TK, class TM> using order_tree = tree<TK, TM,
20   ⇨ less<TK>, rb_tree_tag,
21   ⇨ tree_order_statistics_node_update>;
21 template<class TV> using order_set = order_tree<TV,
22   ⇨ null_type>;
22
23 typedef long long ll;
23 typedef long double ld;
23 typedef pair<int, int> ii;
23 typedef vector<int> vi;
24 typedef vector<vi> vvi;
24 typedef vector<ii> vii;
24
24 #define x first
24 #define y second
24 #define pb push_back
24 #define eb emplace_back
24 #define rep(i,a,b) for(auto i=(a);i!=(b); ++i)
24 #define all(v) (v).begin(), (v).end()
24 #define rs resize
24 #define DBG(x) cerr << __LINE__ << ": " << #x << " = " << (x)
25   ⇨ << endl

```

```
template<class T> using min_queue = priority_queue<T,
↳ vector<T>, greater<T>>;
template<class T> int size(const T &x) { return x.size(); }
↳ // copy the ampersand(&)!

const int INF = 2147483647;
const ll LLINF = ~(1LL<<63); // = 9.223.372.036.854.775.807
const ld PI = acos(-1.0);

void run() {

}

signed main() {
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    (cout << fixed).precision(18);
    run();
    return 0;
}
```

0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Ludo moet **ALLE** opgaves **goed** lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zware) test cases.
- Gebruik ll indien wellicht nodig.

0.2. Wrong Answer.

- (1) Print de oplossing om te debuggen!
- (2) Kijk naar wellicht makkelijkere problemen.
- (3) Bedenk zelf test cases met **randgevallen**!
- (4) Controleer de **precisie**.
- (5) Controleer op **overflow** (gebruik **OVERAL** ll, ld).
Kijk naar overflows in tussenantwoorden bij modulo.
- (6) Controleer op **typo's**.
- (7) Loop de voorbeeld test case accuraat langs.
- (8) Controleer op off-by-one-errors (in indices of lus-grenzen)?

Detecting overflow This GNU builtin checks for over- and under-flow. Result is in res if successful:

```
bool isOverflown = __builtin_[add|mul|sub]-overflow(a, b,
↳ &res);
```

0.3. Covering problems.

Minimum edge cover \iff Maximum independent set

Matching: A set of edges without common vertices (*Maximum is the **largest** such set, maximal is a set which you cannot add more edges to without breaking the property*).

Minimum Vertex Cover: A set vertices (cover) such that each edge in the graph is incident to at least one vertex of the set.

Minimum Edge Cover: A set of edges (cover) such that every vertex is incident to at least one edge of the set.

Maximum Independent Set: A set of vertices in a graph such that no two of them are adjacent.

König's theorem: In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover

0.4. **Game theory.** A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

Nim: Let $X = \bigoplus_{i=1}^n x_i$, then $(x_i)_{i=1}^n$ is a winning position iff $X \neq 0$. Find a move by picking k such that $x_k > x_k \oplus X$.

Misère Nim: Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles.

Staricase Nim: Stones are moved down a staircase and only removed from the last pile. $(x_i)_{i=1}^n$ is an L -position if $(x_{2i-1})_{i=1}^{n/2}$ is (i.e. only look at odd-numbered piles).

Moore's Nim_k: The player may remove from at most k piles (Nim = Nim₁). Expand the piles in base 2, do a carry-less addition in base $k + 1$ (i.e. the number of ones in each column should be divisible by $k + 1$).

Dim⁺: The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is $k + 1$ where 2^k is the largest power of 2 dividing the pile size.

Aliquot game: Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just k .

Nim (at most half): Write $n + 1 = 2^m y$ with m maximal, then the Sprague-Grundy function of n is $(y - 1)/2$.

Lasker's Nim: Players may alternatively split a pile into two new non-empty piles. $g(4k + 1) = 4k + 1$, $g(4k + 2) = 4k + 2$, $g(4k + 3) = 4k + 4$, $g(4k + 4) = 4k + 3$ ($k \geq 0$).

Hackenbush on trees: A tree with stalks $(x_i)_{i=1}^n$ may be replaced with a single stalk with length $\bigoplus_{i=1}^n x_i$.

A useful identity: $\bigoplus_{x=0}^{a-1} x = \{0, a - 1, 1, a\}[a \bmod 4]$.

1. MATH

```
int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }
```

```
// greatest common divisor
ll gcd(ll a, ll b) { while (b) a %= b, swap(a, b); return a;
↳ };
// least common multiple
ll lcm(ll a, ll b) { return a / gcd(a, b) * b; }
ll mod(ll a, ll b) { return (a %= b) < 0 ? a + b : a; }
```

```
// safe multiplication (ab % m) for m <= 4e18 in O(log b)
ll mod_mul(ll a, ll b, ll m) {
    ll r = 0;
    while (b) {
        if (b & 1) r = (r + a) % m; a = (a + a) % m; b >>= 1;
    }
    return r;
}
```

```
// safe exponentation (a^b % m) for m <= 2e9 in O(log b)
ll mod_pow(ll a, ll b, ll m) {
    ll r = 1;
    while (b) {
        if (b & 1) r = (r * a) % m; // r = mod_mul(r, a, m);
        a = (a * a) % m; // a = mod_mul(a, a, m);
    }
}
```

```
b >>= 1;
}
return r;
}

// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
    ll xx = y = 0, yy = x = 1;
    while (b) {
        x -= a / b * xx; swap(x, xx);
        y -= a / b * yy; swap(y, yy);
        a %= b; swap(a, b);
    }
    return a;
}
```

```
// Chinese remainder theorem: returns (u, v) s.t.: x = u (mod
↳ v) <=> x = a (mod n) and x = b (mod m) for n,m <= 1e9
pair<ll, ll> crt(ll a, ll n, ll b, ll m) {
    ll s, t, d = egcd(n, m, s, t);
    if (mod(a - b, d)) return { 0, -1 };
    return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}
```

```
// phi[i] = #{ 0 < j <= i | gcd(i, j) = 1 }
vi totient(int N) {
    vi phi(N);
    for (int i = 0; i < N; i++) phi[i] = i;
    for (int i = 2; i < N; i++)
        if (phi[i] == i)
            for (int j = i; j < N; j += i) phi[j] -= phi[j] / i;
    return phi;
}
```

```
// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
    ll ans = 1;
    while (n) {
        ll np = n % p, kp = k % p;
        if (np < kp) return 0;
        ans = mod(ans * binom(np, kp), p); // (np C kp)
        n /= p; k /= p;
    }
    return ans;
}
```

```
// returns if n is prime for n < 3e24 ( > 2^64)
// but use mul_mod for n > 2e9!!!
bool millerRabin(ll n){
    if (n < 2 || n % 2 == 0) return n == 2;
    ll d = n - 1, ad, s = 0, r;
    for (; d % 2 == 0; d /= 2) s++;
    for (int a : { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
↳ 41 }) {
```

```

    if (n == a) return true;
    if ((ad = mod_pow(a, d, n)) == 1) continue;
    for (r = 0; r < s && ad + 1 != n; r++)
        ad = (ad * ad) % n;
    if (r == s) return false;
}
return true;
}

```

1.1. Primitive Root.

```

ll primitive_root(ll m) {
    vector<ll> div;
    for (ll i = 1; i*i < m; i++) {
        if ((m-1) % i == 0) {
            if (i < m) div.pb(i);
            if (m/i < m) div.pb(m/i); } }
    rep(x,2,m) {
        bool ok = true;
        for (ll d : div)
            if (mod_pow(x, d, m) == 1) {
                ok = false; break; }
        if (ok) return x; }
    return -1; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.2. **Tonelli-Shanks algorithm.** Given prime p and integer $1 \leq n < p$, returns the square root r of n modulo p . There is also another solution given by $-r$ modulo p .

```

ll legendre(ll a, ll p) {
    if (a % p == 0) return 0;
    if (p == 2) return 1;
    return mod_pow(a, (p-1)/2, p) == 1 ? 1 : -1; }
ll tonelli_shanks(ll n, ll p) {
    assert(legendre(n,p) == 1);
    if (p == 2) return 1;
    ll s = 0, q = p-1, z = 2;
    while (~q & 1) s++, q >= 1;
    if (s == 1) return mod_pow(n, (p+1)/4, p);
    while (legendre(z,p) != -1) z++;
    ll c = mod_pow(z, q, p),
        r = mod_pow(n, (q+1)/2, p),
        t = mod_pow(n, q, p),
        m = s;
    while (t != 1) {
        ll i = 1, ts = (ll)t*t % p;
        while (ts != 1) i++, ts = ((ll)ts * ts) % p;
        ll b = mod_pow(c, 1LL<<(m-i-1), p);
        r = ((ll)r * b) % p;
        t = ((ll)t * b % p * b % p);
        c = ((ll)b * b % p);
        m = i; }
    return r; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.3. **Numeric Integration.** Numeric integration using Simpson's rule.

```

double integrate(double (*)(double), double a, double b,
    double delta = 1e-6) {

```

```

    if (abs(a - b) < delta)
        return (b-a)/8 *
            (f(a) + 3*f((2*a+b)/3) + 3*f((a+2*b)/3) + f(b));
    return integrate(f, a,
        (a+b)/2, delta) + integrate(f, (a+b)/2, b, delta); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.4. **Fast Hadamard Transform.** Computes the Hadamard transform of the given array. Can be used to compute the XOR-convolution of arrays, exactly like with FFT. For AND-convolution, use $(x+y, y)$ and $(x-y, y)$. For OR-convolution, use $(x, x+y)$ and $(x, -x+y)$. **Note:** Size of array must be a power of 2.

```

void fht(vi &arr, bool inv=false, int l=0, int r=-1) {
    if (r == -1) { fht(arr, inv, 0, size(arr)); return; }
    if (l+1 == r) return;
    int k = (r-l)/2;
    if (!inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k, r);
    rep(i,l,l+k) { int x = arr[i], y = arr[i+k];
        if (!inv) arr[i] = x-y, arr[i+k] = x+y;
        else arr[i] = (x+y)/2, arr[i+k] = (-x+y)/2; }
    if (inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k, r); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.5. **Tridiagonal Matrix Algorithm.** Solves a tridiagonal system of linear equations $a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$ where $a_1 = c_n = 0$. Beware of numerical instability.

```

#define MAXN 5000
long double A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
    C[0] /= B[0]; D[0] /= B[0];
    rep(i,1,n-1) C[i] /= B[i] - A[i]*C[i-1];
    rep(i,1,n)
        D[i] = (D[i] - A[i] * D[i-1]) / (B[i] - A[i] * C[i-1]);
    X[n-1] = D[n-1];
    for (int i = n-2; i>=0; i--)
        X[i] = D[i] - C[i] * X[i+1]; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.6. **Mertens Function.** Mertens function is $M(n) = \sum_{i=1}^n \mu(i)$. Let $L \approx (n \log \log n)^{2/3}$ and the algorithm runs in $O(n^{2/3})$.

```

#define L 9000000
int mob[L], mer[L];
unordered_map<ll,ll> mem;
ll M(ll n) {
    if (n < L) return mer[n];
    if (mem.find(n) != mem.end()) return mem[n];
    ll ans = 0, done = 1;
    for (ll i = 2; i*i <= n; i++) ans += M(n/i), done = i;
    for (ll i = 1; i*i <= n; i++)
        ans += mer[i] * (n/i - max(done, n/(i+1)));
    return mem[n] = 1 - ans; }
void sieve() {
    for (int i = 1; i < L; i++) mer[i] = mob[i] = 1;
    for (int i = 2; i < L; i++) {
        if (mer[i]) {
            mob[i] = -1;
            for (int j = i+i; j < L; j += i)

```

```

        mer[j] = 0, mob[j] = (j/i)%i == 0 ? 0 : -mob[j/i]; }
    mer[i] = mob[i] + mer[i-1]; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.7. **Summatory Phi.** The summatory phi function $\Phi(n) = \sum_{i=1}^n \phi(i)$. Let $L \approx (n \log \log n)^{2/3}$ and the algorithm runs in $O(n^{2/3})$.

```

#define N 10000000
ll sp[N];
unordered_map<ll,ll> mem;
ll sumphi(ll n) {
    if (n < N) return sp[n];
    if (mem.find(n) != mem.end()) return mem[n];
    ll ans = 0, done = 1;
    for (ll i = 2; i*i <= n; i++) ans += sumphi(n/i), done = i;
    for (ll i = 1; i*i <= n; i++)
        ans += sp[i] * (n/i - max(done, n/(i+1)));
    return mem[n] = n*(n+1)/2 - ans; }
void sieve() {
    for (int i = 1; i < N; i++) sp[i] = i;
    for (int i = 2; i < N; i++) {
        if (sp[i] == i) {
            sp[i] = i-1;
            for (int j = i+i; j < N; j += i) sp[j] -= sp[j] / i; }
        sp[i] += sp[i-1]; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.8. **Josephus problem.** Last man standing out of n if every k th is killed. Zero-based, and does not kill 0 on first pass.

```

int J(int n, int k) {
    if (n == 1) return 0;
    if (k == 1) return n-1;
    if (n < k) return (J(n-1,k)+k)%n;
    int np = n - n/k;
    return k*((J(np,k)+np-n*k*np)%np) / (k-1); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.9. **Number of Integer Points under Line.** Count the number of integer solutions to $Ax + By \leq C$, $0 \leq x \leq n$, $0 \leq y$. In other words, evaluate the sum $\sum_{x=0}^n \left\lfloor \frac{C-Ax}{B} + 1 \right\rfloor$. To count all solutions, let $n = \lfloor \frac{C}{A} \rfloor$. In any case, it must hold that $C - nA \geq 0$. Be very careful about overflows.

```

ll floor_sum(ll n, ll a, ll b, ll c) {
    if (c == 0) return 1;
    if (c < 0) return 0;
    if (a % b == 0) return (n+1)*(c/b+1)-n*(n+1)/2*a/b;
    if (a >= b) return floor_sum(n,a%b,b,c)-a/b*n*(n+1)/2;
    ll t = (c-a*n+b)/b;
    return floor_sum((c-b*t)/b,b,a,c-b*t)+t*(n+1); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

1.10. **Numbers and Sequences.** Some random prime numbers:
 1031, 32771, 1048583, 8125344, 33554467, 9982451653 1073741827,
 34359738421, 1099511627791, 35184372088891, 1125899906842679,
 36028797018963971.

More random prime numbers: $10^3 + \{-9, -3, 9, 13\}$, $10^6 + \{-17, 3, 33\}$, $10^9 + \{7, 9, 21, 33, 87\}$.

	840	32
	720 720	240
Some maximal divisor counts:	735 134 400	1344
	963 761 198 400	6 720
	866 421 317 361 600	26 880
	897 612 484 786 617 600	103 680

2. DATASTRUCTURES

2.1. Segment tree $\mathcal{O}(\log n)$. Standard segment tree

```
typedef /* Tree element */ S;
const int n = 1 << 20; S t[2 * n];
```

```
// required axiom: associativity
S combine(S l, S r) { return l + r; } // sum segment tree
S combine(S l, S r) { return max(l, r); } // max segment tree
```

```
void build() { for (int i = n; --i; ) t[i] = combine(t[2 * i - 1], t[2 * i]); }
```

```
// set value v on position i
void update(int i, S v) { for (t[i += n] = v; i /= 2; ) t[i] = combine(t[2 * i - 1], t[2 * i]); }
```

```
// sum on interval [l, r]
S query(int l, int r) {
    S resL, resR;
    for (l += n, r += n; l < r; l /= 2, r /= 2) {
        if (l & 1) resL = combine(resL, t[l++]);
        if (r & 1) resR = combine(t[--r], resR);
    }
    return combine(resL, resR);
}
```

Lazy segment tree

```
struct node {
    int l, r, x, lazy;
    node() {}
    node(int _l, int _r) : l(_l), r(_r), x(INF), lazy(0) {}
    node(int _l, int _r, int _x) : node(_l, _r) { x = _x; }
    node(node a, node b) : node(a.l, b.r) { x = min(a.x, b.x); }
    void update(int v) { x = v; }
    void range_update(int v) { lazy = v; }
    void apply() { x += lazy; lazy = 0; }
    void push(node &u) { u.lazy += lazy; }
};
```

```
struct segment_tree {
    int n;
    vector<node> arr;
    segment_tree() {}
```

```
segment_tree(const vector<ll> &a) : n(size(a)), arr(4*n) {
    mk(a, 0, 0, n-1); }
node mk(const vector<ll> &a, int l, int r) {
    int m = (l+r)/2;
    return arr[l] = l > r ? node(l, r) :
        l == r ? node(l, r, a[l]) :
        node(mk(a, 2*i+1, l, m), mk(a, 2*i+2, m+1, r)); }
node update(int at, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (at < hl || hr < at) return arr[i];
    if (hl == at && at == hr) {
        arr[i].update(v); return arr[i]; }
    return arr[i] =
        node(update(at, v, 2*i+1), update(at, v, 2*i+2)); }
node query(int l, int r, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return node(hl, hr);
    if (l <= hl && hr <= r) return arr[i];
    return node(query(l, r, 2*i+1), query(l, r, 2*i+2)); }
node range_update(int l, int r, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return arr[i];
    if (l <= hl && hr <= r)
        return arr[i].range_update(v), propagate(i), arr[i];
    return arr[i] = node(range_update(l, r, v, 2*i+1),
        range_update(l, r, v, 2*i+2)); }
void propagate(int i) {
    if (arr[i].l < arr[i].r)
        arr[i].push(arr[2*i+1]), arr[i].push(arr[2*i+2]);
    arr[i].apply(); } }
```

Persistent segment tree

```
int segcnt = 0;
struct segment {
    int l, r, lid, rid, sum;
} segs[2000000];
int build(int l, int r) {
    if (l > r) return -1;
    int id = segcnt++;
    segs[id].l = l;
    segs[id].r = r;
    if (l == r) segs[id].lid = -1, segs[id].rid = -1;
    else {
        int m = (l + r) / 2;
        segs[id].lid = build(l, m);
        segs[id].rid = build(m + 1, r);
    }
    segs[id].sum = 0;
    return id; }
int update(int idx, int v, int id) {
    if (id == -1) return -1;
    if (idx < segs[id].l || idx > segs[id].r) return id;
    int nid = segcnt++;
    segs[nid].l = segs[id].l;
    segs[nid].r = segs[id].r;
```

```
segs[nid].lid = update(idx, v, segs[id].lid);
segs[nid].rid = update(idx, v, segs[id].rid);
segs[nid].sum = segs[id].sum + v;
return nid; }
int query(int id, int l, int r) {
    if (r < segs[id].l || segs[id].r < l) return 0;
    if (l <= segs[id].l && segs[id].r <= r) return
        segs[id].sum;
    return query(segs[id].lid, l, r)
        + query(segs[id].rid, l, r); }
```

2.2. Binary Indexed Tree $\mathcal{O}(\log n)$. Use one-based indices ($i > 0$)!

```
int bit[MAXN + 1];

// arr[i] += v
void update(int i, int v) {
    while (i <= MAXN) bit[i] += v, i += i & -i;
}

// returns sum of arr[i], where i: [1, i]
int query(int i) {
    int v = 0; while (i) v += bit[i], i -= i & -i; return v;
}
```

Use this if you add things, which depend on i :

```
struct fenwick_tree {
    int n; vi data;
    fenwick_tree(int _n) : n(_n), data(vi(n)) {}
    void update(int at, int by) {
        while (at < n) data[at] += by, at |= at + 1; }
    int query(int at) {
        int res = 0;
        while (at >= 0) res += data[at], at = (at & (at + 1)) - 1;
        return res; }
    int rsq(int a, int b) { return query(b) - query(a - 1); }
};
struct fenwick_tree_sq {
    int n; fenwick_tree x1, x0;
    fenwick_tree_sq(int _n) : n(_n), x1(fenwick_tree(n)),
        x0(fenwick_tree(n)) {}
    // insert f(y) = my + c if x <= y
    void update(int x, int m, int c) {
        x1.update(x, m); x0.update(x, c); }
    int query(int x) { return x*x1.query(x) + x0.query(x); }
};
void range_update(fenwick_tree_sq &s, int a, int b, int k) {
    s.update(a, k, k * (1 - a)); s.update(b+1, -k, k * b); }
int range_query(fenwick_tree_sq &s, int a, int b) {
    return s.query(b) - s.query(a-1); }
```

2.3. Disjoint-Set / Union-Find $\mathcal{O}(\alpha(n))$.

```
struct dsu {
    vi par, rnk;
    dsu(int n) : par(n, 0), rnk(n, -1) {}
    int find(int i) { return par[i] < 0 ? i : par[i] =
        find(par[i]); }
```

```

void unite(int a, int b) {
    if ((a = find(a)) == (b = find(b))) return;
    if (rnk[a] < rnk[b]) swap(a, b);
    if (rnk[a] == rnk[b]) rnk[a]++;
    par[a] += par[b]; par[b] = a;
}
};

```

2.4. AVL Tree Balanced Binary Search Tree $\mathcal{O}(\log n)/\mathcal{O}(\log n)$.

```

#define AVL_MULTISSET 0
template <class T> struct avl_tree {
    struct node {
        T item; node *p, *l, *r;
        int size, height;
        node(const T &item, node *_p = NULL) : item(item),
        p(_p),
        l(NULL), r(NULL), size(1), height(0) {} };

    node *root;
    avl_tree() : root(NULL) {}
    inline int sz(node *n) const { return n ? n->size : 0; }
    inline int height(node *n) const {
        return n ? n->height : -1; }
    inline bool left_heavy(node *n) const {
        return n && height(n->l) > height(n->r); }
    inline bool right_heavy(node *n) const {
        return n && height(n->r) > height(n->l); }
    inline bool too_heavy(node *n) const {
        return n && abs(height(n->l) - height(n->r)) > 1; }

    void delete_tree(node *n) { if (n) {
        delete_tree(n->l), delete_tree(n->r); delete n; } }
    node*& parent_leg(node *n) {
        if (!n->p) return root;
        if (n->p->l == n) return n->p->l;
        if (n->p->r == n) return n->p->r;
        assert(false); }
    void augment(node *n) {
        if (!n) return;
        n->size = 1 + sz(n->l) + sz(n->r);
        n->height = 1 + max(height(n->l), height(n->r)); }
    #define rotate(l, r) {
        node *l = n->l;
        l->p = n->p;
        parent_leg(n) = l;
        n->l = l->r;
        if (l->r) l->r->p = n;
        l->r = n, n->p = l;
        augment(n), augment(l)
    }
    void left_rotate(node *n) { rotate(r, l); }
    void right_rotate(node *n) { rotate(l, r); }
    void fix(node *n) {
        while (n) { augment(n);
            if (too_heavy(n)) {
                if (left_heavy(n) && right_heavy(n->l))
                    left_rotate(n->l);

```

```

                else if (right_heavy(n) && left_heavy(n->r))
                    right_rotate(n->r);
                if (left_heavy(n)) right_rotate(n);
                else left_rotate(n);
                n = n->p; }
        n = n->p; } }
    inline int size() const { return sz(root); }
    node* find(const T &item) const {
        node *cur = root;
        while (cur) {
            if (cur->item < item) cur = cur->r;
            else if (item < cur->item) cur = cur->l;
            else break; }
        return cur; }
    node* insert(const T &item) {
        node *prev = NULL, **cur = &root;
        while (*cur) {
            prev = *cur;
            if ((*cur)->item < item) cur = &(*cur)->r;
            #if AVL_MULTISSET
            else cur = &(*cur)->l;
            #else
            else if (item < (*cur)->item) cur = &(*cur)->l;
            else return *cur;
            #endif
        }
        node *n = new node(item, prev);
        *cur = n, fix(n); return n; }
    void erase(const T &item) { erase(find(item)); }
    void erase(node *n, bool free = true) {
        if (!n) return;
        if (!n->l && n->r) parent_leg(n) = n->r, n->r->p = n->p;
        else if (n->l && !n->r)
            parent_leg(n) = n->l, n->l->p = n->p;
        else if (n->l && n->r) {
            node *s = successor(n);
            erase(s, false);
            s->p = n->p, s->l = n->l, s->r = n->r;
            if (n->l) n->l->p = s;
            if (n->r) n->r->p = s;
            parent_leg(n) = s, fix(s);
            return;
        } else parent_leg(n) = NULL;
        fix(n->p), n->p = n->l = n->r = NULL;
        if (free) delete n; }
    node* successor(node *n) const {
        if (!n) return NULL;
        if (n->r) return nth(0, n->r);
        node *p = n->p;
        while (p && p->r == n) n = p, p = p->p;
        return p; }
    node* predecessor(node *n) const {
        if (!n) return NULL;
        if (n->l) return nth(n->l->size-1, n->l);
        node *p = n->p;
        while (p && p->l == n) n = p, p = p->p;

```

```

        return p; }
    node* nth(int n, node *cur = NULL) const {
        if (!cur) cur = root;
        while (cur) {
            if (n < sz(cur->l)) cur = cur->l;
            else if (n > sz(cur->l))
                n -= sz(cur->l) + 1, cur = cur->r;
            else break;
        } return cur; }
    int count_less(node *cur) {
        int sum = sz(cur->l);
        while (cur) {
            if (cur->p && cur->p->r == cur) sum += 1 +
            sz(cur->p->l);
            cur = cur->p;
        } return sum; }
    void clear() { delete_tree(root), root = NULL; } };

```

Use this easy implementation for a map:

```

template <class K, class V> struct avl_map {
    struct node {
        K key; V value;
        node(K k, V v) : key(k), value(v) {}
        bool operator <(const node &other) const {
            return key < other.key; } };
    avl_tree<node> tree;
    V& operator [] (K key) {
        typename avl_tree<node>::node *n =
            tree.find(node(key, V(0)));
        if (!n) n = tree.insert(node(key, V(0)));
        return n->value; } };

```

2.5. Cartesian tree.

```

struct node {
    int x, y, sz;
    node *l, *r;
    node(int _x, int _y)
        : x(_x), y(_y), sz(1), l(NULL), r(NULL) {} };
int tsize(node* t) { return t ? t->sz : 0; }
void augment(node *t) {
    t->sz = 1 + tsize(t->l) + tsize(t->r); }
pair<node*, node*> split(node *t, int x) {
    if (!t) return make_pair((node*)NULL, (node*)NULL);
    if (t->x < x) {
        pair<node*, node*> res = split(t->r, x);
        t->r = res.first; augment(t);
        return make_pair(t, res.second); }
    pair<node*, node*> res = split(t->l, x);
    t->l = res.second; augment(t);
    return make_pair(res.first, t); }
node* merge(node *l, node *r) {
    if (!l) return r; if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r); augment(l); return l; }
    r->l = merge(l, r->l); augment(r); return r; }
node* find(node *t, int x) {

```



```

while (t) {
    if (x < t->x) t = t->l;
    else if (t->x < x) t = t->r;
    else return t; }
return NULL; }

node* insert(node *t, int x, int y) {
    if (find(t, x) != NULL) return t;
    pair<node*,node*> res = split(t, x);
    return merge(res.first,
        merge(new node(x, y), res.second)); }

node* erase(node *t, int x) {
    if (!t) return NULL;
    if (t->x < x) t->r = erase(t->r, x);
    else if (x < t->x) t->l = erase(t->l, x);
    else { node *old = t; t = merge(t->l, t->r); delete old; }
    if (t) augment(t); return t; }

int kth(node *t, int k) {
    if (k < tsize(t->l)) return kth(t->l, k);
    else if (k == tsize(t->l)) return t->x;
    else return kth(t->r, k - tsize(t->l) - 1); }

```

2.6. **Heap.** An implementation of a binary heap.

```

#define RESIZE
#define SWP(x,y) tmp = x, x = y, y = tmp
struct default_int_cmp {
    default_int_cmp() {}
    bool operator()(const int &a, const int &b) {
        return a < b; } };

template <class Compare = default_int_cmp> struct heap {
    int len, count, *q, *loc, tmp;
    Compare _cmp;
    inline bool cmp(int i, int j) { return _cmp(q[i], q[j]); }
    inline void swp(int i, int j) {
        SWP(q[i], q[j]), SWP(loc[q[i]], loc[q[j]]); }
    void swim(int i) {
        while (i > 0) {
            int p = (i - 1) / 2;
            if (!cmp(i, p)) break;
            swp(i, p), i = p; } }
    void sink(int i) {
        while (true) {
            int l = 2*i + 1, r = l + 1;
            if (l >= count) break;
            int m = r >= count || cmp(l, r) ? l : r;
            if (!cmp(m, i)) break;
            swp(m, i), i = m; } }

    heap(int init_len = 128)
        : count(0), len(init_len), _cmp(Compare()) {
        q = new int[len], loc = new int[len];
        memset(loc, 255, len << 2); }
    ~heap() { delete[] q; delete[] loc; }
    void push(int n, bool fix = true) {
        if (len == count || n >= len) {
#ifdef RESIZE
            int newlen = 2 * len;
            while (n >= newlen) newlen *= 2;

```

```

            int *newq = new int[newlen], *newloc = new int[newlen];
            rep(i,0,len) newq[i] = q[i], newloc[i] = loc[i];
            memset(newloc + len, 255, (newlen - len) << 2);
            delete[] q, delete[] loc;
            loc = newloc, q = newq, len = newlen;
        }
        #else
            assert(false);
        #endif
    }
    assert(loc[n] == -1);
    loc[n] = count, q[count++] = n;
    if (fix) swim(count-1); }
    void pop(bool fix = true) {
        assert(count > 0);
        loc[q[0]] = -1, q[0] = q[--count], loc[q[0]] = 0;
        if (fix) sink(0);
    }
    int top() { assert(count > 0); return q[0]; }
    void heapify() { for (int i = count - 1; i > 0; i--)
        if (cmp(i, (i - 1) / 2)) swp(i, (i - 1) / 2); }
    void update_key(int n) {
        assert(loc[n] != -1), swim(loc[n]), sink(loc[n]); }
    bool empty() { return count == 0; }
    int size() { return count; }
    void clear() { count = 0, memset(loc, 255, len << 2); }; }

```

2.7. **Dancing Links.** An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```

template <class T>
struct dancing_links {
    struct node {
        T item;
        node *l, *r;
        node(const T &_item, node *_l = NULL, node *_r = NULL)
            : item(_item), l(_l), r(_r) {
            if (l) l->r = this;
            if (r) r->l = this; } };
    node *front, *back;
    dancing_links() { front = back = NULL; }
    node *push_back(const T &item) {
        back = new node(item, back, NULL);
        if (!front) front = back;
        return back; }
    node *push_front(const T &item) {
        front = new node(item, NULL, front);
        if (!back) back = front;
        return front; }
    void erase(node *n) {
        if (!n->l) front = n->r; else n->l->r = n->r;
        if (!n->r) back = n->l; else n->r->l = n->l; }
    void restore(node *n) {
        if (!n->l) front = n; else n->l->r = n;
        if (!n->r) back = n; else n->r->l = n; } };

```

2.8. **Misof Tree.** A simple tree data structure for inserting, erasing, and querying the n th largest element.

```

#define BITS 15
struct misof_tree {
    int cnt[BITS][1<<BITS];
    misof_tree() { memset(cnt, 0, sizeof(cnt)); }
    void insert(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]++, x >>= 1); }
    void erase(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]--, x >>= 1); }
    int nth(int n) {
        int res = 0;
        for (int i = BITS-1; i >= 0; i--)
            if (cnt[i][res <= 1] <= n) n -= cnt[i][res], res |= 1;
        return res; } };

```

2.9. **k-d Tree.** A k -dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```

#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
    struct pt {
        double coord[K];
        pt() {}
        pt(double c[K]) { rep(i,0,K) coord[i] = c[i]; }
        double dist(const pt &other) const {
            double sum = 0.0;
            rep(i,0,K) sum += pow(coord[i] - other.coord[i], 2.0);
            return sqrt(sum); } };
    struct cmp {
        int c;
        cmp(int _c) : c(_c) {}
        bool operator()(const pt &a, const pt &b) {
            for (int i = 0, cc; i <= K; i++) {
                cc = i == 0 ? c : i - 1;
                if (abs(a.coord[cc] - b.coord[cc]) > EPS)
                    return a.coord[cc] < b.coord[cc];
            }
            return false; } };
    struct bb {
        pt from, to;
        bb(pt _from, pt _to) : from(_from), to(_to) {}
        double dist(const pt &p) {
            double sum = 0.0;
            rep(i,0,K) {
                if (p.coord[i] < from.coord[i])
                    sum += pow(from.coord[i] - p.coord[i], 2.0);
                else if (p.coord[i] > to.coord[i])
                    sum += pow(p.coord[i] - to.coord[i], 2.0);
            }
            return sqrt(sum); }
        bb bound(double l, int c, bool left) {
            pt nf(from.coord), nt(to.coord);
            if (left) nt.coord[c] = min(nt.coord[c], l);
            else nf.coord[c] = max(nf.coord[c], l);
            return bb(nf, nt); } };
    struct node {

```

```

    pt p; node *l, *r;
    node(pt _p, node *_l, node *_r)
        : p(_p), l(_l), r(_r) { } };
node *root;
// kd_tree() : root(NULL) { }
kd_tree(vector<pt> pts) {
    root = construct(pts, 0, size(pts) - 1, 0); }
node* construct(vector<pt> &pts, int from, int to, int c) {
    if (from > to) return NULL;
    int mid = from + (to - from) / 2;
    nth_element(pts.begin() + from, pts.begin() + mid,
        pts.begin() + to + 1, cmp(c));
    return new node(pts[mid],
        construct(pts, from, mid - 1, INC(c)),
        construct(pts, mid + 1, to, INC(c))); }
bool contains(const pt &p) { return _con(p, root, 0); }
bool _con(const pt &p, node *n, int c) {
    if (!n) return false;
    if (cmp(c)(p, n->p)) return _con(p, n->l, INC(c));
    if (cmp(c)(n->p, p)) return _con(p, n->r, INC(c));
    return true; }
void insert(const pt &p) { _ins(p, root, 0); }
void _ins(const pt &p, node* &n, int c) {
    if (!n) n = new node(p, NULL, NULL);
    else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
    else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c)); }
void clear() { _clr(root); root = NULL; }
void _clr(node *n) {
    if (n) _clr(n->l), _clr(n->r), delete n; }
pt nearest_neighbour(const pt &p, bool allow_same=true) {
    assert(root);
    double mn = INFINITY, cs[K];
    rep(i,0,K) cs[i] = -INFINITY;
    pt from(cs);
    rep(i,0,K) cs[i] = INFINITY;
    pt to(cs);
    return _nn(p, root, bb(from, to), mn, 0,
        allow_same).first;
}
pair<pt, bool> _nn(const pt &p, node *n, bb b,
    double &mn, int c, bool same) {
    if (!n || b.dist(p) > mn) return make_pair(pt(), false);
    bool found = same || p.dist(n->p) > EPS,
        l1 = true, l2 = false;
    pt resp = n->p;
    if (found) mn = min(mn, p.dist(resp));
    node *n1 = n->l, *n2 = n->r;
    rep(i,0,2) {
        if (i == 1 || cmp(c)(n->p, p))
            swap(n1, n2), swap(l1, l2);
        pair<pt, bool> res = _nn(p, n1,
            b.bound(n->p.coord[c], c, l1), mn, INC(c), same);
        if (res.second &&
            (!found || p.dist(res.first) < p.dist(resp)))
            resp = res.first, found = true;
    }
}

```

```

    }
    return make_pair(resp, found); } };

2.10. Sqrt Decomposition. Design principle that supports many operations in amortized  $\sqrt{n}$  per operation.
struct segment {
    vi arr;
    segment(vi _arr) : arr(_arr) { } };
vector<segment> T;
int K;
void rebuild() {
    int cnt = 0;
    rep(i,0,size(T))
        cnt += size(T[i].arr);
    K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
    vi arr(cnt);
    for (int i = 0, at = 0; i < size(T); i++)
        rep(j,0,size(T[i].arr))
            arr[at++] = T[i].arr[j];
    T.clear();
    for (int i = 0; i < cnt; i += K)
        T.push_back(segment(vi(arr.begin()+i,
            arr.begin()+min(i+K, cnt)))); }
int split(int at) {
    int i = 0;
    while (i < size(T) && at >= size(T[i].arr))
        at -= size(T[i].arr), i++;
    if (i >= size(T)) return size(T);
    if (at == 0) return i;
    T.insert(T.begin() + i + 1,
        segment(vi(T[i].arr.begin() + at, T[i].arr.end())));
    T[i] = segment(vi(T[i].arr.begin(), T[i].arr.begin() +
        at));
    return i + 1; }
void insert(int at, int v) {
    vi arr; arr.push_back(v);
    T.insert(T.begin() + split(at), segment(arr)); }
void erase(int at) {
    int i = split(at); split(at + 1);
    T.erase(T.begin() + i); }
// vim: cc=60 ts=2 sts=2 sw=2:

2.11. Monotonic Queue. A queue that supports querying for the minimum element. Useful for sliding window algorithms.
struct min_stack {
    stack<int> S, M;
    void push(int x) {
        S.push(x);
        M.push(M.empty() ? x : min(M.top(), x)); }
    int top() { return S.top(); }
    int mn() { return M.top(); }
    void pop() { S.pop(); M.pop(); }
    bool empty() { return S.empty(); } };
struct min_queue {
    min_stack inp, outp;
    void push(int x) { inp.push(x); }
}

```

```

void fix() {
    if (outp.empty()) while (!inp.empty())
        outp.push(inp.top(), inp.pop()); }
int top() { fix(); return outp.top(); }
int mn() {
    if (inp.empty()) return outp.mn();
    if (outp.empty()) return inp.mn();
    return min(inp.mn(), outp.mn()); }
void pop() { fix(); outp.pop(); }
bool empty() { return inp.empty() && outp.empty(); } };

```

2.12. **Convex Hull Trick.** If converting to integers, look out for division by 0 and $\pm\infty$.

```

struct convex_hull_trick {
    vector<pair<double,double> > h;
    double intersect(int i) {
        return (h[i+1].second-h[i].second) /
            (h[i].first-h[i+1].first); }
    void add(double m, double b) {
        h.push_back(make_pair(m,b));
        while (size(h) >= 3) {
            int n = size(h);
            if (intersect(n-3) < intersect(n-2)) break;
            swap(h[n-2], h[n-1]);
            h.pop_back(); } }
    double get_min(double x) {
        int lo = 0, hi = size(h) - 2, res = -1;
        while (lo <= hi) {
            int mid = lo + (hi - lo) / 2;
            if (intersect(mid) <= x) res = mid, lo = mid + 1;
            else hi = mid - 1; }
        return h[res+1].first * x + h[res+1].second; } };

```

And dynamic variant:

```

const ll is_query = -(1LL<<62);
struct Line {
    ll m, b;
    mutable function<const Line*> succ;
    bool operator<(const Line& rhs) const {
        if (rhs.b != is_query) return m < rhs.m;
        const Line* s = succ();
        if (!s) return 0;
        ll x = rhs.m;
        return b - s->b < (s->m - m) * x; } };
// will maintain upper hull for maximum
struct HullDynamic : public multiset<Line> {
    bool bad(iterator y) {
        auto z = next(y);
        if (y == begin()) {
            if (z == end()) return 0;
            return y->m == z->m && y->b <= z->b; }
        auto x = prev(y);
        if (z == end()) return y->m == x->m && y->b <= x->b;
        return (x->b - y->b)*(z->m - y->m) >=
            (y->b - z->b)*(y->m - x->m); }
    void insert_line(ll m, ll b) {
        auto y = insert({ m, b }); }
}

```

```

y->succ = [=] { return next(y) == end() ? 0 : &*next(y);
↪ };
if (bad(y)) { erase(y); return; }
while (next(y) != end() && bad(next(y))) erase(next(y));
while (y != begin() && bad(prev(y))) erase(prev(y)); }
ll eval(ll x) {
    auto l = *lower_bound((Line) { x, is_query });
    return l.m * x + l.b; } };

```

2.13. Sparse Table.

```

struct sparse_table { vvi m;
    sparse_table(vi arr) {
        m.push_back(arr);
        for (int k = 0; (1<<(++k)) <= size(arr); ) {
            m.push_back(vi(size(arr)-(1<<k)+1));
            rep(i,0,size(arr)-(1<<k)+1)
                m[k][i] = min(m[k-1][i], m[k-1][i+(1<<(k-1))]); } }
    int query(int l, int r) {
        int k = 0; while (1<<(k+1) <= r-l+1) k++;
        return min(m[k][l], m[k][r-(1<<k)+1]); } };

```

3. GRAPH ALGORITHMS

3.1. Maximum matching $\mathcal{O}(nm)$.

```
const int sizeL = 1e4, sizeR = 1e4;
```

```

bool vis[sizeR];
int par[sizeR]; // par : R -> L
vi adj[sizeL]; // adj : L -> (N -> R)

```

```

bool match(int u) {
    for (int v : adj[u]) {
        if (vis[v] continue; vis[v] = true;
        if (par[v] == -1 || match(par[v])) {
            par[v] = u;
            return true;
        }
    }
    return false;
}

```

```
// perfect matching iff ret == sizeL == sizeR
```

```

int maxmatch() {
    fill_n(par, sizeR, -1); int ret = 0;
    for (int i = 0; i < sizeL; i++) {
        fill_n(vis, sizeR, false);
        ret += match(i);
    }
    return ret;
}

```

3.2. Hopcroft-Karp bipartite matching $\mathcal{O}(E\sqrt{V})$.

```

#define MAXN 5000
int dist[MAXN+1], q[MAXN+1];
#define dist(v) dist[v == -1 ? MAXN : v]
struct bipartite_graph {
    int N, M, *L, *R; vi *adj;

```

```

bipartite_graph(int _N, int _M) : N(_N), M(_M),
    L(new int[N]), R(new int[M]), adj(new vi[N]) {}
~bipartite_graph() { delete[] adj; delete[] L; delete[] R;
↪ }
bool bfs() {
    int l = 0, r = 0;
    rep(v,0,N) if(L[v] == -1) dist(v) = 0, q[r++] = v;
    else dist(v) = INF;
    dist(-1) = INF;
    while(l < r) {
        int v = q[l++];
        if(dist(v) < dist(-1)) {
            iter(u, adj[v]) if(dist(R[*u]) == INF)
                dist(R[*u]) = dist(v) + 1, q[r++] = R[*u]; } }
    return dist(-1) != INF; }
bool dfs(int v) {
    if(v != -1) {
        iter(u, adj[v])
            if(dist(R[*u]) == dist(v) + 1)
                if(dfs(R[*u])) {
                    R[*u] = v, L[v] = *u;
                    return true; }
        dist(v) = INF;
        return false; }
    return true; }
void add_edge(int i, int j) { adj[i].push_back(j); }
int maximum_matching() {
    int matching = 0;
    memset(L, -1, sizeof(int) * N);
    memset(R, -1, sizeof(int) * M);
    while(bfs()) rep(i,0,N)
        matching += L[i] == -1 && dfs(i);
    return matching; } };
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.2.1. Minimum Vertex Cover in Bipartite Graphs.

```

#include "hopcroft_karp.cpp"
vector<bool> alt;
void dfs(bipartite_graph &g, int at) {
    alt[at] = true;
    iter(it,g.adj[at]) {
        alt[*it + g.N] = true;
        if (g.R[*it] != -1 && !alt[g.R[*it]]) dfs(g, g.R[*it]); }
↪ }
vi mvc_bipartite(bipartite_graph &g) {
    vi res; g.maximum_matching();
    alt.assign(g.N + g.M, false);
    rep(i,0,g.N) if (g.L[i] == -1) dfs(g, i);
    rep(i,0,g.N) if (!alt[i]) res.push_back(i);
    rep(i,0,g.M) if (alt[g.N + i]) res.push_back(g.N + i);
    return res; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.3. Depth first searches.

3.3.1. Cut Points and Bridges.

```

const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;

void dfs(const vvi &adj, vi &cp, vii &bri, int u, int p) {
    low[u] = num[u] = curnum++;
    int cnt = 0; bool found = false;
    rep(i,0,size(adj[u])) {
        int v = adj[u][i];
        if (num[v] == -1) {
            dfs(adj, cp, bri, v, u);
            low[u] = min(low[u], low[v]);
            cnt++;
            found = found || low[v] >= num[u];
            if (low[v] > num[u]) bri.push_back(ii(u, v));
        } else if (p != v) low[u] = min(low[u], num[v]); }
    if (found && (p != -1 || cnt > 1)) cp.push_back(u); }

pair<vi,vii> cut_points_and_bridges(const vvi &adj) {
    int n = size(adj);
    vi cp; vii bri;
    memset(num, -1, n << 2);
    curnum = 0;
    rep(i,0,n) if (num[i] == -1) dfs(adj, cp, bri, i, -1);
    return make_pair(cp, bri); }

```

3.3.2. Strongly Connected Components $\mathcal{O}(V + E)$.

```

vvi adj, comps;
vi tidx, lnk, cnr, st;
vector<bool> vis;
int age, ncomps;

void tarjan(int v) {
    tidx[v] = lnk[v] = ++age; vis[v] = true; st.pb(v);
    for (int w : adj[v]) {
        if (!tidx[w]) tarjan(w), lnk[v] = min(lnk[v], lnk[w]);
        else if (vis[w]) lnk[v] = min(lnk[v], tidx[w]);
    }
    if (lnk[v] != tidx[v]) return;
    comps.pb(vi());
    int w;
    do {
        vis[w = st.back()] = false; cnr[w] = ncomps;
↪     comps.back().pb(w);
        st.pop_back();
    } while (w != v);
    ncomps++;
}

void findSCC(int n) {
    age = ncomps = 0; vis.assign(n, false); tidx.assign(n, 0);
    lnk.resize(n); cnr.resize(n); comps.clear();
    for (int i = 0; i < n; i++)
        if (tidx[i] == 0) tarjan(i);
}

```


3.3.3. Dominator graph.

```
const int N = 1234567;
```

```
vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], parent[N], sdom[N], p[N], best[N],
↳ idom[N], link[N];
```

```
void dfs(int v) {
    pos[v] = cnt;
    order[cnt++] = v;
    for (int u : g[v]) {
        if (pos[u] == -1) {
            parent[u] = v;
            dfs(u);
        }
    }
}
```

```
int find_best(int x) {
    if (p[x] == x) return best[x];
    int u = find_best(p[x]);
    if (pos[sdom[u]] < pos[sdom[best[x]]])
        best[x] = u;
    p[x] = p[p[x]];
    return best[x];
}
```

```
void dominators(int n, int root) {
    fill_n(pos, n, -1);
    cnt = 0;
    dfs(root);
    for (int i = 0; i < n; i++)
        for (int u : g[i]) g_rev[u].push_back(i);
    for (int i = 0; i < n; i++)
        p[i] = best[i] = sdom[i] = i;
    for (int it = cnt - 1; it >= 1; it--) {
        int w = order[it];
        for (int u : g_rev[w]) {
            int t = find_best(u);
            if (pos[sdom[t]] < pos[sdom[w]])
                sdom[w] = sdom[t];
        }
        bucket[sdom[w]].push_back(w);
        idom[w] = sdom[w];
        for (int u : bucket[parent[w]])
            link[u] = find_best(u);
        bucket[parent[w]].clear();
        p[w] = parent[w];
    }
    for (int it = 1; it < cnt; it++) {
        int w = order[it];
        idom[w] = idom[link[w]];
    }
}
```

3.3.4. 2-SAT $\mathcal{O}(V + E)$. Include findSCC.

```
void init2sat(int n) { adj.assign(2 * n, vi()); }

// vl, vr = true -> variable l, variable r should be negated.
void imply(int xl, bool vl, int xr, bool vr) {
    adj[2 * xl + vl].pb(2 * xr + vr); adj[2 * xr + !vr].pb(2 *
↳ xl + !vl); }

void satOr(int xl, bool vl, int xr, bool vr) { imply(xl, !vl,
↳ xr, vr); }

void satConst(int x, bool v) { imply(x, !v, x, v); }
void satIff(int xl, bool vl, int xr, bool vr) {
    imply(xl, vl, xr, vr); imply(xr, vr, xl, vl);}

bool solve2sat(int n, vector<bool> &sol) {
    findSCC(2 * n);
    for (int i = 0; i < n; i++)
        if (cncr[2 * i] == cncr[2 * i + 1]) return false;
    vector<bool> seen(n, false); sol.assign(n, false);
    for (vi &comp : comps) {
        for (int v : comp) {
            if (seen[v / 2]) continue;
            seen[v / 2] = true; sol[v / 2] = v & 1;
        }
    }
    return true;
}
```

3.4. Cycle Detection $\mathcal{O}(V + E)$.

vvi adj; // assumes bidirected graph, adjust accordingly

```
bool cycle_detection() {
    stack<int> s; vector<bool> vis(MAXN, false); vi par(MAXN,
↳ -1); s.push(0);
    vis[0] = true;
    while(!s.empty()) {
        int cur = s.top(); s.pop();
        for(int i : adj[cur]) {
            if(vis[i] && par[cur] != i) return true;
            s.push(i); par[i] = cur; vis[i] = true;
        }
    }
    return false;}
}
```

3.5. Shortest path.

3.5.1. Dijkstra $\mathcal{O}(|E| \log |V|)$.

```
int *dist, *dad;
struct cmp {
    bool operator()(int a, int b) {
        return dist[a] != dist[b] ? dist[a] < dist[b] : a < b; }
};
pair<int*, int*> dijkstra(int n, int s, vii *adj) {
    dist = new int[n];
    dad = new int[n];
    rep(i,0,n) dist[i] = INF, dad[i] = -1;
    set<int, cmp> pq;
    dist[s] = 0, pq.insert(s);
```

```
while (!pq.empty()) {
    int cur = *pq.begin(); pq.erase(pq.begin());
    rep(i,0,size(adj[cur])) {
        int nxt = adj[cur][i].first;
        ndist = dist[cur] + adj[cur][i].second;
        if (ndist < dist[nxt]) pq.erase(nxt),
            dist[nxt] = ndist, dad[nxt] = cur, pq.insert(nxt);
    }
    return pair<int*, int*>(dist, dad); }
```

3.5.2. Floyd-Warshall $\mathcal{O}(V^3)$.

```
int n = 100; ll d[MAXN][MAXN];
for (int i = 0; i < n; i++) fill_n(d[i], n, 1e18);
// set direct distances from i to j in d[i][j] (and d[j][i])
for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
        for (int k = 0; k < n; k++)
            d[j][k] = min(d[j][k], d[j][i] + d[i][k]);
```

3.5.3. Bellman Ford $\mathcal{O}(VE)$. This is only useful if there are edges with weight $w_{ij} < 0$ in the graph.

```
vector< pair<pii, ll> > edges; // ((from, to), weight)
vector<ll> dist;
```

// when undirected, add back edges

```
bool bellman_ford(int V, int source) {
    dist.assign(V, 1e18); dist[source] = 0;
```

```
bool updated = true; int loops = 0;
while (updated && loops < n) {
    updated = false;
    for (auto e : edges) {
        int alt = dist[e.x.x] + e.y;
        if (alt < dist[e.x.y]) {
            dist[e.x.y] = alt; updated = true;
        }
    }
    return loops < n; // loops >= n: negative cycles
}
```

3.5.4. IDA* algorithm.

```
int n, cur[100], pos;
int calch() {
    int h = 0;
    rep(i,0,n) if (cur[i] != 0) h += abs(i - cur[i]);
    return h; }
int dfs(int d, int g, int prev) {
    int h = calch();
    if (g + h > d) return g + h;
    if (h == 0) return 0;
    int mn = INF;
    rep(di,-2,3) {
        if (di == 0) continue;
        int nxt = pos + di;
```

```

    if (nxt == prev) continue;
    if (0 <= nxt && nxt < n) {
        swap(cur[pos], cur[nxt]);
        swap(pos,nxt);
        mn = min(mn, dfs(d, g+1, nxt));
        swap(pos,nxt);
        swap(cur[pos], cur[nxt]); }
    if (mn == 0) break; }
return mn; }
int idastar() {
    rep(i,0,n) if (cur[i] == 0) pos = i;
    int d = calch();
    while (true) {
        int nd = dfs(d, 0, -1);
        if (nd == 0 || nd == INF) return d;
        d = nd; } }

```

3.6. Maximum Flow Algorithms.

3.6.1. Dinic's Algorithm $\mathcal{O}(V^2E)$.

```

struct edge { ll t, r, c, f; };
int S, T, h[MAXN]; vector<edge> g[MAXN];

```

```

void addEdge(int u, int v, ll c) {
    g[u].pb({v, (ll)g[v].size(), c, 0});
    g[v].pb({u, (ll)g[u].size()-1,0,0});
}
void dinicBfs() {
    fill_n(h, MAXN, 0); h[S] = 1;
    queue<int> q; q.push(S);
    while (!q.empty()) {
        int v = q.front(); q.pop();
        for (edge e : g[v]) if (!h[e.t] && e.f < e.c)
            h[e.t] = h[v]+1, q.push(e.t);
    }
}

```

```

ll dinicDFS(int v, ll mf) {
    if (v == T) return mf;
    ll f = 0; bool sat = true;
    for (edge &e : g[v]) {
        if (h[e.t] != h[v] + 1 || e.f >= e.c) continue;
        ll df = dinicDFS(e.t, min(mf - f, e.c - e.f));
        f += df, e.f += df, g[e.t][e.r].f -= df;
        sat &= e.f == e.c; if (mf == f) break;
    }
    if (sat) h[v] = 0;
    return f;
}

```

```

ll dinicMaxFlow(ll f = 0) {
    while (dinicBfs(), h[T]) f += dinicDFS(S, LLINF);
    return f;
}

```

3.6.2. *Min-cost max-flow*. Find the cheapest possible way of sending a certain amount of flow through a flow network.

```

const int maxn = 300;

struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
    g.pb({u, v, 0, c, w});
    g.pb({v, u, 0, 0, -w});
}

void sp(int s, int t) {
    fill_n(D, V, LLINF); D[s] = 0;
    for (int ng = g.size(), _ = V; --_ > 0; ) {
        bool ok = false;
        for (int i = 0; i < ng; i++)
            if (D[g[i].x] != LLINF && g[i].f < g[i].c && D[g[i].x]
                + g[i].w < D[g[i].y]) {
                D[g[i].y] = D[g[i].x] + g[i].w;
                par[g[i].y] = i; ok = true;
            }
        if (!ok) break;
    }
}

```

```

void minCostMaxFlow(int s, int t, ll &c, ll &f) {
    for (c = f = 0; sp(s, t), D[t] < LLINF; ) {
        ll df = LLINF, dc = 0;
        for (int v = t, e; e = par[v], v != s; v = g[e].x) df =
            min(df, g[e].c - g[e].f);
        for (int v = t, e; e = par[v], v != s; v = g[e].x) g[e].f
            += df, g[e].f -= df, dc += g[e].w;
        f += df; c += dc * df;
    }
}

```

3.6.3. *Gomory-Hu Tree - All Pairs Maximum Flow*. An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in $\mathcal{O}(|V|^2)$ plus $|V| - 1$ times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is $\mathcal{O}(|V|^3|E|)$. NOTE: Not sure if it works correctly with disconnected graphs.

```

#include "dinic.cpp"
bool same[MAXV];
pair<vii, vvi> construct_gh_tree(flow_network &g) {
    int n = g.n, v;
    vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
    rep(s,1,n) {
        int l = 0, r = 0;
        par[s].second = g.max_flow(s, par[s].first, false);
        memset(d, 0, n * sizeof(int));
        memset(same, 0, n * sizeof(bool));
        d[q[r++] = s] = 1;
        while (l < r) {
            same[v = q[l++]] = true;
            for (int i = g.head[v]; i != -1; i = g.e[i].nxt)
                if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
                    d[q[r++] = g.e[i].v] = 1; }
    }
}

```

```

rep(i,s+1,n)
    if (par[i].first == par[s].first && same[i])
        par[i].first = s;
g.reset(); }
rep(i,0,n) {
    int mn = INF, cur = i;
    while (true) {
        cap[cur][i] = mn;
        if (cur == 0) break;
        mn = min(mn, par[cur].second), cur = par[cur].first; }
    }
return make_pair(par, cap); }
int compute_max_flow(int s, int t, const pair<vii, vvi> &gh)
    {
    int cur = INF, at = s;
    while (gh.second[at][t] == -1)
        cur = min(cur, gh.first[at].second),
        at = gh.first[at].first;
    return min(cur, gh.second[at][t]); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.7. Minimal Spanning Tree.

3.7.1. Kruskal $\mathcal{O}(E \log V)$.

```

struct edge { int x, y, w; };
vector<edge> edges;

```

```

ll kruskal(int n) { // n: #vertices
    uf_init(n);
    sort(all(edges), [] (edge a, edge b) -> bool { return a.w <
        b.w; });
    ll ret = 0;
    for (edge e : edges)
        if (uf_find(e.x) != uf_find(e.y))
            ret += e.w, uf_union(e.x, e.y);
    return ret;
}

```

3.8. Topological Sort.

3.8.1. Modified Depth-First Search.

```

void tsort_dfs(int cur, char* color, const vvi& adj,
    stack<int>& res, bool& cyc) {
    color[cur] = 1;
    rep(i,0,size(adj[cur])) {
        int nxt = adj[cur][i];
        if (color[nxt] == 0)
            tsort_dfs(nxt, color, adj, res, cyc);
        else if (color[nxt] == 1)
            cyc = true;
        if (cyc) return; }
    color[cur] = 2;
    res.push(cur); }
vvi tsort(int n, vvi adj, bool& cyc) {
    cyc = false;
    stack<int> S;
}

```

```

vi res;
char* color = new char[n];
memset(color, 0, n);
rep(i,0,n) {
    if (!color[i]) {
        tsort_dfs(i, color, adj, S, cyc);
        if (cyc) return res; } }
while (!S.empty()) res.push_back(S.top()), S.pop();
return res; }

```

3.9. **Euler Path.** Finds an euler path (or circuit) in a directed graph, or reports that none exist.

```

#define MAXV 1000
#define MAXE 5000
vi adj[MAXV];
int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
    int start = -1, end = -1, any = 0, c = 0;
    rep(i,0,n) {
        if (outdeg[i] > 0) any = i;
        if (indeg[i] + 1 == outdeg[i]) start = i, c++;
        else if (indeg[i] == outdeg[i] + 1) end = i, c++;
        else if (indeg[i] != outdeg[i]) return ii(-1,-1); }
    if ((start == -1) != (end == -1) || (c != 2 && c != 0))
        return ii(-1,-1);
    if (start == -1) start = end = any;
    return ii(start, end); }
bool euler_path() {
    ii se = start_end();
    int cur = se.first, at = m + 1;
    if (cur == -1) return false;
    stack<int> s;
    while (true) {
        if (outdeg[cur] == 0) {
            res[--at] = cur;
            if (s.empty()) break;
            cur = s.top(); s.pop();
        } else s.push(cur), cur = adj[cur][--outdeg[cur]]; }
    return at == 0; }

```

And an undirected version, which finds a cycle.

```

multiset<int> adj[1010];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
    if (at == to) return it;
    L.insert(it, at), --it;
    while (!adj[at].empty()) {
        int nxt = *adj[at].begin();
        adj[at].erase(adj[at].find(nxt));
        adj[nxt].erase(adj[nxt].find(at));
        if (to == -1) {
            it = euler(nxt, at, it);
            L.insert(it, at);
            --it;
        } else {
            it = euler(nxt, to, it);

```

```

        to = -1; } }
        return it; }
// euler(0,-1,L.begin())

3.10. Heavy-Light Decomposition.
#include "../data-structures/segment_tree.cpp"
const int ID = 0;
int f(int a, int b) { return a + b; }
struct HLD {
    int n, curhead, curloc;
    vi sz, head, parent, loc;
    vvi adj; segment_tree values;
    HLD(int _n) : n(_n), sz(n, 1), head(n),
        parent(n, -1), loc(n), adj(n) {
        vector<ll> tmp(n, ID); values = segment_tree(tmp); }
    void add_edge(int u, int v) {
        adj[u].push_back(v); adj[v].push_back(u); }
    void update_cost(int u, int v, int c) {
        if (parent[v] == u) swap(u, v); assert(parent[u] == v);
        values.update(loc[u], c); }
    int csz(int u) {
        rep(i,0,size(adj[u])) if (adj[u][i] != parent[u])
            sz[u] += csz(adj[parent[adj[u][i]]] = u[i]);
        return sz[u]; }
    void part(int u) {
        head[u] = curhead; loc[u] = curloc++;
        int best = -1;
        rep(i,0,size(adj[u]))
            if (adj[u][i] != parent[u] &&
                (best == -1 || sz[adj[u][i]] > sz[best]))
                best = adj[u][i];
        if (best != -1) part(best);
        rep(i,0,size(adj[u]))
            if (adj[u][i] != parent[u] && adj[u][i] != best)
                part(curhead = adj[u][i]); }
    void build(int r = 0) {
        curloc = 0, csz(curhead = r), part(r); }
    int lca(int u, int v) {
        vi uat, vat; int res = -1;
        while (u != -1) uat.push_back(u), u = parent[head[u]];
        while (v != -1) vat.push_back(v), v = parent[head[v]];
        u = size(uat) - 1, v = size(vat) - 1;
        while (u >= 0 && v >= 0 && head[uat[u]] == head[vat[v]])
            res = (loc[uat[u]] < loc[vat[v]] ? uat[u] : vat[v]),
            u--, v--;
        return res; }
    int query_upto(int u, int v) { int res = ID;
        while (head[u] != head[v])
            res = f(res, values.query(loc[head[u]], loc[u]).x),
            u = parent[head[u]];
        return f(res, values.query(loc[v] + 1, loc[u]).x); }
    int query(int u, int v) { int l = lca(u, v);
        return f(query_upto(u, l), query_upto(v, l)); } };
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.11. **Centroid Decomposition.**

```

#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
    path[MAXV][LGMAXV],
    sz[MAXV], seph[MAXV],
    shortest[MAXV];
struct centroid_decomposition {
    int n; vvi adj;
    centroid_decomposition(int _n) : n(_n), adj(n) { }
    void add_edge(int a, int b) {
        adj[a].push_back(b); adj[b].push_back(a); }
    int dfs(int u, int p) {
        sz[u] = 1;
        rep(i,0,size(adj[u]))
            rep(i,0,size(adj[u]))
                if (adj[u][i] != p) sz[u] += dfs(adj[u][i], u);
        return sz[u]; }
    void makepaths(int sep, int u, int p, int len) {
        jmp[u][seph[sep]] = sep, path[u][seph[sep]] = len;
        int bad = -1;
        rep(i,0,size(adj[u])) {
            if (adj[u][i] == p) bad = i;
            else makepaths(sep, adj[u][i], u, len + 1);
        }
        if (p == sep)
            swap(adj[u][bad], adj[u].back()), adj[u].pop_back(); }
    void separate(int h=0, int u=0) {
        dfs(u,-1); int sep = u;
        down: iter(nxt,adj[sep])
            if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2) {
                sep = *nxt; goto down; }
        seph[sep] = h, makepaths(sep, sep, -1, 0);
        rep(i,0,size(adj[sep])) separate(h+1, adj[sep][i]); }
    void paint(int u) {
        rep(h,0,seph[u]+1)
            shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
                path[u][h]); }

    int closest(int u) {
        int mn = INF/2;
        rep(h,0,seph[u]+1)
            mn = min(mn, path[u][h] + shortest[jmp[u][h]]);
        return mn; } };
// vim: cc=60 ts=2 sts=2 sw=2:

3.12. Least Common Ancestors, Binary Jumping.
const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
    rep(i, 0, SZ) BP[i][0] = P[i];
    rep(j, 1, LOGSZ) rep(i, 0, SZ)
        BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
    if (H[a] > H[b]) swap(a, b);
    int dh = H[b] - H[a], j = 0;

```

```

rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
while (BP[a][j] != BP[b][j]) j++;
while (--j >= 0) if (BP[a][j] != BP[b][j])
    a = BP[a][j], b = BP[b][j];
return a == b ? a : P[a];
}

```

3.13. Tarjan's Off-line Lowest Common Ancestors Algorithm.

```

#include "../data-structures/union_find.cpp"
struct tarjan_olca {
    int *ancestor;
    vi *adj, answers;
    vii *queries;
    bool *colored;
    union_find uf;
    tarjan_olca(int n, vi *_adj) : adj(_adj), uf(n) {
        colored = new bool[n];
        ancestor = new int[n];
        queries = new vii[n];
        memset(colored, 0, n);
    }
    void query(int x, int y) {
        queries[x].push_back(ii(y, size(answers)));
        queries[y].push_back(ii(x, size(answers)));
        answers.push_back(-1);
    }
    void process(int u) {
        ancestor[u] = u;
        rep(i,0,size(adj[u])) {
            int v = adj[u][i];
            process(v);
            uf.unite(u,v);
            ancestor[uf.find(u)] = u;
        }
        colored[u] = true;
        rep(i,0,size(queries[u])) {
            int v = queries[u][i].first;
            if (colored[v]) {
                answers[queries[u][i].second] = ancestor[uf.find(v)];
            }
        }
    }
};
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.14. **Minimum Mean Weight Cycle.** Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```

double min_mean_cycle(vector<vector<pair<int,double>>> adj){
    int n = size(adj); double mn = INFINITY;
    vector<vector<double>> arr(n+1, vector<double>(n, mn));
    arr[0][0] = 0;
    rep(k,1,n+1) rep(j,0,n) iter(it,adj[j])
        arr[k][it->first] = min(arr[k][it->first],
                                it->second + arr[k-1][j]);

    rep(k,0,n) {
        double mx = -INFINITY;
        rep(i,0,n) mx = max(mx, (arr[n][i]-arr[k][i])/(n-k));
        mn = min(mn, mx);
    }
    return mn;
}
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.15. **Minimum Arborescence.** Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root r to each vertex. Returns a vector of size n , where the i th element is the edge for the i th vertex. The answer for the root is undefined!

```

#include "../data-structures/union_find.cpp"
struct arborescence {
    int n; union_find uf;
    vector<vector<pair<ii,int>>> adj;
    arborescence(int _n) : n(_n), uf(n), adj(n) { }
    void add_edge(int a, int b, int c) {
        adj[b].push_back(make_pair(ii(a,b),c));
    }
    vii find_min(int r) {
        vi vis(n,-1), mn(n,INF); vii par(n);
        rep(i,0,n) {
            if (uf.find(i) != i) continue;
            int at = i;
            while (at != r && vis[at] == -1) {
                vis[at] = i;
                iter(it,adj[at]) if (it->second < mn[at] &&
                    uf.find(it->first.first) != at)
                    mn[at] = it->second, par[at] = it->first;
                if (par[at] == ii(0,0)) return vii();
                at = uf.find(par[at].first);
            }
            if (at == r || vis[at] != i) continue;
            union_find tmp = uf; vi seq;
            do { seq.push_back(at); at = uf.find(par[at].first); } while (at != seq.front());
            iter(it,seq) uf.unite(*it,seq[0]);
            int c = uf.find(seq[0]);
            vector<pair<ii,int>> nw;
            iter(it,seq) iter(jt,adj[*it])
                nw.push_back(make_pair(jt->first,
                    jt->second - mn[*it]));
            adj[c] = nw;
            vii rest = find_min(r);
            if (size(rest) == 0) return rest;
            ii use = rest[c];
            rest[at = tmp.find(use.second)] = use;
            iter(it,seq) if (*it != at)
                rest[*it] = par[*it];
            return rest;
        }
    }
};
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.16. **Blossom algorithm.** Finds a maximum matching in an arbitrary graph in $O(|V|^4)$ time. Be wary of loop edges.

```

#define MAXV 300
bool marked[MAXV], emarked[MAXV][MAXV];
int S[MAXV];
vi find_augmenting_path(const vector<vi> &adj, const vi &m){
    int n = size(adj), s = 0;
    vi par(n,-1), height(n), root(n,-1), q, a, b;
    memset(marked,0,sizeof(marked));
    memset(emarked,0,sizeof(emarked));
    rep(i,0,n) if (m[i] >= 0) emarked[i][m[i]] = true;
}

```

```

    else root[i] = i, S[s++] = i;
while (s) {
    int v = S[--s];
    iter(wt,adj[v]) {
        int w = *wt;
        if (emarked[v][w]) continue;
        if (root[w] == -1) {
            int x = S[s++] = m[w];
            par[w]=v, root[w]=root[v], height[w]=height[v]+1;
            par[x]=w, root[x]=root[w], height[x]=height[w]+1;
        } else if (height[w] % 2 == 0) {
            if (root[v] != root[w]) {
                while (v != -1) q.push_back(v), v = par[v];
                reverse(q.begin(), q.end());
                while (w != -1) q.push_back(w), w = par[w];
                return q;
            } else {
                int c = v;
                while (c != -1) a.push_back(c), c = par[c];
                c = w;
                while (c != -1) b.push_back(c), c = par[c];
                while (!a.empty() && !b.empty() && a.back() == b.back())
                    c = a.back(), a.pop_back(), b.pop_back();
                memset(marked,0,sizeof(marked));
                fill(par.begin(), par.end(), 0);
                iter(it,a) par[*it] = 1; iter(it,b) par[*it] = 1;
                par[c] = s = 1;
                rep(i,0,n) root[par[i] = par[i] ? 0 : s++] = i;
                vector<vi> adj2(s);
                rep(i,0,n) iter(it,adj[i]) {
                    if (par[*it] == 0) continue;
                    if (par[i] == 0) {
                        if (!marked[par[*it]]) {
                            adj2[par[i]].push_back(par[*it]);
                            adj2[par[*it]].push_back(par[i]);
                            marked[par[*it]] = true;
                        }
                        else adj2[par[i]].push_back(par[*it]);
                    }
                }
                vi m2(s, -1);
                if (m[c] != -1) m2[m2[par[m[c]]] = 0] = par[m[c]];
                rep(i,0,n) if (par[i] != 0 && m[i] != -1 && par[m[i]] != 0)
                    m2[par[i]] = par[m[i]];
                vi p = find_augmenting_path(adj2, m2);
                int t = 0;
                while (t < size(p) && p[t]) t++;
                if (t == size(p)) {
                    rep(i,0,size(p)) p[i] = root[p[i]];
                    return p;
                }
                if (!p[0] || (m[c] != -1 && p[t+1] != par[m[c]]))
                    reverse(p.begin(), p.end()), t = size(p)-t-1;
                rep(i,0,t) q.push_back(root[p[i]]);
                iter(it,adj[root[p[t-1]]) {
                    if (par[*it] != (s = 0)) continue;
                    a.push_back(c), reverse(a.begin(), a.end());
                    iter(jt,b) a.push_back(*jt);
                    while (a[s] != *it) s++;
                }
            }
        }
    }
}

```

```

    if ((height[*it] & 1) ^ (s < size(a) - size(b)))
        reverse(a.begin(), a.end()), s = size(a)-s-1;
    while(a[s] != c) q.push_back(a[s]), s = (s+1)%size(a);
    q.push_back(c);
    rep(i,t+1,size(p)) q.push_back(root[p[i]]);
    return q; } } }
    emarked[v][w] = emarked[w][v] = true; }
    marked[v] = true; } return q; }
vii max_matching(const vector<vi> &adj) {
    vi m(size(adj), -1), ap; vii res, es;
    rep(i,0,size(adj)) iter(it,adj[i]) es.emplace_back(i,*it);
    random_shuffle(es.begin(), es.end());
    iter(it,es) if (m[it->first] == -1 && m[it->second] == -1)
        m[it->first] = it->second, m[it->second] = it->first;
    do { ap = find_augmenting_path(adj, m);
        rep(i,0,size(ap)) m[m[ap[i^1]]] = ap[i] = ap[i^1];
    } while (!ap.empty());
    rep(i,0,size(m)) if (i < m[i]) res.emplace_back(i, m[i]);
    return res; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

3.17. **Maximum Density Subgraph.** Given (weighted) undirected graph G . Binary search density. If g is current density, construct flow network: (S, u, m) , $(u, T, m + 2g - d_u)$, $(u, v, 1)$, where m is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty S -component, then maximum density is smaller than g , otherwise it's larger. Distance between valid densities is at least $1/(n(n-1))$. Edge case when density is 0. This also works for weighted graphs by replacing d_u by the weighted degree, and doing more iterations (if weights are not integers).

3.18. **Maximum-Weight Closure.** Given a vertex-weighted directed graph G . Turn the graph into a flow network, adding weight ∞ to each edge. Add vertices S, T . For each vertex v of weight w , add edge (S, v, w) if $w \geq 0$, or edge $(v, T, -w)$ if $w < 0$. Sum of positive weights minus minimum $S - T$ cut is the answer. Vertices reachable from S are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

3.19. **Maximum Weighted Independent Set in a Bipartite Graph.** This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges $(S, u, w(u))$ for $u \in L$, $(v, T, w(v))$ for $v \in R$ and (u, v, ∞) for $(u, v) \in E$. The minimum S, T -cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

3.20. **Synchronizing word problem.** A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

4. STRING ALGORITHMS

4.1. Trie.

```
const int SIGMA = 26;
```

```
struct trie {
    bool word; trie **adj;
```

```

    trie() : word(false), adj(new trie*[SIGMA]) {
        for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
    }

```

```

void addWord(const string &str) {
    trie *cur = this;
    for (char ch : str) {
        int i = ch - 'a';
        if (!cur->adj[i]) cur->adj[i] = new trie();
        cur = cur->adj[i];
    }
    cur->word = true;
}

```

```

bool isWord(const string &str) {
    trie *cur = this;
    for (char ch : str) {
        int i = ch - 'a';
        if (!cur->adj[i]) return false;
        cur = cur->adj[i];
    }
    return cur->word;
}
};

```

4.2. Z-algorithm $\mathcal{O}(n)$.

// $z[i]$ = length of longest substring starting from $s[i]$
 \hookrightarrow which is also a prefix of s .

```

vi z_function(const string &s) {
    int n = (int) s.length();
    vi z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]]) ++z[i];
        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    }
    return z;
}

```

4.3. **Suffix array** $\mathcal{O}(n \log^2 n)$. This creates an array $P[0], P[1], \dots, P[n-1]$ such that the suffix $S[i \dots n]$ is the $P[i]^{th}$ suffix of S when lexicographically sorted.

```
typedef pair<pii, int> tii;
```

```
const int maxlogn = 17, int maxn = 1 << maxlogn;
```

```
tii make_triple(int a, int b, int c) { return tii(pii(a, b),
    c); }
```

```
int p[maxlogn + 1][maxn]; tii L[maxn];
```

```

int suffixArray(string S) {
    int N = S.size(), stp = 1, cnt = 1;
    for (int i = 0; i < N; i++) p[0][i] = S[i];
    for (; cnt < N; stp++, cnt <= 1) {

```

```

        for (int i = 0; i < N; i++)
            L[i] = tii(pii(p[stp-1][i], i + cnt < N ? p[stp-1][i +
    cnt] : -1), i);
        sort(L, L + N);
        for (int i = 0; i < N; i++)
            p[stp][L[i].y] = i > 0 && L[i].x == L[i-1].x ?
    p[stp][L[i-1].y] : i;
    }
    return stp - 1; // result is in p[stp - 1][0 .. (N - 1)]
}

```

4.4. **Longest Common Subsequence** $\mathcal{O}(n^2)$. SUBSTRING: consecutive characters !!!

```
int dp[STR_SIZE][STR_SIZE]; // DP problem
```

```

int lcs(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i < n1; i++) {
        for (int j = 0; j < n2; j++) {
            if (i == 0 || j == 0) dp[i][j] = 0;
            else if (w1[i - 1] == w2[j - 1]) dp[i][j] = dp[i - 1][j
    - 1] + 1;
            else dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);
        }
    }
    return dp[n1][n2];
}

```

// backtrace

```

string getLCS(const string &w1, const string &w2) {
    int i = w1.size(), j = w2.size(); string ret = "";
    while (i > 0 && j > 0) {
        if (w1[i - 1] == w2[j - 1]) ret += w1[--i], j--;
        else if (dp[i][j - 1] > dp[i - 1][j]) j--;
        else i--;
    }
    reverse(ret.begin(), ret.end());
    return ret;
}

```

4.5. **Levenshtein Distance** $\mathcal{O}(n^2)$. Also known as the 'Edit distance'.

```
int dp[MAX_SIZE][MAX_SIZE]; // DP problem
```

```

int levDist(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i <= n1; i++) dp[i][0] = i; // removal
    for (int j = 0; j <= n2; j++) dp[0][j] = j; // insertion
    for (int i = 1; i <= n1; i++)
        for (int j = 1; j <= n2; j++)
            dp[i][j] = min(
                1 + min(dp[i - 1][j], dp[i][j - 1]),
                dp[i - 1][j - 1] + (w1[i - 1] != w2[j - 1])
            );
    return dp[n1][n2];
}

```


4.6. Knuth-Morris-Pratt algorithm $\mathcal{O}(N + M)$.

```
int kmp_search(const string &word, const string &text) {
    int n = word.size();
    vi T(n + 1, 0);
    for (int i = 1, j = 0; i < n; ) {
        if (word[i] == word[j]) T[++i] = ++j; // match
        else if (j > 0) j = T[j]; // fallback
        else i++; // no match, keep zero
    }
    int matches = 0;
    for (int i = 0, j = 0; i < text.size(); ) {
        if (text[i] == word[j]) {
            i++;
            if (++j == n) { // match at interval [i - n, i)
                matches++; j = T[j];
            }
        } else if (j > 0) j = T[j];
        else i++;
    }
    return matches;
}
```

4.7. Aho-Corasick Algorithm $\mathcal{O}(N + \sum_{i=1}^m |S_i|)$. All given P must be unique!

```
const int MAXP = 100, MAXLEN = 200, SIGMA = 26, MAXTRIE =
    MAXP * MAXLEN;

int nP;
string P[MAXP], S;

int pnr[MAXTRIE], to[MAXTRIE][SIGMA], sLink[MAXTRIE],
    dLink[MAXTRIE], nnodes;

void ahoCorasick() {
    fill_n(pnr, MAXTRIE, -1);
    for (int i = 0; i < MAXTRIE; i++) fill_n(to[i], SIGMA, 0);
    fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE, 0);
    nnodes = 1;
    // STEP 1: MAKE A TREE
    for (int i = 0; i < nP; i++) {
        int cur = 0;
        for (char c : P[i]) {
            int i = c - 'a';
            if (to[cur][i] == 0) to[cur][i] = nnodes++;
            cur = to[cur][i];
        }
        pnr[cur] = i;
    }
    // STEP 2: CREATE SUFFIX_LINKS AND DICT_LINKS
    queue<int> q; q.push(0);
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (int c = 0; c < SIGMA; c++) {
            if (to[cur][c]) {
                int sl = sLink[to[cur][c]] = cur == 0 ? 0 :
                    to[sLink[cur]][c];
            }
        }
    }
}
```

```
// if all strings have equal length, remove this:
dLink[to[cur][c]] = pnr[sl] >= 0 ? sl : dLink[sl];
q.push(to[cur][c]);
} else to[cur][c] = to[sLink[cur]][c];
}
}
// STEP 3: TRAVERSE S
for (int cur = 0, i = 0, n = S.size(); i < n; i++) {
    cur = to[cur][S[i] - 'a'];
    for (int hit = pnr[cur] >= 0 ? cur : dLink[cur]; hit; hit
        = dLink[hit]) {
        cerr << P[pnr[hit]] << " found at [" << (i + 1 -
            P[pnr[hit]].size()) << ", " << i << "]" << endl;
    }
}
}
```

4.8. **eerTree**. Constructs an eerTree in $\mathcal{O}(n)$, one character at a time.

```
#define MAXN 100100
#define SIGMA 26
#define BASE 'a'
char *s = new char[MAXN];
struct state {
    int len, link, to[SIGMA];
} *st = new state[MAXN+2];
struct eertree {
    int last, sz, n;
    eertree() : last(1), sz(2), n(0) {
        st[0].len = st[0].link = -1;
        st[1].len = st[1].link = 0;
    }
    int extend() {
        char c = s[n++]; int p = last;
        while (n - st[p].len - 2 < 0 || c != s[n - st[p].len -
            2])
            p = st[p].link;
        if (!st[p].to[c - BASE]) {
            int q = last = sz++;
            st[p].to[c - BASE] = q;
            st[q].len = st[p].len + 2;
            do { p = st[p].link;
            } while (p != -1 && (n < st[p].len + 2 ||
                c != s[n - st[p].len - 2]));
            if (p == -1) st[q].link = 1;
            else st[q].link = st[p].to[c - BASE];
            return 1;
        }
        last = st[p].to[c - BASE];
        return 0;
    }
};
// vim: cc=60 ts=2 sts=2 sw=2:
```

4.9. **Suffix Automaton**. Minimum automata that accepts all suffixes of a string with $\mathcal{O}(n)$ construction. The automata itself is a DAG therefore suitable for DP, examples are counting unique substrings, occurrences of substrings and suffix.

```
// TODO: Add longest common subsring
const int MAXL = 100000;
struct suffix_automaton {
```

```
vi len, link, occur, cnt;
vector<map<char,int>> next;
vector<bool> isclone;
ll *occuratleast;
int sz, last;
string s;
suffix_automaton() : len(MAXL*2), link(MAXL*2),
    occur(MAXL*2), next(MAXL*2), isclone(MAXL*2) { clear(); }
void clear() { sz = 1; last = len[0] = 0; link[0] = -1;
    next[0].clear(); isclone[0] = false; }
bool issubstr(string other){
    for(int i = 0, cur = 0; i < size(other); ++i){
        if(cur == -1) return false; cur = next[cur][other[i]];
    }
    return true;
}
void extend(char c){ int cur = sz++; len[cur] =
    len[last]+1;
    next[cur].clear(); isclone[cur] = false; int p = last;
    for(; p != -1 && !next[p].count(c); p = link[p])
        next[p][c] = cur;
    if(p == -1){ link[cur] = 0; }
    else{ int q = next[p][c];
        if(len[p] + 1 == len[q]){ link[cur] = q; }
        else { int clone = sz++; isclone[clone] = true;
            len[clone] = len[p] + 1;
            link[clone] = link[q]; next[clone] = next[q];
            for(; p != -1 && next[p].count(c) && next[p][c] == q;
                p = link[p]){
                next[p][c] = clone;
                link[q] = link[cur] = clone;
            } } last = cur; }
    void count(){
        cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
        map<char,int>::iterator i;
        while(!S.empty()){
            ii cur = S.top(); S.pop();
            if(cur.second){
                for(i = next[cur.first].begin();
                    i != next[cur.first].end(); ++i){
                    cnt[cur.first] += cnt[(*i).second]; } }
            else if(cnt[cur.first] == -1){
                cnt[cur.first] = 1; S.push(ii(cur.first, 1));
                for(i = next[cur.first].begin();
                    i != next[cur.first].end(); ++i){
                    S.push(ii((*i).second, 0)); } } } }
        string lexicok(ll k){
            int st = 0; string res; map<char,int>::iterator i;
            while(k){
                for(i = next[st].begin(); i != next[st].end(); ++i){
                    if(k <= cnt[(*i).second]){ st = (*i).second;
                        res.push_back((*i).first); k--; break;
                    } else { k -= cnt[(*i).second]; } } }
            return res;
        }
        void countoccur(){
            for(int i = 0; i < sz; ++i){ occur[i] = 1 - isclone[i]; }
            vii states(sz);
```

```

    for(int i = 0; i < sz; ++i){ states[i] = ii(len[i],i); }
    sort(states.begin(), states.end());
    for(int i = size(states)-1; i >= 0; --i){
        int v = states[i].second;
        if(link[v] != -1) { occur[link[v]] += occur[v]; } } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

4.10. **Hashing.** Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```

struct hasher { int b = 311, m; vi h, p;
    hasher(string s, int _m)
        : m(_m), h(size(s)+1), p(size(s)+1) {
        p[0] = 1; h[0] = 0;
        rep(i,0,size(s)) p[i+1] = (ll)p[i] * b % m;
        rep(i,0,size(s)) h[i+1] = ((ll)h[i] * b + s[i]) % m; }
    int hash(int l, int r) {
        return (h[r+1] + m - (ll)h[l] * p[r-l+1] % m) % m; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

5. GEOMETRY

```
const double EPS = 1e-7, PI = acos(-1.0);
```

```

typedef long long NUM; // EITHER double OR long long
typedef pair<NUM, NUM> pt;
typedef pt frac;
typedef pair<double, double> vec;
struct line { NUM a, b, c; } // ax + by + c = 0

```

```

vec getvec(pt p, pt dp, frac t) {
    return vec(p.x + 1. * dp.x * t.x / t.y, p.y + 1. * dp.y *
    ↪ t.x / t.y);
}

```

```

pt operator+(pt p, pt q) { return pt(p.x + q.x, p.y + q.y); }
pt operator-(pt p, pt q) { return pt(p.x - q.x, p.y - q.y); }

```

```

pt& operator+=(pt &p, pt q) { return p = p + q; }
pt& operator-=(pt &p, pt q) { return p = p - q; }

```

```

pt operator*(pt p, NUM l) { return pt(p.x * l, p.y * l); }
pt operator/(pt p, NUM l) { return pt(p.x / l, p.y / l); }

```

```

NUM operator*(pt p, pt q) { return p.x * q.x + p.y * q.y; }
NUM operator^(pt p, pt q) { return p.x * q.y - p.y * q.x; }

```

```

istream& operator>>(istream &in, pt &p) { return in >> p.x >>
    ↪ p.y; }
ostream& operator<<(ostream &out, pt p) { return out << '('
    ↪ << p.x << ", " << p.y << ')'; }

```

```

NUM lenSq(pt p) { return p * p; }
NUM lenSq(pt p, pt q) { return lenSq(p - q); }
double len(pt p) { return sqrt(p * p); }
double len(pt p, pt q) { return len(p - q); }

```

```
double angle(pt a, pt o, pt b) {
```

```

    return acos(a - o) * (b - o) / len(a - o) / len(b - o));
}

```

```

pt rotate(pt p, double rad) {
    return pt(p.x * cos(rad) - p.y * sin(rad),
    ↪ p.x * sin(rad) + p.y * cos(rad));
}

```

```

// ----- LINES -----
// square distance from pt a to line bc
frac distPtLineSq(pt a, pt b, pt c) {
    a -= b, c -= b;
    return frac((a ^ c) * (a ^ c), c * c);
}

```

```

// square distance from pt a to line segment bc
frac distPtSegmentSq(pt a, pt b, pt c) {
    a -= b; c -= b;
    NUM dot = a * c, len = c * c;
    if (dot <= 0) return frac(a * a, 1);
    if (dot >= len) return frac((a - c) * (a - c), 1);
    return frac(a * a * len - dot * dot, len);
}

```

```

// projects pt a onto line segment bc
frac proj(pt a, pt b, pt c) { return frac((a - b) * (c - b),
    ↪ (c - b) * (c - b)); }
vec projv(pt a, pt b, pt c) { return getvec(b, c - b, proj(a,
    ↪ b, c)); }

```

```

bool collinear(pt a, pt b, pt c) { return ((a - b) ^ (a - c))
    ↪ == 0; }

```

```

bool pointOnSegment(pt a, pt b, pt c) {
    NUM dot = (a - b) * (c - b), len = (c - b) * (c - b);
    return collinear(a, b, c) && 0 <= dot && dot <= len;
}

```

```

// true => 1 intersection, false => parallel, so 0 or \infty
    ↪ solutions
bool linesIntersect(pt a, pt b, pt c, pt d) { return ((a - b)
    ↪ ^ (c - d)) != 0; }
vec lineLineIntersection(pt a, pt b, pt c, pt d) {
    double det = (a - b) ^ (c - d); pt ret = (c - d) * (a ^ b)
    ↪ - (a - b) * (c ^ d);
    return vec(ret.x / det, ret.y / det);
}

```

```

// dp, dq are directions from p, q
// intersection at p + t_i dp, for 0 <= i < return value
int segmentIntersection(pt p, pt dp, pt q, pt dq, frac &t0,
    ↪ frac &t1){
    if (dp * dp == 0) swap(p, q), swap(dp, dq); // dq = 0
    if (dp * dp == 0) { t0 = t1 = frac(0, 1); return p == q; }
    ↪ // dp = dq = 0

```

```

    pt dpq = (q - p); NUM c = dp ^ dq, c0 = dpq ^ dp, c1 = dpq
    ↪ ^ dq;
    if (c == 0) { // parallel, dp > 0, dq >= 0
        if (c0 != 0) return 0; // not collinear
        NUM v0 = dpq * dp, v1 = v0 + dq * dp, dp2 = dp * dp;
        if (v1 < v0) swap(v0, v1);
        t0 = frac(v0 = max(v0, (NUM) 0), dp2);
        t1 = frac(v1 = min(v1, dp2), dp2);
        return (v0 <= v1) + (v0 < v1);
    } else if (c < 0) c = -c, c0 = -c0, c1 = -c1;
    t0 = t1 = frac(c1, c);
    return 0 <= min(c0, c1) && max(c0, c1) <= c;
}

```

```

// Returns TWICE the area of a polygon to keep it an integer
NUM polygonTwiceArea(const vector<pt> &pts) {
    NUM area = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j = i++)
        area += pts[i] ^ pts[j];
    return abs(area); // area < 0 <=> pts ccw
}

```

```

bool pointInPolygon(pt p, const vector<pt> &pts) {
    double sum = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j = i++)
    ↪ {
        if (pointOnSegment(p, pts[i], pts[j])) return true; //
    ↪ boundary
        double angle = acos((pts[i] - p) * (pts[j] - p) /
    ↪ len(pts[i], p) / len(pts[j], p));
        sum += ((pts[i] - p) ^ (pts[j] - p)) < 0 ? angle :
    ↪ -angle;
        return abs(abs(sum) - 2 * PI) < EPS;
    }
}

```

5.1. Convex Hull $\mathcal{O}(n \log n)$.

```

// the convex hull consists of: { pts[ret[0]], pts[ret[1]],
    ↪ ... pts[ret.back()] }
vi convexHull(const vector<pt> &pts) {
    if (pts.empty()) return vi();
    vi ret, ord;
    int n = pts.size(), st = min_element(all(pts)) -
    ↪ pts.begin();
    rep(i, 0, n)
        if (pts[i] != pts[st]) ord.pb(i);
    sort(all(ord), [&pts,&st] (int a, int b) {
        pt p = pts[a] - pts[st], q = pts[b] - pts[st];
        return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) < lenSq(q);
    });
    ret.pb(st);
    for (int i : ord) {
        // use '>' to include ALL points on the hull-line
        for (int s = ret.size() - 1; s > 0 && ((pts[ret[s-1]] -
    ↪ pts[ret[s]]) ^ (pts[i] - pts[ret[s]])) >= 0; s--)

```

```

    ret.pop_back();
    ret.pb(i);
}
return ret;
}

5.2. Rotating Calipers  $\mathcal{O}(n)$ . Finds the longest distance between two
points in a convex hull.

NUM rotatingCalipers(vector<pt> &hull) {
    int n = hull.size(), a = 0, b = 1;
    if (n <= 1) return 0.0;
    while (((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
    hull[b])) > 0) b++;
    NUM ret = 0.0;
    while (a < n) {
        ret = max(ret, lenSq(hull[a], hull[b]));
        if (((hull[(a + 1) % n] - hull[a % n]) ^ (hull[(b + 1) %
    n] - hull[b]))) <= 0) a++;
        else if (++b == n) b = 0;
    }
    return ret;
}

5.3. Closest points  $\mathcal{O}(n \log n)$ .

int n; pt pts[maxn];

struct byY {
    bool operator()(int a, int b) const { return pts[a].y <
    pts[b].y; }
};

inline NUM dist(pii p) {
    return hypot(pts[p.x].x - pts[p.y].x, pts[p.x].y -
    pts[p.y].y);
}

pii minpt(pii p1, pii p2) { return (dist(p1) < dist(p2)) ? p1
    : p2; }

// closest pts (by index) inside pts[l ... r], with sorted y
// values in ys
pii closest(int l, int r, vi &ys) {
    if (r - l == 2) { // don't assume 1 here.
        ys = { l, l + 1 };
        return pii(l, l + 1);
    } else if (r - l == 3) { // brute-force
        ys = { l, l + 1, l + 2 };
        sort(ys.begin(), ys.end(), byY());
        return minpt(pii(l, l + 1), minpt(pii(l, l + 2), pii(l +
    1, l + 2)));
    }
    int m = (l + r) / 2; vi yl, yr;
    pii delta = minpt(closest(l, m, yl), closest(m, r, yr));
    NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
    pts[m].x);

```

```

    merge(yl.begin(), yl.end(), yr.begin(), yr.end(),
    back_inserter(ys), byY());
    deque<int> q;
    for (int i : ys) {
        if (abs(pts[i].x - xm) <= ddelta) {
            for (int j : q) delta = minpt(delta, pii(i, j));
            q.pb(i);
            if (q.size() > 8) q.pop_front(); // magic from
    Introduction to Algorithms.
        }
    }
    return delta;
}

5.4. Great-Circle Distance. Computes the distance between two
points (given as latitude/longitude coordinates) on a sphere of radius
r.

double gc_distance(double pLat, double pLong,
    double qLat, double qLong, double r) {
    pLat *= pi / 180; pLong *= pi / 180;
    qLat *= pi / 180; qLong *= pi / 180;
    return r * acos(cos(pLat) * cos(qLat) * cos(pLong - qLong)
    +
    sin(pLat) * sin(qLat)); }
// vim: cc=60 ts=2 sts=2 sw=2:

5.5. 3D Primitives.

#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)
#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
    double x, y, z;
    point3d() : x(0), y(0), z(0) {}
    point3d(double _x, double _y, double _z)
    : x(_x), y(_y), z(_z) {}
    point3d operator+(P(p)) const {
        return point3d(x + p.x, y + p.y, z + p.z); }
    point3d operator-(P(p)) const {
        return point3d(x - p.x, y - p.y, z - p.z); }
    point3d operator-() const {
        return point3d(-x, -y, -z); }
    point3d operator*(double k) const {
        return point3d(x * k, y * k, z * k); }
    point3d operator/(double k) const {
        return point3d(x / k, y / k, z / k); }
    double operator%(P(p)) const {
        return x * p.x + y * p.y + z * p.z; }
    point3d operator*(P(p)) const {
        return point3d(y*p.z - z*p.y,
            z*p.x - x*p.z, x*p.y - y*p.x); }
    double length() const {
        return sqrt(*this % *this); }
    double distTo(P(p)) const {
        return (*this - p).length(); }
    double distTo(P(A), P(B)) const {
        // A and B must be two different points

```

```

    return ((*this - A) * (*this - B)).length() /
    A.distTo(B); }
    point3d normalize(double k = 1) const {
        // length() must not return 0
        return (*this) * (k / length()); }
    point3d getProjection(P(A), P(B)) const {
        point3d v = B - A;
        return A + v.normalize((v % (*this - A)) / v.length()); }
    point3d rotate(P(normal)) const {
        //normal must have length 1 and be orthogonal to the
    vector
        return (*this) * normal; }
    point3d rotate(double alpha, P(normal)) const {
        return (*this) * cos(alpha) + rotate(normal) *
    sin(alpha); }
    point3d rotatePoint(P(O), P(axe), double alpha) const {
        point3d Z = axe.normalize(axe % (*this - O));
        return O + Z + (*this - O - Z).rotate(alpha, O); }
    bool isZero() const {
        return abs(x) < EPS && abs(y) < EPS && abs(z) < EPS; }
    bool isOnLine(L(A, B)) const {
        return ((A - *this) * (B - *this)).isZero(); }
    bool isInSegment(L(A, B)) const {
        return isOnLine(A, B) && ((A - *this) % (B -
    *this)) < EPS; }
    bool isInSegmentStrictly(L(A, B)) const {
        return isOnLine(A, B) && ((A - *this) % (B -
    *this)) < EPS; }
    double getAngle() const {
        return atan2(y, x); }
    double getAngle(P(u)) const {
        return atan2((*this * u).length(), *this % u); }
    bool isOnPlane(PL(A, B, C)) const {
        return
            abs((A - *this) * (B - *this) % (C - *this)) < EPS; }
    };
    int line_line_intersect(L(A, B), L(C, D), point3d &O) {
        if (abs((B - A) * (C - A) % (D - A)) > EPS) return 0;
        if (((A - B) * (C - D)).length() < EPS)
            return A.isOnLine(C, D) ? 2 : 0;
        point3d normal = ((A - B) * (C - B)).normalize();
        double s1 = (C - A) * (D - A) % normal;
        O = A + ((B - A) / (s1 + ((D - B) * (C - B) % normal))) *
    s1;
        return 1; }
    int line_plane_intersect(L(A, B), PL(C, D, E), point3d &O) {
        double V1 = (C - A) * (D - A) % (E - A);
        double V2 = (D - B) * (C - B) % (E - B);
        if (abs(V1 + V2) < EPS)
            return A.isOnPlane(C, D, E) ? 2 : 0;
        O = A + ((B - A) / (V1 + V2)) * V1;
        return 1; }
    bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
    point3d &P, point3d &Q) {
        point3d n = nA * nB;

```

```

    if (n.isZero()) return false;
    point3d v = n * nA;
    P = A + (n * nA) * ((B - A) % nB / (v % nB));
    Q = P + n;
    return true; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

5.6. Polygon Centroid.

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

5.7. Rectilinear Minimum Spanning Tree. Given a set of n points in the plane, and the aim is to find a minimum spanning tree connecting these n points, assuming the Manhattan distance is used. The function `candidates` returns at most $4n$ edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```
#define MAXN 100100
```

```

struct RMST {
    struct point {
        int i; ll x, y;
        point() : i(-1) { }
        ll d1() { return x + y; }
        ll d2() { return x - y; }
        ll dist(point other) {
            return abs(x - other.x) + abs(y - other.y); }
        bool operator <(const point &other) const {
            return y == other.y ? x > other.x : y < other.y; }
    } best[MAXN], arr[MAXN], tmp[MAXN];
    int n;
    RMST() : n(0) { }
    void add_point(int x, int y) {
        arr[arr[n].i = n].x = x, arr[n++].y = y; }
    void rec(int l, int r) {
        if (l >= r) return;
        int m = (l+r)/2;
        rec(l,m), rec(m+1,r);
        point bst;
        for (int i = l, j = m+1, k = l; i <= m || j <= r; k++) {
            if (j > r || (i <= m && arr[i].d1() < arr[j].d1())) {
                tmp[k] = arr[i++];
                if (bst.i != -1 && (best[tmp[k].i].i == -1
                    || best[tmp[k].i].d2() < bst.d2()))
                    best[tmp[k].i] = bst;
            } else {
                tmp[k] = arr[j++];
                if (bst.i == -1 || bst.d2() < tmp[k].d2())
                    bst = tmp[k]; } }
        rep(i,l,r+1) arr[i] = tmp[i]; }
    vector<pair<ll,ii>> candidates() {
        vector<pair<ll, ii>> es;

```

```

        rep(p,0,2) {
            rep(q,0,2) {
                sort(arr, arr+n);
                rep(i,0,n) best[i].i = -1;
                rec(0,n-1);
                rep(i,0,n) {
                    if(best[arr[i].i].i != -1)
                        es.push_back({arr[i].dist(best[arr[i].i]),
                            {arr[i].i, best[arr[i].i].i}});
                    swap(arr[i].x, arr[i].y);
                    arr[i].x *= -1, arr[i].y *= -1; } }
                rep(i,0,n) arr[i].x *= -1; }
            return es; } } };
// vim: cc=60 ts=2 sts=2 sw=2:

```

5.8. Formulas. Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two-dimensional vectors.

- $a \cdot b = |a||b| \cos \theta$, where θ is the angle between a and b .
- $a \times b = |a||b| \sin \theta$, where θ is the signed angle between a and b .
- $a \times b$ is equal to the area of the parallelogram with two of its sides formed by a and b . Half of that is the area of the triangle formed by a and b .
- **Euler's formula:** $V - E + F = 2$
- Side lengths a, b, c can form a triangle iff. $a + b > c$, $b + c > a$ and $a + c > b$.
- Sum of internal angles of a regular convex n -gon is $(n - 2)\pi$.
- **Law of sines:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:** $b^2 = a^2 + c^2 - 2ac \cos B$
- Internal tangents of circles $(c_1, r_1), (c_2, r_2)$ intersect at $(c_1 r_2 + c_2 r_1) / (r_1 + r_2)$, external intersect at $(c_1 r_2 - c_2 r_1) / (r_1 + r_2)$.

6. MISCELLANEOUS

6.1. Binary search $\mathcal{O}(\log(hi - lo))$.

```

bool test(int n);

int search(int lo, int hi) {
    // assert(test(lo) && !test(hi));
    while (hi - lo > 1) {
        int m = (lo + hi) / 2;
        (test(m) ? lo : hi) = m;
    }
    // assert(test(lo) && !test(hi));
    return lo;
}

```

6.2. Fast Fourier Transform $\mathcal{O}(n \log n)$. Given two polynomials $A(x) = a_0 + a_1x + \dots + a_{n/2}x^{n/2}$ and $B(x) = b_0 + b_1x + \dots + b_{n/2}x^{n/2}$, FFT calculates all coefficients of $C(x) = A(x) \cdot B(x) = c_0 + c_1x + \dots + c_nx^n$, with $c_i = \sum_{j=0}^i a_j b_{i-j}$.

```

typedef complex<double> cpx;
const int logmaxn = 20, maxn = 1 << logmaxn;

cpx a[maxn] = {}, b[maxn] = {}, c[maxn];

void fft(cpx *src, cpx *dest) {
    for (int i = 0, rep = 0; i < maxn; i++, rep = 0) {

```

```

        for (int j = i, k = logmaxn; k-- > 1) rep = (rep <<
        ↪ 1) | (j & 1);
        dest[rep] = src[i];
    }
    for (int s = 1, m = 1; m <= maxn; s++, m *= 2) {
        cpx r = exp(cpx(0, 2.0 * PI / m));
        for (int k = 0; k < maxn; k += m) {
            cpx cr(1.0, 0.0);
            for (int j = 0; j < m / 2; j++) {
                cpx t = cr * dest[k + j + m / 2]; dest[k + j + m / 2]
        ↪ = dest[k + j] - t;
                dest[k + j] += t; cr *= r;
            }
        }
    }
}

```

```

void multiply() {
    fft(a, c); fft(b, a);
    for (int i = 0; i < maxn; i++) b[i] = conj(a[i] * c[i]);
    fft(b, c);
    for (int i = 0; i < maxn; i++) c[i] = conj(c[i]) / (1.0 *
    ↪ maxn);
}

```

6.3. Minimum Assignment (Hungarian Algorithm) $\mathcal{O}(n^3)$.

```
int a[MAXN + 1][MAXN + 1]; // matrix, 1-based
```

```
int minimum_assignment(int n, int m) { // n rows, m columns
    vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);

```

```

    for (int i = 1; i <= n; i++) {
        p[0] = i;
        int j0 = 0;
        vi minv(m + 1, INF);
        vector<char> used(m + 1, false);
        do {
            used[j0] = true;
            int i0 = p[j0], delta = INF, j1;
            for (int j = 1; j <= m; j++)
                if (!used[j]) {
                    int cur = a[i0][j] - u[i0] - v[j];
                    if (cur < minv[j]) minv[j] = cur, way[j] = j0;
                    if (minv[j] < delta) delta = minv[j], j1 = j;
                }
            for (int j = 0; j <= m; j++) {
                if(used[j]) u[p[j]] += delta, v[j] -= delta;
                else minv[j] -= delta;
            }
            j0 = j1;
        } while (p[j0] != 0);
        do {
            int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
        } while (j0);
    }
}

```



```

if (!Simplex(2)) return numeric_limits<DOUBLE>::infinity();
x = VD(n);
for (int i = 0; i < m; i++) if (B[i] < n)
    x[B[i]] = D[i][n + 1];
return D[m][n + 1]; } };
// Two-phase simplex algorithm for solving linear programs
// of the form
//      maximize      c^T x
//      subject to    Ax <= b
//                  x >= 0
// INPUT: A -- an m x n matrix
//        b -- an m-dimensional vector
//        c -- an n-dimensional vector
//        x -- a vector where the optimal solution will be
//              stored
// OUTPUT: value of the optimal solution (infinity if
//         unbounded above, nan if infeasible)
// To use this code, create an LPSolver object with A, b,
// and c as arguments. Then, call Solve(x).
// #include <iostream>
// #include <iomanip>
// #include <vector>
// #include <cmath>
// #include <limits>
// using namespace std;
// int main() {
//     const int m = 4;
//     const int n = 3;
//     DOUBLE _A[m][n] = {
//         { 6, -1, 0 },
//         { -1, -5, 0 },
//         { 1, 5, 1 },
//         { -1, -5, -1 }
//     };
//     DOUBLE _b[m] = { 10, -4, 5, -5 };
//     DOUBLE _c[n] = { 1, -1, 0 };
//     VVD A(m);
//     VD b(_b, _b + m);
//     VD c(_c, _c + n);
//     for (int i = 0; i < m; i++) A[i] = VD(_A[i], _A[i] + n);
//     LPSolver solver(A, b, c);
//     VD x;
//     DOUBLE value = solver.Solve(x);
//     cerr << "VALUE: " << value << endl; // VALUE: 1.29032
//     cerr << "SOLUTION:"; // SOLUTION: 1.74194 0.451613 1
//     for (size_t i = 0; i < x.size(); i++) cerr << " " <<
//     x[i];
//     cerr << endl;
//     return 0;
// }
// vim: cc=60 ts=2 sts=2 sw=2:

```

7. GEOMETRY (CP3)

7.1. Points and lines.

```

#define INF 1e9
#define EPS 1e-9
#define PI acos(-1.0) // important constant; alternative
// #define PI (2.0 * acos(0.0))

double DEG_to_RAD(double d) { return d * PI / 180.0; }

double RAD_to_DEG(double r) { return r * 180.0 / PI; }

struct point { double x, y; // only used if more precision
// is needed
point() { x = y = 0.0; } // default
// constructor
point(double _x, double _y) : x(_x), y(_y) {} //
// user-defined
bool operator < (point other) const { // override less than
// operator
if (fabs(x - other.x) > EPS) // useful
// for sorting
return x < other.x; // first criteria , by
// x-coordinate
return y < other.y; } // second criteria, by
// y-coordinate
// use EPS (1e-9) when testing equality of two floating
// points
bool operator == (point other) const {
return (fabs(x - other.x) < EPS && (fabs(y - other.y) <
// EPS)); } };

double dist(point p1, point p2) { // Euclidean
// distance
// hypot(dx, dy) returns sqrt(dx * dx +
// dy * dy)
return hypot(p1.x - p2.x, p1.y - p2.y); } //
// return double

// rotate p by theta degrees CCW w.r.t origin (0, 0)
point rotate(point p, double theta) {
double rad = DEG_to_RAD(theta); // multiply theta with
// PI / 180.0
return point(p.x * cos(rad) - p.y * sin(rad),
p.x * sin(rad) + p.y * cos(rad)); }

struct line { double a, b, c; }; // ax + by + c = 0

// the answer is stored in the third parameter (pass by
// reference)
void pointsToLine(point p1, point p2, line &l) {
if (fabs(p1.x - p2.x) < EPS) { // vertical
// line is fine
l.a = 1.0; l.b = 0.0; l.c = -p1.x; //
// default values
} else {
l.a = -(double)(p1.y - p2.y) / (p1.x - p2.x);

```

```

l.b = 1.0; // IMPORTANT: we fix the value of
// b to 1.0
l.c = -(double)(l.a * p1.x) - p1.y;
} }

bool areParallel(line l1, line l2) { // check
// coefficients a & b
return (fabs(l1.a-l2.a) < EPS) && (fabs(l1.b-l2.b) < EPS);
// }

bool areSame(line l1, line l2) { // also check
// coefficient c
return areParallel(l1, l2) && (fabs(l1.c - l2.c) < EPS); }

// returns true (+ intersection point) if two lines are
// intersect
bool areIntersect(line l1, line l2, point &p) {
if (areParallel(l1, l2)) return false; // no
// intersection
// solve system of 2 linear algebraic equations with 2
// unknowns
p.x = (l2.b * l1.c - l1.b * l2.c) / (l2.a * l1.b - l1.a *
// l2.b);
// special case: test for vertical line to avoid division
// by zero
if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
else p.y = -(l2.a * p.x + l2.c);
return true; }

struct vec { double x, y; // name: `vec' is different from
// STL vector
vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) { // convert 2 points to
// vector a->b
return vec(b.x - a.x, b.y - a.y); }

vec scale(vec v, double s) { // nonnegative s = [<1 ..
// 1 .. >1]
return vec(v.x * s, v.y * s); } //
// shorter.same.longer

point translate(point p, vec v) { // translate p
// according to v
return point(p.x + v.x, p.y + v.y); }

// convert point and gradient/slope to line
void pointSlopeToLine(point p, double m, line &l) {
l.a = -m; //
// always -m
l.b = 1; //
// always 1
l.c = -((l.a * p.x) + (l.b * p.y)); } //
// compute this

```

```

void closestPoint(line l, point p, point &ans) {
    line perpendicular; // perpendicular to l and pass
    // through p
    if (fabs(l.b) < EPS) { // special case 1:
        // vertical line
        ans.x = -(l.c); ans.y = p.y; return; }

    if (fabs(l.a) < EPS) { // special case 2:
        // horizontal line
        ans.x = p.x; ans.y = -(l.c); return; }

    pointSlopeToLine(p, 1 / l.a, perpendicular); //
    // normal line
    // intersect line l with this perpendicular line
    // the intersection point is the closest point
    areIntersect(l, perpendicular, ans); }

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
    point b;
    closestPoint(l, p, b); // similar to
    // distToLine
    vec v = toVec(p, b); // create
    // a vector
    ans = translate(translate(p, v), v); // translate
    // p twice

double dot(vec a, vec b) { return (a.x * b.x + a.y * b.y); }

double norm_sq(vec v) { return v.x * v.x + v.y * v.y; }

// returns the distance from p to the line defined by
// two points a and b (a and b must be different)
// the closest point is stored in the 4th parameter (byref)
double distToLine(point p, point a, point b, point &c) {
    // formula: c = a + u * ab
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    c = translate(a, scale(ab, u)); //
    // translate a to c
    return dist(p, c); } // Euclidean distance
    // between p and c

// returns the distance from p to the line segment ab defined
// by
// two points a and b (still OK if a == b)
// the closest point is stored in the 4th parameter (byref)
double distToLineSegment(point p, point a, point b, point &c)
{
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    if (u < 0.0) { c = point(a.x, a.y); //
    // closer to a

```

```

    return dist(p, a); } // Euclidean distance
    // between p and a
    if (u > 1.0) { c = point(b.x, b.y); //
    // closer to b
    return dist(p, b); } // Euclidean distance
    // between p and b
    return distToLine(p, a, b, c); } // run distToLine
    // as above

double angle(point a, point o, point b) { // returns angle
    // aob in rad
    vec oa = toVec(o, a), ob = toVec(o, b);
    return acos(dot(oa, ob) / sqrt(norm_sq(oa) * norm_sq(ob))); }

double cross(vec a, vec b) { return a.x * b.y - a.y * b.x; }

// note: to accept collinear points, we have to change the `>
// to `0`
// returns true if point r is on the left side of line pq
bool ccw(point p, point q, point r) {
    return cross(toVec(p, q), toVec(p, r)) > 0; }

// returns true if point r is on the same line as the line pq
bool collinear(point p, point q, point r) {
    return fabs(cross(toVec(p, q), toVec(p, r))) < EPS; }

7.2. Polygon.
// returns the perimeter, which is the sum of Euclidian
// distances
// of consecutive line segments (polygon edges)
double perimeter(const vector<point> &P) {
    double result = 0.0;
    for (int i = 0; i < (int)P.size()-1; i++) // remember that
    // P[0] = P[n-1]
        result += dist(P[i], P[i+1]);
    return result; }

// returns the area, which is half the determinant
double area(const vector<point> &P) {
    double result = 0.0, x1, y1, x2, y2;
    for (int i = 0; i < (int)P.size()-1; i++) {
        x1 = P[i].x; x2 = P[i+1].x;
        y1 = P[i].y; y2 = P[i+1].y;
        result += (x1 * y2 - x2 * y1);
    }
    return fabs(result) / 2.0; }

// returns true if we always make the same turn while
// examining
// all the edges of the polygon one by one
bool isConvex(const vector<point> &P) {
    int sz = (int)P.size();
    if (sz <= 3) return false; // a point/sz=2 or a line/sz=3
    // is not convex

```

```

    bool isLeft = ccw(P[0], P[1], P[2]); //
    // remember one result
    for (int i = 1; i < sz-1; i++) // then compare
    // with the others
        if (ccw(P[i], P[i+1], P[(i+2) == sz ? 1 : i+2])) !=
        // isLeft)
            return false; // different sign -> this
    // polygon is concave
    return true; } // this
    // polygon is convex

// returns true if point p is in either convex/concave
// polygon P
bool inPolygon(point pt, const vector<point> &P) {
    if ((int)P.size() == 0) return false;
    double sum = 0; // assume the first vertex is equal to
    // the last vertex
    for (int i = 0; i < (int)P.size()-1; i++) {
        if (ccw(pt, P[i], P[i+1]))
            sum += angle(P[i], pt, P[i+1]); //
    // left turn/ccw
        else sum -= angle(P[i], pt, P[i+1]); } //
    // right turn/cw
    return fabs(fabs(sum) - 2*PI) < EPS; }

// line segment p-q intersect with line A-B.
point lineIntersectSeg(point p, point q, point A, point B) {
    double a = B.y - A.y;
    double b = A.x - B.x;
    double c = B.x * A.y - A.x * B.y;
    double u = fabs(a * p.x + b * p.y + c);
    double v = fabs(a * q.x + b * q.y + c);
    return point((p.x * v + q.x * u) / (u+v), (p.y * v + q.y *
    // u) / (u+v)); }

// cuts polygon Q along the line formed by point a -> point b
// (note: the last point must be the same as the first point)
vector<point> cutPolygon(point a, point b, const
// vector<point> &Q) {
    vector<point> P;
    for (int i = 0; i < (int)Q.size(); i++) {
        double left1 = cross(toVec(a, b), toVec(a, Q[i])), left2
        // = 0;
        if (i != (int)Q.size()-1) left2 = cross(toVec(a, b),
        // toVec(a, Q[i+1]));
        if (left1 > -EPS) P.push_back(Q[i]); // Q[i] is on
        // the left of ab
        if (left1 * left2 < -EPS) // edge (Q[i], Q[i+1])
        // crosses line ab
            P.push_back(lineIntersectSeg(Q[i], Q[i+1], a, b));
    }
    if (!P.empty() && !(P.back() == P.front()))
        P.push_back(P.front()); // make P's first point =
    // P's last point

```

```

    return P; }

point pivot;
bool angleCmp(point a, point b) {           //
    ↪ angle-sorting function
    if (collinear(pivot, a, b))
    ↪ // special case
        return dist(pivot, a) < dist(pivot, b);    // check which
    ↪ one is closer
    double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
    double d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0; }    //
    ↪ compare two angles

vector<point> CH(vector<point> P) {    // the content of P may
    ↪ be reshuffled
    int i, j, n = (int)P.size();
    if (n <= 3) {
        if (!P[0] == P[n-1]) P.push_back(P[0]); // safeguard
    ↪ from corner case
        return P;                                // special case, the
    ↪ CH is P itself
    }

    // first, find P0 = point with lowest Y and if tie:
    ↪ rightmost X
    int P0 = 0;
    for (i = 1; i < n; i++)
        if (P[i].y < P[P0].y || (P[i].y == P[P0].y && P[i].x >
    ↪ P[P0].x))
        P0 = i;

    point temp = P[0]; P[0] = P[P0]; P[P0] = temp;    // swap
    ↪ P[P0] with P[0]

    // second, sort points by angle w.r.t. pivot P0
    pivot = P[0];                                // use this global
    ↪ variable as reference
    sort(++P.begin(), P.end(), angleCmp);            // we do
    ↪ not sort P[0]

    // third, the ccw tests
    vector<point> S;
    S.push_back(P[n-1]); S.push_back(P[0]); S.push_back(P[1]);
    ↪ // initial S
    i = 2;                                          // then, we
    ↪ check the rest
    while (i < n) {                                // note: N must be >= 3 for this
    ↪ method to work
        j = (int)S.size()-1;
        if (ccw(S[j-1], S[j], P[i])) S.push_back(P[i++]); //
    ↪ left turn, accept

```

```

    else S.pop_back(); }    // or pop the top of S until we
    ↪ have a left turn
    return S; }                                           //
    ↪ return the result

7.3. Triangle.
double perimeter(double ab, double bc, double ca) {
    return ab + bc + ca; }

double perimeter(point a, point b, point c) {
    return dist(a, b) + dist(b, c) + dist(c, a); }

double area(double ab, double bc, double ca) {
    // Heron's formula, split sqrt(a * b) into sqrt(a) *
    ↪ sqrt(b); in implementation
    double s = 0.5 * perimeter(ab, bc, ca);
    return sqrt(s) * sqrt(s - ab) * sqrt(s - bc) * sqrt(s -
    ↪ ca); }

double area(point a, point b, point c) {
    return area(dist(a, b), dist(b, c), dist(c, a)); }

double rInCircle(double ab, double bc, double ca) {
    return area(ab, bc, ca) / (0.5 * perimeter(ab, bc, ca)); }

double rInCircle(point a, point b, point c) {
    return rInCircle(dist(a, b), dist(b, c), dist(c, a)); }

// assumption: the required points/lines functions have been
    ↪ written
// returns 1 if there is an inCircle center, returns 0
    ↪ otherwise
// if this function returns 1, ctr will be the inCircle
    ↪ center
// and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point &ctr, double
    ↪ &r) {
    r = rInCircle(p1, p2, p3);
    if (fabs(r) < EPS) return 0;                        // no
    ↪ inCircle center

    line l1, l2;                                        // compute these two angle
    ↪ bisectors
    double ratio = dist(p1, p2) / dist(p1, p3);
    point p = translate(p2, scale(toVec(p2, p3), ratio / (1 +
    ↪ ratio)));
    pointsToLine(p1, p, l1);

    ratio = dist(p2, p1) / dist(p2, p3);
    p = translate(p1, scale(toVec(p1, p3), ratio / (1 +
    ↪ ratio)));
    pointsToLine(p2, p, l2);

    areIntersect(l1, l2, ctr);                          // get their
    ↪ intersection point

```

```

    return 1; }

double rCircumCircle(double ab, double bc, double ca) {
    return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

double rCircumCircle(point a, point b, point c) {
    return rCircumCircle(dist(a, b), dist(b, c), dist(c, a)); }

// assumption: the required points/lines functions have been
    ↪ written
// returns 1 if there is a circumCenter center, returns 0
    ↪ otherwise
// if this function returns 1, ctr will be the circumCircle
    ↪ center
// and r is the same as rCircumCircle
int circumCircle(point p1, point p2, point p3, point &ctr,
    ↪ double &r){
    double a = p2.x - p1.x, b = p2.y - p1.y;
    double c = p3.x - p1.x, d = p3.y - p1.y;
    double e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
    double f = c * (p1.x + p3.x) + d * (p1.y + p3.y);
    double g = 2.0 * (a * (p3.y - p2.y) - b * (p3.x - p2.x));
    if (fabs(g) < EPS) return 0;

    ctr.x = (d*e - b*f) / g;
    ctr.y = (a*f - c*e) / g;
    r = dist(p1, ctr); // r = distance from center to 1 of the
    ↪ 3 points
    return 1; }

// returns true if point d is inside the circumCircle defined
    ↪ by a,b,c
int inCircumCircle(point a, point b, point c, point d) {
    return (a.x - d.x) * (b.y - d.y) * ((c.x - d.x) * (c.x -
    ↪ d.x) + (c.y - d.y) * (c.y - d.y)) +
        (a.y - d.y) * ((b.x - d.x) * (b.x - d.x) + (b.y -
    ↪ d.y) * (b.y - d.y)) * (c.x - d.x) +
        ((a.x - d.x) * (a.x - d.x) + (a.y - d.y) * (a.y -
    ↪ d.y)) * (b.x - d.x) * (c.y - d.y) -
        ((a.x - d.x) * (a.x - d.x) + (a.y - d.y) * (a.y -
    ↪ d.y)) * (b.y - d.y) * (c.x - d.x) -
        (a.y - d.y) * (b.x - d.x) * ((c.x - d.x) * (c.x -
    ↪ d.x) + (c.y - d.y) * (c.y - d.y)) -
        (a.x - d.x) * ((b.x - d.x) * (b.x - d.x) + (b.y -
    ↪ d.y) * (b.y - d.y)) * (c.y - d.y) > 0 ? 1 : 0;
}

bool canFormTriangle(double a, double b, double c) {
    return (a + b > c) && (a + c > b) && (b + c > a); }

7.4. Circle.
int insideCircle(point_i p, point_i c, int r) { // all
    ↪ integer version
    int dx = p.x - c.x, dy = p.y - c.y;

```

```
int Euc = dx * dx + dy * dy, rSq = r * r; //
↪ all integer
return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
↪ //inside/border/outside

bool circle2PtsRad(point p1, point p2, double r, point &c) {
    double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
                (p1.y - p2.y) * (p1.y - p2.y);
    double det = r * r / d2 - 0.25;
    if (det < 0.0) return false;
    double h = sqrt(det);
    c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
    c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
    return true; } // to get the other center, reverse
↪ p1 and p2
```

8. COMBINATORICS

Catalan	$C_0 = 1, C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \frac{4n-2}{n+1} C_{n-1}$	
Stirling 1st kind	$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0, \begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$	#perms of n objs with exactly k cycles
Stirling 2nd kind	$\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1, \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$	#ways to partition n objs into k nonempty sets
Euler	$\left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\rangle = 1, \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (n-k) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle$	#perms of n objs with exactly k ascents
Euler 2nd Order	$\left\langle\!\!\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle\!\!\right\rangle = (k+1) \left\langle\!\!\left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle\!\!\right\rangle + (2n-k-1) \left\langle\!\!\left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle\!\!\right\rangle$	#perms of $1, 1, 2, 2, \dots, n, n$ with exactly k ascents
Bell	$B_1 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	#partitions of $1..n$ (Stirling 2nd, no limit on k)

#labeled rooted trees	n^{n-1}
#labeled unrooted trees	n^{n-2}
#forests of k rooted trees	$\frac{k}{n} \binom{n}{k} n^{n-k}$
$\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$	$\sum_{i=1}^n i^3 = n^2(n+1)^2/4$
$!n = n \times !(n-1) + (-1)^n$	$!n = (n-1)!(n-1) + (n-2)!$
$\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}$	$\sum_i \binom{n-i}{i} = F_{n+1}$
$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$	$x^k = \sum_{i=0}^k i! \left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\} \binom{x}{i} = \sum_{i=0}^k \left\langle \begin{smallmatrix} k \\ i \end{smallmatrix} \right\rangle \binom{x+i}{k}$
$a \equiv b \pmod{x, y} \Rightarrow a \equiv b \pmod{\text{lcm}(x, y)}$	$\sum_{d n} \phi(d) = n$
$ac \equiv bc \pmod{m} \Rightarrow a \equiv b \pmod{\frac{m}{\text{gcd}(c, m)}}$	$(\sum_{d n} \sigma_0(d))^2 = \sum_{d n} \sigma_0(d)^3$
p prime $\Leftrightarrow (p-1)! \equiv -1 \pmod{p}$	$\text{gcd}(n^a - 1, n^b - 1) = n^{\text{gcd}(a, b)} - 1$
$\sigma_x(n) = \prod_{i=0}^r \frac{p_i^{(a_i+1)x} - 1}{p_i^x - 1}$	$\sigma_0(n) = \prod_{i=0}^r (a_i + 1)$
$\sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$	$\sum_{i=1}^n 2^{\omega(i)} = O(n \log n)$
$2^{\omega(n)} = O(\sqrt{n})$	$v_f^2 = v_i^2 + 2ad$
$d = v_i t + \frac{1}{2} a t^2$	$d = \frac{v_i + v_f}{2} t$
$v_f = v_i + at$	

8.1. The Twelvelfold Way. Putting n balls into k boxes.

Balls	same	distinct	same	distinct	
Boxes	same	same	distinct	distinct	Remarks
-	$p_k(n)$	$\sum_{i=0}^k \left\{ \begin{smallmatrix} n \\ i \end{smallmatrix} \right\}$	$\binom{n+k-1}{k-1}$	k^n	$p_k(n)$: #partitions of n into $\leq k$ positive parts
size ≥ 1	$p(n, k)$	$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	$\binom{n-1}{k-1}$	$k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	$p(n, k)$: #partitions of n into k positive parts
size ≤ 1	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n! \binom{k}{n}$	$[cond]$: 1 if $cond = true$, else 0

9. USEFUL INFORMATION

10. Misc

10.1. Debugging Tips.

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
 - Getting **NaN**? Make sure **acos** etc. are not getting values out of their range (perhaps **1+eps**).
 - Rounding negative numbers?
 - Outputting in scientific notation?
- Wrong Answer?
 - Read the problem statement again!
 - Are multiple test cases being handled correctly? Try repeating the same test case many times.
 - Integer overflow?
 - Think very carefully about boundaries of all input parameters
 - Try out possible edge cases:
 - * $n = 0, n = -1, n = 1, n = 2^{31} - 1$ or $n = -2^{31}$
 - * List is empty, or contains a single element
 - * n is even, n is odd
 - * Graph is empty, or contains a single vertex
 - * Graph is a multigraph (loops or multiple edges)
 - * Polygon is concave or non-simple
 - Is initial condition wrong for small cases?
 - Are you sure the algorithm is correct?
 - Explain your solution to someone.
 - Are you using any functions that you don't completely understand? Maybe STL functions?
 - Maybe you (or someone else) should rewrite the solution?
 - Can the input line be empty?
- Run-Time Error?
 - Is it actually Memory Limit Exceeded?

10.2. Solution Ideas.

- Dynamic Programming
 - Parsing CFGs: CYK Algorithm
 - Drop a parameter, recover from others
 - Swap answer and a parameter
 - When grouping: try splitting in two
 - 2^k trick
 - When optimizing
 - * Convex hull optimization
 - $dp[i] = \min_{j < i} \{dp[j] + b[j] \times a[i]\}$
 - $b[j] \geq b[j + 1]$
 - optionally $a[i] \leq a[i + 1]$
 - $O(n^2)$ to $O(n)$
 - * Divide and conquer optimization
 - $dp[i][j] = \min_{k < j} \{dp[i - 1][k] + C[k][j]\}$
 - $A[i][j] \leq A[i][j + 1]$
 - $O(kn^2)$ to $O(kn \log n)$
 - sufficient: $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$, $a \leq b \leq c \leq d$ (QI)
 - * Knuth optimization
 - $dp[i][j] = \min_{i < k < j} \{dp[i][k] + dp[k][j] + C[i][j]\}$
 - $A[i][j - 1] \leq A[i][j] \leq A[i + 1][j]$
 - $O(n^3)$ to $O(n^2)$

• sufficient: QI and $C[b][c] \leq C[a][d]$, $a \leq b \leq c \leq d$

- Greedy
- Randomized
- Optimizations
 - Use bitset (/64)
 - Switch order of loops (cache locality)
- Process queries offline
 - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
 - Mo's algorithm
 - Sqrt decomposition
 - Store 2^k jump pointers
- Data structure techniques
 - Sqrt buckets
 - Store 2^k jump pointers
 - 2^k merging trick
- Counting
 - Inclusion-exclusion principle
 - Generating functions
- Graphs
 - Can we model the problem as a graph?
 - Can we use any properties of the graph?
 - Strongly connected components
 - Cycles (or odd cycles)
 - Bipartite (no odd cycles)
 - * Bipartite matching
 - * Hall's marriage theorem
 - * Stable Marriage
 - Cut vertex/bridge
 - Biconnected components
 - Degrees of vertices (odd/even)
 - Trees
 - * Heavy-light decomposition
 - * Centroid decomposition
 - * Least common ancestor
 - * Centers of the tree
 - Eulerian path/circuit
 - Chinese postman problem
 - Topological sort
 - (Min-Cost) Max Flow
 - Min Cut
 - * Maximum Density Subgraph
 - Huffman Coding
 - Min-Cost Arborescence
 - Steiner Tree
 - Kirchhoff's matrix tree theorem
 - Prüfer sequences
 - Lovász Toggle
 - Look at the DFS tree (which has no cross-edges)
 - Is the graph a DFA or NFA?
 - * Is it the Synchronizing word problem?
- math
 - Is the function multiplicative?
 - Look for a pattern

- Permutations
 - * Consider the cycles of the permutation
- Functions
 - * Sum of piecewise-linear functions is a piecewise-linear function
 - * Sum of convex (concave) functions is convex (concave)
- Modular arithmetic
 - * Chinese Remainder Theorem
 - * Linear Congruence
- Sieve
- System of linear equations
- Values too big to represent?
 - * Compute using the logarithm
 - * Divide everything by some large value
- Linear programming
 - * Is the dual problem easier to solve?
- Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
 - 2-SAT
 - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ($\log(n)$)
- Strings
 - Trie (maybe over something weird, like bits)
 - Suffix array
 - Suffix automaton (+DP?)
 - Aho-Corasick
 - eertree
 - Work with $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
 - Lazy propagation
 - Persistent
 - Implicit
 - Segment tree of X
- Geometry
 - Minkowski sum (of convex sets)
 - Rotating calipers
 - Sweep line (horizontally or vertically?)
 - Sweep angle
 - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem

- Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)
- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

11. FORMULAS

- **Legendre symbol:** $\left(\frac{a}{b}\right) = a^{(b-1)/2} \pmod{b}$, b odd prime.
- **Heron’s formula:** A triangle with side lengths a, b, c has area $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.
- **Pick’s theorem:** A polygon on an integer grid strictly containing i lattice points and having b lattice points on the boundary has area $i + \frac{b}{2} - 1$. (Nothing similar in higher dimensions)
- **Euler’s totient:** The number of integers less than n that are coprime to n are $n \prod_{p|n} \left(1 - \frac{1}{p}\right)$ where each p is a distinct prime factor of n .
- **König’s theorem:** In any bipartite graph $G = (L \cup R, E)$, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover. Let U be the set of unmatched vertices in L , and Z be the set of vertices that are either in U or are connected to U by an alternating path. Then $K = (L \setminus Z) \cup (R \cap Z)$ is the minimum vertex cover.
- A minumum Steiner tree for n vertices requires at most $n - 2$ additional Steiner vertices.
- The number of vertices of a graph is equal to its minimum vertex cover number plus the size of a maximum independent set.
- **Lagrange polynomial** through points $(x_0, y_0), \dots, (x_k, y_k)$ is $L(x) = \sum_{j=0}^k y_j \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$
- **Hook length formula:** If λ is a Young diagram and $h_\lambda(i, j)$ is the hook-length of cell (i, j) , then then the number of Young tableaux $d_\lambda = n! / \prod h_\lambda(i, j)$.
- **Möbius inversion formula:** If $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(d) f(n/d)$. If $f(n) = \sum_{m=1}^n g(\lfloor n/m \rfloor)$, then $g(n) = \sum_{m=1}^n \mu(m) f(\lfloor \frac{n}{m} \rfloor)$.
- #primitive pythagorean triples with hypotenuse $< n$ approx $n/(2\pi)$.
- **Frobenius Number:** largest number which can’t be expressed as a linear combination of numbers a_1, \dots, a_n with non-negative coefficients. $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$, $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d - 1)$. An integer $x > (\max_i a_i)^2$ can be expressed in such a way iff. $x \mid \gcd(a_1, \dots, a_n)$.

11.1. Physics.

- **Snell’s law:** $\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$

11.2. **Markov Chains.** A Markov Chain can be represented as a weighted directed graph of states, where the weight of an edge represents the probability of transitioning over that edge in one timestep. Let $P^{(m)} = (p_{ij}^{(m)})$ be the probability matrix of transitioning from state i to state j in m timesteps, and note that $P^{(1)}$ is the adjacency matrix of the graph. **Chapman-Kolmogorov:** $p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$. It follows that $P^{(m+n)} = P^{(m)} P^{(n)}$ and $P^{(m)} = P^m$. If $p^{(0)}$ is the initial probability distribution (a vector), then $p^{(0)} P^{(m)}$ is the probability distribution after m timesteps.

The return times of a state i is $R_i = \{m \mid p_{ii}^{(m)} > 0\}$, and i is *aperiodic* if $\gcd(R_i) = 1$. A MC is aperiodic if any of its vertices is aperiodic. A MC is *irreducible* if the corresponding graph is strongly connected. A distribution π is stationary if $\pi P = \pi$. If MC is irreducible then $\pi_i = 1/\mathbb{E}[T_i]$, where T_i is the expected time between two visits at i . π_j/π_i is the expected number of visits at j in between two consecutive visits at i . A MC is *ergodic* if $\lim_{m \rightarrow \infty} p^{(0)} P^m = \pi$. A MC is ergodic iff. it is irreducible and aperiodic.

A MC for a random walk in an undirected weighted graph (un-weighted graph can be made weighted by adding 1-weights) has $p_{uv} = w_{uv} / \sum_x w_{ux}$. If the graph is connected, then $\pi_u = \sum_x w_{ux} / \sum_v \sum_x w_{vx}$. Such a random walk is aperiodic iff. the graph is not bipartite.

An *absorbing* MC is of the form $P = \begin{pmatrix} Q & R \\ 0 & I_r \end{pmatrix}$. Let $N = \sum_{m=0}^\infty Q^m = (I_t - Q)^{-1}$. Then, if starting in state i , the expected number of steps till absorption is the i -th entry in $N1$. If starting in state i , the probability of being absorbed in state j is the (i, j) -th entry of NR .

Many problems on MC can be formulated in terms of a system of recurrence relations, and then solved using Gaussian elimination.

11.3. **Burnside’s Lemma.** Let G be a finite group that acts on a set X . For each g in G let X^g denote the set of elements in X that are fixed by g . Then the number of orbits

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

$$Z(S_n) = \frac{1}{n} \sum_{l=1}^n a_l Z(S_{n-l})$$

11.4. **Bézout’s identity.** If (x, y) is any solution to $ax + by = d$ (e.g. found by the Extended Euclidean Algorithm), then all solutions are given by

$$\left(x + k \frac{b}{\gcd(a, b)}, y - k \frac{a}{\gcd(a, b)}\right)$$

11.5. Misc.

11.5.1. *Binomial transform.* If $a_n = \sum_{k=0}^n \binom{n}{k} b_k$, then $b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$.

11.5.2. *Generating functions.* Ordinary (o.g.f.): $A(x) := \sum_{n=0}^\infty a_n x^n$. Exponential (e.g.f.): $A(x) := \sum_{n=0}^\infty a_n x^n / n!$. o.g.f. convolution: $c_n = \sum_{k=0}^n a_k b_{n-k}$ (use FFT for $\mathcal{O}(n \log n)$). e.g.f. convolution: $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$ (use FFT for $\mathcal{O}(n \log n)$).

11.5.3. *General linear recurrences.* If $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$, then $A(x) = \frac{a_0}{1-B(x)}$. We also can compute all a_n with Divide-and-Conquer algorithm in $\mathcal{O}(n \log^2 nn)$.

11.5.4. *Inverse polynomial modulo x^l .* Given $A(x)$, find $B(x)$ such that $A(x)B(x) = 1 + x^l Q(x)$ for some $Q(x)$. Step 1: Start with $B_0(x) = 1/a_0$ Step 2: $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \pmod{x^{2^{k+1}}}$.

11.5.5. *Fast subset convolution.* Given array a_i of size 2^k calculate $b_i = \sum_{j \& i = i} a_j$.

```
for (int b = 1; b < (1 << k); b <= 1)
    for (int i = 0; i < (1 << k); i++)
        if (!(i & b)) a[i | b] += a[i];
// inv: if (!(i & b)) a[i | b] -= a[i];
```

11.5.6. *Determinants and PM.*

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}$$

$$\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i, \sigma(i)}$$

$$\begin{aligned} pf(A) &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)} \\ &= \sum_{M \in \text{PM}(n)} \text{sgn}(M) \prod_{(i,j) \in M} a_{i,j} \end{aligned}$$

11.5.7. *BEST Theorem.* Count directed Eulerian cycles. Number of OST given by Kirchoff’s Theorem (remove r/c with root) $\# \text{OST}(G, r) \cdot \prod_v (d_v - 1)!$

11.5.8. *Primitive Roots.* Only exists when n is $2, 4, p^k, 2p^k$, where p odd prime. Assume n prime. Number of primitive roots $\phi(\phi(n))$ Let g be primitive root. All primitive roots are of the form g^k where $k, \phi(p)$ are coprime. k -roots: $g^{i \cdot \phi(n)/k}$ for $0 \leq i < k$

11.5.9. *Sum of primes.* For any multiplicative f : $S(n, p) = S(n, p - 1) - f(p) \cdot (S(n/p, p - 1) - S(p - 1, p - 1))$

11.5.10. *Floor.*

$$\begin{aligned} \lfloor \lfloor x/y \rfloor / z \rfloor &= \lfloor x/(yz) \rfloor \\ x \% y &= x - y \lfloor x/y \rfloor \end{aligned}$$

PRACTICE CONTEST CHECKLIST

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use printf/scanf with long long/long double?
- Are `__int128` and `__float128` available?
- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: `i?python[23]`, `factor`.
- Try submitting with `assert(false)` and `assert(true)`.
- Return-value from `main`.
- Look for directory with sample test cases.
- Make sure printing works.
- Remove this page from the notebook (optionally).