TCR

git merge -s octopus solution cup

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Contents

.bashrc

```
alias gg='g++ -std=c++17 -Wall -Wshadow'
alias g='gg -DDEBUG -g -fsanitize=address,undefined'
```

.vimrc

```
set nu rnu sw=4 ts=4 sts=4 noet ai hls shcf=-ic
sy on | colo slate
```

Test script (usage: ./test.sh A/B/..)

```
g++ -g -Wall -fsanitize=address,undefined
↪  -Wfatal-errors -std=c++17 $1.cc || exit
for i in $1/*.in
do
  j="${i/.in/.ans}"
  ./a.out < $i > output
  diff output $j || echo "!!WA on $i!!"
done
```

template.cc

```
#include <bits/stdc++.h>
using namespace std;

typedef long long ll;
typedef long double ld;
typedef pair<ll, ll> ii;
typedef vector<ll> vi;
typedef vector<vi> vvi;
typedef vector<ii> vii;

#define x first
#define y second
#define pb push_back
#define eb emplace_back
#define rep(i,a,b) for(auto i=(a); i<(b); ++i)
#define REP(i,n) rep(i,0,n)
#define all(v) begin(v), end(v)
#define sz(v) ((int) (v).size())
#define rs resize
#define DBG(x) cerr << __LINE__ << ": " \
    << #x<< " = " << (x) << endl

template<class T> ostream& operator<<(ostream &os,
    const vector<T> &v) {
  os << "\n[";
  for(const T &x : v) os << x << ',';
  return os << "]\n";
}

template<class T1, class T2>
struct hash<pair<T1,T2>> { public:
  size_t operator()(const pair<T1,T2> &p) const {
    size_t x = hash<T1>()(p.x), y = hash<T2>()(p.y);
    return x ^ (y + 0x9e3779b9 + (x<<6) + (x>>2));
  }
};

void run() {

}
```

```
signed main() {
  // DON'T MIX "scanf" and "cin"!
  ios_base::sync_with_stdio(false);
  cin.tie(NULL);
  cout << fixed << setprecision(20);
  run();
  return 0;
}
```

template.py

```
# reading input:
from sys import *
n,m = [ int(x) for x in
↪  stdin.readline().rstrip().split() ]
stdout.write( str(n*m)+"\n" )
# set operations:
from itertools import *
for (x,y) in product(range(3),repeat=2):
  stdout.write( str(3*x+y)+" " )
print()
for L in combinations(range(4),2):
  stdout.write( str(L)+" " )
print()
# fancy lambdas:
from functools import *
y = reduce( lambda x,y: x+y, map( lambda x: x*x,
↪  range(4) ), -3 )
print(y)
# formatting:
from math import *
stdout.write( "{0:.2f}\n".format(pi) )
```

0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Iedereen (incl. Ludo) moet **ALLE** opgaves **goed** lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zware) test cases.
- Gebruik `ll`.

0.2. Wrong Answer.

- Print de oplossing om te debuggen!
- Kijk naar wellicht makkelijkere problemen.
- Bedenk zelf test cases met **randgevallen**!
- Controleer de **precisie**.
- Controleer op **overflow** (gebruik **OVERAL** `ll`, `ld`). *Kijk naar overflows in tussenantwoorden bij modulo.*
- Controleer op **typo's**.
- Loop de voorbeeld test case accuraat langs.
- Controleer op off-by-one-errors (in indices of lus-grenzen)?

Detecting overflow: This GNU builtin checks for over- and underflow. Result is in `res` if successful:

```
bool isOverflown =
↪  __builtin_[add|mul|sub]_overflow(a, b, &res);
```

1. Mathematics

XOR sum: $\bigoplus_{x=0}^{a-1} x = \{0, a-1, 1, a\}[a \bmod 4]$.

```
int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }

// greatest common divisor
ll gcd(ll a,ll b){while(b)a%=b,swap(a,b);return a;};
// least common multiple
ll lcm(ll a, ll b) { return a/gcd(a, b)*b; }
ll mod(ll a, ll b) { return (a%=b) < 0 ? a+b : a; }

// ab % m for m <= 4e18 in O(log b)
ll mod_mul(ll a, ll b, ll m) {
  ll r = 0;
  while(b) {
    if (b & 1) r = mod(r+a,m);
    a = mod(a+a,m); b >>= 1;
  }
  return r;
}

// a^b % m for m <= 2e9 in O(log b)
ll mod_pow(ll a, ll b, ll m) {
  ll r = 1;
  while(b) {
    if (b & 1) r = (r * a) % m; // mod_mul
    a = (a * a) % m; // mod_mul
    b >>= 1;
  }
  return r;
}

// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
  ll xx = y = 0, yy = x = 1;
  while (b) {
    x -= a / b * xx; swap(x, xx);
    y -= a / b * yy; swap(y, yy);
    a %= b; swap(a, b);
  }
  return a;
}

// Chinese Remainder Theorem: returns (u, v) s.t.
// x=u (mod v) <=> x=a (mod n) and x=b (mod m)
pair<ll, ll> crt(ll a, ll n, ll b, ll m) {
  ll s, t, d = egcd(n, m, s, t); //n,m<=1e9
  if (mod(a - b, d)) return { 0, -1 };
  return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}
```

```
// phi[i] = #{ 0 < j <= i | gcd(i, j) = 1 } sieve
vi totient(int N) {
  vi phi(N);
  for (int i = 0; i < N; i++) phi[i] = i;
  for (int i = 2; i < N; i++) if (phi[i] == i)
    for (int j = i; j < N; j+=i) phi[j] -= phi[j]/i;
  return phi;
}

// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
  ll ans = 1;
  while (n) {
    ll np = n % p, kp = k % p;
    if (np < kp) return 0;
    ans = mod(ans * binom(np, kp), p); // (np C kp)
    n /= p; k /= p;
  }
  return ans;
}

// returns if n is prime for n < 3e24 (>2^64)
// but use mul_mod for n > 2e9.
bool millerRabin(ll n){
  if (n < 2 || n % 2 == 0) return n == 2;
  ll d = n - 1, ad, s = 0, r;
  for (; d % 2 == 0; d /= 2) s++;
  for (int a : {  2,  3,  5,  7, 11, 13,
               17, 19, 23, 29, 31, 37, 41 }) {
    if (n == a) return true;
    if ((ad = mod_pow(a, d, n)) == 1) continue;
    for (r = 0; r < s && ad + 1 != n; r++)
      ad = (ad * ad) % n;
    if (r == s) return false;
  }
  return true;
}
```

1.1. Primitive Root $O(\sqrt{m})$.

Returns a generator of $\mathbb{F}_m^*$. If m not prime, replace $m-1$ by totient of m.

```
ll primitive_root(ll m) {
  vector<ll> div;
  for (ll i = 1; i*i < m; i++)
    if ((m-1) % i == 0) {
      if (i < m-1) div.pb(i);
      if ((m-1)/i < m) div.pb((m-1)/i);
    }
  rep(x,2,m) {
    bool ok = true;
    for (ll d : div) if (mod_pow(x, d, m) == 1)
      { ok = false; break; }
    if (ok) return x;
  }
  return -1;
}
```

1.2. Tonelli-Shanks algorithm.

Given prime p and integer $1 \le n < p$, returns the square root r of n modulo p. There is also another solution given by $-r$ modulo p.

```
ll legendre(ll a, ll p) {
  if (a % p == 0) return 0;
  return p == 2 || mod_pow(a, (p-1)/2, p) == 1 ? 1 :
  ↪  -1;
}
ll tonelli_shanks(ll n, ll p) {
  assert(legendre(n,p) == 1);
  if (p == 2) return 1;
  ll s = 0, q = p-1, z = 2;
  while (~q & 1) s++, q >>= 1;
  if (s == 1) return mod_pow(n, (p+1)/4, p);
  while (legendre(z,p) != -1) z++;
  ll c = mod_pow(z, q, p),
     r = mod_pow(n, (q+1)/2, p),
     t = mod_pow(n, q, p),
     m = s;
  while (t != 1) {
    ll i = 1, ts = (ll)t*t % p;
    while (ts != 1) i++, ts = ((ll)ts * ts) % p;
    ll b = mod_pow(c, 1LL<<(m-i-1), p);
    r = (ll)r * b % p;
    t = (ll)t * b % p * b % p;
    c = (ll)b * b % p;
    m = i;
  }
  return r;
}
```

1.3. Numeric Integration.

Numeric integration using Simpson's rule.

```
ld numint(ld (*f)(ld), ld a, ld b, ld EPS = 1e-6) {
  ld ba = b - a, m=(a+b)/2;
  return abs(ba) < EPS
    ? ba/8*(f(a)+f(b)+f(a+ba/3)*3+f(b-ba/3)*3)
    : numint(f,a,m,EPS) + numint(f,m,b,EPS);
}
```

1.4. Fast Hadamard Transform.

Computes XOR-convolutions in $O(k2^k)$ on k bits.
For AND-convolution, use $(x+y,\ y),\quad (x-y,\ y)$.
For OR-convolution, use $(x,\ x+y),\quad (x,\ -x+y)$.
Note: The array size must be a power of 2.

```
void fht(vi &A, bool inv=false, int l, int r) {
  if (l+1 == r) return;
  int k = (r-l)/2;
  if (!inv) fht(A, inv, l, l+k), fht(A, inv, l+k,
  ↪  r);
  rep(i,l,l+k) {
    int x = A[i], y = A[i+k];
    if (!inv) A[i] =  x-y,    A[i+k] =  x+y;
    else      A[i] = (x+y)/2, A[i+k] = (-x+y)/2;
  }
  if (inv) fht(A, inv, l, l+k), fht(A, inv, l+k, r);
}
```

1.5. Tridiagonal Matrix Algorithm.

Solves a tridiagonal system of linear equations

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$$

where $a_1 = c_n = 0$. Beware of numerical instability.

```
#define MAXN 5000
ld A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
  C[0] /= B[0]; D[0] /= B[0];
  rep(i,1,n-1) C[i] /= B[i] - A[i]*C[i-1];
  rep(i,1,n) D[i] =
    (D[i] - A[i]*D[i-1]) / (B[i] - A[i]*C[i-1]);
  X[n-1] = D[n-1];
  for (int i = n-1; i--;) X[i] = D[i] - C[i]*X[i+1];
}
```

1.6. Number of Integer Points under Line.

Count the number of integer solutions to $Ax + By \le C$, $0 \le x \le n$, $0 \le y$. In other words, evaluate the sum $\sum_{x=0}^{n} \left\lfloor \frac{C-Ax}{B} + 1 \right\rfloor$. To count all solutions, let $n = \left\lfloor \frac{c}{a} \right\rfloor$. In any case, it must hold that $C - nA \ge 0$. Be very careful about overflows.

```
ll floor_sum(ll n, ll a, ll b, ll c) {
  if (c == 0) return 1;
  if (c < 0) return 0;
  if (a % b == 0) return
  ↪  (n+1)*(c/b+1)-n*(n+1)/2*a/b;
  if (a >= b) return
  ↪  floor_sum(n,a%b,b,c)-a/b*n*(n+1)/2;
  ll t = (c-a*n+b)/b;
  return floor_sum((c-b*t)/b,b,a,c-b*t)+t*(n+1); }
```

1.7. Solving linear recurrences.

Given some brute-forced sequence $s[0], s[1], \ldots, s[2n-1]$, Berlekamp-Massey finds the shortest possible recurrence relation in $\mathcal{O}(n^2)$. After that, `lin_rec` finds $s[k]$ in $\mathcal{O}(n^2 \log k)$.

```
// Given a sequence s[0], ..., s[2n-1] finds the
↪  smallest linear recurrence
// of size <= n compatible with s.
vl BerlekampMassey(const vl &s, ll mod) {
  int n = sz(s), L = 0, m = 0;
  vl C(n), B(n), T;
  C[0] = B[0] = 1;
  ll b = 1;
  REP(i, n) {
    ++m;
    ll d = s[i] % mod;
    rep(j, 1, L+1) d = (d + C[j] * s[i - j]) % mod;
    if (!d) continue;
    T = C;
    ll coef = d * modpow(b, mod-2, mod) % mod;
    rep(j,m,n) C[j] = (C[j] - coef * B[j-m]) % mod;
    if (2 * L > i) continue;
    L = i + 1 - L;
    B = T; b = d; m = 0;
  }
  C.resize(L + 1);
  C.erase(C.begin());
  for (auto &x : C) x = (mod - x) % mod;
  return C;
}

// Input: A[0,...,n-1], C[0,...,n-1] satisfying
//     A[i] = \sum_{j=1}^{n} C[j-1] A[i-j],
// Outputs A[k]
ll lin_rec(const vl &A, const vl &C, ll k, ll mod){
  int n = sz(A);
  auto combine = [&](vl a, vl b) {
    vl res(sz(a) + sz(b) - 1, 0);
    REP(i, sz(a)) REP(j, sz(b))
      res[i+j] = (res[i+j] + a[i]*b[j]) % mod;
    for (int i = 2*n; i > n; --i) REP(j,n)
      res[i-1-j] = (res[i-1-j] + res[i]*C[j]) % mod;
    res.resize(n + 1);
    return res;
  };
  vl pol(n + 1), e(pol);
  pol[0] = e[1] = 1;
  for (++k; k; k /= 2) {
    if (k % 2) pol = combine(pol, e);
    e = combine(e, e);
  }
  ll res = 0;
  REP(i, n) res = (res + pol[i + 1] * A[i]) % mod;
  return res;
}
```

1.8. Misc.

1.8.1. *Josephus problem.*

Last man standing out of n if every kth is killed. Zero-based, and does not kill 0 on first pass.

```
int J(int n, int k) {
  if (n == 1 || k == 1) return n-1;
  if (n < k) return (J(n-1,k)+k)%n;
  int np = n - n/k;
  return k*((J(np,k)+np-n%k%np)%np) / (k-1); }
```

- **Prime numbers**:
  1031, 32771, 1048583, 8125344, 33554467, 9982451653, 1073741827, 34359738421, 1099511627791, 35184372088891, 1125899906842679, 36028797018963971.
  $10^3 + \{-9, -3, 9, 13\}$, $10^6 + \{-17, 3, 33\}$, $10^9 + \{7, 9, 21, 33, 87\}$.
- **Generating functions**: Ordinary (ogf): $A(x) := \sum_{n=0}^{\infty} a_i x^i$.
  Calculate product $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ with FFT.
  Exponential (e.g.f.): $A(x) := \sum_{n=0}^{\infty} a_i x^i / i!$,
  $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$ (use FFT).

- **General linear recurrences:** If $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$, then $A(x) = \frac{a_0}{1-B(x)}$.
- **Inverse polynomial modulo x^l:** Given $A(x)$, find $B(x)$ such that $A(x)B(x) = 1 + x^l Q(x)$ for some $Q(x)$.
  Step 1: Start with $B_0(x) = 1/a_0$
  Step 2: $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \mod x^{2^{k+1}}$.
- **Fast subset convolution:** Given array a_i of size 2^k calculate $b_i = \sum_{j \& i = i} a_j$.

```
for (int b = 1; b < (1 << k); b <<= 1)
  for (int i = 0; i < (1<<k); i++)
    if (!(i & b)) a[i | b] += a[i];
// inv: if (!(i & b)) a[i | b] -= a[i];
```

- **Primitive Roots:** It only exists when n is $2, 4, p^k, 2p^k$, where p odd prime. If g is a primitive root, all primitive roots are of the form g^k where $k, \phi(p)$ are coprime (hence there are $\phi(\phi(p))$ primitive roots).
- **Maximum number of divisors:**

$\leq N$	10^3	10^6	10^9	10^{12}	10^{18}
m	840	720720	735134400	963761198400	
$\sigma_0(m)$	32	240	1344	6270	103680

For $n = 10^{18}, m = 897612484786617600$.

2. Datastructures

2.1. Order tree.

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
template<class TK,class TM> using order_tree =
↪ tree<TK,TM,greater<TK>,rb_tree_tag,
↪ tree_order_statistics_node_update>;
template<class TK> using order_set =
↪ order_tree<TK,null_type>;

vi s;
order_set<ii> t;
void update( ll k, ll v ) {
  t.erase( ii{ s[k], k } );
  s[k] = v;
  t.insert( ii{ s[k], k } );
}

signed main() {
  ll n = 4;
  s.resize(n,0);
  rep(i,0,n) t.insert(ii{0,i});
  update( 2, 3 );
  cout << t.find_by_order( 2 )->y << endl;
  cout << t.order_of_key( ii{s[3],3} ) << endl;
}
```

2.2. Segment tree $\mathcal{O}(\log n)$.

2.2.1. *Standard segment tree.*

```
typedef int S; // or define your own object
const int n = 1 << 20;
S t[2 * n];

// combine must be an associative function!
S combine(S l, S r){ return l+r; } //or max(l,r) etc

void build() {
  for (int i = n; --i; )
    t[i] = combine(t[2 * i], t[2 * i + 1]);
}

// set value v on position i
void update(int i, S v) {
  for (t[i+=n] = v; i /= 2; )
    t[i] = combine(t[2 * i], t[2 * i + 1]);
}

// sum on interval [l, r)
S query(int l, int r) {
  S resL = 0, resR = 0;
  for (l += n, r += n; l < r; l /= 2, r /= 2) {
    if (l & 1) resL = combine(resL, t[l++]);
    if (r & 1) resR = combine(t[--r], resR);
  }
  return combine(resL, resR);
}
```

2.2.2. *Lazy segment tree.*

Be careful: all intervals are right-closed $[\ell, r]$.

```
struct node {
  int l, r, x, lazy;
  node() {}
  node(int _l, int _r) : l(_l), r(_r), x(INT_MAX),
  ↪ lazy(0){}
  node(int _l, int _r, int _x) : node(_l,_r){x=_x;}
  node(node a,node b):node(a.l,b.r){x=min(a.x,b.x);}
  void update(int v) { x = v; }
  void range_update(int v) { lazy = v; }
  void apply() { x += lazy; lazy = 0; }
  void push(node &u) { u.lazy += lazy; }
};

struct segment_tree {
  int n;
  vector<node> arr;
  segment_tree() { }
  segment_tree(const vi &a) : n(sz(a)), arr(4*n) {
    mk(a,0,0,n-1); }
  node mk(const vi &a, int i, int l, int r) {
    int m = (l+r)/2;
    return arr[i] = l > r ? node(l,r) :
      l == r ? node(l,r,a[l]) :
      node(mk(a,2*i+1,l,m),mk(a,2*i+2,m+1,r));
  }
  node update(int at, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (at < hl || hr < at) return arr[i];
    if (hl == at && at == hr) {
      arr[i].update(v); return arr[i]; }
    return arr[i] =
      node(update(at,v,2*i+1),update(at,v,2*i+2));
  }
  node query(int l, int r, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return node(hl,hr);
    if (l <= hl && hr <= r) return arr[i];
    return node(query(l,r,2*i+1),query(l,r,2*i+2));
  }
  node range_update(int l, int r, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return arr[i];
    if (l <= hl && hr <= r) {
      arr[i].range_update(v);
      propagate(i);
      return arr[i];
    }
    return arr[i] = node(range_update(l,r,v,2*i+1),
        range_update(l,r,v,2*i+2));
  }
  void propagate(int i) {
    if (arr[i].l < arr[i].r) {
      arr[i].push(arr[2*i+1]);
      arr[i].push(arr[2*i+2]);
    }
    arr[i].apply();
  }
};
```

2.2.3. *Persistent segment tree.*

Be careful: all intervals are right-closed $[\ell, r]$, including build.

```
int segcnt = 0;
struct segment {
  int l, r, lid, rid, sum;
} S[2000000];

int build(int l, int r) {
  if (l > r) return -1;
  int id = segcnt++;
  S[id].l = l;
  S[id].r = r;
  if (l == r) S[id].lid = -1, S[id].rid = -1;
  else {
    int m = (l + r) / 2;
    S[id].lid = build(l , m);
    S[id].rid = build(m + 1, r);
  }
  S[id].sum = 0;
  return id;
```

```
}
int update(int idx, int v, int id) {
  if (id == -1) return -1;
  if (idx < S[id].l || idx > S[id].r) return id;
  int nid = segcnt++;
  S[nid].l = S[id].l;
  S[nid].r = S[id].r;
  S[nid].lid = update(idx, v, S[id].lid);
  S[nid].rid = update(idx, v, S[id].rid);
  S[nid].sum = S[id].sum + v;
  return nid;
}
int query(int id, int l, int r) {
  if (r < S[id].l || S[id].r < l) return 0;
  if (l<=S[id].l && S[id].r<=r) return S[id].sum;
  return query(S[id].lid,l,r)+query(S[id].rid,l,r);
}
```

2.3. **Binary Indexed Tree $\mathcal{O}(\log n)$.** Use one-based indices ($i > 0$)!

```
struct BIT {
  int n; vi A;
  BIT(int _n) : n(_n), A(_n+1, 0) {}
  BIT(vi &v) : n(sz(v)), A(1) {
    for (auto x:v) A.pb(x);
    for (int i=1, j; j=i&-i, i<=n; i++)
      if (i+j <= n) A[i+j] += A[i];
  }
  void update(int i, ll v) { // a[i] += v
    while (i <= n) A[i] += v, i += i&-i;
  }
  ll query(int i) { // sum_{j<=i} a[j]
    ll v = 0;
    while (i) v += A[i], i -= i&-i;
    return v;
  }
};

struct rangeBIT {
  int n; BIT b1, b2;
  rangeBIT(int _n) : n(_n), b1(_n), b2(_n+1) {}
  rangeBIT(vi &v) : n(sz(v)), b1(v), b2(sz(v)+1) {}
  void pupdate(int i, ll v) { b1.update(i, v); }
  void rupdate(int i, int j, ll v) { // a[i,..,j]+=v
    b2.update(i, v);
    b2.update(j+1, -v);
    b1.update(j+1, v*j);
    b1.update(i, (1-i)*v);
  }
  ll query(int i){return b1.query(i)+b2.query(i)*i;}
};
```

2.4. **Disjoint-Set / Union-Find $\mathcal{O}(\alpha(n))$.**

```
struct dsu { vi p; dsu(int n) : p(n, -1) {}
  int find(int i) {
    return p[i] < 0 ? i : p[i] = find(p[i]); }
  void unite(int a, int b) {
    if ((a = find(a)) == (b = find(b))) return;
    if (p[a] > p[b]) swap(a, b);
    p[a] += p[b]; p[b] = a;
  }
};
```

2.5. **Cartesian tree.**

```
struct node {
  int x, y, sz;
  node *l, *r;
  node(int _x, int _y)
    : x(_x), y(_y), sz(1), l(NULL), r(NULL) { } };
int tsize(node* t) { return t ? t->sz : 0; }
void augment(node *t) {
  t->sz = 1 + tsize(t->l) + tsize(t->r); }
pair<node*,node*> split(node *t, int x) {
  if (!t) return make_pair((node*)NULL,(node*)NULL);
  if (t->x < x) {
    pair<node*,node*> res = split(t->r, x);
    t->r = res.x; augment(t);
    return make_pair(t, res.y); }
  pair<node*,node*> res = split(t->l, x);
  t->l = res.y; augment(t);
  return make_pair(res.x, t); }
node* merge(node *l, node *r) {
  if (!l) return r; if (!r) return l;
  if (l->y > r->y) {
    l->r = merge(l->r, r); augment(l); return l; }
  r->l = merge(l, r->l); augment(r); return r; }
node* find(node *t, int x) {
  while (t) {
    if (x < t->x) t = t->l;
    else if (t->x < x) t = t->r;
    else return t; }
  return NULL; }
node* insert(node *t, int x, int y) {
  if (find(t, x) != NULL) return t;
  pair<node*,node*> res = split(t, x);
  return merge(res.x, merge(new node(x, y), res.y));
}
node* erase(node *t, int x) {
  if (!t) return NULL;
  if (t->x < x) t->r = erase(t->r, x);
  else if (x < t->x) t->l = erase(t->l, x);
  else{node *old=t; t=merge(t->l,t->r); delete old;}
  if (t) augment(t); return t;
}
int kth(node *t, int k) {
  if (k < tsize(t->l)) return kth(t->l, k);
  else if (k == tsize(t->l)) return t->x;
  else return kth(t->r, k - tsize(t->l) - 1);
}
```

2.6. **Heap.** An implementation of a binary heap.

```
template <class Comp = less<int>> struct heap {
  vi q, loc; Comp op;
  heap() : op(Comp()) {}
  bool cmp(int i, int j) { return op(q[i], q[j]); }
  void swp(int i, int j) {
    swap(q[i], q[j]), swap(loc[q[i]], loc[q[j]]);
  }
  void swim(int i) {
    for (int p; i; swp(i, p), i = p)
      if (!cmp(i, p=(i-1)/2)) break;
  }
  void sink(int i) {
    for (int j; (j=2*i+1)<sz(q); swp(j, i), i=j) {
      if (j+1 < sz(q) && cmp(j+1, j)) ++j;
      if (!cmp(j, i)) break;
    }
  }
  void push(int n) {
    while (n >= sz(loc)) loc.pb(-1);
    assert(loc[n] == -1);
    loc[n] = sz(q), q.pb(n);
    swim(sz(q) - 1);
  }
  int top() { assert(!empty()); return q[0]; }
  int pop() {
    int res = top();
    q[0] = q.back(), q.pop_back();
    loc[q[0]]=0, loc[res] = -1;
    sink(0); return res;
  }
  void heapify() {
    for (int i=sz(q); --i; )
      if (cmp(i, (i-1)/2)) swp(i, (i-1)/2);
  }
  void update_key(int n) {
    assert(loc[n] != -1);
    swim(loc[n]), sink(loc[n]);
  }
  int size() { return sz(q); }
  bool empty() { return !size(); }
  void clear() { q.clear(), loc.clear(); }
};
```

2.7. **Dancing Links.** An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```
template <class T>
struct dancing_links {
  struct node {
    T item;
    node *l, *r;
  node(const T &_item, node *_l=NULL, node *_r=NULL)
      : item(_item), l(_l), r(_r) {
      if (l) l->r = this;
      if (r) r->l = this; } };
  node *front, *back;
  dancing_links() { front = back = NULL; }
  node *push_back(const T &item) {
    back = new node(item, back, NULL);
    if (!front) front = back;
    return back; }
```

```
  node *push_front(const T &item) {
    front = new node(item, NULL, front);
    if (!back) back = front;
    return front; }
  void erase(node *n) {
    if (!n->l) front = n->r; else n->l->r = n->r;
    if (!n->r) back = n->l; else n->r->l = n->l; }
  void restore(node *n) {
    if (!n->l) front = n; else n->l->r = n;
    if (!n->r) back = n; else n->r->l = n; } };
```

2.8. **Misof Tree.** A simple tree data structure for inserting, erasing, and querying the nth largest element.

```
const int BITS = 15;
struct misof_tree {
  int cnt[BITS][1<<BITS];
  misof_tree() { memset(cnt,0,sizeof(cnt)); }
  void insert(int x) {
    for (int i=0; i<BITS; cnt[i++][x]++, x >>= 1); }
  void erase(int x) {
    for (int i=0; i<BITS; cnt[i++][x]--, x >>= 1); }
  int nth(int n) {
    int res = 0;
    for (int i = BITS-1; i >= 0; i--)
      if (cnt[i][res <<= 1] <= n)
        n -= cnt[i][res], res |= 1;
    return res;
  }
};
```

2.9. ***k*-d Tree.** A k-dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```
#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
  struct pt {
    double coord[K];
    pt() {}
    pt(double c[K]) { REP(i,K) coord[i] = c[i]; }
    double dist(const pt &other) const {
      double sum = 0.0;
      REP(i,K) sum +=
      ↪ pow(coord[i]-other.coord[i],2);
      return sqrt(sum); } };
  struct cmp {
    int c;
    cmp(int _c) : c(_c) {}
    bool operator ()(const pt &a, const pt &b) {
      for (int i = 0, cc; i <= K; i++) {
        cc = i == 0 ? c : i - 1;
        if (abs(a.coord[cc] - b.coord[cc]) > EPS)
          return a.coord[cc] < b.coord[cc];
      }
      return false; } };
  struct bb {
    pt from, to;
    bb(pt _from, pt _to) : from(_from), to(_to) {}
    double dist(const pt &p) {
      double sum = 0.0;
      REP(i,K) {
        if (p.coord[i] < from.coord[i])
          sum += pow(from.coord[i] - p.coord[i],
          ↪ 2.0);
        else if (p.coord[i] > to.coord[i])
          sum += pow(p.coord[i] - to.coord[i], 2.0);
      }
      return sqrt(sum); }
    bb bound(double l, int c, bool left) {
      pt nf(from.coord), nt(to.coord);
      if (left) nt.coord[c] = min(nt.coord[c], l);
      else nf.coord[c] = max(nf.coord[c], l);
      return bb(nf, nt); } };
  struct node {
    pt p; node *l, *r;
    node(pt _p, node *_l, node *_r)
      : p(_p), l(_l), r(_r) { } };
  node *root;

  // kd_tree() : root(NULL) { }
  kd_tree(vector<pt> pts) {
    root = construct(pts, 0, size(pts) - 1, 0); }
  node* construct(vector<pt> &pts, int fr, int to,
  ↪ int c) {
    if (fr > to) return NULL;
    int mid = fr + (to-fr) / 2;
    nth_element(pts.begin() + fr, pts.begin() + mid,
          pts.begin() + to + 1, cmp(c));
    return new node(pts[mid],
            construct(pts, fr, mid - 1, INC(c)),
            construct(pts, mid + 1, to, INC(c))); }
  bool contains(const pt &p){ return
  ↪ _con(p,root,0);}
  bool _con(const pt &p, node *n, int c) {
    if (!n) return false;
    if (cmp(c)(p, n->p)) return _con(p,n->l,INC(c));
    if (cmp(c)(n->p, p)) return _con(p,n->r,INC(c));
    return true; }
  void insert(const pt &p) { _ins(p, root, 0); }
  void _ins(const pt &p, node* &n, int c) {
    if (!n) n = new node(p, NULL, NULL);
    else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
    else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c));
  }
  void clear() { _clr(root); root = NULL; }
  void _clr(node *n) {
    if (n) _clr(n->l), _clr(n->r), delete n; }
  pt nearest_neighbour(const pt &p, bool same=true)
  ↪ {
    assert(root);
    double mn = INFINITY, cs[K];
    REP(i,K) cs[i] = -INFINITY;
    pt from(cs);
    REP(i,K) cs[i] = INFINITY;
    pt to(cs);
    return _nn(p, root, bb(from, to), mn, 0,
    ↪ same).x;
  }
  pair<pt, bool> _nn(const pt &p, node *n, bb b,
      double &mn, int c, bool same) {
    if (!n || b.dist(p) > mn)
      return make_pair(pt(), false);
    bool found = same || p.dist(n->p) > EPS,
         l1 = true, l2 = false;
    pt resp = n->p;
    if (found) mn = min(mn, p.dist(resp));
    node *n1 = n->l, *n2 = n->r;
    REP(i,2) {
      if (i == 1 || cmp(c)(n->p, p))
        swap(n1, n2), swap(l1, l2);
      auto res = _nn(p, n1, b.bound(n->p.coord[c],
      ↪ c, l1), mn, INC(c), same);
      if (res.y && (!found || p.dist(res.x) <
      ↪ p.dist(resp)))
        resp = res.x, found = true;
    }
    return make_pair(resp, found); } };
```

2.10. **Sqrt Decomposition.** Design principle that supports many operations in amortized $\sqrt{n}$ per operation.

```
struct segment {
  vi arr;
  segment(vi _arr) : arr(_arr) { } };
vector&lt;segment&gt; T;
int K;
void rebuild() {
  int cnt = 0;
  rep(i,0,size(T))
    cnt += size(T[i].arr);
  K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
  vi arr(cnt);
  for (int i = 0, at = 0; i < size(T); i++)
    rep(j,0,size(T[i].arr))
      arr[at++] = T[i].arr[j];
  T.clear();
  for (int i = 0; i < cnt; i += K)
    T.push_back(segment(vi(arr.begin()+i,
                           arr.begin()+min(i+K,
                           ↪ cnt)))); }
int split(int at) {
  int i = 0;
  while (i < size(T) && at >= size(T[i].arr))
    at -= size(T[i].arr), i++;
  if (i >= size(T)) return size(T);
  if (at == 0) return i;
  T.insert(T.begin() + i + 1,
      segment(vi(T[i].arr.begin() + at,
      ↪ T[i].arr.end())));
  T[i] = segment(vi(T[i].arr.begin(),
  ↪ T[i].arr.begin() + at));
  return i + 1; }
```

```
void insert(int at, int v) {
  vi arr; arr.push_back(v);
  T.insert(T.begin() + split(at), segment(arr)); }
void erase(int at) {
  int i = split(at); split(at + 1);
  T.erase(T.begin() + i); }
```

2.11. Monotonic Queue. A queue that supports querying for the minimum element. Useful for sliding window algorithms.

```
struct min_stack {
  stack<int> S, M;
  void push(int x) {
    S.push(x);
    M.push(M.empty() ? x : min(M.top(), x)); }
  int top() { return S.top(); }
  int mn() { return M.top(); }
  void pop() { S.pop(); M.pop(); }
  bool empty() { return S.empty(); } };
struct min_queue {
  min_stack inp, outp;
  void push(int x) { inp.push(x); }
  void fix() {
    if (outp.empty()) while (!inp.empty())
      outp.push(inp.top()), inp.pop(); }
  int top() { fix(); return outp.top(); }
  int mn() {
    if (inp.empty()) return outp.mn();
    if (outp.empty()) return inp.mn();
    return min(inp.mn(), outp.mn()); }
  void pop() { fix(); outp.pop(); }
  bool empty() { return inp.empty()&&outp.empty(); }
};
```

2.12. Line container à la 'Convex Hull Trick' $\mathcal{O}(n \log n)$. Container where you can add lines of the form $y_i(x) = k_i x + m_i$ and query $\max_i y_i(x)$.

```
bool Q;
struct Line {
  mutable ll k, m, p;
  bool operator<(const Line& o) const {
    return Q ? p < o.p : k < o.k;
  }
};
struct LineContainer : multiset<Line> {
  // (for doubles, use inf = 1/.0, div(a,b) = a/b)
  const ll inf = LLONG_MAX;
  ll div(ll a, ll b) { // floored division
    return a / b - ((a ^ b) < 0 && a % b); }
  bool isect(iterator x, iterator y) {
    if (y == end()) { x->p = inf; return false; }
    if (x->k == y->k)
      x->p = x->m > y->m ? inf : -inf;
    else
      x->p = div(y->m - x->m, x->k - y->k);
    return x->p >= y->p;
  }
  void add(ll k, ll m) {
    auto z = insert({k, m, 0}), y = z++, x = y;
    while (isect(y, z)) z = erase(z);
    if (x != begin() && isect(--x, y))
      isect(x, y = erase(y));
    while ((y = x) != begin() && (--x)->p >= y->p)
      isect(x, erase(y));
  }
  ll query(ll x) {
    assert(!empty());
    Q=1; auto l = *lower_bound({0,0,x}); Q=0;
    return l.k * x + l.m;
  }
};
```

2.13. Sparse Table $O(\log n)$ per query.

```
struct sparse_table {
  vvi m;
  sparse_table(vi arr) {
    m.pb(arr);
    for (int k=0; (1<<(++k)) <= sz(arr); ) {
      int w = (1<<k), hw = w/2;
      m.pb(vi(sz(arr) - w + 1));
      for (int i = 0; i+w <= sz(arr); i++) {
        m[k][i] = min(m[k-1][i], m[k-1][i+hw]);
      }
    }
  int query(int l, int r) { // query min in [l,r]
    int k = 31 - __builtin_clz(r-l); // k = 0;
    // while (1<<(k+1) <= r-l+1) k++;
    return min(m[k][l], m[k][r-(1<<k)+1]);
  }
};
```

3. Graph Algorithms

3.1. Shortest path.

3.1.1. *Dijkstra* $\mathcal{O}(|E| \log |V|)$.

```
const ll INFTY = -1;
vi dijkstra( vector<vii> G, ll s ) {
  vi d( G.size(), INFTY );
  priority_queue<ii,vector<ii>,greater<ii>> Q;
  Q.emplace(0,s);
  while(!Q.empty()){
    ll c = Q.top().x, a = Q.top().y;
    Q.pop();
    if(d[a] != INFTY)
      continue;
    d[a] = c;
    for(ii e : G[a])
      Q.emplace(d[a] + e.y, e.x);
  }
  return d;
}
```

3.1.2. *Floyd-Warshall* $\mathcal{O}(V^3)$.

Be careful with negative edges! Note: $|d[i][j]|$ can grow exponentially, and INFTY + negative < INFTY.

```
const ll INF = 1LL << 61;
void floyd_warshall( vvi& d ) {
  ll n = d.size();
  REP(i,n) REP(j,n) REP(k,n)
    if(d[j][i] < INF && d[i][k] < INF) // neg edges!
      d[j][k] = max(-INF,
        min(d[j][k], d[j][i] + d[i][k]));
}

vvi d(n,vi(n,INF));
REP(i,n) d[i][i] = 0;
```

3.1.3. *Bellman Ford* $\mathcal{O}(VE)$.

This is only useful if there are edges with weight $w_{ij} < 0$ in the graph.

```
const ll INF = 1LL << 61;
// G[u] = { (v,w) | edge u->v, cost w }
vi bellman_ford(vector<vii> G, ll s) {
  ll n = G.size();
  vi d(n, INF); d[s] = 0;
  REP(loops, n) REP(u, n) if(d[u] != INF)
    for(ii e : G[u]) if(d[u] + e.y < d[e.x])
      d[e.x] = d[u] + e.y;
  // detect paths of -INF length
  for( ll change = 1; change--; )
    REP(u, n) if(d[u] != INF)
      for(ii e : G[u]) if(d[e.x] != -INF)
        if(d[u] + e.y < d[e.x])
          d[e.x] = -INF, change = 1;
  return d;
}
```

3.1.4. *IDA* algorithm.*

```
int n, cur[100], pos;
int calch() {
  int h = 0;
  rep(i,0,n) if (cur[i] != 0) h += abs(i - cur[i]);
  return h; }
int dfs(int d, int g, int prev) {
  int h = calch();
  if (g + h > d) return g + h;
  if (h == 0) return 0;
  int mn = INT_MAX;
  rep(di,-2,3) {
    if (di == 0) continue;
    int nxt = pos + di;
    if (nxt == prev) continue;
    if (0 <= nxt && nxt < n) {
      swap(cur[pos], cur[nxt]);
      swap(pos,nxt);
      mn = min(mn, dfs(d, g+1, nxt));
      swap(pos,nxt);
      swap(cur[pos], cur[nxt]); }
    if (mn == 0) break; }
  return mn; }
```

```
int idastar() {
  rep(i,0,n) if (cur[i] == 0) pos = i;
  int d = calch();
  while (true) {
    int nd = dfs(d, 0, -1);
    if (nd == 0 || nd == INT_MAX) return d;
    d = nd; } }
```

3.2. Maximum Matching.

Matching: A set of edges without common vertices *(Maximum is the* ***largest*** *such set, maximal is a set which you cannot add more edges to without breaking the property).*

Minimum Vertex Cover: A set of vertices such that each edge in the graph is incident to at least one vertex of the set.

Minimum Edge Cover: A set of edges such that every vertex is incident to at least one edge of the set.

Maximum Independent Set: A set of vertices in a graph such that no two of them are adjacent.

Minimum edge cover $\iff$ Maximum independent set.

König's theorem: In any bipartite graph $G = (L \cup R, E)$, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover. Let U be the set of unmatched vertices in L, and Z be the set of vertices that are either in U or are connected to U by an alternating path. Then $K = (L \setminus Z) \cup (R \cap Z)$ is the minimum vertex cover.

In any bipartite graph,

$$\text{maxmatch} = \text{MVC} = V - \text{MIS}.$$

See 3.2.2.

3.2.1. *Standard bipartite matching $\mathcal{O}(|L| \cdot |R|)$.*

```
vector<bool> vis; vi par; vvi adj; // L->{R,...}

bool match(int u) {
  for (int v : adj[u]) {
    if (vis[v]) continue;
    vis[v] = true;
    if (par[v] == -1 || match(par[v]))
      { par[v] = u; return true; }
  }
  return false;
}

// perfect matching iff ret == L == R
int maxmatch(int L, int R) {
  par.assign(R, -1);
  int ret = 0;
  REP(i, L) vis.assign(R, false), ret += match(i);
  return ret;
}
```

3.2.2. *Hopcroft-Karp bipartite matching $\mathcal{O}(E\sqrt{V})$.*

```
struct bi_graph {
  int n, m, s; vvi G; vi L, R, d;
  bi_graph(int _n, int _m) : n(_n), m(_m), s(0),
      G(n), L(n,-1), R(m,n), d(n+1) {}
  void add_edge(int a, int b) { G[a].pb(b); }
  bool bfs() {
    queue<int> q; d[n] = LLONG_MAX;
    REP(v, n)
      if (L[v] < 0) d[v] = 0, q.push(v);
      else d[v] = LLONG_MAX;
    while (!q.empty()) {
      int v = q.front(); q.pop();
      if (d[v] >= d[n]) continue;
      for (int u : G[v]) if (d[R[u]] == LLONG_MAX)
        d[R[u]] = d[v]+1, q.push(R[u]);
    }
    return d[n] != LLONG_MAX;
  }
  bool dfs(int v) {
    if (v == n) return true;
    for (int u : G[v])
      if (d[R[u]] == d[v]+1 && dfs(R[u])) {
        R[u] = v; L[v] = u; return true;
      }
    d[v] = LLONG_MAX; return false;
  }
  int max_match() {
    while (bfs()) REP(i,n) s += L[i]<0 && dfs(i);
    return s;
  }

  void dfs2(int v, vector<bool> &alt) {
    alt[v] = true;
    for (int u : G[v]) {
      alt[u+n] = true;
      if (R[u] != n && !alt[R[u]]) dfs2(R[u], alt);
    }
  }

  vi min_vertex_cover() {
    vector<bool> alt(n+m, false); vi res;
    max_match();
    REP(i, n) if (L[i] < 0) dfs2(i, alt);
    // !alt[i] (i<n) OR alt[i] (i >= n)
    REP(i, n+m) if (alt[i] != (i<n)) res.pb(i);
    return res;
  }
};
```

3.2.3. *Stable marriage.*

With n men, $m \geq n$ women, n preference lists of women for each men, and for every woman j an preference of men defined by `pref[][j]` (lower is better) find for every man a women such that no pair of a men and a woman want to run off together.

```
// n = aantal mannen, m = aantal vrouwen
// voor een man i, is order[i] de prefere
vi stable(int n, int m, vvi order, vvi pref) {
  queue<int> q;
  REP(i, n) q.push(i);
  vi mas(m,-1), mak(n,-1), p(n,0);
  while (!q.empty()) {
    int k = q.front();
    q.pop();
    int s = order[k][p[k]], k2 = mas[s];
    if (mas[s] == -1) {
      mas[s] = k;
      mak[k] = s;
    } else if (pref[k][s] < pref[k2][s]) {
      mas[s] = k;
      mak[k] = s;
      mak[k2] = -1;
      q.push(k2);
    } else {
      q.push(k);
    }
    p[k]++;
  }
  return mak;
}
```

3.3. Cycle Detection $\mathcal{O}(V + E)$.

```
bool cycle_detection(const vvi &adj) {
  int n = sz(adj); // undirected graph
  stack<int> s;
  vector<bool> vis(n, false); vi par(n, -1);
  s.push(0); vis[0] = true;
  while (!s.empty()) {
    int cur = s.top(); s.pop();
    for (int i : adj[cur]) {
      if (vis[i] && par[cur] != i) return true;
      s.push(i); par[i] = cur; vis[i] = true;
    }
  }
  return false;
}
```

3.4. Depth first searches.

3.4.1. *Topological Sort $O(V + E)$.*

```
vi topo(vvi &adj) { // requires C++14
  int n=sz(adj); vector<bool> vis(n,0); vi ans;
  auto dfs = [&](int v, const auto& f)->void {
    vis[v] = true;
    for (int w : adj[v]) if (!vis[w]) f(w, f);
    ans.pb(v);
  };
  REP(i, n) if (!vis[i]) dfs(i, dfs);
  reverse(all(ans));
  return ans;
}
```

3.4.2. *Cut Points and Bridges $O(V + E)$.*

```
const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;

void dfs(vvi &adj, vi &cp, vii &bs, int u, int p) {
  low[u] = num[u] = curnum++;
  int cnt = 0; bool found = false;
  REP(i, sz(adj[u])) {
    int v = adj[u][i];
    if (num[v] == -1) {
      dfs(adj, cp, bs, v, u);
      low[u] = min(low[u], low[v]);
      cnt++;
      found = found || low[v] >= num[u];
      if (low[v] > num[u]) bs.eb(u, v);
    } else if (p != v) low[u] = min(low[u], num[v]);
  }
  if (found && (p != -1 || cnt > 1)) cp.pb(u);
}

pair<vi,vii> cut_points_and_bridges(vvi &adj) {
  int n = size(adj);
  vi cp; vii bs;
  memset(num, -1, n << 2);
  curnum = 0;
  REP(i,n) if (num[i] < 0) dfs(adj, cp, bs, i, -1);
  return make_pair(cp, bs);
}
```

3.4.3. *Strongly Connected Components $\mathcal{O}(V + E)$.*

```
struct SCC {
  int n, age=0, ncomps=0; vvi adj, comps;
  vi tidx, lnk, cnr, st; vector<bool> vis;
  SCC(vvi &_adj) : n(sz(_adj)), adj(_adj),
      tidx(n, 0), lnk(n), cnr(n), vis(n, false) {
    REP(i, n) if (!tidx[i]) dfs(i);
  }

  void dfs(int v) {
    tidx[v] = lnk[v] = ++age;
    vis[v] = true; st.pb(v);
    for (int w : adj[v]) {
      if (!tidx[w])
        dfs(w), lnk[v] = min(lnk[v], lnk[w]);
      else if (vis[w]) lnk[v] = min(lnk[v],
      ↪  tidx[w]);
    }
    if (lnk[v] != tidx[v]) return;
    comps.pb(vi());
    int w;
    do {
      vis[w = st.back()] = false; cnr[w] = ncomps;
      comps.back().pb(w);
      st.pop_back();
    } while (w != v);
    ncomps++;
  }
};
```

3.4.4. *2-SAT $\mathcal{O}(V + E)$.* Uses SCC.

```
struct TwoSat {
  int n; SCC *scc = nullptr; vvi adj;
  TwoSat(int _n) : n(_n), adj(_n*2, vi()) {}
  ~TwoSat() { delete scc; }

  // l => r, i.e. r is true or ~l
  void imply(int l, int r) {
    adj[n+l].pb(n+r); adj[n+(~r)].pb(n+(~l)); }
  void OR(int a, int b) { imply(~a, b); }
  void CONST(int a) { OR(a, a); }
  void IFF(int a, int b) { imply(a,b); imply(b,a); }

  bool solve(vector<bool> &sol) {
    delete scc; scc = new SCC(adj);
    REP(i, n) if (scc->cnr[n+i] == scc->cnr[n+(~i)])
      return false;
    vector<bool> seen(n, false);
    sol.assign(n, false);
    for (vi &cc : scc->comps) for (int v : cc) {
      int i = v<n ? n + (~v) : v - n;
      if (!seen[i]) seen[i]=true, sol[i] = v>=n;
    }
    return true;
  }
};
```

3.4.5. *Dominator graph.*

- A node d dominates a node n if every path from the entry node to n must go through d.
- The immediate dominator (idom) of a node n is the unique node that strictly dominates n but does not strictly dominate any other node that strictly dominates n.

```
const int N = 1e6;
vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], par[N], sdom[N];
int p[N], best[N], idom[N], link[N];

void dfs(int v) {
  pos[v] = cnt;
  order[cnt++] = v;
  for (int u : g[v])
    if (pos[u] < 0) par[u] = v, dfs(u);
}

int find_best(int x) {
  if (p[x] == x) return best[x];
  int u = find_best(p[x]);
  if (pos[sdom[u]] < pos[sdom[best[x]]])
    best[x] = u;
  p[x] = p[p[x]];
  return best[x];
}

void dominators(int n, int root) {
  fill_n(pos, n, -1);
  cnt = 0;
  dfs(root);
  REP(i, n) for (int u : g[i]) g_rev[u].pb(i);
  REP(i, n) p[i] = best[i] = sdom[i] = i;

  for (int it = cnt - 1; it >= 1; it--) {
    int w = order[it];
    for (int u : g_rev[w]) {
      int t = find_best(u);
      if (pos[sdom[t]] < pos[sdom[w]])
        sdom[w] = sdom[t];
    }
    bucket[sdom[w]].pb(w);
    idom[w] = sdom[w];
    for (int u : bucket[par[w]])
      link[u] = find_best(u);
    bucket[par[w]].clear();
    p[w] = par[w];
  }

  for (int it = 1; it < cnt; it++) {
    int w = order[it];
    idom[w] = idom[link[w]];
  }
}
```

3.5. Min Cut / Max Flow.

3.5.1. *Dinic's Algorithm $\mathcal{O}(V^2E)$.*

```
struct Edge { int t; ll c, f; };
struct Dinic {
  vi H, P; vvi E;
  vector<Edge> G;
  Dinic(int n) : H(n), P(n), E(n) {}

  void addEdge(int u, int v, ll c) {
    E[u].pb(G.size()); G.pb({v, c, 0LL});
    E[v].pb(G.size()); G.pb({u, 0LL, 0LL});
  }
  ll dfs(int t, int v, ll f) {
    if (v == t || !f) return f;
    for ( ; P[v] < (int) E[v].size(); P[v]++) {
      int e = E[v][P[v]], w = G[e].t;
      if (H[w] != H[v] + 1) continue;
      ll df = dfs(t, w, min(f, G[e].c - G[e].f));
      if (df > 0) {
        G[e].f += df, G[e ^ 1].f -= df;
        return df;
      }
    } return 0;
  }
  ll maxflow(int s, int t, ll f = 0) {
    while (1) {
      fill(all(H), 0); H[s] = 1;
      queue<int> q; q.push(s);
```

```
      while (!q.empty()) {
        int v = q.front(); q.pop();
        for (int w : E[v])
          if (G[w].f < G[w].c && !H[G[w].t])
            H[G[w].t] = H[v] + 1, q.push(G[w].t);
      }
      if (!H[t]) return f;
      fill(all(P), 0);
      while (ll df = dfs(t, s, LLONG_MAX)) f += df;
    }
  }
};
```

3.5.2. *Min-cost max-flow $O(n^2m^2)$.* Find the cheapest possible way of sending a certain amount of flow through a flow network.

```
const int maxn = 300;

struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
  g.pb({u, v, 0, c, w});
  g.pb({v, u, 0, 0, -w});
}

void sp(int s, int t) {
  fill_n(D, V, LLONG_MAX); D[s] = 0;
  for (int ng = g.size(), _ = V; _--; ) {
    bool ok = false;
    for (int i = 0; i < ng; i++)
      if (D[g[i].x] != LLONG_MAX && g[i].f < g[i].c && D[g[i].x] + g[i].w < D[g[i].y]) {
        D[g[i].y] = D[g[i].x] + g[i].w;
        par[g[i].y] = i; ok = true;
      }
    if (!ok) break;
  }
}

void minCostMaxFlow(int s, int t, ll &c, ll &f) {
  for (c = f = 0; sp(s, t), D[t] < LLONG_MAX; ) {
    ll df = LLONG_MAX, dc = 0;
    for (int v = t, e; e = par[v], v != s; v = g[e].x) df = min(df, g[e].c - g[e].f);
    for (int v = t, e; e = par[v], v != s; v = g[e].x) g[e].f += df, g[e^1].f -= df, dc += g[e].w;
    f += df; c += dc * df;
  }
}
```

3.5.3. *Gomory-Hu Tree - All Pairs Maximum Flow.* An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in $O(|V|^2)$ plus $|V|-1$ times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is $O(|V|^3|E|)$. NOTE: Not sure if it works correctly with disconnected graphs.

```
#include "dinic.cpp"
bool same[MAXV];
pair<vii, vvi> construct_gh_tree(flow_network &g) {
  int n = g.n, v;
  vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
  rep(s,1,n) {
    int l = 0, r = 0;
    par[s].second = g.max_flow(s, par[s].first, false);
    memset(d, 0, n * sizeof(int));
    memset(same, 0, n * sizeof(bool));
    d[q[r++] = s] = 1;
    while (l < r) {
      same[v = q[l++]] = true;
      for (int i = g.head[v]; i != -1; i = g.e[i].nxt)
        if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
          d[q[r++] = g.e[i].v] = 1; }
    rep(i,s+1,n)
      if (par[i].first == par[s].first && same[i])
        par[i].first = s;
    g.reset(); }
  rep(i,0,n) {
    int mn = INT_MAX, cur = i;
    while (true) {
      cap[cur][i] = mn;
      if (cur == 0) break;
      mn = min(mn, par[cur].second), cur = par[cur].first; } }
  return make_pair(par, cap); }
int compute_max_flow(int s, int t, const pair<vii, vvi> &gh) {
  int cur = INT_MAX, at = s;
  while (gh.second[at][t] == -1)
    cur = min(cur, gh.first[at].second),
    at = gh.first[at].first;
  return min(cur, gh.second[at][t]); }
```

3.6. Minimal Spanning Tree $\mathcal{O}(E \log V)$.

```
struct edge { int x, y; ll w; };
ll kruskal(int n, vector<edge> edges) {
  dsu D(n);
  sort(all(edges), [] (edge a, edge b) -> bool {
    return a.w < b.w; });
  ll ret = 0;
  for (edge e : edges)
    if (D.find(e.x) != D.find(e.y))
      ret += e.w, D.unite(e.x, e.y);
  return ret;
}
```

3.7. Euler Path $O(V+E)$ hopefully.

Finds an Euler Path (or circuit) in a *directed* graph iff one exists.

```
const int MAXV = 1000, MAXE = 5000;
vi adj[MAXV];
int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
  int start = -1, end = -1, any = 0, c = 0;
  REP(i, n) {
    if(outdeg[i] > 0) any = i;
    if(indeg[i] + 1 == outdeg[i]) start = i, c++;
    else if(indeg[i] == outdeg[i] + 1) end = i, c++;
    else if(indeg[i] != outdeg[i]) return ii(-1,-1);
  }
  if ((start == -1) != (end == -1) || (c != 2 && c))
    return ii(-1,-1);
  if (start == -1) start = end = any;
  return ii(start, end); }
bool euler_path() {
  ii se = start_end();
  int cur = se.first, at = m + 1;
  if (cur == -1) return false;
  stack<int> s;
  while (true) {
    if (outdeg[cur] == 0) {
      res[--at] = cur;
      if (s.empty()) break;
      cur = s.top(); s.pop();
    } else s.push(cur), cur = adj[cur][--outdeg[cur]];
  }
  return at == 0;
}
```

Finds an Euler *cycle* in a *undirected* graph:

```
const int MAXV = 1000;
multiset<int> adj[MAXV];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
  if (at == to) return it;
  L.insert(it, at), --it;
  while (!adj[at].empty()) {
    int nxt = *adj[at].begin();
    adj[at].erase(adj[at].find(nxt));
    adj[nxt].erase(adj[nxt].find(at));
    if (to == -1) {
      it = euler(nxt, at, it);
      L.insert(it, at);
      --it;
    } else {
      it = euler(nxt, to, it);
      to = -1; } }
  return it; }
// usage: euler(0,-1,L.begin());
```

3.8. Heavy-Light Decomposition.

```
struct HLD {
  vvi adj; int cur_pos = 0;
  vi par, dep, hvy, head, pos;

  HLD(int n, const vvi &A) : adj(all(A)), par(n),
      dep(n), hvy(n,-1), head(n), pos(n) {
```

```
    cur_pos = 0; dfs(0); decomp(0, 0);
  }

  int dfs(int v) { // determine parent/depth/sizes
    int wei = 1, mw = 0;
    for (int c : adj[v]) if (c != par[v]) {
      par[c] = v, dep[c] = dep[v]+1;
      int w = dfs(c);
      wei += w;
      if (w > mw) mw = w, hvy[v] = c;
    }
    return wei;
  }

  // pos: index in SegmentTree, head: root of path
  void decomp(int v, int h) {
    head[v] = h, pos[v] = cur_pos++;
    if (hvy[v] != -1) decomp(hvy[v], h);
    for (int c : adj[v])
      if (c != par[v] && c != hvy[v]) decomp(c, c);
  }

  // requires queryST(a, b) = max{A[i] | a≤i<b }.
  int query(int a, int b) {
    int res = 0;
    for (; head[a] != head[b]; b = par[head[b]]) {
      if (dep[head[a]] > dep[head[b]]) swap(a, b);
      res= max(res, queryST(pos[head[b]],pos[b]+1));
    }
    if (dep[a] > dep[b]) swap(a, b);
    return max(res, queryST(pos[a], pos[b]+1));
  }
};
```

3.9. Centroid Decomposition.

```
#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
  path[MAXV][LGMAXV],
  sz[MAXV], seph[MAXV],
  shortest[MAXV];
struct centroid_decomposition {
  int n; vvi adj;
  centroid_decomposition(int _n) : n(_n), adj(n) { }
  void add_edge(int a, int b) {
    adj[a].push_back(b); adj[b].push_back(a); }
  int dfs(int u, int p) {
    sz[u] = 1;
    rep(i,0,size(adj[u]))
      if (adj[u][i] != p) sz[u] += dfs(adj[u][i],
      ↪ u);
    return sz[u]; }
  void makepaths(int sep, int u, int p, int len) {
    jmp[u][seph[sep]] = sep, path[u][seph[sep]] =
    ↪ len;
    int bad = -1;
    rep(i,0,size(adj[u])) {
      if (adj[u][i] == p) bad = i;
      else makepaths(sep, adj[u][i], u, len + 1);
    }
    if (p == sep)
      swap(adj[u][bad], adj[u].back()),
      ↪ adj[u].pop_back(); }
  void separate(int h=0, int u=0) {
    dfs(u,-1); int sep = u;
    down: iter(nxt,adj[sep])
      if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2)
      ↪ {
        sep = *nxt; goto down; }
    seph[sep] = h, makepaths(sep, sep, -1, 0);
    rep(i,0,size(adj[sep])) separate(h+1,
    ↪ adj[sep][i]); }
  void paint(int u) {
    rep(h,0,seph[u]+1)
      shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
                                path[u][h]); }
  int closest(int u) {
    int mn = INT_MAX/2;
    rep(h,0,seph[u]+1)
      mn = min(mn, path[u][h] +
      ↪ shortest[jmp[u][h]]);
    return mn; } };
```

3.10. Least Common Ancestors, Binary Jumping.

```
const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
  rep(i, 0, SZ) BP[i][0] = P[i];
  rep(j, 1, LOGSZ) rep(i, 0, SZ)
    BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
  if (H[a] > H[b]) swap(a, b);
  int dh = H[b] - H[a], j = 0;
  rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
  while (BP[a][j] != BP[b][j]) j++;
  while (--j >= 0) if (BP[a][j] != BP[b][j])
    a = BP[a][j], b = BP[b][j];
  return a == b ? a : P[a];
}
```

3.11. Miscellaneous.

3.11.1. *Misra-Gries $D+1$-edge coloring.* Finds a $\max_i \deg(i)+1$-edge coloring where there all incident edges have distinct colors. Finding a D-edge coloring is NP-hard.

```
struct Edge { int to, col, rev; };

struct MisraGries {
  int N, K=0; vvi F;
  vector<vector<Edge>> G;

  MisraGries(int n) : N(n), G(n) {}

  // add an undirected edge, NO DUPLICATES ALLOWED
  void addEdge(int u, int v) {
    G[u].pb({v, -1, (int) G[v].size()});
    G[v].pb({u, -1, (int) G[u].size()-1});
  }

  void color(int v, int i) {
    vi fan = { i };
    vector<bool> used(G[v].size());
    used[i] = true;
    for (int j = 0; j < (int) G[v].size(); j++)
      if (!used[j] && G[v][j].col >= 0 &&
      ↪ F[G[v][fan.back()].to][G[v][j].col] < 0)
        used[j] = true, fan.pb(j), j = -1;
    int c = 0; while (F[v][c] >= 0) c++;
    int d = 0; while (F[G[v][fan.back()].to][d] >=
    ↪ 0) d++;
    int w = v, a = d, k = 0, ccol;
    while (true) {
      swap(F[w][c], F[w][d]);
      if (F[w][c] >= 0) G[w][F[w][c]].col = c;
      if (F[w][d] >= 0) G[w][F[w][d]].col = d;
      if (F[w][a^=c^d] < 0) break;
      w = G[w][F[w][a]].to;
    }
    do {
      Edge &e = G[v][fan[k]];
      ccol = F[e.to][d] < 0 ? d :
      ↪ G[v][fan[k+1]].col;
      if (e.col >= 0) F[e.to][e.col] = -1;
      F[e.to][ccol] = e.rev;
      F[v][ccol] = fan[k];
      e.col = G[e.to][e.rev].col = ccol;
      k++;
    } while (ccol != d);
  }
  // finds a K-edge-coloring
  void color() {
    REP(v, N) K = max(K, (int) G[v].size() + 1);
    F = vvi(N, vi(K, -1));
    REP(v, N) for (int i = G[v].size(); i--; )
      if (G[v][i].col < 0) color(v, i);
  }
};
```

3.11.2. *Minimum Mean Weight Cycle.* Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```
double
↪ min_mean_cycle(vector<vector<pair<int,double>>>
↪ adj){
  int n = size(adj); double mn = INFINITY;
  vector<vector<double> > arr(n+1, vector<double>(n,
  ↪ mn));
```

```
  arr[0][0] = 0;
  rep(k,1,n+1) rep(j,0,n) iter(it,adj[j])
    arr[k][it->first] = min(arr[k][it->first],
                            it->second +
                            ↪  arr[k-1][j]);
  rep(k,0,n) {
    double mx = -INFINITY;
    rep(i,0,n) mx = max(mx,
    ↪  (arr[n][i]-arr[k][i])/(n-k));
    mn = min(mn, mx); }
  return mn; }
```

3.11.3. *Minimum Arborescence.* Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root r to each vertex. Returns a vector of size n, where the ith element is the edge for the ith vertex. The answer for the root is undefined!

$\mathcal{O}(EV)$ runtime and $\mathcal{O}(E)$ memory:

```
#include "../datastructures/union_find.cpp"
struct arborescence {
  int n; union_find uf;
  vector<vector<pair<ii,int> > > adj;
  arborescence(int _n) : n(_n), uf(n), adj(n) { }
  void add_edge(int a, int b, int c) {
    adj[b].eb(ii(a,b),c); }
  vii find_min(int r) {
    vi vis(n,-1), mn(n,INT_MAX); vii par(n);
    REP(i, n) {
      if (uf.find(i) != i) continue;
      int at = i;
      while (at != r && vis[at] == -1) {
        vis[at] = i;
        for (auto it : adj[at])
          if (it.y < mn[at] && uf.find(it.x.x) !=
          ↪  at)
            mn[at] = it.y, par[at] = it.x;
        if (par[at] == ii(0,0)) return vii();
        at = uf.find(par[at].x);
      }
      if (at == r || vis[at] != i) continue;
      union_find tmp = uf;
      vi seq;
      do seq.pb(at), at = uf.find(par[at].x);
      while (at != seq.front());
      int c = uf.find(seq[0]);
      for (auto it : seq) uf.unite(it, c);
      for (auto &jt : adj[c]) jt.y -= mn[c];
      for (auto it : seq) {
        if (it == c) continue;
        for (auto jt : adj[it])
          adj[c].eb(jt.x, jt.y - mn[it]);
        adj[it].clear();
      }
      vii rest = find_min(r);
      if (rest.empty()) return rest;
      ii use = rest[c];
      rest[at = tmp.find(use.y)] = use;
      for (int it : seq) if (it != at)
        rest[it] = par[it];
      return rest;
    }
    return par; } };
```

$\mathcal{O}(V^2 \log V)$ runtime and $\mathcal{O}(E)$ memory:

```
const int oo = 0x3f3f3f3f, MAXN = 4024;

// N = #V, R = root
int N, R;
// for each node a list of pairs (predecessor,
↪  cost):
vector<pii> g[MAXN];
int pred[MAXN], label[MAXN], node[MAXN],
↪  helper[MAXN];

int get_node(int n) {
  return node[n] == n ? n :
      (node[n] = get_node(node[n]));
}

int update_node(int n) {
  int m = oo;
  for (auto ed : g[n]) m = min(m, ed.y);
  REP(j, sz(g[n])) {
    g[n][j].y -= m;
    if (g[n][j].y == 0)
      pred[n] = g[n][j].x;
  }
  return m;
}

ll cycle(vi &active, int n, int &cend) {
  n = get_node(n);
  if (label[n] == 1) return false;
  if (label[n] == 2) { cend = n; return 0; }

  active.pb(n);
  label[n] = 2;
  auto res = cycle(active, pred[n], cend);
  if (cend == n) {
    int F = find(all(active), n)-active.begin();
    vi todo(active.begin() + F, active.end());
    active.resize(F);
    vii> newg;
    for (auto i: todo) node[i] = n;
    for (auto i: todo) for(auto &ed : g[i])
      helper[ed.x = get_node(ed.x)] = ed.y;
    for (auto i: todo) for(auto ed : g[i])
      helper[ed.x] = min(ed.y, helper[ed.x]);
    for (auto i: todo) for(auto ed: g[i]) {
      if (helper[ed.x] != oo && ed.x != n) {
        newg.eb(ed.x, helper[ed.x]);
        helper[ed.x] = oo;
      }
    }
    g[n] = newg;
    res += update_node(n);
    label[n] = 0;
    cend = -1;
    return cycle(active, n, cend) + res;
  }
  if (cend == -1) {
    active.pop_back();
    label[n] = 1;
  }
  return res;
}

// Calculates value of minimal arborescence from R,
// assuming it exists.
// NOTE: N, R must be initialized at this point!!!
// Algo changes g!!
ll min_arbor() {
  ll res = 0;
  REP(i, N) {
    node[i] = i;
    if (i != R) res += update_node(i);
  }
  REP(i, N) label[i] = (i==R);
  REP(i, N) {
    if (label[i] == 1 || get_node(i) != i)
      continue;
    vi active;
    int cend = -1;
    res += cycle(active, i, cend);
  }
  return res;
}
```

3.11.4. *Maximum Density Subgraph.* Given (weighted) undirected graph G. Binary search density. If g is current density, construct flow network: (S, u, m), $(u, T, m + 2g - d_u)$, $(u, v, 1)$, where m is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty S-component, then maximum density is smaller than g, otherwise it's larger. Distance between valid densities is at least $1/(n(n-1))$. Edge case when density is 0. This also works for weighted graphs by replacing d_u by the weighted degree, and doing more iterations (if weights are not integers).

3.11.5. *Maximum-Weight Closure.* Given a vertex-weighted directed graph G. Turn the graph into a flow network, adding weight ∞ to each edge. Add vertices S, T. For each vertex v of weight w, add edge (S, v, w) if $w \geq 0$, or edge $(v, T, -w)$ if $w < 0$. Sum of positive weights minus minimum $S - T$ cut is the answer. Vertices reachable from S are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

3.11.6. *Maximum Weighted Independent Set in a Bipartite Graph.* This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges $(S, u, w(u))$ for $u \in L$, $(v, T, w(v))$ for $v \in R$ and (u, v, ∞) for $(u, v) \in E$. The minimum S, T-cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

3.11.7. *Synchronizing word problem.* A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

4. String algorithms

4.1. Trie.

```
const int SIGMA = 26;

struct trie {
  bool word; trie **adj;

  trie() : word(false), adj(new trie*[SIGMA]) {
    for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
  }

  void addWord(const string &str) {
    trie *cur = this;
    for (char ch : str) {
      int i = ch - 'a';
      if (!cur->adj[i]) cur->adj[i] = new trie();
      cur = cur->adj[i];
    }
    cur->word = true;
  }

  bool isWord(const string &str) {
    trie *cur = this;
    for (char ch : str) {
      int i = ch - 'a';
      if (!cur->adj[i]) return false;
      cur = cur->adj[i];
    }
    return cur->word;
  }
};
```

4.2. Z-algorithm $\mathcal{O}(n)$.

```
// z[i] = length of longest substring starting from
↪  s[i] which is also a prefix of s.
vi z_function(const string &s) {
  int n = (int) s.length();
  vi z(n);
  for (int i = 1, l = 0, r = 0; i < n; ++i) {
    if (i <= r) z[i] = min (r - i + 1, z[i - l]);
    while(i+z[i] < n && s[z[i]] == s[i+z[i]])
    ↪  ++z[i];
    if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
  }
  return z;
}
```

4.3. Suffix array $\mathcal{O}(n \log n)$.

Lexicographically sorts the cyclic shifts of S where $p[0]$ is the index of the smallest string, etc.

```
vi sort_cyclic_shifts(const string &s) {
  const int alphabet = 256, n = sz(s);

  vi p(n), c(n), cnt(max(alphabet, n), 0);
  REP(i, n) cnt[s[i]]++;
  partial_sum(all(cnt), cnt.begin());
  REP(i, n) p[--cnt[s[i]]] = i;
  c[p[0]] = 0;
  int cl = 1;
  rep(i,1,n) {
    if (s[p[i]] != s[p[i-1]]) cl++;
    c[p[i]] = cl - 1;
  }

  vi pn(n), cn(n);
  for (int h = 0, l = 1; l < n; l*=2, ++h) {
    REP(i, n) {
      pn[i] = p[i] - (1 << h);
      if (pn[i] < 0) pn[i] += n;
    }
    fill(cnt.begin(), cnt.begin() + cl, 0);
    REP(i, n) cnt[c[pn[i]]]++;
    rep(i,1,cl) cnt[i] += cnt[i-1];
    for (int i = n-1; i >= 0; i--)
      p[--cnt[c[pn[i]]]] = pn[i];
    cn[p[0]] = 0;
    cl = 1;
    rep(i, 1, n) {
      if (c[p[i]] != c[p[i-1]] || c[(p[i]+l)%n]
          != c[(p[i-1]+l)%n]) cl++;
      cn[p[i]] = cl - 1;
    }
    c.swap(cn);
  }
  return p;
}

vi suffix_array(string s) {
  s += '\0';
  vi v = sort_cyclic_shifts(s);
  v.erase(v.begin());
  return v;
}
```

4.4. Longest Common Subsequence $\mathcal{O}(n^2)$.

Substring: *consecutive characters* !!!

```
int dp[STR_SIZE][STR_SIZE]; // DP problem

int lcs(const string &w1, const string &w2) {
  int n1 = w1.size(), n2 = w2.size();
  for (int i = 0; i < n1; i++) {
    for (int j = 0; j < n2; j++) {
      if (i == 0 || j == 0) dp[i][j] = 0;
      else if (w1[i-1] == w2[j-1])
        dp[i][j] = dp[i-1][j-1]+1;
      else dp[i][j] = max(dp[i-1][j], dp[i][j-1]);
    }
  }
  return dp[n1][n2];
}

// backtrace
string getLCS(const string &w1, const string &w2) {
  int i = w1.size(), j = w2.size(); string ret = "";
  while (i > 0 && j > 0) {
    if (w1[i - 1] == w2[j - 1]) ret += w1[--i], j--;
    else if (dp[i][j - 1] > dp[i - 1][j]) j--;
    else i--;
  }
  reverse(ret.begin(), ret.end());
  return ret;
}
```

4.5. Levenshtein Distance $\mathcal{O}(n^2)$.

Minimal number of insertions, removals and edits required to transform one string in the other.

```
int dp[MAX_SIZE][MAX_SIZE]; // DP problem

int levDist(const string &w1, const string &w2) {
  int n1 = sz(w1)+1, n2 = sz(w2)+1;
  REP(i, n1) dp[i][0] = i; // removal
  REP(j, n2) dp[0][j] = j; // insertion
  rep(i,1,n1) rep(j,1,n2)
    dp[i][j] = min(
      1 + min(dp[i-1][j], dp[i][j-1]),
      dp[i-1][j-1] + (w1[i-1] != w2[j-1])
    );
  return dp[n1][n2];
}
```

4.6. Knuth-Morris-Pratt algorithm $\mathcal{O}(N + M)$.

```
int kmp(const string &word, const string &text) {
  int n = word.size();
  vi T(n + 1, 0);
  for (int i = 1, j = 0; i < n; ) {
    if (word[i] == word[j]) T[++i] = ++j; // match
    else if (j > 0) j = T[j]; // fallback
    else i++; // no match, keep zero
  }
  int matches = 0;
  for (int i = 0, j = 0; i < text.size(); ) {
    if (text[i] == word[j]) {
      i++;
      if (++j == n) // match at interval [i - n, i)
        matches++, j = T[j];
    } else if (j > 0) j = T[j];
```

```
    else i++;
  }
  return matches;
}
```

4.7. Aho-Corasick Algorithm $\mathcal{O}(N+\sum_{i=1}^{m}|S_i|)$.

Dictionary substring matching as automaton. All given P must be unique!

```
const int MAXP = 100, MAXLEN = 200, SIGMA = 26, MAXTRIE = MAXP * MAXLEN;

int nP;
string P[MAXP], S;

int pnr[MAXTRIE], to[MAXTRIE][SIGMA], sLink[MAXTRIE], dLink[MAXTRIE], nnodes;

void ahoCorasick() {
  fill_n(pnr, MAXTRIE, -1);
  for (int i = 0; i < MAXTRIE; i++) fill_n(to[i], SIGMA, 0);
  fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE, 0);
  nnodes = 1;
  // STEP 1: MAKE A TREE
  for (int i = 0; i < nP; i++) {
    int cur = 0;
    for (char c : P[i]) {
      int i = c - 'a';
      if (to[cur][i] == 0) to[cur][i] = nnodes++;
      cur = to[cur][i];
    }
    pnr[cur] = i;
  }
  // STEP 2: CREATE SUFFIX_LINKS AND DICT_LINKS
  queue<int> q; q.push(0);
  while (!q.empty()) {
    int cur = q.front(); q.pop();
    for (int c = 0; c < SIGMA; c++) {
      if (to[cur][c]) {
        int sl = sLink[to[cur][c]] = cur == 0 ? 0 : to[sLink[cur]][c];
        // if all strings have equal length, remove this:
        dLink[to[cur][c]] = pnr[sl] >= 0 ? sl : dLink[sl];
        q.push(to[cur][c]);
      } else to[cur][c] = to[sLink[cur]][c];
    }
  }
  // STEP 3: TRAVERSE S
  for (int cur = 0, i = 0, n = S.size(); i < n; i++) {
    cur = to[cur][S[i] - 'a'];
    for (int hit = pnr[cur] >= 0 ? cur : dLink[cur]; hit; hit = dLink[hit]) {
      cerr << P[pnr[hit]] << " found at [" << (i + 1 - P[pnr[hit]].size()) << ", " << i << "]" << endl;
    }
  }
}
```

4.8. eerTree.

Constructs an eerTree in $O(n)$, one character at a time.

```
#define MAXN 100100
#define SIGMA 26
#define BASE 'a'
char *s = new char[MAXN];
struct state {
  int len, link, to[SIGMA];
} *st = new state[MAXN+2];
struct eertree {
  int last, sz, n;
  eertree() : last(1), sz(2), n(0) {
    st[0].len = st[0].link = -1;
    st[1].len = st[1].link = 0; }
  int extend() {
    char c = s[n++]; int p = last;
    while (n - st[p].len - 2 < 0 || c != s[n - st[p].len - 2])
      p = st[p].link;
    if (!st[p].to[c-BASE]) {
      int q = last = sz++;
      st[p].to[c-BASE] = q;
      st[q].len = st[p].len + 2;
      do { p = st[p].link;
      } while (p != -1 && (n < st[p].len + 2 ||
               c != s[n - st[p].len - 2]));
      if (p == -1) st[q].link = 1;
      else st[q].link = st[p].to[c-BASE];
      return 1; }
    last = st[p].to[c-BASE];
    return 0; } };
```

4.9. Suffix Automaton.

Minimum automata that accepts all suffixes of a string with $O(n)$ construction. The automata itself is a DAG therefore suitable for DP, examples are counting unique substrings, occurrences of substrings and suffix.

```
// TODO: Add longest common substring
const int MAXL = 100000;
struct suffix_automaton {
  vi len, link, occur, cnt;
  vector<map<char,int> > next;
  vector<bool> isclone;
  ll *occuratleast;
  int sz, last;
  string s;
  suffix_automaton() : len(MAXL*2), link(MAXL*2),
    occur(MAXL*2), next(MAXL*2), isclone(MAXL*2) { clear(); }
  void clear() { sz = 1; last = len[0] = 0; link[0] = -1;
          next[0].clear(); isclone[0] = false; }
bool issubstr(string other){
  for(int i = 0, cur = 0; i < size(other); ++i){
    if(cur == -1) return false; cur = next[cur][other[i]]; }
  return true; }
void extend(char c){ int cur = sz++; len[cur] = len[last]+1;
  next[cur].clear(); isclone[cur] = false; int p = last;
  for(; p != -1 && !next[p].count(c); p = link[p])
    next[p][c] = cur;
  if(p == -1){ link[cur] = 0; }
  else{ int q = next[p][c];
    if(len[p] + 1 == len[q]){ link[cur] = q; }
    else { int clone = sz++; isclone[clone] = true;
      len[clone] = len[p] + 1;
      link[clone] = link[q]; next[clone] = next[q];
      for(; p != -1 && next[p].count(c) && next[p][c] == q;
          p = link[p]){
        next[p][c] = clone; }
      link[q] = link[cur] = clone;
  } } last = cur; }
void count(){
  cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
  map<char,int>::iterator i;
  while(!S.empty()){
    ii cur = S.top(); S.pop();
    if(cur.y){
      for(i = next[cur.x].begin();
          i != next[cur.x].end();++i){
        cnt[cur.x] += cnt[(*i).y]; } }
    else if(cnt[cur.x] == -1){
      cnt[cur.x] = 1; S.push(ii(cur.x, 1));
      for(i = next[cur.x].begin();
          i != next[cur.x].end();++i){
        S.push(ii((*i).y, 0)); } } } }
string lexicok(ll k){
  int st=0; string res; map<char,int>::iterator i;
  while(k){
    for(i = next[st].begin(); i != next[st].end(); ++i){
      if(k <= cnt[(*i).y]){ st = (*i).y;
        res.push_back((*i).x); k--; break;
      } else { k -= cnt[(*i).y]; } } }
  return res; }
void countoccur(){
  REP(i, sz) occur[i] = 1 - isclone[i];
  vii states(sz);
  REP(i, sz) states[i] = ii(len[i],i);
  sort(states.begin(), states.end());
  for(int i = size(states)-1; i >= 0; --i){
    int v = states[i].y;
```

```
      if (link[v] != -1)
        occur[link[v]] += occur[v]; }}};
```

4.10. **Hashing.** Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```
struct hasher {
  int b = 311, m; vi h, p;
  hasher(string s, int _m) :
      m(_m), h(sz(s)+1), p(sz(s)+1) {
    p[0] = 1; h[0] = 0;
    REP(i,sz(s)) p[i+1] = (ll)p[i] * b % m;
    REP(i,sz(s)) h[i+1] = ((ll)h[i] * b + s[i]) % m;
  }
  int hash(int l, int r) {
    return (h[r+1] + m - (ll)h[l]*p[r-l+1] % m) % m;
  }
};
```

5. Geometry

```
const ld EPS = 1e-7, PI = acos(-1.0);
typedef ld NUM; // EITHER ld OR ll
typedef pair<NUM, NUM> pt;

pt operator+(pt p,pt q){ return {p.x+q.x,p.y+q.y}; }
pt operator-(pt p,pt q){ return {p.x-q.x,p.y-q.y}; }
pt operator*(pt p, NUM n) { return {p.x*n, p.y*n}; }

pt& operator+=(pt &p, pt q) { return p = p+q; }
pt& operator-=(pt &p, pt q) { return p = p-q; }

NUM operator*(pt p, pt q){ return p.x*q.x+p.y*q.y; }
NUM operator^(pt p, pt q){ return p.x*q.y-p.y*q.x; }

// square distance from p to line ab
ld distPtLineSq(pt p, pt a, pt b) {
  p -= a; b -= a;
  return ld(p^b) * (p^b) / (b*b);
}
// square distance from p to linesegment ab
ld distPtSegmentSq(pt p, pt a, pt b) {
  p -= a; b -= a;
  NUM dot = p*b, len = b*b;
  if (dot <= 0) return p*p;
  if (dot >= len) return (p-b)*(p-b);
  return p*p - ld(dot)*dot/len;
}
// Test if p is on line segment ab
bool segmentHasPoint(pt p, pt a, pt b) {
  pt u = p-a, v = p-b;
  return abs(u^v) < EPS && u*v <= 0;
}

// projects p onto the line ab
pair<ld,ld> proj(pt p, pt a, pt b) {
  p -= a; b -= a;
  return a + b*(ld(b*p) / (b*b));
}

bool col(pt a, pt b, pt c) {
  return abs((a-b) ^ (a-c)) < EPS;
}

// true => 1 intersection, false => parallel or same
bool linesIntersect(pt a, pt b, pt c, pt d) {
  return abs((a-b) ^ (c-d)) > EPS;
}

pair<ld,ld> lineLineIntersection(pt a, pt b, pt c,
↪ pt d) {
  ld det = (a-b) ^ (c-d);
  assert(abs(det) > EPS);
  return ((c-d)*(a^b) - (a-b)*(c^d)) *
  ↪ (ld(1.0)/det);
}

// dp, dq are directions from p, q
// intersection at p + t_i dp, for 0 <= i < return
↪ value
int segmentIntersection(pt p, pt dp, pt q, pt dq,
    pt &A, pt &B) {
  if (abs(dp * dp)<EPS)
    swap(p,q), swap(dp,dq); // dq=0
  if (abs(dp * dp)<EPS) {
    A = p; // dp = dq = 0
    return p == q;
  }

  pt dpq = q-p;
  NUM c = dp^dq, c0 = dpq^dp, c1 = dpq^dq;
  if (abs(c) < EPS) { // parallel, dp > 0, dq >= 0
    if (abs(c0) > EPS) return 0; // not collinear
    NUM v0 = dpq*dp, v1 = v0 + dq*dp, dp2 = dp*dp;
    if (v1 < v0) swap(v0, v1);

    v0 = max(v0, NUM(0));
    v1 = min(v1, dp2);

    A = p + dp * (ld(v0) / dp2);
    B = p + dp * (ld(v1) / dp2);

    return (v0 <= v1) + (v0 < v1);
  }

  if (c < 0) {
    c = -c; c0 = -c0; c1 = -c1;
  }

  A = p + dp * (ld(c1)/c);
  return 0 <= min(c0,c1) && max(c0,c1) <= c;
}

// Returns TWICE the area of a polygon (for
↪ integers)
NUM polygonTwiceArea(const vector<pt> &p) {
  NUM area = 0;
  for (int n = sz(p), i=0, j=n-1; i<n; j = i++)
    area += p[i] ^ p[j];
  return abs(area); // area < 0 <=> p ccw
}

bool insidePolygon(const vector<pt> &pts, pt p, bool
↪ strict = true) {
  int n = 0;
  for (int N = sz(pts), i = 0, j = N - 1; i < N; j =
  ↪ i++) {
    // if p is on edge of polygon
    if (segmentHasPoint(p, pts[i], pts[j])) return
    ↪ !strict;
    // or: if(distPtSegmentSq(p, pts[i], pts[j]) <=
    ↪ EPS) return !strict;

    // increment n if segment intersects line from p
    n += (max(pts[i].y, pts[j].y) > p.y &&
    ↪ min(pts[i].y, pts[j].y) <= p.y &&
      (((pts[j] - pts[i])^(p-pts[i])) > 0) ==
      ↪ (pts[i].y <= p.y));
  }
  return n & 1; // inside if odd number of
  ↪ intersections
}
```

5.1. Convex Hull $\mathcal{O}(n \log n)$.

```
// the convex hull consists of: { pts[ret[0]],
↪ pts[ret[1]], ... pts[ret.back()] }
vi convexHull(const vector<pt> &pts) {
  if (pts.empty()) return vi();
  vi ret, ord;
  int n = pts.size(), st = min_element(all(pts)) -
  ↪ pts.begin();
  rep(i, 0, n)
    if (pts[i] != pts[st]) ord.pb(i);
  sort(all(ord), [&pts,&st] (int a, int b) {
    pt p = pts[a] - pts[st], q = pts[b] - pts[st];
    return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) <
    ↪ lenSq(q);
  });
  ord.pb(st); ret.pb(st);
  for (int i : ord) {
    // use '>' to include ALL points on the
    ↪ hull-line
    for (int s = ret.size() - 1; s > 0 &&
    ↪ ((pts[ret[s-1]] - pts[ret[s]]) ^ (pts[i] -
    ↪ pts[ret[s]])) >= 0; s--)
      ret.pop_back();
    ret.pb(i);
  }
  ret.pop_back();
  return ret;
}
```

5.2. Rotating Calipers $\mathcal{O}(n)$. Finds the longest distance between two points in a convex hull.

```
NUM rotatingCalipers(vector<pt> &hull) {
  int n = hull.size(), a = 0, b = 1;
  if (n <= 1) return 0.0;
  while (((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
  ↪ hull[b])) > 0) b++;
  NUM ret = 0.0;
  while (a < n) {
    ret = max(ret, lenSq(hull[a], hull[b]));
    if (((hull[(a + 1) % n] - hull[a % n]) ^
    ↪ (hull[(b + 1) % n] - hull[b])) <= 0) a++;
    else if (++b == n) b = 0;
  }
  return ret;
}
```

5.3. Closest points $\mathcal{O}(n \log n)$.

```
int n; pt pts[maxn];

struct byY {
  bool operator()(int a, int b) const { return
  ↪ pts[a].y < pts[b].y; }
};

inline NUM dist(ii p) { return hypot(pts[p.x].x -
↪ pts[p.y].x, pts[p.x].y - pts[p.y].y); }

ii minpt(ii p1, ii p2) { return dist(p1) < dist(p2)
↪ ? p1 : p2; }

// closest pts (by index) inside pts[l ... r], with
↪ sorted y values in ys
ii closest(int l, int r, vi &ys) {
  if (r - l == 2) { // don't assume 1 here.
    ys = { l, l + 1 };
    return ii(l, l + 1);
  } else if (r - l == 3) { // brute-force
    ys = { l, l + 1, l + 2 };
    sort(all(ys), byY());
    return minpt(ii(l, l + 1), minpt(ii(l, l + 2),
    ↪ ii(l + 1, l + 2)));
  }
  int m = (l + r) / 2; vi yl, yr;
  ii delta = minpt(closest(l, m, yl), closest(m, r,
  ↪ yr));
  NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
  ↪ pts[m].x);
  merge(all(yl), all(yr), back_inserter(ys), byY());
  deque<int> q;
  for (int i : ys) if (abs(pts[i].x - xm) <= ddelta)
  ↪ {
    for (int j : q) delta = minpt(delta, ii(i, j));
    q.pb(i);
    if (q.size() > 8) q.pop_front(); // magic from
    ↪ Introduction to Algorithms.
  }
  return delta;
}
```

5.4. Great-Circle Distance. Computes the distance between two points (given as latitude/longitude coordinates) on a sphere of radius r.

```
ld gc_distance(ld pLat, ld pLong, ld qLat, ld qLong,
↪ ld r) {
  pLat *= pi / 180; pLong *= pi / 180;
  qLat *= pi / 180; qLong *= pi / 180;
  return r * acos(cos(pLat)*cos(qLat)*cos(pLong -
  ↪ qLong) + sin(pLat)*sin(qLat)); }
```

5.5. Delaunay triangulation.

```
// https://cp-algorithms.com/geometry/delaunay.html
typedef long long ll;

bool ge(const ll& a, const ll& b) { return a >= b; }
bool le(const ll& a, const ll& b) { return a <= b; }
bool eq(const ll& a, const ll& b) { return a == b; }
bool gt(const ll& a, const ll& b) { return a > b; }
bool lt(const ll& a, const ll& b) { return a < b; }
int sgn(const ll& a) { return (a>0) - (a<0); }

struct pt {
  ll x, y;
  pt() { }
  pt(ll _x, ll _y) : x(_x), y(_y) { }
  pt operator-(const pt& p) const {
    return pt(x - p.x, y - p.y); }
  ll cross(const pt& p) const {
    return x*p.y - y*p.x; }
  ll cross(const pt& a, const pt& b) const {
    return (a - *this).cross(b - *this); }
  ll dot(const pt& p) const {
    return x*p.x + y*p.y; }
  ll dot(const pt& a, const pt& b) const {
    return (a - *this).dot(b - *this); }
  ll lenSq() const { return this->dot(*this); }
  bool operator==(const pt& p) const {
    return eq(x, p.x) && eq(y, p.y); }
};

const pt inf_pt = pt(1e18, 1e18);

struct Quad { // `QuadEdge` originally
  pt O; // origin
  Quad *rot = nullptr, *onext = nullptr;
  bool used = false;
  Quad* rev() const { return rot->rot; }
  Quad* lnext() const {
    return rot->rev()->onext->rot; }
  Quad* oprev() const {
    return rot->onext->rot; }
  pt dest() const { return rev()->O; }
};

Quad* make_edge(pt from, pt to) {
  Quad* e1 = new Quad, e2 = new Quad;
  Quad* e3 = new Quad, e4 = new Quad;
  e1->O = from; e2->O = to;
  e3->O = e4->O = inf_pt;
  e1->rot = e3; e2->rot = e4;
  e3->rot = e2; e4->rot = e1;
  e1->onext = e1; e2->onext = e2;
  e3->onext = e4; e4->onext = e3;
  return e1;
}

void splice(Quad* a, Quad* b) {
  swap(a->onext->rot->onext, b->onext->rot->onext);
  swap(a->onext, b->onext);
}

void delete_edge(Quad* e) {
  splice(e, e->oprev());
  splice(e->rev(), e->rev()->oprev());
  delete e->rot; delete e->rev()->rot;
  delete e; delete e->rev();
}

Quad* connect(Quad* a, Quad* b) {
  Quad* e = make_edge(a->dest(), b->O);
  splice(e, a->lnext());
  splice(e->rev(), b);
  return e;
}

bool left_of(pt p, Quad* e) {
  return gt(p.cross(e->O, e->dest()), 0); }
bool right_of(pt p, Quad* e) {
  return lt(p.cross(e->O, e->dest()), 0); }

template <class T> T det3(T a1, T a2, T a3,
    T b1, T b2, T b3, T c1, T c2, T c3) {
  return a1*(b2*c3 - c2*b3) - a2*(b1*c3 - c1*b3)
    + a3*(b1*c2 - c1*b2);
}

// Calculate directly with __int128, or with angles
bool in_circle(pt a, pt b, pt c, pt d) {
#if defined(__LP64__) || defined(_WIN64)
  __int128 det = 0;
  det -= det3<__int128>(b.x,b.y,b.lenSq(),
    c.x,c.y,c.lenSq(), d.x,d.y,d.lenSq());
  det += det3<__int128>(a.x,a.y,a.lenSq(),
    c.x,c.y,c.lenSq(), d.x,d.y,d.lenSq());
  det -= det3<__int128>(a.x,a.y,a.lenSq(),
    b.x,b.y,b.lenSq(), d.x,d.y,d.lenSq());
  det += det3<__int128>(a.x,a.y,a.lenSq(),
    b.x,b.y,b.lenSq(), c.x,c.y,c.lenSq());
  return det > 0;
#else
  auto ang = [](pt l, pt mid, pt r) {
```

```
    ll x = mid.dot(l, r), y = mid.cross(l, r);
    return atan2((ld) x, (ld) y);
  };
  return (ang(a,b,c) + ang(c,d,a)
    - ang(b,c,d) - ang(d,a,b)) > 1e-8;
#endif
}

pair<Quad*, Quad*> build_tr(int l, int r,
    vector<pt>& p) {
  if (r - l + 1 == 2) {
    Quad* res = make_edge(p[l], p[r]);
    return make_pair(res, res->rev());
  }
  if (r - l + 1 == 3) {
    Quad *a = make_edge(p[l], p[l+1]);
    Quad *b = make_edge(p[l+1], p[r]);
    splice(a->rev(), b);
    int sg = sgn(p[l].cross(p[l + 1], p[r]));
    if (sg == 0) return make_pair(a, b->rev());
    Quad* c = connect(b, a);
    if (sg == 1) return make_pair(a, b->rev());
    return make_pair(c->rev(), c);
  }
  int mid = (l + r) / 2;
  Quad *ldo, *ldi, *rdo, *rdi;
  tie(ldo, ldi) = build_tr(l, mid, p);
  tie(rdi, rdo) = build_tr(mid + 1, r, p);
  while (true) {
    if (left_of(rdi->O, ldi)) {
      ldi = ldi->lnext(); continue; }
    if (right_of(ldi->O, rdi)) {
      rdi = rdi->rev()->onext; continue; }
    break;
  }
  Quad* B = connect(rdi->rev(), ldi);
  auto valid = [&B](Quad* e) {
    return right_of(e->dest(), B); };

  if (ldi->O == ldo->O) ldo = B->rev();
  if (rdi->O == rdo->O) rdo = B;
  while (true) {
    Quad* lc = B->rev()->onext; // left candidate
    if (valid(lc)) {
      while (in_circle(B->dest(), B->O,
          lc->dest(), lc->onext->dest())) {
        Quad* t = lc->onext;
        delete_edge(lc);
        lc = t;
      }
    }
    Quad* rc = B->oprev(); // right candidate
    if (valid(rc)) {
      while (in_circle(B->dest(), B->O,
          rc->dest(), rc->oprev()->dest())) {
        Quad* t = rc->oprev();
        delete_edge(rc);
        rc = t;
      }
    }
    if (!valid(lc) && !valid(rc)) break;
    if (!valid(lc) || (valid(rc) && in_circle(
        lc->dest(), lc->O, rc->O, rc->dest())))
      B = connect(rc, B->rev());
    else B = connect(B->rev(), lc->rev());
  }
  return make_pair(ldo, rdo);
}

vector<tuple<pt, pt, pt>> delaunay(vector<pt> p) {
  sort(all(p), [](const pt& a, const pt& b) {
    return lt(a.x, b.x) ||
        (eq(a.x, b.x) && lt(a.y, b.y));
  });
  auto res = build_tr(0, sz(p) - 1, p);
  Quad* e = res.first;
  vector<Quad*> edges = {e};
  while(lt(e->onext->dest().cross(e->dest(),e->O),0))
    e = e->onext;
  auto add = [&p, &e, &edges]() {
    Quad* cur = e;
    do {
      cur->used = true;
      p.pb(cur->O);
      edges.pb(cur->rev());
      cur = cur->lnext();
    } while (cur != e);
  };
  add(); p.clear();

  int kek = 0;
  while (kek < sz(edges))
    if (!(e = edges[kek++])->used) add();
  vector<tuple<pt, pt, pt>> ans;
  for (int i = 0; i < sz(p); i += 3)
    ans.pb(make_tuple(p[i], p[i + 1], p[i + 2]));
  return ans;
}
```

5.6. 3D Primitives.

```
#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)
#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
  double x, y, z;
  point3d() : x(0), y(0), z(0) {}
  point3d(double _x, double _y, double _z)
    : x(_x), y(_y), z(_z) {}
  point3d operator+(P(p)) const {
    return point3d(x + p.x, y + p.y, z + p.z); }
  point3d operator-(P(p)) const {
    return point3d(x - p.x, y - p.y, z - p.z); }
  point3d operator-() const {
    return point3d(-x, -y, -z); }
  point3d operator*(double k) const {
    return point3d(x * k, y * k, z * k); }
  point3d operator/(double k) const {
    return point3d(x / k, y / k, z / k); }
  double operator%(P(p)) const {
    return x * p.x + y * p.y + z * p.z; }
  point3d operator*(P(p)) const {
    return point3d(y*p.z - z*p.y,
                   z*p.x - x*p.z, x*p.y - y*p.x); }
  double length() const {
    return sqrt(*this % *this); }
  double distTo(P(p)) const {
    return (*this - p).length(); }
  double distTo(P(A), P(B)) const {
    // A and B must be two different points
    return ((*this - A) * (*this - B)).length() /
    ↪ A.distTo(B);}
  point3d normalize(double k = 1) const {
    // length() must not return 0
    return (*this) * (k / length()); }
  point3d getProjection(P(A), P(B)) const {
    point3d v = B - A;
    return A + v.normalize((v % (*this - A)) /
    ↪ v.length()); }
  point3d rotate(P(normal)) const {
    //normal must have length 1 and be orthogonal to
    ↪ the vector
    return (*this) * normal; }
  point3d rotate(double alpha, P(normal)) const {
    return (*this) * cos(alpha) + rotate(normal) *
    ↪ sin(alpha);}
  point3d rotatePoint(P(O), P(axe), double alpha)
  ↪ const{
    point3d Z = axe.normalize(axe % (*this - O));
    return O + Z + (*this - O - Z).rotate(alpha, O);
    ↪ }
  bool isZero() const {
    return abs(x) < EPS && abs(y) < EPS && abs(z) <
    ↪ EPS; }
  bool isOnLine(L(A, B)) const {
    return ((A - *this) * (B - *this)).isZero(); }
  bool isInSegment(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
    ↪ *this))<EPS;}
  bool isInSegmentStrictly(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
    ↪ *this))<-EPS;}
  double getAngle() const {
    return atan2(y, x); }
  double getAngle(P(u)) const {
    return atan2((*this * u).length(), *this % u); }
  bool isOnPlane(PL(A, B, C)) const {
    return
      abs((A - *this) * (B - *this) % (C - *this)) <
      ↪ EPS; } };
int line_line_intersect(L(A, B), L(C, D), point3d
↪ &O){
```

```
  if (abs((B - A) * (C - A) % (D - A)) > EPS) return
  ↪  0;
  if (((A - B) * (C - D)).length() < EPS)
    return A.isOnLine(C, D) ? 2 : 0;
  point3d normal = ((A - B) * (C - B)).normalize();
  double s1 = (C - A) * (D - A) % normal;
  O = A + ((B - A) / (s1 + ((D - B) * (C - B) %
  ↪  normal))) * s1;
  return 1; }
int line_plane_intersect(L(A, B), PL(C, D, E),
↪  point3d & O) {
  double V1 = (C - A) * (D - A) % (E - A);
  double V2 = (D - B) * (C - B) % (E - B);
  if (abs(V1 + V2) < EPS)
    return A.isOnPlane(C, D, E) ? 2 : 0;
  O = A + ((B - A) / (V1 + V2)) * V1;
  return 1; }
bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
    point3d &P, point3d &Q) {
  point3d n = nA * nB;
  if (n.isZero()) return false;
  point3d v = n * nA;
  P = A + (n * nA) * ((B - A) % nB / (v % nB));
  Q = P + n;
  return true; }
```

5.7. Polygon Centroid.

$$C_x = \frac{1}{6A}\sum_{i=0}^{n-1}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A}\sum_{i=0}^{n-1}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2}\sum_{i=0}^{n-1}(x_i y_{i+1} - x_{i+1} y_i)$$

5.8. Rectilinear Minimum Spanning Tree.

Given a set of n points in the plane, and the aim is to find a minimum spanning tree connecting these n points, assuming the Manhattan distance is used. The function `candidates` returns at most $4n$ edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```
#define MAXN 100100
struct RMST {
  struct point {
    int i; ll x, y;
    point() : i(-1) { }
    ll d1() { return x + y; }
    ll d2() { return x - y; }
    ll dist(point other) {
      return abs(x - other.x) + abs(y - other.y); }
    bool operator <(const point &other) const {
      return y==other.y ? x > other.x : y < other.y;
    }
  } best[MAXN], A[MAXN], tmp[MAXN];
  int n;
  RMST() : n(0) {}
  void add_point(int x, int y) {
    A[A[n].i = n].x = x, A[n++].y = y; }
  void rec(int l, int r) {
    if (l >= r) return;
    int m = (l+r)/2;
    rec(l,m), rec(m+1,r);
    point bst;
    for(int i=l, j=m+1, k=l; i <= m || j <= r; k++){
      if(j>r || (i <= m && A[i].d1() < A[j].d1())){
        tmp[k] = A[i++];
        if (bst.i != -1 && (best[tmp[k].i].i == -1
            || best[tmp[k].i].d2() < bst.d2()))
          best[tmp[k].i] = bst;
      } else {
        tmp[k] = A[j++];
        if (bst.i == -1 || bst.d2() < tmp[k].d2())
          bst = tmp[k]; } }
    rep(i,l,r+1) A[i] = tmp[i]; }
  vector<pair<ll,ii> > candidates() {
    vector<pair<ll, ii> > es;
    REP(p, 2) {
      REP(q, 2) {
        sort(A, A+n);
        REP(i, n) best[i].i = -1;
        rec(0, n-1);
        REP(i, n) {
          if(best[A[i].i].i != -1)
            es.pb({A[i].dist(best[A[i].i]),
                  {A[i].i, best[A[i].i].i}});
          swap(A[i].x, A[i].y);
          A[i].x *= -1, A[i].y *= -1; } }
      REP(i, n) A[i].x *= -1; }
    return es; } };
```

5.9. Points and lines (CP3).

```
const ld EPS = 1e-9;

ld DEG_to_RAD(ld d) { return d*PI/180.0; }
ld RAD_to_DEG(ld r) { return r*180.0/PI; }

struct point { ld x, y;
  point() { x = y = 0.0; }
  point(ld _x, ld _y) : x(_x), y(_y) {}
  // useful for sorting
  bool operator < (point other) const {
    if (fabs(x - other.x) > EPS)
      return x < other.x;
    return y < other.y; }
  // use EPS (1e-9) when testing for equality
  bool operator == (point other) const {
   return fabs(x-other.x)<EPS &&
   ↪  fabs(y-other.y)<EPS;
  }
};
ld dist(point p1, point p2) {
  // hypot(dx, dy) returns sqrt(dx * dx + dy * dy)
  return hypot(p1.x - p2.x, p1.y - p2.y);
}
// rotate p by rad RADIANS CCW w.r.t origin (0, 0)
point rotate(point p, ld rad) {
  return point(p.x*cos(rad) - p.y*sin(rad),
               p.x*sin(rad) + p.y*cos(rad));
}

// lines are (x,y) s.t. ax + by = c. AND b=0,1.
struct line { ld a, b, c; };

// gives line throuhg p1, p2
line pointsToLine(point p1, point p2) {
  if (fabs(p1.x - p2.x) < EPS) // vertical line
    return { 1.0, 0.0, -p1.x };
  else return {
    -(ld)(p1.y - p2.y) / (p1.x - p2.x),
    1.0,
    -(ld)(l.a * p1.x) - p1.y;
  };
}

bool areParallel(line l1, line l2) {
  return fabs(l1.a-l2.a)<EPS && fabs(l1.b-l2.b)<EPS;
}

bool areSame(line l1, line l2) {
  return areParallel(l1,l2) && fabs(l1.c-l2.c)<EPS;
}

// returns true (+ intersection) if l1,l2 intersect
bool areIntersect(line l1, line l2, point &p) {
  if (areParallel(l1, l2)) return false; // 0 or inf
  // solve two equations:
  p.x = (l2.b * l1.c - l1.b * l2.c)
      / (l2.a * l1.b - l1.a * l2.b);
  // special case: test for vertical line:
  if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
  else                  p.y = -(l2.a * p.x + l2.c);
  return true;
}

// name: `vec' is different from STL vector
struct vec { ld x, y;
  vec(ld _x, ld _y) : x(_x), y(_y) {} };
// convert 2 points to vector a->b
vec toVec(point a, point b) {
  return vec(b.x - a.x, b.y - a.y); }
vec scale(vec v, ld s) { return vec(v.x*s, v.y*s); }
// translate p according to v
point translate(point p, vec v) {
  return point(p.x + v.x , p.y + v.y); }

// convert point and gradient/slope to line
void pointSlopeToLine(point p, ld m, line &l) {
```

```
  l.a = -m; // always -m
  l.b = 1; // always 1
  l.c = -((l.a * p.x) + (l.b * p.y)); }

void closestPoint(line l, point p, point &ans) {
  if (fabs(l.b) < EPS) { // case 1: vertical line
    ans.x = -(l.c); ans.y = p.y; return; }

  if (fabs(l.a) < EPS) { // case 2: horizontal line
    ans.x = p.x; ans.y = -(l.c); return; }
  // normal line:
  line perpendicular;
  pointSlopeToLine(p, 1 / l.a, perpendicular);
  // intersect line l with this perpendicular line
  // the intersection point is the closest point
  areIntersect(l, perpendicular, ans); }

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
  point b;
  closestPoint(l, p, b); // similar to distToLine
  return point(2*b.x - p.x, 2*b.y - p.y);

ld dot(vec a, vec b) { return a.x*b.x + a.y*b.y; }
ld cross(vec a,vec b){ return a.x*b.y - a.y*b.x; }
ld norm_sq(vec v)    { return v.x*v.x + v.y*v.y; }

// returns the distance from p to the line defined
// by points a and b (a != b), closest point in c.
ld distToLine(point p, point a, point b, point &c) {
 // formula: c = a + u * ab
  vec ap = toVec(a, p), ab = toVec(a, b);
  ld u = dot(ap, ab) / norm_sq(ab);
  c = translate(a, scale(ab, u));
  return dist(p, c); }

// returns the distance from p to the line segment
// ab defined by points a and b (still OK if a == b)
// the closest point is stored in c byref.
ld distToLineSegment(point p, point a, point b,
↪  point &c) {
  vec ap = toVec(a, p), ab = toVec(a, b);
  ld u = dot(ap, ab) / norm_sq(ab);
  if (u < 0.0) { c = point(a.x, a.y);
    return dist(p, a); } // closer to a
  if (u > 1.0) { c = point(b.x, b.y);
    return dist(p, b); } // closer to b
  // otherwise closest is perp to line:
  return distToLine(p, a, b, c); }

// returns angle aob in rad
ld angle(point a, point o, point b) {
  vec oa = toVec(o, a), ob = toVec(o, b);
  return acos(dot(oa, ob)
      / sqrt(norm_sq(oa) * norm_sq(ob)));
}

// note: to accept collinear points, change `> 0'
// returns true if r is on the left side of line pq
bool ccw(point p, point q, point r) {
  return cross(toVec(p, q), toVec(p, r)) > 0; }

// returns true if r is on the same line as line pq
bool collinear(point p, point q, point r) {
  return fabs(cross(toVec(p,q), toVec(p,r))) < EPS;
}
```

5.10. **Polygon (CP3).** Polygons have $P_0 = P_{n-1}$ here.

```
typedef vector<point> poly;

// returns the perimeter: sum of Euclidean distances
// of consecutive line segments (polygon edges)
ld perimeter(const poly &P) {
  ld result = 0.0;
  REP(i, sz(P)-1) // remember that P[0] = P[n-1]
    result += dist(P[i], P[i+1]);
  return result; }

// returns the area, which is half the determinant
ld area(const poly &P) {
  ld result = 0.0;
  REP(i, sz(P)-1)
    result += P[i].x*P[i+1].y - P[i+1].x*P[i].y;
  return result;
}

// returns true if we always make the same turn
// throughout the polygon
bool isConvex(const poly &P) {
  int n = sz(P);
  if (n <= 3) return false; // point=2; line=3
  bool isLeft = ccw(P[0], P[1], P[2]);
  rep(i, n-2) if (ccw(P[i], P[i+1],
        P[(i+2) == n ? 1 : i+2]) != isLeft)
    return false; // different sign -> concave
  return true; } // convex

// returns true if pt is in polygon P
bool inPolygon(point pt, const poly &P) {
  if (sz(P) == 0) return false;
  ld sum = 0; // Assume P[0] == P[n-1]
  REP(i, sz(P)-1) {
    if (ccw(pt, P[i], P[i+1]))
         sum += angle(P[i], pt, P[i+1]);
    else sum -= angle(P[i], pt, P[i+1]); }
  return fabs(fabs(sum) - 2*PI) < EPS;
}

// line segment p-q intersect with line A-B.
point lineIntersectSeg(point p, point q,
      point A, point B) {
  ld a = B.y - A.y;
  ld b = A.x - B.x;
  ld c = B.x * A.y - A.x * B.y;
  ld u = fabs(a * p.x + b * p.y + c);
  ld v = fabs(a * q.x + b * q.y + c);
  return point((p.x*v + q.x*u) / (u+v),
               (p.y*v + q.y*u) / (u+v)); }

// cuts polygon Q along the line formed by a -> b
// (note: Q[0] == Q[n-1] is assumed)
poly cutPolygon(point a, point b, const poly &Q) {
  poly P;
  REP(i, sz(Q)) {
    ld left1 = cross(toVec(a,b), toVec(a,Q[i]));
    ld left2 = 0;
    if (i != sz(Q)-1)
      left2 = cross(toVec(a, b), toVec(a, Q[i+1]));
    if (left1 > -EPS)
      P.pb(Q[i]); // Q[i] is left of ab
    if (left1 * left2 < -EPS)
      // edge Q[i]--Q[i+1] crosses line ab
      P.pb(lineIntersectSeg(Q[i], Q[i+1], a, b));
  }
  if (!P.empty() && !(P.back() == P.front()))
    P.pb(P.front()); // make P[0] == P[n-1]
  return P; }

point pivot; // sorts points by angle around pivot
bool angleCmp(point a, point b) {
  if (collinear(pivot, a, b)) // special case
    return dist(pivot, a) < dist(pivot, b);
  ld d1x = a.x - pivot.x, d1y = a.y - pivot.y;
  ld d2x = b.x - pivot.x, d2y = b.y - pivot.y;
  return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0;
}

poly CH(poly P) { // no order of P assumed!
  int i, j, n = sz(P)
  if (n <= 3) {
    // safeguard from corner case
    if (!(P[0] == P[n-1])) P.pb(P[0]);
    return P; // special case, the CH is P itself
  }

 // P0 = point with lowest Y (if tie rightmost X)
  int P0 = 0;
  rep(i, 1, n) if (P[i].y < P[P0].y
        || (P[i].y == P[P0].y && P[i].x > P[P0].x))
    P0 = i;
  // swap P[P0] with P[0]:
  point temp = P[0]; P[0] = P[P0]; P[P0] = temp;

  // second, sort points by angle w.r.t. pivot P0
  pivot = P[0];
  sort(++P.begin(), P.end(), angleCmp); // keep P[0]

  // third, the ccw tests
  poly S = { P[n-1], P[0], P[1] }; // initial S
  i = 2; // then, we check the rest
  while (i < n) { // required: N must be >= 3
    j = sz(S) - 1;
```

```
    if (ccw(S[j-1], S[j], P[i]))
      S.pb(P[i++]); // left turn, accept
    else // pop top of S when right turn
      S.pop_back();
  }
  return S;
}
```

5.11. Triangle (CP3).

```
ld perimeter(point a, point b, point c) {
  return dist(a, b) + dist(b, c) + dist(c, a); }

ld area(ld ab, ld bc, ld ca) {
 // Heron's formula
  ld s = 0.5 * (ab+bc+ca);
  return sqrt(s)*sqrt(s-ab)*sqrt(s-bc)*sqrt(s-ca);
}
ld area(point a, point b, point c) {
  return area(dist(a, b), dist(b, c), dist(c, a));
}
ld rInCircle(ld ab, ld bc, ld ca) {
  return area(ab,bc,ca)*2.0 / (ab+bc+ca);
}

ld rInCircle(point a, point b, point c) {
  return rInCircle(dist(a,b),dist(b,c),dist(c,a));
}
// assumption: the required points/lines functions
// have been written.
// Returns if there is an inCircle center
// if it returns TRUE, ctr will be the inCircle
// center and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point
↪  &ctr, ld &r) {
  r = rInCircle(p1, p2, p3);
  if (fabs(r) < EPS) return false;

  line l1, l2; // compute these two angle bisectors
  ld ratio = dist(p1, p2) / dist(p1, p3);
  point p = translate(p2,
    scale(toVec(p2, p3), ratio / (1 + ratio)));
  pointsToLine(p1, p, l1);

  ratio = dist(p2, p1) / dist(p2, p3);
  p = translate(p1,
    scale(toVec(p1, p3), ratio / (1 + ratio)));
  pointsToLine(p2, p, l2);
  // get their intersection point:
  areIntersect(l1, l2, ctr);
  return true;
}

ld rCircumCircle(ld ab, ld bc, ld ca) {
  return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

ld rCircumCircle(point a, point b, point c) {
  return rCircumCircle(
      dist(a,b), dist(b,c), dist(c,a));
}

// assumption: the required points/lines functions
// have been written.
// Returns 1 iff there is a circumCenter center
// if this function returns 1, ctr will be the
// circumCircle center and r = rCircumCircle
bool circumCircle(point p1, point p2, point p3,
↪  point &ctr, ld &r){
  ld a = p2.x - p1.x, b = p2.y - p1.y;
  ld c = p3.x - p1.x, d = p3.y - p1.y;
  ld e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
  ld f = c * (p1.x + p3.x) + d * (p1.y + p3.y);
  ld g = 2.0 * (a * (p3.y-p2.y) - b * (p3.x-p2.x));
  if (fabs(g) < EPS) return false;

  ctr.x = (d*e - b*f) / g;
  ctr.y = (a*f - c*e) / g;
  r = dist(p1, ctr); // r = dist(center, p_i)
  return true;
}

// returns if pt d is inside the circumCircle
// defined by a,b,c
bool inCircumCircle(point a, point b,
    point c, point d) {
  vec va=toVec(a,d), vb=toVec(b,d), vc=toVec(c,d);
  return 0 <
   va.x * vb.y * (vc.x*vc.x + vc.y*vc.y) +
   va.y * (vb.x*vb.x + vb.y*vb.y) * vc.x +
   (va.x*va.x + va.y*va.y) * vb.x * vc.y -
   (va.x*va.x + va.y*va.y) * vb.y * vc.x -
   va.y * vb.x * (vc.x*vc.x + vc.y*vc.y) -
   va.x * (vb.x*vb.x+vb.y*vb.y) * vc.y;
}

bool canFormTriangle(ld a, ld b, ld c) {
  return a+b > c && a+c > b && b+c > a; }
```

5.12. Circle (CP3).

```
int insideCircle(point_i p, point_i c, int r) { //
↪  all integer version
  int dx = p.x - c.x, dy = p.y - c.y;
  int Euc = dx * dx + dy * dy, rSq = r * r;
  ↪  // all integer
  return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
  ↪  //inside/border/outside

bool circle2PtsRad(point p1, point p2, double r,
↪  point &c) {
  double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
              (p1.y - p2.y) * (p1.y - p2.y);
  double det = r * r / d2 - 0.25;
  if (det < 0.0) return false;
  double h = sqrt(det);
  c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
  c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
  return true; }          // to get the other center,
  ↪  reverse p1 and p2
```

5.13. Formulas.

Let $a = (a_x, a_y)$ and $b = (b_x, b_y)$ be two-dimensional vectors.

- $a \cdot b = |a||b|\cos\theta$, where θ is the angle between a and b.
- $a \times b = |a||b|\sin\theta$, where θ is the signed angle between a and b.
- $a \times b$ is equal to the area of the parallelogram with two of its sides formed by a and b. Half of that is the area of the triangle formed by a and b.
- **Euler's formula:** $V - E + F = 2$
- Side lengths a, b, c can form a triangle iff. $a + b > c$, $b + c > a$ and $a + c > b$.
- Sum of internal angles of a regular convex n-gon is $(n-2)\pi$.
- **Law of sines:** $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:** $b^2 = a^2 + c^2 - 2ac\cos B$
- Internal tangents of circles $(c_1, r_1), (c_2, r_2)$ intersect at $(c_1 r_2 + c_2 r_1)/(r_1 + r_2)$, external intersect at $(c_1 r_2 - c_2 r_1)/(r_1 + r_2)$.

6. Miscellaneous

6.1. Binary search $\mathcal{O}(\log(hi - lo))$.

```
bool test(int n);

int search(int lo, int hi) {
  assert(test(lo) && !test(hi)); // BE CERTAIN
  while (hi - lo > 1) {
    int m = (lo + hi) / 2;
    (test(m) ? lo : hi) = m;
  }
  // assert(test(lo) && !test(hi));
  return lo;
}
```

6.2. Fast Fourier Transform $\mathcal{O}(n \log n)$.

Given two polynomials $A(x) = a_0 + a_1 x + \cdots + a_{n/2} x^{n/2}$ and $B(x) = b_0 + b_1 x + \cdots + b_{n/2} x^{n/2}$, FFT calculates all coefficients of $C(x) = A(x) \cdot B(x) = c_0 + c_1 x + \ldots c_n x^n$, with $c_i = \sum_{j=0}^{i} a_j b_{i-j}$.

```
typedef complex<double> cpx;
const int LOGN = 19, MAXN = 1 << LOGN;

int rev[MAXN];
cpx rt[MAXN], a[MAXN] = {}, b[MAXN] = {};

void fft(cpx *A) {
  REP(i, MAXN) if (i < rev[i]) swap(A[i],
  ↪  A[rev[i]]);
  for (int k = 1; k < MAXN; k *= 2)
    for (int i = 0; i < MAXN; i += 2*k) REP(j, k) {
        cpx t = rt[j + k] * A[i + j + k];
        A[i + j + k] = A[i + j] - t;
```

```
        A[i + j] += t;
      }
}

void multiply() { // a = convolution of a * b
  const ld PI = acos(-1.0);
  rev[0] = 0; rt[1] = cpx(1, 0);
  REP(i, MAXN) rev[i] = (rev[i/2] | (i&1)<<LOGN)/2;
  for (int k = 2; k < MAXN; k *= 2) {
    cpx z(cos(PI/k), sin(PI/k));
    rep(i, k/2, k) rt[2*i]=rt[i], rt[2*i+1]=rt[i]*z;
  }
  fft(a); fft(b);
  REP(i, MAXN) a[i] *= b[i] / (double)MAXN;
  reverse(a+1,a+MAXN); fft(a);
}
```

6.3. Minimum Assignment (Hungarian Algorithm) $\mathcal{O}(n^3)$.

```
int a[MAXN + 1][MAXM + 1]; // matrix, 1-based
int minimum_assignment(int n, int m) { // n rows, m columns
  vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);
  for (int i = 1; i <= n; i++) {
    p[0] = i;
    int j0 = 0;
    vi mv(m + 1, INT_MAX);
    vector<char> used(m + 1, false);
    do {
      used[j0] = true;
      int i0 = p[j0], delta = INT_MAX, j1;
      for (int j = 1; j <= m; j++)
        if (!used[j]) {
          int cur = a[i0][j] - u[i0] - v[j];
          if (cur < mv[j]) mv[j] = cur, way[j] = j0;
          if (mv[j] < delta) delta = mv[j], j1 = j;
        }
      for (int j = 0; j <= m; j++) {
        if(used[j]) u[p[j]] += delta, v[j] -= delta;
        else mv[j] -= delta;
      }
      j0 = j1;
    } while (p[j0] != 0);
    do {
      int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
    } while (j0);
  }
  // column j is assigned to row p[j]
  return -v[0];
}
```

6.4. Partial linear equation solver $\mathcal{O}(N^3)$.

```
typedef double NUM;
const int ROWS = 200, COLS = 200;
const NUM EPS = 1e-5;

// F2: bitset<COLS+1> M[ROWS]; bitset<ROWS> vals;
NUM M[ROWS][COLS + 1], vals[COLS];
bool hasval[COLS];

bool is0(NUM a) { return -EPS < a && a < EPS; }

// finds x such that Ax = b
// A_ij is M[i][j], b_i is M[i][m]
int solveM(int n, int m) {
  // F2: vals.reset();
  int pr = 0, pc = 0;
  while (pc < m) {
    int r = pr, c;
    while (r < n && is0(M[r][pc])) r++;
    if (r == n) { pc++; continue; }

    // F2: M[pr]^=M[r]; M[r]^=M[pr]; M[pr]^=M[r];
    for (c = 0; c <= m; c++)
      swap(M[pr][c], M[r][c]);

    r = pr++; c = pc++;
    // F2: vals.set(pc, M[pr][m]);
    NUM div = M[r][c];
    for (int col = c; col <= m; col++)
      M[r][col] /= div;
    REP(row, n) {
      if (row == r) continue;
      // F2: if (M[row].test(c)) M[row] ^= M[r];
      NUM times = -M[row][c];
      for (int col = c; col <= m; col++)
        M[row][col] += times * M[r][col];
    }
  } // now M is in RREF

  for (int r = pr; r < n; r++)
    if (!is0(M[r][m])) return 0;
  // F2: return 1;
  fill_n(hasval, n, false);
  for (int col = 0, row; col < m; col++) {
    hasval[col] = !is0(M[row][col]);
    if (!hasval[col]) continue;
    for (int c = col + 1; c < m; c++) {
      if (!is0(M[row][c])) hasval[col] = false;
    }
    if (hasval[col]) vals[col] = M[row][m];
    row++;
  }
  REP(i, n) if (!hasval[i]) return 2;
  return 1;
}
```

6.5. Cycle-Finding.

```
ii find_cycle(int x0, int (*f)(int)) {
  int t = f(x0), h = f(t), mu = 0, lam = 1;
  while (t != h) t = f(t), h = f(f(h));
  h = x0;
  while (t != h) t = f(t), h = f(h), mu++;
  h = f(t);
  while (t != h) h = f(h), lam++;
  return ii(mu, lam); }
```

6.6. Longest Increasing Subsequence.

```
vi lis(vi arr) {
  vi seq, back(sz(arr)), ans;
  REP(i, sz(arr)) {
    int res = 0, lo = 1, hi = sz(seq);
    while (lo <= hi) {
      int mid = (lo+hi)/2;
      if (arr[seq[mid-1]] < arr[i]) res = mid, lo = mid + 1;
      else hi = mid - 1;
    }
    if (res < sz(seq)) seq[res] = i;
    else seq.pb(i);
    back[i] = res == 0 ? -1 : seq[res-1];
  }
  int at = seq.back();
  while (at != -1) ans.pb(at), at = back[at];
  reverse(all(ans));
  return ans;
}
```

6.7. Dates.

```
int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
  return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
    367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
    3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
    d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
  int x, n, i, j;
  x = jd + 68569;
  n = 4 * x / 146097;
  x -= (146097 * n + 3) / 4;
  i = (4000 * (x + 1)) / 1461001;
  x -= 1461 * i / 4 - 31;
  j = 80 * x / 2447;
  d = x - 2447 * j / 80;
  x = j / 11;
  m = j + 2 - 12 * x;
  y = 100 * (n - 49) + i + x; }
```

6.8. Simplex.

```
typedef vector<ld> VD;
typedef vector<VD> VVD;
const ld EPS = 1e-9;
struct LPSolver {
 int m, n; vi B, N; VVD D;
 LPSolver(const VVD &A, const VD &b, const VD &c) :
     m(b.size()), n(c.size()),
     N(n + 1), B(m), D(m + 2, VD(n + 2)) {
  REP(i, m) REP(j, n) D[i][j] = A[i][j];
  REP(i, m) { B[i] = n + i; D[i][n] = -1;
    D[i][n + 1] = b[i]; }
  REP(j, n) N[j] = j, D[m][j] = -c[j];
  N[n] = -1; D[m + 1][n] = 1;
```

```
 }
 void Pivot(int r, int s) {
  double inv = 1.0 / D[r][s];
  REP(i, m+2) if (i != r) REP(j, n+2) if (j != s)
    D[i][j] -= D[r][j] * D[i][s] * inv;
  REP(j, n+2) if (j != s) D[r][j] *= inv;
  REP(i, m+2) if (i != r) D[i][s] *= -inv;
  D[r][s] = inv;
  swap(B[r], N[s]); }
 bool Simplex(int phase) {
  int x = phase == 1 ? m + 1 : m;
  while (true) {
   int s = -1;
   for (int j = 0; j <= n; j++) {
    if (phase == 2 && N[j] == -1) continue;
    if (s == -1 || D[x][j] < D[x][s] ||
        D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
   if (D[x][s] > -EPS) return true;
   int r = -1;
   REP(i, m) {
    if (D[i][s] < EPS) continue;
    if (r == -1 || D[i][n + 1] / D[i][s] < D[r][n +
    ↪  1] /
        D[r][s] || (D[i][n + 1] / D[i][s]) ==
        ↪  (D[r][n + 1] /
        D[r][s]) && B[i] < B[r]) r = i; }
   if (r == -1) return false;
   Pivot(r, s); } }
 ld Solve(VD &x) {
  int r = 0;
  rep(i, 1, m) if (D[i][n+1] < D[r][n+1]) r = i;
  if (D[r][n + 1] < -EPS) {
   Pivot(r, n);
   if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
     return -numeric_limits<ld>::infinity();
   REP(i, m) if (B[i] == -1) {
    int s = -1;
    for (int j = 0; j <= n; j++)
     if (s == -1 || D[i][j] < D[i][s] ||
         D[i][j] == D[i][s] && N[j] < N[s])
       s = j;
    Pivot(i, s); }
  }
  if (!Simplex(2)) return
  ↪  numeric_limits<ld>::infinity();
  x = VD(n);
  for (int i = 0; i < m; i++) if (B[i] < n)
    x[B[i]] = D[i][n + 1];
  return D[m][n + 1]; } };
// 2-phase simplex solves linear system:
//     maximize     c^T x
//     subject to   Ax <= b, x >= 0
// INPUT: A -- an m x n matrix
//        b -- an m-dimensional vector
//        c -- an n-dimensional vector
//        x -- optimal solution (by reference)
// OUTPUT: c^T x (inf. if unbounded above, nan if
↪  infeasible)
// *** Example ***
// const int m = 4, n = 3;
// ld _A[m][n] = {{6,-1,0}, {-1,-5,0},
//    {1,5,1}, {-1,-5,-1}};
// ld _b[m] = {10,-4,5,-5}, _c[n]= {1,-1,0};
// VVD A(m);
// VD b(_b, _b + m), c(_c, _c + n), x;
// REP(i, m) A[i] = VD(_A[i], _A[i] + n);
// LPSolver solver(A, b, c);
// ld value = solver.Solve(x);
// cerr << "VALUE: " << value << endl; // 1.29032
// cerr << "SOLUTION:"; // 1.74194 0.451613 1
// REP(i, sz(x)) cerr << " " << x[i];
// cerr << endl;
```

7. Combinatorics

- Catalan numbers (valid bracket seq's of length $2n$):
  $C_0 = 1, C_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1}$.
- Stirling 1^{th} kind ($\#\pi \in \mathfrak{S}_n$ with exactly k cycles):
  $\genfrac{[}{]}{0pt}{}{n}{0} = \genfrac{[}{]}{0pt}{}{0}{n} = \delta_{0n}, \genfrac{[}{]}{0pt}{}{n}{k} = (n-1)\genfrac{[}{]}{0pt}{}{n-1}{k} + \genfrac{[}{]}{0pt}{}{n-1}{k-1}$.
- Stirling 2^{nd} kind (k-partitions of $[n]$):
  $\genfrac{\{}{\}}{0pt}{}{n}{1} = \genfrac{\{}{\}}{0pt}{}{n}{n} = 1, \genfrac{\{}{\}}{0pt}{}{n}{k} = k\genfrac{\{}{\}}{0pt}{}{n-1}{k} + \genfrac{\{}{\}}{0pt}{}{n-1}{k-1}$.
- Bell numbers (partitions of $[n]$):
  $B_0 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^{n} \genfrac{\{}{\}}{0pt}{}{n}{k}$.
- Euler ($\#\pi \in \mathfrak{S}_n$ with exactly k ascents):
  $\genfrac{\langle}{\rangle}{0pt}{}{n}{0} = \genfrac{\langle}{\rangle}{0pt}{}{n}{n-1} = 1, \genfrac{\langle}{\rangle}{0pt}{}{n}{k} = (k+1)\genfrac{\langle}{\rangle}{0pt}{}{n-1}{k} + (n-k)\genfrac{\langle}{\rangle}{0pt}{}{n-1}{k-1}$.
- Euler 2^{nd} order (nr perms of $1, 1, 2, 2, \ldots, n, n$ with exactly k ascents):
  $\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = (k+1)\left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n-k-1)\left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle$.
- Rooted trees: n^{n-1}, unrooted: n^{n-2}.
- Forests of k rooted trees: $\binom{n}{k} k \cdot n^{n-k-1}$.
- $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad 1^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n}, \quad \sum_i \binom{n-i}{i} = F_{n+1}$
- $\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}, \quad x^k = \sum_{i=0}^{k} i! \genfrac{\{}{\}}{0pt}{}{k}{i} \binom{x}{i} = \sum_{i=0}^{k} \genfrac{\langle}{\rangle}{0pt}{}{k}{i} \binom{x+i}{k}$
- $a \equiv b \pmod{x, y} \Leftrightarrow a \equiv b \pmod{\operatorname{lcm}(x, y)}$.
- $ac \equiv bc \pmod{m} \Leftrightarrow a \equiv b \pmod{m / \gcd(c, m)}$.
- $\gcd(n^a - 1, n^b - 1) = \gcd(a, b) - 1$.
- **Möbius inversion formula:** If $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(d) f(n/d)$. If $f(n) = \sum_{m=1}^{n} g(\lfloor n/m \rfloor)$, then $g(n) = \sum_{m=1}^{n} \mu(m) f(\lfloor \frac{n}{m} \rfloor)$.
- **Inclusion-Exclusion:** If $g(T) = \sum_{S \subseteq T} f(S)$, then
  $$f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} g(T).$$
  Corollary: $b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \Longleftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k$.

- **The Twelvefold Way:** Putting n balls into k boxes. $\mathrm{p}(n, k)$ is # partitions of n in k parts, each > 0. $\mathrm{p}_k(n) = \sum_{i=0}^{k} \mathrm{p}(n, k)$.

Balls Boxes	same same	distinct same	same distinct	distinct distinct
-	$\mathrm{p}_k(n)$	$\sum_{i=0}^{k} \genfrac{\{}{\}}{0pt}{}{n}{i}$	$\binom{n+k-1}{k-1}$	k^n
size ≥ 1	$\mathrm{p}(n, k)$	$\genfrac{\{}{\}}{0pt}{}{n}{k}$	$\binom{n-1}{k-1}$	$k!\genfrac{\{}{\}}{0pt}{}{n}{k}$
size ≤ 1	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n!\binom{k}{n}$

8. Formulas

- **Legendre symbol:** $\left(\frac{a}{b}\right) = a^{(b-1)/2} \pmod{b}$, b odd prime.
- **Heron's formula:** A triangle with side lengths a, b, c has area $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$.
- **Shoelace formula:** $A = \frac{1}{2} | \sum_{i=0}^{n-1} x_i y_{i+1} - x_{i+1} y_i |$.
- **Pick's theorem:** A polygon on an integer grid strictly containing i lattice points and having b lattice points on the boundary has area $i + \frac{b}{2} - 1$. (Nothing similar in higher dimensions)
- **Absorption probabilities** A random walk on $[0, n]$ with probability p to increase and q to decrease, starting at k has at n absorption probability $\frac{(q/p)^k - 1}{(q/p)^n - 1}$ if $q \neq p$, and k/n if $q = p$.
- A minimum Steiner tree for n vertices requires at most $n - 2$ additional Steiner vertices.
- **Lagrange polynomial** through points $(x_0, y_0), \ldots, (x_k, y_k)$ is
  $$L(x) = \sum_{j=0}^{k} y_j \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}.$$
- **Hook length formula:** If λ is a Young diagram and $h_\lambda(i, j)$ is the hook-length of cell (i, j), then then the number of Young tableux $d_\lambda = n! / \prod h_\lambda(i, j)$.
- #primitive pythagorean triples with hypotenuse $< n$ approx $n/(2\pi)$.
- **Frobenius Number:** largest number which can't be expressed as a linear combination of numbers $a_1, \ldots, a_n$ with non-negative coefficients. $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$, $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d - 1)$. An integer $x > (\max_i a_i)^2$ can be expressed in such a way iff. $x \mid \gcd(a_1, \ldots, a_n)$.
- **Snell's law:** $v_2 \sin\theta_1 = v_1 \sin\theta_2$ gives the shortest path between two media.

- **BEST theorem:** The number of Eulerian cycles in a *directed* graph G is:
$$t_w(G)\prod_{v\in G}(\deg v-1)!,$$
where $t_w(G)$ is the number of arborescences ("directed spanning" tree) rooted at w: $t_w(G)=\det(q_{ij})_{i,j\neq w}$, with $q_{ij}=[i=j]\mathrm{indeg}(i)-\#\{(i,j)\in E\}$.
- **Burnside's Lemma:** Let a finite group G act on a set X. Denote $X^g=\{x\in X\mid gx=x\}$. For each g in G let X^g denote the set of elements in X that are fixed by g. Then the number of orbits is:
$$|X/G|=\frac{1}{|G|}\sum_{g\in G}|X^g|.$$
- **Bézout's identity:** If (x,y) is a solution to $ax+by=d$ (x,y can be found with EGCD), then all solutions are given by
$$(x+k\cdot\mathrm{lcm}(a,b)/a,\ y-k\cdot\mathrm{lcm}(a,b)/b),\quad k\in\mathbb{Z}$$

9. Game Theory

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

- **Nim:** Let $X=\bigoplus_{i=1}^n x_i$, then $(x_i)_{i=1}^n$ is a winning position iff $X\neq 0$. Find a move by picking k such that $x_k>x_k\oplus X$.
- **Misère Nim:** Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles. The second player wins $(a_1,\dots,a_n)$ if 1) there is a pile $a_i>1$ and $\oplus_{i=1}^n a_i=0$ or 2) all $a_i\le 1$ and $\oplus_{i=1}^n a_i=1$.
- **Staircase Nim:** Stones are moved down a staircase and only removed from the last pile. $(x_i)_{i=1}^n$ is an L-position if $(x_{2i-1})_{i=1}^{n/2}$ is (i.e. only look at odd-numbered piles).
- **Moore's Nim$_k$:** The player may remove from at most k piles (Nim = Nim$_1$). Expand the piles in base 2, do a carry-less addition in base $k+1$ (i.e. the number of ones in each column should be divisible by $k+1$).
- **Dim$^+$:** The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is $k+1$ where 2^k is the largest power of 2 dividing the pile size.
- **Aliquot game:** Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just k.
- **Nim (at most half):** Write $n+1=2^m y$ with m maximal, then the Sprague-Grundy function of n is $(y-1)/2$.
- **Lasker's Nim:** Players may alternatively split a pile into two new non-empty piles. $g(4k+1)=4k+1$, $g(4k+2)=4k+2$, $g(4k+3)=4k+4$, $g(4k+4)=4k+3$ ($k\ge 0$).
- **Hackenbush on trees:** A tree with stalks $(x_i)_{i=1}^n$ may be replaced with a single stalk with length $\bigoplus_{i=1}^n x_i$.

10. Scheduling Theory

Let p_j be the time task j takes on a machine, d_j the deadline, C_j the time it is completed, $L_j=C_j-d_j$ the lateness, $T_j=\max(L_j,0)$ the tardiness, $U_j=1$ iff $T_j>0$ and else 0.

- One machine, minimise $L_{\max}$: do the tasks in increasing deadline
- One machine, minimise $\sum_j w_jC_j$: do the task increasing in p_j/w_j
- One machine, minimise $\sum_{j=1}^n C_j$ under the condition that all tasks can be done on time:
  (1) Initialise $k=n,\tau=\sum_j p_j, J=[n]$
  (2) Take $i_k\in J$ with $d_{i_k}\ge\tau$ and $p_{i_k}\ge p_\ell$ for $\ell\in J$ with $d_\ell\ge\tau$
  (3) $\tau:=\tau-p_{i_k}, k:=k-1, J:=J-\{i_k\}$. If $k\neq 0$, go to step 2.
  (4) The optimale schedule is $i_1,\dots,i_n$.
- One machine, minimise $\sum_j U_j$. Add all tasks in order of increasing deadline; if adding a task makes it contrary with its deadline, remove the processed task with the highest processing time.
- Two machines (all tasks have to be done on both machines, in any order), minimise $C_{\max}$: a greedy algorithm, when a machine is free it picks a task that hasn't been done yet on either machine and has longest processing time on the other machine.
- Two machines (all tasks have to be done first on machine 1, then machine 2), minimise $C_{\max}$. There is an optimal schedule with on both machines the same order of tasks. Take $X=\{j:p_{1j}\le p_{2j}\}$ and Y the complement. Sort X increasing in p_{1j} and Y decreasing in p_{2j}. Then X,Y is an optimal schedule.
- Two machines (all tasks have to be done first on machine 1, then on 2, or vice versa), minimise $C_{\max}$: let J_{12} be the tasks that have to be done first on machine 1, then on 2 and similar J_{21}. Use the above algorithm to find S_{12},S_{21} optimal for J_{12},J_{21}. Then optimal is S_{12},S_{21} for M1 and S_{21},S_{12} for M2. (If there are tasks that have to be done on only one machine, do them in the middle.)

11. Debugging Tips

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
  - Getting `NaN`? Make sure `acos` etc. are not getting values out of their range (perhaps `1+eps`).
  - Rounding negative numbers?
  - Outputting in scientific notation?
- Wrong Answer?
  - Read the problem statement again!
  - Are multiple test cases being handled correctly? Try repeating the same test case many times.
  - Integer overflow?
  - Think very carefully about boundaries of all input parameters
  - Try out possible edge cases:
    * $n=0,n=-1,n=1,n=2^{31}-1$ or $n=-2^{31}$
    * List is empty, or contains a single element
    * n is even, n is odd
    * Graph is empty, or contains a single vertex
    * Graph is a multigraph (loops or multiple edges)
    * Polygon is concave or non-simple
  - Is initial condition wrong for small cases?
  - Are you sure the algorithm is correct?
  - Explain your solution to someone.
  - Are you using any functions that you don't completely understand? Maybe STL functions?
  - Maybe you (or someone else) should rewrite the solution?
  - Can the input line be empty?
- Run-Time Error?
  - Is it actually Memory Limit Exceeded?

11.1. Dynamic programming optimizations.

- Convex Hull
  - $\mathrm{dp}[i]=\min_{j<i}\{\mathrm{dp}[j]+b[j]\times a[i]\}$
  - $b[j]\ge b[j+1]$
  - optionally $a[i]\le a[i+1]$
  - $O(n^2)$ to $O(n)$ (see 2.12).
- Divide & Conquer
  - $\mathrm{dp}[i][j]=\min_{k<j}\{\mathrm{dp}[i-1][k]+C[k][j]\}$
  - $A[i][j]\le A[i][j+1]$
  - sufficient:
$$C[a][c]+C[b][d]\le C[a][d]+C[b][c],(a\le b\le c\le d)\quad\text{(QI)}$$
  - $O(kn^2)$ to $O(kn\log n)$

```
vvi A; // A[i][j] is voor [i,j)
```

```
void divco(ll ls, ll rs, ll lt, ll rt, vi &t, vi
↪  &s) {
  // berekent t|_{[lt,rt)}
  if(lt >= rt) return;
  ll ms = ls, mt = (lt + rt)/2;
  t[mt] = -INF;
  rep(i,ls,rs) {
    if (i >= mt) break;
    if (s[i] + A[i][mt] > t[mt]) {
      t[mt] = s[i] + A[i][mt];
      ms = i;
    }
  }
  divco(ls,ms+1,lt,mt,t,s);
  divco(ms,rs,mt+1,rt,t,s);
}
```

- Knuth
  - $\text{dp}[i][j] = \min_{i<k<j}\{\text{dp}[i][k] + \text{dp}[k][j] + C[i][j]\}$
  - $A[i][j-1] \leq A[i][j] \leq A[i+1][j]$
  - $O(n^3)$ to $O(n^2)$
  - sufficient: QI and $C[b][c] \leq C[a][d]$, $a \leq b \leq c \leq d$

11.2. **Solution Ideas.**

- Dynamic Programming
  - Parsing CFGs: CYK Algorithm
  - Drop a parameter, recover from others
  - Swap answer and a parameter
  - When grouping: try splitting in two
  - 2^k trick
- Greedy
- Randomized
- Optimizations
  - Use bitset (/64)
  - Switch order of loops (cache locality)
- Process queries offline
  - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
  - Mo's algorithm
  - Sqrt decomposition
  - Store 2^k jump pointers
- Data structure techniques
  - Sqrt buckets
  - Store 2^k jump pointers
  - 2^k merging trick
- Counting
  - Inclusion-exclusion principle
  - Generating functions
- Graphs
  - Can we model the problem as a graph?
  - Can we use any properties of the graph?
  - Strongly connected components
  - Cycles (or odd cycles)
  - Bipartite (no odd cycles)
    - Bipartite matching
    - Hall's marriage theorem
    - Stable Marriage
  - Cut vertex/bridge
  - Biconnected components
  - Degrees of vertices (odd/even)
  - Trees
    - Heavy-light decomposition
    - Centroid decomposition
    - Least common ancestor
    - Centers of the tree
  - Eulerian path/circuit
  - Chinese postman problem
  - Topological sort
  - (Min-Cost) Max Flow
  - Min Cut
    - Maximum Density Subgraph
  - Huffman Coding
  - Min-Cost Arborescence
  - Steiner Tree
  - Kirchoff's matrix tree theorem
  - Prüfer sequences
  - Lovász Toggle
  - Look at the DFS tree (which has no cross-edges)
  - Is the graph a DFA or NFA?
    - Is it the Synchronizing word problem?
- math
  - Is the function multiplicative?
  - Look for a pattern
  - Permutations
    - Consider the cycles of the permutation
  - Functions
    - Sum of piecewise-linear functions is a piecewise-linear function
    - Sum of convex (concave) functions is convex (concave)
  - Modular arithmetic
    - Chinese Remainder Theorem
    - Linear Congruence
  - Sieve
  - System of linear equations
  - Values too big to represent?
    - Compute using the logarithm
    - Divide everything by some large value
  - Linear programming
    - Is the dual problem easier to solve?
  - Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
  - 2-SAT
  - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ($\log(n)$)
- Strings
  - Trie (maybe over something weird, like bits)
  - Suffix array
  - Suffix automaton (+DP?)
  - Aho-Corasick
  - eerTree
  - Work with $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
  - Lazy propagation
  - Persistent
  - Implicit
  - Segment tree of X
- Geometry
  - Minkowski sum (of convex sets)
  - Rotating calipers
  - Sweep line (horizontally or vertically?)
  - Sweep angle
  - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem
  - Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)

- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

Practice Contest Checklist

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use printf/scanf with long long/long double?
- Are `__int128` and `__float128` available?
- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: i?python[23], factor.
- Try submitting with `assert(false)` and `assert(true)`.
- Omitting `return 0;` still works?
- Look for directory with sample test cases.
- Make sure printing works.