

# TCR

git diff solution (Jens Heuseveldt, Ludo Pulles, Pim Spelier)

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	Practice Contest Checklist

At the start of a contest, create the following files in the home-dir:  
.vimrc:

```

18 set nu sw=4 ts=4 sts=4 noet ai hls shcf=-ic
2 sy on | colo slate
13 .bashrc:
13 alias gsubmit='g++ -Wall -Wshadow -std=c++14'
13 alias gll='gsubmit -DLOCAL -g'
13 gsettings set
13 → org.compiz.core:/org/compiz/profiles/unity/plugins/core/
13 → vsize 3
14 gsettings set
14 → org.compiz.core:/org/compiz/profiles/unity/plugins/core/
14 → hsize 3
14 Test script (usage: ./test.sh A/B/..)
14 g++ $1.cpp
14 for i in $1/*.in
15 do
15     j="${i/.in/.ans}"
16     ./a.out < $i > output
16     diff output $j || echo "WA on $i"
16 done
16                                     template.cpp
17 #include<bits/stdc++.h>
17 using namespace std;
17 using namespace __gnu_pbds;
17
17 // BBST + order statistics (if supported by judge)
17 // iterator find_by_order(int r) (zero based)
17 // int order_of_key(TK v)
18 template<class TK, class TM> using order_tree = tree<TK, TM,
18 → less<TK>, rb_tree_tag,
18 → tree_order_statistics_node_update>;
18 template<class TV> using order_set = order_tree<TV,
18 → null_type>;
19
19 typedef long long ll;
20 typedef long double ld;
21 typedef pair<int, int> ii;
22 typedef vector<int> vi;
22 typedef vector<vi> vvi;
22 typedef vector<ii> vii;
23
23 #define x first
23 #define y second
23 #define pb push_back
24 #define eb emplace_back
24 #define rep(i,a,b) for(auto i=(a);i!=(b); ++i)
24 #define REP(i,n) rep(i,0,n)
24 #define all(v) (v).begin(), (v).end()
24 #define rs resize
24 #define DBG(x) cerr << __LINE__ << " : " << #x << " = " << (x)
25 → << endl
25
25 template<class T> using min_queue = priority_queue<T,
25 → vector<T>, greater<T>>;
25 template<class T> int size(const T &x) { return x.size(); }
25 → // copy the ampersand(&)!

```

```
const int INF = 2147483647;
const ll LLINF = ~(1LL<<63); // = 9.223.372.036.854.775.807
const ld PI = acos(-1.0);
```

```
void run() {
}
```

```
signed main() {
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    (cout << fixed).precision(18);
    run();
    return 0;
}
```

### 0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Ludo moet **ALLE** opgaves **goed** lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zware) test cases.
- Gebruik ll indien wellicht nodig.

### 0.2. Wrong Answer.

- (1) Print de oplossing om te debuggen!
- (2) Kijk naar wellicht makkelijkere problemen.
- (3) Bedenk zelf test cases met **randgevallen**!
- (4) Controleer de **precisie**.
- (5) Controleer op **overflow** (gebruik **OVERAL** ll, ld).  
*Kijk naar overflows in tussenantwoorden bij modulo.*
- (6) Controleer op **typo's**.
- (7) Loop de voorbeeld test case accuraat langs.
- (8) Controleer op off-by-one-errors (in indices of lus-grenzen)?

**Detecting overflow** This GNU builtin checks for over- and under-flow. Result is in res if successful:

```
bool isOverflown = __builtin_[add|mul|sub]_overflow(a, b,
↪ &res);
```

### 0.3. Covering problems.

*Minimum edge cover  $\iff$  Maximum independent set*

**Matching:** A set of edges without common vertices (*Maximum is the largest such set, maximal is a set which you cannot add more edges to without breaking the property*).

**Minimum Vertex Cover:** A set vertices (cover) such that each edge in the graph is incident to at least one vertex of the set.

**Minimum Edge Cover:** A set of edges (cover) such that every vertex is incident to at least one edge of the set.

**Maximum Independent Set:** A set of vertices in a graph such that no two of them are adjacent.

**König's theorem:** In any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover

**0.4. Game theory.** A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

**Nim:** Let  $X = \bigoplus_{i=1}^n x_i$ , then  $(x_i)_{i=1}^n$  is a winning position iff  $X \neq 0$ . Find a move by picking  $k$  such that  $x_k > x_k \oplus X$ .

**Misère Nim:** Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles.

**Staircase Nim:** Stones are moved down a staircase and only removed from the last pile.  $(x_i)_{i=1}^n$  is an  $L$ -position if  $(x_{2i-1})_{i=1}^{n/2}$  is (i.e. only look at odd-numbered piles).

**Moore's Nim<sub>k</sub>:** The player may remove from at most  $k$  piles (Nim = Nim<sub>1</sub>). Expand the piles in base 2, do a carry-less addition in base  $k+1$  (i.e. the number of ones in each column should be divisible by  $k+1$ ).

**Dim<sup>+</sup>:** The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is  $k+1$  where  $2^k$  is the largest power of 2 dividing the pile size.

**Aliquot game:** Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just  $k$ .

**Nim (at most half):** Write  $n+1 = 2^m y$  with  $m$  maximal, then the Sprague-Grundy function of  $n$  is  $(y-1)/2$ .

**Lasker's Nim:** Players may alternatively split a pile into two new non-empty piles.  $g(4k+1) = 4k+1$ ,  $g(4k+2) = 4k+2$ ,  $g(4k+3) = 4k+4$ ,  $g(4k+4) = 4k+3$  ( $k \geq 0$ ).

**Hackenbush on trees:** A tree with stalks  $(x_i)_{i=1}^n$  may be replaced with a single stalk with length  $\bigoplus_{i=1}^n x_i$ .

**A useful identity:**  $\bigoplus_{x=0}^{a-1} x = \{0, a-1, 1, a\}[a \bmod 4]$ .

#### 1. MATH

```
int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }
```

```
// greatest common divisor
ll gcd(ll a, ll b) { while (b) a %= b, swap(a, b); return a;
↪ };
// least common multiple
ll lcm(ll a, ll b) { return a / gcd(a, b) * b; }
ll mod(ll a, ll b) { return (a %= b) < 0 ? a + b : a; }
```

```
// safe multiplication (ab % m) for m <= 4e18 in O(log b)
ll mod_mul(ll a, ll b, ll m) {
    ll r = 0;
    while (b) {
        if (b & 1) r = (r + a) % m; a = (a + a) % m; b >>= 1;
    }
    return r;
}
```

```
// safe exponentiation (a^b % m) for m <= 2e9 in O(log b)
ll mod_pow(ll a, ll b, ll m) {
    ll r = 1;
    while (b) {
        if (b & 1) r = (r * a) % m; // r = mod_mul(r, a, m);
        a = (a * a) % m; // a = mod_mul(a, a, m);
        b >>= 1;
    }
}
```

```
return r;
}
```

```
// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
    ll xx = y = 0, yy = x = 1;
    while (b) {
        x -= a / b * xx; swap(x, xx);
        y -= a / b * yy; swap(y, yy);
        a %= b; swap(a, b);
    }
    return a;
}
```

```
// Chinese remainder theorem: returns (u, v) s.t.: x = u (mod
↪ v) <=> x = a (mod n) and x = b (mod m) for n,m <= 1e9
pair<ll, ll> crt(ll a, ll n, ll b, ll m) {
    ll s, t, d = egcd(n, m, s, t);
    if (mod(a - b, d)) return { 0, -1 };
    return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}
```

```
// phi[i] = #{ 0 < j <= i | gcd(i, j) = 1 }
vi totient(int N) {
    vi phi(N);
    for (int i = 0; i < N; i++) phi[i] = i;
    for (int i = 2; i < N; i++)
        if (phi[i] == i)
            for (int j = i; j < N; j += i) phi[j] -= phi[j] / i;
    return phi;
}
```

```
// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
    ll ans = 1;
    while (n) {
        ll np = n % p, kp = k % p;
        if (np < kp) return 0;
        ans = mod(ans * binom(np, kp), p); // (np C kp)
        n /= p; k /= p;
    }
    return ans;
}
```

```
// returns if n is prime for n < 3e24 ( > 2^64)
// but use mul_mod for n > 2e9!!!
bool millerRabin(ll n){
    if (n < 2 || n % 2 == 0) return n == 2;
    ll d = n - 1, ad, s = 0, r;
    for (; d % 2 == 0; d /= 2) s++;
    for (int a : { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
↪ 41 }) {
        if (n == a) return true;
        if ((ad = mod_pow(a, d, n)) == 1) continue;
        for (r = 0; r < s && ad + 1 != n; r++)
            ad = (ad * ad) % n;
    }
}
```

```

    if (r == s) return false;
}
return true;
}

```

### 1.1. Primitive Root.

```

ll primitive_root(ll m) {
    vector<ll> div;
    for (ll i = 1; i*i < m; i++) {
        if ((m-1) % i == 0) {
            if (i < m) div.pb(i);
            if (m/i < m) div.pb(m/i); } }
    rep(x,2,m) {
        bool ok = true;
        for (ll d : div)
            if (mod_pow(x, d, m) == 1) {
                ok = false; break; }
        if (ok) return x; }
    return -1; }

```

**1.2. Tonelli-Shanks algorithm.** Given prime  $p$  and integer  $1 \leq n < p$ , returns the square root  $r$  of  $n$  modulo  $p$ . There is also another solution given by  $-r$  modulo  $p$ .

```

ll legendre(ll a, ll p) {
    if (a % p == 0) return 0;
    if (p == 2) return 1;
    return mod_pow(a, (p-1)/2, p) == 1 ? 1 : -1; }
ll tonelli_shanks(ll n, ll p) {
    assert(legendre(n,p) == 1);
    if (p == 2) return 1;
    ll s = 0, q = p-1, z = 2;
    while (~q & 1) s++, q >>= 1;
    if (s == 1) return mod_pow(n, (p+1)/4, p);
    while (legendre(z,p) != -1) z++;
    ll c = mod_pow(z, q, p),
        r = mod_pow(n, (q+1)/2, p),
        t = mod_pow(n, q, p),
        m = s;
    while (t != 1) {
        ll i = 1, ts = (ll)t*t % p;
        while (ts != 1) i++, ts = ((ll)ts * ts) % p;
        ll b = mod_pow(c, 1LL<<(m-i-1), p);
        r = (ll)r * b % p;
        t = (ll)t * b % p * b % p;
        c = (ll)b * b % p;
        m = i; }
    return r; }

```

**1.3. Numeric Integration.** Numeric integration using Simpson's rule.

```

ld numint(ld (*)(ld), ld a, ld b, ld EPS = 1e-6) {
    ld ba = b - a, m=(a+b)/2;
    return abs(ba) < EPS ?
        ↪ ba/8*(f(a)+f(b)+f(a+ba/3)*3+f(b-ba/3)*3)
        : numint(f,a,m,EPS) + numint(f,m,b,EPS);
}

```

**1.4. Fast Hadamard Transform.** Computes the Hadamard transform of the given array. Can be used to compute the XOR-convolution of arrays, exactly like with FFT. For AND-convolution, use  $(x+y, y)$  and  $(x-y, y)$ . For OR-convolution, use  $(x, x+y)$  and  $(x, -x+y)$ . **Note:** Size of array must be a power of 2.

```

void fht(vi &arr, bool inv=false, int l, int r) {
    if (l+1 == r) return;
    int k = (r-l)/2;
    if (!inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k, r);
    rep(i,l,l+k) { int x = arr[i], y = arr[i+k];
        if (!inv) arr[i] = x-y, arr[i+k] = x+y;
        else arr[i] = (x+y)/2, arr[i+k] = (-x+y)/2; }
    if (inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k, r); }

```

**1.5. Tridiagonal Matrix Algorithm.** Solves a tridiagonal system of linear equations  $a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$  where  $a_1 = c_n = 0$ . Beware of numerical instability.

```

#define MAXN 5000
ld A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
    C[0] /= B[0]; D[0] /= B[0];
    rep(i,1,n-1) C[i] /= B[i] - A[i]*C[i-1];
    rep(i,1,n) D[i] = (D[i] - A[i]*D[i-1]) / (B[i] -
        ↪ A[i]*C[i-1]);
    X[n-1] = D[n-1];
    for (int i = n-1; i--;)
        X[i] = D[i] - C[i] * X[i+1]; }

```

**1.6. Mertens Function.** Mertens function is  $M(n) = \sum_{i=1}^n \mu(i)$ . Let  $L \approx (n \log \log n)^{2/3}$  and the algorithm runs in  $O(n^{2/3})$ .

```

#define L 9000000
int mob[L], mer[L];
unordered_map<ll,ll> mem;
ll M(ll n) {
    if (n < L) return mer[n];
    if (mem.find(n) != mem.end()) return mem[n];
    ll ans = 0, done = 1;
    for (ll i = 2; i*i <= n; i++) ans += M(n/i), done = i;
    for (ll i = 1; i*i <= n; i++)
        ans += mer[i] * (n/i - max(done, n/(i+1)));
    return mem[n] = 1 - ans; }
void sieve() {
    for (int i = 1; i < L; i++) mer[i] = mob[i] = 1;
    for (int i = 2; i < L; i++) {
        if (mer[i]) {
            mob[i] = -1;
            for (int j = i+i; j < L; j += i)
                mer[j] = 0, mob[j] = (j/i)%i == 0 ? 0 : -mob[j/i]; }
        mer[i] = mob[i] + mer[i-1]; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

**1.7. Summatory Phi.** The summatory phi function  $\Phi(n) = \sum_{i=1}^n \phi(i)$ . Let  $L \approx (n \log \log n)^{2/3}$  and the algorithm runs in  $O(n^{2/3})$ .

```

#define N 10000000
ll sp[N];
unordered_map<ll,ll> mem;

```

```

ll sumphi(ll n) {
    if (n < N) return sp[n];
    if (mem.find(n) != mem.end()) return mem[n];
    ll ans = 0, done = 1;
    for (ll i = 2; i*i <= n; i++) ans += sumphi(n/i), done = i;
    for (ll i = 1; i*i <= n; i++)
        ans += sp[i] * (n/i - max(done, n/(i+1)));
    return mem[n] = n*(n+1)/2 - ans; }
void sieve() {
    for (int i = 1; i < N; i++) sp[i] = i;
    for (int i = 2; i < N; i++) {
        if (sp[i] == i) {
            sp[i] = i-1;
            for (int j = i+i; j < N; j += i) sp[j] -= sp[j] / i; }
        sp[i] += sp[i-1]; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

**1.8. Josephus problem.** Last man standing out of  $n$  if every  $k$ th is killed. Zero-based, and does not kill 0 on first pass.

```

int J(int n, int k) {
    if (n == 1) return 0;
    if (k == 1) return n-1;
    if (n < k) return (J(n-1,k)+k)%n;
    int np = n - n/k;
    return k*((J(np,k)+np-n*k*np)%np) / (k-1); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

**1.9. Number of Integer Points under Line.** Count the number of integer solutions to  $Ax + By \leq C$ ,  $0 \leq x \leq n$ ,  $0 \leq y$ . In other words, evaluate the sum  $\sum_{x=0}^n \left\lfloor \frac{C-Ax}{B} + 1 \right\rfloor$ . To count all solutions, let  $n = \lfloor \frac{C}{A} \rfloor$ . In any case, it must hold that  $C - nA \geq 0$ . Be very careful about overflows.

```

ll floor_sum(ll n, ll a, ll b, ll c) {
    if (c == 0) return 1;
    if (c < 0) return 0;
    if (a % b == 0) return (n+1)*(c/b+1)-n*(n+1)/2*a/b;
    if (a >= b) return floor_sum(n,a%b,b,c)-a/b*n*(n+1)/2;
    ll t = (c-a*n+b)/b;
    return floor_sum((c-b*t)/b,b,a,c-b*t)+t*(n+1); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

**1.10. Numbers and Sequences.** Some random prime numbers: 1031, 32771, 1048583, 8125344, 33554467, 9982451653 1073741827, 34359738421, 1099511627791, 35184372088891, 1125899906842679, 36028797018963971.

More random prime numbers:  $10^3 + \{-9, -3, 9, 13\}$ ,  $10^6 + \{-17, 3, 33\}$ ,  $10^9 + \{7, 9, 21, 33, 87\}$ .

	840	32
	720 720	240
Some maximal divisor counts:	735 134 400	1 344
	963 761 198 400	6 720
	866 421 317 361 600	26 880
	897 612 484 786 617 600	103 680

## 2. DATASTRUCTURES

2.1. Segment tree  $\mathcal{O}(\log n)$ . Standard segment tree

```
typedef /* Tree element */ S;
const int n = 1 << 20; S t[2 * n];

// required axiom: associativity
S combine(S l, S r) { return l + r; } // sum segment tree
S combine(S l, S r) { return max(l, r); } // max segment tree

void build() { for (int i = n; --i; ) t[i] = combine(t[2 * i], t[2 * i + 1]); }

// set value v on position i
void update(int i, S v) { for (t[i += n] = v; i /= 2; ) t[i] = combine(t[2 * i], t[2 * i + 1]); }

// sum on interval [l, r)
S query(int l, int r) {
    S resL, resR;
    for (l += n, r += n; l < r; l /= 2, r /= 2) {
        if (l & 1) resL = combine(resL, t[l++]);
        if (r & 1) resR = combine(t[--r], resR);
    }
    return combine(resL, resR);
}
```

## Lazy segment tree

```
struct node {
    int l, r, x, lazy;
    node() {}
    node(int _l, int _r) : l(_l), r(_r), x(INF), lazy(0) {}
    node(int _l, int _r, int _x) : node(_l, _r) { x = _x; }
    node(node a, node b) : node(a.l, b.r) { x = min(a.x, b.x); }
    void update(int v) { x = v; }
    void range_update(int v) { lazy = v; }
    void apply() { x += lazy; lazy = 0; }
    void push(node &u) { u.lazy += lazy; };
}
```

```
struct segment_tree {
    int n;
    vector<node> arr;
    segment_tree() {}
    segment_tree(const vector<ll> &a) : n(size(a)), arr(4*n) {
        mk(a, 0, 0, n-1);
    }
    node mk(const vector<ll> &a, int i, int l, int r) {
        int m = (l+r)/2;
        return arr[i] = l > r ? node(l, r) :
            l == r ? node(l, r, a[l]) :
            node(mk(a, 2*i+1, l, m), mk(a, 2*i+2, m+1, r));
    }
    node update(int at, ll v, int i=0) {
        propagate(i);
        int hl = arr[i].l, hr = arr[i].r;
        if (at < hl || hr < at) return arr[i];
        if (hl == at && at == hr) {
            arr[i].update(v); return arr[i];
        }
    }
}
```

```
return arr[i] =
    node(update(at, v, 2*i+1), update(at, v, 2*i+2)); }
node query(int l, int r, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return node(hl, hr);
    if (l <= hl && hr <= r) return arr[i];
    return node(query(l, r, 2*i+1), query(l, r, 2*i+2)); }
node range_update(int l, int r, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return arr[i];
    if (l <= hl && hr <= r)
        return arr[i].range_update(v), propagate(i), arr[i];
    return arr[i] = node(range_update(l, r, v, 2*i+1),
        range_update(l, r, v, 2*i+2)); }
void propagate(int i) {
    if (arr[i].l < arr[i].r)
        arr[i].push(arr[2*i+1]), arr[i].push(arr[2*i+2]);
    arr[i].apply(); } }
```

## Persistent segment tree

```
int segcnt = 0;
struct segment {
    int l, r, lid, rid, sum;
} segs[2000000];
int build(int l, int r) {
    if (l > r) return -1;
    int id = segcnt++;
    segs[id].l = l;
    segs[id].r = r;
    if (l == r) segs[id].lid = -1, segs[id].rid = -1;
    else {
        int m = (l + r) / 2;
        segs[id].lid = build(l, m);
        segs[id].rid = build(m + 1, r);
    }
    segs[id].sum = 0;
    return id;
}
int update(int idx, int v, int id) {
    if (id == -1) return -1;
    if (idx < segs[id].l || idx > segs[id].r) return id;
    int nid = segcnt++;
    segs[nid].l = segs[id].l;
    segs[nid].r = segs[id].r;
    segs[nid].lid = update(idx, v, segs[id].lid);
    segs[nid].rid = update(idx, v, segs[id].rid);
    segs[nid].sum = segs[id].sum + v;
    return nid;
}
int query(int id, int l, int r) {
    if (r < segs[id].l || segs[id].r < l) return 0;
    if (l <= segs[id].l && segs[id].r <= r) return
        segs[id].sum;
    return query(segs[id].lid, l, r)
        + query(segs[id].rid, l, r);
}
```

2.2. Binary Indexed Tree  $\mathcal{O}(\log n)$ . Use one-based indices ( $i > 0$ )!

```
int bit[MAXN + 1];

// arr[i] += v
void update(int i, int v) {
    while (i <= MAXN) bit[i] += v, i += i & -i;
}

// returns sum of arr[i], where i: [1, i]
int query(int i) {
    int v = 0; while (i) v += bit[i], i -= i & -i; return v;
}
```

Use this if you add things, which depend on  $i$ :

```
struct fenwick_tree {
    int n; vi data;
    fenwick_tree(int _n) : n(_n), data(vi(n)) {}
    void update(int at, int by) {
        while (at < n) data[at] += by, at |= at + 1;
    }
    int query(int at) {
        int res = 0;
        while (at >= 0) res += data[at], at = (at & (at + 1)) - 1;
        return res;
    }
    int rsq(int a, int b) { return query(b) - query(a - 1); }
};

struct fenwick_tree_sq {
    int n; fenwick_tree x1, x0;
    fenwick_tree_sq(int _n) : n(_n), x1(fenwick_tree(n)),
        x0(fenwick_tree(n)) {}
    // insert f(y) = my + c if x <= y
    void update(int x, int m, int c) {
        x1.update(x, m); x0.update(x, c);
    }
    int query(int x) { return x*x1.query(x) + x0.query(x); }
};

void range_update(fenwick_tree_sq &s, int a, int b, int k) {
    s.update(a, k, k * (1 - a)); s.update(b+1, -k, k * b);
}
int range_query(fenwick_tree_sq &s, int a, int b) {
    return s.query(b) - s.query(a-1);
}
```

2.3. Disjoint-Set / Union-Find  $\mathcal{O}(\alpha(n))$ .

```
struct dsu {
    vi par, rnk;
    dsu(int n) : par(n, -1), rnk(n, 0) {}
    int find(int i) { return par[i] < 0 ? i : par[i] =
        find(par[i]); }
    void unite(int a, int b) {
        if ((a = find(a)) == (b = find(b))) return;
        if (rnk[a] < rnk[b]) swap(a, b);
        if (rnk[a] == rnk[b]) rnk[a]++;
        par[a] += par[b]; par[b] = a;
    }
};
```

2.4. AVL Tree Balanced Binary Search Tree  $\mathcal{O}(\log n)/\mathcal{O}(\log n)$ .

```
#define AVL_MULTISET 0
template <class T> struct avl_tree {
```



```

struct node {
    T item; node *p, *l, *r;
    int size, height;
    node(const T &item, node *_p = NULL) : item(_item),
        ↪ p(_p),
        l(NULL), r(NULL), size(1), height(0) { } };

node *root;
avl_tree() : root(NULL) { }
inline int sz(node *n) const { return n ? n->size : 0; }
inline int height(node *n) const {
    return n ? n->height : -1; }
inline bool left_heavy(node *n) const {
    return n && height(n->l) > height(n->r); }
inline bool right_heavy(node *n) const {
    return n && height(n->r) > height(n->l); }
inline bool too_heavy(node *n) const {
    return n && abs(height(n->l) - height(n->r)) > 1; }

void delete_tree(node *n) { if (n) {
    delete_tree(n->l), delete_tree(n->r); delete n; } }
node*& parent_leg(node *n) {
    if (!n->p) return root;
    if (n->p->l == n) return n->p->l;
    if (n->p->r == n) return n->p->r;
    assert(false); }
void augment(node *n) {
    if (!n) return;
    n->size = 1 + sz(n->l) + sz(n->r);
    n->height = 1 + max(height(n->l), height(n->r)); }
#define rotate(l, r) \
    node *l = n->l; \
    l->p = n->p; \
    parent_leg(n) = l; \
    n->l = l->r; \
    if (l->r) l->r->p = n; \
    l->r = n, n->p = l; \
    augment(n), augment(l)
void left_rotate(node *n) { rotate(r, l); }
void right_rotate(node *n) { rotate(l, r); }
void fix(node *n) {
    while (n) { augment(n);
        if (too_heavy(n)) {
            if (left_heavy(n) && right_heavy(n->l))
                left_rotate(n->l);
            else if (right_heavy(n) && left_heavy(n->r))
                right_rotate(n->r);
            if (left_heavy(n)) right_rotate(n);
            else left_rotate(n);
            n = n->p; }
        n = n->p; } }
inline int size() const { return sz(root); }
node* find(const T &item) const {
    node *cur = root;
    while (cur) {
        if (cur->item < item) cur = cur->r;

```

```

        else if (item < cur->item) cur = cur->l;
        else break; }
    return cur; }
node* insert(const T &item) {
    node *prev = NULL, **cur = &root;
    while (*cur) {
        prev = *cur;
        if ((*cur)->item < item) cur = &((*cur)->r);
        ↪ if AVL_MULTISSET
        else cur = &((*cur)->l);
    }
    else
        else if (item < (*cur)->item) cur = &((*cur)->l);
        else return *cur;
    #endif
    }
    node *n = new node(item, prev);
    *cur = n, fix(n); return n; }
void erase(const T &item) { erase(find(item)); }
void erase(node *n, bool free = true) {
    if (!n) return;
    if (!n->l && n->r) parent_leg(n) = n->r, n->r->p = n->p;
    else if (n->l && !n->r)
        parent_leg(n) = n->l, n->l->p = n->p;
    else if (n->l && n->r) {
        node *s = successor(n);
        erase(s, false);
        s->p = n->p, s->l = n->l, s->r = n->r;
        if (n->l) n->l->p = s;
        if (n->r) n->r->p = s;
        parent_leg(n) = s, fix(s);
        return;
    } else parent_leg(n) = NULL;
    fix(n->p), n->p = n->l = n->r = NULL;
    if (free) delete n; }
node* successor(node *n) const {
    if (!n) return NULL;
    if (n->r) return nth(0, n->r);
    node *p = n->p;
    while (p && p->r == n) n = p, p = p->p;
    return p; }
node* predecessor(node *n) const {
    if (!n) return NULL;
    if (n->l) return nth(n->l->size-1, n->l);
    node *p = n->p;
    while (p && p->l == n) n = p, p = p->p;
    return p; }
node* nth(int n, node *cur = NULL) const {
    if (!cur) cur = root;
    while (cur) {
        if (n < sz(cur->l)) cur = cur->l;
        else if (n > sz(cur->l))
            n -= sz(cur->l) + 1, cur = cur->r;
        else break;
    } return cur; }
int count_less(node *cur) {
    int sum = sz(cur->l);

```

```

    while (cur) {
        if (cur->p && cur->p->r == cur) sum += 1 +
            ↪ sz(cur->p->l);
        cur = cur->p;
    } return sum; }
void clear() { delete_tree(root), root = NULL; } };

```

Use this easy implementation for a map:

```

template <class K, class V> struct avl_map {
    struct node {
        K key; V value;
        node(K k, V v) : key(k), value(v) { }
        bool operator <(const node &other) const {
            return key < other.key; } };
    avl_tree<node> tree;
    V& operator [] (K key) {
        typename avl_tree<node>::node *n =
            tree.find(node(key, V(0)));
        if (!n) n = tree.insert(node(key, V(0)));
        return n->item.value; } };

```

## 2.5. Cartesian tree.

```

struct node {
    int x, y, sz;
    node *l, *r;
    node(int _x, int _y)
        : x(_x), y(_y), sz(1), l(NULL), r(NULL) { } };
int tsize(node* t) { return t ? t->sz : 0; }
void augment(node *t) {
    t->sz = 1 + tsize(t->l) + tsize(t->r); }
pair<node*,node*> split(node *t, int x) {
    if (!t) return make_pair((node*)NULL, (node*)NULL);
    if (t->x < x) {
        pair<node*,node*> res = split(t->r, x);
        t->r = res.first; augment(t);
        return make_pair(t, res.second); }
    pair<node*,node*> res = split(t->l, x);
    t->l = res.second; augment(t);
    return make_pair(res.first, t); }
node* merge(node *l, node *r) {
    if (!l) return r; if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r); augment(l); return l; }
    r->l = merge(l, r->l); augment(r); return r; }
node* find(node *t, int x) {
    while (t) {
        if (x < t->x) t = t->l;
        else if (t->x < x) t = t->r;
        else return t; }
    return NULL; }
node* insert(node *t, int x, int y) {
    if (find(t, x) != NULL) return t;
    pair<node*,node*> res = split(t, x);
    return merge(res.first,
        merge(new node(x, y), res.second)); }

```

```
node* erase(node *t, int x) {
    if (!t) return NULL;
    if (t->x < x) t->r = erase(t->r, x);
    else if (x < t->x) t->l = erase(t->l, x);
    else { node *old = t; t = merge(t->l, t->r); delete old; }
    if (t) augment(t); return t; }

int kth(node *t, int k) {
    if (k < tsize(t->l)) return kth(t->l, k);
    else if (k == tsize(t->l)) return t->x;
    else return kth(t->r, k - tsize(t->l) - 1); }
```

2.6. **Heap.** An implementation of a binary heap.

```
#define RESIZE
#define SWP(x,y) tmp = x, x = y, y = tmp
struct default_int_cmp {
    default_int_cmp() {}
    bool operator()(const int &a, const int &b) {
        return a < b; } };
template <class Compare = default_int_cmp> struct heap {
    int len, count, *q, *loc, tmp;
    Compare _cmp;
    inline bool cmp(int i, int j) { return _cmp(q[i], q[j]); }
    inline void swp(int i, int j) {
        SWP(q[i], q[j]), SWP(loc[q[i]], loc[q[j]]); }
    void swim(int i) {
        while (i > 0) {
            int p = (i - 1) / 2;
            if (!cmp(i, p)) break;
            swp(i, p), i = p; } }
    void sink(int i) {
        while (true) {
            int l = 2*i + 1, r = l + 1;
            if (l >= count) break;
            int m = r >= count || cmp(l, r) ? l : r;
            if (!cmp(m, i)) break;
            swp(m, i), i = m; } }
    heap(int init_len = 128)
        : count(0), len(init_len), _cmp(Compare()) {
        q = new int[len], loc = new int[len];
        memset(loc, 255, len << 2); }
    ~heap() { delete[] q; delete[] loc; }
    void push(int n, bool fix = true) {
        if (len == count || n >= len) {
#ifdef RESIZE
            int newlen = 2 * len;
            while (n >= newlen) newlen *= 2;
            int *newq = new int[newlen], *newloc = new int[newlen];
            rep(i, 0, len) newq[i] = q[i], newloc[i] = loc[i];
            memset(newloc + len, 255, (newlen - len) << 2);
            delete[] q, delete[] loc;
            loc = newloc, q = newq, len = newlen;
#else
            assert(false);
#endif
        }
        assert(loc[n] == -1); }
```

```
loc[n] = count, q[count++] = n;
if (fix) swim(count-1); }
void pop(bool fix = true) {
    assert(count > 0);
    loc[q[0]] = -1, q[0] = q[--count], loc[q[0]] = 0;
    if (fix) sink(0);
}
int top() { assert(count > 0); return q[0]; }
void heapify() { for (int i = count - 1; i > 0; i--)
    if (cmp(i, (i - 1) / 2)) swp(i, (i - 1) / 2); }
void update_key(int n) {
    assert(loc[n] != -1), swim(loc[n]), sink(loc[n]); }
bool empty() { return count == 0; }
int size() { return count; }
void clear() { count = 0, memset(loc, 255, len << 2); };
```

2.7. **Dancing Links.** An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```
template <class T>
struct dancing_links {
    struct node {
        T item;
        node *l, *r;
        node(const T &item, node *_l = NULL, node *_r = NULL)
            : item(_item), l(_l), r(_r) {
            if (l) l->r = this;
            if (r) r->l = this; } };
    node *front, *back;
    dancing_links() { front = back = NULL; }
    node *push_back(const T &item) {
        back = new node(item, back, NULL);
        if (!front) front = back;
        return back; }
    node *push_front(const T &item) {
        front = new node(item, NULL, front);
        if (!back) back = front;
        return front; }
    void erase(node *n) {
        if (!n->l) front = n->r; else n->l->r = n->r;
        if (!n->r) back = n->l; else n->r->l = n->l; }
    void restore(node *n) {
        if (!n->l) front = n; else n->l->r = n;
        if (!n->r) back = n; else n->r->l = n; } };
```

2.8. **Misof Tree.** A simple tree data structure for inserting, erasing, and querying the  $n$ th largest element.

```
#define BITS 15
struct misof_tree {
    int cnt[BITS][1<<BITS];
    misof_tree() { memset(cnt, 0, sizeof(cnt)); }
    void insert(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]++, x >>= 1); }
    void erase(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]--, x >>= 1); }
    int nth(int n) {
        int res = 0; }
```

```
for (int i = BITS-1; i >= 0; i--)
    if (cnt[i][res <= 1] <= n) n -= cnt[i][res], res |= 1;
return res; } };
```

2.9. **k-d Tree.** A  $k$ -dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```
#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
    struct pt {
        double coord[K];
        pt() {}
        pt(double c[K]) { rep(i, 0, K) coord[i] = c[i]; }
        double dist(const pt &other) const {
            double sum = 0.0;
            rep(i, 0, K) sum += pow(coord[i] - other.coord[i], 2.0);
            return sqrt(sum); } };
    struct cmp {
        int c;
        cmp(int _c) : c(_c) {}
        bool operator()(const pt &a, const pt &b) {
            for (int i = 0, cc; i <= K; i++) {
                cc = i == 0 ? c : i - 1;
                if (abs(a.coord[cc] - b.coord[cc]) > EPS)
                    return a.coord[cc] < b.coord[cc];
            }
            return false; } };
    struct bb {
        pt from, to;
        bb(pt _from, pt _to) : from(_from), to(_to) {}
        double dist(const pt &p) {
            double sum = 0.0;
            rep(i, 0, K) {
                if (p.coord[i] < from.coord[i])
                    sum += pow(from.coord[i] - p.coord[i], 2.0);
                else if (p.coord[i] > to.coord[i])
                    sum += pow(p.coord[i] - to.coord[i], 2.0);
            }
            return sqrt(sum); }
        bb bound(double l, int c, bool left) {
            pt nf(from.coord), nt(to.coord);
            if (left) nt.coord[c] = min(nt.coord[c], l);
            else nf.coord[c] = max(nf.coord[c], l);
            return bb(nf, nt); } };
    struct node {
        pt p; node *l, *r;
        node(pt _p, node *_l, node *_r)
            : p(_p), l(_l), r(_r) {} };
    node *root;
    // kd_tree() : root(NULL) {}
    kd_tree(vector<pt> pts) {
        root = construct(pts, 0, size(pts) - 1, 0); }
    node* construct(vector<pt> &pts, int from, int to, int c) {
        if (from > to) return NULL;
        int mid = from + (to - from) / 2; }
```

```

    nth_element(pts.begin() + from, pts.begin() + mid,
        pts.begin() + to + 1, cmp(c));
    return new node(pts[mid],
        construct(pts, from, mid - 1, INC(c)),
        construct(pts, mid + 1, to, INC(c))); }
bool contains(const pt &p) { return _con(p, root, 0); }
bool _con(const pt &p, node *n, int c) {
    if (!n) return false;
    if (cmp(c)(p, n->p)) return _con(p, n->l, INC(c));
    if (cmp(c)(n->p, p)) return _con(p, n->r, INC(c));
    return true; }
void insert(const pt &p) { _ins(p, root, 0); }
void _ins(const pt &p, node* &n, int c) {
    if (!n) n = new node(p, NULL, NULL);
    else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
    else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c)); }
void clear() { _clr(root); root = NULL; }
void _clr(node *n) {
    if (n) _clr(n->l), _clr(n->r), delete n; }
pt nearest_neighbour(const pt &p, bool allow_same=true) {
    assert(root);
    double mn = INFINITY, cs[K];
    rep(i,0,K) cs[i] = -INFINITY;
    pt from(cs);
    rep(i,0,K) cs[i] = INFINITY;
    pt to(cs);
    return _nn(p, root, bb(from, to), mn, 0,
        ↪ allow_same).first;
}
pair<pt, bool> _nn(const pt &p, node *n, bb b,
    double &mn, int c, bool same) {
    if (!n || b.dist(p) > mn) return make_pair(pt(), false);
    bool found = same || p.dist(n->p) > EPS,
        l1 = true, l2 = false;
    pt resp = n->p;
    if (found) mn = min(mn, p.dist(resp));
    node *n1 = n->l, *n2 = n->r;
    rep(i,0,2) {
        if (i == 1 || cmp(c)(n->p, p))
            swap(n1, n2), swap(l1, l2);
        pair<pt, bool> res = _nn(p, n1,
            b.bound(n->p.coord[c], c, l1), mn, INC(c), same);
        if (res.second &&
            (!found || p.dist(res.first) < p.dist(resp)))
            resp = res.first, found = true;
    }
    return make_pair(resp, found); } };
```

2.10. **Sqrt Decomposition.** Design principle that supports many operations in amortized  $\sqrt{n}$  per operation.

```

struct segment {
    vi arr;
    segment(vi _arr) : arr(_arr) { } };
vector<segment> T;
int K;
void rebuild() {
```

```

    int cnt = 0;
    rep(i,0,size(T))
        cnt += size(T[i].arr);
    K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
    vi arr(cnt);
    for (int i = 0, at = 0; i < size(T); i++)
        rep(j,0,size(T[i].arr))
            arr[at++] = T[i].arr[j];
    T.clear();
    for (int i = 0; i < cnt; i += K)
        T.push_back(segment(vi(arr.begin()+i,
            arr.begin()+min(i+K, cnt)))); }
int split(int at) {
    int i = 0;
    while (i < size(T) && at >= size(T[i].arr))
        at -= size(T[i].arr), i++;
    if (i >= size(T)) return size(T);
    if (at == 0) return i;
    T.insert(T.begin() + i + 1,
        segment(vi(T[i].arr.begin() + at, T[i].arr.end())));
    T[i] = segment(vi(T[i].arr.begin(), T[i].arr.begin() +
        ↪ at));
    return i + 1; }
void insert(int at, int v) {
    vi arr; arr.push_back(v);
    T.insert(T.begin() + split(at), segment(arr)); }
void erase(int at) {
    int i = split(at); split(at + 1);
    T.erase(T.begin() + i); }
// vim: cc=60 ts=2 sts=2 sw=2:
```

2.11. **Monotonic Queue.** A queue that supports querying for the minimum element. Useful for sliding window algorithms.

```

struct min_stack {
    stack<int> S, M;
    void push(int x) {
        S.push(x);
        M.push(M.empty() ? x : min(M.top(), x)); }
    int top() { return S.top(); }
    int mn() { return M.top(); }
    void pop() { S.pop(); M.pop(); }
    bool empty() { return S.empty(); } } };
struct min_queue {
    min_stack inp, outp;
    void push(int x) { inp.push(x); }
    void fix() {
        if (outp.empty()) while (!inp.empty())
            outp.push(inp.top()), inp.pop(); }
    int top() { fix(); return outp.top(); }
    int mn() {
        if (inp.empty()) return outp.mn();
        if (outp.empty()) return inp.mn();
        return min(inp.mn(), outp.mn()); }
    void pop() { fix(); outp.pop(); }
    bool empty() { return inp.empty() && outp.empty(); } };
```

2.12. **Convex Hull Trick.** If converting to integers, look out for division by 0 and  $\pm\infty$ .

```

struct convex_hull_trick {
    vector<pair<double,double> > h;
    double intersect(int i) {
        return (h[i+1].second-h[i].second) /
            (h[i].first-h[i+1].first); }
    void add(double m, double b) {
        h.push_back(make_pair(m,b));
        while (size(h) >= 3) {
            int n = size(h);
            if (intersect(n-3) < intersect(n-2)) break;
            swap(h[n-2], h[n-1]);
            h.pop_back(); } }
    double get_min(double x) {
        int lo = 0, hi = size(h) - 2, res = -1;
        while (lo <= hi) {
            int mid = lo + (hi - lo) / 2;
            if (intersect(mid) <= x) res = mid, lo = mid + 1;
            else hi = mid - 1; }
        return h[res+1].first * x + h[res+1].second; } };
```

And dynamic variant:

```

const ll is_query = -(1LL<<62);
struct Line {
    ll m, b;
    mutable function<const Line*> succ;
    bool operator<(const Line& rhs) const {
        if (rhs.b != is_query) return m < rhs.m;
        const Line* s = succ();
        if (!s) return 0;
        ll x = rhs.m;
        return b - s->b < (s->m - m) * x; } };
// will maintain upper hull for maximum
struct HullDynamic : public multiset<Line> {
    bool bad(iterator y) {
        auto z = next(y);
        if (y == begin()) {
            if (z == end()) return 0;
            return y->m == z->m && y->b <= z->b; }
        auto x = prev(y);
        if (z == end()) return y->m == x->m && y->b <= x->b;
        return (x->b - y->b)*(z->m - y->m) >=
            (y->b - z->b)*(y->m - x->m); }
    void insert_line(ll m, ll b) {
        auto y = insert({ m, b });
        y->succ = [=] { return next(y) == end() ? 0 : &*next(y);
            ↪ };
        if (bad(y)) { erase(y); return; }
        while (next(y) != end() && bad(next(y))) erase(next(y));
        while (y != begin() && bad(prev(y))) erase(prev(y)); }
    ll eval(ll x) {
        auto l = *lower_bound((Line) { x, is_query });
        return l.m * x + l.b; } };
```

## 2.13. Sparse Table.

```
struct sparse_table { vii m;
    sparse_table(vi arr) {
        m.push_back(arr);
        for (int k = 0; (1<<(+k)) <= size(arr); ) {
            m.push_back(vi(size(arr)-(1<<k)+1));
            rep(i,0,size(arr)-(1<<k)+1)
                m[k][i] = min(m[k-1][i], m[k-1][i+(1<<(k-1))]); } }
    int query(int l, int r) {
        int k = 0; while (1<<(k+1) <= r-l+1) k++;
        return min(m[k][l], m[k][r-(1<<k)+1]); } };
```

## 3. GRAPH ALGORITHMS

## 3.1. Shortest path.

3.1.1. Dijkstra  $\mathcal{O}(|E|\log|V|)$ .

```
int *dist, *dad;
struct cmp {
    bool operator()(int a, int b) {
        return dist[a] != dist[b] ? dist[a] < dist[b] : a < b; }
};
pair<int*, int*> dijkstra(int n, int s, vii *adj) {
    dist = new int[n];
    dad = new int[n];
    rep(i,0,n) dist[i] = INF, dad[i] = -1;
    set<int, cmp> pq;
    dist[s] = 0, pq.insert(s);
    while (!pq.empty()) {
        int cur = *pq.begin(); pq.erase(pq.begin());
        rep(i,0,size(adj[cur])) {
            int nxt = adj[cur][i].first;
            ndist = dist[cur] + adj[cur][i].second;
            if (ndist < dist[nxt]) pq.erase(nxt),
                dist[nxt] = ndist, dad[nxt] = cur, pq.insert(nxt);
        } }
    return pair<int*, int*>(dist, dad); }
```

3.1.2. Floyd-Warshall  $\mathcal{O}(V^3)$ .

```
int n = 100; ll d[MAXN][MAXN];
for (int i = 0; i < n; i++) fill_n(d[i], n, 1e18);
// set direct distances from i to j in d[i][j] (and d[j][i])
for (int i = 0; i < n; i++)
    for (int j = 0; j < n; j++)
        for (int k = 0; k < n; k++)
            d[j][k] = min(d[j][k], d[j][i] + d[i][k]);
```

3.1.3. Bellman Ford  $\mathcal{O}(VE)$ . This is only useful if there are edges with weight  $w_{ij} < 0$  in the graph.

```
vector< pair<pii, ll> > edges; // ((from, to), weight)
vector<ll> dist;
```

// when undirected, add back edges

```
bool bellman_ford(int V, int source) {
    dist.assign(V, 1e18); dist[source] = 0;
```

```
    bool updated = true; int loops = 0;
```

```
while (updated && loops < n) {
    updated = false;
    for (auto e : edges) {
        int alt = dist[e.x.x] + e.y;
        if (alt < dist[e.x.y]) {
            dist[e.x.y] = alt; updated = true;
        }
    }
    return loops < n; // loops >= n: negative cycles
}
```

## 3.1.4. IDA\* algorithm.

```
int n, cur[100], pos;
int calch() {
    int h = 0;
    rep(i,0,n) if (cur[i] != 0) h += abs(i - cur[i]);
    return h; }
int dfs(int d, int g, int prev) {
    int h = calch();
    if (g + h > d) return g + h;
    if (h == 0) return 0;
    int mn = INF;
    rep(di,-2,3) {
        if (di == 0) continue;
        int nxt = pos + di;
        if (nxt == prev) continue;
        if (0 <= nxt && nxt < n) {
            swap(cur[pos], cur[nxt]);
            swap(pos,nxt);
            mn = min(mn, dfs(d, g+1, nxt));
            swap(pos,nxt);
            swap(cur[pos], cur[nxt]); }
        if (mn == 0) break; }
    return mn; }
int idastar() {
    rep(i,0,n) if (cur[i] == 0) pos = i;
    int d = calch();
    while (true) {
        int nd = dfs(d, 0, -1);
        if (nd == 0 || nd == INF) return d;
        d = nd; } }
```

3.2. Maximum matching  $\mathcal{O}(nm)$ .

```
const int sizeL = 1e4, sizeR = 1e4;
```

```
bool vis[sizeR];
int par[sizeR]; // par : R -> L
vi adj[sizeL]; // adj : L -> (N -> R)
```

```
bool match(int u) {
    for (int v : adj[u]) {
        if (vis[v]) continue; vis[v] = true;
        if (par[v] == -1 || match(par[v])) {
            par[v] = u;
            return true;
        }
    }
}
```

```
}
return false;
}

// perfect matching iff ret == sizeL == sizeR
int maxmatch() {
    fill_n(par, sizeR, -1); int ret = 0;
    for (int i = 0; i < sizeL; i++) {
        fill_n(vis, sizeR, false);
        ret += match(i);
    }
    return ret;
}
```

3.3. Hopcroft-Karp bipartite matching  $\mathcal{O}(E\sqrt{V})$ .

```
#define MAXN 5000
int dist[MAXN+1], q[MAXN+1];
#define dist(v) dist[v == -1 ? MAXN : v]
struct bipartite_graph {
    int N, M, *L, *R; vi *adj;
    bipartite_graph(int _N, int _M) : N(_N), M(_M),
        L(new int[N]), R(new int[M]), adj(new vi[N]) {}
    ~bipartite_graph() { delete[] adj; delete[] L; delete[] R;
        ~ }
    bool bfs() {
        int l = 0, r = 0;
        rep(v,0,N) if (L[v] == -1) dist(v) = 0, q[r++] = v;
        else dist(v) = INF;
        dist(-1) = INF;
        while (l < r) {
            int v = q[l++];
            if (dist(v) < dist(-1)) {
                iter(u, adj[v]) if (dist(R[*u]) == INF)
                    dist(R[*u]) = dist(v) + 1, q[r++] = R[*u]; } }
        return dist(-1) != INF; }
    bool dfs(int v) {
        if (v != -1) {
            iter(u, adj[v])
                if (dist(R[*u]) == dist(v) + 1)
                    if (dfs(R[*u])) {
                        R[*u] = v, L[v] = *u;
                        return true; }
            dist(v) = INF;
            return false; }
        return true; }
    void add_edge(int i, int j) { adj[i].push_back(j); }
    int maximum_matching() {
        int matching = 0;
        memset(L, -1, sizeof(int) * N);
        memset(R, -1, sizeof(int) * M);
        while (bfs()) rep(i,0,N)
            matching += L[i] == -1 && dfs(i);
        return matching; } };
// vim: cc=60 ts=2 sts=2 sw=2:
```



## 3.3.1. Minimum Vertex Cover in Bipartite Graphs.

#include "hopcroft\_karp.cpp"

vector&lt;bool&gt; alt;

```
void dfs(bipartite_graph &g, int at) {
    alt[at] = true;
    iter(it, g.adj[at]) {
        alt[*it + g.N] = true;
        if (g.R[*it] != -1 && !alt[g.R[*it]]) dfs(g, g.R[*it]); }
    ↪ }
```

```
vi mvc_bipartite(bipartite_graph &g) {
    vi res; g.maximum_matching();
    alt.assign(g.N + g.M, false);
    rep(i, 0, g.N) if (g.L[i] == -1) dfs(g, i);
    rep(i, 0, g.N) if (!alt[i]) res.push_back(i);
    rep(i, 0, g.M) if (alt[g.N + i]) res.push_back(g.N + i);
    return res; }
```

// vim: cc=60 ts=2 sts=2 sw=2:

## 3.4. Depth first searches.

## 3.4.1. Cut Points and Bridges.

```
const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;
```

```
void dfs(const vvi &adj, vi &cp, vii &bri, int u, int p) {
    low[u] = num[u] = curnum++;
    int cnt = 0; bool found = false;
    rep(i, 0, size(adj[u])) {
        int v = adj[u][i];
        if (num[v] == -1) {
            dfs(adj, cp, bri, v, u);
            low[u] = min(low[u], low[v]);
            cnt++;
            found = found || low[v] >= num[u];
            if (low[v] > num[u]) bri.push_back(ii(u, v));
        } else if (p != v) low[u] = min(low[u], num[v]); }
    if (found && (p != -1 || cnt > 1)) cp.push_back(u); }
```

```
pair<vi, vii> cut_points_and_bridges(const vvi &adj) {
    int n = size(adj);
    vi cp; vii bri;
    memset(num, -1, n << 2);
    curnum = 0;
    rep(i, 0, n) if (num[i] == -1) dfs(adj, cp, bri, i, -1);
    return make_pair(cp, bri); }
```

3.4.2. Strongly Connected Components  $\mathcal{O}(V + E)$ .

```
vvi adj, comps;
vi tidx, lnk, cnr, st;
vector<bool> vis;
int age, ncomps;
```

```
void tarjan(int v) {
    tidx[v] = lnk[v] = ++age; vis[v] = true; st.pb(v);
    for (int w : adj[v]) {
        if (!tidx[w]) tarjan(w), lnk[v] = min(lnk[v], lnk[w]);
        else if (vis[w]) lnk[v] = min(lnk[v], tidx[w]);
    }
```

```
}
if (lnk[v] != tidx[v]) return;
comps.pb(vi());
int w;
do {
    vis[w = st.back()] = false; cnr[w] = ncomps;
    ↪ comps.back().pb(w);
    st.pop_back();
} while (w != v);
ncomps++;
}
```

```
void findSCC(int n) {
    age = ncomps = 0; vis.assign(n, false); tidx.assign(n, 0);
    lnk.resize(n); cnr.resize(n); comps.clear();
    for (int i = 0; i < n; i++)
        if (tidx[i] == 0) tarjan(i);
}
```

## 3.4.3. Dominator graph.

const int N = 1234567;

```
vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], parent[N], sdom[N], p[N], best[N],
↪ idom[N], link[N];
```

```
void dfs(int v) {
    pos[v] = cnt;
    order[cnt++] = v;
    for (int u : g[v]) {
        if (pos[u] == -1) {
            parent[u] = v;
            dfs(u);
        }
    }
}
```

```
int find_best(int x) {
    if (p[x] == x) return best[x];
    int u = find_best(p[x]);
    if (pos[sdom[u]] < pos[sdom[best[x]]])
        best[x] = u;
    p[x] = p[p[x]];
    return best[x];
}
```

```
void dominators(int n, int root) {
    fill_n(pos, n, -1);
    cnt = 0;
    dfs(root);
    for (int i = 0; i < n; i++)
        for (int u : g[i]) g_rev[u].push_back(i);
    for (int i = 0; i < n; i++)
        p[i] = best[i] = sdom[i] = i;
    for (int it = cnt - 1; it >= 1; it--) {
        int w = order[it];
```

```
for (int u : g_rev[w]) {
    int t = find_best(u);
    if (pos[sdom[t]] < pos[sdom[w]])
        sdom[w] = sdom[t];
}
bucket[sdom[w]].push_back(w);
idom[w] = sdom[w];
for (int u : bucket[parent[w]])
    link[u] = find_best(u);
bucket[parent[w]].clear();
p[w] = parent[w];
}
for (int it = 1; it < cnt; it++) {
    int w = order[it];
    idom[w] = idom[link[w]];
}
}
```

3.4.4. 2-SAT  $\mathcal{O}(V + E)$ . Include findSCC.

void init2sat(int n) { adj.assign(2 \* n, vi()); }

// vl, vr = true -&gt; variable l, variable r should be negated.

```
void imply(int xl, bool vl, int xr, bool vr) {
    adj[2 * xl + vl].pb(2 * xr + vr); adj[2 * xr + !vr].pb(2 *
    ↪ xl + !vl); }
```

```
void satOr(int xl, bool vl, int xr, bool vr) { imply(xl, !vl,
    ↪ xr, vr); }
void satConst(int x, bool v) { imply(x, !v, x, v); }
void satIfff(int xl, bool vl, int xr, bool vr) {
    imply(xl, vl, xr, vr); imply(xr, vr, xl, vl); }
```

```
bool solve2sat(int n, vector<bool> &sol) {
    findSCC(2 * n);
    for (int i = 0; i < n; i++)
        if (cnr[2 * i] == cnr[2 * i + 1]) return false;
    vector<bool> seen(n, false); sol.assign(n, false);
    for (vi &comp : comps) {
        for (int v : comp) {
            if (seen[v / 2]) continue;
            seen[v / 2] = true; sol[v / 2] = v & 1;
        }
    }
    return true;
}
```

3.5. Cycle Detection  $\mathcal{O}(V + E)$ .

vvi adj; // assumes bidirected graph, adjust accordingly

```
bool cycle_detection() {
    stack<int> s; vector<bool> vis(MAXN, false); vi par(MAXN,
    ↪ -1); s.push(0);
    vis[0] = true;
    while(!s.empty()) {
        int cur = s.top(); s.pop();
        for(int i : adj[cur]) {
```

```

    if(vis[i] && par[cur] != i) return true;
    s.push(i); par[i] = cur; vis[i] = true;
}
}
return false;}

```

### 3.6. Maximum Flow Algorithms.

#### 3.6.1. Dinic's Algorithm $\mathcal{O}(V^2E)$ .

```

struct edge { ll t, r, c, f; };
int S, T, h[MAXN]; vector<edge> g[MAXN];

```

```

void addEdge(int u, int v, ll c) {
    g[u].pb({v, (ll)g[v].size(), c, 0});
    g[v].pb({u, (ll)g[u].size()-1,0,0});
}

void dinicBfs() {
    fill_n(h, MAXN, 0); h[S] = 1;
    queue<int> q; q.push(S);
    while (!q.empty()) {
        int v = q.front(); q.pop();
        for (edge e : g[v]) if (!h[e.t] && e.f < e.c)
            h[e.t] = h[v]+1, q.push(e.t);
    }
}

```

```

ll dinicDFS(int v, ll mf) {
    if (v == T) return mf;
    ll f = 0; bool sat = true;
    for (edge &e : g[v]) {
        if (h[e.t] != h[v] + 1 || e.f >= e.c) continue;
        ll df = dinicDFS(e.t, min(mf - f, e.c - e.f));
        f += df, e.f -= df, g[e.t][e.r].f -= df;
        sat &= e.f == e.c; if (mf == f) break;
    }
    if (sat) h[v] = 0;
    return f;
}

```

```

ll dinicMaxFlow(ll f = 0) {
    while (dinicBfs(), h[T]) f += dinicDFS(S, LLINF);
    return f;
}

```

3.6.2. *Min-cost max-flow*. Find the cheapest possible way of sending a certain amount of flow through a flow network.

```
const int maxn = 300;
```

```

struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
    g.pb({u, v, 0, c, w});
    g.pb({v, u, 0, 0, -w});
}

```

```

void sp(int s, int t) {
    fill_n(D, V, LLINF); D[s] = 0;

```

```

    for (int ng = g.size(), _ = V; _--;) {
        bool ok = false;
        for (int i = 0; i < ng; i++)
            if (D[g[i].x] != LLINF && g[i].f < g[i].c && D[g[i].x]
                ↪ + g[i].w < D[g[i].y]) {
                D[g[i].y] = D[g[i].x] + g[i].w;
                par[g[i].y] = i; ok = true;
            }
        if (!ok) break;
    }
}

```

```

void minCostMaxFlow(int s, int t, ll &c, ll &f) {
    for (c = f = 0; sp(s, t), D[t] < LLINF; ) {
        ll df = LLINF, dc = 0;
        for (int v = t, e; e = par[v], v != s; v = g[e].x) df =
            ↪ min(df, g[e].c - g[e].f);
        for (int v = t, e; e = par[v], v != s; v = g[e].x) g[e].f
            ↪ += df, g[e].f -= df, dc += g[e].w;
        f += df; c += dc * df;
    }
}

```

3.6.3. *Gomory-Hu Tree - All Pairs Maximum Flow*. An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in  $\mathcal{O}(|V|^2)$  plus  $|V| - 1$  times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is  $\mathcal{O}(|V|^3|E|)$ . NOTE: Not sure if it works correctly with disconnected graphs.

```
#include "dinic.cpp"
```

```
bool same[MAXV];
```

```
pair<vii, vvi> construct_gh_tree(flow_network &g) {
```

```

    int n = g.n, v;
    vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
    rep(s,1,n) {
        int l = 0, r = 0;
        par[s].second = g.max_flow(s, par[s].first, false);
        memset(d, 0, n * sizeof(int));
        memset(same, 0, n * sizeof(bool));
        d[q[r++]] = s] = 1;
        while (l < r) {
            same[v = q[l++]] = true;
            for (int i = g.head[v]; i != -1; i = g.e[i].nxt)
                if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
                    d[q[r++]] = g.e[i].v] = 1; }
            rep(i,s+1,n)
                if (par[i].first == par[s].first && same[i])
                    par[i].first = s;
            g.reset(); }
        rep(i,0,n) {
            int mn = INF, cur = i;
            while (true) {
                cap[cur][i] = mn;
                if (cur == 0) break;
                mn = min(mn, par[cur].second), cur = par[cur].first; }
            ↪ }
    }
}

```

```

    return make_pair(par, cap); }
int compute_max_flow(int s, int t, const pair<vii, vvi> &gh)
    ↪ {
    int cur = INF, at = s;
    while (gh.second[at][t] == -1)
        cur = min(cur, gh.first[at].second),
        at = gh.first[at].first;
    return min(cur, gh.second[at][t]); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

### 3.7. Minimal Spanning Tree.

#### 3.7.1. Kruskal $\mathcal{O}(E \log V)$ .

```

struct edge { int x, y, w; };
vector<edge> edges;

```

```

ll kruskal(int n) { // n: #vertices
    uf_init(n);
    sort(all(edges), [] (edge a, edge b) -> bool { return a.w <
        ↪ b.w; });
    ll ret = 0;
    for (edge e : edges)
        if (uf_find(e.x) != uf_find(e.y))
            ret += e.w, uf_union(e.x, e.y);
    return ret;
}

```

### 3.8. Topological Sort.

#### 3.8.1. Modified Depth-First Search.

```

void tsort_dfs(int cur, char* color, const vvi& adj,
    stack<int>& res, bool& cyc) {
    color[cur] = 1;
    rep(i,0,size(adj[cur])) {
        int nxt = adj[cur][i];
        if (color[nxt] == 0)
            tsort_dfs(nxt, color, adj, res, cyc);
        else if (color[nxt] == 1)
            cyc = true;
        if (cyc) return; }
    color[cur] = 2;
    res.push(cur); }

vi tsort(int n, vvi adj, bool& cyc) {
    cyc = false;
    stack<int> S;
    vi res;
    char* color = new char[n];
    memset(color, 0, n);
    rep(i,0,n) {
        if (!color[i]) {
            tsort_dfs(i, color, adj, S, cyc);
            if (cyc) return res; } }
    while (!S.empty()) res.push_back(S.top()), S.pop();
    return res; }

```

3.9. **Euler Path.** Finds an euler path (or circuit) in a directed graph, or reports that none exist.

```
#define MAXV 1000
#define MAXE 5000
vi adj[MAXV];
int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
    int start = -1, end = -1, any = 0, c = 0;
    rep(i,0,n) {
        if (outdeg[i] > 0) any = i;
        if (indeg[i] + 1 == outdeg[i]) start = i, c++;
        else if (indeg[i] == outdeg[i] + 1) end = i, c++;
        else if (indeg[i] != outdeg[i]) return ii(-1,-1);
    }
    if ((start == -1) != (end == -1) || (c != 2 && c != 0))
        return ii(-1,-1);
    if (start == -1) start = end = any;
    return ii(start, end);
}
bool euler_path() {
    ii se = start_end();
    int cur = se.first, at = m + 1;
    if (cur == -1) return false;
    stack<int> s;
    while (true) {
        if (outdeg[cur] == 0) {
            res[--at] = cur;
            if (s.empty()) break;
            cur = s.top(); s.pop();
        } else s.push(cur), cur = adj[cur][--outdeg[cur]];
    }
    return at == 0;
}
```

And an undirected version, which finds a cycle.

```
multiset<int> adj[1010];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
    if (at == to) return it;
    L.insert(it, at), --it;
    while (!adj[at].empty()) {
        int nxt = *adj[at].begin();
        adj[at].erase(adj[at].find(nxt));
        adj[nxt].erase(adj[nxt].find(at));
        if (to == -1) {
            it = euler(nxt, at, it);
            L.insert(it, at);
            --it;
        } else {
            it = euler(nxt, to, it);
            to = -1;
        }
    }
    return it;
}
// euler(0,-1,L.begin())
```

3.10. **Heavy-Light Decomposition.**

```
#include "../data-structures/segment_tree.cpp"
const int ID = 0;
int f(int a, int b) { return a + b; }
struct HLD {
    int n, curhead, curloc;
```

```
vi sz, head, parent, loc;
vvi adj; segment_tree values;
HLD(int _n) : n(_n), sz(n, 1), head(n),
    parent(n, -1), loc(n), adj(n) {
    vector<ll> tmp(n, ID); values = segment_tree(tmp); }
void add_edge(int u, int v) {
    adj[u].push_back(v); adj[v].push_back(u); }
void update_cost(int u, int v, int c) {
    if (parent[v] == u) swap(u, v); assert(parent[u] == v);
    values.update(loc[u], c); }
int csz(int u) {
    rep(i,0,size(adj[u])) if (adj[u][i] != parent[u])
        sz[u] += csz(adj[parent[adj[u][i]] = u][i]);
    return sz[u]; }
void part(int u) {
    head[u] = curhead; loc[u] = curloc++;
    int best = -1;
    rep(i,0,size(adj[u]))
        if (adj[u][i] != parent[u] &&
            (best == -1 || sz[adj[u][i]] > sz[best]))
            best = adj[u][i];
    if (best != -1) part(best);
    rep(i,0,size(adj[u]))
        if (adj[u][i] != parent[u] && adj[u][i] != best)
            part(curhead = adj[u][i]); }
void build(int r = 0) {
    curloc = 0, csz(curhead = r), part(r); }
int lca(int u, int v) {
    vi uat, vat; int res = -1;
    while (u != -1) uat.push_back(u), u = parent[head[u]];
    while (v != -1) vat.push_back(v), v = parent[head[v]];
    u = size(uat) - 1, v = size(vat) - 1;
    while (u >= 0 && v >= 0 && head[uat[u]] == head[vat[v]])
        res = (loc[uat[u]] < loc[vat[v]] ? uat[u] : vat[v]),
        u--, v--;
    return res; }
int query_upto(int u, int v) { int res = ID;
    while (head[u] != head[v])
        res = f(res, values.query(loc[head[u]], loc[u]).x),
        u = parent[head[u]];
    return f(res, values.query(loc[v] + 1, loc[u]).x); }
int query(int u, int v) { int l = lca(u, v);
    return f(query_upto(u, l), query_upto(v, l)); } }
// vim: cc=60 ts=2 sts=2 sw=2:
```

3.11. **Centroid Decomposition.**

```
#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
    path[MAXV][LGMAXV],
    sz[MAXV], seph[MAXV],
    shortest[MAXV];
struct centroid_decomposition {
    int n; vvi adj;
    centroid_decomposition(int _n) : n(_n), adj(n) { }
    void add_edge(int a, int b) {
```

```
    adj[a].push_back(b); adj[b].push_back(a); }
    int dfs(int u, int p) {
        sz[u] = 1;
        rep(i,0,size(adj[u]))
            if (adj[u][i] != p) sz[u] += dfs(adj[u][i], u);
        return sz[u]; }
    void makepaths(int sep, int u, int p, int len) {
        jmp[u][seph[sep]] = sep, path[u][seph[sep]] = len;
        int bad = -1;
        rep(i,0,size(adj[u])) {
            if (adj[u][i] == p) bad = i;
            else makepaths(sep, adj[u][i], u, len + 1);
        }
        if (p == sep)
            swap(adj[u][bad], adj[u].back()), adj[u].pop_back(); }
    void separate(int h=0, int u=0) {
        dfs(u,-1); int sep = u;
        down: iter(nxt,adj[sep])
            if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2) {
                sep = *nxt; goto down; }
        seph[sep] = h, makepaths(sep, sep, -1, 0);
        rep(i,0,size(adj[sep])) separate(h+1, adj[sep][i]); }
    void paint(int u) {
        rep(h,0,seph[u]+1)
            shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
                path[u][h]); }
    int closest(int u) {
        int mn = INF/2;
        rep(h,0,seph[u]+1)
            mn = min(mn, path[u][h] + shortest[jmp[u][h]]);
        return mn; } };
// vim: cc=60 ts=2 sts=2 sw=2:
```

3.12. **Least Common Ancestors, Binary Jumping.**

```
const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
    rep(i, 0, SZ) BP[i][0] = P[i];
    rep(j, 1, LOGSZ) rep(i, 0, SZ)
        BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
    if (H[a] > H[b]) swap(a, b);
    int dh = H[b] - H[a], j = 0;
    rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
    while (BP[a][j] != BP[b][j]) j++;
    while (--j >= 0) if (BP[a][j] != BP[b][j])
        a = BP[a][j], b = BP[b][j];
    return a == b ? a : P[a];
}
```

## 3.13. Tarjan's Off-line Lowest Common Ancestors Algorithm.

```
#include "../data-structures/union_find.cpp"
struct tarjan_olca {
    int *ancestor;
    vi *adj, answers;
    vii *queries;
    bool *colored;
    union_find uf;
    tarjan_olca(int n, vi *_adj) : adj(_adj), uf(n) {
        colored = new bool[n];
        ancestor = new int[n];
        queries = new vii[n];
        memset(colored, 0, n);
    }
    void query(int x, int y) {
        queries[x].push_back(ii(y, size(answers)));
        queries[y].push_back(ii(x, size(answers)));
        answers.push_back(-1);
    }
    void process(int u) {
        ancestor[u] = u;
        rep(i, 0, size(adj[u])) {
            int v = adj[u][i];
            process(v);
            uf.unite(u, v);
            ancestor[uf.find(u)] = u;
        }
        colored[u] = true;
        rep(i, 0, size(queries[u])) {
            int v = queries[u][i].first;
            if (colored[v]) {
                answers[queries[u][i].second] = ancestor[uf.find(v)];
            }
        }
    }
};
// vim: cc=60 ts=2 sts=2 sw=2:
```

3.14. Misra-Gries  $D+1$ -edge coloring.

```
struct Edge { int to, col, rev; };

struct MisraGries {
    int N, K=0; vvi F;
    vector<vector<Edge>> G;

    MisraGries(int n) : N(n), G(n) {}
    // add an undirected edge, NO DUPLICATES ALLOWED
    void addEdge(int u, int v) {
        G[u].pb({v, -1, (int) G[v].size()});
        G[v].pb({u, -1, (int) G[u].size()-1});
    }
};
```

```
void color(int v, int i) {
    vi fan = { i };
    vector<bool> used(G[v].size());
    used[i] = true;
    for (int j = 0; j < (int) G[v].size(); j++)
        if (!used[j] && G[v][j].col >= 0 &&
            F[G[v][fan.back()].to][G[v][j].col] < 0)
            used[j] = true, fan.pb(j), j = -1;
    int c = 0; while (F[v][c] >= 0) c++;
    int d = 0; while (F[G[v][fan.back()].to][d] >= 0) d++;
```

```
int w = v, a = d, k = 0, ccol;
while (true) {
    swap(F[w][c], F[w][d]);
    if (F[w][c] >= 0) G[w][F[w][c]].col = c;
    if (F[w][d] >= 0) G[w][F[w][d]].col = d;
    if (F[w][a^c^d] < 0) break;
    w = G[w][F[w][a]].to;
}
do {
    Edge &e = G[v][fan[k]];
    ccol = F[e.to][d] < 0 ? d : G[v][fan[k+1]].col;
    if (e.col >= 0) F[e.to][e.col] = -1;
    F[e.to][ccol] = e.rev;
    F[v][ccol] = fan[k];
    e.col = G[e.to][e.rev].col = ccol;
    k++;
} while (ccol != d);
}
// finds a K-edge-coloring
void color() {
    REP(v, N) K = max(K, (int) G[v].size() + 1);
    F = vvi(N, vi(K, -1));
    REP(v, N) for (int i = G[v].size(); i--;)
        if (G[v][i].col < 0) color(v, i);
}
};
```

3.15. **Minimum Mean Weight Cycle.** Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```
double min_mean_cycle(vector<vector<pair<int,double>>> adj){
    int n = size(adj); double mn = INFINITY;
    vector<vector<double>> arr(n+1, vector<double>(n, mn));
    arr[0][0] = 0;
    rep(k, 1, n+1) rep(j, 0, n) iter(it, adj[j])
        arr[k][it->first] = min(arr[k][it->first],
                                it->second + arr[k-1][j]);
    rep(k, 0, n) {
        double mx = -INFINITY;
        rep(i, 0, n) mx = max(mx, (arr[n][i]-arr[k][i])/(n-k));
        mn = min(mn, mx);
    }
    return mn;
}
// vim: cc=60 ts=2 sts=2 sw=2:
```

3.16. **Minimum Arborescence.** Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root  $r$  to each vertex. Returns a vector of size  $n$ , where the  $i$ th element is the edge for the  $i$ th vertex. The answer for the root is undefined!

```
#include "../data-structures/union_find.cpp"
struct arborescence {
    int n; union_find uf;
    vector<vector<pair<ii,int>>> adj;
    arborescence(int _n) : n(_n), uf(n), adj(n) {}
    void add_edge(int a, int b, int c) {
        adj[b].push_back(make_pair(ii(a,b),c));
    }
};
```

```
vii find_min(int r) {
    vi vis(n,-1), mn(n,INF); vii par(n);
    rep(i, 0, n) {
        if (uf.find(i) != i) continue;
        int at = i;
        while (at != r && vis[at] == -1) {
            vis[at] = i;
            iter(it, adj[at]) if (it->second < mn[at] &&
                uf.find(it->first.first) != at)
                mn[at] = it->second, par[at] = it->first;
            if (par[at] == ii(0,0)) return vii();
            at = uf.find(par[at].first);
        }
        if (at == r || vis[at] != i) continue;
        union_find tmp = uf; vi seq;
        do { seq.push_back(at); at = uf.find(par[at].first);
        } while (at != seq.front());
        iter(it, seq) uf.unite(*it, seq[0]);
        int c = uf.find(seq[0]);
        vector<pair<ii,int>> nw;
        iter(it, seq) iter(jt, adj[*it])
            nw.push_back(make_pair(jt->first,
                                    jt->second - mn[*it]));
        adj[c] = nw;
        vii rest = find_min(r);
        if (size(rest) == 0) return rest;
        ii use = rest[c];
        rest[at = tmp.find(use.second)] = use;
        iter(it, seq) if (*it != at)
            rest[*it] = par[*it];
        return rest;
    }
    return par;
}
// vim: cc=60 ts=2 sts=2 sw=2:
```

3.17. **Blossom algorithm.** Finds a maximum matching in an arbitrary graph in  $O(|V|^4)$  time. Be vary of loop edges.

```
#define MAXV 300
bool marked[MAXV], emarked[MAXV][MAXV];
int S[MAXV];
vi find_augmenting_path(const vector<vi> &adj, const vi &m){
    int n = size(adj), s = 0;
    vi par(n,-1), height(n), root(n,-1), q, a, b;
    memset(marked, 0, sizeof(marked));
    memset(emarked, 0, sizeof(emarked));
    rep(i, 0, n) if (m[i] >= 0) emarked[i][m[i]] = true;
    else root[i] = i, S[s++] = i;
    while (s) {
        int v = S[--s];
        iter(wt, adj[v]) {
            int w = *wt;
            if (emarked[v][w]) continue;
            if (root[w] == -1) {
                int x = S[s++] = m[w];
                par[w]=v, root[w]=root[v], height[w]=height[v]+1;
                par[x]=w, root[x]=root[w], height[x]=height[w]+1;
            } else if (height[w] % 2 == 0) {
```

```

if (root[v] != root[w]) {
    while (v != -1) q.push_back(v), v = par[v];
    reverse(q.begin(), q.end());
    while (w != -1) q.push_back(w), w = par[w];
    return q;
} else {
    int c = v;
    while (c != -1) a.push_back(c), c = par[c];
    c = w;
    while (c != -1) b.push_back(c), c = par[c];
    while (!a.empty() && !b.empty() && a.back() == b.back())
        c = a.back(), a.pop_back(), b.pop_back();
    memset(marked, 0, sizeof(marked));
    fill(par.begin(), par.end(), 0);
    iter(it, a) par[*it] = 1; iter(it, b) par[*it] = 1;
    par[c] = s = 1;
    rep(i, 0, n) root[par[i] = par[i] ? 0 : s++] = i;
    vector<vi> adj2(s);
    rep(i, 0, n) iter(it, adj[i]) {
        if (par[*it] == 0) continue;
        if (par[i] == 0) {
            if (!marked[par[*it]]) {
                adj2[par[i]].push_back(par[*it]);
                adj2[par[*it]].push_back(par[i]);
                marked[par[*it]] = true;
            }
            else adj2[par[i]].push_back(par[*it]);
        }
    }
    vi m2(s, -1);
    if (m[c] != -1) m2[m2[par[m[c]]] = 0] = par[m[c]];
    rep(i, 0, n) if (par[i] != 0 && m[i] != -1 && par[m[i]] != 0)
        m2[par[i]] = par[m[i]];
    vi p = find_augmenting_path(adj2, m2);
    int t = 0;
    while (t < size(p) && p[t]) t++;
    if (t == size(p)) {
        rep(i, 0, size(p)) p[i] = root[p[i]];
        return p;
    }
    if (!p[0] || (m[c] != -1 && p[t+1] != par[m[c]]))
        reverse(p.begin(), p.end()), t = size(p) - t - 1;
    rep(i, 0, t) q.push_back(root[p[i]]);
    iter(it, adj[root[p[t-1]]) {
        if (par[*it] != (s = 0)) continue;
        a.push_back(c), reverse(a.begin(), a.end());
        iter(jt, a) a.push_back(*jt);
        while (a[s] != *it) s++;
        if ((height[*it] & 1) ^ (s < size(a) - size(b)))
            reverse(a.begin(), a.end()), s = size(a) - s - 1;
        while (a[s] != c) q.push_back(a[s]), s = (s+1)%size(a);
        q.push_back(c);
        rep(i, t+1, size(p)) q.push_back(root[p[i]]);
        return q;
    } }
    emarked[v][w] = emarked[w][v] = true;
    marked[v] = true;
} return q;
}
vii max_matching(const vector<vi> &adj) {
    vi m(size(adj), -1), ap; vii res, es;
    rep(i, 0, size(adj)) iter(it, adj[i]) es.emplace_back(i, *it);
    random_shuffle(es.begin(), es.end());

```

```

iter(it, es) if (m[it->first] == -1 && m[it->second] == -1)
    m[it->first] = it->second, m[it->second] = it->first;
do { ap = find_augmenting_path(adj, m);
    rep(i, 0, size(ap)) m[m[ap[i^1]] = ap[i]] = ap[i^1];
} while (!ap.empty());
rep(i, 0, size(m)) if (i < m[i]) res.emplace_back(i, m[i]);
return res;
}

```

// vim: cc=60 ts=2 sts=2 sw=2:

**3.18. Maximum Density Subgraph.** Given (weighted) undirected graph  $G$ . Binary search density. If  $g$  is current density, construct flow network:  $(S, u, m), (u, T, m + 2g - d_u), (u, v, 1), (v, 1)$ , where  $m$  is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty  $S$ -component, then maximum density is smaller than  $g$ , otherwise it's larger. Distance between valid densities is at least  $1/(n(n-1))$ . Edge case when density is 0. This also works for weighted graphs by replacing  $d_u$  by the weighted degree, and doing more iterations (if weights are not integers).

**3.19. Maximum-Weight Closure.** Given a vertex-weighted directed graph  $G$ . Turn the graph into a flow network, adding weight  $\infty$  to each edge. Add vertices  $S, T$ . For each vertex  $v$  of weight  $w$ , add edge  $(S, v, w)$  if  $w \geq 0$ , or edge  $(v, T, -w)$  if  $w < 0$ . Sum of positive weights minus minimum  $S - T$  cut is the answer. Vertices reachable from  $S$  are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

**3.20. Maximum Weighted Independent Set in a Bipartite Graph.** This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges  $(S, u, w(u))$  for  $u \in L, (v, T, w(v))$  for  $v \in R$  and  $(u, v, \infty)$  for  $(u, v) \in E$ . The minimum  $S, T$ -cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

**3.21. Synchronizing word problem.** A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

#### 4. STRING ALGORITHMS

##### 4.1. Trie.

const int SIGMA = 26;

```

struct trie {
    bool word; trie **adj;

    trie() : word(false), adj(new trie*[SIGMA]) {
        for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
    }

    void addWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) cur->adj[i] = new trie();
            cur = cur->adj[i];
        }
    }
}

```

```

cur->word = true;
}

bool isWord(const string &str) {
    trie *cur = this;
    for (char ch : str) {
        int i = ch - 'a';
        if (!cur->adj[i]) return false;
        cur = cur->adj[i];
    }
    return cur->word;
}
};

```

##### 4.2. Z-algorithm $\mathcal{O}(n)$ .

//  $z[i]$  = length of longest substring starting from  $s[i]$   
 $\hookrightarrow$  which is also a prefix of  $s$ .  
vii z\_function(const string &s) {  
 int n = (int) s.length();  
 vi z(n);  
 for (int i = 1, l = 0, r = 0; i < n; ++i) {  
 if (i <= r) z[i] = min(r - i + 1, z[i - l]);  
 while (i + z[i] < n && s[z[i]] == s[i + z[i]]) ++z[i];  
 if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;  
 }  
 return z;  
}

**4.3. Suffix array**  $\mathcal{O}(n \log^2 n)$ . This creates an array  $P[0], P[1], \dots, P[n-1]$  such that the suffix  $S[i \dots n]$  is the  $P[i]^{th}$  suffix of  $S$  when lexicographically sorted.

```

typedef pair<ii, int> tii;
const int maxlogn = 17, maxn = 1 << maxlogn;

int p[maxlogn + 1][maxn]; tii L[maxn];

int suffixArray(string S) {
    int N = S.size(), stp = 1, cnt = 1;
    REP(i, N) p[0][i] = S[i];
    for (; cnt < N; stp++, cnt <= 1) {
        REP(i, N)
            L[i] = tii(ii(p[stp-1][i], i + cnt < N ? p[stp-1][i + cnt] : -1), i);
        sort(L, L + N);
        REP(i, N)
            p[stp][L[i].y] = i > 0 && L[i].x == L[i-1].x ?
                p[stp][L[i-1].y] : i;
    }
    return stp - 1; // result is in p[stp - 1][0 .. (N - 1)]
}

```



4.4. **Longest Common Subsequence**  $\mathcal{O}(n^2)$ . SUBSTRING: *consecutive characters* !!!

**int** dp[STR\_SIZE][STR\_SIZE]; // DP problem

```
int lcs(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i < n1; i++) {
        for (int j = 0; j < n2; j++) {
            if (i == 0 || j == 0) dp[i][j] = 0;
            else if (w1[i - 1] == w2[j - 1]) dp[i][j] = dp[i - 1][j]
                ↳ - 1] + 1;
            else dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);
        }
    }
    return dp[n1][n2];
}
```

// *backtrace*

```
string getLCS(const string &w1, const string &w2) {
    int i = w1.size(), j = w2.size(); string ret = "";
    while (i > 0 && j > 0) {
        if (w1[i - 1] == w2[j - 1]) ret += w1[--i], j--;
        else if (dp[i][j - 1] > dp[i - 1][j]) j--;
        else i--;
    }
    reverse(ret.begin(), ret.end());
    return ret;
}
```

4.5. **Levenshtein Distance**  $\mathcal{O}(n^2)$ . Also known as the 'Edit distance'.

**int** dp[MAX\_SIZE][MAX\_SIZE]; // DP problem

```
int levDist(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i <= n1; i++) dp[i][0] = i; // removal
    for (int j = 0; j <= n2; j++) dp[0][j] = j; // insertion
    for (int i = 1; i <= n1; i++)
        for (int j = 1; j <= n2; j++)
            dp[i][j] = min(
                1 + min(dp[i - 1][j], dp[i][j - 1]),
                dp[i - 1][j - 1] + (w1[i - 1] != w2[j - 1])
            );
    return dp[n1][n2];
}
```

4.6. **Knuth-Morris-Pratt algorithm**  $\mathcal{O}(N + M)$ .

```
int kmp_search(const string &word, const string &text) {
    int n = word.size();
    vi T(n + 1, 0);
    for (int i = 1, j = 0; i < n; ) {
        if (word[i] == word[j]) T[++i] = ++j; // match
        else if (j > 0) j = T[j]; // fallback
        else i++; // no match, keep zero
    }
    int matches = 0;
    for (int i = 0, j = 0; i < text.size(); ) {
```

```
    if (text[i] == word[j]) {
        i++;
        if (++j == n) // match at interval [i - n, i)
            matches++, j = T[j];
        else if (j > 0) j = T[j];
        else i++;
    }
    return matches;
}
```

4.7. **Aho-Corasick Algorithm**  $\mathcal{O}(N + \sum_{i=1}^m |S_i|)$ . All given P must be unique!

**const int** MAXP = 100, MAXLEN = 200, SIGMA = 26, MAXTRIE =  
↳ MAXP \* MAXLEN;

**int** nP;  
string P[MAXP], S;

**int** pnr[MAXTRIE], to[MAXTRIE][SIGMA], sLink[MAXTRIE],  
↳ dLink[MAXTRIE], nnodes;

```
void ahoCorasick() {
    fill_n(pnr, MAXTRIE, -1);
    for (int i = 0; i < MAXTRIE; i++) fill_n(to[i], SIGMA, 0);
    fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE, 0);
    nnodes = 1;
    // STEP 1: MAKE A TREE
    for (int i = 0; i < nP; i++) {
        int cur = 0;
        for (char c : P[i]) {
            int i = c - 'a';
            if (to[cur][i] == 0) to[cur][i] = nnodes++;
            cur = to[cur][i];
        }
        pnr[cur] = i;
    }
}
```

// *STEP 2: CREATE SUFFIX\_LINKS AND DICT\_LINKS*

```
queue<int> q; q.push(0);
while (!q.empty()) {
    int cur = q.front(); q.pop();
    for (int c = 0; c < SIGMA; c++) {
        if (to[cur][c]) {
            int sl = sLink[to[cur][c]] = cur == 0 ? 0 :
                ↳ to[sLink[cur]][c];
            // if all strings have equal length, remove this:
            dLink[to[cur][c]] = pnr[sl] >= 0 ? sl : dLink[sl];
            q.push(to[cur][c]);
        } else to[cur][c] = to[sLink[cur]][c];
    }
}
```

// *STEP 3: TRAVERSE S*

```
for (int cur = 0, i = 0, n = S.size(); i < n; i++) {
    cur = to[cur][S[i] - 'a'];
    for (int hit = pnr[cur] >= 0 ? cur : dLink[cur]; hit; hit
        ↳ = dLink[hit]) {
```

```
        cerr << P[pnr[hit]] << " found at [" << (i + 1 -
            ↳ P[pnr[hit]].size()) << ", " << i << "]" << endl;
    }
}
```

4.8. **eerTree**. Constructs an eerTree in  $\mathcal{O}(n)$ , one character at a time.

#define MAXN 100100

#define SIGMA 26

#define BASE 'a'

**char** \*s = new **char**[MAXN];

**struct** state {

**int** len, link, to[SIGMA];

} \*st = new state[MAXN+2];

**struct** eertree {

**int** last, sz, n;

eertree() : last(1), sz(2), n(0) {

st[0].len = st[0].link = -1;

st[1].len = st[1].link = 0; }

**int** extend() {

**char** c = s[n++]; **int** p = last;

**while** (n - st[p].len - 2 < 0 || c != s[n - st[p].len -  
↳ 2])

p = st[p].link;

**if** (!st[p].to[c-BASE]) {

**int** q = last = sz++;

st[p].to[c-BASE] = q;

st[q].len = st[p].len + 2;

**do** { p = st[p].link;

} **while** (p != -1 && (n < st[p].len + 2 ||  
 c != s[n - st[p].len - 2]));

**if** (p == -1) st[q].link = 1;

**else** st[q].link = st[p].to[c-BASE];

**return** 1; }

last = st[p].to[c-BASE];

**return** 0; } };

// vim: cc=60 ts=2 sts=2 sw=2:

4.9. **Suffix Automaton**. Minimum automata that accepts all suffixes of a string with  $\mathcal{O}(n)$  construction. The automata itself is a DAG therefore suitable for DP, examples are counting unique substrings, occurrences of substrings and suffix.

// *TODO: Add longest common subsring*

**const int** MAXL = 100000;

**struct** suffix\_automaton {

**vi** len, link, occur, cnt;

vector<map<**char**, **int**>> next;

vector<**bool**> isclone;

ll \*occuratleast;

**int** sz, last;

string s;

suffix\_automaton() : len(MAXL\*2), link(MAXL\*2),

occur(MAXL\*2), next(MAXL\*2), isclone(MAXL\*2) { clear(); }

**void** clear() { sz = 1; last = len[0] = 0; link[0] = -1;

next[0].clear(); isclone[0] = **false**; }

**bool** issubstr(string other){

```

    for(int i = 0, cur = 0; i < size(other); ++i){
        if(cur == -1) return false; cur = next[cur][other[i]];
        ↪ }
    return true; }
void extend(char c){ int cur = sz++; len[cur] =
    ↪ len[last]+1;
    next[cur].clear(); isclone[cur] = false; int p = last;
    for(; p != -1 && !next[p].count(c); p = link[p])
        next[p][c] = cur;
    if(p == -1){ link[cur] = 0; }
    else{ int q = next[p][c];
        if(len[p] + 1 == len[q]){ link[cur] = q; }
        else { int clone = sz++; isclone[clone] = true;
            len[clone] = len[p] + 1;
            link[clone] = link[q]; next[clone] = next[q];
            for(; p != -1 && next[p].count(c) && next[p][c] == q;
                p = link[p]){
                next[p][c] = clone; }
            link[q] = link[cur] = clone;
        } } last = cur; }
void count(){
    cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
    map<char,int>::iterator i;
    while(!S.empty()){
        ii cur = S.top(); S.pop();
        if(cur.second){
            for(i = next[cur.first].begin();
                i != next[cur.first].end();++i){
                cnt[cur.first] += cnt[(*i).second]; } }
        else if(cnt[cur.first] == -1){
            cnt[cur.first] = 1; S.push(ii(cur.first, 1));
            for(i = next[cur.first].begin();
                i != next[cur.first].end();++i){
                S.push(ii((*i).second, 0)); } } } }
    string lexicok(ll k){
        int st = 0; string res; map<char,int>::iterator i;
        while(k){
            for(i = next[st].begin(); i != next[st].end(); ++i){
                if(k <= cnt[(*i).second]){ st = (*i).second;
                    res.push_back((*i).first); k--; break;
                } else { k -= cnt[(*i).second]; } } }
        return res; }
void countoccur(){
    for(int i = 0; i < sz; ++i){ occur[i] = 1 - isclone[i]; }
    vii states(sz);
    for(int i = 0; i < sz; ++i){ states[i] = ii(len[i],i); }
    sort(states.begin(), states.end());
    for(int i = size(states)-1; i >= 0; --i){
        int v = states[i].second;
        if(link[v] != -1) { occur[link[v]] += occur[v]; } } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

4.10. **Hashing.** Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```

struct hasher { int b = 311, m; vi h, p;
    hasher(string s, int _m)

```

```

    : m(_m), h(size(s)+1), p(size(s)+1) {
    p[0] = 1; h[0] = 0;
    rep(i,0,size(s)) p[i+1] = (ll)p[i] * b % m;
    rep(i,0,size(s)) h[i+1] = ((ll)h[i] * b + s[i]) % m; }
    int hash(int l, int r) {
        return (h[r+1] + m - (ll)h[l] * p[r-l+1] % m) % m; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

## 5. GEOMETRY

```

const double EPS = 1e-7, PI = acos(-1.0);

typedef long long NUM; // EITHER double OR long long
typedef pair<NUM, NUM> pt;
#define x first
#define y second

pt operator+(pt p, pt q) { return pt(p.x + q.x, p.y + q.y); }
pt operator-(pt p, pt q) { return pt(p.x - q.x, p.y - q.y); }

pt& operator+=(pt &p, pt q) { return p = p + q; }
pt& operator-=(pt &p, pt q) { return p = p - q; }

pt operator*(pt p, NUM l) { return pt(p.x * l, p.y * l); }
pt operator/(pt p, NUM l) { return pt(p.x / l, p.y / l); }

NUM operator*(pt p, pt q) { return p.x * q.x + p.y * q.y; }
NUM operator^(pt p, pt q) { return p.x * q.y - p.y * q.x; }

istream& operator>>(istream &in, pt &p) { return in >> p.x >>
    ↪ p.y; }
ostream& operator<<(ostream &out, pt p) { return out << '('
    ↪ << p.x << ", " << p.y << ')'; }

NUM lenSq(pt p) { return p * p; }
NUM lenSq(pt p, pt q) { return lenSq(p - q); }
double len(pt p) { return hypot(p.x, p.y); } // more overflow
    ↪ safe
double len(pt p, pt q) { return len(p - q); }

typedef pt frac;
typedef pair<double, double> vec;
vec getvec(pt p, pt dp, frac t) { return vec(p.x + 1. * dp.x
    ↪ * t.x / t.y, p.y + 1. * dp.y * t.x / t.y); }

// square distance from pt a to line bc
frac distPtLineSq(pt a, pt b, pt c) {
    a -= b, c -= b;
    return frac((a ^ c) * (a ^ c), c * c);
}

// square distance from pt a to linesegment bc
frac distPtSegmentSq(pt a, pt b, pt c) {
    a -= b; c -= b;
    NUM dot = a * c, len = c * c;
    if (dot <= 0) return frac(a * a, 1);

```

```

    if (dot >= len) return frac((a - c) * (a - c), 1);
    return frac(a * a * len - dot * dot, len);
}

// projects pt a onto linesegment bc
frac proj(pt a, pt b, pt c) { return frac((a - b) * (c - b),
    ↪ (c - b) * (c - b)); }
vec projv(pt a, pt b, pt c) { return getvec(b, c - b, proj(a,
    ↪ b, c)); }

bool collinear(pt a, pt b, pt c) { return ((a - b) ^ (a - c))
    ↪ == 0; }

bool pointOnSegment(pt a, pt b, pt c) {
    NUM dot = (a - b) * (c - b), len = (c - b) * (c - b);
    return collinear(a, b, c) && 0 <= dot && dot <= len;
}

// true => 1 intersection, false => parallel, so 0 or \infy
    ↪ solutions
bool linesIntersect(pt a, pt b, pt c, pt d) { return ((a - b)
    ↪ ^ (c - d)) != 0; }
vec lineLineIntersection(pt a, pt b, pt c, pt d) {
    double det = (a - b) ^ (c - d); pt ret = (c - d) * (a ^ b)
    ↪ - (a - b) * (c ^ d);
    return vec(ret.x / det, ret.y / det);
}

// dp, dq are directions from p, q
// intersection at p + t_i dp, for 0 <= i < return value
int segmentIntersection(pt p, pt dp, pt q, pt dq, frac &t0,
    ↪ frac &t1){
    if (dp * dp == 0) swap(p, q), swap(dp, dq); // dq = 0
    if (dp * dp == 0) { t0 = t1 = frac(0, 1); return p == q; }
    ↪ // dp = dq = 0
    pt dpq = (q - p); NUM c = dp ^ dq, c0 = dpq ^ dp, c1 = dpq
    ↪ ^ dq;
    if (c == 0) { // parallel, dp > 0, dq >= 0
        if (c0 != 0) return 0; // not collinear
        NUM v0 = dpq * dp, v1 = v0 + dq * dp, dp2 = dp * dp;
        if (v1 < v0) swap(v0, v1);
        t0 = frac(v0 = max(v0, (NUM) 0), dp2);
        t1 = frac(v1 = min(v1, dp2), dp2);
        return (v0 <= v1) + (v0 < v1);
    } else if (c < 0) c = -c, c0 = -c0, c1 = -c1;
    t0 = t1 = frac(c1, c);
    return 0 <= min(c0, c1) && max(c0, c1) <= c;
}

// Returns TWICE the area of a polygon to keep it an integer
NUM polygonTwiceArea(const vector<pt> &pts) {
    NUM area = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j = i++)
        area += pts[i] ^ pts[j];
}

```

```

    return abs(area); // area < 0 ==> pts ccw
}

bool pointInPolygon(pt p, const vector<pt> &pts) {
    double sum = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j = i++)
        ↪ {
            if (pointOnSegment(p, pts[i], pts[j])) return true; //
            ↪ boundary
            double angle = acos((pts[i] - p) * (pts[j] - p) /
            ↪ len(pts[i], p) / len(pts[j], p));
            sum += ((pts[i] - p) ^ (pts[j] - p)) < 0 ? angle :
            ↪ -angle;
        }
    return abs(abs(sum) - 2 * PI) < EPS;
}

```

### 5.1. Convex Hull $\mathcal{O}(n \log n)$ .

```

// the convex hull consists of: { pts[ret[0]], pts[ret[1]],
↪ ... pts[ret.back()] }
vi convexHull(const vector<pt> &pts) {
    if (pts.empty()) return vi();
    vi ret, ord;
    int n = pts.size(), st = min_element(all(pts)) -
    ↪ pts.begin();
    rep(i, 0, n)
        if (pts[i] != pts[st]) ord.pb(i);
    sort(all(ord), [&pts, &st] (int a, int b) {
        pt p = pts[a] - pts[st], q = pts[b] - pts[st];
        return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) < lenSq(q);
    });
    ret.pb(st);
    for (int i : ord) {
        // use '>' to include ALL points on the hull-line
        for (int s = ret.size() - 1; s > 0 && ((pts[ret[s-1]] -
        ↪ pts[ret[s]]) ^ (pts[i] - pts[ret[s]])) >= 0; s--)
            ret.pop_back();
        ret.pb(i);
    }
    return ret;
}

```

### 5.2. Rotating Calipers $\mathcal{O}(n)$ . Finds the longest distance between two points in a convex hull.

```

NUM rotatingCalipers(vector<pt> &hull) {
    int n = hull.size(), a = 0, b = 1;
    if (n <= 1) return 0.0;
    while (((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
    ↪ hull[b])) > 0) b++;
    NUM ret = 0.0;
    while (a < n) {
        ret = max(ret, lenSq(hull[a], hull[b]));
        if (((hull[(a + 1) % n] - hull[a % n]) ^ (hull[(b + 1) %
        ↪ n] - hull[b])) <= 0) a++;
        else if (++b == n) b = 0;
    }
}

```

```

    return ret;
}

5.3. Closest points  $\mathcal{O}(n \log n)$ .
int n; pt pts[maxn];

struct byY {
    bool operator()(int a, int b) const { return pts[a].y <
    ↪ pts[b].y; }
};

inline NUM dist(ii p) { return hypot(pts[p.x].x - pts[p.y].x,
↪ pts[p.x].y - pts[p.y].y); }

ii minpt(ii p1, ii p2) { return dist(p1) < dist(p2) ? p1 :
↪ p2; }

// closest pts (by index) inside pts[l ... r], with sorted y
↪ values in ys
ii closest(int l, int r, vi &ys) {
    if (r - l == 2) { // don't assume 1 here.
        ys = { l, l + 1 };
        return ii(l, l + 1);
    } else if (r - l == 3) { // brute-force
        ys = { l, l + 1, l + 2 };
        sort(all(ys), byY());
        return minpt(ii(l, l + 1), minpt(ii(l, l + 2), ii(l + 1,
        ↪ l + 2)));
    }
    int m = (l + r) / 2; vi yl, yr;
    ii delta = minpt(closest(l, m, yl), closest(m, r, yr));
    NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
    ↪ pts[m].x);
    merge(all(yl), all(yr), back_inserter(ys), byY());
    deque<int> q;
    for (int i : ys) if (abs(pts[i].x - xm) <= ddelta) {
        for (int j : q) delta = minpt(delta, ii(i, j));
        q.pb(i);
        if (q.size() > 8) q.pop_front(); // magic from
        ↪ Introduction to Algorithms.
    }
    return delta;
}

```

### 5.4. Great-Circle Distance. Computes the distance between two points (given as latitude/longitude coordinates) on a sphere of radius $r$ .

```

ld gc_distance(ld pLat, ld pLong, ld qLat, ld qLong, ld r) {
    pLat *= pi / 180; pLong *= pi / 180;
    qLat *= pi / 180; qLong *= pi / 180;
    return r * acos(cos(pLat)*cos(qLat)*cos(pLong - qLong) +
    ↪ sin(pLat)*sin(qLat)); }

```

### 5.5. 3D Primitives.

```

#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)

```

```

#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
    double x, y, z;
    point3d() : x(0), y(0), z(0) {}
    point3d(double _x, double _y, double _z)
        : x(_x), y(_y), z(_z) {}
    point3d operator+(P(p)) const {
        return point3d(x + p.x, y + p.y, z + p.z); }
    point3d operator-(P(p)) const {
        return point3d(x - p.x, y - p.y, z - p.z); }
    point3d operator-() const {
        return point3d(-x, -y, -z); }
    point3d operator*(double k) const {
        return point3d(x * k, y * k, z * k); }
    point3d operator/(double k) const {
        return point3d(x / k, y / k, z / k); }
    double operator%(P(p)) const {
        return x * p.x + y * p.y + z * p.z; }
    point3d operator*(P(p)) const {
        return point3d(y*p.z - z*p.y,
        ↪ z*p.x - x*p.z, x*p.y - y*p.x); }
    double length() const {
        return sqrt(*this % *this); }
    double distTo(P(p)) const {
        return (*this - p).length(); }
    double distTo(P(A), P(B)) const {
        // A and B must be two different points
        return ((*this - A) * (*this - B)).length() /
        ↪ A.distTo(B); }
    point3d normalize(double k = 1) const {
        // length() must not return 0
        return (*this) * (k / length()); }
    point3d getProjection(P(A), P(B)) const {
        point3d v = B - A;
        return A + v.normalize((v % (*this - A)) / v.length()); }
    point3d rotate(P(normal)) const {
        //normal must have length 1 and be orthogonal to the
        ↪ vector
        return (*this) * normal; }
    point3d rotate(double alpha, P(normal)) const {
        return (*this) * cos(alpha) + rotate(normal) *
        ↪ sin(alpha); }
    point3d rotatePoint(P(0), P(axe), double alpha) const {
        point3d Z = axe.normalize(axe % (*this - 0));
        return 0 + Z + (*this - 0 - Z).rotate(alpha, 0); }
    bool isZero() const {
        return abs(x) < EPS && abs(y) < EPS && abs(z) < EPS; }
    bool isOnLine(L(A, B)) const {
        return ((A - *this) * (B - *this)).isZero(); }
    bool isInSegment(L(A, B)) const {
        return isOnLine(A, B) && ((A - *this) % (B -
        ↪ *this)) < EPS; }
    bool isInSegmentStrictly(L(A, B)) const {
        return isOnLine(A, B) && ((A - *this) % (B -
        ↪ *this)) < -EPS; }
}

```

```

double getAngle() const {
    return atan2(y, x); }
double getAngle(P(u)) const {
    return atan2((*this * u).length(), *this % u); }
bool isOnPlane(PL(A, B, C)) const {
    return
        abs((A - *this) * (B - *this) % (C - *this)) < EPS; }
    ↪ };
int line_line_intersect(L(A, B), L(C, D), point3d &O){
    if (abs((B - A) * (C - A) % (D - A)) > EPS) return 0;
    if (((A - B) * (C - D)).length() < EPS)
        return A.isOnLine(C, D) ? 2 : 0;
    point3d normal = ((A - B) * (C - B)).normalize();
    double s1 = (C - A) * (D - A) % normal;
    0 = A + ((B - A) / (s1 + ((D - B) * (C - B) % normal))) *
    ↪ s1;
    return 1; }
int line_plane_intersect(L(A, B), PL(C, D, E), point3d &O) {
    double V1 = (C - A) * (D - A) % (E - A);
    double V2 = (D - B) * (C - B) % (E - B);
    if (abs(V1 + V2) < EPS)
        return A.isOnPlane(C, D, E) ? 2 : 0;
    0 = A + ((B - A) / (V1 + V2)) * V1;
    return 1; }
bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
    point3d &P, point3d &Q) {
    point3d n = nA * nB;
    if (n.isZero()) return false;
    point3d v = n * nA;
    P = A + (n * nA) * ((B - A) % nB / (v % nB));
    Q = P + n;
    return true; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

### 5.6. Polygon Centroid.

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

**5.7. Rectilinear Minimum Spanning Tree.** Given a set of  $n$  points in the plane, and the aim is to find a minimum spanning tree connecting these  $n$  points, assuming the Manhattan distance is used. The function candidates returns at most  $4n$  edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```
#define MAXN 100100
```

```

struct RMST {
    struct point {
        int i; ll x, y;
        point() : i(-1) { }
        ll d1() { return x + y; }
        ll d2() { return x - y; }
    };
};

```

```

ll dist(point other) {
    return abs(x - other.x) + abs(y - other.y); }
bool operator <(const point &other) const {
    return y == other.y ? x > other.x : y < other.y; }
} best[MAXN], arr[MAXN], tmp[MAXN];
int n;
RMST() : n(0) {}
void add_point(int x, int y) {
    arr[arr[n].i = n].x = x, arr[n++].y = y; }
void rec(int l, int r) {
    if (l >= r) return;
    int m = (l+r)/2;
    rec(l,m), rec(m+1,r);
    point bst;
    for (int i = l, j = m+1, k = l; i <= m || j <= r; k++) {
        if (j > r || (i <= m && arr[i].d1() < arr[j].d1())) {
            tmp[k] = arr[i++];
            if (bst.i != -1 && (best[tmp[k].i].i == -1
                || best[tmp[k].i].d2() < bst.d2()))
                best[tmp[k].i] = bst;
        } else {
            tmp[k] = arr[j++];
            if (bst.i == -1 || bst.d2() < tmp[k].d2())
                bst = tmp[k]; } }
    rep(i,l,r+1) arr[i] = tmp[i]; }
vector<pair<ll,ii> > candidates() {
    vector<pair<ll, ii> > es;
    rep(p,0,2) {
        rep(q,0,2) {
            sort(arr, arr+n);
            rep(i,0,n) best[i].i = -1;
            rec(0,n-1);
            rep(i,0,n) {
                if (best[arr[i].i].i != -1)
                    es.push_back({arr[i].dist(best[arr[i].i].i),
                        {arr[i].i, best[arr[i].i].i}});
                swap(arr[i].x, arr[i].y);
                arr[i].x *= -1, arr[i].y *= -1; } }
            rep(i,0,n) arr[i].x *= -1; }
        return es; } }
// vim: cc=60 ts=2 sts=2 sw=2:

```

**5.8. Formulas.** Let  $a = (a_x, a_y)$  and  $b = (b_x, b_y)$  be two-dimensional vectors.

- $a \cdot b = |a||b| \cos \theta$ , where  $\theta$  is the angle between  $a$  and  $b$ .
- $a \times b = |a||b| \sin \theta$ , where  $\theta$  is the signed angle between  $a$  and  $b$ .
- $a \times b$  is equal to the area of the parallelogram with two of its sides formed by  $a$  and  $b$ . Half of that is the area of the triangle formed by  $a$  and  $b$ .
- **Euler's formula:**  $V - E + F = 2$
- Side lengths  $a, b, c$  can form a triangle iff.  $a + b > c$ ,  $b + c > a$  and  $a + c > b$ .
- Sum of internal angles of a regular convex  $n$ -gon is  $(n - 2)\pi$ .
- **Law of sines:**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:**  $b^2 = a^2 + c^2 - 2ac \cos B$

- Internal tangents of circles  $(c_1, r_1), (c_2, r_2)$  intersect at  $(c_1 r_2 + c_2 r_1) / (r_1 + r_2)$ , external intersect at  $(c_1 r_2 - c_2 r_1) / (r_1 + r_2)$ .

## 6. MISCELLANEOUS

6.1. **Binary search**  $\mathcal{O}(\log(hi - lo))$ .

```

bool test(int n);

int search(int lo, int hi) {
    // assert(test(lo) && !test(hi));
    while (hi - lo > 1) {
        int m = (lo + hi) / 2;
        (test(m) ? lo : hi) = m;
    }
    // assert(test(lo) && !test(hi));
    return lo;
}

```

6.2. **Fast Fourier Transform**  $\mathcal{O}(n \log n)$ . Given two polynomials  $A(x) = a_0 + a_1 x + \dots + a_n / 2 x^{n/2}$  and  $B(x) = b_0 + b_1 x + \dots + b_n / 2 x^{n/2}$ , FFT calculates all coefficients of  $C(x) = A(x) \cdot B(x) = c_0 + c_1 x + \dots + c_n x^n$ , with  $c_i = \sum_{j=0}^i a_j b_{i-j}$ .

```

typedef complex<double> cpx;
const int logmaxn = 20, maxn = 1 << logmaxn;

```

```
cpx a[maxn] = {}, b[maxn] = {}, c[maxn];
```

```

void fft(cpx *src, cpx *dest) {
    for (int i = 0, rep = 0; i < maxn; i++, rep = 0) {
        for (int j = i, k = logmaxn; k--; j >= 1)
            rep = (rep << 1) | (j & 1);
        dest[rep] = src[i];
    }
    for (int s = 1, m = 1; m <= maxn; s++, m *= 2) {
        cpx r = exp(cpx(0, 2.0 * PI / m));
        for (int k = 0; k < maxn; k += m) {
            cpx cr(1.0, 0.0);
            for (int j = 0; j < m / 2; j++) {
                cpx t = cr * dest[k + j + m / 2];
                dest[k + j + m / 2] = dest[k + j] - t;
                dest[k + j] += t; cr *= r;
            }
        }
    }
}

```

```

void multiply() {
    fft(a, c); fft(b, a);
    REP(i, maxn) b[i] = conj(a[i] * c[i]);
    fft(b, c);
    REP(i, maxn) c[i] = conj(c[i]) / (1.0 * maxn);
}

```

6.3. Minimum Assignment (Hungarian Algorithm)  $\mathcal{O}(n^3)$ .

```

int a[MAXN + 1][MAXM + 1]; // matrix, 1-based
int minimum_assignment(int n, int m) { // n rows, m columns
    vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);
    for (int i = 1; i <= n; i++) {
        p[0] = i;
        int j0 = 0;
        vi minv(m + 1, INF);
        vector<char> used(m + 1, false);
        do {
            used[j0] = true;
            int i0 = p[j0], delta = INF, j1;
            for (int j = 1; j <= m; j++)
                if (!used[j]) {
                    int cur = a[i0][j] - u[i0] - v[j];
                    if (cur < minv[j]) minv[j] = cur, way[j] = j0;
                    if (minv[j] < delta) delta = minv[j], j1 = j;
                }
            for (int j = 0; j <= m; j++) {
                if (used[j]) u[p[j]] += delta, v[j] -= delta;
                else minv[j] -= delta;
            }
            j0 = j1;
        } while (p[j0] != 0);
        do {
            int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
        } while (j0);
    }
    // column j is assigned to row p[j]
    return -v[0];
}

```

6.4. Partial linear equation solver  $\mathcal{O}(N^3)$ .

```

typedef double NUM;
const int MAXROWS = 200, MAXCOLS = 200;
const NUM EPS = 1e-5;

// F2: bitset<MAXCOLS+1> mat[MAXROWS]; bitset<MAXROWS> vals;
NUM mat[MAXROWS][MAXCOLS + 1], vals[MAXCOLS]; bool
    hasval[MAXCOLS];
bool is0(NUM a) { return -EPS < a && a < EPS; }

```

```

// finds x such that Ax = b
// A_ij is mat[i][j], b_i is mat[i][m]
int solvemmat(int n, int m) {
    // F2: vals.reset();
    int pr = 0, pc = 0;
    while (pc < m) {
        int r = pr, c;
        while (r < n && is0(mat[r][pc])) r++;
        if (r == n) { pc++; continue; }

        // F2: mat[pr] ^= mat[r]; mat[r] ^= mat[pr]; mat[pr] ^=
        // mat[r];
        for (c = 0; c <= m; c++) swap(mat[pr][c], mat[r][c]);
    }
}

```

```

    r = pr++; c = pc++;
    // F2: vals.set(pc, mat[pr][m]);
    NUM div = mat[r][c];
    for (int col = c; col <= m; col++) mat[r][col] /= div;
    REP(row, n) {
        if (row == r) continue;
        // F2: if (mat[row].test(c)) mat[row] ^= mat[r];
        NUM times = -mat[row][c];
        for (int col = c; col <= m; col++)
            mat[row][col] += times * mat[r][col];
    }
} // now mat is in RREF

for (int r = pr; r < n; r++)
    if (!is0(mat[r][n])) return 0;
// F2: return 1;
fill_n(hasval, n, false);
for (int col = 0, row; col < m; col++) {
    hasval[col] = !is0(mat[row][col]);
    if (!hasval[col]) continue;
    for (int c = col + 1; c < m; c++) {
        if (!is0(mat[row][c])) hasval[col] = false;
    }
    if (hasval[col]) vals[col] = mat[row][n];
    row++;
}
REP(i, n) if (!hasval[i]) return 2;
return 1;
}

```

## 6.5. Cycle-Finding.

```

ii find_cycle(int x0, int (*f)(int)) {
    int t = f(x0), h = f(t), mu = 0, lam = 1;
    while (t != h) t = f(t), h = f(f(h));
    h = x0;
    while (t != h) t = f(t), h = f(h), mu++;
    h = f(t);
    while (t != h) h = f(h), lam++;
    return ii(mu, lam); }
// vim: cc=60 ts=2 sts=2 sw=2:

```

## 6.6. Longest Increasing Subsequence.

```

vi lis(vi arr) {
    vi seq, back(size(arr)), ans;
    rep(i, 0, size(arr)) {
        int res = 0, lo = 1, hi = size(seq);
        while (lo <= hi) {
            int mid = (lo+hi)/2;
            if (arr[seq[mid-1]] < arr[i]) res = mid, lo = mid + 1;
            else hi = mid - 1; }
        if (res < size(seq)) seq[res] = i;
        else seq.push_back(i);
        back[i] = res == 0 ? -1 : seq[res-1]; }
    int at = seq.back();
    while (at != -1) ans.push_back(at), at = back[at];
    reverse(ans.begin(), ans.end());
}

```

```

    return ans; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

## 6.7. Dates.

```

int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
    return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
        367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
        3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
        d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
    int x, n, i, j;
    x = jd + 68569;
    n = 4 * x / 146097;
    x -= (146097 * n + 3) / 4;
    i = (4000 * (x + 1)) / 1461001;
    x -= 1461 * i / 4 - 31;
    j = 80 * x / 2447;
    d = x - 2447 * j / 80;
    x = j / 11;
    m = j + 2 - 12 * x;
    y = 100 * (n - 49) + i + x; }
// vim: cc=60 ts=2 sts=2 sw=2:

```

## 6.8. Simplex.

```

typedef long double DOUBLE;
typedef vector<DOUBLE> VD;
typedef vector<VD> VVD;
typedef vector<int> VI;
const DOUBLE EPS = 1e-9;
struct LPSolver {
    int m, n;
    VI B, N;
    VVD D;
    LPSolver(const VVD &A, const VD &b, const VD &c) :
        m(b.size()), n(c.size()),
        N(n + 1), B(m), D(m + 2, VD(n + 2)) {
        for (int i = 0; i < m; i++) for (int j = 0; j < n; j++)
            D[i][j] = A[i][j];
        for (int i = 0; i < m; i++) { B[i] = n + i; D[i][n] = -1;
            D[i][n + 1] = b[i]; }
        for (int j = 0; j < n; j++) { N[j] = j; D[m][j] = -c[j]; }
        N[n] = -1; D[m + 1][n] = 1; }
    void Pivot(int r, int s) {
        double inv = 1.0 / D[r][s];
        for (int i = 0; i < m + 2; i++) if (i != r)
            for (int j = 0; j < n + 2; j++) if (j != s)
                D[i][j] -= D[r][j] * D[i][s] * inv;
        for (int j = 0; j < n + 2; j++) if (j != s) D[r][j] *= inv;
        for (int i = 0; i < m + 2; i++) if (i != r) D[i][s] *=
            -inv;
        D[r][s] = inv;
        swap(B[r], N[s]); }
    bool Simplex(int phase) {
        int x = phase == 1 ? m + 1 : m;
        while (true) {

```



```

int s = -1;
for (int j = 0; j <= n; j++) {
    if (phase == 2 && N[j] == -1) continue;
    if (s == -1 || D[x][j] < D[x][s] ||
        D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
if (D[x][s] > -EPS) return true;
int r = -1;
for (int i = 0; i < m; i++) {
    if (D[i][s] < EPS) continue;
    if (r == -1 || D[i][n+1] / D[i][s] < D[r][n+1] /
        D[r][s] || (D[i][n+1] / D[i][s]) == (D[r][n+1] /
        D[r][s]) && B[i] < B[r]) r = i; }
if (r == -1) return false;
Pivot(r, s); } }
DOUBLE Solve(VD &x) {
    int r = 0;
    for (int i = 1; i < m; i++) if (D[i][n+1] < D[r][n+1])
        r = i;
    if (D[r][n+1] < -EPS) {
        Pivot(r, n);
        if (!Simplex(1) || D[m+1][n+1] < -EPS)
            return -numeric_limits<DOUBLE>::infinity();
        for (int i = 0; i < m; i++) if (B[i] == -1) {
            int s = -1;
            for (int j = 0; j <= n; j++)
                if (s == -1 || D[i][j] < D[i][s] ||
                    D[i][j] == D[i][s] && N[j] < N[s])
                    s = j;
            Pivot(i, s); } }
    if (!Simplex(2)) return numeric_limits<DOUBLE>::infinity();
    x = VD(n);
    for (int i = 0; i < m; i++) if (B[i] < n)
        x[B[i]] = D[i][n+1];
    return D[m][n+1]; } }
// Two-phase simplex algorithm for solving linear programs
// of the form
//      maximize      c^T x
//      subject to    Ax <= b
//                  x >= 0
// INPUT: A -- an m x n matrix
//        b -- an m-dimensional vector
//        c -- an n-dimensional vector
//        x -- a vector where the optimal solution will be
//              stored
// OUTPUT: value of the optimal solution (infinity if
//         unbounded above, nan if infeasible)
// To use this code, create an LPSolver object with A, b,
// and c as arguments. Then, call Solve(x).
// #include <iostream>
// #include <iomanip>
// #include <vector>
// #include <cmath>
// #include <limits>
// using namespace std;
// int main() {
//     const int m = 4;

```

```

//     const int n = 3;
//     DOUBLE _A[m][n] = {
//         { 6, -1, 0 },
//         { -1, -5, 0 },
//         { 1, 5, 1 },
//         { -1, -5, -1 }
//     };
//     DOUBLE _b[m] = { 10, -4, 5, -5 };
//     DOUBLE _c[n] = { 1, -1, 0 };
//     VVD A(m);
//     VD b(_b, _b + m);
//     VD c(_c, _c + n);
//     for (int i = 0; i < m; i++) A[i] = VD(_A[i], _A[i] + n);
//     LPSolver solver(A, b, c);
//     VD x;
//     DOUBLE value = solver.Solve(x);
//     cerr << "VALUE: " << value << endl; // VALUE: 1.29032
//     cerr << "SOLUTION:"; // SOLUTION: 1.74194 0.451613 1
//     for (size_t i = 0; i < x.size(); i++) cerr << " " <<
//         x[i];
//     cerr << endl;
//     return 0;
// }
// vim: cc=60 ts=2 sts=2 sw=2:

```

## 7. GEOMETRY (CP3)

### 7.1. Points and lines.

```

#define INF 1e9
#define EPS 1e-9
#define PI acos(-1.0) // important constant; alternative
// #define PI (2.0 * acos(0.0))

double DEG_to_RAD(double d) { return d * PI / 180.0; }

double RAD_to_DEG(double r) { return r * 180.0 / PI; }

struct point { double x, y; // only used if more precision
// is needed
    point() { x = y = 0.0; } // default
// constructor
    point(double _x, double _y) : x(_x), y(_y) {} //
// user-defined
    bool operator < (point other) const { // override less than
// operator
        if (fabs(x - other.x) > EPS) // useful
            return x < other.x; // first criteria, by
// x-coordinate
        return y < other.y; } // second criteria, by
// y-coordinate
// use EPS (1e-9) when testing equality of two floating
// points
    bool operator == (point other) const {
        return (fabs(x - other.x) < EPS && (fabs(y - other.y) <
// EPS)); } };

```

```

double dist(point p1, point p2) { // Euclidean
// distance
// hypot(dx, dy) returns sqrt(dx * dx +
// dy * dy)
    return hypot(p1.x - p2.x, p1.y - p2.y); } //
// return double

// rotate p by theta degrees CCW w.r.t origin (0, 0)
point rotate(point p, double theta) {
    double rad = DEG_to_RAD(theta); // multiply theta with
// PI / 180.0
    return point(p.x * cos(rad) - p.y * sin(rad),
        p.x * sin(rad) + p.y * cos(rad)); }

struct line { double a, b, c; }; // a way to
// represent a line

// the answer is stored in the third parameter (pass by
// reference)
void pointsToLine(point p1, point p2, line &l) {
    if (fabs(p1.x - p2.x) < EPS) { // vertical
// line is fine
        l.a = 1.0; l.b = 0.0; l.c = -p1.x; //
// default values
    } else {
        l.a = -(double)(p1.y - p2.y) / (p1.x - p2.x);
        l.b = 1.0; // IMPORTANT: we fix the value of
// b to 1.0
        l.c = -(double)(l.a * p1.x) - p1.y;
    } }

bool areParallel(line l1, line l2) { // check
// coefficients a & b
    return (fabs(l1.a-l2.a) < EPS) && (fabs(l1.b-l2.b) < EPS);
// }

bool areSame(line l1, line l2) { // also check
// coefficient c
    return areParallel(l1, l2) && (fabs(l1.c - l2.c) < EPS); }

// returns true (+ intersection point) if two lines are
// intersect
bool areIntersect(line l1, line l2, point &p) {
    if (areParallel(l1, l2)) return false; // no
// intersection
// solve system of 2 linear algebraic equations with 2
// unknowns
    p.x = (l2.b * l1.c - l1.b * l2.c) / (l2.a * l1.b - l1.a *
// l2.b);
// special case: test for vertical line to avoid division
// by zero
    if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
}

```

```

else p.y = -(l2.a * p.x + l2.c);
return true; }

struct vec { double x, y; // name: `vec` is different from
↳ STL vector
vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) { // convert 2 points to
↳ vector a->b
return vec(b.x - a.x, b.y - a.y); }

vec scale(vec v, double s) { // nonnegative s = [<1 ..
↳ 1 .. >1]
return vec(v.x * s, v.y * s); } //
↳ shorter.same.longer

point translate(point p, vec v) { // translate p
↳ according to v
return point(p.x + v.x, p.y + v.y); }

// convert point and gradient/slope to line
void pointSlopeToLine(point p, double m, line &l) {
l.a = -m; //
↳ always -m
l.b = 1; //
↳ always 1
l.c = -((l.a * p.x) + (l.b * p.y)); //
↳ compute this

void closestPoint(line l, point p, point &ans) {
line perpendicular; // perpendicular to l and pass
↳ through p
if (fabs(l.b) < EPS) { // special case 1:
↳ vertical line
ans.x = -(l.c); ans.y = p.y; return; }

if (fabs(l.a) < EPS) { // special case 2:
↳ horizontal line
ans.x = p.x; ans.y = -(l.c); return; }

pointSlopeToLine(p, 1 / l.a, perpendicular); //
↳ normal line
// intersect line l with this perpendicular line
// the intersection point is the closest point
areIntersect(l, perpendicular, ans); }

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
point b;
closestPoint(l, p, b); // similar to
↳ distToLine
vec v = toVec(p, b); // create
↳ a vector
ans = translate(translate(p, v), v); // translate
↳ p twice

```

```

double dot(vec a, vec b) { return (a.x * b.x + a.y * b.y); }

double norm_sq(vec v) { return v.x * v.x + v.y * v.y; }

// returns the distance from p to the line defined by
// two points a and b (a and b must be different)
// the closest point is stored in the 4th parameter (byref)
double distToLine(point p, point a, point b, point &c) {
// formula: c = a + u * ab
vec ap = toVec(a, p), ab = toVec(a, b);
double u = dot(ap, ab) / norm_sq(ab);
c = translate(a, scale(ab, u)); //
↳ translate a to c
return dist(p, c); } // Euclidean distance
↳ between p and c

// returns the distance from p to the line segment ab defined
↳ by
// two points a and b (still OK if a == b)
// the closest point is stored in the 4th parameter (byref)
double distToLineSegment(point p, point a, point b, point &c)
↳ {
vec ap = toVec(a, p), ab = toVec(a, b);
double u = dot(ap, ab) / norm_sq(ab);
if (u < 0.0) { c = point(a.x, a.y); //
↳ closer to a
return dist(p, a); } // Euclidean distance
↳ between p and a
if (u > 1.0) { c = point(b.x, b.y); //
↳ closer to b
return dist(p, b); } // Euclidean distance
↳ between p and b
return distToLine(p, a, b, c); } // run distToLine
↳ as above

double angle(point a, point o, point b) { // returns angle
↳ aob in rad
vec oa = toVec(o, a), ob = toVec(o, b);
return acos(dot(oa, ob) / sqrt(norm_sq(oa) * norm_sq(ob)));
↳ }

double cross(vec a, vec b) { return a.x * b.y - a.y * b.x; }

// note: to accept collinear points, we have to change the `>
↳ 0`
// returns true if point r is on the left side of line pq
bool ccw(point p, point q, point r) {
return cross(toVec(p, q), toVec(p, r)) > 0; }

// returns true if point r is on the same line as the line pq
bool collinear(point p, point q, point r) {
return fabs(cross(toVec(p, q), toVec(p, r))) < EPS; }

```

## 7.2. Polygon.

```

// returns the perimeter, which is the sum of Euclidian
↳ distances
// of consecutive line segments (polygon edges)
double perimeter(const vector<point> &P) {
double result = 0.0;
for (int i = 0; i < (int)P.size()-1; i++) // remember that
↳ P[0] = P[n-1]
result += dist(P[i], P[i+1]);
return result; }

// returns the area, which is half the determinant
double area(const vector<point> &P) {
double result = 0.0, x1, y1, x2, y2;
for (int i = 0; i < (int)P.size()-1; i++) {
x1 = P[i].x; x2 = P[i+1].x;
y1 = P[i].y; y2 = P[i+1].y;
result += (x1 * y2 - x2 * y1);
}
return fabs(result) / 2.0; }

// returns true if we always make the same turn while
↳ examining
// all the edges of the polygon one by one
bool isConvex(const vector<point> &P) {
int sz = (int)P.size();
if (sz <= 3) return false; // a point/sz=2 or a line/sz=3
↳ is not convex
bool isLeft = ccw(P[0], P[1], P[2]); //
↳ remember one result
for (int i = 1; i < sz-1; i++) // then compare
↳ with the others
if (ccw(P[i], P[i+1], P[(i+2) == sz ? 1 : i+2])) !=
↳ isLeft
return false; // different sign -> this
↳ polygon is concave
return true; } // this
↳ polygon is convex

// returns true if point p is in either convex/concave
↳ polygon P
bool inPolygon(point pt, const vector<point> &P) {
if ((int)P.size() == 0) return false;
double sum = 0; // assume the first vertex is equal to
↳ the last vertex
for (int i = 0; i < (int)P.size()-1; i++) {
if (ccw(pt, P[i], P[i+1]))
sum += angle(P[i], pt, P[i+1]); //
↳ left turn/ccw
else sum -= angle(P[i], pt, P[i+1]); } //
↳ right turn/cw
return fabs(fabs(sum) - 2*PI) < EPS; }

// line segment p-q intersect with line A-B.

```

```

point lineIntersectSeg(point p, point q, point A, point B) {
    double a = B.y - A.y;
    double b = A.x - B.x;
    double c = B.x * A.y - A.x * B.y;
    double u = fabs(a * p.x + b * p.y + c);
    double v = fabs(a * q.x + b * q.y + c);
    return point((p.x * v + q.x * u) / (u+v), (p.y * v + q.y *
    ↪ u) / (u+v)); }

// cuts polygon Q along the line formed by point a -> point b
// (note: the last point must be the same as the first point)
vector<point> cutPolygon(point a, point b, const
    ↪ vector<point> &Q) {
    vector<point> P;
    for (int i = 0; i < (int)Q.size(); i++) {
        double left1 = cross(toVec(a, b), toVec(a, Q[i])), left2
        ↪ = 0;
        if (i != (int)Q.size()-1) left2 = cross(toVec(a, b),
        ↪ toVec(a, Q[i+1]));
        if (left1 > -EPS) P.push_back(Q[i]); // Q[i] is on
        ↪ the left of ab
        if (left1 * left2 < -EPS) // edge (Q[i], Q[i+1])
        ↪ crosses line ab
            P.push_back(lineIntersectSeg(Q[i], Q[i+1], a, b));
    }
    if (!P.empty() && !(P.back() == P.front()))
        P.push_back(P.front()); // make P's first point =
        ↪ P's last point
    return P; }

point pivot;
bool angleCmp(point a, point b) { //
    ↪ angle-sorting function
    if (collinear(pivot, a, b))
        ↪ // special case
        return dist(pivot, a) < dist(pivot, b); // check which
        ↪ one is closer
    double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
    double d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0; } //
    ↪ compare two angles

vector<point> CH(vector<point> P) { // the content of P may
    ↪ be reshuffled
    int i, j, n = (int)P.size();
    if (n <= 3) {
        if (!(P[0] == P[n-1])) P.push_back(P[0]); // safeguard
        ↪ from corner case
        return P; // special case, the
        ↪ CH is P itself
    }

    // first, find P0 = point with lowest Y and if tie:
    ↪ rightmost X

```

```

int P0 = 0;
for (i = 1; i < n; i++)
    if (P[i].y < P[P0].y || (P[i].y == P[P0].y && P[i].x >
    ↪ P[P0].x))
        P0 = i;

point temp = P[0]; P[0] = P[P0]; P[P0] = temp; // swap
    ↪ P[P0] with P[0]

// second, sort points by angle w.r.t. pivot P0
pivot = P[0]; // use this global
    ↪ variable as reference
sort(++P.begin(), P.end(), angleCmp); // we do
    ↪ not sort P[0]

// third, the ccw tests
vector<point> S;
S.push_back(P[n-1]); S.push_back(P[0]); S.push_back(P[1]);
    ↪ // initial S
i = 2; // then, we
    ↪ check the rest
while (i < n) { // note: N must be >= 3 for this
    ↪ method to work
        j = (int)S.size()-1;
        if (ccw(S[j-1], S[j], P[i])) S.push_back(P[i++]); //
        ↪ left turn, accept
        else S.pop_back(); } // or pop the top of S until we
        ↪ have a left turn
return S; } //
    ↪ return the result

```

### 7.3. Triangle.

```

double perimeter(double ab, double bc, double ca) {
    return ab + bc + ca; }

double perimeter(point a, point b, point c) {
    return dist(a, b) + dist(b, c) + dist(c, a); }

double area(double ab, double bc, double ca) {
    // Heron's formula, split sqrt(a * b) into sqrt(a) *
    ↪ sqrt(b); in implementation
    double s = 0.5 * perimeter(ab, bc, ca);
    return sqrt(s) * sqrt(s - ab) * sqrt(s - bc) * sqrt(s -
    ↪ ca); }

double area(point a, point b, point c) {
    return area(dist(a, b), dist(b, c), dist(c, a)); }

double rInCircle(double ab, double bc, double ca) {
    return area(ab, bc, ca) / (0.5 * perimeter(ab, bc, ca)); }

double rInCircle(point a, point b, point c) {
    return rInCircle(dist(a, b), dist(b, c), dist(c, a)); }

```

```

// assumption: the required points/lines functions have been
    ↪ written
// returns 1 if there is an inCircle center, returns 0
    ↪ otherwise
// if this function returns 1, ctr will be the inCircle
    ↪ center
// and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point &ctr, double
    ↪ &r) {
    r = rInCircle(p1, p2, p3);
    if (fabs(r) < EPS) return 0; // no
    ↪ inCircle center

    line l1, l2; // compute these two angle
    ↪ bisectors
    double ratio = dist(p1, p2) / dist(p1, p3);
    point p = translate(p2, scale(toVec(p2, p3), ratio / (1 +
    ↪ ratio)));
    pointsToLine(p1, p, l1);

    ratio = dist(p2, p1) / dist(p2, p3);
    p = translate(p1, scale(toVec(p1, p3), ratio / (1 +
    ↪ ratio)));
    pointsToLine(p2, p, l2);

    areIntersect(l1, l2, ctr); // get their
    ↪ intersection point
    return 1; }

double rCircumCircle(double ab, double bc, double ca) {
    return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

double rCircumCircle(point a, point b, point c) {
    return rCircumCircle(dist(a, b), dist(b, c), dist(c, a)); }

// assumption: the required points/lines functions have been
    ↪ written
// returns 1 if there is a circumCenter center, returns 0
    ↪ otherwise
// if this function returns 1, ctr will be the circumCircle
    ↪ center
// and r is the same as rCircumCircle
int circumCircle(point p1, point p2, point p3, point &ctr,
    ↪ double &r){
    double a = p2.x - p1.x, b = p2.y - p1.y;
    double c = p3.x - p1.x, d = p3.y - p1.y;
    double e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
    double f = c * (p1.x + p3.x) + d * (p1.y + p3.y);
    double g = 2.0 * (a * (p3.y - p2.y) - b * (p3.x - p2.x));
    if (fabs(g) < EPS) return 0;

    ctr.x = (d*e - b*f) / g;
    ctr.y = (a*f - c*e) / g;
    r = dist(p1, ctr); // r = distance from center to 1 of the
    ↪ 3 points

```

```
return 1; }

// returns true if point d is inside the circumCircle defined
↳ by a,b,c
int inCircumCircle(point a, point b, point c, point d) {
    return (a.x - d.x) * (b.y - d.y) * ((c.x - d.x) * (c.x -
↳ d.x) + (c.y - d.y) * (c.y - d.y)) +
        (a.y - d.y) * ((b.x - d.x) * (b.x - d.x) + (b.y -
↳ d.y) * (b.y - d.y)) * (c.x - d.x) +
        ((a.x - d.x) * (a.x - d.x) + (a.y - d.y) * (a.y -
↳ d.y)) * (b.x - d.x) * (c.y - d.y) -
        ((a.x - d.x) * (a.x - d.x) + (a.y - d.y) * (a.y -
↳ d.y)) * (b.y - d.y) * (c.x - d.x) -
        (a.y - d.y) * (b.x - d.x) * ((c.x - d.x) * (c.x -
↳ d.x) + (c.y - d.y) * (c.y - d.y)) -
        (a.x - d.x) * ((b.x - d.x) * (b.x - d.x) + (b.y -
↳ d.y) * (b.y - d.y)) * (c.y - d.y) > 0 ? 1 : 0;
}
```

```
bool canFormTriangle(double a, double b, double c) {
    return (a + b > c) && (a + c > b) && (b + c > a); }
```

7.4. Circle.

```
int insideCircle(point_i p, point_i c, int r) { // all
↳ integer version
    int dx = p.x - c.x, dy = p.y - c.y;
    int Euc = dx * dx + dy * dy, rSq = r * r;
    ↳ all integer
    return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
    ↳ //inside/border/outside
```

```
bool circle2PtsRad(point p1, point p2, double r, point &c) {
    double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
        (p1.y - p2.y) * (p1.y - p2.y);
    double det = r * r / d2 - 0.25;
    if (det < 0.0) return false;
    double h = sqrt(det);
    c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
    c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
    return true; } // to get the other center, reverse
↳ p1 and p2
```

#labeled rooted trees

$n^{n-1}$

#labeled unrooted trees

$n^{n-2}$

#forests of  $k$  rooted trees

$\frac{k}{n} \binom{n}{k} n^{n-k}$

$\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$

$\sum_{i=1}^n i^3 = n^2(n+1)^2/4$

$n! = n \times (n-1)! + (-1)^n$

$\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}$

$\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$

$a \equiv b \pmod{x, y} \Rightarrow a \equiv b \pmod{\text{lcm}(x, y)}$

$ac \equiv bc \pmod{m} \Rightarrow a \equiv b \pmod{\frac{m}{\gcd(c, m)}}$

$p \text{ prime} \Leftrightarrow (p-1)! \equiv -1 \pmod{p}$

$\sigma_x(n) = \prod_{i=0}^r \frac{p_i^{(a_i+1)x} - 1}{p_i^x - 1}$

$\sum_{k=0}^m (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$

$2^{\omega(n)} = O(\sqrt{n})$

$d = v_i t + \frac{1}{2} a t^2$

$v_f = v_i + a t$

$n^{n-1}$

$n^{n-2}$

$\frac{k}{n} \binom{n}{k} n^{n-k}$

$\sum_{i=1}^n i^3 = n^2(n+1)^2/4$

$n! = (n-1)!(n-1) + (n-2)!$

$\sum_i \binom{n-i}{i} = F_{n+1}$

$x^k = \sum_{i=0}^k i! \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \binom{x}{i} = \sum_{i=0}^k \left\langle \begin{matrix} k \\ i \end{matrix} \right\rangle \binom{x+i}{k}$

$\sum_{d|n} \phi(d) = n$

$(\sum_{d|n} \sigma_0(d))^2 = \sum_{d|n} \sigma_0(d)^3$

$\gcd(n^a - 1, n^b - 1) = n^{\gcd(a, b)} - 1$

$\sigma_0(n) = \prod_{i=0}^r (a_i + 1)$

$\sum_{i=1}^n 2^{\omega(i)} = O(n \log n)$

$v_f^2 = v_i^2 + 2 a d$

$d = \frac{v_i + v_f}{2} t$

8.1. The Twelvefold Way. Putting  $n$  balls into  $k$  boxes.

Balls	same	distinct	same	distinct	
Boxes	same	same	distinct	distinct	Remarks
-	$p_k(n)$	$\sum_{i=0}^k \left\{ \begin{matrix} n \\ i \end{matrix} \right\}$	$\binom{n+k-1}{k-1}$	$k^n$	$p_k(n)$ : #partitions of $n$ into $\leq k$ positive parts
size $\geq 1$	$p(n, k)$	$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	$\binom{n-1}{k-1}$	$k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	$p(n, k)$ : #partitions of $n$ into $k$ positive parts
size $\leq 1$	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n! \binom{k}{n}$	$[cond]$ : 1 if $cond = true$ , else 0

8. COMBINATORICS

Catalan	$C_0 = 1, C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \frac{4n-2}{n+1} C_{n-1}$	
Stirling 1st kind	$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 0, \begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$	#perms of $n$ objs with exactly $k$ cycles
Stirling 2nd kind	$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1, \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$	#ways to partition $n$ objs into $k$ nonempty sets
Euler	$\langle n \rangle = \left\langle \begin{matrix} n \\ n-1 \end{matrix} \right\rangle = 1, \langle n \rangle = (k+1) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (n-k) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle$	#perms of $n$ objs with exactly $k$ ascents
Euler 2nd Order	$\left\langle\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\right\rangle = (k+1) \left\langle\left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle\right\rangle + (2n-k-1) \left\langle\left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle\right\rangle$	#perms of $1, 1, 2, 2, \dots, n, n$ with exactly $k$ ascents
Bell	$B_1 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	#partitions of $1..n$ (Stirling 2nd, no limit on $k$ )

## 9. USEFUL INFORMATION

## 10. Misc

## 10.1. Debugging Tips.

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
  - Getting NaN? Make sure `acos` etc. are not getting values out of their range (perhaps `1+eps`).
  - Rounding negative numbers?
  - Outputting in scientific notation?
- Wrong Answer?
  - Read the problem statement again!
  - Are multiple test cases being handled correctly? Try repeating the same test case many times.
  - Integer overflow?
  - Think very carefully about boundaries of all input parameters
  - Try out possible edge cases:
    - \*  $n = 0, n = -1, n = 1, n = 2^{31} - 1$  or  $n = -2^{31}$
    - \* List is empty, or contains a single element
    - \*  $n$  is even,  $n$  is odd
    - \* Graph is empty, or contains a single vertex
    - \* Graph is a multigraph (loops or multiple edges)
    - \* Polygon is concave or non-simple
  - Is initial condition wrong for small cases?
  - Are you sure the algorithm is correct?
  - Explain your solution to someone.
  - Are you using any functions that you don't completely understand? Maybe STL functions?
  - Maybe you (or someone else) should rewrite the solution?
  - Can the input line be empty?
- Run-Time Error?
  - Is it actually Memory Limit Exceeded?

## 10.2. Solution Ideas.

- Dynamic Programming
  - Parsing CFGs: CYK Algorithm
  - Drop a parameter, recover from others
  - Swap answer and a parameter
  - When grouping: try splitting in two
  - $2^k$  trick
  - When optimizing
    - \* Convex hull optimization
      - $dp[i] = \min_{j < i} \{dp[j] + b[j] \times a[i]\}$
      - $b[j] \geq b[j + 1]$
      - optionally  $a[i] \leq a[i + 1]$
      - $O(n^2)$  to  $O(n)$
    - \* Divide and conquer optimization
      - $dp[i][j] = \min_{k < j} \{dp[i - 1][k] + C[k][j]\}$
      - $A[i][j] \leq A[i][j + 1]$
      - $O(kn^2)$  to  $O(kn \log n)$
      - sufficient:  $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$ ,  $a \leq b \leq c \leq d$  (QI)
    - \* Knuth optimization
      - $dp[i][j] = \min_{i < k < j} \{dp[i][k] + dp[k][j] + C[i][j]\}$
      - $A[i][j - 1] \leq A[i][j] \leq A[i + 1][j]$
      - $O(n^3)$  to  $O(n^2)$

• sufficient: QI and  $C[b][c] \leq C[a][d]$ ,  $a \leq b \leq c \leq d$

- Greedy
- Randomized
- Optimizations
  - Use bitset (/64)
  - Switch order of loops (cache locality)
- Process queries offline
  - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
  - Mo's algorithm
  - Sqrt decomposition
  - Store  $2^k$  jump pointers
- Data structure techniques
  - Sqrt buckets
  - Store  $2^k$  jump pointers
  - $2^k$  merging trick
- Counting
  - Inclusion-exclusion principle
  - Generating functions
- Graphs
  - Can we model the problem as a graph?
  - Can we use any properties of the graph?
  - Strongly connected components
  - Cycles (or odd cycles)
  - Bipartite (no odd cycles)
    - \* Bipartite matching
    - \* Hall's marriage theorem
    - \* Stable Marriage
  - Cut vertex/bridge
  - Biconnected components
  - Degrees of vertices (odd/even)
  - Trees
    - \* Heavy-light decomposition
    - \* Centroid decomposition
    - \* Least common ancestor
    - \* Centers of the tree
  - Eulerian path/circuit
  - Chinese postman problem
  - Topological sort
  - (Min-Cost) Max Flow
  - Min Cut
    - \* Maximum Density Subgraph
  - Huffman Coding
  - Min-Cost Arborescence
  - Steiner Tree
  - Kirchoff's matrix tree theorem
  - Prüfer sequences
  - Lovász Toggle
  - Look at the DFS tree (which has no cross-edges)
  - Is the graph a DFA or NFA?
    - \* Is it the Synchronizing word problem?
- math
  - Is the function multiplicative?
  - Look for a pattern

- Permutations
  - \* Consider the cycles of the permutation
- Functions
  - \* Sum of piecewise-linear functions is a piecewise-linear function
  - \* Sum of convex (concave) functions is convex (concave)
- Modular arithmetic
  - \* Chinese Remainder Theorem
  - \* Linear Congruence
- Sieve
- System of linear equations
- Values too big to represent?
  - \* Compute using the logarithm
  - \* Divide everything by some large value
- Linear programming
  - \* Is the dual problem easier to solve?
- Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
  - 2-SAT
  - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ( $\log(n)$ )
- Strings
  - Trie (maybe over something weird, like bits)
  - Suffix array
  - Suffix automaton (+DP?)
  - Aho-Corasick
  - `eerTree`
  - Work with  $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
  - Lazy propagation
  - Persistent
  - Implicit
  - Segment tree of X
- Geometry
  - Minkowski sum (of convex sets)
  - Rotating calipers
  - Sweep line (horizontally or vertically?)
  - Sweep angle
  - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem



- Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)
- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

11. FORMULAS

- **Legendre symbol:**  $\left(\frac{a}{b}\right) = a^{(b-1)/2} \pmod{b}$ ,  $b$  odd prime.
- **Heron’s formula:** A triangle with side lengths  $a, b, c$  has area  $\sqrt{s(s-a)(s-b)(s-c)}$  where  $s = \frac{a+b+c}{2}$ .
- **Pick’s theorem:** A polygon on an integer grid strictly containing  $i$  lattice points and having  $b$  lattice points on the boundary has area  $i + \frac{b}{2} - 1$ . (Nothing similar in higher dimensions)
- **Euler’s totient:** The number of integers less than  $n$  that are coprime to  $n$  are  $n \prod_{p|n} \left(1 - \frac{1}{p}\right)$  where each  $p$  is a distinct prime factor of  $n$ .
- **König’s theorem:** In any bipartite graph  $G = (L \cup R, E)$ , the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover. Let  $U$  be the set of unmatched vertices in  $L$ , and  $Z$  be the set of vertices that are either in  $U$  or are connected to  $U$  by an alternating path. Then  $K = (L \setminus Z) \cup (R \cap Z)$  is the minimum vertex cover.
- A minumum Steiner tree for  $n$  vertices requires at most  $n - 2$  additional Steiner vertices.
- The number of vertices of a graph is equal to its minimum vertex cover number plus the size of a maximum independent set.
- **Lagrange polynomial** through points  $(x_0, y_0), \dots, (x_k, y_k)$  is  $L(x) = \sum_{j=0}^k y_j \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}$
- **Hook length formula:** If  $\lambda$  is a Young diagram and  $h_\lambda(i, j)$  is the hook-length of cell  $(i, j)$ , then then the number of Young tableaux  $d_\lambda = n! / \prod h_\lambda(i, j)$ .
- **Möbius inversion formula:** If  $f(n) = \sum_{d|n} g(d)$ , then  $g(n) = \sum_{d|n} \mu(d) f(n/d)$ . If  $f(n) = \sum_{m=1}^n g(\lfloor n/m \rfloor)$ , then  $g(n) = \sum_{m=1}^n \mu(m) f(\lfloor \frac{n}{m} \rfloor)$ .
- #primitive pythagorean triples with hypotenuse  $< n$  approx  $n/(2\pi)$ .
- **Frobenius Number:** largest number which can’t be expressed as a linear combination of numbers  $a_1, \dots, a_n$  with non-negative coefficients.  $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$ ,  $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$ .  $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d - 1)$ . An integer  $x > (\max_i a_i)^2$  can be expressed in such a way iff.  $x \mid \gcd(a_1, \dots, a_n)$ .

11.1. Physics.

- **Snell’s law:**  $\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$

11.2. **Markov Chains.** A Markov Chain can be represented as a weighted directed graph of states, where the weight of an edge represents the probability of transitioning over that edge in one timestep. Let  $P^{(m)} = (p_{ij}^{(m)})$  be the probability matrix of transitioning from state  $i$  to state  $j$  in  $m$  timesteps, and note that  $P^{(1)}$  is the adjacency matrix of the graph. **Chapman-Kolmogorov:**  $p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)}$ . It follows that  $P^{(m+n)} = P^{(m)} P^{(n)}$  and  $P^{(m)} = P^m$ . If  $p^{(0)}$  is the initial probability distribution (a vector), then  $p^{(0)} P^{(m)}$  is the probability distribution after  $m$  timesteps.

The return times of a state  $i$  is  $R_i = \{m \mid p_{ii}^{(m)} > 0\}$ , and  $i$  is *aperiodic* if  $\gcd(R_i) = 1$ . A MC is aperiodic if any of its vertices is aperiodic. A MC is *irreducible* if the corresponding graph is strongly connected. A distribution  $\pi$  is stationary if  $\pi P = \pi$ . If MC is irreducible then  $\pi_i = 1/\mathbb{E}[T_i]$ , where  $T_i$  is the expected time between two visits at  $i$ .  $\pi_j/\pi_i$  is the expected number of visits at  $j$  in between two consecutive visits at  $i$ . A MC is *ergodic* if  $\lim_{m \rightarrow \infty} p^{(0)} P^m = \pi$ . A MC is ergodic iff. it is irreducible and aperiodic.

A MC for a random walk in an undirected weighted graph (un-weighted graph can be made weighted by adding 1-weights) has  $p_{uv} = w_{uv} / \sum_x w_{ux}$ . If the graph is connected, then  $\pi_u = \sum_x w_{ux} / \sum_v \sum_x w_{vx}$ . Such a random walk is aperiodic iff. the graph is not bipartite.

An *absorbing* MC is of the form  $P = \begin{pmatrix} Q & R \\ 0 & I_r \end{pmatrix}$ . Let  $N = \sum_{m=0}^\infty Q^m = (I_t - Q)^{-1}$ . Then, if starting in state  $i$ , the expected number of steps till absorption is the  $i$ -th entry in  $N1$ . If starting in state  $i$ , the probability of being absorbed in state  $j$  is the  $(i, j)$ -th entry of  $NR$ .

Many problems on MC can be formulated in terms of a system of recurrence relations, and then solved using Gaussian elimination.

11.3. **Burnside’s Lemma.** Let  $G$  be a finite group that acts on a set  $X$ . For each  $g$  in  $G$  let  $X^g$  denote the set of elements in  $X$  that are fixed by  $g$ . Then the number of orbits

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

$$Z(S_n) = \frac{1}{n} \sum_{l=1}^n a_l Z(S_{n-l})$$

11.4. **Bézout’s identity.** If  $(x, y)$  is any solution to  $ax + by = d$  (e.g. found by the Extended Euclidean Algorithm), then all solutions are given by

$$\left(x + k \frac{b}{\gcd(a, b)}, y - k \frac{a}{\gcd(a, b)}\right)$$

11.5. Misc.

11.5.1. *Binomial transform.* If  $a_n = \sum_{k=0}^n \binom{n}{k} b_k$ , then  $b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$ .

11.5.2. *Generating functions.* Ordinary (o.g.f.):  $A(x) := \sum_{n=0}^\infty a_n x^n$ . Exponential (e.g.f.):  $A(x) := \sum_{n=0}^\infty a_n x^n / n!$ . o.g.f. convolution:  $c_n = \sum_{k=0}^n a_k b_{n-k}$  (use FFT for  $\mathcal{O}(n \log n)$ ). e.g.f. convolution:  $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$  (use FFT for  $\mathcal{O}(n \log n)$ ).

11.5.3. *General linear recurrences.* If  $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$ , then  $A(x) = \frac{a_0}{1-B(x)}$ . We also can compute all  $a_n$  with Divide-and-Conquer algorithm in  $\mathcal{O}(n \log^2 nn)$ .

11.5.4. *Inverse polynomial modulo  $x^l$ .* Given  $A(x)$ , find  $B(x)$  such that  $A(x)B(x) = 1 + x^l Q(x)$  for some  $Q(x)$ . Step 1: Start with  $B_0(x) = 1/a_0$  Step 2:  $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \pmod{x^{2^{k+1}}}$ .

11.5.5. *Fast subset convolution.* Given array  $a_i$  of size  $2^k$  calculate  $b_i = \sum_{j \& i = i} a_j$ .

```
for (int b = 1; b < (1 << k); b <= 1)
    for (int i = 0; i < (1 << k); i++)
        if (!(i & b)) a[i | b] += a[i];
// inv: if (!(i & b)) a[i | b] -= a[i];
```

11.5.6. *Determinants and PM.*

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}$$

$$\text{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i, \sigma(i)}$$

$$\begin{aligned} pf(A) &= \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)} \\ &= \sum_{M \in \text{PM}(n)} \text{sgn}(M) \prod_{(i,j) \in M} a_{i,j} \end{aligned}$$

11.5.7. *BEST Theorem.* Count directed Eulerian cycles. Number of OST given by Kirchoff’s Theorem (remove r/c with root)  $\# \text{OST}(G, r) \cdot \prod_v (d_v - 1)!$

11.5.8. *Primitive Roots.* Only exists when  $n$  is  $2, 4, p^k, 2p^k$ , where  $p$  odd prime. Assume  $n$  prime. Number of primitive roots  $\phi(\phi(n))$  Let  $g$  be primitive root. All primitive roots are of the form  $g^k$  where  $k, \phi(p)$  are coprime.  $k$ -roots:  $g^{i \cdot \phi(n)/k}$  for  $0 \leq i < k$

11.5.9. *Sum of primes.* For any multiplicative  $f$ :  $S(n, p) = S(n, p - 1) - f(p) \cdot (S(n/p, p - 1) - S(p - 1, p - 1))$

11.5.10. *Floor.*

$$\begin{aligned} \lfloor \lfloor x/y \rfloor / z \rfloor &= \lfloor x/(yz) \rfloor \\ x \% y = x - y \lfloor x/y \rfloor \end{aligned}$$

PRACTICE CONTEST CHECKLIST

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use printf/scanf with long long/long double?
- Are `__int128` and `__float128` available?
- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: `i?python[23]`, `factor`.
- Try submitting with `assert(false)` and `assert(true)`.
- Return-value from `main`.
- Look for directory with sample test cases.
- Make sure printing works.
- Remove this page from the notebook (optionally).