

## TCR

git merge -s octopus solution cup(Ludo Pulles, Reinier Schmiermann, Pim Spelier)

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## Practice Contest Checklist

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**Test script** (usage: ./test.sh A/B/..)

```

g++ -Wall -Wshadow -Wfatal-errors -Wpedantic
↪ -std=c++17 $1.cc || exit
for i in $(ls *.in)
do
    j="$(echo $i | sed 's/.in/.ans/')"
    ./a.out < $i > output
    diff output $j || echo "!!!WA on $i!!!"
done

template.cc
#include <bits/stdc++.h>
using namespace std;

typedef long long ll;
typedef long double ld;
typedef pair<ll, ll> ii;
typedef vector<ll> vi;
typedef vector<vi> vvi;
typedef vector<ii> vii;

#define x first
#define y second
#define pb push_back
#define eb emplace_back
#define rep(i,a,b) for(auto i=(a);i!=(b); ++i)
#define REP(i,n) rep(i,0,n)
#define all(v) (v).begin(), (v).end()
#define rs resize
#define DBG(x) cerr << __LINE__ << ": " << #x << " = " << (x) << endl

template<class T> using min_queue = priority_queue<T,
↪ vector<T>, greater<T>>;
template<class T> int size(const T &x) { return
↪ x.size(); } // copy the ampersand(&)!

template<class T> ostream& operator<<(ostream&
↪ os,vector<T>& v) {
    os << "\n["; for( T& x : v ) os << x << ", "; return
↪ os << " ]";
}

const ld PI = acos(-1.0);

void run() {

}

signed main() {
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    (cout << fixed).precision(18);

```

```

run();
return 0;
}

template.py

from sys import *
n,m = [ int(x) for x in
↳ stdin.readline().rstrip().split() ]
stdout.write( str(n*m)+"\n" )

from itertools import *
for (x,y) in product(range(3),repeat=2):
    stdout.write( str(3*x+y)+" " )
stdout.write( "\n" )
for L in combinations(range(4),2):
    stdout.write( str(L)+" " )
stdout.write( "\n" )

from functools import *
y = reduce( lambda x,y: x+y, map( lambda x: x*x,
↳ range(4) ), -3 )
stdout.write( str(y)+"\n" )

from math import *
stdout.write( "{0:.2f}\n".format(pi) )

```

### 0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Iedereen (incl. Ludo) moet **ALLE** opgaves goed lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zware) test cases.
- Gebruik ll.

### 0.2. Wrong Answer.

- (1) Print de oplossing om te debuggen!
- (2) Kijk naar wellicht makkelijkere problemen.
- (3) Bedenk zelf test cases met **randgevallen**!
- (4) Controleer de **precisie**.
- (5) Controleer op **overflow** (gebruik **OVERAL** ll, ld).  
Kijk naar overflows in tussenantwoorden bij modulo.
- (6) Controleer op **typo's**.
- (7) Loop de voorbeeld test case accuraat langs.
- (8) Controleer op off-by-one-errors (in indices of lus-grenzen)?

**Detecting overflow** This GNU builtin checks for over- and underflow. Result is in res if successful:

```

bool isOverflown =
↳ __builtin_[add|mul|sub]_overflow(a, b, &res);

```

### 0.3. Covering problems.

Minimum edge cover  $\iff$  Maximum independent set

**Matching:** A set of edges without common vertices (*Maximum is the largest such set, maximal is a set which you cannot add more edges to without breaking the property*).

**Minimum Vertex Cover:** A set vertices (cover) such that each edge in the graph is incident to at least one vertex of the set.

**Minimum Edge Cover:** A set of edges (cover) such that every vertex is incident to at least one edge of the set.

**Maximum Independent Set:** A set of vertices in a graph such that no two of them are adjacent.

**König's theorem:** In any bipartite graph, MCBM = MVC = V - MIS.

**A useful identity:**  $\bigoplus_{x=0}^{a-1} x = \{0, a-1, 1, a\}[a \bmod 4]$ .

#### 1. MATHEMATICS

```

int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }

// greatest common divisor
ll gcd(ll a, ll b) { while(b) a%=b, swap(a,b); return a; };
// least common multiple
ll lcm(ll a, ll b) { return a/gcd(a, b)*b; }
ll mod(ll a, ll b) { return (a % b) < 0 ? a+b : a; }

// a*b % m for m <= 4e18 in O(log b)
ll mod_mul(ll a, ll b, ll m) {
    ll r = 0;
    while(b) {
        if (b & 1) r = mod(r+a,m);
        a = mod(a+a,m); b >>= 1;
    }
    return r;
}

// a^b % m for m <= 2e9 in O(log b)
ll mod_pow(ll a, ll b, ll m) {
    ll r = 1;
    while(b) {
        if (b & 1) r = (r * a) % m; // mod_mul
        a = (a * a) % m; // mod_mul
        b >>= 1;
    }
    return r;
}

// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
    ll xx = y = 0, yy = x = 1;
    while (b) {
        x -= a / b * xx; swap(x, xx);
        y -= a / b * yy; swap(y, yy);
        a %= b; swap(a, b);
    }
    return a;
}

// Chinese Remainder Theorem: returns (u, v) s.t.
// x=u (mod v) <=> x=a (mod n) and x=b (mod m)
pair<ll, ll> crt(ll a, ll n, ll b, ll m) { //n,m<=1e9
    ll s, t, d = egcd(n, m, s, t);

```

```

if (mod(a - b, d)) return { 0, -1 };
return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}

// phi[i] = #{ 0 < j <= i | gcd(i, j) = 1 } sieve
vi totient(int N) {
    vi phi(N);
    for (int i = 0; i < N; i++) phi[i] = i;
    for (int i = 2; i < N; i++) if (phi[i] == i)
        for (int j = i; j < N; j+=i) phi[j] -= phi[j]/i;
    return phi;
}

// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
    ll ans = 1;
    while (n) {
        ll np = n % p, kp = k % p;
        if (np < kp) return 0;
        ans = mod(ans * binom(np, kp), p); // (np C kp)
        n /= p; k /= p;
    }
    return ans;
}

// returns if n is prime for n < 3e24 (>2^64)
// but use mul_mod for n > 2e9.
bool millerRabin(ll n){
    if (n < 2 || n % 2 == 0) return n == 2;
    ll d = n - 1, ad, s = 0, r;
    for (; d % 2 == 0; d /= 2) s++;
    for (int a : { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 }) {
        if (n == a) return true;
        if ((ad = mod_pow(a, d, n)) == 1) continue;
        for (r = 0; r < s && ad + 1 != n; r++)
            ad = (ad * ad) % n;
        if (r == s) return false;
    }
    return true;
}

1.1. Primitive Root  $O(\sqrt{m})$ . Returns a generator of  $\mathbb{F}_m^*$ . If  $m$  not prime, replace  $m-1$  by totient of  $m$ .
ll primitive_root(ll m) {
    vector<ll> div;
    for (ll i = 1; i*i < m; i++) {
        if ((m-1) % i == 0) {
            if (i < m-1) div.pb(i);
            if ((m-1)/i < m) div.pb((m-1)/i); } }
    rep(x,2,m) {
        bool ok = true;
        for (ll d : div)
            if (mod_pow(x, d, m) == 1) {
                ok = false; break; }
        if (ok) return x; }
    return -1; }

```

**1.2. Tonelli-Shanks algorithm.** Given prime  $p$  and integer  $1 \leq n < p$ , returns the square root  $r$  of  $n$  modulo  $p$ . There is also another solution given by  $-r$  modulo  $p$ .

```
11 legendre(11 a, 11 p) {
    if (a % p == 0) return 0;
    if (p == 2) return 1;
    return mod_pow(a, (p-1)/2, p) == 1 ? 1 : -1; }
11 tonelli_shanks(11 n, 11 p) {
    assert(legendre(n,p) == 1);
    if (p == 2) return 1;
    11 s = 0, q = p-1, z = 2;
    while (~q & 1) s++, q >>= 1;
    if (s == 1) return mod_pow(n, (p+1)/4, p);
    while (legendre(z,p) != -1) z++;
    11 c = mod_pow(z, q, p),
        r = mod_pow(n, (q+1)/2, p),
        t = mod_pow(n, q, p),
        m = s;
    while (t != 1) {
        11 i = 1, ts = (11)t*t % p;
        while (ts != 1) i++, ts = ((11)ts * ts) % p;
        11 b = mod_pow(c, 1LL<<(m-i-1), p);
        r = (11)r * b % p;
        t = (11)t * b % p * b % p;
        c = (11)b * b % p;
        m = i; }
    return r; }
```

**1.3. Numeric Integration.** Numeric integration using Simpson's rule.

```
1d numint(1d (*f)(1d), 1d a, 1d b, 1d EPS = 1e-6) {
    1d ba = b - a, m=(a+b)/2;
    return abs(ba) < EPS ?
        ↪ ba/8*(f(a)+f(b)+f(a+ba/3)*3+f(b-ba/3)*3)
        : numint(f,a,m,EPS) + numint(f,m,b,EPS);
}
```

**1.4. Fast Hadamard Transform.** Computes the Hadamard transform of the given array. Can be used to compute the XOR-convolution of arrays, exactly like with FFT. For AND-convolution, replace  $(x - y, x + y), ((x + y)/2, (-x + y)/2)$  with  $(x + y, y), (x - y, y)$ . For OR-convolution, use  $(x, x + y)$  and  $(x, -x + y)$ . **Note:** Size of array must be a power of 2.

```
void fht(vi &arr, bool inv=false, int l, int r) {
    if (l+1 == r) return;
    int k = (r-l)/2;
    if (!inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k,
        ↪ r);
    rep(i,l,l+k) { int x = arr[i], y = arr[i+k];
        if (!inv) arr[i] = x-y, arr[i+k] = x+y;
        else arr[i] = (x+y)/2, arr[i+k] = (-x+y)/2; }
    if (inv) fht(arr, inv, l, l+k), fht(arr, inv, l+k,
        ↪ r); }
```

**1.5. Tridiagonal Matrix Algorithm.** Solves a tridiagonal system of linear equations  $a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$  where  $a_1 = c_n = 0$ . Beware of numerical instability.

```
#define MAXN 5000
1d A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
    C[0] /= B[0]; D[0] /= B[0];
    rep(i,1,n-1) C[i] /= B[i] - A[i]*C[i-1];
    rep(i,1,n) D[i] = (D[i] - A[i]*D[i-1]) / (B[i] -
        ↪ A[i]*C[i-1]);
    X[n-1] = D[n-1];
    for (int i = n-1; i--;)
        X[i] = D[i] - C[i] * X[i+1]; }
```

**1.6. Josephus problem.** Last man standing out of  $n$  if every  $k$ th is killed. Zero-based, and does not kill 0 on first pass.

```
int J(int n, int k) {
    if (n == 1 || k == 1) return n-1;
    if (n < k) return (J(n-1,k)+k)%n;
    int np = n - n/k;
    return k*((J(np,k)+np-n*k*np)%np) / (k-1); }
```

**1.7. Number of Integer Points under Line.** Count the number of integer solutions to  $Ax + By \leq C$ ,  $0 \leq x \leq n$ ,  $0 \leq y$ . In other words, evaluate the sum  $\sum_{x=0}^n \left\lfloor \frac{C-Ax}{B} + 1 \right\rfloor$ . To count all solutions, let  $n = \left\lfloor \frac{C}{a} \right\rfloor$ . In any case, it must hold that  $C - nA \geq 0$ . Be very careful about overflows.

```
11 floor_sum(11 n, 11 a, 11 b, 11 c) {
    if (c == 0) return 1;
    if (c < 0) return 0;
    if (a % b == 0) return (n+1)*(c/b+1)-n*(n+1)/2*a/b;
    if (a >= b) return
        ↪ floor_sum(n,a%b,b,c)-a/b*n*(n+1)/2;
    11 t = (c-a*n+b)/b;
    return floor_sum((c-b*t)/b,b,a,c-b*t)+t*(n+1); }
```

**1.8. Misc.**

• **Prime numbers:**

1031, 32771, 1048583, 8125344, 33554467, 9982451653, 1073741827, 34359738421, 1099511627791, 35184372088891, 1125899906842679, 36028797018963971.

• **Generating functions:** Ordinary (ogf):  $A(x) := \sum_{n=0}^{\infty} a_n x^n$ . Calculate product  $c_n = \sum_{k=0}^n a_k b_{n-k}$  with FFT. Exponential (e.g.f.):  $A(x) := \sum_{n=0}^{\infty} a_n x^n / n!$ ,  $c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$  (use FFT).

• **General linear recurrences:** If  $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$ , then  $A(x) = \frac{a_0}{1-B(x)}$ .

• **Inverse polynomial modulo  $x^l$ :** Given  $A(x)$ , find  $B(x)$  such that  $A(x)B(x) = 1 + x^l Q(x)$  for some  $Q(x)$ .

Step 1: Start with  $B_0(x) = 1/a_0$

Step 2:  $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \bmod x^{2^{k+1}}$ .

• **Fast subset convolution:** Given array  $a_i$  of size  $2^k$  calculate  $b_i = \sum_{j \& i = i} a_j$ .

```
for (int b = 1; b < (1 << k); b <= 1)
    for (int i = 0; i < (1 << k); i++)
```

```
if (!(i & b)) a[i | b] += a[i];
// inv: if (!(i & b)) a[i | b] -= a[i];
```

• **Primitive Roots:** It only exists when  $n$  is  $2, 4, p^k, 2p^k$ , where  $p$  odd prime. If  $g$  is a primitive root, all primitive roots are of the form  $g^k$  where  $k, \phi(p)$  are coprime (hence there are  $\phi(\phi(p))$  primitive roots).

$\leq N$	$10^3$	$10^6$	$10^9$	$10^{12}$	$10^{18}$
$m$	840	720720	735134400	963761198400	
$d(m)$	32	240	1344	6270	103680

For  $n = 10^{18}$ ,  $m = 897612484786617600$ .

## 2. DATASTRUCTURES

**2.1. Order tree.**

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
template<class TK, class TM> using order_tree =
    ↪ tree<TK, TM, greater<TK>, rb_tree_tag,
    ↪ tree_order_statistics_node_update>;
template<class TK> using order_set =
    ↪ order_tree<TK, null_type>;
```

```
vi s;
order_set<ii> t;
void update( 11 k, 11 v ) {
    t.erase( ii{ s[k], k } );
    s[k] = v;
    t.insert( ii{ s[k], k } );
}
```

```
signed main() {
    11 n = 4;
    s.resize(n,0);
    rep(i,0,n) t.insert(ii{0,i});
    update( 2, 3 );
    cout << t.find_by_order( 2 )->y << endl;
    cout << t.order_of_key( ii{s[3],3} ) << endl;
}
```

**2.2. Segment tree  $\mathcal{O}(\log n)$ . Standard segment tree**

```
typedef /* Tree element */ S;
const int n = 1 << 20; S t[2 * n];

// required axiom: associativity
S combine(S l, S r) { return l + r; } // sum segment
↪ tree
S combine(S l, S r) { return max(l, r); } // max
↪ segment tree
```

```
void build() { for (int i = n; --i; ) t[i] =
    ↪ combine(t[2 * i], t[2 * i + 1]); }
```

```
// set value v on position i
void update(int i, S v) { for (t[i += n] = v; i /= 2;
    ↪ ) t[i] = combine(t[2 * i], t[2 * i + 1]); }
```

```
// sum on interval [l, r]
S query(int l, int r) {
    S resL = 0, resR = 0;
    for (l += n, r += n; l < r; l /= 2, r /= 2) {
        if (l & 1) resL = combine(resL, t[l++]);
        if (r & 1) resR = combine(t[--r], resR);
    }
    return combine(resL, resR);
}
```

### Lazy segment tree

Be careful: all intervals are right-closed  $[\ell, r]$ .

```
struct node {
    int l, r, x, lazy;
    node() {}
    node(int _l, int _r) : l(_l), r(_r), x(INT_MAX),
        lazy(0) {}
    node(int _l, int _r, int _x) : node(_l, _r) { x =
        _x; }
    node(node a, node b) : node(a.l, b.r) { x = min(a.x,
        b.x); }
    void update(int v) { x = v; }
    void range_update(int v) { lazy = v; }
    void apply() { x += lazy; lazy = 0; }
    void push(node &u) { u.lazy += lazy; };
};

struct segment_tree {
    int n;
    vector<node> arr;
    segment_tree() {}
    segment_tree(const vector<ll> &a) : n(size(a)),
        arr(4*n) {
        mk(a, 0, 0, n-1); }
    node mk(const vector<ll> &a, int i, int l, int r) {
        int m = (l+r)/2;
        return arr[i] = l > r ? node(l, r) :
            l == r ? node(l, r, a[l]) :
            node(mk(a, 2*i+1, l, m), mk(a, 2*i+2, m+1, r)); }
    node update(int at, ll v, int i=0) {
        propagate(i);
        int hl = arr[i].l, hr = arr[i].r;
        if (at < hl || hr < at) return arr[i];
        if (hl == at && at == hr) {
            arr[i].update(v); return arr[i]; }
        return arr[i] =
            node(update(at, v, 2*i+1), update(at, v, 2*i+2)); }
    node query(int l, int r, int i=0) {
        propagate(i);
        int hl = arr[i].l, hr = arr[i].r;
        if (r < hl || hr < l) return node(hl, hr);
        if (l <= hl && hr <= r) return arr[i];
        return node(query(l, r, 2*i+1), query(l, r, 2*i+2)); }
    node range_update(int l, int r, ll v, int i=0) {
        propagate(i);
        int hl = arr[i].l, hr = arr[i].r;
        if (r < hl || hr < l) return arr[i];
        if (l <= hl && hr <= r)
```

```
return arr[i].range_update(v), propagate(i),
    arr[i];
return arr[i] = node(range_update(l, r, v, 2*i+1),
    range_update(l, r, v, 2*i+2)); }
void propagate(int i) {
    if (arr[i].l < arr[i].r)
        arr[i].push(arr[2*i+1]),
        arr[i].push(arr[2*i+2]);
    arr[i].apply(); }
};
```

### Persistent segment tree

Be careful: all intervals are right-closed  $[\ell, r]$ , including build.

```
int segcnt = 0;
struct segment {
    int l, r, lid, rid, sum;
} segs[2000000];
int build(int l, int r) {
    if (l > r) return -1;
    int id = segcnt++;
    segs[id].l = l;
    segs[id].r = r;
    if (l == r) segs[id].lid = -1, segs[id].rid = -1;
    else {
        int m = (l + r) / 2;
        segs[id].lid = build(l, m);
        segs[id].rid = build(m + 1, r); }
    segs[id].sum = 0;
    return id; }
int update(int idx, int v, int id) {
    if (id == -1) return -1;
    if (idx < segs[id].l || idx > segs[id].r) return
        id;
    int nid = segcnt++;
    segs[nid].l = segs[id].l;
    segs[nid].r = segs[id].r;
    segs[nid].lid = update(idx, v, segs[id].lid);
    segs[nid].rid = update(idx, v, segs[id].rid);
    segs[nid].sum = segs[id].sum + v;
    return nid; }
int query(int id, int l, int r) {
    if (r < segs[id].l || segs[id].r < l) return 0;
    if (l <= segs[id].l && segs[id].r <= r) return
        segs[id].sum;
    return query(segs[id].lid, l, r)
        + query(segs[id].rid, l, r); }
```

### 2.3. Binary Indexed Tree $\mathcal{O}(\log n)$ . Use one-based indices ( $i > 0$ )!

```
struct BIT {
    int n;
    vector<ll> A;

    BIT(int _n) : n(_n), A(n, 0) {}
    // A[i] += v
    void update(int i, ll v) {
        while (i < n) A[i] += v, i += i & -i;
    }
    // returns sum_{0 < j <= i} A[j]
    ll query(int i) {
```

```
ll v = 0; while (i > 0) v += A[i], i -= i & -i;
    return v;
};

Use this if you add things, which depend on i:
struct fenwick_tree {
    int n; vi data;
    fenwick_tree(int _n) : n(_n), data(vi(n)) {}
    void update(int at, int by) {
        while (at < n) data[at] += by, at |= at + 1; }
    int query(int at) {
        int res = 0;
        while (at >= 0) res += data[at], at = (at & (at +
            1)) - 1;
        return res; }
    int rsq(int a, int b) { return query(b) - query(a -
        1); }
};

struct fenwick_tree_sq {
    int n; fenwick_tree x1, x0;
    fenwick_tree_sq(int _n) : n(_n),
        x1(fenwick_tree(n)),
        x0(fenwick_tree(n)) {}
    // insert f(y) = my + c if x <= y
    void update(int x, int m, int c) {
        x1.update(x, m); x0.update(x, c); }
    int query(int x) { return x*x1.query(x) +
        x0.query(x); }
};

void range_update(fenwick_tree_sq &s, int a, int b,
    int k) {
    s.update(a, k, k * (1 - a)); s.update(b+1, -k, k *
        b); }
int range_query(fenwick_tree_sq &s, int a, int b) {
    return s.query(b) - s.query(a-1); }

2.4. Disjoint-Set / Union-Find  $\mathcal{O}(\alpha(n))$ .
struct dsu {
    vi par, rnk;
    dsu(int n) : par(n, -1), rnk(n, 0) {}
    int find(int i) { return
        par[i] < 0 ? i : par[i] = find(par[i]); }
    void unite(int a, int b) {
        if ((a = find(a)) == (b = find(b))) return;
        if (rnk[a] < rnk[b]) swap(a, b);
        if (rnk[a] == rnk[b]) rnk[a]++;
        par[a] += par[b]; par[b] = a;
    }
};

2.5. Cartesian tree.
struct node {
    int x, y, sz;
    node *l, *r;
    node(int _x, int _y)
        : x(_x), y(_y), sz(1), l(NULL), r(NULL) {} };
int tsize(node* t) { return t ? t->sz : 0; }
```

```

void augment(node *t) {
    t->sz = 1 + tsize(t->l) + tsize(t->r); }
pair<node*,node*> split(node *t, int x) {
    if (!t) return make_pair((node*)NULL, (node*)NULL);
    if (t->x < x) {
        pair<node*,node*> res = split(t->r, x);
        t->r = res.first; augment(t);
        return make_pair(t, res.second); }
    pair<node*,node*> res = split(t->l, x);
    t->l = res.second; augment(t);
    return make_pair(res.first, t); }
node* merge(node *l, node *r) {
    if (!l) return r; if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r); augment(l); return l; }
    r->l = merge(l, r->l); augment(r); return r; }
node* find(node *t, int x) {
    while (t) {
        if (x < t->x) t = t->l;
        else if (t->x < x) t = t->r;
        else return t; }
    return NULL; }
node* insert(node *t, int x, int y) {
    if (find(t, x) != NULL) return t;
    pair<node*,node*> res = split(t, x);
    return merge(res.first,
        merge(new node(x, y), res.second)); }
node* erase(node *t, int x) {
    if (!t) return NULL;
    if (t->x < x) t->r = erase(t->r, x);
    else if (x < t->x) t->l = erase(t->l, x);
    else { node *old = t; t = merge(t->l, t->r); delete
        ↪ old; }
    if (t) augment(t); return t; }
int kth(node *t, int k) {
    if (k < tsize(t->l)) return kth(t->l, k);
    else if (k == tsize(t->l)) return t->x;
    else return kth(t->r, k - tsize(t->l) - 1); }

```

2.6. **Heap.** An implementation of a binary heap.

```

#define RESIZE
#define SWP(x,y) tmp = x, x = y, y = tmp
struct default_int_cmp {
    default_int_cmp() {}
    bool operator ()(const int &a, const int &b) {
        return a < b; } };
template <class Compare = default_int_cmp> struct
    ↪ heap {
    int len, count, *q, *loc, tmp;
    Compare _cmp;
    inline bool cmp(int i, int j) { return _cmp(q[i],
        ↪ q[j]); }
    inline void swp(int i, int j) {
        SWP(q[i], q[j]), SWP(loc[q[i]], loc[q[j]]); }
    void swim(int i) {
        while (i > 0) {
            int p = (i - 1) / 2;
            if (!cmp(i, p)) break;

```

```

        swp(i, p), i = p; } }
void sink(int i) {
    while (true) {
        int l = 2*i + 1, r = l + 1;
        if (l >= count) break;
        int m = r >= count || cmp(l, r) ? l : r;
        if (!cmp(m, i)) break;
        swp(m, i), i = m; } }
heap(int init_len = 128)
    : count(0), len(init_len), _cmp(Compare()) {
    q = new int[len], loc = new int[len];
    memset(loc, 255, len << 2); }
~heap() { delete[] q; delete[] loc; }
void push(int n, bool fix = true) {
    if (len == count || n >= len) {
#ifdef RESIZE
        int newlen = 2 * len;
        while (n >= newlen) newlen *= 2;
        int *newq = new int[newlen], *newloc = new
            ↪ int[newlen];
        rep(i,0,len) newq[i] = q[i], newloc[i] =
            ↪ loc[i];
        memset(newloc + len, 255, (newlen - len) << 2);
        delete[] q, delete[] loc;
        loc = newloc, q = newq, len = newlen;
#else
        assert(false);
#endif
    }
    assert(loc[n] == -1);
    loc[n] = count, q[count++] = n;
    if (fix) swim(count-1); }
void pop(bool fix = true) {
    assert(count > 0);
    loc[q[0]] = -1, q[0] = q[--count], loc[q[0]] = 0;
    if (fix) sink(0);
}
int top() { assert(count > 0); return q[0]; }
void heapify() { for (int i = count - 1; i > 0;
    ↪ i--)
    if (cmp(i, (i - 1) / 2)) swp(i, (i - 1) / 2); }
void update_key(int n) {
    assert(loc[n] != -1), swim(loc[n]), sink(loc[n]);
    ↪ }
bool empty() { return count == 0; }
int size() { return count; }
void clear() { count = 0, memset(loc, 255, len <<
    ↪ 2); } };

```

2.7. **Dancing Links.** An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```

template <class T>
struct dancing_links {
    struct node {
        T item;
        node *l, *r;

```

```

        node(const T &_item, node *_l = NULL, node *_r =
            ↪ NULL)
            : item(_item), l(_l), r(_r) {
                if (l) l->r = this;
                if (r) r->l = this; } };
node *front, *back;
dancing_links() { front = back = NULL; }
node *push_back(const T &item) {
    back = new node(item, back, NULL);
    if (!front) front = back;
    return back; }
node *push_front(const T &item) {
    front = new node(item, NULL, front);
    if (!back) back = front;
    return front; }
void erase(node *n) {
    if (!n->l) front = n->r; else n->l->r = n->r;
    if (!n->r) back = n->l; else n->r->l = n->l; }
void restore(node *n) {
    if (!n->l) front = n; else n->l->r = n;
    if (!n->r) back = n; else n->r->l = n; } };

```

2.8. **Misof Tree.** A simple tree data structure for inserting, erasing, and querying the  $n$ th largest element.

```

#define BITS 15
struct misof_tree {
    int cnt[BITS][1<<BITS];
    misof_tree() { memset(cnt, 0, sizeof(cnt)); }
    void insert(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]++, x >>=
            ↪ 1); }
    void erase(int x) {
        for (int i = 0; i < BITS; cnt[i++][x]--, x >>=
            ↪ 1); }
    int nth(int n) {
        int res = 0;
        for (int i = BITS-1; i >= 0; i--)
            if (cnt[i][res <= 1] <= n) n -= cnt[i][res],
                ↪ res |= 1;
        return res; } };

```

2.9.  **$k$ -d Tree.** A  $k$ -dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```

#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
    struct pt {
        double coord[K];
        pt() {}
        pt(double c[K]) { rep(i,0,K) coord[i] = c[i]; }
        double dist(const pt &other) const {
            double sum = 0.0;
            rep(i,0,K) sum += pow(coord[i] -
                ↪ other.coord[i], 2.0);
            return sqrt(sum); } };
    struct cmp {

```



```

int c;
cmp(int _c) : c(_c) {}
bool operator () (const pt &a, const pt &b) {
    for (int i = 0, cc; i <= K; i++) {
        cc = i == 0 ? c : i - 1;
        if (abs(a.coord[cc] - b.coord[cc]) > EPS)
            return a.coord[cc] < b.coord[cc];
    }
    return false; } };
struct bb {
    pt from, to;
    bb(pt _from, pt _to) : from(_from), to(_to) {}
    double dist(const pt &p) {
        double sum = 0.0;
        rep(i, 0, K) {
            if (p.coord[i] < from.coord[i])
                sum += pow(from.coord[i] - p.coord[i],
                    ↪ 2.0);
            else if (p.coord[i] > to.coord[i])
                sum += pow(p.coord[i] - to.coord[i], 2.0);
        }
        return sqrt(sum); }
    bb bound(double l, int c, bool left) {
        pt nf(from.coord), nt(to.coord);
        if (left) nt.coord[c] = min(nt.coord[c], l);
        else nf.coord[c] = max(nf.coord[c], l);
        return bb(nf, nt); } };
struct node {
    pt p; node *l, *r;
    node(pt _p, node *_l, node *_r)
        : p(_p), l(_l), r(_r) { } };
node *root;
// kd_tree() : root(NULL) { }
kd_tree(vector<pt> pts) {
    root = construct(pts, 0, size(pts) - 1, 0);
    node* construct(vector<pt> &pts, int from, int to,
    ↪ int c) {
        if (from > to) return NULL;
        int mid = from + (to - from) / 2;
        nth_element(pts.begin() + from, pts.begin() +
        ↪ mid,
            pts.begin() + to + 1, cmp(c));
        return new node(pts[mid],
            construct(pts, from, mid - 1, INC(c)),
            construct(pts, mid + 1, to, INC(c))); }
    bool contains(const pt &p) { return _con(p, root,
    ↪ 0); }
    bool _con(const pt &p, node *n, int c) {
        if (!n) return false;
        if (cmp(c)(p, n->p)) return _con(p, n->l,
        ↪ INC(c));
        if (cmp(c)(n->p, p)) return _con(p, n->r,
        ↪ INC(c));
        return true; }
    void insert(const pt &p) { _ins(p, root, 0); }
    void _ins(const pt &p, node* &n, int c) {
        if (!n) n = new node(p, NULL, NULL);

```

```

        else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
        else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c));
        ↪ }
    void clear() { _clr(root); root = NULL; }
    void _clr(node *n) {
        if (n) _clr(n->l), _clr(n->r), delete n; }
    pt nearest_neighbour(const pt &p, bool
    ↪ allow_same=true) {
        assert(root);
        double mn = INFINITY, cs[K];
        rep(i, 0, K) cs[i] = -INFINITY;
        pt from(cs);
        rep(i, 0, K) cs[i] = INFINITY;
        pt to(cs);
        return _nn(p, root, bb(from, to), mn, 0,
        ↪ allow_same).first;
    }
    pair<pt, bool> _nn(const pt &p, node *n, bb b,
        double &mn, int c, bool same) {
        if (!n || b.dist(p) > mn) return make_pair(pt(),
        ↪ false);
        bool found = same || p.dist(n->p) > EPS,
            l1 = true, l2 = false;
        pt resp = n->p;
        if (found) mn = min(mn, p.dist(resp));
        node *n1 = n->l, *n2 = n->r;
        rep(i, 0, 2) {
            if (i == 1 || cmp(c)(n->p, p))
                swap(n1, n2), swap(l1, l2);
            pair<pt, bool> res = _nn(p, n1,
                b.bound(n->p.coord[c], c, l1), mn, INC(c),
                ↪ same);
            if (res.second &&
                (!found || p.dist(res.first) <
                ↪ p.dist(resp)))
                resp = res.first, found = true;
        }
        return make_pair(resp, found); } };

```

2.10. Sqrt Decomposition. Design principle that supports many operations in amortized  $\sqrt{n}$  per operation.

```

struct segment {
    vi arr;
    segment(vi _arr) : arr(_arr) { } };
vector<segment> T;
int K;
void rebuild() {
    int cnt = 0;
    rep(i, 0, size(T))
        cnt += size(T[i].arr);
    K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
    vi arr(cnt);
    for (int i = 0, at = 0; i < size(T); i++)
        rep(j, 0, size(T[i].arr))
            arr[at++] = T[i].arr[j];
    T.clear();
    for (int i = 0; i < cnt; i += K)

```

```

        T.push_back(segment(vi(arr.begin()+i,
            arr.begin()+min(i+K,
            ↪ cnt)))); }
    int split(int at) {
        int i = 0;
        while (i < size(T) && at >= size(T[i].arr))
            at -= size(T[i].arr), i++;
        if (i >= size(T)) return size(T);
        if (at == 0) return i;
        T.insert(T.begin() + i + 1,
            segment(vi(T[i].arr.begin() + at,
            ↪ T[i].arr.end())));
        T[i] = segment(vi(T[i].arr.begin(),
            ↪ T[i].arr.begin() + at));
        return i + 1; }
    void insert(int at, int v) {
        vi arr; arr.push_back(v);
        T.insert(T.begin() + split(at), segment(arr)); }
    void erase(int at) {
        int i = split(at); split(at + 1);
        T.erase(T.begin() + i); }

```

2.11. Monotonic Queue. A queue that supports querying for the minimum element. Useful for sliding window algorithms.

```

struct min_stack {
    stack<int> S, M;
    void push(int x) {
        S.push(x);
        M.push(M.empty() ? x : min(M.top(), x)); }
    int top() { return S.top(); }
    int mn() { return M.top(); }
    void pop() { S.pop(); M.pop(); }
    bool empty() { return S.empty(); } };
struct min_queue {
    min_stack inp, outp;
    void push(int x) { inp.push(x); }
    void fix() {
        if (outp.empty()) while (!inp.empty())
            outp.push(inp.top(), inp.pop());
    }
    int top() { fix(); return outp.top(); }
    int mn() {
        if (inp.empty()) return outp.mn();
        if (outp.empty()) return inp.mn();
        return min(inp.mn(), outp.mn()); }
    void pop() { fix(); outp.pop(); }
    bool empty() { return inp.empty() && outp.empty(); }
    ↪ } };

```

2.12. Convex Hull Trick (replace  $O(n^2)$  by respectively  $O(n)$  and  $O(n \log n)$ ). If converting to integers, look out for division by 0 and  $\pm\infty$ .

```

struct convex_hull_trick {
    vector<pair<double, double>> > h;
    double intersect(int i) {
        return (h[i+1].second-h[i].second) /
            (h[i].first-h[i+1].first); }
    void add(double m, double b) {

```

```

h.push_back(make_pair(m,b));
while (size(h) >= 3) {
    int n = size(h);
    if (intersect(n-3) < intersect(n-2)) break;
    swap(h[n-2], h[n-1]);
    h.pop_back(); } }
double get_min(double x) {
    int lo = 0, hi = size(h) - 2, res = -1;
    while (lo <= hi) {
        int mid = lo + (hi - lo) / 2;
        if (intersect(mid) <= x) res = mid, lo = mid + 1;
        else hi = mid - 1; }
    return h[res+1].first * x + h[res+1].second; } };

```

And dynamic variant:

```

const ll is_query = -(1LL<<62);
struct Line {
    ll m, b;
    mutable function<const Line*> succ;
    bool operator<(const Line& rhs) const {
        if (rhs.b != is_query) return m < rhs.m;
        const Line* s = succ();
        if (!s) return 0;
        ll x = rhs.m;
        return b - s->b < (s->m - m) * x; } };
// will maintain upper hull for maximum
struct HullDynamic : public multiset<Line> {
    bool bad(iterator y) {
        auto z = next(y);
        if (y == begin()) {
            if (z == end()) return 0;
            return y->m == z->m && y->b <= z->b; }
        auto x = prev(y);
        if (z == end()) return y->m == x->m && y->b <=
            x->b;
        return (x->b - y->b) * (z->m - y->m) >=
            (y->b - z->b) * (y->m - x->m); }
    void insert_line(ll m, ll b) {
        auto y = insert({ m, b });
        y->succ = [=] { return next(y) == end() ? 0 :
            &*next(y); };
        if (bad(y)) { erase(y); return; }
        while (next(y) != end() && bad(next(y)))
            erase(next(y));
        while (y != begin() && bad(prev(y)))
            erase(prev(y)); }
    ll eval(ll x) {
        auto l = *lower_bound((Line) { x, is_query });
        return l.m * x + l.b; } };

```

### 2.13. Sparse Table $O(\log n)$ per query.

```

struct sparse_table { vvi m;
    sparse_table(vi arr) {
        m.push_back(arr);
        for (int k = 0; (1<<(+k)) <= size(arr); ) {
            m.push_back(vi(size(arr)-(1<<k)+1));
            rep(i,0,size(arr)-(1<<k)+1)

```

```

            m[k][i] = min(m[k-1][i],
                m[k-1][i+(1<<(k-1))]); } }
    int query(int l, int r) {
        int k = 0; while (1<<(k+1) <= r-l+1) k++;
        return min(m[k][l], m[k][r-(1<<k)+1]); } };

```

## 3. GRAPH ALGORITHMS

### 3.1. Shortest path.

#### 3.1.1. Dijkstra $O(|E| \log |V|)$ .

```

const ll INFTY = -1;
vi dijkstra( vector<vii> G, ll s ) {
    vi d( G.size(), INFTY );
    priority_queue<ii, vector<ii>, greater<ii>> Q;
    Q.emplace(0, s);
    while(!Q.empty()){
        ll c = Q.top().x, a = Q.top().y;
        Q.pop();
        if(d[a] != INFTY) continue;
        d[a] = c;
        for(ii e : G[a])
            Q.emplace(d[a] + e.y, e.x);
    }
    return d;
}

```

3.1.2. *Stable marriage*. With  $n$  men,  $m \geq n$  women,  $n$  preference lists of women for each men, and for every woman  $j$  an preference of men defined by  $\text{pref}[i][j]$  (lower is better) find for every man a women such that no pair of a men and a woman want to run off together.

```

// n = aantal mannen, m = aantal vrouwen
// voor een man i, is order[i] de prefere
vi stable(int n, int m, vvi order, vvi pref) {
    queue<int> q;
    rep(i,0,n)
        q.push(i);
    vi mas(m,-1), mak(n,-1), p(n,0);
    while (!q.empty()) {
        int k = q.front();
        q.pop();
        int s = order[k][p[k]], k2 = mas[s];
        if (mas[s] == -1) {
            mas[s] = k;
            mak[k] = s;
        } else if (pref[k][s] < pref[k2][s]) {
            mas[s] = k;
            mak[k] = s;
            mak[k2] = -1;
            q.push(k2);
        } else {
            q.push(k);
        }
        p[k]++;
    }
    return mak;
}

```

3.1.3. *Floyd-Warshall*  $O(V^3)$ . Be careful with negative edges! Note:  $|d[i][j]|$  can grow exponentially, and  $\text{INFTY} + \text{negative} < \text{INFTY}$ .

```

#define INFTY (1LL<<61LL)
void floyd_warshall( vvi& d ) {
    ll n = d.size();
    rep(i,0,n) rep(j,0,n) rep(k,0,n)
        if( d[j][i] < INFTY and d[i][k] < INFTY ) // !!!
            d[j][k] =
                max(-INFTY, min(d[j][k], d[j][i]+d[i][k]));
}
vvi d(n,vi(n,INFTY)); rep(i,0,n) d[i][i] = 0;

```

3.1.4. *Bellman Ford*  $O(VE)$ . This is only useful if there are edges with weight  $w_{ij} < 0$  in the graph.

```

#define INFTY (1LL<<61LL)
// G undirected, (v,w) in G[u] 'n edge van u naar v
// lengte w
vi bellman_ford( vector<vii> G, ll s ) {
    ll n = G.size();
    vi d(n,INFTY); d[s] = 0;
    rep( loops, 0, n )
        rep( u, 0, n ) if( d[u] != INFTY )
            for( ii e : G[u] )
                if( d[u] + e.y < d[e.x] )
                    d[e.x] = d[u] + e.y;
    // detect paths of -INFTY length
    for( ll change = 1; change--> 0; )
        rep( u, 0, n ) if( d[u] != INFTY )
            for( ii e : G[u] ) if( d[e.x] != -INFTY )
                if( d[u] + e.y < d[e.x] )
                    d[e.x] = -INFTY, change = 1;
    return d;
}

```

#### 3.1.5. IDA\* algorithm.

```

int n, cur[100], pos;
int calch() {
    int h = 0;
    rep(i,0,n) if (cur[i] != 0) h += abs(i - cur[i]);
    return h; }
int dfs(int d, int g, int prev) {
    int h = calch();
    if (g + h > d) return g + h;
    if (h == 0) return 0;
    int mn = INT_MAX;
    rep(di,-2,3) {
        if (di == 0) continue;
        int nxt = pos + di;
        if (nxt == prev) continue;
        if (0 <= nxt && nxt < n) {
            swap(cur[pos], cur[nxt]);
            swap(pos, nxt);
            mn = min(mn, dfs(d, g+1, nxt));
            swap(pos, nxt);
            swap(cur[pos], cur[nxt]); }
    }
}

```

```

    if (mn == 0) break; }
    return mn; }
int idastar() {
    rep(i,0,n) if (cur[i] == 0) pos = i;
    int d = calch();
    while (true) {
        int nd = dfs(d, 0, -1);
        if (nd == 0 || nd == INT_MAX) return d;
        d = nd; } }

```

### 3.2. Maximum matching $\mathcal{O}(nm)$ .

```
const int sizeL = 1e4, sizeR = 1e4;
```

```

bool vis[sizeR];
int par[sizeR]; // par : R -> L
vi adj[sizeL]; // adj : L -> (N -> R)

bool match(int u) {
    for (int v : adj[u]) {
        if (vis[v]) continue; vis[v] = true;
        if (par[v] == -1 || match(par[v])) {
            par[v] = u;
            return true;
        }
    }
    return false;
}

```

// perfect matching iff ret == sizeL == sizeR

```

int maxmatch() {
    fill_n(par, sizeR, -1); int ret = 0;
    for (int i = 0; i < sizeL; i++) {
        fill_n(vis, sizeR, false);
        ret += match(i);
    }
    return ret;
}

```

### 3.3. Hopcroft-Karp bipartite matching $\mathcal{O}(E\sqrt{V})$ .

```
const ll INFY = (1LL<<61LL);
```

```

struct bi_graph {
    ll n, m;
    vvi adj;
    vi L, R, d;
    queue<ll> q;
    bi_graph( ll _n, ll _m ) : n(_n), m(_m),
        adj(n, L(n,-1), R(m,n), d(n+1)) {}
    ll add_edge( ll a, ll b ) { adj[a].pb(b); }
    ll bfs() {
        rep(v,0,n)
            if( L[v] == -1 ) d[v] = 0, q.push(v);
            else d[v] = INFY;
        d[n] = INFY;
        while( !q.empty() ) {
            ll v = q.front(); q.pop();
            if( d[v] < d[n] )
                for( ll u : adj[v] ) if( d[R[u]] == INFY )

```

```

                d[R[u]] = d[v]+1, q.push(R[u]);
            }
            return d[n] != INFY;
        }
        ll dfs( ll v ) {
            if( v == n ) return true;
            for( ll u : adj[v] )
                if( d[R[u]] == d[v] + 1 and dfs(R[u]) ) {
                    R[u] = v; L[v] = u;
                    return true;
                }
            d[v] = INFY;
            return false;
        }
        ll maximum_matching() {
            ll s = 0;
            while( bfs() ) rep(i,0,n)
                s += L[i] == -1 && dfs( i );
            return s;
        }
    };

```

#### 3.3.1. Minimum Vertex Cover in Bipartite Graphs.

```

#include "hopcroft_karp.cpp"
vi alt;
void dfs( bi_graph &G, ll v ) {
    alt[v] = 1;
    for( ll u : G.adj[v] ) {
        alt[u+G.n] = 1;
        if( G.R[u] != G.n && !alt[G.R[u]] )
            dfs(G,G.R[u]);
    }
}
vi mvc_bipartite( bi_graph &G ) {
    vi res; G.maximum_matching();
    alt.assign( G.n + G.m, 0 );
    rep(i,0,G.n) if( G.L[i] == -1 ) dfs(G,i);
    rep(i,0,G.n) if( !alt[i] ) res.pb(i);
    rep(i,0,G.n) if( alt[G.n+i] ) res.pb(G.n+i);
    return res;
}

```

### 3.4. Depth first searches.

#### 3.4.1. Cut Points and Bridges $\mathcal{O}(V + E)$ .

```

const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;

```

```

void dfs(const vvi &adj, vi &cp, vii &bri, int u, int
    p) {
    low[u] = num[u] = curnum++;
    int cnt = 0; bool found = false;
    rep(i,0,size(adj[u])) {
        int v = adj[u][i];
        if (num[v] == -1) {
            dfs(adj, cp, bri, v, u);
            low[u] = min(low[u], low[v]);
            cnt++;
            found = found || low[v] >= num[u];

```

```

        if (low[v] > num[u]) bri.push_back(ii(u, v));
        } else if (p != v) low[u] = min(low[u], num[v]);
        }
    }
    if (found && (p != -1 || cnt > 1)) cp.push_back(u);
    }
}

```

```

pair<vi,vii> cut_points_and_bridges(const vvi &adj) {
    int n = size(adj);
    vi cp; vii bri;
    memset(num, -1, n << 2);
    curnum = 0;
    rep(i,0,n) if (num[i] == -1) dfs(adj, cp, bri, i,
        -1);
    return make_pair(cp, bri); }

```

#### 3.4.2. Strongly Connected Components $\mathcal{O}(V + E)$ .

```

vvi adj, comps;
vi tidx, lnk, cnr, st;
vector<bool> vis;
int age, ncomps;

void tarjan(int v) {
    tidx[v] = lnk[v] = ++age; vis[v] = true; st.pb(v);
    for (int w : adj[v]) {
        if (!tidx[w]) tarjan(w), lnk[v] = min(lnk[v],
            lnk[w]);
        else if (vis[w]) lnk[v] = min(lnk[v], tidx[w]);
    }
    if (lnk[v] != tidx[v]) return;
    comps.pb(vi());
    int w;
    do {
        vis[w = st.back()] = false; cnr[w] = ncomps;
        st.pop_back();
    } while (w != v);
    ncomps++;
}

```

```

void findSCC(int n) {
    age = ncomps = 0; vis.assign(n, false);
    tidx.assign(n, 0);
    lnk.resize(n); cnr.resize(n); comps.clear();
    for (int i = 0; i < n; i++)
        if (tidx[i] == 0) tarjan(i);
}

```

#### 3.4.3. Dominator graph.

```
const int N = 1234567;
```

```

vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], parent[N], sdom[N], p[N],
    best[N], idom[N], link[N];

```

```

void dfs(int v) {
    pos[v] = cnt;
    order[cnt++] = v;
}

```



```

for (int u : g[v]) {
    if (pos[u] == -1) {
        parent[u] = v;
        dfs(u);
    }
}

int find_best(int x) {
    if (p[x] == x) return best[x];
    int u = find_best(p[x]);
    if (pos[sdom[u]] < pos[sdom[best[x]]])
        best[x] = u;
    p[x] = p[p[x]];
    return best[x];
}

void dominators(int n, int root) {
    fill_n(pos, n, -1);
    cnt = 0;
    dfs(root);
    for (int i = 0; i < n; i++)
        for (int u : g[i]) g_rev[u].push_back(i);
    for (int i = 0; i < n; i++)
        p[i] = best[i] = sdom[i] = i;
    for (int it = cnt - 1; it >= 1; it--) {
        int w = order[it];
        for (int u : g_rev[w]) {
            int t = find_best(u);
            if (pos[sdom[t]] < pos[sdom[w]])
                sdom[w] = sdom[t];
        }
        bucket[sdom[w]].push_back(w);
        idom[w] = sdom[w];
        for (int u : bucket[parent[w]])
            link[u] = find_best(u);
        bucket[parent[w]].clear();
        p[w] = parent[w];
    }
    for (int it = 1; it < cnt; it++) {
        int w = order[it];
        idom[w] = idom[link[w]];
    }
}

3.4.4. 2-SAT  $\mathcal{O}(V + E)$ . Include findSCC.
void init2sat(int n) { adj.assign(2 * n, vi()); }

// vl, vr = true -> variable l, variable r should be
// negated.
void imply(int xl, bool vl, int xr, bool vr) {
    adj[2 * xl + vl].pb(2 * xr + vr); adj[2 * xr
    -> +!vr].pb(2 * xl +!vl); }

void satOr(int xl, bool vl, int xr, bool vr) {
    -> imply(xl, !vl, xr, vr); }
void satConst(int x, bool v) { imply(x, !v, x, v); }
void satIff(int xl, bool vl, int xr, bool vr) {

```

```

imply(xl, vl, xr, vr); imply(xr, vr, xl, vl);}

bool solve2sat(int n, vector<bool> &sol) {
    findSCC(2 * n);
    for (int i = 0; i < n; i++)
        if (cnr[2 * i] == cnr[2 * i + 1]) return false;
    vector<bool> seen(n, false); sol.assign(n, false);
    for (vi &comp : comps) {
        for (int v : comp) {
            if (seen[v / 2]) continue;
            if (seen[v / 2] = true; sol[v / 2] = v & 1;
        }
    }
    return true;
}

```

### 3.5. Cycle Detection $\mathcal{O}(V + E)$ .

```

vvi adj; // assumes bidirected graph, adjust
-> accordingly

bool cycle_detection() {
    stack<int> s; vector<bool> vis(MAXN, false); vi
    -> par(MAXN, -1); s.push(0);
    vis[0] = true;
    while (!s.empty()) {
        int cur = s.top(); s.pop();
        for (int i : adj[cur]) {
            if (vis[i] && par[cur] != i) return true;
            s.push(i); par[i] = cur; vis[i] = true;
        }
    }
    return false;
}

```

### 3.6. Maximum Flow Algorithms.

#### 3.6.1. Dinic's Algorithm $\mathcal{O}(V^2E)$ .

```

struct Edge { int t; ll c, f; };
struct Dinic {
    vi H, P; vvi E;
    vector<Edge> G;
    Dinic(int n) : H(n), P(n), E(n) {}

    void addEdge(int u, int v, ll c) {
        E[u].pb(G.size()); G.pb({v, c, 0LL});
        E[v].pb(G.size()); G.pb({u, 0LL, 0LL});
    }

    ll dfs(int t, int v, ll f) {
        if (v == t || !f) return f;
        for ( ; P[v] < (int) E[v].size(); P[v]++) {
            int e = E[v][P[v]], w = G[e].t;
            if (H[w] != H[v] + 1) continue;
            ll df = dfs(t, w, min(f, G[e].c - G[e].f));
            if (df > 0) {
                G[e].f += df, G[e ^ 1].f -= df;
                return df;
            }
        }
        return 0;
    }
}

```

```

ll maxflow(int s, int t, ll f = 0) {
    while (1) {
        fill(all(H), 0); H[s] = 1;
        queue<int> q; q.push(s);
        while (!q.empty()) {
            int v = q.front(); q.pop();
            for (int w : E[v]) if (G[w].f < G[w].c &&
                -> !H[G[w].t])
                H[G[w].t] = H[v] + 1, q.push(G[w].t);
        }
        if (!H[t]) return f;
        fill(all(P), 0);
        while (ll df = dfs(t, s, LLONG_MAX)) f += df;
    }
}

```

3.6.2. *Min-cost max-flow*  $\mathcal{O}(n^2m^2)$ . Find the cheapest possible way of sending a certain amount of flow through a flow network.

```

const int maxn = 300;

struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
    g.pb({u, v, 0, c, w});
    g.pb({v, u, 0, 0, -w});
}

void sp(int s, int t) {
    fill_n(D, V, LLONG_MAX); D[s] = 0;
    for (int ng = g.size(), _ = V; _--;) {
        bool ok = false;
        for (int i = 0; i < ng; i++)
            if (D[g[i].x] != LLONG_MAX && g[i].f < g[i].c
                -> && D[g[i].x] + g[i].w < D[g[i].y]) {
                D[g[i].y] = D[g[i].x] + g[i].w;
                par[g[i].y] = i; ok = true;
            }
        if (!ok) break;
    }
}

void minCostMaxFlow(int s, int t, ll &c, ll &f) {
    for (c = f = 0; sp(s, t), D[t] < LLONG_MAX; ) {
        ll df = LLONG_MAX, dc = 0;
        for (int v = t, e; e = par[v], v != s; v =
            -> g[e].x) df = min(df, g[e].c - g[e].f);
        for (int v = t, e; e = par[v], v != s; v =
            -> g[e].x) g[e].f += df, g[e ^ 1].f -= df, dc +=
            -> g[e].w;
        f += df; c += dc * df;
    }
}

```

3.6.3. *Gomory-Hu Tree - All Pairs Maximum Flow*. An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in  $O(|V|^2)$  plus  $|V| - 1$  times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is  $O(|V|^3|E|)$ . NOTE: Not sure if it works correctly with disconnected graphs.

```
#include "dinic.cpp"
bool same[MAXV];
pair<vii, vvi> construct_gh_tree(flow_network &g) {
    int n = g.n, v;
    vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
    rep(s, 1, n) {
        int l = 0, r = 0;
        par[s].second = g.max_flow(s, par[s].first,
            ↳ false);
        memset(d, 0, n * sizeof(int));
        memset(same, 0, n * sizeof(bool));
        d[q[r++]] = s; l = 1;
        while (l < r) {
            same[v = q[l++]] = true;
            for (int i = g.head[v]; i != -1; i =
                ↳ g.e[i].nxt)
                if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
                    d[q[r++]] = g.e[i].v; l = 1;
            rep(i, s+1, n)
                if (par[i].first == par[s].first && same[i])
                    par[i].first = s;
            g.reset();
        }
        rep(i, 0, n) {
            int mn = INT_MAX, cur = i;
            while (true) {
                cap[cur][i] = mn;
                if (cur == 0) break;
                mn = min(mn, par[cur].second), cur =
                    ↳ par[cur].first;
            }
            return make_pair(par, cap);
        }
    }
    int compute_max_flow(int s, int t, const pair<vii,
        ↳ vvi> &gh) {
        int cur = INT_MAX, at = s;
        while (gh.second[at][t] == -1)
            cur = min(cur, gh.first[at].second),
            at = gh.first[at].first;
        return min(cur, gh.second[at][t]);
    }
}
```

### 3.7. Minimal Spanning Tree.

#### 3.7.1. *Kruskal* $O(E \log V)$ .

```
struct edge { int x, y; ll w; };
ll kruskal(int n, vector<edge> edges) {
    dsu D(n);
    sort(all(edges), [] (edge a, edge b) -> bool {
        return a.w < b.w; });
    ll ret = 0;
    for (edge e : edges) if (D.find(e.x) !=
        ↳ D.find(e.y))
        ret += e.w, D.unite(e.x, e.y);
}
```

```
return ret;
}
```

### 3.8. Topological Sort $O(V + E)$ .

#### 3.8.1. *Modified Depth-First Search*.

```
void tsort_dfs(int cur, char* color, const vvi& adj,
    stack<int>& res, bool& cyc) {
    color[cur] = 1;
    rep(i, 0, size(adj[cur])) {
        int nxt = adj[cur][i];
        if (color[nxt] == 0)
            tsort_dfs(nxt, color, adj, res, cyc);
        else if (color[nxt] == 1)
            cyc = true;
        if (cyc) return;
    }
    color[cur] = 2;
    res.push(cur);
}
vi tsort(int n, vvi adj, bool& cyc) {
    cyc = false;
    stack<int> S;
    vi res;
    char* color = new char[n];
    memset(color, 0, n);
    rep(i, 0, n) {
        if (!color[i]) {
            tsort_dfs(i, color, adj, S, cyc);
            if (cyc) return res;
        }
    }
    while (!S.empty()) res.push_back(S.top()), S.pop();
    return res;
}
```

3.9. *Euler Path*  $O(V + E)$  hopefully. Finds an Euler Path (or circuit) in a directed graph iff one exists.

```
const int MAXV = 1000, MAXE = 5000;
vi adj[MAXV];
int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
    int start = -1, end = -1, any = 0, c = 0;
    rep(i, 0, n) {
        if (outdeg[i] > 0) any = i;
        if (indeg[i] + 1 == outdeg[i]) start = i, c++;
        else if (indeg[i] == outdeg[i] + 1) end = i, c++;
        else if (indeg[i] != outdeg[i]) return ii(-1, -1);
        ↳
    }
    if ((start == -1) != (end == -1) || (c != 2 && c !=
        ↳ 0))
        return ii(-1, -1);
    if (start == -1) start = end = any;
    return ii(start, end);
}
bool euler_path() {
    ii se = start_end();
    int cur = se.first, at = m + 1;
    if (cur == -1) return false;
    stack<int> s;
    while (true) {
        if (outdeg[cur] == 0) {
            res[--at] = cur;
            if (s.empty()) break;
        }
    }
}
```

```
cur = s.top(); s.pop();
} else s.push(cur), cur =
    ↳ adj[cur][--outdeg[cur]];
return at == 0; }
```

Finds an Euler cycle in a undirected graph:

```
const int MAXV = 1000;
multiset<int> adj[MAXV];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
    if (at == to) return it;
    L.insert(it, at), --it;
    while (!adj[at].empty()) {
        int nxt = *adj[at].begin();
        adj[at].erase(adj[at].find(nxt));
        adj[nxt].erase(adj[nxt].find(at));
        if (to == -1) {
            it = euler(nxt, at, it);
            L.insert(it, at);
            --it;
        } else {
            it = euler(nxt, to, it);
            to = -1;
        }
    }
    return it;
}
// usage: euler(0, -1, L.begin());
```

### 3.10. Heavy-Light Decomposition.

```
#include "../data-structures/segment_tree.cpp"
const int ID = 0;
int f(int a, int b) { return a + b; }
struct HLD {
    int n, curhead, curloc;
    vi sz, head, parent, loc;
    vvi adj; segment_tree values;
    HLD(int _n) : n(_n), sz(n, 1), head(n),
        parent(n, -1), loc(n), adj(n) {
        vector<ll> tmp(n, ID); values =
            ↳ segment_tree(tmp);
    }
    void add_edge(int u, int v) {
        adj[u].push_back(v); adj[v].push_back(u);
    }
    void update_cost(int u, int v, int c) {
        if (parent[v] == u) swap(u, v); assert(parent[u]
            ↳ == v);
        values.update(loc[u], c);
    }
    int csz(int u) {
        rep(i, 0, size(adj[u])) if (adj[u][i] != parent[u])
            sz[u] += csz(adj[parent[adj[u][i]] = u][i]);
        return sz[u];
    }
    void part(int u) {
        head[u] = curhead; loc[u] = curloc++;
        int best = -1;
        rep(i, 0, size(adj[u]))
            if (adj[u][i] != parent[u] &&
                (best == -1 || sz[adj[u][i]] > sz[best]))
                best = adj[u][i];
        if (best != -1) part(best);
        rep(i, 0, size(adj[u]))
    }
}
```

```

    if (adj[u][i] != parent[u] && adj[u][i] !=
        ↪ best)
        part(curhead = adj[u][i]); }
void build(int r = 0) {
    curloc = 0, csz(curhead = r), part(r); }
int lca(int u, int v) {
    vi uat, vat; int res = -1;
    while (u != -1) uat.push_back(u), u =
        ↪ parent[head[u]];
    while (v != -1) vat.push_back(v), v =
        ↪ parent[head[v]];
    u = size(uat) - 1, v = size(vat) - 1;
    while (u >= 0 && v >= 0 && head[uat[u]] ==
        ↪ head[vat[v]])
        res = (loc[uat[u]] < loc[vat[v]] ? uat[u] :
            ↪ vat[v]),
        u--, v--;
    return res; }
int query_upto(int u, int v) { int res = ID;
    while (head[u] != head[v])
        res = f(res, values.query(loc[head[u]],
            ↪ loc[u]).x),
        u = parent[head[u]];
    return f(res, values.query(loc[v] + 1,
        ↪ loc[u]).x); }
int query(int u, int v) { int l = lca(u, v);
    return f(query_upto(u, l), query_upto(v, l)); }
    ↪ };

```

### 3.11. Centroid Decomposition.

```

#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
    path[MAXV][LGMAXV],
    sz[MAXV], seph[MAXV],
    shortest[MAXV];
struct centroid_decomposition {
    int n; vvi adj;
    centroid_decomposition(int _n) : n(_n), adj(n) {}
    void add_edge(int a, int b) {
        adj[a].push_back(b); adj[b].push_back(a); }
    int dfs(int u, int p) {
        sz[u] = 1;
        rep(i, 0, size(adj[u]))
            if (adj[u][i] != p) sz[u] += dfs(adj[u][i], u);
        return sz[u]; }
    void makepaths(int sep, int u, int p, int len) {
        jmp[u][seph[sep]] = sep, path[u][seph[sep]] =
            ↪ len;
        int bad = -1;
        rep(i, 0, size(adj[u])) {
            if (adj[u][i] == p) bad = i;
            else makepaths(sep, adj[u][i], u, len + 1);
        }
        if (p == sep)
            swap(adj[u][bad], adj[u].back()),
            ↪ adj[u].pop_back(); }

```

```

void separate(int h=0, int u=0) {
    dfs(u, -1); int sep = u;
    down: iter(nxt, adj[sep])
        if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2) {
            sep = *nxt; goto down; }
    seph[sep] = h, makepaths(sep, sep, -1, 0);
    rep(i, 0, size(adj[sep])) separate(h+1,
        ↪ adj[sep][i]); }
void paint(int u) {
    rep(h, 0, seph[u]+1)
        shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
            ↪ path[u][h]); }
int closest(int u) {
    int mn = INT_MAX/2;
    rep(h, 0, seph[u]+1)
        mn = min(mn, path[u][h] + shortest[jmp[u][h]]);
    return mn; } };

```

### 3.12. Least Common Ancestors, Binary Jumping.

```

const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
    rep(i, 0, SZ) BP[i][0] = P[i];
    rep(j, 1, LOGSZ) rep(i, 0, SZ)
        BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
    if (H[a] > H[b]) swap(a, b);
    int dh = H[b] - H[a], j = 0;
    rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
    while (BP[a][j] != BP[b][j]) j++;
    while (--j >= 0) if (BP[a][j] != BP[b][j])
        a = BP[a][j], b = BP[b][j];
    return a == b ? a : P[a];
}

```

### 3.13. Tarjan's Off-line Lowest Common Ancestors Algorithm.

```

#include "../data-structures/union_find.cpp"
struct tarjan_olca {
    int *ancestor;
    vi *adj, answers;
    vii *queries;
    bool *colored;
    union_find uf;
    tarjan_olca(int n, vi *_adj) : adj(_adj), uf(n) {
        colored = new bool[n];
        ancestor = new int[n];
        queries = new vii[n];
        memset(colored, 0, n); }
    void query(int x, int y) {
        queries[x].push_back(ii(y, size(answers)));
        queries[y].push_back(ii(x, size(answers)));
        answers.push_back(-1); }
    void process(int u) {
        ancestor[u] = u;
        rep(i, 0, size(adj[u])) {
            int v = adj[u][i];
            process(v);
            uf.unite(u, v);
            ancestor[uf.find(u)] = u;
            colored[u] = true;
            rep(i, 0, size(queries[u])) {
                int v = queries[u][i].first;
                if (colored[v]) {
                    answers[queries[u][i].second] =
                        ↪ ancestor[uf.find(v)];
                } } } }
}

3.14. Misra-Gries  $D+1$ -edge coloring. Finds a  $\max_i \deg(i) + 1$ -
edge coloring where there all incident edges have distinct colors. Find-
ing a  $D$ -edge coloring is NP-hard.

struct Edge { int to, col, rev; };

struct MisraGries {
    int N, K=0; vvi F;
    vector<vector<Edge>> G;

    MisraGries(int n) : N(n), G(n) {}
    // add an undirected edge, NO DUPLICATES ALLOWED
    void addEdge(int u, int v) {
        G[u].pb({v, -1, (int) G[v].size()});
        G[v].pb({u, -1, (int) G[u].size()-1});
    }

    void color(int v, int i) {
        vi fan = { i };
        vector<bool> used(G[v].size());
        used[i] = true;
        for (int j = 0; j < (int) G[v].size(); j++)
            if (!used[j] && G[v][j].col >= 0 &&
                ↪ F[G[v][fan.back()]].to[G[v][j].col] < 0)
                used[j] = true, fan.pb(j), j = -1;
        int c = 0; while (F[v][c] >= 0) c++;
        int d = 0; while (F[G[v][fan.back()]].to[d] >= 0)
            ↪ d++;
        int w = v, a = d, k = 0, ccol;
        while (true) {
            swap(F[w][c], F[w][d]);
            if (F[w][c] >= 0) G[w][F[w][c]].col = c;
            if (F[w][d] >= 0) G[w][F[w][d]].col = d;
            if (F[w][a^c^d] < 0) break;
            w = G[w][F[w][a]].to;
        }
        do {
            Edge &e = G[v][fan[k]];
            ccol = F[e.to][d] < 0 ? d : G[v][fan[k+1]].col;
            if (e.col >= 0) F[e.to][e.col] = -1;
            F[e.to][ccol] = e.rev;
            F[v][ccol] = fan[k];
            e.col = G[e.to][e.rev].col = ccol;
            k++;
        } while (ccol != d);
    }
}

```

```

}
// finds a K-edge-coloring
void color() {
    REP(v, N) K = max(K, (int) G[v].size() + 1);
    F = vv(N, vi(K, -1));
    REP(v, N) for (int i = G[v].size(); i--;)
        if (G[v][i].col < 0) color(v, i);
}
};

```

**3.15. Minimum Mean Weight Cycle.** Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```

double
↳ min_mean_cycle(vector<vector<pair<int,double>>>
↳ adj){
    int n = size(adj); double mn = INFINITY;
    vector<vector<double>> arr(n+1, vector<double>(n,
    ↳ mn));
    arr[0][0] = 0;
    rep(k,1,n+1) rep(j,0,n) iter(it,adj[j])
        arr[k][it->first] = min(arr[k][it->first],
            it->second +
            ↳ arr[k-1][j]);
    rep(k,0,n) {
        double mx = -INFINITY;
        rep(i,0,n) mx = max(mx,
            ↳ (arr[n][i]-arr[k][i])/(n-k));
        mn = min(mn, mx); }
    return mn; }

```

**3.16. Minimum Arborescence.** Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root  $r$  to each vertex. Returns a vector of size  $n$ , where the  $i$ th element is the edge for the  $i$ th vertex. The answer for the root is undefined!

```

#include "../data-structures/union_find.cpp"
struct arborescence {
    int n; union_find uf;
    vector<vector<pair<ii,int>>> adj;
    arborescence(int _n) : n(_n), uf(n), adj(n) { }
    void add_edge(int a, int b, int c) {
        adj[b].push_back(make_pair(ii(a,b),c)); }
    vii find_min(int r) {
        vi vis(n,-1), mn(n,INT_MAX); vii par(n);
        rep(i,0,n) {
            if (uf.find(i) != i) continue;
            int at = i;
            while (at != r && vis[at] == -1) {
                vis[at] = i;
                iter(it,adj[at]) if (it->second < mn[at] &&
                    uf.find(it->first.first) != at)
                    mn[at] = it->second, par[at] = it->first;
                if (par[at] == ii(0,0)) return vii();
                at = uf.find(par[at].first); }
            if (at == r || vis[at] != i) continue;

```

```

        union_find tmp = uf; vi seq;
        do { seq.push_back(at); at =
            ↳ uf.find(par[at].first);
        } while (at != seq.front());
        iter(it,seq) uf.unite(*it,seq[0]);
        int c = uf.find(seq[0]);
        vector<pair<ii,int>> > nw;
        iter(it,seq) iter(jt,adj[*it])
            nw.push_back(make_pair(jt->first,
                jt->second - mn[*it]));
        adj[c] = nw;
        vii rest = find_min(r);
        if (size(rest) == 0) return rest;
        ii use = rest[c];
        rest[at = tmp.find(use.second)] = use;
        iter(it,seq) if (*it != at)
            rest[*it] = par[*it];
        return rest; }
    return par; } }

```

**3.17. Blossom algorithm.** Finds a maximum matching in an arbitrary graph in  $O(|V|^4)$  time. Be aware of loop edges.

```

#define MAXV 300
bool marked[MAXV], emarked[MAXV][MAXV];
int S[MAXV];
vi find_augmenting_path(const vector<vi> &adj, const
↳ vi &m){
    int n = size(adj), s = 0;
    vi par(n,-1), height(n), root(n,-1), q, a, b;
    memset(marked,0,sizeof(marked));
    memset(emarked,0,sizeof(emarked));
    rep(i,0,n) if (m[i] >= 0) emarked[i][m[i]] = true;
        else root[i] = i, S[s++] = i;
    while (s) {
        int v = S[--s];
        iter(wt,adj[v]) {
            int w = *wt;
            if (emarked[v][w]) continue;
            if (root[w] == -1) {
                int x = S[s++] = m[w];
                par[w]=v, root[w]=root[v],
                ↳ height[w]=height[v]+1;
                par[x]=w, root[x]=root[w],
                ↳ height[x]=height[w]+1;
            } else if (height[w] % 2 == 0) {
                if (root[v] != root[w]) {
                    while(v != -1) q.push_back(v), v = par[v];
                    reverse(q.begin(), q.end());
                    while(w != -1) q.push_back(w), w = par[w];
                    return q;
                } else {
                    int c = v;
                    while(c != -1) a.push_back(c), c = par[c];
                    c = w;
                    while(c != -1) b.push_back(c), c = par[c];
                    ↳ while(!a.empty() && !b.empty() && a.back() == b.back())

```

```

        c = a.back(), a.pop_back(), b.pop_back();
        memset(marked,0,sizeof(marked));
        fill(par.begin(), par.end(), 0);
        iter(it,a) par[*it] = 1; iter(it,b)
            ↳ par[*it] = 1;
        par[c] = s = 1;
        rep(i,0,n) root[par[i]] = par[i] ? 0 : s++;
            ↳ = i;
        vector<vi> adj2(s);
        rep(i,0,n) iter(it,adj[i]) {
            if (par[*it] == 0) continue;
            if (par[i] == 0) {
                if (!marked[par[*it]]) {
                    adj2[par[i]].push_back(par[*it]);
                    adj2[par[*it]].push_back(par[i]);
                    marked[par[*it]] = true; }
                } else adj2[par[i]].push_back(par[*it]);
            ↳ }
        vi m2(s, -1);
        if (m[c] != -1) m2[m2[par[m[c]]] = 0] =
            ↳ par[m[c]];
        rep(i,0,n)
            ↳ if (par[i] != 0 && m[i] != -1 && par[m[i]] != 0)
                m2[par[i]] = par[m[i]];
        vi p = find_augmenting_path(adj2, m2);
        int t = 0;
        while (t < size(p) && p[t]) t++;
        if (t == size(p)) {
            rep(i,0,size(p)) p[i] = root[p[i]];
            return p; }
        if (!p[0] || (m[c] != -1 && p[t+1] !=
            ↳ par[m[c]]))
            reverse(p.begin(), p.end()), t =
            ↳ size(p)-t-1;
        rep(i,0,t) q.push_back(root[p[i]]);
        iter(it,adj[root[p[t-1]]) {
            if (par[*it] != (s = 0)) continue;
            a.push_back(c), reverse(a.begin(),
            ↳ a.end());
            iter(jt,b) a.push_back(*jt);
            while (a[s] != *it) s++;
            if ((height[*it] & 1) ^ (s < size(a) -
            ↳ size(b)))
                reverse(a.begin(), a.end()), s =
                ↳ size(a)-s-1;
            ↳ while(a[s] != c) q.push_back(a[s]), s=(s+1)%size(a);
            q.push_back(c);
            rep(i,t+1,size(p))
                ↳ q.push_back(root[p[i]]);
            return q; } }
        emarked[v][w] = emarked[w][v] = true; }
        marked[v] = true; } return q; }
    vii max_matching(const vector<vi> &adj) {
        vi m(size(adj), -1), ap; vii res, es;

```

```

rep(i,0,size(adj)) iter(it,adj[i])
↳ es.emplace_back(i,*it);
random_shuffle(es.begin(), es.end());
iter(it,es) if (m[it->first] == -1 && m[it->second]
↳ == -1)
    m[it->first] = it->second, m[it->second] =
    ↳ it->first;
do { ap = find_augmenting_path(adj, m);
    rep(i,0,size(ap)) m[m[ap[i^1]] = ap[i]] =
    ↳ ap[i^1];
} while (!ap.empty());
rep(i,0,size(m)) if (i < m[i]) res.emplace_back(i,
↳ m[i]);
return res; }

```

**3.18. Maximum Density Subgraph.** Given (weighted) undirected graph  $G$ . Binary search density. If  $g$  is current density, construct flow network:  $(S, u, m), (u, T, m+2g-d_u), (u, v, 1)$ , where  $m$  is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty  $S$ -component, then maximum density is smaller than  $g$ , otherwise it's larger. Distance between valid densities is at least  $1/(n(n-1))$ . Edge case when density is 0. This also works for weighted graphs by replacing  $d_u$  by the weighted degree, and doing more iterations (if weights are not integers).

**3.19. Maximum-Weight Closure.** Given a vertex-weighted directed graph  $G$ . Turn the graph into a flow network, adding weight  $\infty$  to each edge. Add vertices  $S, T$ . For each vertex  $v$  of weight  $w$ , add edge  $(S, v, w)$  if  $w \geq 0$ , or edge  $(v, T, -w)$  if  $w < 0$ . Sum of positive weights minus minimum  $S-T$  cut is the answer. Vertices reachable from  $S$  are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

**3.20. Maximum Weighted Independent Set in a Bipartite Graph.** This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges  $(S, u, w(u))$  for  $u \in L, (v, T, w(v))$  for  $v \in R$  and  $(u, v, \infty)$  for  $(u, v) \in E$ . The minimum  $S, T$ -cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

**3.21. Synchronizing word problem.** A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

#### 4. STRING ALGORITHMS

##### 4.1. Trie.

```

const int SIGMA = 26;

struct trie {
    bool word; trie **adj;

    trie() : word(false), adj(new trie*[SIGMA]) {

```

```

        for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
    }

    void addWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) cur->adj[i] = new trie();
            cur = cur->adj[i];
        }
        cur->word = true;
    }

    bool isWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) return false;
            cur = cur->adj[i];
        }
        return cur->word;
    }
};

```

##### 4.2. Z-algorithm $\mathcal{O}(n)$ .

```

// z[i] = length of longest substring starting from
↳ s[i] which is also a prefix of s.
vi z_function(const string &s) {
    int n = (int) s.length();
    vi z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]])
            ↳ ++z[i];
        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    }
    return z;
}

```

**4.3. Suffix array  $\mathcal{O}(n \log^2 n)$ .** This creates an array  $P[0], P[1], \dots, P[n-1]$  such that the suffix  $S[i \dots n]$  is the  $P[i]^{th}$  suffix of  $S$  when lexicographically sorted.

```

typedef pair<ii, int> tii;
const int maxlogn = 17, maxn = 1 << maxlogn;

int p[maxlogn + 1][maxn]; tii L[maxn];

int suffixArray(string S) {
    int N = S.size(), stp = 1, cnt = 1;
    REP(i, N) p[0][i] = S[i];
    for (; cnt < N; stp++, cnt <= 1) {
        REP(i, N)
            L[i] = tii(ii(p[stp-1][i], i + cnt < N ?
            ↳ p[stp-1][i + cnt] : -1), i);
        sort(L, L + N);
        REP(i, N)
            p[stp][L[i].y] = i > 0 && L[i].x == L[i-1].x ?
            ↳ p[stp][L[i-1].y] : i;
    }
}

```

```

    }
    return stp - 1; // result is in p[stp - 1][0 .. (N
    ↳ - 1)]
}

```

**4.4. Longest Common Subsequence  $\mathcal{O}(n^2)$ .** SUBSTRING: consecutive characters!!!

```

int dp[STR_SIZE][STR_SIZE]; // DP problem

int lcs(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i < n1; i++) {
        for (int j = 0; j < n2; j++) {
            if (i == 0 || j == 0) dp[i][j] = 0;
            else if (w1[i - 1] == w2[j - 1]) dp[i][j] =
            ↳ dp[i - 1][j - 1] + 1;
            else dp[i][j] = max(dp[i - 1][j], dp[i][j -
            ↳ 1]);
        }
    }
    return dp[n1][n2];
}

// backtrace
string getLCS(const string &w1, const string &w2) {
    int i = w1.size(), j = w2.size(); string ret = "";
    while (i > 0 && j > 0) {
        if (w1[i - 1] == w2[j - 1]) ret += w1[--i], j--;
        else if (dp[i][j - 1] > dp[i - 1][j]) j--;
        else i--;
    }
    reverse(ret.begin(), ret.end());
    return ret;
}

```

**4.5. Levenshtein Distance  $\mathcal{O}(n^2)$ .** Minimal number of insertions, removals and edits required to transform one string in the other.

```

int dp[MAX_SIZE][MAX_SIZE]; // DP problem

int levDist(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i <= n1; i++) dp[i][0] = i; //
    ↳ removal
    for (int j = 0; j <= n2; j++) dp[0][j] = j; //
    ↳ insertion
    for (int i = 1; i <= n1; i++)
        for (int j = 1; j <= n2; j++)
            dp[i][j] = min(
                1 + min(dp[i - 1][j], dp[i][j - 1]),
                dp[i - 1][j - 1] + (w1[i - 1] != w2[j - 1])
            );
    return dp[n1][n2];
}

```



**4.6. Knuth-Morris-Pratt algorithm  $\mathcal{O}(N+M)$ .**

```

int kmp_search(const string &word, const string
↳ &text) {
    int n = word.size();
    vi T(n + 1, 0);
    for (int i = 1, j = 0; i < n; ) {
        if (word[i] == word[j]) T[++i] = ++j; // match
        else if (j > 0) j = T[j]; // fallback
        else i++; // no match, keep zero
    }
    int matches = 0;
    for (int i = 0, j = 0; i < text.size(); ) {
        if (text[i] == word[j]) {
            i++;
            if (++j == n) // match at interval [i - n, i)
                matches++, j = T[j];
        } else if (j > 0) j = T[j];
        else i++;
    }
    return matches;
}

```

**4.7. Aho-Corasick Algorithm  $\mathcal{O}(N + \sum_{i=1}^m |S_i|)$ .** Dictionary sub-string matching as automaton. All given P must be unique!

```

const int MAXP = 100, MAXLEN = 200, SIGMA = 26,
↳ MAXTRIE = MAXP * MAXLEN;

int nP;
string P[MAXP], S;

int pnr[MAXTRIE], to[MAXTRIE][SIGMA], sLink[MAXTRIE],
↳ dLink[MAXTRIE], nnodes;

void ahoCorasick() {
    fill_n(pnr, MAXTRIE, -1);
    for (int i = 0; i < MAXTRIE; i++) fill_n(to[i],
↳ SIGMA, 0);
    fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE,
↳ 0);
    nnodes = 1;
    // STEP 1: MAKE A TREE
    for (int i = 0; i < nP; i++) {
        int cur = 0;
        for (char c : P[i]) {
            int i = c - 'a';
            if (to[cur][i] == 0) to[cur][i] = nnodes++;
            cur = to[cur][i];
        }
        pnr[cur] = i;
    }
    // STEP 2: CREATE SUFFIX_LINKS AND DICT_LINKS
    queue<int> q; q.push(0);
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (int c = 0; c < SIGMA; c++) {
            if (to[cur][c]) {

```

```

        int sl = sLink[to[cur][c]] = cur == 0 ? 0 :
↳ to[sLink[cur]][c];
        // if all strings have equal length, remove
        ↳ this:
        dLink[to[cur][c]] = pnr[sl] >= 0 ? sl :
↳ dLink[sl];
        q.push(to[cur][c]);
        } else to[cur][c] = to[sLink[cur]][c];
    }
}
// STEP 3: TRAVERSE S
for (int cur = 0, i = 0, n = S.size(); i < n; i++)
↳ {
    cur = to[cur][S[i] - 'a'];
    for (int hit = pnr[cur] >= 0 ? cur : dLink[cur];
↳ hit; hit = dLink[hit]) {
        cerr << P[pnr[hit]] << " found at [" << (i + 1)
↳ - P[pnr[hit]].size()) << ", " << i << "]"
↳ << endl;
    }
}
}

```

**4.8. eerTree.** Constructs an eerTree in  $\mathcal{O}(n)$ , one character at a time.

```

#define MAXN 100100
#define SIGMA 26
#define BASE 'a'
char *s = new char[MAXN];
struct state {
    int len, link, to[SIGMA];
} *st = new state[MAXN+2];
struct eertree {
    int last, sz, n;
    eertree() : last(1), sz(2), n(0) {
        st[0].len = st[0].link = -1;
        st[1].len = st[1].link = 0;
    }
    int extend() {
        char c = s[n++]; int p = last;
        while (n - st[p].len - 2 < 0 || c != s[n -
↳ st[p].len - 2])
            p = st[p].link;
        if (!st[p].to[c-BASE]) {
            int q = last = sz++;
            st[p].to[c-BASE] = q;
            st[q].len = st[p].len + 2;
            do { p = st[p].link;
            } while (p != -1 && (n < st[p].len + 2 ||
↳ c != s[n - st[p].len - 2]));
            if (p == -1) st[q].link = 1;
            else st[q].link = st[p].to[c-BASE];
            return 1;
        }
        last = st[p].to[c-BASE];
        return 0;
    }
};

```

**4.9. Suffix Automaton.** Minimum automata that accepts all suffixes of a string with  $\mathcal{O}(n)$  construction. The automata itself is a

DAG therefore suitable for DP, examples are counting unique substrings, occurrences of substrings and suffix.

```

// TODO: Add longest common subsring
const int MAXL = 100000;
struct suffix_automaton {
    vi len, link, occur, cnt;
    vector<map<char,int>> next;
    vector<bool> isclone;
    ll *occuratleast;
    int sz, last;
    string s;
    suffix_automaton() : len(MAXL*2), link(MAXL*2),
        occur(MAXL*2), next(MAXL*2), isclone(MAXL*2) {
        clear();
    }
    void clear() { sz = 1; last = len[0] = 0; link[0] =
↳ -1;
        next[0].clear(); isclone[0] = false;
        clear();
    }
    bool issubstr(string other) {
        for (int i = 0, cur = 0; i < size(other); ++i) {
            if (cur == -1) return false; cur =
↳ next[cur][other[i]];
        }
        return true;
    }
    void extend(char c) { int cur = sz++; len[cur] =
↳ len[last]+1;
        next[cur].clear(); isclone[cur] = false; int p =
↳ last;
        for (; p != -1 && !next[p].count(c); p = link[p])
            next[p][c] = cur;
        if (p == -1) { link[cur] = 0; }
        else { int q = next[p][c];
            if (len[p] + 1 == len[q]) { link[cur] = q; }
            else { int clone = sz++; isclone[clone] = true;
                len[clone] = len[p] + 1;
                link[clone] = link[q]; next[clone] = next[q];
                for (; p != -1 && next[p].count(c) &&
↳ next[p][c] == q;
                    p = link[p]) {
                    next[p][c] = clone;
                    link[q] = link[cur] = clone;
                }
                last = cur;
            }
        }
    }
    void count() {
        cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
        map<char,int>::iterator i;
        while (!S.empty()) {
            ii cur = S.top(); S.pop();
            if (cur.second) {
                for (i = next[cur.first].begin();
                    i != next[cur.first].end(); ++i) {
                    cnt[cur.first] += cnt[(*i).second];
                }
            }
            else if (cnt[cur.first] == -1) {
                cnt[cur.first] = 1; S.push(ii(cur.first, 1));
                for (i = next[cur.first].begin();
                    i != next[cur.first].end(); ++i) {
                    S.push(ii((*i).second, 0));
                }
            }
        }
    }
    string lexicok(ll k) {

```

```

int st = 0; string res; map<char,int>::iterator
↳ i;
while(k){
    for(i = next[st].begin(); i != next[st].end();
        ↳ ++i){
        if(k <= cnt[(*)].second){ st = (*i).second;
            res.push_back((*)].first); k--; break;
        } else { k -= cnt[(*)].second; } } }
return res; }
void countoccur(){
    for(int i = 0; i < sz; ++i){ occur[i] = 1 -
        ↳ isclone[i]; }
    vii states(sz);
    for(int i = 0; i < sz; ++i){ states[i] =
        ↳ ii(len[i],i); }
    sort(states.begin(), states.end());
    for(int i = size(states)-1; i >= 0; --i){
        int v = states[i].second;
        if(link[v] != -1) { occur[link[v]] += occur[v];
            ↳ } } } } }

```

4.10. **Hashing.** Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```

struct hasher { int b = 311, m; vi h, p;
    hasher(string s, int _m)
        : m(_m), h(size(s)+1), p(size(s)+1) {
        p[0] = 1; h[0] = 0;
        rep(i,0,size(s)) p[i+1] = (1ll)p[i] * b % m;
        rep(i,0,size(s)) h[i+1] = ((1ll)h[i] * b + s[i]) %
            ↳ m; }
    int hash(int l, int r) {
        return (h[r+1] + m - (1ll)h[l] * p[r-l+1] % m) %
            ↳ m; } }

```

## 5. GEOMETRY

```
const double EPS = 1e-7, PI = acos(-1.0);
```

```

typedef long long NUM; // EITHER double OR long long
typedef pair<NUM, NUM> pt;
#define x first
#define y second

```

```

pt operator+(pt p,pt q){return pt(p.x+q.x, p.y+q.y);}
pt operator-(pt p,pt q){return pt(p.x-q.x, p.y-q.y);}

```

```

pt& operator+=(pt &p, pt q) { return p = p+q; }
pt& operator-=(pt &p, pt q) { return p = p-q; }

```

```

pt operator*(pt p,NUM l) { return pt(p.x*l, p.y*l); }
pt operator/(pt p,NUM l) { return pt(p.x/l, p.y/l); }

```

```

NUM operator*(pt p, pt q) { return p.x*q.x+p.y*q.y; }
NUM operator^(pt p, pt q) { return p.x*q.y-p.y*q.x; }

```

```

NUM lenSq(pt p) { return p * p; }
NUM lenSq(pt p, pt q) { return lenSq(p - q); }
double len(pt p) { return hypot(p.x, p.y); }

```

```

double len(pt p, pt q) { return len(p - q); }

typedef pt frac;
typedef pair<double, double> vec;
vec getvec(pt p, pt dp, frac t) { return vec(p.x + 1.
    ↳ * dp.x * t.x / t.y, p.y + 1. * dp.y * t.x / t.y); }

```

```

// square distance from pt a to line bc
frac distPtLineSq(pt a, pt b, pt c) {
    a -= b, c -= b;
    return frac((a ^ c) * (a ^ c), c * c);
}

```

```

// square distance from pt a to linesegment bc
frac distPtSegmentSq(pt a, pt b, pt c) {
    a -= b; c -= b;
    NUM dot = a * c, len = c * c;
    if (dot <= 0) return frac(a * a, 1);
    if (dot >= len) return frac((a - c) * (a - c), 1);
    return frac(a * a * len - dot * dot, len);
}

```

```

// projects pt a onto linesegment bc
frac proj(pt a, pt b, pt c) { return frac((a - b) *
    ↳ (c - b), (c - b) * (c - b)); }
vec projv(pt a, pt b, pt c) { return getvec(b, c - b,
    ↳ proj(a, b, c)); }

```

```

bool collinear(pt a, pt b, pt c) { return ((a - b) ^
    ↳ (a - c)) == 0; }

```

```

// true => 1 intersection, false => parallel, so 0 or
↳ \infy solutions
bool linesIntersect(pt a, pt b, pt c, pt d) { return
    ↳ ((a - b) ^ (c - d)) != 0; }
vec lineLineIntersection(pt a, pt b, pt c, pt d) {
    double det = (a - b) ^ (c - d); pt ret = (c - d) *
        ↳ (a ^ b) - (a - b) * (c ^ d);
    return vec(ret.x / det, ret.y / det);
}

```

```

// dp, dq are directions from p, q
// intersection at p + t_i dp, for 0 <= i < return
↳ value
int segmentIntersection(pt p, pt dp, pt q, pt dq,
    ↳ frac &t0, frac &t1){
    if (dp * dp == 0) swap(p, q), swap(dp, dq); // dq =
        ↳ 0
    if (dp * dp == 0) { t0 = t1 = frac(0, 1); return p
        ↳ == q; } // dp = dq = 0
    pt dpq = (q - p); NUM c = dp ^ dq, c0 = dpq ^ dp,
        ↳ c1 = dpq ^ dq;
    if (c == 0) { // parallel, dp > 0, dq >= 0
        if (c0 != 0) return 0; // not collinear
        NUM v0 = dpq * dp, v1 = v0 + dq * dp, dp2 = dp *
            ↳ dp;

```

```

    if (v1 < v0) swap(v0, v1);
    t0 = frac(v0 = max(v0, (NUM) 0), dp2);
    t1 = frac(v1 = min(v1, dp2), dp2);
    return (v0 <= v1) + (v0 < v1);
} else if (c < 0) c = -c, c0 = -c0, c1 = -c1;
t0 = t1 = frac(c1, c);
return 0 <= min(c0, c1) && max(c0, c1) <= c;
}

```

```

// Returns TWICE the area of a polygon to keep it an
↳ integer
NUM polygonTwiceArea(const vector<pt> &pts) {
    NUM area = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j
        ↳ = i++)
        area += pts[i] ^ pts[j];
    return abs(area); // area < 0 ==> pts ccw
}

```

```

bool segmenthaspt(pt s, pt e, pt p) {
    pt ds = p-s, de = p-e;
    return (ds ^ de) == 0LL && (ds * de) <= 0LL;
}

```

```

bool insidePolygon(const vector<pt> &pts, pt p, bool
    ↳ strict = true) {
    int n = 0;
    for (int N = pts.size(), i = 0, j = N - 1; i < N; j
        ↳ = i++) {
        // if p is on edge of polygon
        if (segmenthaspt(pts[i], pts[j], p)) return
            ↳ !strict;
        // or: if(distPtSegmentSq(p, pts[i], pts[j]) <=
            ↳ EPS) return !strict;

        // increment n if segment intersects line from p
        n += (max(pts[i].y, pts[j].y) > p.y &&
            ↳ min(pts[i].y, pts[j].y) <= p.y &&
            ↳ ((pts[j] - pts[i])^(p-pts[i]) > 0) ==
            ↳ (pts[i].y <= p.y));
    }
    return n & 1; // inside if odd number of
    ↳ intersections
}

```

## 5.1. Convex Hull $\mathcal{O}(n \log n)$ .

```

// the convex hull consists of: { pts[ret[0]],
    ↳ pts[ret[1]], ... pts[ret.back()] }
vi convexHull(const vector<pt> &pts) {
    if (pts.empty()) return vi();
    vi ret, ord;
    int n = pts.size(), st = min_element(all(pts)) -
        ↳ pts.begin();
    rep(i, 0, n)
        if (pts[i] != pts[st]) ord.pb(i);

```

```

sort(all(ord), [&pts,&st] (int a, int b) {
    pt p = pts[a] - pts[st], q = pts[b] - pts[st];
    return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) <
        ↪ lenSq(q);
});
ord.pb(st); ret.pb(st);
for (int i : ord) {
    // use '>' to include ALL points on the hull-line
    for (int s = ret.size() - 1; s > 0 &&
        ↪ ((pts[ret[s-1]] - pts[ret[s]]) ^ (pts[i] -
        ↪ pts[ret[s]])) >= 0; s--)
        ret.pop_back();
    ret.pb(i);
}
ret.pop_back();
return ret;
}

```

5.2. Rotating Calipers  $\mathcal{O}(n)$ . Finds the longest distance between two points in a convex hull.

```

NUM rotatingCalipers(vector<pt> &hull) {
    int n = hull.size(), a = 0, b = 1;
    if (n <= 1) return 0.0;
    while ((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
        ↪ hull[b])) > 0) b++;
    NUM ret = 0.0;
    while (a < n) {
        ret = max(ret, lenSq(hull[a], hull[b]));
        if (((hull[(a + 1) % n] - hull[a % n]) ^ (hull[(b
        ↪ + 1) % n] - hull[b])) <= 0) a++;
        else if (++b == n) b = 0;
    }
    return ret;
}

```

5.3. Closest points  $\mathcal{O}(n \log n)$ .

```

int n; pt pts[maxn];

struct byY {
    bool operator()(int a, int b) const { return
        ↪ pts[a].y < pts[b].y; }
};

inline NUM dist(ii p) { return hypot(pts[p.x].x -
    ↪ pts[p.y].x, pts[p.x].y - pts[p.y].y); }

ii minpt(ii p1, ii p2) { return dist(p1) < dist(p2) ?
    ↪ p1 : p2; }

// closest pts (by index) inside pts[l ... r], with
    ↪ sorted y values in ys
ii closest(int l, int r, vi &ys) {
    if (r - l == 2) { // don't assume 1 here.
        ys = { l, l + 1 };
        return ii(l, l + 1);
    } else if (r - l == 3) { // brute-force
        ys = { l, l + 1, l + 2 };
    }
}

```

```

sort(all(ys), byY());
return minpt(ii(l, l + 1), minpt(ii(l, l + 2),
    ↪ ii(l + 1, l + 2)));
}
int m = (l + r) / 2; vi yl, yr;
ii delta = minpt(closest(l, m, yl), closest(m, r,
    ↪ yr));
NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
    ↪ pts[m].x);
merge(all(yl), all(yr), back_inserter(ys), byY());
deque<int> q;
for (int i : ys) if (abs(pts[i].x - xm) <= ddelta)
    ↪ {
        for (int j : q) delta = minpt(delta, ii(i, j));
        q.pb(i);
        if (q.size() > 8) q.pop_front(); // magic from
            ↪ Introduction to Algorithms.
    }
return delta;
}

```

5.4. Great-Circle Distance. Computes the distance between two points (given as latitude/longitude coordinates) on a sphere of radius  $r$ .

```

ld gc_distance(ld pLat, ld pLong, ld qLat, ld qLong,
    ↪ ld r) {
    pLat *= pi / 180; pLong *= pi / 180;
    qLat *= pi / 180; qLong *= pi / 180;
    return r * acos(cos(pLat)*cos(qLat)*cos(pLong -
        ↪ qLong) + sin(pLat)*sin(qLat)); }

```

5.5. 3D Primitives.

```

#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)
#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
    double x, y, z;
    point3d() : x(0), y(0), z(0) {}
    point3d(double _x, double _y, double _z)
        : x(_x), y(_y), z(_z) {}
    point3d operator+(P(p)) const {
        return point3d(x + p.x, y + p.y, z + p.z); }
    point3d operator-(P(p)) const {
        return point3d(x - p.x, y - p.y, z - p.z); }
    point3d operator-() const {
        return point3d(-x, -y, -z); }
    point3d operator*(double k) const {
        return point3d(x * k, y * k, z * k); }
    point3d operator/(double k) const {
        return point3d(x / k, y / k, z / k); }
    double operator%(P(p)) const {
        return x * p.x + y * p.y + z * p.z; }
    point3d operator*(P(p)) const {
        return point3d(y*p.z - z*p.y,
            ↪ z*p.x - x*p.z, x*p.y - y*p.x); }
    double length() const {
        return sqrt(*this % *this); }
}

```

```

double distTo(P(p)) const {
    return (*this - p).length(); }
double distTo(P(A), P(B)) const {
    // A and B must be two different points
    return ((*this - A) * (*this - B)).length() /
        ↪ A.distTo(B); }
point3d normalize(double k = 1) const {
    // length() must not return 0
    return (*this) * (k / length()); }
point3d getProjection(P(A), P(B)) const {
    point3d v = B - A;
    return A + v.normalize((v % (*this - A)) /
        ↪ v.length()); }
point3d rotate(P(normal)) const {
    //normal must have length 1 and be orthogonal to
        ↪ the vector
    return (*this) * normal; }
point3d rotate(double alpha, P(normal)) const {
    return (*this) * cos(alpha) + rotate(normal) *
        ↪ sin(alpha); }
point3d rotatePoint(P(O), P(axe), double alpha)
    ↪ const {
    point3d Z = axe.normalize(axe % (*this - O));
    return O + Z + (*this - O - Z).rotate(alpha, O);
    ↪ }
bool isZero() const {
    return abs(x) < EPS && abs(y) < EPS && abs(z) <
        ↪ EPS; }
bool isOnLine(L(A, B)) const {
    return ((A - *this) * (B - *this)).isZero(); }
bool isInSegment(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
        ↪ *this)) < EPS; }
bool isInSegmentStrictly(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
        ↪ *this)) < -EPS; }
double getAngle() const {
    return atan2(y, x); }
double getAngle(P(u)) const {
    return atan2((*this * u).length(), *this % u); }
bool isOnPlane(PL(A, B, C)) const {
    return
        ↪ abs((A - *this) * (B - *this) % (C - *this)) <
        ↪ EPS; } }
int line_line_intersect(L(A, B), L(C, D), point3d
    ↪ &O) {
    if (abs((B - A) * (C - A) % (D - A)) > EPS) return
        ↪ 0;
    if (((A - B) * (C - D)).length() < EPS)
        ↪ return A.isOnLine(C, D) ? 2 : 0;
    point3d normal = ((A - B) * (C - B)).normalize();
    double s1 = (C - A) * (D - A) % normal;
    O = A + ((B - A) / (s1 + ((D - B) * (C - B) %
        ↪ normal))) * s1;
    return 1; }

```

```
int line_plane_intersect(L(A, B), PL(C, D, E),
    ⇐ point3d &O) {
    double V1 = (C - A) * (D - A) % (E - A);
    double V2 = (D - B) * (C - B) % (E - B);
    if (abs(V1 + V2) < EPS)
        return A.isOnPlane(C, D, E) ? 2 : 0;
    O = A + ((B - A) / (V1 + V2)) * V1;
    return 1; }

bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
    point3d &P, point3d &Q) {
    point3d n = nA * nB;
    if (n.isZero()) return false;
    point3d v = n * nA;
    P = A + (n * nA) * ((B - A) % nB / (v % nB));
    Q = P + n;
    return true; }
```

### 5.6. Polygon Centroid.

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

**5.7. Rectilinear Minimum Spanning Tree.** Given a set of  $n$  points in the plane, and the aim is to find a minimum spanning tree connecting these  $n$  points, assuming the Manhattan distance is used. The function `candidates` returns at most  $4n$  edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```
#define MAXN 100100
struct RMST {
    struct point {
        int i; ll x, y;
        point() : i(-1) {}
        ll d1() { return x + y; }
        ll d2() { return x - y; }
        ll dist(point other) {
            return abs(x - other.x) + abs(y - other.y); }
        bool operator <(const point &other) const {
            return y == other.y ? x > other.x : y <
                ⇐ other.y; }
    } best[MAXN], arr[MAXN], tmp[MAXN];
    int n;
    RMST() : n(0) {}
    void add_point(int x, int y) {
        arr[arr[n].i = n].x = x, arr[n++].y = y; }
    void rec(int l, int r) {
        if (l >= r) return;
        int m = (l+r)/2;
        rec(l,m), rec(m+1,r);
        point bst;
```

```

for (int i = 1, j = m+1, k = 1; i <= m || j <= r;
    ↪ k++) {
    if (j > r || (i <= m && arr[i].d1() <
        ↪ arr[j].d1())) {
        tmp[k] = arr[i++];
        if (bst.i != -1 && (best[tmp[k].i].i == -1
            || best[tmp[k].i].d2() <
                ↪ bst.d2()))
            best[tmp[k].i] = bst;
    } else {
        tmp[k] = arr[j++];
        if (bst.i == -1 || bst.d2() < tmp[k].d2())
            bst = tmp[k]; } }
rep(i,1,r+1) arr[i].i = tmp[i]; }
vector<pair<ll,ii> > candidates() {
    vector<pair<ll, ii> > es;
    rep(p,0,2) {
        rep(q,0,2) {
            sort(arr, arr+n);
            rep(i,0,n) best[i].i = -1;
            rec(0,n-1);
            rep(i,0,n) {
                if(best[arr[i].i].i != -1)
                    ↪ es.push_back({arr[i].dist(best[arr[i].i]),
                        {arr[i].i,
                            ↪ best[arr[i].i].i}}});
                swap(arr[i].x, arr[i].y);
                arr[i].x *= -1, arr[i].y *= -1; } }
            rep(i,0,n) arr[i].x *= -1; }
    return es; } }

```

**5.8. Formulas.** Let  $a = (a_x, a_y)$  and  $b = (b_x, b_y)$  be two-dimensional vectors.

- $a \cdot b = |a||b| \cos \theta$ , where  $\theta$  is the angle between  $a$  and  $b$ .
- $a \times b = |a||b| \sin \theta$ , where  $\theta$  is the signed angle between  $a$  and  $b$ .
- $a \times b$  is equal to the area of the parallelogram with two of its sides formed by  $a$  and  $b$ . Half of that is the area of the triangle formed by  $a$  and  $b$ .
- **Euler's formula:**  $V - E + F = 2$
- Side lengths  $a, b, c$  can form a triangle iff.  $a + b > c$ ,  $b + c > a$  and  $a + c > b$ .
- Sum of internal angles of a regular convex  $n$ -gon is  $(n - 2)\pi$ .
- **Law of sines:**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:**  $b^2 = a^2 + c^2 - 2ac \cos B$
- Internal tangents of circles  $(c_1, r_1), (c_2, r_2)$  intersect at  $(c_1 r_2 + c_2 r_1) / (r_1 + r_2)$ , external intersect at  $(c_1 r_2 - c_2 r_1) / (r_1 + r_2)$ .

## 6. MISCELLANEOUS

### 6.1. Binary search $\mathcal{O}(\log(hi - lo))$ .

```
bool test(int n);
```

```
int search(int lo, int hi) {
```

```
assert(test(lo) && !test(hi)); // BE CERTAIN
while (hi - lo > 1) {
    int m = (lo + hi) / 2;
    (test(m) ? lo : hi) = m;
}
// assert(test(lo) && !test(hi));
return lo;
}
```

**6.2. Fast Fourier Transform**  $\mathcal{O}(n \log n)$ . Given two polynomials  $A(x) = a_0 + a_1x + \dots + a_{n/2}x^{n/2}$  and  $B(x) = b_0 + b_1x + \dots + b_{n/2}x^{n/2}$ , FFT calculates all coefficients of  $C(x) = A(x) \cdot B(x) = c_0 + c_1x + \dots + c_nx^n$ , with  $c_i = \sum_{j=0}^i a_j b_{i-j}$ .

```
typedef complex<double> cpx;
const int LOGN = 19, MAXN = 1 << LOGN;
```

```
int rev[MAXN];
cpx rt[MAXN], a[MAXN] = {}, b[MAXN] = {};
```

```
void fft(cpx *A) {
    REP(i, MAXN) if (i < rev[i]) swap(A[i], A[rev[i]]);
    for (int k = 1; k < MAXN; k *= 2)
        for (int i = 0; i < MAXN; i += 2*k) REP(j, k) {
            cpx t = rt[j + k] * A[i + j + k];
            A[i + j + k] = A[i + j] - t;
            A[i + j] += t;
        }
}
```

```
void multiply() { // a = convolution of a * b
    rev[0] = 0; rt[1] = cpx(1, 0);
    REP(i, MAXN) rev[i] = (rev[i/2] | (i&1)<<LOGN)/2;
    for (int k = 2; k < MAXN; k *= 2) {
        cpx z(cos(PI/k), sin(PI/k));
        rep(i, k/2, k) rt[2*i]=rt[i], rt[2*i+1]=rt[i]*z;
    }
    fft(a); fft(b);
    REP(i, MAXN) a[i] *= b[i] / (double)MAXN;
    reverse(a+1,a+MAXN); fft(a);
}
```

### 6.3. Minimum Assignment (Hungarian Algorithm) $\mathcal{O}(n^3)$ .

```
int a[MAXN + 1][MAXM + 1]; // matrix, 1-based
int minimum_assignment(int n, int m) { // n rows, m
    ↪ columns
    vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);
    for (int i = 1; i <= n; i++) {
        p[0] = i;
        int j0 = 0;
        vi minv(m + 1, INT_MAX);
        vector<char> used(m + 1, false);
        do {
            used[j0] = true;
            int i0 = p[j0], delta = INT_MAX, j1;
            for (int j = 1; j <= m; j++)
                if (!used[j]) {
                    int cur = a[i0][j] - u[i0] - v[j];
```

```

    if (cur < minv[j]) minv[j] = cur, way[j] =
        ↪ j0;
    if (minv[j] < delta) delta = minv[j], j1 =
        ↪ j;
}
for (int j = 0; j <= m; j++) {
    if (used[j]) u[p[j]] += delta, v[j] -= delta;
    else minv[j] -= delta;
}
j0 = j1;
while (p[j0] != 0);
do {
    int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
} while (j0);
// column j is assigned to row p[j]
return -v[0];
}

```

#### 6.4. Partial linear equation solver $\mathcal{O}(N^3)$ .

```

typedef double NUM;
const int MAXROWS = 200, MAXCOLS = 200;
const NUM EPS = 1e-5;

// F2: bitset<MAXCOLS+1> mat[MAXROWS];
↪ bitset<MAXROWS> vals;
NUM mat[MAXROWS][MAXCOLS + 1], vals[MAXCOLS]; bool
↪ hasval[MAXCOLS];
bool is0(NUM a) { return -EPS < a && a < EPS; }

// finds x such that Ax = b
// A_ij is mat[i][j], b_i is mat[i][m]
int solvemmat(int n, int m) {
    // F2: vals.reset();
    int pr = 0, pc = 0;
    while (pc < m) {
        int r = pr, c;
        while (r < n && is0(mat[r][pc])) r++;
        if (r == n) { pc++; continue; }

        // F2: mat[pr] ^= mat[r]; mat[r] ^= mat[pr];
        ↪ mat[pr] ^= mat[r];
        for (c = 0; c <= m; c++) swap(mat[pr][c],
            ↪ mat[r][c]);

        r = pr++; c = pc++;
        // F2: vals.set(pc, mat[pr][m]);
        NUM div = mat[r][c];
        for (int col = c; col <= m; col++) mat[r][col] /=
            ↪ div;
        REP(row, n) {
            if (row == r) continue;
            // F2: if (mat[row].test(c)) mat[row] ^=
            ↪ mat[r];
            NUM times = -mat[row][c];
            for (int col = c; col <= m; col++)
                mat[row][col] += times * mat[r][col];
        }
    }
}

```

```

    }
} // now mat is in RREF

for (int r = pr; r < n; r++)
    if (!is0(mat[r][m])) return 0;
// F2: return 1;
fill_n(hasval, n, false);
for (int col = 0, row; col < m; col++) {
    hasval[col] = !is0(mat[row][col]);
    if (!hasval[col]) continue;
    for (int c = col + 1; c < m; c++) {
        if (!is0(mat[row][c])) hasval[col] = false;
    }
    if (hasval[col]) vals[col] = mat[row][m];
    row++;
}
REP(i, n) if (!hasval[i]) return 2;
return 1;
}

```

#### 6.5. Cycle-Finding.

```

ii find_cycle(int x0, int (*f)(int)) {
    int t = f(x0), h = f(t), mu = 0, lam = 1;
    while (t != h) t = f(t), h = f(f(h));
    h = x0;
    while (t != h) t = f(t), h = f(h), mu++;
    h = f(t);
    while (t != h) h = f(h), lam++;
    return ii(mu, lam); }

```

#### 6.6. Longest Increasing Subsequence.

```

vi lis(vi arr) {
    vi seq, back(size(arr)), ans;
    rep(i, 0, size(arr)) {
        int res = 0, lo = 1, hi = size(seq);
        while (lo <= hi) {
            int mid = (lo+hi)/2;
            if (arr[seq[mid-1]] < arr[i]) res = mid, lo =
                ↪ mid + 1;
            else hi = mid - 1; }
        if (res < size(seq)) seq[res] = i;
        else seq.push_back(i);
        back[i] = res == 0 ? -1 : seq[res-1]; }
    int at = seq.back();
    while (at != -1) ans.push_back(at), at = back[at];
    reverse(ans.begin(), ans.end());
    return ans; }

```

#### 6.7. Dates.

```

int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
    return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
        367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
        3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
        d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
    int x, n, i, j;
    x = jd + 68569;
}

```

```

n = 4 * x / 146097;
x -= (146097 * n + 3) / 4;
i = (4000 * (x + 1)) / 1461001;
x -= 1461 * i / 4 - 31;
j = 80 * x / 2447;
d = x - 2447 * j / 80;
x = j / 11;
m = j + 2 - 12 * x;
y = 100 * (n - 49) + i + x; }

```

#### 6.8. Simplex.

```

typedef long double DOUBLE;
typedef vector<DOUBLE> VD;
typedef vector<VD> VVD;
typedef vector<int> VI;
const DOUBLE EPS = 1e-9;
struct LPSolver {
    int m, n;
    VI B, N;
    VVD D;
    LPSolver(const VVD &A, const VD &b, const VD &c) :
        m(b.size()), n(c.size()),
        N(n + 1), B(m), D(m + 2, VD(n + 2)) {
        for (int i = 0; i < m; i++) for (int j = 0; j < n;
            ↪ j++)
            D[i][j] = A[i][j];
        for (int i = 0; i < m; i++) { B[i] = n + i; D[i][n]
            ↪ = -1;
            D[i][n + 1] = b[i]; }
        for (int j = 0; j < n; j++) { N[j] = j; D[m][j] =
            ↪ -c[j]; }
        N[n] = -1; D[m + 1][n] = 1; }
    void Pivot(int r, int s) {
        double inv = 1.0 / D[r][s];
        for (int i = 0; i < m + 2; i++) if (i != r)
            for (int j = 0; j < n + 2; j++) if (j != s)
                D[i][j] -= D[r][j] * D[i][s] * inv;
        for (int j = 0; j < n + 2; j++) if (j != s) D[r][j]
            ↪ *= inv;
        for (int i = 0; i < m + 2; i++) if (i != r) D[i][s]
            ↪ *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]); }
    bool Simplex(int phase) {
        int x = phase == 1 ? m + 1 : m;
        while (true) {
            int s = -1;
            for (int j = 0; j <= n; j++) {
                if (phase == 2 && N[j] == -1) continue;
                if (s == -1 || D[x][j] < D[x][s] ||
                    D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
            if (D[x][s] > -EPS) return true;
            int r = -1;
            for (int i = 0; i < m; i++) {
                if (D[i][s] < EPS) continue;
                if (r == -1 || D[i][n + 1] / D[i][s] < D[r][n +
                    ↪ 1] /

```



```

    D[r][s] || (D[i][n + 1] / D[i][s]) == (D[r][n
    ↪ + 1] /
    D[r][s]) && B[i] < B[r]) r = i; }
    if (r == -1) return false;
    Pivot(r, s); } }
DOUBLE Solve(VD &x) {
    int r = 0;
    for (int i = 1; i < m; i++) if (D[i][n + 1] <
    ↪ D[r][n + 1])
        r = i;
    if (D[r][n + 1] < -EPS) {
        Pivot(r, n);
        if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
            return -numeric_limits<DOUBLE>::infinity();
        for (int i = 0; i < m; i++) if (B[i] == -1) {
            int s = -1;
            for (int j = 0; j <= n; j++)
                if (s == -1 || D[i][j] < D[i][s] ||
                    D[i][j] == D[i][s] && N[j] < N[s])
                    s = j;
            Pivot(i, s); } }
    if (!Simplex(2)) return
    ↪ numeric_limits<DOUBLE>::infinity();
    x = VD(n);
    for (int i = 0; i < m; i++) if (B[i] < n)
        x[B[i]] = D[i][n + 1];
    return D[m][n + 1]; } }
// Two-phase simplex algorithm for solving linear
// programs
// of the form
//      maximize      c^T x
//      subject to    Ax <= b
//                  x >= 0
// INPUT: A -- an m x n matrix
//        b -- an m-dimensional vector
//        c -- an n-dimensional vector
//        x -- a vector where the optimal solution
// will be
//      stored
// OUTPUT: value of the optimal solution (infinity if
//        unbounded above, nan if
//        infeasible)
// To use this code, create an LPSolver object with
// A, b,
// and c as arguments. Then, call Solve(x).
// #include <iostream>
// #include <iomanip>
// #include <vector>
// #include <cmath>
// #include <limits>
// using namespace std;
// int main() {
//     const int m = 4;
//     const int n = 3;
//     DOUBLE _A[m][n] = {
//         { 6, -1, 0 },
//         { -1, -5, 0 },

```

```

//     { 1, 5, 1 },
//     { -1, -5, -1 }
// };
// DOUBLE _b[m] = { 10, -4, 5, -5 };
// DOUBLE _c[n] = { 1, -1, 0 };
// VVD A(m);
// VD b(_b, _b + m);
// VD c(_c, _c + n);
// for (int i = 0; i < m; i++) A[i] = VD(_A[i],
// ↪ _A[i] + n);
// LPSolver solver(A, b, c);
// VD x;
// DOUBLE value = solver.Solve(x);
// cerr << "VALUE: " << value << endl; // VALUE:
// 1.29032
// cerr << "SOLUTION:"; // SOLUTION: 1.74194
// 0.451613 1
// for (size_t i = 0; i < x.size(); i++) cerr << "
// << x[i];
// cerr << endl;
// return 0;
// }

```

## 7. GEOMETRY (CP3)

### 7.1. Points and lines.

```

#define EPS 1e-9
#define PI acos(-1.0) // important constant;
// alternative #define PI (2.0 * acos(0.0))

double DEG_to_RAD(double d) { return d * PI / 180.0;
// }

double RAD_to_DEG(double r) { return r * 180.0 / PI;
// }

struct point { double x, y; // only used if more
// precision is needed
point() { x = y = 0.0; } //
// default constructor
point(double _x, double _y) : x(_x), y(_y) {}
// user-defined
bool operator < (point other) const { // override
// less than operator
    if (fabs(x - other.x) > EPS) //
        ↪ useful for sorting
        return x < other.x; // first criteria
        ↪ , by x-coordinate
        return y < other.y; } // second
        ↪ criteria, by y-coordinate
// use EPS (1e-9) when testing equality of two
// floating points
bool operator == (point other) const {
    return (fabs(x - other.x) < EPS && (fabs(y -
    ↪ other.y) < EPS)); } }

```

```

double dist(point p1, point p2) { //
// Euclidean distance
// hypot(dx, dy) returns
// ↪ sqrt(dx * dx + dy * dy)
return hypot(p1.x - p2.x, p1.y - p2.y); }
// ↪ return double

// rotate p by theta degrees CCW w.r.t origin (0, 0)
point rotate(point p, double theta) {
    double rad = DEG_to_RAD(theta); // multiply
    ↪ theta with PI / 180.0
    return point(p.x * cos(rad) - p.y * sin(rad),
        p.x * sin(rad) + p.y * cos(rad)); }

struct line { double a, b, c; }; // a way to
// represent a line

// the answer is stored in the third parameter (pass
// by reference)
void pointsToLine(point p1, point p2, line &l) {
    if (fabs(p1.x - p2.x) < EPS) { //
        ↪ vertical line is fine
        l.a = 1.0; l.b = 0.0; l.c = -p1.x;
        ↪ // default values
    } else {
        l.a = -(double)(p1.y - p2.y) / (p1.x - p2.x);
        l.b = 1.0; // IMPORTANT: we fix the
        ↪ value of b to 1.0
        l.c = -(double)(l.a * p1.x) - p1.y;
    } }

bool areParallel(line l1, line l2) { // check
// coefficients a & b
return (fabs(l1.a - l2.a) < EPS) && (fabs(l1.b - l2.b)
// < EPS); }

bool areSame(line l1, line l2) { // also
// check coefficient c
return areParallel(l1, l2) && (fabs(l1.c - l2.c) <
// EPS); }

// returns true (+ intersection point) if two lines
// are intersect
bool areIntersect(line l1, line l2, point &p) {
    if (areParallel(l1, l2)) return false;
    ↪ // no intersection
    // solve system of 2 linear algebraic equations
    // with 2 unknowns
    p.x = (l2.b * l1.c - l1.b * l2.c) / (l2.a * l1.b -
    ↪ l1.a * l2.b);
    // special case: test for vertical line to avoid
    ↪ division by zero
    if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
    else p.y = -(l2.a * p.x + l2.c);
    return true; }

```

```

struct vec { double x, y; // name: 'vec' is
    ↪ different from STL vector
    vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) { // convert 2
    ↪ points to vector a->b
    return vec(b.x - a.x, b.y - a.y); }

vec scale(vec v, double s) { // nonnegative s
    ↪ = [ <1 .. 1 .. >1 ]
    return vec(v.x * s, v.y * s); } //
    ↪ shorter.same.longer

point translate(point p, vec v) { // translate
    ↪ p according to v
    return point(p.x + v.x, p.y + v.y); }

// convert point and gradient/slope to line
void pointSlopeToLine(point p, double m, line &l) {
    l.a = -m;
    ↪ // always -m
    l.b = 1;
    ↪ // always 1
    l.c = -((l.a * p.x) + (l.b * p.y)); }
    ↪ // compute this

void closestPoint(line l, point p, point &ans) {
    line perpendicular; // perpendicular to l
    ↪ and pass through p
    if (fabs(l.b) < EPS) { // special case
        ↪ 1: vertical line
        ans.x = -(l.c); ans.y = p.y; return; }

    if (fabs(l.a) < EPS) { // special case
        ↪ 2: horizontal line
        ans.x = p.x; ans.y = -(l.c); return; }

    pointSlopeToLine(p, 1 / l.a, perpendicular);
    ↪ // normal line
    // intersect line l with this perpendicular line
    // the intersection point is the closest point
    areIntersect(l, perpendicular, ans); }

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
    point b;
    closestPoint(l, p, b); //
    ↪ similar to distToLine
    vec v = toVec(p, b); //
    ↪ create a vector
    ans = translate(translate(p, v), v); //
    ↪ translate p twice

double dot(vec a, vec b) { return (a.x * b.x + a.y *
    ↪ b.y); }

```

```

double norm_sq(vec v) { return v.x * v.x + v.y * v.y;
    ↪ }

// returns the distance from p to the line defined by
// two points a and b (a and b must be different)
// the closest point is stored in the 4th parameter
    ↪ (byref)
double distToLine(point p, point a, point b, point
    ↪ &c) {
    // formula: c = a + u * ab
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    c = translate(a, scale(ab, u)); //
    ↪ translate a to c
    return dist(p, c); } // Euclidean
    ↪ distance between p and c

// returns the distance from p to the line segment ab
    ↪ defined by
// two points a and b (still OK if a == b)
// the closest point is stored in the 4th parameter
    ↪ (byref)
double distToLineSegment(point p, point a, point b,
    ↪ point &c) {
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    if (u < 0.0) { c = point(a.x, a.y);
        ↪ // closer to a
        return dist(p, a); } // Euclidean
        ↪ distance between p and a
    if (u > 1.0) { c = point(b.x, b.y);
        ↪ // closer to b
        return dist(p, b); } // Euclidean
        ↪ distance between p and b
    return distToLine(p, a, b, c); // run
    ↪ distToLine as above

double angle(point a, point o, point b) { // returns
    ↪ angle aob in rad
    vec oa = toVec(o, a), ob = toVec(o, b);
    return acos(dot(oa, ob) / sqrt(norm_sq(oa) *
        ↪ norm_sq(ob))); }

double cross(vec a, vec b) { return a.x * b.y - a.y *
    ↪ b.x; }

// note: to accept collinear points, we have to
    ↪ change the '> 0'
// returns true if point r is on the left side of
    ↪ line pq
bool ccw(point p, point q, point r) {
    return cross(toVec(p, q), toVec(p, r)) > 0; }

// returns true if point r is on the same line as the
    ↪ line pq

```

```

bool collinear(point p, point q, point r) {
    return fabs(cross(toVec(p, q), toVec(p, r))) < EPS;
    ↪ }

```

## 7.2. Polygon.

```

// returns the perimeter, which is the sum of
    ↪ Euclidian distances
// of consecutive line segments (polygon edges)
double perimeter(const vector<point> &P) {
    double result = 0.0;
    for (int i = 0; i < (int)P.size()-1; i++) //
        ↪ remember that P[0] = P[n-1]
        result += dist(P[i], P[i+1]);
    return result; }

// returns the area, which is half the determinant
double area(const vector<point> &P) {
    double result = 0.0, x1, y1, x2, y2;
    for (int i = 0; i < (int)P.size()-1; i++) {
        x1 = P[i].x; x2 = P[i+1].x;
        y1 = P[i].y; y2 = P[i+1].y;
        result += (x1 * y2 - x2 * y1);
    }
    return fabs(result) / 2.0; }

// returns true if we always make the same turn while
    ↪ examining
// all the edges of the polygon one by one
bool isConvex(const vector<point> &P) {
    int sz = (int)P.size();
    if (sz <= 3) return false; // a point/sz=2 or a
        ↪ line/sz=3 is not convex
    bool isLeft = ccw(P[0], P[1], P[2]);
    ↪ // remember one result
    for (int i = 1; i < sz-1; i++) // then
        ↪ compare with the others
        if (ccw(P[i], P[i+1], P[(i+2) == sz ? 1 : i+2])
            ↪ != isLeft)
            return false; // different sign ->
            ↪ this polygon is concave
    return true; } //
    ↪ this polygon is convex

// returns true if point p is in either
    ↪ convex/concave polygon P
bool inPolygon(point pt, const vector<point> &P) {
    if ((int)P.size() == 0) return false;
    double sum = 0; // assume the first vertex is
        ↪ equal to the last vertex
    for (int i = 0; i < (int)P.size()-1; i++) {
        if (ccw(pt, P[i], P[i+1]))
            sum += angle(P[i], pt, P[i+1]);
        ↪ // left turn/ccw
        else sum -= angle(P[i], pt, P[i+1]); }
    ↪ // right turn/cw

```

```

    return fabs(fabs(sum) - 2*PI) < EPS; }

// line segment p-q intersect with line A-B.
point lineIntersectSeg(point p, point q, point A,
    ↪ point B) {
    double a = B.y - A.y;
    double b = A.x - B.x;
    double c = B.x * A.y - A.x * B.y;
    double u = fabs(a * p.x + b * p.y + c);
    double v = fabs(a * q.x + b * q.y + c);
    return point((p.x * v + q.x * u) / (u+v), (p.y * v
    ↪ + q.y * u) / (u+v)); }

// cuts polygon Q along the line formed by point a ->
    ↪ point b
// (note: the last point must be the same as the
    ↪ first point)
vector<point> cutPolygon(point a, point b, const
    ↪ vector<point> &Q) {
    vector<point> P;
    for (int i = 0; i < (int)Q.size(); i++) {
        double left1 = cross(toVec(a, b), toVec(a,
            ↪ Q[i])), left2 = 0;
        if (i != (int)Q.size()-1) left2 = cross(toVec(a,
            ↪ b), toVec(a, Q[i+1]));
        if (left1 > -EPS) P.push_back(Q[i]); //
            ↪ Q[i] is on the left of ab
        if (left1 * left2 < -EPS) // edge (Q[i],
            ↪ Q[i+1]) crosses line ab
            P.push_back(lineIntersectSeg(Q[i], Q[i+1], a,
            ↪ b));
    }
    if (!P.empty() && !(P.back() == P.front()))
        P.push_back(P.front()); // make P's first
        ↪ point = P's last point
    return P; }

point pivot;
bool angleCmp(point a, point b) { //
    ↪ angle-sorting function
    if (collinear(pivot, a, b))
        ↪ // special case
        return dist(pivot, a) < dist(pivot, b); //
            ↪ check which one is closer
    double dlx = a.x - pivot.x, dly = a.y - pivot.y;
    double d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (atan2(dly, dlx) - atan2(d2y, d2x)) < 0; }
    ↪ // compare two angles

vector<point> CH(vector<point> P) { // the content
    ↪ of P may be reshuffled
    int i, j, n = (int)P.size();
    if (n <= 3) {
        if (!(P[0] == P[n-1])) P.push_back(P[0]); //
        ↪ safeguard from corner case

```

```

    return P; // special
    ↪ case, the CH is P itself
}

// first, find P0 = point with lowest Y and if tie:
    ↪ rightmost X
int P0 = 0;
for (i = 1; i < n; i++)
    if (P[i].y < P[P0].y || (P[i].y == P[P0].y &&
        ↪ P[i].x > P[P0].x))
        P0 = i;

point temp = P[0]; P[0] = P[P0]; P[P0] = temp;
    ↪ // swap P[P0] with P[0]

// second, sort points by angle w.r.t. pivot P0
pivot = P[0]; // use this global
    ↪ variable as reference
sort(++P.begin(), P.end(), angleCmp);
    ↪ // we do not sort P[0]

// third, the ccw tests
vector<point> S;
S.push_back(P[n-1]); S.push_back(P[0]);
    ↪ S.push_back(P[1]); // initial S
i = 2; //
    ↪ then, we check the rest
while (i < n) { // note: N must be >= 3
    ↪ for this method to work
        j = (int)S.size()-1;
        if (ccw(S[j-1], S[j], P[i])) S.push_back(P[i++]);
            ↪ // left turn, accept
        else S.pop_back(); } // or pop the top of S
    ↪ until we have a left turn
return S; }
    ↪ // return the result

```

### 7.3. Triangle.

```

double perimeter(double ab, double bc, double ca) {
    return ab + bc + ca; }

double perimeter(point a, point b, point c) {
    return dist(a, b) + dist(b, c) + dist(c, a); }

double area(double ab, double bc, double ca) {
    // Heron's formula, split sqrt(a * b) into sqrt(a)
    ↪ * sqrt(b); in implementation
    double s = 0.5 * perimeter(ab, bc, ca);
    return sqrt(s) * sqrt(s - ab) * sqrt(s - bc) *
    ↪ sqrt(s - ca); }

double area(point a, point b, point c) {
    return area(dist(a, b), dist(b, c), dist(c, a)); }

double rInCircle(double ab, double bc, double ca) {
    return area(ab, bc, ca) / (0.5 * perimeter(ab, bc,
    ↪ ca)); }

```

```

double rInCircle(point a, point b, point c) {
    return rInCircle(dist(a, b), dist(b, c), dist(c,
    ↪ a)); }

// assumption: the required points/lines functions
    ↪ have been written
// returns 1 if there is an inCircle center, returns
    ↪ 0 otherwise
// if this function returns 1, ctr will be the
    ↪ inCircle center
// and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point
    ↪ &ctr, double &r) {
    r = rInCircle(p1, p2, p3);
    if (fabs(r) < EPS) return 0; //
        ↪ no inCircle center

    line l1, l2; // compute these
        ↪ two angle bisectors
    double ratio = dist(p1, p2) / dist(p1, p3);
    point p = translate(p2, scale(toVec(p2, p3), ratio
        ↪ / (1 + ratio)));
    pointsToLine(p1, p, l1);

    ratio = dist(p2, p1) / dist(p2, p3);
    p = translate(p1, scale(toVec(p1, p3), ratio / (1 +
        ↪ ratio)));
    pointsToLine(p2, p, l2);

    areIntersect(l1, l2, ctr); // get their
        ↪ intersection point
    return 1; }

double rCircumCircle(double ab, double bc, double ca)
    ↪ {
    return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

double rCircumCircle(point a, point b, point c) {
    return rCircumCircle(dist(a, b), dist(b, c),
    ↪ dist(c, a)); }

// assumption: the required points/lines functions
    ↪ have been written
// returns 1 if there is a circumCenter center,
    ↪ returns 0 otherwise
// if this function returns 1, ctr will be the
    ↪ circumCircle center
// and r is the same as rCircumCircle
int circumCircle(point p1, point p2, point p3, point
    ↪ &ctr, double &r) {
    double a = p2.x - p1.x, b = p2.y - p1.y;
    double c = p3.x - p1.x, d = p3.y - p1.y;
    double e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
    double f = c * (p1.x + p3.x) + d * (p1.y + p3.y);

```

```
double g = 2.0 * (a * (p3.y - p2.y) - b * (p3.x -
    ↪ p2.x));
if (fabs(g) < EPS) return 0;

ctr.x = (d*e - b*f) / g;
ctr.y = (a*f - c*e) / g;
r = dist(p1, ctr); // r = distance from center to
    ↪ 1 of the 3 points
return 1; }
```

```
// returns if pt d is inside the circumCircle defined
    ↪ by a,b,c
```

```
bool inCircumCircle(point a, point b, point c, point d) {
    vec va=toVec(a, d), vb=toVec(b, d), vc=toVec(c, d);
    return 0 <
        (va.x)*(vb.y)*(vc.x)*(vc.x)+(vc.y)*(vc.y)+(va.y)*
        (vb.x)*(vb.x)+(vb.y)*(vb.y)+(vc.x)+(va.y)*(vb.x)+
        (va.y)*(va.x)+(va.y)*(va.y)+(vb.x)*(vc.y)-
        ((va.x)*(va.x)+(va.y)*(va.y))*(vb.y)*(vc.x)-
        (va.y)*(vb.x)*(vc.x)+(vc.x)+(vc.y)*(vc.y)-
        (va.x)*(vb.x)*(vb.x)+(vb.y)*(vb.y)*(vc.y));
}
```

```
bool canFormTriangle(double a, double b, double c) {
    return (a + b > c) && (a + c > b) && (b + c > a); }
```

#### 7.4. Circle.

```
int insideCircle(point_i p, point_i c, int r) { //
    ↪ all integer version
    int dx = p.x - c.x, dy = p.y - c.y;
    int Euc = dx * dx + dy * dy, rSq = r * r;
    ↪ // all integer
    return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
    ↪ //inside/border/outside
```

```
bool circle2PtsRad(point p1, point p2, double r,
    ↪ point &c) {
    double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
        (p1.y - p2.y) * (p1.y - p2.y);
    double det = r * r / d2 - 0.25;
    if (det < 0.0) return false;
    double h = sqrt(det);
    c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
    c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
    return true; } // to get the other center,
    ↪ reverse p1 and p2
```

#### 8. COMBINATORICS

- Catalan numbers (valid bracket seq's of length  $2n$ ):  
 $C_0 = 1, C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1}$ .
- Stirling 1<sup>th</sup> kind ( $\#\pi \in \mathfrak{S}_n$  with exactly  $k$  cycles):  
 $\left[ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right] = \left[ \begin{smallmatrix} 0 \\ n \end{smallmatrix} \right] = \delta_{0n}, \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (n-1) \left[ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right] + \left[ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right]$ .
- Stirling 2<sup>nd</sup> kind ( $k$ -partitions of  $[n]$ ):  
 $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1, \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$ .

- Bell numbers (partitions of  $[n]$ ):  
 $B_0 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ .
- Euler ( $\#\pi \in \mathfrak{S}_n$  with exactly  $k$  ascents):  
 $\left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\rangle = 1, \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (n-k) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle$ .
- Euler 2<sup>nd</sup> order (nr perms of  $1, 1, 2, 2, \dots, n, n$  with exactly  $k$  ascents):  
 $\left\langle\!\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle\!\right\rangle = (k+1) \left\langle\!\left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle\!\right\rangle + (2n-k-1) \left\langle\!\left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle\!\right\rangle$ .
- Rooted trees:  $n^{n-1}$ , unrooted:  $n^{n-2}$ .
- Forests of  $k$  rooted trees:  $\binom{n}{k} k \cdot n^{n-k-1}$ .
- $1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad 1^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}, \quad \sum_i \binom{n-i}{i} = F_{n+1}$
- $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad x^k = \sum_{i=0}^k i! \left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\} \binom{x}{i} = \sum_{i=0}^k \left\langle \begin{smallmatrix} k \\ i \end{smallmatrix} \right\rangle \binom{x+i}{i}$
- $a \equiv b \pmod{x, y} \Leftrightarrow a \equiv b \pmod{\text{lcm}(x, y)}$ .
- $ac \equiv bc \pmod{m} \Leftrightarrow a \equiv b \pmod{m/\text{gcd}(c, m)}$ .
- $\text{gcd}(n^a - 1, n^b - 1) = \text{gcd}(a, b) - 1$ .
- **Möbius inversion formula:** If  $f(n) = \sum_{d|n} g(d)$ , then  $g(n) = \sum_{d|n} \mu(d) f(n/d)$ . If  $f(n) = \sum_{m=1}^n g(\lfloor n/m \rfloor)$ , then  $g(n) = \sum_{m=1}^n \mu(m) f(\lfloor \frac{n}{m} \rfloor)$ .
- **Inclusion-Exclusion:** If  $g(T) = \sum_{S \subseteq T} f(S)$ , then

$$f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} g(T).$$

Corollary:  $b_n = \sum_{k=0}^n \binom{n}{k} a_k \Leftrightarrow a_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} b_k$ .

8.1. **The Twelfold Way.** Putting  $n$  balls into  $k$  boxes.  $p(n, k)$  is # partitions of  $n$  in  $k$  parts, each  $> 0$ .  $p_k(n) = \sum_{i=0}^k p(n, k)$ .

Balls	same	distinct	same	distinct
Boxes	same	same	distinct	distinct
-	$p_k(n)$	$\sum_{i=0}^k \left\{ \begin{smallmatrix} n \\ i \end{smallmatrix} \right\}$	$\binom{n+k-1}{k-1}$	$k^n$
size $\geq 1$	$p(n, k)$	$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	$\binom{n-1}{k-1}$	$k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$
size $\leq 1$	$[n \leq k]$	$[n \leq k]$	$\binom{n}{k}$	$n! \binom{k}{n}$

#### 9. FORMULAS

- **Legendre symbol:**  $\left( \frac{a}{b} \right) = a^{(b-1)/2} \pmod{b}$ ,  $b$  odd prime.
- **Heron's formula:** A triangle with side lengths  $a, b, c$  has area  $\sqrt{s(s-a)(s-b)(s-c)}$  where  $s = \frac{a+b+c}{2}$ .
- **Shoelace formula:**  $A = \frac{1}{2} |\sum_{i=0}^{n-1} x_i y_{i+1} - x_{i+1} y_i|$ .
- **Pick's theorem:** A polygon on an integer grid strictly containing  $i$  lattice points and having  $b$  lattice points on the boundary has area  $i + \frac{b}{2} - 1$ . (Nothing similar in higher dimensions)
- **Absorption probabilities** A random walk on  $[0, n]$  with probability  $p$  to increase and  $q$  to decrease, starting at  $k$  has at  $n$  absorption probability  $\frac{(q/p)^k - 1}{(q/p)^n - 1}$  if  $q \neq p$ , and  $k/n$  if  $q = p$ .
- **König's theorem:** In any bipartite graph  $G = (L \cup R, E)$ , the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover. Let  $U$  be the set of unmatched vertices in  $L$ , and  $Z$  be the set of vertices that are either in  $U$  or are

connected to  $U$  by an alternating path. Then  $K = (L \setminus Z) \cup (R \cap Z)$  is the minimum vertex cover.

- A minimum Steiner tree for  $n$  vertices requires at most  $n - 2$  additional Steiner vertices.
- The number of vertices of a graph is equal to its minimum vertex cover number plus the size of a maximum independent set.
- **Lagrange polynomial** through points  $(x_0, y_0), \dots, (x_k, y_k)$  is  $L(x) = \sum_{j=0}^k y_j \prod_{\substack{m=0 \\ m \neq j}}^k \frac{x - x_m}{x_j - x_m}$
- **Hook length formula:** If  $\lambda$  is a Young diagram and  $h_\lambda(i, j)$  is the hook-length of cell  $(i, j)$ , then the number of Young tableaux  $d_\lambda = n! / \prod h_\lambda(i, j)$ .
- #primitive pythagorean triples with hypotenuse  $< n$  approx  $n/(2\pi)$ .
- **Frobenius Number:** largest number which can't be expressed as a linear combination of numbers  $a_1, \dots, a_n$  with non-negative coefficients.  $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$ ,  $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$ .  $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d - 1)$ . An integer  $x > (\max_i a_i)^2$  can be expressed in such a way iff.  $x \mid \text{gcd}(a_1, \dots, a_n)$ .
- **Snell's law:**  $v_2 \sin \theta_1 = v_1 \sin \theta_2$  gives the shortest path between two media.
- **BEST theorem:** The number of Eulerian cycles in a directed graph  $G$  is:

$$t_w(G) \prod_{v \in G} (\deg v - 1)!,$$

where  $t_w(G)$  is the number of arborescences ("directed spanning" tree) rooted at  $w$ :  $t_w(G) = \det(q_{ij})_{i,j \neq w}$ , with  $q_{ij} = [i = j] \text{indeg}(i) - \#\{(i, j) \in E\}$ .

9.1. **Burnside's Lemma.** Let a finite group  $G$  act on a set  $X$ . Denote  $X^g = \{x \in X \mid gx = x\}$ . For each  $g$  in  $G$  let  $X^g$  denote the set of elements in  $X$  that are fixed by  $g$ . Then the number of orbits is:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

9.2. **Bézout's identity.** If  $(x, y)$  is a solution to  $ax + by = d$  ( $x, y$  can be found with EGCD), then all solutions are given by

$$(x + k \cdot \text{lcm}(a, b)/a, y - k \cdot \text{lcm}(a, b)/b), \quad k \in \mathbb{Z}$$

#### 10. GAME THEORY

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

- **Nim:** Let  $X = \bigoplus_{i=1}^n x_i$ , then  $(x_i)_{i=1}^n$  is a winning position iff  $X \neq 0$ . Find a move by picking  $k$  such that  $x_k > x_k \oplus X$ .
- **Misère Nim:** Regular Nim, except that the last player to move loses. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles. The second player wins  $(a_1, \dots, a_n)$  if 1) there is a pile  $a_i > 1$  and  $\bigoplus_{i=1}^n a_i = 0$  or 2) all  $a_i \leq 1$  and  $\bigoplus_{i=1}^n a_i = 1$ .
- **Staircase Nim:** Stones are moved down a staircase and only removed from the last pile.  $(x_i)_{i=1}^n$  is an  $L$ -position if  $(x_{2i-1})_{i=1}^{n/2}$  is (i.e. only look at odd-numbered piles).

- **Moore's Nim<sub>k</sub>**: The player may remove from at most  $k$  piles (Nim = Nim<sub>1</sub>). Expand the piles in base 2, do a carry-less addition in base  $k + 1$  (i.e. the number of ones in each column should be divisible by  $k + 1$ ).
- **Dim<sup>+</sup>**: The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is  $k + 1$  where  $2^k$  is the largest power of 2 dividing the pile size.
- **Aliquot game**: Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just  $k$ .
- **Nim (at most half)**: Write  $n + 1 = 2^m y$  with  $m$  maximal, then the Sprague-Grundy function of  $n$  is  $(y - 1)/2$ .
- **Lasker's Nim**: Players may alternatively split a pile into two new non-empty piles.  $g(4k + 1) = 4k + 1$ ,  $g(4k + 2) = 4k + 2$ ,  $g(4k + 3) = 4k + 4$ ,  $g(4k + 4) = 4k + 3$  ( $k \geq 0$ ).
- **Hackenbush on trees**: A tree with stalks  $(x_i)_{i=1}^n$  may be replaced with a single stalk with length  $\bigoplus_{i=1}^n x_i$ .

## 11. DEBUGGING TIPS

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
  - Getting NaN? Make sure  $\text{acos}$  etc. are not getting values out of their range (perhaps  $1+\text{eps}$ ).
  - Rounding negative numbers?
  - Outputting in scientific notation?
- Wrong Answer?
  - Read the problem statement again!
  - Are multiple test cases being handled correctly? Try repeating the same test case many times.
  - Integer overflow?
  - Think very carefully about boundaries of all input parameters
  - Try out possible edge cases:
    - \*  $n = 0, n = -1, n = 1, n = 2^{31} - 1$  or  $n = -2^{31}$
    - \* List is empty, or contains a single element
    - \*  $n$  is even,  $n$  is odd
    - \* Graph is empty, or contains a single vertex
    - \* Graph is a multigraph (loops or multiple edges)
    - \* Polygon is concave or non-simple
  - Is initial condition wrong for small cases?
  - Are you sure the algorithm is correct?
  - Explain your solution to someone.
  - Are you using any functions that you don't completely understand? Maybe STL functions?
  - Maybe you (or someone else) should rewrite the solution?
  - Can the input line be empty?
- Run-Time Error?
  - Is it actually Memory Limit Exceeded?

### 11.1. Solution Ideas.

- Dynamic Programming
  - Parsing CFGs: CYK Algorithm
  - Drop a parameter, recover from others

- Swap answer and a parameter
  - When grouping: try splitting in two
  - $2^k$  trick
  - When optimizing
    - \* Convex hull optimization
      - $\text{dp}[i] = \min_{j < i} \{ \text{dp}[j] + b[j] \times a[i] \}$
      - $b[j] \geq b[j + 1]$
      - optionally  $a[i] \leq a[i + 1]$
      - $O(n^2)$  to  $O(n)$
    - \* Divide and conquer optimization
      - $\text{dp}[i][j] = \min_{k < j} \{ \text{dp}[i - 1][k] + C[k][j] \}$
      - $A[i][j] \leq A[i][j + 1]$
      - $O(kn^2)$  to  $O(kn \log n)$
      - sufficient:  $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$ ,  $a \leq b \leq c \leq d$  (QI)
- ```
vvi A; // A[i][j] is voor [i, j]

void divco(ll ls, ll rs, ll lt, ll rt, vi &t,
  ↳ vi &s) { // berekent t[_][lt, rt] }
  if (lt >= rt)
    return;
  ll ms = ls, mt = (lt + rt) / 2;
  t[mt] = -INF;
  rep(i, ls, rs) {
    if (i >= mt) {
      break;
    }
    if (s[i] + A[i][mt] > t[mt]) {
      t[mt] = s[i] + A[i][mt];
      ms = i;
    }
  }
  divco(ls, ms + 1, lt, mt, t, s);
  divco(ms, rs, mt + 1, rt, t, s);
}
```
- \* Knuth optimization
    - $\text{dp}[i][j] = \min_{i < k < j} \{ \text{dp}[i][k] + \text{dp}[k][j] + C[i][j] \}$
    - $A[i][j - 1] \leq A[i][j] \leq A[i + 1][j]$
    - $O(n^3)$  to  $O(n^2)$
    - sufficient: QI and  $C[b][c] \leq C[a][d]$ ,  $a \leq b \leq c \leq d$

- Greedy
- Randomized
- Optimizations
  - Use bitset (/64)
  - Switch order of loops (cache locality)
- Process queries offline
  - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
  - Mo's algorithm
  - Sqrt decomposition
  - Store  $2^k$  jump pointers
- Data structure techniques

- Sqrt buckets
- Store  $2^k$  jump pointers
- $2^k$  merging trick
- Counting
  - Inclusion-exclusion principle
  - Generating functions
- Graphs
  - Can we model the problem as a graph?
  - Can we use any properties of the graph?
  - Strongly connected components
  - Cycles (or odd cycles)
  - Bipartite (no odd cycles)
    - \* Bipartite matching
    - \* Hall's marriage theorem
    - \* Stable Marriage
  - Cut vertex/bridge
  - Biconnected components
  - Degrees of vertices (odd/even)
  - Trees
    - \* Heavy-light decomposition
    - \* Centroid decomposition
    - \* Least common ancestor
    - \* Centers of the tree
  - Eulerian path/circuit
  - Chinese postman problem
  - Topological sort
  - (Min-Cost) Max Flow
  - Min Cut
    - \* Maximum Density Subgraph
  - Huffman Coding
  - Min-Cost Arborescence
  - Steiner Tree
  - Kirchhoff's matrix tree theorem
  - Prüfer sequences
  - Lovász Toggle
  - Look at the DFS tree (which has no cross-edges)
  - Is the graph a DFA or NFA?
    - \* Is it the Synchronizing word problem?
- math
  - Is the function multiplicative?
  - Look for a pattern
  - Permutations
    - \* Consider the cycles of the permutation
  - Functions
    - \* Sum of piecewise-linear functions is a piecewise-linear function
    - \* Sum of convex (concave) functions is convex (concave)
  - Modular arithmetic
    - \* Chinese Remainder Theorem
    - \* Linear Congruence
  - Sieve
  - System of linear equations



- Values too big to represent?
  - \* Compute using the logarithm
  - \* Divide everything by some large value
- Linear programming
  - \* Is the dual problem easier to solve?
- Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
  - 2-SAT
  - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ( $\log(n)$ )
- Strings
  - Trie (maybe over something weird, like bits)
  - Suffix array
  - Suffix automaton (+DP?)
  - Aho-Corasick
  - eertree
  - Work with  $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
  - Lazy propagation
  - Persistent
  - Implicit
  - Segment tree of X
- Geometry
  - Minkowski sum (of convex sets)
  - Rotating calipers
  - Sweep line (horizontally or vertically?)
  - Sweep angle
  - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem
  - Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)
- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

## PRACTICE CONTEST CHECKLIST

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use printf/scanf with long long/long double?
- Are `__int128` and `__float128` available?
- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: `i?python[23]`, `factor`.
- Try submitting with `assert(false)` and `assert(true)`.
- Omitting `return 0;` still works?
- Look for directory with sample test cases.
- Make sure printing works.