

# TCR

## git merge -s octopus solution cup

Ludo Pulles, Reinier Schmiermann, Pim Spelier

### CONTENTS

0.1.	De winnende aanpak	2
0.2.	Wrong Answer	2
0.3.	Covering problems	2
1.	Mathematics	2
1.1.	Primitive Root $O(\sqrt{m})$	3
1.2.	Tonelli-Shanks algorithm	3
1.3.	Numeric Integration	3
1.4.	Fast Hadamard Transform	3
1.5.	Tridiagonal Matrix Algorithm	3
1.6.	Josephus problem	3
1.7.	Number of Integer Points under Line	3
1.8.	Solving linear recurrences	3
1.9.	Misc	4
2.	Datastructures	4
2.1.	Order tree	4
2.2.	Segment tree $O(\log n)$	4
2.3.	Binary Indexed Tree $O(\log n)$	5
2.4.	Disjoint-Set / Union-Find $O(\alpha(n))$	5
2.5.	Cartesian tree	5
2.6.	Heap	5
2.7.	Dancing Links	6
2.8.	Misof Tree	6
2.9.	$k$ -d Tree	6
2.10.	Sqrt Decomposition	7
2.11.	Monotonic Queue	7
2.12.	Convex Hull Trick (replace $O(n^2)$ by respectively $O(n)$ and $O(n \log n)$ )	7
2.13.	Sparse Table $O(\log n)$ per query	7
3.	Graph Algorithms	7
3.1.	Shortest path	7
3.2.	Maximum matching $O(nm)$	8
3.3.	Hopcroft-Karp bipartite matching $O(E\sqrt{V})$	8
3.4.	Depth first searches	9
3.5.	Cycle Detection $O(V + E)$	10
3.6.	Maximum Flow Algorithms	10
3.7.	Minimal Spanning Tree	10
3.8.	Topological Sort $O(V + E)$	10
3.9.	Euler Path $O(V + E)$ hopefully	11
3.10.	Heavy-Light Decomposition	11
3.11.	Centroid Decomposition	11
3.12.	Least Common Ancestors, Binary Jumping	12
3.13.	Misra-Gries $D + 1$ -edge coloring	12
3.14.	Minimum Mean Weight Cycle	12

3.15.	Minimum Arborescence	12
3.16.	Maximum Density Subgraph	12
3.17.	Maximum-Weight Closure	12
3.18.	Maximum Weighted Independent Set in a Bipartite Graph	13
3.19.	Synchronizing word problem	13
4.	String algorithms	13
4.1.	Trie	13
4.2.	Z-algorithm $O(n)$	13
4.3.	Suffix array $O(n \log n)$	13
4.4.	Longest Common Subsequence $O(n^2)$	13
4.5.	Levenshtein Distance $O(n^2)$	13
4.6.	Knuth-Morris-Pratt algorithm $O(N + M)$	13
4.7.	Aho-Corasick Algorithm $O(N + \sum_{i=1}^m  S_i )$	14
4.8.	eerTree	14
4.9.	Suffix Automaton	14
4.10.	Hashing	15
5.	Geometry	15
5.1.	Convex Hull $O(n \log n)$	15
5.2.	Rotating Calipers $O(n)$	15
5.3.	Closest points $O(n \log n)$	16
5.4.	Great-Circle Distance	16
5.5.	3D Primitives	16
5.6.	Polygon Centroid	17
5.7.	Rectilinear Minimum Spanning Tree	17
5.8.	Formulas	17
6.	Geometry (CP3)	17
6.1.	Points and lines	17
6.2.	Polygon	17
6.3.	Triangle	18
6.4.	Circle	19
7.	Miscellaneous	20
7.1.	Binary search $O(\log(hi - lo))$	20
7.2.	Fast Fourier Transform $O(n \log n)$	20
7.3.	Minimum Assignment (Hungarian Algorithm) $O(n^3)$	20
7.4.	Partial linear equation solver $O(N^3)$	20
7.5.	Cycle-Finding	20
7.6.	Longest Increasing Subsequence	21
7.7.	Dates	21
7.8.	Simplex	21
8.	Combinatorics	21
8.1.	The Twelffold Way	22
9.	Formulas	22
9.1.	Burnside's Lemma	22
9.2.	Bézout's identity	22
10.	Game Theory	22
11.	Debugging Tips	22
11.1.	Solution Ideas	23
	Practice Contest Checklist	24

```

.bashrc

alias gg='g++ -std=c++17 -Wall -Wconversion
↳ -Wshadow'
alias g='gg -DDEBUG -g -fsanitize=address,undefined'

.vimrc

set nu rnu sw=4 ts=4 sts=4 noet ai hls shcf=-ic
sy on | colo slate

Test script (usage: ./test.sh A/B/..)
g++ -g -Wall -fsanitize=address,undefined
↳ -Wfatal-error -std=c++17 $1.cc || exit
for i in $(ls *.in)
do
    j="$(echo $i | sed 's/.in/.ans/')"
    ./a.out < $i > output
    diff output $j || echo "!!WA on $i!!"
done

template.cc

#include <bits/stdc++.h>
using namespace std;

typedef long long ll;
typedef long double ld;
typedef pair<ll, ll> ii;
typedef vector<ll> vi;
typedef vector<vi> vvi;
typedef vector<ii> vii;

#define x first
#define y second
#define pb push_back
#define eb emplace_back
#define rep(i,a,b) for(auto i=(a); i<(b); ++i)
#define REP(i,n) rep(i,0,n)
#define all(v) (v).begin(), (v).end()
#define rs resize
#define DBG(x) cerr << __LINE__ << ": " << #x << " = " << (x) << endl

const ld PI = acos(-1.0);
template<class T> using min_queue =
    priority_queue<T, vector<T>, greater<T>>;
template<class T> int sz(const T &x) {
    return (int) x.size(); // copy the ampersand(&)!
}

template<class T> ostream& operator<<(ostream &os,
↳ vector<T> &v) {
    os << "\n[";
    for(T &x : v) os << x << ", ";
    return os << "]\n";
}

struct pairhash {
public:

```

```

template<typename T1, typename T2>
size_t operator()(const pair<T1, T2> &p) const {
    size_t lhs = hash<T1>()(p.x);
    size_t rhs = hash<T2>()(p.y);
    return lhs ^ (rhs+0x9e3779b9+(lhs<<6)+(lhs>>2));
}

void run() {}

signed main() {
    // DON'T MIX "scanf" and "cin"!
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    (cout << fixed).precision(18);
    run();
    return 0;
}

```

template.py

```

from sys import *
n,m = [ int(x) for x in
    ↪ stdin.readline().rstrip().split() ]
stdout.write( str(n*m)+"\n" )

from itertools import *
for (x,y) in product(range(3),repeat=2):
    stdout.write( str(3*x+y)+" " )
stdout.write( "\n" )
for L in combinations(range(4),2):
    stdout.write( str(L)+" " )
stdout.write( "\n" )

from functools import *
y = reduce( lambda x,y: x+y, map( lambda x: x*x,
    ↪ range(4) ), -3 )
stdout.write( str(y)+"\n" )

from math import *
stdout.write( "{0:.2f}\n".format(pi) )

```

### 0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Iedereen (incl. Ludo) moet **ALLE** opgaves goed lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zware) test cases.
- Gebruik ll.

### 0.2. Wrong Answer.

- Print de oplossing om te debuggen!
- Kijk naar wellicht makkelijkere problemen.
- Bedenk zelf test cases met **randgevallen**!
- Controleer de **precisie**.

- Controleer op **overflow** (gebruik **OVERAL** ll, ld).  
*Kijk naar overflows in tussenantwoorden bij modulo.*
- Controleer op **typo's**.
- Loop de voorbeeld test case accuraat langs.
- Controleer op off-by-one-errors (in indices of lus-grenzen)?

**Detecting overflow:** This GNU builtin checks for over- and underflow. Result is in res if successful:

```

bool isOverflown =
    ↪ __builtin_[add|mul|sub]_overflow(a, b, &res);

```

### 0.3. Covering problems.

*Minimum edge cover*  $\iff$  *Maximum independent set*

**Matching:** A set of edges without common vertices (*Maximum is the largest such set, maximal is a set which you cannot add more edges to without breaking the property*).

**Minimum Vertex Cover:** A set vertices (cover) such that each edge in the graph is incident to at least one vertex of the set.

**Minimum Edge Cover:** A set of edges (cover) such that every vertex is incident to at least one edge of the set.

**Maximum Independent Set:** A set of vertices in a graph such that no two of them are adjacent.

**König's theorem:** In any bipartite graph, MCBM = MVC = V - MIS.

### 1. MATHEMATICS

**XOR sum:**  $\bigoplus_{x=0}^{a-1} x = \{0, a-1, a\} [a \bmod 4]$ .

```

int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }

```

```

// greatest common divisor
ll gcd(ll a, ll b) { while(b) a%=b, swap(a,b); return a; }
// least common multiple
ll lcm(ll a, ll b) { return a/gcd(a, b)*b; }
ll mod(ll a, ll b) { return (a%b) < 0 ? a+b : a; }

```

*// ab % m for m <= 4e18 in O(log b)*

```

ll mod_mul(ll a, ll b, ll m) {
    ll r = 0;
    while(b) {
        if (b & 1) r = mod(r+a,m);
        a = mod(a+a,m); b >>= 1;
    }
    return r;
}

```

*// a^b % m for m <= 2e9 in O(log b)*

```

ll mod_pow(ll a, ll b, ll m) {
    ll r = 1;
    while(b) {
        if (b & 1) r = (r * a) % m; // mod_mul
        a = (a * a) % m; // mod_mul
        b >>= 1;
    }
    return r;
}

```

```

}

// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
    ll xx = y = 0, yy = x = 1;
    while (b) {
        x -= a / b * xx; swap(x, xx);
        y -= a / b * yy; swap(y, yy);
        a %= b; swap(a, b);
    }
    return a;
}

// Chinese Remainder Theorem: returns (u, v) s.t.
// x=u (mod v) <=> x=a (mod n) and x=b (mod m)
pair<ll, ll> crt(ll a, ll n, ll b, ll m) {
    ll s, t, d = egcd(n, m, s, t); //n,m<=1e9
    if (mod(a - b, d)) return { 0, -1 };
    return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}

// phi[i] = #{ 0 < j <= i | gcd(i, j) = 1 } sieve
vi totient(int N) {
    vi phi(N);
    for (int i = 0; i < N; i++) phi[i] = i;
    for (int i = 2; i < N; i++) if (phi[i] == i)
        for (int j = i; j < N; j+=i) phi[j] -= phi[j]/i;
    return phi;
}

// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
    ll ans = 1;
    while (n) {
        ll np = n % p, kp = k % p;
        if (np < kp) return 0;
        ans = mod(ans * binom(np, kp), p); // (np C kp)
        n /= p; k /= p;
    }
    return ans;
}

// returns if n is prime for n < 3e24 (>2^64)
// but use mul_mod for n > 2e9.
bool millerRabin(ll n) {
    if (n < 2 || n % 2 == 0) return n == 2;
    ll d = n - 1, ad, s = 0, r;
    for (; d % 2 == 0; d /= 2) s++;
    for (int a : { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 }) {
        if (n == a) return true;
        if ((ad = mod_pow(a, d, n)) == 1) continue;
        for (r = 0; r < s && ad + 1 != n; r++)
            ad = (ad * ad) % n;
        if (r == s) return false;
    }
    return true;
}

```

1.1. **Primitive Root**  $O(\sqrt{m})$ . Returns a generator of  $\mathbb{F}_m^*$ . If  $m$  not prime, replace  $m-1$  by totient of  $m$ .

```
ll primitive_root(ll m) {
    vector<ll> div;
    for (ll i = 1; i*i < m; i++)
        if ((m-1) % i == 0) {
            if (i < m-1) div.pb(i);
            if ((m-1)/i < m) div.pb((m-1)/i);
        }
    rep(x,2,m) {
        bool ok = true;
        for (ll d : div) if (mod_pow(x, d, m) == 1)
            { ok = false; break; }
        if (ok) return x;
    }
    return -1;
}
```

1.2. **Tonelli-Shanks algorithm**. Given prime  $p$  and integer  $1 \leq n < p$ , returns the square root  $r$  of  $n$  modulo  $p$ . There is also another solution given by  $-r$  modulo  $p$ .

```
ll legendre(ll a, ll p) {
    if (a % p == 0) return 0;
    return p == 2 || mod_pow(a, (p-1)/2, p) == 1 ? 1 :
        -1;
}
ll tonelli_shanks(ll n, ll p) {
    assert(legendre(n,p) == 1);
    if (p == 2) return 1;
    ll s = 0, q = p-1, z = 2;
    while (~q & 1) s++, q >>= 1;
    if (s == 1) return mod_pow(n, (p+1)/4, p);
    while (legendre(z,p) != -1) z++;
    ll c = mod_pow(z, q, p),
        r = mod_pow(n, (q+1)/2, p),
        t = mod_pow(n, q, p),
        m = s;
    while (t != 1) {
        ll i = 1, ts = (ll)t*t % p;
        while (ts != 1) i++, ts = ((ll)ts * ts) % p;
        ll b = mod_pow(c, 1LL<<(m-i-1), p);
        r = (ll)r * b % p;
        t = (ll)t * b % p * b % p;
        c = (ll)b * b % p;
        m = i;
    }
    return r;
}
```

1.3. **Numeric Integration**. Numeric integration using Simpson's rule.

```
ld numint(ld (*f)(ld), ld a, ld b, ld EPS = 1e-6) {
    ld ba = b - a, m=(a+b)/2;
    return abs(ba) < EPS
        ? ba/8*(f(a)+f(b)+f(a+ba/3)*3+f(b-ba/3)*3)
        : numint(f,a,m,EPS) + numint(f,m,b,EPS);
}
```

1.4. **Fast Hadamard Transform**. Computes XOR-convolutions in  $O(k2^k)$  on  $k$  bits.

For AND-convolution, use  $(x+y, y), (x-y, y)$ .  
For OR-convolution, use  $(x, x+y), (x, -x+y)$ .

**Note:** The array size must be a power of 2.

```
void fht(vi &A, bool inv=false, int l, int r) {
    if (l+1 == r) return;
    int k = (r-l)/2;
    if (!inv) fht(A, inv, l, l+k), fht(A, inv, l+k,
        -> r);
    rep(i,l,l+k) {
        int x = A[i], y = A[i+k];
        if (!inv) A[i] = x-y, A[i+k] = x+y;
        else A[i] = (x+y)/2, A[i+k] = (-x+y)/2;
    }
    if (inv) fht(A, inv, l, l+k), fht(A, inv, l+k, r);
}
```

1.5. **Tridiagonal Matrix Algorithm**. Solves a tridiagonal system of linear equations

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$$

where  $a_1 = c_n = 0$ . Beware of numerical instability.

```
#define MAXN 5000
ld A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
    C[0] /= B[0]; D[0] /= B[0];
    rep(i,1,n-1) C[i] /= B[i] - A[i]*C[i-1];
    rep(i,1,n) D[i] =
        (D[i] - A[i]*D[i-1]) / (B[i] - A[i]*C[i-1]);
    X[n-1] = D[n-1];
    for (int i = n-1; i--;) X[i] = D[i] - C[i]*X[i+1];
}
```

1.6. **Josephus problem**. Last man standing out of  $n$  if every  $k$ th is killed. Zero-based, and does not kill 0 on first pass.

```
int J(int n, int k) {
    if (n == 1 || k == 1) return n-1;
    if (n < k) return (J(n-1,k)+k)%n;
    int np = n - n/k;
    return k*((J(np,k)+np-n*k*np)%np) / (k-1);
}
```

1.7. **Number of Integer Points under Line**. Count the number of integer solutions to  $Ax + By \leq C$ ,  $0 \leq x \leq n$ ,  $0 \leq y$ . In other words, evaluate the sum  $\sum_{x=0}^n \left\lfloor \frac{C-Ax}{B} + 1 \right\rfloor$ . To count all solutions, let  $n = \left\lfloor \frac{C}{a} \right\rfloor$ . In any case, it must hold that  $C - nA \geq 0$ . Be very careful about overflows.

```
ll floor_sum(ll n, ll a, ll b, ll c) {
    if (c == 0) return 1;
    if (c < 0) return 0;
    if (a % b == 0) return
        -> (n+1)*(c/b+1)-n*(n+1)/2*a/b;
    if (a >= b) return
        -> floor_sum(n,a%b,b,c)-a/b*n*(n+1)/2;
    ll t = (c-a*n+b)/b;
    return floor_sum((c-b*t)/b,b,a,c-b*t)+t*(n+1);
}
```

1.8. **Solving linear recurrences**. Given some brute-forced sequence  $s[0], s[1], \dots, s[2n-1]$ , Berlekamp-Massey finds the shortest possible recurrence relation in  $\mathcal{O}(n^2)$ . After that, `lin_rec` finds  $s[k]$  in  $\mathcal{O}(n^2 \log k)$ .

```
// Given a sequence s[0], ..., s[2n-1] finds the
// smallest linear recurrence
// of size <= n compatible with s.
vl BerlekampMassey(const vl &s, ll mod) {
    int n = sz(s), L = 0, m = 0;
    vl C(n), B(n), T;
    C[0] = B[0] = 1;
    ll b = 1;
    REP(i, n) {
        ++m;
        ll d = s[i] % mod;
        rep(j, 1, L+1) d = (d + C[j] * s[i - j]) % mod;
        if (!d) continue;
        T = C;
        ll coef = d * modpow(b, mod-2, mod) % mod;
        rep(j,m,n) C[j] = (C[j] - coef * B[j-m]) % mod;
        if (2 * L > i) continue;
        L = i + 1 - L;
        B = T; b = d; m = 0;
    }
    C.resize(L + 1);
    C.erase(C.begin());
    for (auto &x : C) x = (mod - x) % mod;
    return C;
}
```

```
// Input: A[0,...,n-1], C[0,...,n-1] satisfying
// A[i] = \sum_{j=1}^{n-1} C[j-1] A[i-j],
// Outputs A[k]
ll lin_rec(const vl &A, const vl &C, ll k, ll mod){
    int n = sz(A);
    auto combine = [&](vl a, vl b) {
        vl res(sz(a) + sz(b) - 1, 0);
        REP(i, sz(a)) REP(j, sz(b))
            res[i+j] = (res[i+j] + a[i]*b[j]) % mod;
        for (int i = 2*n; i > n; --i) REP(j,n)
            res[i-1-j] = (res[i-1-j] + res[i]*C[j]) % mod;
        res.resize(n + 1);
        return res;
    };
    vl pol(n + 1, e(pol));
    pol[0] = e[1] = 1;
    for (++k; k; k /= 2) {
        if (k % 2) pol = combine(pol, e);
        e = combine(e, e);
    }
    ll res = 0;
    REP(i, n) res = (res + pol[i + 1] * A[i]) % mod;
    return res;
}
```

## 1.9. Misc.

## • Prime numbers:

1031, 32771, 1048583, 8125344, 33554467, 9982451653, 1073741827, 34359738421, 1099511627791, 35184372088891, 112589906842679, 36028797018963971.

$10^3 + \{-9, -3, 9, 13\}$ ,  $10^6 + \{-17, 3, 33\}$ ,  $10^9 + \{7, 9, 21, 33, 87\}$ .

• Generating functions: Ordinary (ogf):  $A(x) := \sum_{n=0}^{\infty} a_n x^n$ .

Calculate product  $c_n = \sum_{k=0}^n a_k b_{n-k}$  with FFT.

Exponential (e.g.f.):  $A(x) := \sum_{n=0}^{\infty} a_n x^n / n!$ ,

$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$  (use FFT).

• General linear recurrences: If  $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$ , then  $A(x) = \frac{a_0}{1-B(x)}$ .• Inverse polynomial modulo  $x^l$ : Given  $A(x)$ , find  $B(x)$  such that  $A(x)B(x) = 1 + x^l Q(x)$  for some  $Q(x)$ .

Step 1: Start with  $B_0(x) = 1/a_0$

Step 2:  $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \bmod x^{2^{k+1}}$ .

• Fast subset convolution: Given array  $a_i$  of size  $2^k$  calculate

$b_i = \sum_{j \& i = i} a_j$ .

for (int b = 1; b < (1 << k); b <= 1)

for (int i = 0; i < (1 << k); i++)

if (!(i & b)) a[i | b] += a[i];

// inv: if (!(i & b)) a[i | b] -= a[i];

• Primitive Roots: It only exists when  $n$  is  $2, 4, p^k, 2p^k$ , where  $p$  odd prime. If  $g$  is a primitive root, all primitive roots are of the form  $g^k$  where  $k, \phi(p)$  are coprime (hence there are  $\phi(\phi(p))$  primitive roots).

$\leq N$	$10^3$	$10^6$	$10^9$	$10^{12}$	$10^{18}$
$m$	840	720720	735134400	963761198400	
$d(m)$	32	240	1344	6270	103680

For  $n = 10^{18}$ ,  $m = 897612484786617600$ .

## 2. DATASTRUCTURES

## 2.1. Order tree.

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
template<class TK, class TM> using order_tree =
    tree<TK, TM, greater<TK>, rb_tree_tag,
    tree_order_statistics_node_update>;
template<class TK> using order_set =
    order_tree<TK, null_type>;
```

```
vi s;
order_set<ii> t;
void update( ll k, ll v ) {
    t.erase( ii{ s[k], k } );
    s[k] = v;
    t.insert( ii{ s[k], k } );
}
```

```
signed main() {
    ll n = 4;
    s.resize(n, 0);
```

```
rep(i, 0, n) t.insert(ii{0, i});
update( 2, 3 );
cout << t.find_by_order( 2 )->y << endl;
cout << t.order_of_key( ii{s[3], 3} ) << endl;
}
```

2.2. Segment tree  $\mathcal{O}(\log n)$ . Standard segment tree

```
typedef int S; // or define your own object
```

```
const int n = 1 << 20;
```

```
S t[2 * n];
```

```
// combine must be an associative function!
```

```
S combine(S l, S r) { return l+r; } //or max(l, r)
    ↪ etc
```

```
void build() {
    for (int i = n; --i; )
        t[i] = combine(t[2 * i], t[2 * i + 1]);
}
```

```
// set value v on position i
```

```
void update(int i, S v) {
    for (t[i+n] = v; i /= 2; )
        t[i] = combine(t[2 * i], t[2 * i + 1]);
}
```

```
// sum on interval [l, r)
```

```
S query(int l, int r) {
    S resL = 0, resR = 0;
    for (l += n, r += n; l < r; l /= 2, r /= 2) {
        if (l & 1) resL = combine(resL, t[l++]);
        if (r & 1) resR = combine(t[--r], resR);
    }
    return combine(resL, resR);
}
```

## Lazy segment tree

Be careful: all intervals are right-closed  $[\ell, r]$ .

```
struct node {
    int l, r, x, lazy;
    node() {}
    node(int _l, int _r) :
        l(_l), r(_r), x(INT_MAX), lazy(0) {}
    node(int _l, int _r, int _x) : node(_l, _r) { x = _x; }
    node(node a, node
        ↪ b) : node(a.l, b.r) { x = min(a.x, b.x); }
    void update(int v) { x = v; }
    void range_update(int v) { lazy = v; }
    void apply() { x += lazy; lazy = 0; }
    void push(node &u) { u.lazy += lazy; }
};
```

```
struct segment_tree {
    int n;
    vector<node> arr;
    segment_tree() {}
    segment_tree(const vi &a) : n(sz(a)), arr(4*n) {
```

```
mk(a, 0, 0, n-1); }
node mk(const vi &a, int i, int l, int r) {
    int m = (l+r)/2;
    return arr[i] = l > r ? node(l, r) :
        l == r ? node(l, r, a[l]) :
            node(mk(a, 2*i+1, l, m), mk(a, 2*i+2, m+1, r));
}
node update(int at, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (at < hl || hr < at) return arr[i];
    if (hl == at && at == hr) {
        arr[i].update(v); return arr[i]; }
    return arr[i] =
        node(update(at, v, 2*i+1), update(at, v, 2*i+2));
}
node query(int l, int r, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return node(hl, hr);
    if (l <= hl && hr <= r) return arr[i];
    return node(query(l, r, 2*i+1), query(l, r, 2*i+2));
}
node range_update(int l, int r, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return arr[i];
    if (l <= hl && hr <= r) {
        arr[i].range_update(v);
        propagate(i);
        return arr[i];
    }
    return arr[i] = node(range_update(l, r, v, 2*i+1),
        range_update(l, r, v, 2*i+2));
}
void propagate(int i) {
    if (arr[i].l < arr[i].r) {
        arr[i].push(arr[2*i+1]);
        arr[i].push(arr[2*i+2]);
    }
    arr[i].apply();
}
};
```

## Persistent segment tree

Be careful: all intervals are right-closed  $[\ell, r]$ , including build.

```
int segcnt = 0;
struct segment {
    int l, r, lid, rid, sum;
} S[2000000];

int build(int l, int r) {
    if (l > r) return -1;
    int id = segcnt++;
    S[id].l = l;
    S[id].r = r;
    if (l == r) S[id].lid = -1, S[id].rid = -1;
    else {
```

```

    int m = (l + r) / 2;
    S[id].lid = build(l, m);
    S[id].rid = build(m + 1, r);
}
S[id].sum = 0;
return id;
}
int update(int idx, int v, int id) {
    if (id == -1) return -1;
    if (idx < S[id].l || idx > S[id].r) return id;
    int nid = segcnt++;
    S[nid].l = S[id].l;
    S[nid].r = S[id].r;
    S[nid].lid = update(idx, v, S[id].lid);
    S[nid].rid = update(idx, v, S[id].rid);
    S[nid].sum = S[id].sum + v;
    return nid;
}
int query(int id, int l, int r) {
    if (r < S[id].l || S[id].r < l) return 0;
    if (l <= S[id].l && S[id].r <= r) return
        ↪ S[id].sum;
    return query(S[id].lid, l, r) +
        ↪ query(S[id].rid, l, r);
}

```

### 2.3. Binary Indexed Tree $\mathcal{O}(\log n)$ . Use one-based indices ( $i > 0$ )!

```

struct BIT {
    int n; vi A;
    BIT(int _n) : n(_n), A(_n+1, 0) {}
    BIT(vi &v) : n(sz(v)), A(1) {
        for (auto x:v) A.pb(x);
        for (int i=1, j; j=i&-i, i<=n; i++)
            if (i+j <= n) A[i+j] += A[i];
    }
    void update(int i, ll v) { // a[i] += v
        while (i <= n) A[i] += v, i += i&-i;
    }
    ll query(int i) { // sum_{j<=i} a[j]
        ll v = 0;
        while (i) v += A[i], i -= i&-i;
        return v;
    }
};

struct rangeBIT {
    int n; BIT b1, b2;
    rangeBIT(int _n) : n(_n), b1(_n), b2(_n+1) {}
    rangeBIT(vi &v) : n(sz(v)), b1(v), b2(sz(v)+1) {}
    void pupdate(int i, ll v) { b1.update(i, v); }
    void rupdate(int i, int j, ll v) { // a[i,...,j] += v
        b2.update(i, v);
        b2.update(j+1, -v);
        b1.update(j+1, v*j);
        b1.update(i, (1-i)*v);
    }
}

```

```

ll query(int i) {return
    ↪ b1.query(i)+b2.query(i)*i;}
};

```

### 2.4. Disjoint-Set / Union-Find $\mathcal{O}(\alpha(n))$ .

```

struct dsu {
    vi par, rnk;
    dsu(int n) : par(n, -1), rnk(n, 0) {}
    int find(int i) { return
        par[i] < 0 ? i : par[i] = find(par[i]); }
    void unite(int a, int b) {
        if ((a = find(a)) == (b = find(b))) return;
        if (rnk[a] < rnk[b]) swap(a, b);
        if (rnk[a] == rnk[b]) rnk[a]++;
        par[a] += par[b]; par[b] = a;
    }
};

```

### 2.5. Cartesian tree.

```

struct node {
    int x, y, sz;
    node *l, *r;
    node(int _x, int _y)
        : x(_x), y(_y), sz(1), l(NULL), r(NULL) {} };
int tsize(node* t) { return t ? t->sz : 0; }
void augment(node *t) {
    t->sz = 1 + tsize(t->l) + tsize(t->r); }
pair<node*, node*> split(node *t, int x) {
    if (!t) return make_pair((node*)NULL, (node*)NULL);
    if (t->x < x) {
        pair<node*, node*> res = split(t->r, x);
        t->r = res.x; augment(t);
        return make_pair(t, res.y); }
    pair<node*, node*> res = split(t->l, x);
    t->l = res.y; augment(t);
    return make_pair(res.x, t); }
node* merge(node *l, node *r) {
    if (!l) return r; if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r); augment(l); return l; }
    r->l = merge(l, r->l); augment(r); return r; }
node* find(node *t, int x) {
    while (t) {
        if (x < t->x) t = t->l;
        else if (t->x < x) t = t->r;
        else return t; }
    return NULL; }
node* insert(node *t, int x, int y) {
    if (find(t, x) != NULL) return t;
    pair<node*, node*> res = split(t, x);
    return merge(res.x, merge(new node(x, y), res.y)); }
node* erase(node *t, int x) {
    if (!t) return NULL;
    if (t->x < x) t->r = erase(t->r, x);
    else if (x < t->x) t->l = erase(t->l, x);
    else {node *old=t; t=merge(t->l, t->r); delete old;
        ↪ }
}

```

```

    if (t) augment(t); return t;
}
int kth(node *t, int k) {
    if (k < tsize(t->l)) return kth(t->l, k);
    else if (k == tsize(t->l)) return t->x;
    else return kth(t->r, k - tsize(t->l) - 1);
}

```

### 2.6. Heap. An implementation of a binary heap.

```

#define RESIZE
#define SWP(x,y) tmp = x, x = y, y = tmp
struct int_less {
    int_less() {}
    bool operator () (const int &a, const int &b) {
        return a < b;
    }
};
template <class Compare = int_less> struct heap {
    int cap, len, *q, *loc, tmp;
    Compare _cmp;
    inline bool cmp(int i, int j) {
        return _cmp(q[i], q[j]);
    }
    inline void swp(int i, int j) {
        SWP(q[i], q[j]), SWP(loc[q[i]], loc[q[j]]);
    }
    void swim(int i) {
        while (i > 0) {
            int p = (i - 1) / 2;
            if (!cmp(i, p)) break;
            swp(i, p), i = p;
        }
    }
    void sink(int i) {
        while (true) {
            int l = 2*i + 1, r = l + 1;
            if (l >= len) break;
            int m = r >= len || cmp(l, r) ? l : r;
            if (!cmp(m, i)) break;
            swp(m, i), i = m;
        }
    }
    heap(int C=128) : len(0), cap(C), _cmp(Compare())
        ↪ {
        q = new int[C]; loc = new int[C];
        memset(loc, 255, cap << 2);
    }
    ~heap() {
        delete[] q; delete[] loc;
    }
    void push(int n, bool fix = true) {
        if (cap == len || n >= cap) {
            #ifndef RESIZE
            int newcap = 2 * cap;
            while (n >= newcap) newcap *= 2;
            int *newq = new int[newcap], *newloc = new
                ↪ int[newcap];
            REP(i, cap) newq[i] = q[i], newloc[i] = loc[i];

```



```

memset(newloc+cap, 255, (newcap-cap) << 2);
delete[] q, delete[] loc;
loc = newloc, q = newq, cap = newcap;
#else
assert(false);
#endif
}
assert(loc[n] == -1);
loc[n] = len, q[len++] = n;
if (fix) swim(len-1);
}
void pop(bool fix = true) {
assert(len > 0);
loc[q[0]] = -1, q[0] = q[--len], loc[q[0]] = 0;
if (fix) sink(0);
}
int top() { assert(len > 0); return q[0]; }
void heapify() {
for (int i = len - 1; i > 0; i--)
if (cmp(i, (i-1)/2)) swp(i, (i-1)/2);
}
void update_key(int n) {
assert(loc[n] != -1); swim(loc[n]); sink(loc[n]);
}
bool empty() { return len == 0; }
int size() { return len; }
void clear() {
len = 0; memset(loc, 255, cap << 2);
}
};

```

**2.7. Dancing Links.** An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```

template <class T>
struct dancing_links {
struct node {
T item;
node *l, *r;
node(const T &item, node *l=NULL, node *r=NULL)
: item(_item), l(_l), r(_r) {
if (l) l->r = this;
if (r) r->l = this; } };
node *front, *back;
dancing_links() { front = back = NULL; }
node *push_back(const T &item) {
back = new node(item, back, NULL);
if (!front) front = back;
return back; }
node *push_front(const T &item) {
front = new node(item, NULL, front);
if (!back) back = front;
return front; }
void erase(node *n) {
if (!n->l) front = n->r; else n->l->r = n->r;
if (!n->r) back = n->l; else n->r->l = n->l; }
void restore(node *n) {

```

```

if (!n->l) front = n; else n->l->r = n;
if (!n->r) back = n; else n->r->l = n; } }

```

**2.8. Misof Tree.** A simple tree data structure for inserting, erasing, and querying the  $n$ th largest element.

```

const int BITS = 15;
struct misof_tree {
int cnt[BITS][1<<BITS];
misof_tree() { memset(cnt, 0, sizeof(cnt)); }
void insert(int x) {
for (int i=0; i<BITS; cnt[i++][x]++, x >= 1); }
void erase(int x) {
for (int i=0; i<BITS; cnt[i++][x]--, x >= 1); }
int nth(int n) {
int res = 0;
for (int i = BITS-1; i >= 0; i--)
if (cnt[i][res <= 1] <= n)
n -= cnt[i][res], res |= 1;
return res;
}
};

```

**2.9.  $k$ -d Tree.** A  $k$ -dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```

#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
struct pt {
double coord[K];
pt() {}
pt(double c[K]) { REP(i,K) coord[i] = c[i]; }
double dist(const pt &other) const {
double sum = 0.0;
REP(i,K) sum +=
pow(coord[i]-other.coord[i], 2);
return sqrt(sum); } };
struct cmp {
int c;
cmp(int _c) : c(_c) {}
bool operator () (const pt &a, const pt &b) {
for (int i = 0, cc; i <= K; i++) {
cc = i == 0 ? c : i - 1;
if (abs(a.coord[cc] - b.coord[cc]) > EPS)
return a.coord[cc] < b.coord[cc];
}
return false; } };
struct bb {
pt from, to;
bb(pt _from, pt _to) : from(_from), to(_to) {}
double dist(const pt &p) {
double sum = 0.0;
REP(i,K) {
if (p.coord[i] < from.coord[i])
sum += pow(from.coord[i] - p.coord[i],
2.0);
else if (p.coord[i] > to.coord[i])
sum += pow(p.coord[i] - to.coord[i], 2.0);
}
}
}

```

```

return sqrt(sum); }
bb bound(double l, int c, bool left) {
pt nf(from.coord), nt(to.coord);
if (left) nt.coord[c] = min(nt.coord[c], l);
else nf.coord[c] = max(nf.coord[c], l);
return bb(nf, nt); } };
struct node {
pt p; node *l, *r;
node(pt _p, node *_l, node *_r)
: p(_p), l(_l), r(_r) { } };
node *root;

// kd_tree() : root(NULL) { }
kd_tree(vector<pt> pts) {
root = construct(pts, 0, size(pts) - 1, 0); }
node* construct(vector<pt> &pts, int fr, int to,
int c) {
if (fr > to) return NULL;
int mid = fr + (to-fr) / 2;
nth_element(pts.begin() + fr, pts.begin() + mid,
pts.begin() + to + 1, cmp(c));
return new node(pts[mid],
construct(pts, fr, mid - 1, INC(c)),
construct(pts, mid + 1, to, INC(c))); }
bool contains(const pt &p) { return
_con(p, root, 0); }
bool _con(const pt &p, node *n, int c) {
if (!n) return false;
if (cmp(c)(p, n->p)) return _con(p, n->l, INC(c));
if (cmp(c)(n->p, p)) return _con(p, n->r, INC(c));
return true; }
void insert(const pt &p) { _ins(p, root, 0); }
void _ins(const pt &p, node* &n, int c) {
if (!n) n = new node(p, NULL, NULL);
else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c));
}
void clear() { _clr(root); root = NULL; }
void _clr(node *n) {
if (n) _clr(n->l), _clr(n->r), delete n; }
pt nearest_neighbour(const pt &p, bool same=true)
{
assert(root);
double mn = INFINITY, cs[K];
REP(i,K) cs[i] = -INFINITY;
pt from(cs);
REP(i,K) cs[i] = INFINITY;
pt to(cs);
return _nn(p, root, bb(from, to), mn, 0,
same).x;
}
pair<pt, bool> _nn(const pt &p, node *n, bb b,
double &mn, int c, bool same) {
if (!n || b.dist(p) > mn)
return make_pair(pt(), false);
bool found = same || p.dist(n->p) > EPS,
l1 = true, l2 = false;

```

```

pt resp = n->p;
if (found) mn = min(mn, p.dist(resp));
node *n1 = n->l, *n2 = n->r;
REP(i,2) {
    if (i == 1 || cmp(c)(n->p, p))
        swap(n1, n2), swap(l1, l2);
    auto res = _nn(p, n1, b.bound(n->p.coord[c],
        ↪ c, l1), mn, INC(c), same);
    if (res.y && (!found || p.dist(res.x) <
        ↪ p.dist(resp)))
        resp = res.x, found = true;
}
return make_pair(resp, found); } };
```

2.10. **Sqrt Decomposition.** Design principle that supports many operations in amortized  $\sqrt{n}$  per operation.

```

struct segment {
    vi arr;
    segment(vi _arr) : arr(_arr) { } };
vector<segment> T;
int K;
void rebuild() {
    int cnt = 0;
    rep(i,0,size(T))
        cnt += size(T[i].arr);
    K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
    vi arr(cnt);
    for (int i = 0, at = 0; i < size(T); i++)
        rep(j,0,size(T[i].arr))
            arr[at++] = T[i].arr[j];
    T.clear();
    for (int i = 0; i < cnt; i += K)
        T.push_back(segment(vi(arr.begin()+i,
            arr.begin()+min(i+K,
            ↪ cnt)))); }

int split(int at) {
    int i = 0;
    while (i < size(T) && at >= size(T[i].arr))
        at -= size(T[i].arr), i++;
    if (i >= size(T)) return size(T);
    if (at == 0) return i;
    T.insert(T.begin() + i + 1,
        segment(vi(T[i].arr.begin() + at,
            ↪ T[i].arr.end())));
    T[i] = segment(vi(T[i].arr.begin(),
        ↪ T[i].arr.begin() + at));
    return i + 1; }

void insert(int at, int v) {
    vi arr; arr.push_back(v);
    T.insert(T.begin() + split(at), segment(arr)); }

void erase(int at) {
    int i = split(at); split(at + 1);
    T.erase(T.begin() + i); }
```

2.11. **Monotonic Queue.** A queue that supports querying for the minimum element. Useful for sliding window algorithms.

```

struct min_stack {
    stack<int> S, M;
    void push(int x) {
        S.push(x);
        M.push(M.empty() ? x : min(M.top(), x)); }
    int top() { return S.top(); }
    int mn() { return M.top(); }
    void pop() { S.pop(); M.pop(); }
    bool empty() { return S.empty(); } };

struct min_queue {
    min_stack inp, outp;
    void push(int x) { inp.push(x); }
    void fix() {
        if (outp.empty()) while (!inp.empty())
            outp.push(inp.top(), inp.pop()); }
    int top() { fix(); return outp.top(); }
    int mn() {
        if (inp.empty()) return outp.mn();
        if (outp.empty()) return inp.mn();
        return min(inp.mn(), outp.mn()); }
    void pop() { fix(); outp.pop(); }
    bool empty() { return inp.empty() && outp.empty(); } };
};
```

2.12. **Convex Hull Trick (replace  $O(n^2)$  by respectively  $O(n)$  and  $O(n \log n)$ ).** If converting to integers, look out for division by 0 and  $\pm\infty$ .

```

struct convex_hull_trick {
    vector<pair<double,double>> h;
    double intersect(int i) {
        return (h[i+1].second-h[i].second) /
            (h[i].first-h[i+1].first); }
    void add(double m, double b) {
        h.push_back(make_pair(m,b));
        while (size(h) >= 3) {
            int n = size(h);
            if (intersect(n-3) < intersect(n-2)) break;
            swap(h[n-2], h[n-1]);
            h.pop_back(); } }
    double get_min(double x) {
        int lo = 0, hi = size(h) - 2, res = -1;
        while (lo <= hi) {
            int mid = lo + (hi - lo) / 2;
            if (intersect(mid) <= x) res = mid, lo =
                ↪ mid+1;
            else hi = mid - 1; }
        return h[res+1].first * x + h[res+1].second; }
    ↪ };
```

And dynamic variant:

```

const ll is_query = -(1LL<<62);
struct Line {
    ll m, b;
    mutable function<const Line*> succ;
    bool operator<(const Line& rhs) const {
        if (rhs.b != is_query) return m < rhs.m;
        const Line* s = succ();
        if (!s) return 0;
        if (!s) return 0;
```

```

    ll x = rhs.m;
    return b - s->b < (s->m - m) * x; } };
// will maintain upper hull for maximum
struct HullDynamic : public multiset<Line> {
    bool bad(iterator y) {
        auto z = next(y);
        if (y == begin()) {
            if (z == end()) return 0;
            return y->m == z->m && y->b <= z->b; }
        auto x = prev(y);
        if (z==end()) return y->m == x->m && y->b <=
            ↪ x->b;
        return (x->b - y->b)*(z->m - y->m) >=
            (y->b - z->b)*(y->m - x->m); }
    void insert_line(ll m, ll b) {
        auto y = insert({ m, b });
        y->succ = [=]{return
            ↪ next(y)==end()?0:&*next(y);};
        if (bad(y)) { erase(y); return; }
        while (next(y) != end() && bad(next(y)))
            ↪ erase(next(y));
        while (y != begin() && bad(prev(y)))
            ↪ erase(prev(y)); }
    ll eval(ll x) {
        auto l = *lower_bound((Line) { x, is_query });
        return l.m * x + l.b; } };
```

2.13. **Sparse Table  $O(\log n)$  per query.**

```

struct sparse_table {
    vvi m;
    sparse_table(vi arr) {
        m.pb(arr);
        for (int k=0; (1<<(+k)) <= sz(arr); ) {
            int w = (1<<k), hw = w/2;
            m.pb(vi(sz(arr) - w + 1);
            for (int i = 0; i+w <= sz(arr); i++) {
                m[k][i] = min(m[k-1][i], m[k-1][i+hw]);
            }
        }
        int query(int l, int r) { // query min in [l,r]
            int k = 31 - __builtin_clz(r-l); // k = 0;
            // while (1<<(k+1) <= r-l+1) k++;
            return min(m[k][l], m[k][r-(1<<k)+1]);
        }
    };
};
```

### 3. GRAPH ALGORITHMS

3.1. **Shortest path.**

3.1.1. *Dijkstra*  $O(|E| \log |V|)$ .

```

const ll INFY = -1;
vi dijkstra( vector<vii> G, ll s ) {
    vi d( G.size(), INFY );
    priority_queue<ii,vector<ii>,greater<ii>> Q;
    Q.emplace(0,s);
    while(!Q.empty()){
        ll c = Q.top().x, a = Q.top().y;
```

```

Q.pop();
if(d[a] != INFTY)
    continue;
d[a] = c;
for(ii e : G[a])
    Q.emplace(d[a] + e.y, e.x);
}
return d;
}

```

3.1.2. *Stable marriage*. With  $n$  men,  $m \geq n$  women,  $n$  preference lists of women for each men, and for every woman  $j$  an preference of men defined by `pref[][j]` (lower is better) find for every man a women such that no pair of a men and a woman want to run off together.

```

// n = aantal mannen, m = aantal vrouwen
// voor een man i, is order[i] de prefere
vi stable(int n, int m, vvi order, vvi pref) {
    queue<int> q;
    REP(i, n) q.push(i);
    vi mas(m, -1), mak(n, -1), p(n, 0);
    while (!q.empty()) {
        int k = q.front();
        q.pop();
        int s = order[k][p[k]], k2 = mas[s];
        if (mas[s] == -1) {
            mas[s] = k;
            mak[k] = s;
        } else if (pref[k][s] < pref[k2][s]) {
            mas[s] = k;
            mak[k] = s;
            mak[k2] = -1;
            q.push(k2);
        } else {
            q.push(k);
        }
        p[k]++;
    }
    return mak;
}

```

3.1.3. *Floyd-Warshall*  $\mathcal{O}(V^3)$ . Be careful with negative edges! Note:  $|d[i][j]|$  can grow exponentially, and `INFTY + negative < INFTY`.

```

const ll INF = 1LL << 61;
void floyd_warshall(vvi& d) {
    ll n = d.size();
    REP(i, n) REP(j, n) REP(k, n)
        if(d[j][i] < INF && d[i][k] < INF) // neg edges!
            d[j][k] = max(-INF,
                min(d[j][k], d[j][i] + d[i][k]));
}

vvi d(n, vi(n, INF));
REP(i, n) d[i][i] = 0;

```

3.1.4. *Bellman Ford*  $\mathcal{O}(VE)$ . This is only useful if there are edges with weight  $w_{ij} < 0$  in the graph.

```

const ll INF = 1LL << 61;
// G[u] = { (v,w) | edge u->v, cost w }
vi bellman_ford(vector<vii> G, ll s) {
    ll n = G.size();
    vi d(n, INF); d[s] = 0;
    REP(loops, n) REP(u, n) if(d[u] != INF)
        for(ii e : G[u]) if(d[u] + e.y < d[e.x])
            d[e.x] = d[u] + e.y;
    // detect paths of -INF length
    for( ll change = 1; change--;)
        REP(u, n) if(d[u] != INF)
            for(ii e : G[u]) if(d[e.x] != -INF)
                if(d[u] + e.y < d[e.x])
                    d[e.x] = -INF, change = 1;
    return d;
}

```

3.1.5. *IDA\* algorithm*.

```

int n, cur[100], pos;
int calch() {
    int h = 0;
    rep(i, 0, n) if (cur[i] != 0) h += abs(i - cur[i]);
    return h;
}
int dfs(int d, int g, int prev) {
    int h = calch();
    if (g + h > d) return g + h;
    if (h == 0) return 0;
    int mn = INT_MAX;
    rep(di, -2, 3) {
        if (di == 0) continue;
        int nxt = pos + di;
        if (nxt == prev) continue;
        if (0 <= nxt && nxt < n) {
            swap(cur[pos], cur[nxt]);
            swap(pos, nxt);
            mn = min(mn, dfs(d, g+1, nxt));
            swap(pos, nxt);
            swap(cur[pos], cur[nxt]);
        }
        if (mn == 0) break;
    }
    return mn;
}
int idastar() {
    rep(i, 0, n) if (cur[i] == 0) pos = i;
    int d = calch();
    while (true) {
        int nd = dfs(d, 0, -1);
        if (nd == 0 || nd == INT_MAX) return d;
        d = nd;
    }
}

```

3.2. *Maximum matching*  $\mathcal{O}(nm)$ .

```

const int sizeL = 1e4, sizeR = 1e4;

bool vis[sizeR];
int par[sizeR]; // par : R -> L
vi adj[sizeL]; // adj : L -> (N -> R)

bool match(int u) {
    for (int v : adj[u]) {
        if (vis[v]) continue; vis[v] = true;

```

```

        if (par[v] == -1 || match(par[v])) {
            par[v] = u;
            return true;
        }
    }
    return false;
}

// perfect matching iff ret == sizeL == sizeR
int maxmatch() {
    fill_n(par, sizeR, -1); int ret = 0;
    for (int i = 0; i < sizeL; i++) {
        fill_n(vis, sizeR, false);
        ret += match(i);
    }
    return ret;
}

```

3.3. *Hopcroft-Karp bipartite matching*  $\mathcal{O}(E\sqrt{V})$ .

```

const ll INFTY = (1LL<<61LL);

struct bi_graph {
    ll n, m;
    vvi adj;
    vi L, R, d;
    queue<ll> q;
    bi_graph(ll _n, ll _m) : n(_n), m(_m),
        adj(n, L(n, -1), R(m, n), d(n+1)) {}
    ll add_edge(ll a, ll b) { adj[a].pb(b); }
    ll bfs() {
        rep(v, 0, n)
            if (L[v] == -1) d[v] = 0, q.push(v);
            else d[v] = INFTY;
        d[n] = INFTY;
        while (!q.empty()) {
            ll v = q.front(); q.pop();
            if (d[v] < d[n])
                for (ll u : adj[v]) if (d[R[u]] == INFTY)
                    d[R[u]] = d[v]+1, q.push(R[u]);
        }
        return d[n] != INFTY;
    }
    ll dfs(ll v) {
        if (v == n) return true;
        for (ll u : adj[v])
            if (d[R[u]] == d[v] + 1 and dfs(R[u])) {
                R[u] = v; L[v] = u;
                return true;
            }
        d[v] = INFTY;
        return false;
    }
    ll maximum_matching() {
        ll s = 0;
        while (bfs()) rep(i, 0, n)
            s += L[i] == -1 && dfs(i);
        return s;
    }
}

```



```

}
};

```

### 3.3.1. Minimum Vertex Cover in Bipartite Graphs.

```

#include "hopcroft_karp.cpp"
vi alt;
void dfs( bi_graph &G, ll v ) {
    alt[v] = 1;
    for( ll u : G.adj[v] ) {
        alt[u+G.n] = 1;
        if( G.R[u] != G.n && !alt[G.R[u]] )
            dfs(G,G.R[u]);
    }
}
vi mvc_bipartite( bi_graph &G ) {
    vi res; G.maximum_matching();
    alt.assign( G.n + G.m, 0 );
    rep(i,0,G.n) if( G.L[i] == -1 ) dfs(G,i);
    rep(i,0,G.n) if( !alt[i] ) res.pb(i);
    rep(i,0,G.n) if( alt[G.n+i] ) res.pb(G.n+i);
    return res;
}

```

## 3.4. Depth first searches.

### 3.4.1. Cut Points and Bridges $O(V + E)$ .

```

const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;

void dfs(const vvi &adj, vi &cp, vii &bri, int u, int
    ↪ p){
    low[u] = num[u] = curnum++;
    int cnt = 0; bool found = false;
    REP(i, sz(adj[u])) {
        int v = adj[u][i];
        if (num[v] == -1) {
            dfs(adj, cp, bri, v, u);
            low[u] = min(low[u], low[v]);
            cnt++;
            found = found || low[v] >= num[u];
            if (low[v] > num[u]) bri.eb(u, v);
        } else if (p != v) low[u] = min(low[u], num[v]);
    }
    if (found && (p != -1 || cnt > 1)) cp.pb(u);
}

pair<vi,vii> cut_points_and_bridges(const vvi &adj)
    ↪ {
    int n = size(adj);
    vi cp; vii bri;
    memset(num, -1, n << 2);
    curnum = 0;
    REP(i,n) if(num[i] == -1) dfs(adj, cp, bri, i,
        ↪ -1);
    return make_pair(cp, bri);
}

```

### 3.4.2. Strongly Connected Components $O(V + E)$ .

```

vvi adj, comps;
vi tidx, lnk, cnr, st;
vector<bool> vis;
int age, ncomps;

void tarjan(int v) {
    tidx[v] = lnk[v] = ++age; vis[v] = true; st.pb(v);
    for (int w : adj[v]) {
        if(!tidx[w])
            ↪ tarjan(w), lnk[v]=min(lnk[v], lnk[w]);
        else if(vis[w]) lnk[v] = min(lnk[v], tidx[w]);
    }
    if (lnk[v] != tidx[v]) return;
    comps.pb(vi());
    int w;
    do {
        vis[w = st.back()] = false; cnr[w] = ncomps;
        comps.back().pb(w);
        st.pop_back();
    } while (w != v);
    ncomps++;
}

void findSCC(int n) {
    age = ncomps = 0;
    vis.assign(n, false);
    tidx.assign(n, 0);
    lnk.resize(n); cnr.resize(n); comps.clear();
    for (int i = 0; i < n; i++)
        if (tidx[i] == 0) tarjan(i);
}

```

### 3.4.3. 2-SAT $O(V + E)$ . Include findSCC.

```

void init2sat(int n) { adj.assign(2 * n, vi()); }

// (var x1 = v1) ==> (var xr = vr)
void imply(int x1, bool v1, int xr, bool vr) {
    adj[2 * x1 + v1].pb(2 * xr + vr);
    adj[2 * xr + !vr].pb(2 * x1 + !v1);
}

void satOr(int x1, bool v1, int xr, bool vr) {
    imply(x1, !v1, xr, vr);
}

void satConst(int x, bool v) { imply(x, !v, x, v); }
void satIfF(int x1, bool v1, int xr, bool vr) {
    imply(x1, v1, xr, vr); imply(xr, vr, x1, v1);
}

bool solve2sat(int n, vector<bool> &sol) {
    findSCC(2 * n);
    for (int i = 0; i < n; i++)
        if (cnr[2 * i] == cnr[2 * i + 1]) return false;
    vector<bool> seen(n, false); sol.assign(n, false);
    for (vi &comp : comps) {
        for (int v : comp) {
            if (seen[v / 2]) continue;
            seen[v / 2] = true;

```

```

        sol[v / 2] = v & 1;
    }
}
return true;
}

```

### 3.4.4. Dominator graph.

```

const int N = 1234567;

vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], parent[N], sdом[N], p[N],
    ↪ best[N], idom[N], link[N];

void dfs(int v) {
    pos[v] = cnt;
    order[cnt++] = v;
    for (int u : g[v]) {
        if (pos[u] == -1) {
            parent[u] = v;
            dfs(u);
        }
    }
}

int find_best(int x) {
    if (p[x] == x) return best[x];
    int u = find_best(p[x]);
    if (pos[sdom[u]] < pos[sdom[best[x]]])
        best[x] = u;
    p[x] = p[p[x]];
    return best[x];
}

void dominators(int n, int root) {
    fill_n(pos, n, -1);
    cnt = 0;
    dfs(root);
    for (int i = 0; i < n; i++)
        for (int u : g[i]) g_rev[u].push_back(i);
    for (int i = 0; i < n; i++)
        p[i] = best[i] = sdом[i] = i;
    for (int it = cnt - 1; it >= 1; it--) {
        int w = order[it];
        for (int u : g_rev[w]) {
            int t = find_best(u);
            if (pos[sdom[t]] < pos[sdom[w]])
                sdом[w] = sdом[t];
        }
        bucket[sdom[w]].push_back(w);
        idom[w] = sdом[w];
        for (int u : bucket[parent[w]])
            link[u] = find_best(u);
        bucket[parent[w]].clear();
        p[w] = parent[w];
    }
    for (int it = 1; it < cnt; it++) {
        int w = order[it];
        idom[w] = idom[link[w]];
    }
}

```

```

}
}

```

### 3.5. Cycle Detection $\mathcal{O}(V + E)$ .

vvi adj; // assumes a bidirected graph

```

bool cycle_detection() {
    stack<int> s; vector<bool> vis(MAXN, false);
    vi par(MAXN, -1); s.push(0);
    vis[0] = true;
    while (!s.empty()) {
        int cur = s.top(); s.pop();
        for (int i : adj[cur]) {
            if (vis[i] && par[cur] != i) return true;
            s.push(i); par[i] = cur; vis[i] = true;
        }
    }
    return false;
}

```

### 3.6. Maximum Flow Algorithms.

#### 3.6.1. Dinic's Algorithm $\mathcal{O}(V^2E)$ .

```

struct Edge { int t; ll c, f; };
struct Dinic {
    vi H, P; vvi E;
    vector<Edge> G;
    Dinic(int n) : H(n), P(n), E(n) {}

    void addEdge(int u, int v, ll c) {
        E[u].pb(G.size()); G.pb({v, c, 0LL});
        E[v].pb(G.size()); G.pb({u, 0LL, 0LL});
    }

    ll dfs(int t, int v, ll f) {
        if (v == t || !f) return f;
        for ( ; P[v] < (int) E[v].size(); P[v]++) {
            int e = E[v][P[v]], w = G[e].t;
            if (H[w] != H[v] + 1) continue;
            ll df = dfs(t, w, min(f, G[e].c - G[e].f));
            if (df > 0) {
                G[e].f += df, G[e ^ 1].f -= df;
                return df;
            }
        }
        return 0;
    }

    ll maxflow(int s, int t, ll f = 0) {
        while (1) {
            fill(all(H), 0); H[s] = 1;
            queue<int> q; q.push(s);
            while (!q.empty()) {
                int v = q.front(); q.pop();
                for (int w : E[v])
                    if (G[w].f < G[w].c && !H[G[w].t])
                        H[G[w].t] = H[v] + 1, q.push(G[w].t);
            }
            if (!H[t]) return f;
            fill(all(P), 0);
            while (ll df = dfs(t, s, LLONG_MAX)) f += df;
        }
    }
}

```

```

}
};

```

#### 3.6.2. Min-cost max-flow $\mathcal{O}(n^2m^2)$ . Find the cheapest possible way of sending a certain amount of flow through a flow network.

```

const int maxn = 300;

struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
    g.pb({u, v, 0, c, w});
    g.pb({v, u, 0, 0, -w});
}

void sp(int s, int t) {
    fill_n(D, V, LLONG_MAX); D[s] = 0;
    for (int ng = g.size(), _ = V; _--;) {
        bool ok = false;
        for (int i = 0; i < ng; i++)
            if (D[g[i].x] != LLONG_MAX && g[i].f < g[i].c
                && D[g[i].x] + g[i].w < D[g[i].y]) {
                D[g[i].y] = D[g[i].x] + g[i].w;
                par[g[i].y] = i; ok = true;
            }
        if (!ok) break;
    }
}

void minCostMaxFlow(int s, int t, ll &c, ll &f) {
    for (c = f = 0; sp(s, t), D[t] < LLONG_MAX; ) {
        ll df = LLONG_MAX, dc = 0;
        for (int v = t, e; e = par[v], v != s; v =
            <- g[e].x) df = min(df, g[e].c - g[e].f);
        for (int v = t, e; e = par[v], v != s; v =
            <- g[e].x) g[e].f += df, g[e ^ 1].f -= df, dc +=
            <- g[e].w;
        f += df; c += dc * df;
    }
}

```

3.6.3. Gomory-Hu Tree - All Pairs Maximum Flow. An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in  $\mathcal{O}(|V|^2)$  plus  $|V| - 1$  times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is  $\mathcal{O}(|V|^3|E|)$ . NOTE: Not sure if it works correctly with disconnected graphs.

```

#include "dinic.cpp"
bool same[MAKV];
pair<vii, vvi> construct_gh_tree(flow_network &g) {
    int n = g.n, v;
    vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
    rep(s, 1, n) {
        int l = 0, r = 0;
        par[s].second = g.max_flow(s, par[s].first,
            <- false);
        memset(d, 0, n * sizeof(int));
        memset(same, 0, n * sizeof(bool));
    }
}

```

```

d[q[r++]] = s = 1;
while (l < r) {
    same[v = q[l++]] = true;
    for (int i = g.head[v]; i != -1; i =
        <- g.e[i].nxt)
        if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
            d[q[r++]] = g.e[i].v = 1; }
    rep(i, s+1, n)
        if (par[i].first == par[s].first && same[i])
            par[i].first = s;
    g.reset(); }
    rep(i, 0, n) {
        int mn = INT_MAX, cur = i;
        while (true) {
            cap[cur][i] = mn;
            if (cur == 0) break;
            mn = min(mn, par[cur].second), cur =
                <- par[cur].first; } }
    return make_pair(par, cap); }

int compute_max_flow(int s, int t, const pair<vii,
    <- vvi> &gh) {
    int cur = INT_MAX, at = s;
    while (gh.second[at][t] == -1)
        cur = min(cur, gh.first[at].second),
        at = gh.first[at].first;
    return min(cur, gh.second[at][t]); }

```

### 3.7. Minimal Spanning Tree.

#### 3.7.1. Kruskal $\mathcal{O}(E \log V)$ .

```

struct edge { int x, y; ll w; };
ll kruskal(int n, vector<edge> edges) {
    dsu D(n);
    sort(all(edges), [] (edge a, edge b) -> bool {
        return a.w < b.w; });
    ll ret = 0;
    for (edge e : edges)
        if (D.find(e.x) != D.find(e.y))
            ret += e.w, D.unite(e.x, e.y);
    return ret;
}

```

### 3.8. Topological Sort $\mathcal{O}(V + E)$ .

#### 3.8.1. Modified Depth-First Search.

```

void tsort_dfs(int cur, char* color, const vvi& adj,
    stack<int>& res, bool& cyc) {
    color[cur] = 1;
    rep(i, 0, size(adj[cur])) {
        int nxt = adj[cur][i];
        if (color[nxt] == 0)
            tsort_dfs(nxt, color, adj, res, cyc);
        else if (color[nxt] == 1)
            cyc = true;
        if (cyc) return; }
    color[cur] = 2;
    res.push(cur); }

vi tsort(int n, vvi adj, bool& cyc) {

```

```

cyc = false;
stack<int> S;
vi res;
char* color = new char[n];
memset(color, 0, n);
rep(i,0,n) {
    if (!color[i]) {
        tsort_dfs(i, color, adj, S, cyc);
        if (cyc) return res; } }
while (!S.empty()) res.push_back(S.top());
↪ S.pop();
return res; }

```

3.9. Euler Path  $O(V + E)$  hopefully. Finds an Euler Path (or circuit) in a *directed* graph iff one exists.

```

const int MAXV = 1000, MAXE = 5000;
vi adj[MAXV];
int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
    int start = -1, end = -1, any = 0, c = 0;
    REP(i, n) {
        if (outdeg[i] > 0) any = i;
        if (indeg[i] + 1 == outdeg[i]) start = i, c++;
        else if (indeg[i] == outdeg[i] + 1) end = i, c++;
        else if (indeg[i] != outdeg[i]) return ii(-1,-1);
    }
    if ((start == -1) != (end == -1) || (c != 2 && c))
        return ii(-1,-1);
    if (start == -1) start = end = any;
    return ii(start, end); }
bool euler_path() {
    ii se = start_end();
    int cur = se.first, at = m + 1;
    if (cur == -1) return false;
    stack<int> s;
    while (true) {
        if (outdeg[cur] == 0) {
            res[--at] = cur;
            if (s.empty()) break;
            cur = s.top(); s.pop();
        } else s.push(cur), cur =
        ↪ adj[cur][--outdeg[cur]];
    }
    return at == 0;
}

```

Finds an Euler *cycle* in a *undirected* graph:

```

const int MAXV = 1000;
multiset<int> adj[MAXV];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
    if (at == to) return it;
    L.insert(it, at), --it;
    while (!adj[at].empty()) {
        int nxt = *adj[at].begin();
        adj[at].erase(adj[at].find(nxt));
        adj[nxt].erase(adj[nxt].find(at));
    }
}

```

```

if (to == -1) {
    it = euler(nxt, at, it);
    L.insert(it, at);
    --it;
} else {
    it = euler(nxt, to, it);
    to = -1; } }
return it; }
// usage: euler(0,-1,L.begin());

```

### 3.10. Heavy-Light Decomposition.

```

#include "../data-structures/segment_tree.cpp"
const int ID = 0;
int f(int a, int b) { return a + b; }
struct HLD {
    int n, curhead, curloc;
    vi sz, head, parent, loc;
    vvi adj; segment_tree values;
    HLD(int _n) : n(_n), sz(n, 1), head(n),
        parent(n, -1), loc(n), adj(n) {
        vector<ll> tmp(n, ID); values =
        ↪ segment_tree(tmp); }
    void add_edge(int u, int v) {
        adj[u].push_back(v); adj[v].push_back(u); }
    void update_cost(int u, int v, int c) {
        if (parent[v] == u) swap(u, v); assert(parent[u]
        ↪ == v);
        values.update(loc[u], c); }
    int csz(int u) {
        rep(i,0,size(adj[u])) if (adj[u][i] !=
        ↪ parent[u])
            sz[u] += csz(adj[parent[adj[u][i]]] = u[i]);
        return sz[u]; }
    void part(int u) {
        head[u] = curhead; loc[u] = curloc++;
        int best = -1;
        rep(i,0,size(adj[u]))
            if (adj[u][i] != parent[u] &&
                (best == -1 || sz[adj[u][i]] > sz[best]))
                best = adj[u][i];
        if (best != -1) part(best);
        rep(i,0,size(adj[u]))
            if (adj[u][i] != parent[u] && adj[u][i] !=
            ↪ best)
                part(curhead = adj[u][i]); }
    void build(int r = 0) {
        curloc = 0, csz(curhead = r), part(r); }
    int lca(int u, int v) {
        vi uat, vat; int res = -1;
        while (u != -1) uat.push_back(u), u =
        ↪ parent[head[u]];
        while (v != -1) vat.push_back(v), v =
        ↪ parent[head[v]];
        u = size(uat) - 1, v = size(vat) - 1;
        while (u >= 0 && v >= 0 && head[uat[u]] ==
        ↪ head[vat[v]])

```

```

        res = (loc[uat[u]] < loc[vat[v]] ? uat[u] :
        ↪ vat[v]),
        u--, v--;
    return res; }
int query_upto(int u, int v) { int res = ID;
    while (head[u] != head[v])
        res = f(res, values.query(loc[head[u]],
        ↪ loc[u]).x),
        u = parent[head[u]];
    return f(res, values.query(loc[v] + 1,
    ↪ loc[u]).x); }
int query(int u, int v) { int l = lca(u, v);
    return f(query_upto(u, l), query_upto(v, l)); }
↪ };

```

### 3.11. Centroid Decomposition.

```

#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
    path[MAXV][LGMAXV],
    sz[MAXV], seph[MAXV],
    shortest[MAXV];
struct centroid_decomposition {
    int n; vvi adj;
    centroid_decomposition(int _n) : n(_n), adj(n) { }
    void add_edge(int a, int b) {
        adj[a].push_back(b); adj[b].push_back(a); }
    int dfs(int u, int p) {
        sz[u] = 1;
        rep(i,0,size(adj[u]))
            if (adj[u][i] != p) sz[u] += dfs(adj[u][i],
            ↪ u);
        return sz[u]; }
    void makepaths(int sep, int u, int p, int len) {
        jmp[u][seph[sep]] = sep, path[u][seph[sep]] =
        ↪ len;
        int bad = -1;
        rep(i,0,size(adj[u])) {
            if (adj[u][i] == p) bad = i;
            else makepaths(sep, adj[u][i], u, len + 1);
        }
        if (p == sep)
            swap(adj[u][bad], adj[u].back()),
            ↪ adj[u].pop_back(); }
    void separate(int h=0, int u=0) {
        dfs(u,-1); int sep = u;
        down: iter(nxt,adj[sep])
            if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2)
            ↪ {
                sep = *nxt; goto down; }
        seph[sep] = h, makepaths(sep, sep, -1, 0);
        rep(i,0,size(adj[sep])) separate(h+1,
        ↪ adj[sep][i]); }
    void paint(int u) {
        rep(h,0,seph[u]+1)
            shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
            ↪ path[u][h]); }
}

```

```

int closest(int u) {
    int mn = INT_MAX/2;
    rep(h, 0, seph[u]+1)
        mn = min(mn, path[u][h] +
            ↪ shortest[jmp[u][h]]);
    return mn; }

```

### 3.12. Least Common Ancestors, Binary Jumping.

```

const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
    rep(i, 0, SZ) BP[i][0] = P[i];
    rep(j, 1, LOGSZ) rep(i, 0, SZ)
        BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
    if (H[a] > H[b]) swap(a, b);
    int dh = H[b] - H[a], j = 0;
    rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
    while (BP[a][j] != BP[b][j]) j++;
    while (--j >= 0) if (BP[a][j] != BP[b][j])
        a = BP[a][j], b = BP[b][j];
    return a == b ? a : P[a];
}

```

3.13. Misra-Gries  $D+1$ -edge coloring. Finds a  $\max_i \deg(i) + 1$ -edge coloring where there all incident edges have distinct colors. Finding a  $D$ -edge coloring is NP-hard.

```

struct Edge { int to, col, rev; };

```

```

struct MisraGries {
    int N, K=0; vvi F;
    vector<vector<Edge>> G;

    MisraGries(int n) : N(n), G(n) {}
    // add an undirected edge, NO DUPLICATES ALLOWED
    void addEdge(int u, int v) {
        G[u].pb({v, -1, (int) G[v].size()});
        G[v].pb({u, -1, (int) G[u].size()-1});
    }
}

```

```

void color(int v, int i) {
    vi fan = { i };
    vector<bool> used(G[v].size());
    used[i] = true;
    for (int j = 0; j < (int) G[v].size(); j++)
        if (!used[j] && G[v][j].col >= 0 &&
            ↪ F[G[v][fan.back()].to][G[v][j].col] < 0)
            used[j] = true, fan.pb(j), j = -1;
    int c = 0; while (F[v][c] >= 0) c++;
    int d = 0; while (F[G[v][fan.back()].to][d] >=
        ↪ 0) d++;
    int w = v, a = d, k = 0, ccol;
    while (true) {
        swap(F[w][c], F[w][d]);
        if (F[w][c] >= 0) G[w][F[w][c]].col = c;

```

```

        if (F[w][d] >= 0) G[w][F[w][d]].col = d;
        if (F[w][a^c^d] < 0) break;
        w = G[w][F[w][a]].to;
    }
    do {
        Edge &e = G[v][fan[k]];
        ccol = F[e.to][d] < 0 ? d :
            ↪ G[v][fan[k+1]].col;
        if (e.col >= 0) F[e.to][e.col] = -1;
        F[e.to][ccol] = e.rev;
        F[v][ccol] = fan[k];
        e.col = G[e.to][e.rev].col = ccol;
        k++;
    } while (ccol != d);
}
// finds a K-edge-coloring
void color() {
    REP(v, N) K = max(K, (int) G[v].size() + 1);
    F = vvi(N, vi(K, -1));
    REP(v, N) for (int i = G[v].size(); i--;)
        if (G[v][i].col < 0) color(v, i);
}
};

```

3.14. Minimum Mean Weight Cycle. Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```

double
↪ min_mean_cycle(vector<vector<pair<int,double>>>
↪ adj){
    int n = size(adj); double mn = INFINITY;
    vector<vector<double>> arr(n+1, vector<double>(n,
        ↪ mn));
    arr[0][0] = 0;
    rep(k, 1, n+1) rep(j, 0, n) iter(it, adj[j])
        arr[k][it->first] = min(arr[k][it->first],
            it->second +
            ↪ arr[k-1][j]);
    rep(k, 0, n) {
        double mx = -INFINITY;
        rep(i, 0, n) mx = max(mx,
            ↪ (arr[n][i] - arr[k][i]) / (n-k));
        mn = min(mn, mx);
    }
    return mn; }

```

3.15. Minimum Arborescence. Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root  $r$  to each vertex. Returns a vector of size  $n$ , where the  $i$ th element is the edge for the  $i$ th vertex. The answer for the root is undefined!

```

#include "../data-structures/union_find.cpp"
struct arborescence {
    int n; union_find uf;
    vector<vector<pair<ii,int>>> adj;
    arborescence(int _n) : n(_n), uf(n), adj(n) {}
    void add_edge(int a, int b, int c) {

```

```

        adj[b].push_back(make_pair(ii(a,b),c)); }
    vii find_min(int r) {
        vi vis(n,-1), mn(n,INT_MAX); vii par(n);
        rep(i,0,n) {
            if (uf.find(i) != i) continue;
            int at = i;
            while (at != r && vis[at] == -1) {
                vis[at] = i;
                iter(it,adj[at]) if (it->second < mn[at] &&
                    uf.find(it->first.first) != at)
                    mn[at] = it->second, par[at] = it->first;
                if (par[at] == ii(0,0)) return vii();
                at = uf.find(par[at].first);
            }
            if (at == r || vis[at] != i) continue;
            union_find tmp = uf; vi seq;
            do { seq.push_back(at); at =
                ↪ uf.find(par[at].first);
            } while (at != seq.front());
            iter(it,seq) uf.unite(*it,seq[0]);
            int c = uf.find(seq[0]);
            vector<pair<ii,int>> > nw;
            iter(it,seq) iter(jt,adj[*it])
                nw.push_back(make_pair(jt->first,
                    jt->second - mn[*it]));
            adj[c] = nw;
            vii rest = find_min(r);
            if (size(rest) == 0) return rest;
            ii use = rest[c];
            rest[at = tmp.find(use.second)] = use;
            iter(it,seq) if (*it != at)
                rest[*it] = par[*it];
            return rest; }
        return par; } };

```

3.16. Maximum Density Subgraph. Given (weighted) undirected graph  $G$ . Binary search density. If  $g$  is current density, construct flow network:  $(S, u, m)$ ,  $(u, T, m+2g-d_u)$ ,  $(u, v, 1)$ , where  $m$  is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty  $S$ -component, then maximum density is smaller than  $g$ , otherwise it's larger. Distance between valid densities is at least  $1/(n(n-1))$ . Edge case when density is 0. This also works for weighted graphs by replacing  $d_u$  by the weighted degree, and doing more iterations (if weights are not integers).

3.17. Maximum-Weight Closure. Given a vertex-weighted directed graph  $G$ . Turn the graph into a flow network, adding weight  $\infty$  to each edge. Add vertices  $S, T$ . For each vertex  $v$  of weight  $w$ , add edge  $(S, v, w)$  if  $w \geq 0$ , or edge  $(v, T, -w)$  if  $w < 0$ . Sum of positive weights minus minimum  $S-T$  cut is the answer. Vertices reachable from  $S$  are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

**3.18. Maximum Weighted Independent Set in a Bipartite Graph.** This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges  $(S, u, w(u))$  for  $u \in L$ ,  $(v, T, w(v))$  for  $v \in R$  and  $(u, v, \infty)$  for  $(u, v) \in E$ . The minimum  $S, T$ -cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

**3.19. Synchronizing word problem.** A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

#### 4. STRING ALGORITHMS

##### 4.1. Trie.

```
const int SIGMA = 26;

struct trie {
    bool word; trie **adj;

    trie() : word(false), adj(new trie*[SIGMA]) {
        for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
    }

    void addWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) cur->adj[i] = new trie();
            cur = cur->adj[i];
        }
        cur->word = true;
    }

    bool isWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) return false;
            cur = cur->adj[i];
        }
        return cur->word;
    }
};
```

##### 4.2. Z-algorithm $\mathcal{O}(n)$ .

//  $z[i]$  = length of longest substring starting from  $s[i]$  which is also a prefix of  $s$ .

```
vi z_function(const string &s) {
    int n = (int) s.length();
    vi z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]])
            ++z[i];
        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    }
}
```

```
}
return z;
}
```

**4.3. Suffix array  $\mathcal{O}(n \log n)$ .** Lexicographically sorts the cyclic shifts of  $S$  where  $p[0]$  is the index of the smallest string, etc.

```
vi sort_cyclic_shifts(const string &s) {
    const int alphabet = 256, n = sz(s);

    vi p(n), c(n), cnt(max(alphabet, n), 0);
    REP(i, n) cnt[s[i]]++;
    partial_sum(cnt.begin(), cnt.begin() + n, cnt.begin());
    REP(i, n) p[--cnt[s[i]]] = i;
    c[p[0]] = 0;
    int cl = 1;
    rep(i, 1, n) {
        if (s[p[i]] != s[p[i - 1]]) cl++;
        c[p[i]] = cl - 1;
    }

    vi pn(n), cn(n);
    for (int h = 0, l = 1; l < n; l *= 2, ++h) {
        REP(i, n) {
            pn[i] = p[i] - (l << h);
            if (pn[i] < 0) pn[i] += n;
        }
        fill(cnt.begin(), cnt.begin() + n, 0);
        REP(i, n) cnt[c[pn[i]]]++;
        rep(i, 1, n) cnt[i] += cnt[i - 1];
        for (int i = n - 1; i >= 0; i--)
            p[--cnt[c[pn[i]]]] = pn[i];
        cn[p[0]] = 0;
        cl = 1;
        rep(i, 1, n) {
            if (c[p[i]] != c[p[i - 1]] || c[(p[i] + l) % n]
                != c[(p[i - 1] + l) % n]) cl++;
            cn[p[i]] = cl - 1;
        }
        c.swap(cn);
    }
    return p;
}
```

```
vi suffix_array(string s) {
    s += '\0';
    vi v = sort_cyclic_shifts(s);
    v.erase(v.begin());
    return v;
}
```

**4.4. Longest Common Subsequence  $\mathcal{O}(n^2)$ .** SUBSTRING: consecutive characters!!!

```
int dp[STR_SIZE][STR_SIZE]; // DP problem

int lcs(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i < n1; i++) {
        for (int j = 0; j < n2; j++) {
```

```
            if (i == 0 || j == 0) dp[i][j] = 0;
            else if (w1[i - 1] == w2[j - 1])
                dp[i][j] = dp[i - 1][j - 1] + 1;
            else dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);
        }
    }
    return dp[n1][n2];
}

// backtrack
string getLCS(const string &w1, const string &w2) {
    int i = w1.size(), j = w2.size(); string ret = "";
    while (i > 0 && j > 0) {
        if (w1[i - 1] == w2[j - 1]) ret += w1[--i], j--;
        else if (dp[i][j - 1] > dp[i - 1][j]) j--;
        else i--;
    }
    reverse(ret.begin(), ret.end());
    return ret;
}
```

**4.5. Levenshtein Distance  $\mathcal{O}(n^2)$ .** Minimal number of insertions, removals and edits required to transform one string in the other.

```
int dp[MAX_SIZE][MAX_SIZE]; // DP problem

int levDist(const string &w1, const string &w2) {
    int n1 = sz(w1) + 1, n2 = sz(w2) + 1;
    REP(i, n1) dp[i][0] = i; // removal
    REP(j, n2) dp[0][j] = j; // insertion
    rep(i, 1, n1) rep(j, 1, n2)
        dp[i][j] = min(
            1 + min(dp[i - 1][j], dp[i][j - 1]),
            dp[i - 1][j - 1] + (w1[i - 1] != w2[j - 1])
        );
    return dp[n1][n2];
}
```

**4.6. Knuth-Morris-Pratt algorithm  $\mathcal{O}(N + M)$ .**

```
int kmp(const string &word, const string &text) {
    int n = word.size();
    vi T(n + 1, 0);
    for (int i = 1, j = 0; i < n; ) {
        if (word[i] == word[j]) T[++i] = ++j; // match
        else if (j > 0) j = T[j]; // fallback
        else i++; // no match, keep zero
    }
    int matches = 0;
    for (int i = 0, j = 0; i < text.size(); ) {
        if (text[i] == word[j]) {
            i++;
            if (++j == n) // match at interval [i - n, i)
                matches++, j = T[j];
        } else if (j > 0) j = T[j];
        else i++;
    }
    return matches;
}
```



4.7. **Aho-Corasick Algorithm**  $\mathcal{O}(N + \sum_{i=1}^m |S_i|)$ . Dictionary substring matching as automaton. All given P must be unique!

```
const int MAXP = 100, MAXLEN = 200, SIGMA = 26,
↳ MAXTRIE = MAXP * MAXLEN;

int nP;
string P[MAXP], S;

int pnr[MAXTRIE], to[MAXTRIE][SIGMA],
↳ sLink[MAXTRIE], dLink[MAXTRIE], nnodes;

void ahoCorasick() {
    fill_n(pnr, MAXTRIE, -1);
    for (int i = 0; i < MAXTRIE; i++) fill_n(to[i],
↳ SIGMA, 0);
    fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE,
↳ 0);
    nnodes = 1;
    // STEP 1: MAKE A TREE
    for (int i = 0; i < nP; i++) {
        int cur = 0;
        for (char c : P[i]) {
            int i = c - 'a';
            if (to[cur][i] == 0) to[cur][i] = nnodes++;
            cur = to[cur][i];
        }
        pnr[cur] = i;
    }
    // STEP 2: CREATE SUFFIX_LINKS AND DICT_LINKS
    queue<int> q; q.push(0);
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (int c = 0; c < SIGMA; c++) {
            if (to[cur][c]) {
                int sl = sLink[to[cur][c]] = cur == 0 ? 0 :
↳ to[sLink[cur]][c];
                // if all strings have equal length, remove
↳ this:
                dLink[to[cur][c]] = pnr[sl] >= 0 ? sl :
↳ dLink[sl];
                q.push(to[cur][c]);
            } else to[cur][c] = to[sLink[cur]][c];
        }
    }
    // STEP 3: TRAVERSE S
    for (int cur = 0, i = 0, n = S.size(); i < n; i++)
↳ {
        cur = to[cur][S[i] - 'a'];
        for (int hit = pnr[cur] >= 0 ? cur : dLink[cur];
↳ hit; hit = dLink[hit]) {
            cerr << P[pnr[hit]] << " found at [" << (i + 1
↳ - P[pnr[hit]].size()) << ", " << i << "]"
            << endl;
        }
    }
}
```

4.8. **eerTree**. Constructs an eerTree in  $\mathcal{O}(n)$ , one character at a time.

```
#define MAXN 100100
#define SIGMA 26
#define BASE 'a'
char *s = new char[MAXN];
struct state {
    int len, link, to[SIGMA];
} *st = new state[MAXN+2];
struct eerTree {
    int last, sz, n;
    eerTree() : last(1), sz(2), n(0) {
        st[0].len = st[0].link = -1;
        st[1].len = st[1].link = 0; }
    int extend() {
        char c = s[n++]; int p = last;
        while (n - st[p].len - 2 < 0 || c != s[n -
↳ st[p].len - 2])
            p = st[p].link;
        if (!st[p].to[c-BASE]) {
            int q = last = sz++;
            st[p].to[c-BASE] = q;
            st[q].len = st[p].len + 2;
            do { p = st[p].link;
                } while (p != -1 && (n < st[p].len + 2 ||
                c != s[n - st[p].len - 2]));
            if (p == -1) st[q].link = 1;
            else st[q].link = st[p].to[c-BASE];
            return 1; }
        last = st[p].to[c-BASE];
        return 0; } };
```

4.9. **Suffix Automaton**. Minimum automata that accepts all suffixes of a string with  $\mathcal{O}(n)$  construction. The automata itself is a DAG therefore suitable for DP, examples are counting unique substrings, occurrences of substrings and suffix.

```
// TODO: Add longest common substring
const int MAXL = 100000;
struct suffix_automaton {
    vi len, link, occur, cnt;
    vector<map<char, int>> next;
    vector<bool> isclone;
    ll *occuratleast;
    int sz, last;
    string s;
    suffix_automaton() : len(MAXL*2), link(MAXL*2),
        occur(MAXL*2), next(MAXL*2), isclone(MAXL*2) {
        clear(); }
    void clear() { sz = 1; last = len[0] = 0; link[0]
↳ = -1;
        next[0].clear(); isclone[0] =
        false; }
    bool issubstr(string other) {
        for (int i = 0, cur = 0; i < size(other); ++i) {
            if (cur == -1) return false; cur =
↳ next[cur][other[i]]; }
        return true; }
```

```
void extend(char c) { int cur = sz++; len[cur] =
↳ len[last]+1;
    next[cur].clear(); isclone[cur] = false; int p =
↳ last;
    for (; p != -1 && !next[p].count(c); p = link[p])
        next[p][c] = cur;
    if (p == -1) { link[cur] = 0; }
    else { int q = next[p][c];
        if (len[p] + 1 == len[q]) { link[cur] = q; }
        else { int clone = sz++; isclone[clone] =
↳ true;
            len[clone] = len[p] + 1;
            link[clone] = link[q]; next[clone] =
↳ next[q];
            for (; p != -1 && next[p].count(c) &&
↳ next[p][c] == q;
                p = link[p]) {
                next[p][c] = clone; }
            link[q] = link[cur] = clone;
        } } last = cur; }

void count() {
    cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
    map<char, int>::iterator i;
    while (!S.empty()) {
        ii cur = S.top(); S.pop();
        if (cur.second) {
            for (i = next[cur.first].begin();
                i != next[cur.first].end(); ++i) {
                cnt[cur.first] += cnt[(i).second]; } }
            else if (cnt[cur.first] == -1) {
                cnt[cur.first] = 1; S.push(ii(cur.first,
↳ 1));
            for (i = next[cur.first].begin();
                i != next[cur.first].end(); ++i) {
                S.push(ii((i).second, 0)); } } } }

string lexicok(ll k) {
    int st = 0; string res; map<char, int>::iterator
↳ i;
    while (k) {
        for (i = next[st].begin(); i != next[st].end();
↳ ++i) {
            if (k <= cnt[(i).second]) { st = (i).second;
                res.push_back((i).first); k--; break;
            } else { k -= cnt[(i).second]; } } }
        return res; }

void countoccur() {
    for (int i = 0; i < sz; ++i) { occur[i] = 1 -
↳ isclone[i]; }
    vii states(sz);
    for (int i = 0; i < sz; ++i) { states[i] =
↳ ii(len[i], i); }
    sort(states.begin(), states.end());
    for (int i = size(states)-1; i >= 0; --i) {
        int v = states[i].second;
        if (link[v] != -1) { occur[link[v]] +=
↳ occur[v]; } } }
```

4.10. **Hashing.** Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```
struct hasher {
    int b = 311, m; vi h, p;
    hasher(string s, int _m) :
        m(_m), h(sz(s)+1), p(sz(s)+1) {
        p[0] = 1; h[0] = 0;
        REP(i, sz(s)) p[i+1] = (1ll)p[i] * b % m;
        REP(i, sz(s)) h[i+1] = ((1ll)h[i] * b + s[i]) % m;
    }
    int hash(int l, int r) {
        return (h[r+1] + m - (1ll)h[l]*p[r-l+1] % m) % m;
    }
};
```

## 5. GEOMETRY

```
const ld EPS = 1e-7, PI = acos(-1.0);
typedef ld NUM; // EITHER ld OR ll
typedef pair<NUM, NUM> pt;
```

```
pt operator+(pt p, pt q) { return {p.x+q.x, p.y+q.y}; }
pt operator-(pt p, pt q) { return {p.x-q.x, p.y-q.y}; }
pt operator*(pt p, NUM n) { return {p.x*n, p.y*n}; }
```

```
pt& operator+=(pt &p, pt q) { return p = p+q; }
pt& operator-=(pt &p, pt q) { return p = p-q; }
```

```
NUM operator*(pt p, pt q) { return p.x*q.x+p.y*q.y; }
NUM operator^(pt p, pt q) { return p.x*q.y-p.y*q.x; }
```

// square distance from p to line ab

```
ld distPtLineSq(pt p, pt a, pt b) {
    p -= a; b -= a;
    return ld(p^b) * (p^b) / (b*b);
}
```

// square distance from p to line segment ab

```
ld distPtSegmentSq(pt p, pt a, pt b) {
    p -= a; b -= a;
    NUM dot = p*b, len = b*b;
    if (dot <= 0) return p*p;
    if (dot >= len) return (p-b)*(p-b);
    return p*p - ld(dot)*dot/len;
}
```

// Test if p is on line segment ab

```
bool segmentHasPoint(pt p, pt a, pt b) {
    pt u = p-a, v = p-b;
    return abs(u^v) < EPS && u*v <= 0;
}
```

// projects p onto the line ab

```
pair<ld, ld> proj(pt p, pt a, pt b) {
    p -= a; b -= a;
    return a + b*(ld(b*p) / (b*b));
}
```

```
bool col(pt a, pt b, pt c) {
    return abs((a-b) ^ (a-c)) < EPS;
```

```
}

// true => 1 intersection, false => parallel or same
bool linesIntersect(pt a, pt b, pt c, pt d) {
    return abs((a-b) ^ (c-d)) > EPS;
}
```

```
pair<ld, ld> lineLineIntersection(pt a, pt b, pt c,
    pt d) {
    ld det = (a-b) ^ (c-d);
    assert(abs(det) > EPS);
    return ((c-d)*(a^b) - (a-b)*(c^d)) *
        (ld(1.0)/det);
}
```

// dp, dq are directions from p, q  
// intersection at p + t\_i dp, for 0 <= i < return  
value

```
int segmentIntersection(pt p, pt dp, pt q, pt dq,
    pt &A, pt &B) {
    if (abs(dp * dp) < EPS)
        swap(p, q), swap(dp, dq); // dq=0
    if (abs(dp * dp) < EPS) {
        A = p; // dp = dq = 0
        return p == q;
    }
```

```
pt dpq = q-p;
NUM c = dp^dq, c0 = dpq^dp, c1 = dpq^dq;
if (abs(c) < EPS) { // parallel, dp > 0, dq >= 0
    if (abs(c0) > EPS) return 0; // not collinear
    NUM v0 = dpq*dp, v1 = v0 + dq*dp, dp2 = dp*dp;
    if (v1 < v0) swap(v0, v1);
```

```
v0 = max(v0, NUM(0));
v1 = min(v1, dp2);
```

```
A = p + dp * (ld(v0) / dp2);
B = p + dp * (ld(v1) / dp2);
```

```
return (v0 <= v1) + (v0 < v1);
}
```

```
if (c < 0) {
    c = -c; c0 = -c0; c1 = -c1;
}
```

```
A = p + dp * (ld(c1)/c);
return 0 <= min(c0, c1) && max(c0, c1) <= c;
}
```

// Returns TWICE the area of a polygon (for  
integers)

```
NUM polygonTwiceArea(const vector<pt> &p) {
    NUM area = 0;
    for (int n = sz(p), i=0, j=n-1; i<n; j = i++)
        area += p[i] ^ p[j];
```

```
return abs(area); // area < 0 <=> p ccw
}
```

```
bool insidePolygon(const vector<pt> &pts, pt p, bool
    strict = true) {
    int n = 0;
    for (int N = sz(pts), i = 0, j = N - 1; i < N; j =
        i++) {
        // if p is on edge of polygon
        if (segmentHasPoint(p, pts[i], pts[j])) return
            !strict;
        // or: if(distPtSegmentSq(p, pts[i], pts[j])) <=
            EPS) return !strict;
```

```
// increment n if segment intersects line from p
n += (max(pts[i].y, pts[j].y) > p.y &&
    min(pts[i].y, pts[j].y) <= p.y &&
    ((pts[j] - pts[i])^(p-pts[i])) > 0) ==
    (pts[i].y <= p.y));
}
return n & 1; // inside if odd number of
    intersections
}
```

## 5.1. Convex Hull $\mathcal{O}(n \log n)$ .

// the convex hull consists of: { pts[ret[0]],  
pts[ret[1]], ... pts[ret.back()] }

```
vi convexHull(const vector<pt> &pts) {
    if (pts.empty()) return vi();
    vi ret, ord;
    int n = pts.size(), st = min_element(all(pts)) -
        pts.begin();
    rep(i, 0, n)
        if (pts[i] != pts[st]) ord.pb(i);
    sort(all(ord), [&pts, &st] (int a, int b) {
        pt p = pts[a] - pts[st], q = pts[b] - pts[st];
        return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) <
            lenSq(q);
    });
    ord.pb(st); ret.pb(st);
    for (int i : ord) {
        // use '>' to include ALL points on the
            hull-line
        for (int s = ret.size() - 1; s > 0 &&
            ((pts[ret[s-1]] - pts[ret[s]]) ^ (pts[i] -
                pts[ret[s]])) >= 0; s--)
            ret.pop_back();
        ret.pb(i);
    }
    ret.pop_back();
    return ret;
}
```

## 5.2. Rotating Calipers $\mathcal{O}(n)$ . Finds the longest distance between two points in a convex hull.

```
NUM rotatingCalipers(vector<pt> &hull) {
    int n = hull.size(), a = 0, b = 1;
```

```

if (n <= 1) return 0.0;
while ((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
    ↪ hull[b])) > 0) b++;
NUM ret = 0.0;
while (a < n) {
    ret = max(ret, lenSq(hull[a], hull[b]));
    if ((hull[(a + 1) % n] - hull[a % n]) ^
    ↪ (hull[(b + 1) % n] - hull[b])) <= 0) a++;
    else if (++b == n) b = 0;
}
return ret;
}

```

### 5.3. Closest points $\mathcal{O}(n \log n)$ .

```

int n; pt pts[maxn];

struct byY {
    bool operator()(int a, int b) const { return
    ↪ pts[a].y < pts[b].y; }
};

inline NUM dist(ii p) { return hypot(pts[p.x].x -
    ↪ pts[p.y].x, pts[p.x].y - pts[p.y].y); }

ii minpt(ii p1, ii p2) { return dist(p1) < dist(p2)
    ↪ ? p1 : p2; }

// closest pts (by index) inside pts[l ... r], with
    ↪ sorted y values in ys
ii closest(int l, int r, vi &ys) {
    if (r - l == 2) { // don't assume 1 here.
        ys = { l, l + 1 };
        return ii(l, l + 1);
    } else if (r - l == 3) { // brute-force
        ys = { l, l + 1, l + 2 };
        sort(all(ys), byY());
        return minpt(ii(l, l + 1), minpt(ii(l, l + 2),
            ↪ ii(l + 1, l + 2)));
    }
    int m = (l + r) / 2; vi yl, yr;
    ii delta = minpt(closest(l, m, yl), closest(m, r,
        ↪ yr));
    NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
        ↪ pts[m].x);
    merge(all(yl), all(yr), back_inserter(ys), byY());
    deque<int> q;
    for (int i : ys) if (abs(pts[i].x - xm) <= ddelta)
        ↪ {
        for (int j : q) delta = minpt(delta, ii(i, j));
        q.pb(i);
        if (q.size() > 8) q.pop_front(); // magic from
            ↪ Introduction to Algorithms.
        }
    return delta;
}

```

5.4. **Great-Circle Distance.** Computes the distance between two points (given as latitude/longitude coordinates) on a sphere of radius  $r$ .

```

ld gc_distance(ld pLat, ld pLong, ld qLat, ld qLong,
    ↪ ld r) {
    pLat *= pi / 180; pLong *= pi / 180;
    qLat *= pi / 180; qLong *= pi / 180;
    return r * acos(cos(pLat)*cos(qLat)*cos(pLong -
        ↪ qLong) + sin(pLat)*sin(qLat)); }

```

### 5.5. 3D Primitives.

```

#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)
#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
    double x, y, z;
    point3d() : x(0), y(0), z(0) {}
    point3d(double _x, double _y, double _z)
        : x(_x), y(_y), z(_z) {}
    point3d operator+(P(p)) const {
        return point3d(x + p.x, y + p.y, z + p.z); }
    point3d operator-(P(p)) const {
        return point3d(x - p.x, y - p.y, z - p.z); }
    point3d operator-() const {
        return point3d(-x, -y, -z); }
    point3d operator*(double k) const {
        return point3d(x * k, y * k, z * k); }
    point3d operator/(double k) const {
        return point3d(x / k, y / k, z / k); }
    double operator%(P(p)) const {
        return x * p.x + y * p.y + z * p.z; }
    point3d operator*(P(p)) const {
        return point3d(y*p.z - z*p.y,
            ↪ z*p.x - x*p.z, x*p.y - y*p.x); }
    double length() const {
        return sqrt(*this % *this); }
    double distTo(P(p)) const {
        return (*this - p).length(); }
    double distTo(P(A), P(B)) const {
        // A and B must be two different points
        return ((*this - A) * (*this - B)).length() /
            ↪ A.distTo(B); }
    point3d normalize(double k = 1) const {
        // length() must not return 0
        return (*this) * (k / length()); }
    point3d getProjection(P(A), P(B)) const {
        point3d v = B - A;
        return A + v.normalize((v % (*this - A)) /
            ↪ v.length()); }
    point3d rotate(P(normal)) const {
        //normal must have length 1 and be orthogonal to
        ↪ the vector
        return (*this) * normal; }
    point3d rotate(double alpha, P(normal)) const {
        return (*this) * cos(alpha) + rotate(normal) *
            ↪ sin(alpha); }

```

```

point3d rotatePoint(P(O), P(axe), double alpha)
    ↪ const {
    point3d Z = axe.normalize(axe % (*this - O));
    return O + Z + (*this - O - Z).rotate(alpha, O);
    ↪ }
bool isZero() const {
    return abs(x) < EPS && abs(y) < EPS && abs(z) <
    ↪ EPS; }
bool isOnLine(L(A, B)) const {
    return ((A - *this) * (B - *this)).isZero(); }
bool isInSegment(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
        ↪ *this)) < EPS; }
bool isInSegmentStrictly(L(A, B)) const {
    return isOnLine(A, B) && ((A - *this) % (B -
        ↪ *this)) < -EPS; }
double getAngle() const {
    return atan2(y, x); }
double getAngle(P(u)) const {
    return atan2((*this * u).length(), *this % u); }
bool isOnPlane(PL(A, B, C)) const {
    return
        ↪ abs((A - *this) * (B - *this) % (C - *this)) <
            ↪ EPS; } }
int line_line_intersect(L(A, B), L(C, D), point3d
    ↪ &O) {
    if (abs((B - A) * (C - A) % (D - A)) > EPS) return
        ↪ 0;
    if (((A - B) * (C - D)).length() < EPS)
        ↪ return A.isOnLine(C, D) ? 2 : 0;
    point3d normal = ((A - B) * (C - D)).normalize();
    double s1 = (C - A) * (D - A) % normal;
    O = A + ((B - A) / (s1 + ((D - B) * (C - B) %
        ↪ normal))) * s1;
    return 1; }
int line_plane_intersect(L(A, B), PL(C, D, E),
    ↪ point3d &O) {
    double V1 = (C - A) * (D - A) % (E - A);
    double V2 = (D - B) * (C - B) % (E - B);
    if (abs(V1 + V2) < EPS)
        ↪ return A.isOnPlane(C, D, E) ? 2 : 0;
    O = A + ((B - A) / (V1 + V2)) * V1;
    return 1; }
bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
    ↪ point3d &P, point3d &Q) {
    point3d n = nA * nB;
    if (n.isZero()) return false;
    point3d v = n * nA;
    P = A + (n * nA) * ((B - A) % nB / (v % nB));
    Q = P + n;
    return true; }

```

## 5.6. Polygon Centroid.

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

**5.7. Rectilinear Minimum Spanning Tree.** Given a set of  $n$  points in the plane, and the aim is to find a minimum spanning tree connecting these  $n$  points, assuming the Manhattan distance is used. The function candidates returns at most  $4n$  edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```
#define MAXN 100100
struct RMST {
    struct point {
        int i; ll x, y;
        point() : i(-1) { }
        ll d1() { return x + y; }
        ll d2() { return x - y; }
        ll dist(point other) {
            return abs(x - other.x) + abs(y - other.y); }
        bool operator < (const point &other) const {
            return y == other.y ? x > other.x : y <
                other.y; }
    } best[MAXN], arr[MAXN], tmp[MAXN];
    int n;
    RMST() : n(0) {}
    void add_point(int x, int y) {
        arr[arr[n].i = n].x = x, arr[n++].y = y; }
    void rec(int l, int r) {
        if (l >= r) return;
        int m = (l+r)/2;
        rec(l,m), rec(m+1,r);
        point bst;
        for (int i = l, j = m+1, k = 1; i <= m || j <=
            r; k++) {
            if (j > r || (i <= m && arr[i].d1() <
                arr[j].d1())) {
                tmp[k] = arr[i++];
                if (bst.i != -1 && (best[tmp[k].i].i == -1
                    || best[tmp[k].i].d2() <
                        bst.d2()))
                    best[tmp[k].i] = bst;
            } else {
                tmp[k] = arr[j++];
                if (bst.i == -1 || bst.d2() < tmp[k].d2())
                    bst = tmp[k]; }
        rep(i,l,r+1) arr[i] = tmp[i]; }
    vector<pair<ll,ii> > candidates() {
        vector<pair<ll, ii> > es;
        rep(p,0,2) {
```

```
rep(q,0,2) {
    sort(arr, arr+n);
    rep(i,0,n) best[i].i = -1;
    rec(0,n-1);
    rep(i,0,n) {
        if (best[arr[i].i].i != -1)
            es.push_back({arr[i].dist(best[arr[i].i]),
                {arr[i].i,
                    best[arr[i].i].i}});
        swap(arr[i].x, arr[i].y);
        arr[i].x *= -1, arr[i].y *= -1; } }
    rep(i,0,n) arr[i].x *= -1; }
    return es; } }
```

**5.8. Formulas.** Let  $a = (a_x, a_y)$  and  $b = (b_x, b_y)$  be two-dimensional vectors.

- $a \cdot b = |a||b| \cos \theta$ , where  $\theta$  is the angle between  $a$  and  $b$ .
- $a \times b = |a||b| \sin \theta$ , where  $\theta$  is the signed angle between  $a$  and  $b$ .
- $a \times b$  is equal to the area of the parallelogram with two of its sides formed by  $a$  and  $b$ . Half of that is the area of the triangle formed by  $a$  and  $b$ .
- **Euler's formula:**  $V - E + F = 2$
- Side lengths  $a, b, c$  can form a triangle iff.  $a + b > c$ ,  $b + c > a$  and  $a + c > b$ .
- Sum of internal angles of a regular convex  $n$ -gon is  $(n - 2)\pi$ .
- **Law of sines:**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:**  $b^2 = a^2 + c^2 - 2ac \cos B$
- Internal tangents of circles  $(c_1, r_1), (c_2, r_2)$  intersect at  $(c_1 r_2 + c_2 r_1) / (r_1 + r_2)$ , external intersect at  $(c_1 r_2 - c_2 r_1) / (r_1 + r_2)$ .

## 6. GEOMETRY (CP3)

## 6.1. Points and lines.

```
#define EPS 1e-9
#define PI acos(-1.0) // important constant;
// alternative #define PI (2.0 * acos(0.0))

double DEG_to_RAD(double d) { return d * PI / 180.0; }
// }

double RAD_to_DEG(double r) { return r * 180.0 / PI; }
// }

struct point { double x, y; // only used if more
// precision is needed
    point() { x = y = 0.0; } //
    // default constructor
    point(double _x, double _y) : x(_x), y(_y) { }
    // user-defined
    bool operator < (point other) const { // override
    // less than operator
        if (fabs(x - other.x) > EPS) //
            // useful for sorting
```

```
return x < other.x; // first criteria
// }
// by x-coordinate
return y < other.y; } // second
// criteria, by y-coordinate
// use EPS (1e-9) when testing equality of two
// floating points
bool operator == (point other) const {
    return (fabs(x - other.x) < EPS && (fabs(y -
        other.y) < EPS)); } };

double dist(point p1, point p2) { //
    // Euclidean distance
    // hypot(dx, dy) returns
    // sqrt(dx * dx + dy * dy)
    return hypot(p1.x - p2.x, p1.y - p2.y); }
// }

// rotate p by theta degrees CCW w.r.t origin (0, 0)
point rotate(point p, double theta) {
    double rad = DEG_to_RAD(theta); // multiply
    // theta with PI / 180.0
    return point(p.x * cos(rad) - p.y * sin(rad),
        p.x * sin(rad) + p.y * cos(rad)); }

struct line { double a, b, c; }; // a way
// to represent a line

// the answer is stored in the third parameter (pass
// by reference)
void pointsToLine(point p1, point p2, line &l) {
    if (fabs(p1.x - p2.x) < EPS) { //
        // vertical line is fine
        l.a = 1.0; l.b = 0.0; l.c = -p1.x;
        // default values
    } else {
        l.a = -(double) (p1.y - p2.y) / (p1.x - p2.x);
        l.b = 1.0; // IMPORTANT: we fix the
        // value of b to 1.0
        l.c = -(double) (l.a * p1.x) - p1.y;
    } }

bool areParallel(line l1, line l2) { // check
    // coefficients a & b
    return (fabs(l1.a - l2.a) < EPS) && (fabs(l1.b - l2.b)
        < EPS); }

bool areSame(line l1, line l2) { // also
    // check coefficient c
    return areParallel(l1, l2) && (fabs(l1.c - l2.c) <
        EPS); }

// returns true (+ intersection point) if two lines
// are intersect
bool areIntersect(line l1, line l2, point &p) {
    if (areParallel(l1, l2)) return false;
    // }
    // no intersection
```

```
// solve system of 2 linear algebraic equations
↳ with 2 unknowns
p.x = (l2.b * l1.c - l1.b * l2.c) / (l2.a * l1.b -
↳ l1.a * l2.b);
// special case: test for vertical line to avoid
↳ division by zero
if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
else
    p.y = -(l2.a * p.x + l2.c);
return true; }

struct vec { double x, y; // name: 'vec' is
↳ different from STL vector
    vec(double _x, double _y) : x(_x), y(_y) {} };

vec toVec(point a, point b) { // convert 2
↳ points to vector a->b
    return vec(b.x - a.x, b.y - a.y); }

vec scale(vec v, double s) { // nonnegative s
↳ = [<1 .. 1 .. >1]
    return vec(v.x * s, v.y * s); } //
↳ shorter.same.longer

point translate(point p, vec v) { //
↳ translate p according to v
    return point(p.x + v.x, p.y + v.y); }

// convert point and gradient/slope to line
void pointSlopeToLine(point p, double m, line &l) {
    l.a = -m;
    ↳ // always -m
    l.b = 1;
    ↳ // always 1
    l.c = -((l.a * p.x) + (l.b * p.y)); }
    ↳ // compute this

void closestPoint(line l, point p, point &ans) {
    line perpendicular; // perpendicular to l
    ↳ and pass through p
    if (fabs(l.b) < EPS) { // special
    ↳ case 1: vertical line
        ans.x = -(l.c); ans.y = p.y; return; }

    if (fabs(l.a) < EPS) { // special case
    ↳ 2: horizontal line
        ans.x = p.x; ans.y = -(l.c); return; }

    pointSlopeToLine(p, 1 / l.a, perpendicular);
    ↳ // normal line
    // intersect line l with this perpendicular line
    // the intersection point is the closest point
    areIntersect(l, perpendicular, ans); }

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
    point b;
```

```
closestPoint(l, p, b); //
↳ similar to distToLine
vec v = toVec(p, b);
↳ // create a vector
ans = translate(translate(p, v), v); } //
↳ translate p twice

double dot(vec a, vec b) { return (a.x * b.x + a.y *
↳ b.y); }

double norm_sq(vec v) { return v.x * v.x + v.y *
↳ v.y; }

// returns the distance from p to the line defined
↳ by
// two points a and b (a and b must be different)
// the closest point is stored in the 4th parameter
↳ (byref)
double distToLine(point p, point a, point b, point
↳ &c) {
    // formula: c = a + u * ab
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    c = translate(a, scale(ab, u));
    ↳ // translate a to c
    return dist(p, c); } // Euclidean
    ↳ distance between p and c

// returns the distance from p to the line segment
↳ ab defined by
// two points a and b (still OK if a == b)
// the closest point is stored in the 4th parameter
↳ (byref)
double distToLineSegment(point p, point a, point b,
↳ point &c) {
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    if (u < 0.0) { c = point(a.x, a.y);
    ↳ // closer to a
        return dist(p, a); } // Euclidean
    ↳ distance between p and a
    if (u > 1.0) { c = point(b.x, b.y);
    ↳ // closer to b
        return dist(p, b); } // Euclidean
    ↳ distance between p and b
    return distToLine(p, a, b, c); } // run
    ↳ distToLine as above

double angle(point a, point o, point b) { //
↳ returns angle aob in rad
    vec oa = toVec(o, a), ob = toVec(o, b);
    return acos(dot(oa, ob) / sqrt(norm_sq(oa) *
↳ norm_sq(ob))); }

double cross(vec a, vec b) { return a.x * b.y - a.y
↳ * b.x; }
```

```
// note: to accept collinear points, we have to
↳ change the '> 0'
// returns true if point r is on the left side of
↳ line pq
bool ccw(point p, point q, point r) {
    return cross(toVec(p, q), toVec(p, r)) > 0; }

// returns true if point r is on the same line as
↳ the line pq
bool collinear(point p, point q, point r) {
    return fabs(cross(toVec(p, q), toVec(p, r))) <
↳ EPS; }
```

## 6.2. Polygon.

```
// returns the perimeter, which is the sum of
↳ Euclidian distances
// of consecutive line segments (polygon edges)
double perimeter(const vector<point> &P) {
    double result = 0.0;
    for (int i = 0; i < (int)P.size()-1; i++) //
    ↳ remember that P[0] = P[n-1]
        result += dist(P[i], P[i+1]);
    return result; }

// returns the area, which is half the determinant
double area(const vector<point> &P) {
    double result = 0.0, x1, y1, x2, y2;
    for (int i = 0; i < (int)P.size()-1; i++) {
        x1 = P[i].x; x2 = P[i+1].x;
        y1 = P[i].y; y2 = P[i+1].y;
        result += (x1 * y2 - x2 * y1);
    }
    return fabs(result) / 2.0; }

// returns true if we always make the same turn
↳ while examining
// all the edges of the polygon one by one
bool isConvex(const vector<point> &P) {
    int sz = (int)P.size();
    if (sz <= 3) return false; // a point/sz=2 or a
    ↳ line/sz=3 is not convex
    bool isLeft = ccw(P[0], P[1], P[2]);
    ↳ // remember one result
    for (int i = 1; i < sz-1; i++) // then
    ↳ compare with the others
        if (ccw(P[i], P[i+1], P[(i+2) == sz ? 1 : i+2])
        ↳ != isLeft)
            return false; // different sign ->
    ↳ this polygon is concave
    return true; } //
    ↳ this polygon is convex

// returns true if point p is in either
↳ convex/concave polygon P
```



```

bool inPolygon(point pt, const vector<point> &P) {
    if ((int)P.size() == 0) return false;
    double sum = 0; // assume the first vertex is
    ↪ equal to the last vertex
    for (int i = 0; i < (int)P.size()-1; i++) {
        if (ccw(pt, P[i], P[i+1]))
            sum += angle(P[i], pt, P[i+1]);
            ↪ // left turn/ccw
        else sum -= angle(P[i], pt, P[i+1]); }
        ↪ // right turn/cw
    return fabs(fabs(sum) - 2*PI) < EPS; }

// line segment p-q intersect with line A-B.
point lineIntersectSeg(point p, point q, point A,
    ↪ point B) {
    double a = B.y - A.y;
    double b = A.x - B.x;
    double c = B.x * A.y - A.x * B.y;
    double u = fabs(a * p.x + b * p.y + c);
    double v = fabs(a * q.x + b * q.y + c);
    return point((p.x * v + q.x * u) / (u+v), (p.y * v
    ↪ + q.y * u) / (u+v)); }

// cuts polygon Q along the line formed by point a
    ↪ -> point b
// (note: the last point must be the same as the
    ↪ first point)
vector<point> cutPolygon(point a, point b, const
    ↪ vector<point> &Q) {
    vector<point> P;
    for (int i = 0; i < (int)Q.size(); i++) {
        double left1 = cross(toVec(a, b), toVec(a,
            ↪ Q[i])), left2 = 0;
        if (i != (int)Q.size()-1) left2 = cross(toVec(a,
            ↪ b), toVec(a, Q[i+1]));
        if (left1 > -EPS) P.push_back(Q[i]); //
            ↪ Q[i] is on the left of ab
        if (left1 * left2 < -EPS) // edge (Q[i],
            ↪ Q[i+1]) crosses line ab
            P.push_back(lineIntersectSeg(Q[i], Q[i+1], a,
            ↪ b));
    }
    if (!P.empty() && !(P.back() == P.front()))
        P.push_back(P.front()); // make P's first
        ↪ point = P's last point
    return P; }

point pivot;
bool angleCmp(point a, point b) { //
    ↪ angle-sorting function
    if (collinear(pivot, a, b))
        ↪ // special case
        return dist(pivot, a) < dist(pivot, b); //
            ↪ check which one is closer
    double d1x = a.x - pivot.x, d1y = a.y - pivot.y;
    double d2x = b.x - pivot.x, d2y = b.y - pivot.y;

```

```

    return (atan2(d1y, d1x) - atan2(d2y, d2x)) < 0; }
    ↪ // compare two angles

vector<point> CH(vector<point> P) { // the content
    ↪ of P may be reshuffled
    int i, j, n = (int)P.size();
    if (n <= 3) {
        if (!(P[0] == P[n-1])) P.push_back(P[0]); //
            ↪ safeguard from corner case
        return P; // special
            ↪ case, the CH is P itself
    }

    // first, find P0 = point with lowest Y and if
    ↪ tie: rightmost X
    int P0 = 0;
    for (i = 1; i < n; i++)
        if (P[i].y < P[P0].y || (P[i].y == P[P0].y &&
            ↪ P[i].x > P[P0].x))
            P0 = i;

    point temp = P[0]; P[0] = P[P0]; P[P0] = temp;
    ↪ // swap P[P0] with P[0]

    // second, sort points by angle w.r.t. pivot P0
    pivot = P[0]; // use this
    ↪ global variable as reference
    sort(++P.begin(), P.end(), angleCmp);
    ↪ // we do not sort P[0]

    // third, the ccw tests
    vector<point> S;
    S.push_back(P[n-1]); S.push_back(P[0]);
    ↪ S.push_back(P[1]); // initial S
    i = 2; //
    ↪ then, we check the rest
    while (i < n) { // note: N must be >= 3
        ↪ for this method to work
        j = (int)S.size()-1;
        if (ccw(S[j-1], S[j], P[i]))
            ↪ S.push_back(P[i++]); // left turn, accept
        else S.pop_back(); } // or pop the top of S
        ↪ until we have a left turn
    return S; }
    ↪ // return the result

```

### 6.3. Triangle.

```

double perimeter(double ab, double bc, double ca) {
    return ab + bc + ca; }

double perimeter(point a, point b, point c) {
    return dist(a, b) + dist(b, c) + dist(c, a); }

double area(double ab, double bc, double ca) {
    // Heron's formula, split sqrt(a * b) into sqrt(a)
    ↪ * sqrt(b); in implementation

```

```

double s = 0.5 * perimeter(ab, bc, ca);
return sqrt(s) * sqrt(s - ab) * sqrt(s - bc) *
    ↪ sqrt(s - ca); }

double area(point a, point b, point c) {
    return area(dist(a, b), dist(b, c), dist(c, a)); }

double rInCircle(double ab, double bc, double ca) {
    return area(ab, bc, ca) / (0.5 * perimeter(ab, bc,
    ↪ ca)); }

double rInCircle(point a, point b, point c) {
    return rInCircle(dist(a, b), dist(b, c), dist(c,
    ↪ a)); }

// assumption: the required points/lines functions
    ↪ have been written
// returns 1 if there is an inCircle center, returns
    ↪ 0 otherwise
// if this function returns 1, ctr will be the
    ↪ inCircle center
// and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point
    ↪ &ctr, double &r) {
    r = rInCircle(p1, p2, p3);
    if (fabs(r) < EPS) return 0; //
        ↪ no inCircle center

    line l1, l2; // compute these
        ↪ two angle bisectors
    double ratio = dist(p1, p2) / dist(p1, p3);
    point p = translate(p2, scale(toVec(p2, p3), ratio
        ↪ / (1 + ratio)));
    pointsToLine(p1, p, l1);

    ratio = dist(p2, p1) / dist(p2, p3);
    p = translate(p1, scale(toVec(p1, p3), ratio / (1
        ↪ + ratio)));
    pointsToLine(p2, p, l2);

    areIntersect(l1, l2, ctr); // get their
        ↪ intersection point
    return 1; }

double rCircumCircle(double ab, double bc, double
    ↪ ca) {
    return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

double rCircumCircle(point a, point b, point c) {
    return rCircumCircle(dist(a, b), dist(b, c),
    ↪ dist(c, a)); }

// assumption: the required points/lines functions
    ↪ have been written
// returns 1 if there is a circumCenter center,
    ↪ returns 0 otherwise

```

```
// if this function returns 1, ctr will be the
↳ circumCircle center
// and r is the same as rCircumCircle
int circumCircle(point p1, point p2, point p3, point
↳ &ctr, double &r){
    double a = p2.x - p1.x, b = p2.y - p1.y;
    double c = p3.x - p1.x, d = p3.y - p1.y;
    double e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
    double f = c * (p1.x + p3.x) + d * (p1.y + p3.y);
    double g = 2.0 * (a * (p3.y - p2.y) - b * (p3.x -
↳ p2.y));
    if (fabs(g) < EPS) return 0;

    ctr.x = (d*e - b*f) / g;
    ctr.y = (a*f - c*e) / g;
    r = dist(p1, ctr); // r = distance from center to
↳ 1 of the 3 points
    return 1; }
```

```
// returns if pt d is inside the circumCircle
↳ defined by a,b,c
bool inCircumCircle(point a,point b,point c,point
↳ d){
    vec va=toVec(a, d), vb=toVec(b, d), vc=toVec(c,
↳ d);
    return 0 <
    (va.x)*(vb.y)*((vc.x)*(vc.x)+(vc.y)*(vc.y))+
    (va.y)*((vb.x)*(vb.x)+(vb.y)*(vb.y))*(vc.x)+
    ((va.x)*(va.x)+(va.y)*(va.y))*(vb.x)*(vc.y)-
    ((va.x)*(va.x)+(va.y)*(va.y))*(vb.y)*(vc.x)-
    (va.y)*(vb.x)*((vc.x)*(vc.x)+(vc.y)*(vc.y))-
    (va.x)*((vb.x)*(vb.x)+(vb.y)*(vb.y))*(vc.y);
}
```

```
bool canFormTriangle(double a, double b, double c) {
    return (a + b > c) && (a + c > b) && (b + c > a);
↳ }
```

#### 6.4. Circle.

```
int insideCircle(point_i p, point_i c, int r) { //
↳ all integer version
    int dx = p.x - c.x, dy = p.y - c.y;
    int Euc = dx * dx + dy * dy, rSq = r * r;
    ↳ // all integer
    return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
↳ //inside/border/outside
```

```
bool circle2PtsRad(point p1, point p2, double r,
↳ point &c) {
    double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
    (p1.y - p2.y) * (p1.y - p2.y);
    double det = r * r / d2 - 0.25;
    if (det < 0.0) return false;
    double h = sqrt(det);
    c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
```

```
c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
return true; } // to get the other center,
↳ reverse p1 and p2
```

#### 7. MISCELLANEOUS

##### 7.1. Binary search $\mathcal{O}(\log(hi - lo))$ .

```
bool test(int n);

int search(int lo, int hi) {
    assert(test(lo) && !test(hi)); // BE CERTAIN
    while (hi - lo > 1) {
        int m = (lo + hi) / 2;
        (test(m) ? lo : hi) = m;
    }
    // assert(test(lo) && !test(hi));
    return lo;
}
```

7.2. Fast Fourier Transform  $\mathcal{O}(n \log n)$ . Given two polynomials  $A(x) = a_0 + a_1x + \dots + a_{n/2}x^{n/2}$  and  $B(x) = b_0 + b_1x + \dots + b_{n/2}x^{n/2}$ , FFT calculates all coefficients of  $C(x) = A(x) \cdot B(x) = c_0 + c_1x + \dots + c_nx^n$ , with  $c_i = \sum_{j=0}^i a_j b_{i-j}$ .

```
typedef complex<double> cpx;
const int LOGN = 19, MAXN = 1 << LOGN;

int rev[MAXN];
cpx rt[MAXN], a[MAXN] = {}, b[MAXN] = {};
```

```
void fft(cpx *A) {
    REP(i, MAXN) if (i < rev[i]) swap(A[i],
↳ A[rev[i]]);
    for (int k = 1; k < MAXN; k *= 2)
        for (int i = 0; i < MAXN; i += 2*k) REP(j, k) {
            cpx t = rt[j + k] * A[i + j + k];
            A[i + j + k] = A[i + j] - t;
            A[i + j] += t;
        }
}
```

```
void multiply() { // a = convolution of a * b
    rev[0] = 0; rt[1] = cpx(1, 0);
    REP(i, MAXN) rev[i] = (rev[i/2] | ((i&1)<<LOGN)/2);
    for (int k = 2; k < MAXN; k *= 2) {
        cpx z(cos(PI/k), sin(PI/k));
        rep(i, k/2, k) rt[2*i]=rt[i], rt[2*i+1]=rt[i]*z;
    }
    fft(a); fft(b);
    REP(i, MAXN) a[i] *= b[i] / (double)MAXN;
    reverse(a+1, a+MAXN); fft(a);
}
```

##### 7.3. Minimum Assignment (Hungarian Algorithm) $\mathcal{O}(n^3)$ .

```
int a[MAXN + 1][MAXM + 1]; // matrix, 1-based
int minimum_assignment(int n, int m) { // n rows, m
↳ columns
    vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);
```

```
for (int i = 1; i <= n; i++) {
    p[0] = i;
    int j0 = 0;
    vi minv(m + 1, INT_MAX);
    vector<char> used(m + 1, false);
    do {
        used[j0] = true;
        int i0 = p[j0], delta = INT_MAX, j1;
        for (int j = 1; j <= m; j++)
            if (!used[j]) {
                int cur = a[i0][j] - u[i0] - v[j];
                if (cur < minv[j]) minv[j] = cur, way[j] =
↳ j0;
                if (minv[j] < delta) delta = minv[j], j1 =
↳ j;
            }
        for (int j = 0; j <= m; j++) {
            if(used[j]) u[p[j]] += delta, v[j] -= delta;
            else minv[j] -= delta;
        }
        j0 = j1;
    } while (p[j0] != 0);
    do {
        int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
    } while (j0);
    // column j is assigned to row p[j]
    return -v[0];
}
```

##### 7.4. Partial linear equation solver $\mathcal{O}(N^3)$ .

```
typedef double NUM;
const int MAXROWS = 200, MAXCOLS = 200;
const NUM EPS = 1e-5;

// F2: bitset<MAXCOLS+1> mat[MAXROWS];
↳ bitset<MAXROWS> vals;
NUM mat[MAXROWS][MAXCOLS + 1], vals[MAXCOLS]; bool
↳ hasval[MAXCOLS];
bool is0(NUM a) { return -EPS < a && a < EPS; }
```

```
// finds x such that Ax = b
// A_ij is mat[i][j], b_i is mat[i][m]
int solvemmat(int n, int m) {
    // F2: vals.reset();
    int pr = 0, pc = 0;
    while (pc < m) {
        int r = pr, c;
        while (r < n && is0(mat[r][pc])) r++;
        if (r == n) { pc++; continue; }

        // F2: mat[pr] ^= mat[r]; mat[r] ^= mat[pr];
        ↳ mat[pr] ^= mat[r];
        for (c = 0; c <= m; c++) swap(mat[pr][c],
↳ mat[r][c]);

        r = pr++; c = pc++;
```

```
// F2: vals.set(pc, mat[pr][m]);
NUM div = mat[r][c];
for (int col = c; col <= m; col++) mat[r][col]
    /= div;
REP(row, n) {
    if (row == r) continue;
    // F2: if (mat[row].test(c)) mat[row] ^=
    // mat[r];
    NUM times = -mat[row][c];
    for (int col = c; col <= m; col++)
        mat[row][col] += times * mat[r][col];
} // now mat is in RREF

for (int r = pr; r < n; r++)
    if (!is0(mat[r][m])) return 0;
// F2: return 1;
fill_n(hasval, n, false);
for (int col = 0, row; col < m; col++) {
    hasval[col] = !is0(mat[row][col]);
    if (!hasval[col]) continue;
    for (int c = col + 1; c < m; c++) {
        if (!is0(mat[row][c])) hasval[c] = false;
    }
    if (hasval[col]) vals[col] = mat[row][m];
    row++;
}
REP(i, n) if (!hasval[i]) return 2;
return 1;
}
```

### 7.5. Cycle-Finding.

```
ii find_cycle(int x0, int (*f)(int)) {
    int t = f(x0), h = f(t), mu = 0, lam = 1;
    while (t != h) t = f(t), h = f(f(h));
    h = x0;
    while (t != h) t = f(t), h = f(h), mu++;
    h = f(t);
    while (t != h) h = f(h), lam++;
    return ii(mu, lam); }
}
```

### 7.6. Longest Increasing Subsequence.

```
vi lis(vi arr) {
    vi seq, back(size(arr)), ans;
    rep(i, 0, size(arr)) {
        int res = 0, lo = 1, hi = size(seq);
        while (lo <= hi) {
            int mid = (lo+hi)/2;
            if (arr[seq[mid-1]] < arr[i]) res = mid, lo =
                mid + 1;
            else hi = mid - 1; }
        if (res < size(seq)) seq[res] = i;
        else seq.push_back(i);
        back[i] = res == 0 ? -1 : seq[res-1]; }
    int at = seq.back();
    while (at != -1) ans.push_back(at), at = back[at];
    reverse(ans.begin(), ans.end());
    return ans; }
```

### 7.7. Dates.

```
int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
    return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
        367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
        3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
        d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
    int x, n, i, j;
    x = jd + 68569;
    n = 4 * x / 146097;
    x -= (146097 * n + 3) / 4;
    i = (4000 * (x + 1)) / 1461001;
    x -= 1461 * i / 4 - 31;
    j = 80 * x / 2447;
    d = x - 2447 * j / 80;
    x = j / 11;
    m = j + 2 - 12 * x;
    y = 100 * (n - 49) + i + x; }
```

### 7.8. Simplex.

```
typedef vector<ld> VD;
typedef vector<VD> VVD;
const ld EPS = 1e-9;
struct LPSolver {
    int m, n; vi B, N; VVD D;
    LPSolver(const VVD &A, const VD &b, const VD &c) :
        m(b.size()), n(c.size()),
        N(n + 1), B(m), D(m + 2, VD(n + 2)) {
        REP(i, m) REP(j, n) D[i][j] = A[i][j];
        REP(i, m) { B[i] = n + i; D[i][n] = -1;
            D[i][n + 1] = b[i]; }
        REP(j, n) N[j] = j, D[m][j] = -c[j];
        N[n] = -1; D[m + 1][n] = 1;
    }
    void Pivot(int r, int s) {
        double inv = 1.0 / D[r][s];
        REP(i, m+2) if (i != r) REP(j, n+2) if (j != s)
            D[i][j] -= D[r][j] * D[i][s] * inv;
        REP(j, n+2) if (j != s) D[r][j] *= inv;
        REP(i, m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]); }
    bool Simplex(int phase) {
        int x = phase == 1 ? m + 1 : m;
        while (true) {
            int s = -1;
            for (int j = 0; j <= n; j++) {
                if (phase == 2 && N[j] == -1) continue;
                if (s == -1 || D[x][j] < D[x][s] ||
                    D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
            if (D[x][s] > -EPS) return true;
            int r = -1;
            REP(i, m) {
                if (D[i][s] < EPS) continue;
                if (r == -1 || D[i][n + 1] / D[i][s] < D[r][n +
                    1] /
```

```
            D[r][s] || (D[i][n + 1] / D[i][s]) ==
                (D[r][n + 1] /
                    D[r][s]) && B[i] < B[r]) r = i; }
            if (r == -1) return false;
            Pivot(r, s); } }
ld Solve(VD &x) {
    int r = 0;
    rep(i, 1, m) if (D[i][n+1] < D[r][n+1]) r = i;
    if (D[r][n + 1] < -EPS) {
        Pivot(r, n);
        if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
            return -numeric_limits<ld>::infinity();
        REP(i, m) if (B[i] == -1) {
            int s = -1;
            for (int j = 0; j <= n; j++)
                if (s == -1 || D[i][j] < D[i][s] ||
                    D[i][j] == D[i][s] && N[j] < N[s])
                    s = j;
            Pivot(i, s); }
    }
    if (!Simplex(2)) return
        numeric_limits<ld>::infinity();
    x = VD(n);
    for (int i = 0; i < m; i++) if (B[i] < n)
        x[B[i]] = D[i][n + 1];
    return D[m][n + 1]; } }
// 2-phase simplex solves linear system:
// maximize c^T x
// subject to Ax <= b, x >= 0
// INPUT: A -- an m x n matrix
// b -- an m-dimensional vector
// c -- an n-dimensional vector
// x -- optimal solution (by reference)
// OUTPUT: c^T x (inf. if unbounded above, nan if
// infeasible)
// *** Example ***
// const int m = 4, n = 3;
// ld _A[m][n] = {{6,-1,0}, {-1,-5,0},
// {1,5,1}, {-1,-5,-1}};
// ld _b[m] = {10,-4,5,-5}, _c[n] = {1,-1,0};
// VVD A(m);
// VD b(_b, _b + m), c(_c, _c + n), x;
// REP(i, m) A[i] = VD(_A[i], _A[i] + n);
// LPSolver solver(A, b, c);
// ld value = solver.Solve(x);
// cerr << "VALUE: " << value << endl; // 1.29032
// cerr << "SOLUTION:"; // 1.74194 0.451613 1
// REP(i, sz(x)) cerr << " " << x[i];
// cerr << endl;
```

### 8. COMBINATORICS

- Catalan numbers (valid bracket seq's of length  $2n$ ):  
 $C_0 = 1, C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$
- Stirling 1<sup>th</sup> kind ( $\# \pi \in \mathfrak{S}_n$  with exactly  $k$  cycles):  
 $\left[ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right] = \delta_{0n}, \left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (n-1) \left[ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right] + \left[ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right].$

- Stirling 2<sup>nd</sup> kind ( $k$ -partitions of  $[n]$ ):  
 $\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1, \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}.$
- Bell numbers (partitions of  $[n]$ ):  
 $B_0 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}.$
- Euler ( $\#\pi \in \mathfrak{S}_n$  with exactly  $k$  ascents):  
 $\left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle = \left\langle \begin{matrix} n \\ n-1 \end{matrix} \right\rangle = 1, \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = (k+1) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (n-k) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle.$
- Euler 2<sup>nd</sup> order (nr perms of  $1, 1, 2, 2, \dots, n, n$  with exactly  $k$  ascents):  
 $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = (k+1) \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (2n-k-1) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle.$
- Rooted trees:  $n^{n-1}$ , unrooted:  $n^{n-2}$ .
- Forests of  $k$  rooted trees:  $\binom{n}{k} k \cdot n^{n-k-1}$ .
- $1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad 1^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}, \quad \sum_i \binom{n-i}{i} = F_{n+1}$
- $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad x^k = \sum_{i=0}^k i! \left\{ \begin{matrix} k \\ i \end{matrix} \right\} (x) = \sum_{i=0}^k \left\langle \begin{matrix} k \\ i \end{matrix} \right\rangle \binom{x+i}{k}$
- $a \equiv b \pmod{x, y} \Leftrightarrow a \equiv b \pmod{\text{lcm}(x, y)}$ .
- $ac \equiv bc \pmod{m} \Leftrightarrow a \equiv b \pmod{m/\text{gcd}(c, m)}$ .
- $\text{gcd}(n^a - 1, n^b - 1) = \text{gcd}(a, b) - 1$ .
- **Möbius inversion formula:** If  $f(n) = \sum_{d|n} g(d)$ , then  $g(n) = \sum_{d|n} \mu(d) f(n/d)$ . If  $f(n) = \sum_{m=1}^n g(\lfloor n/m \rfloor)$ , then  $g(n) = \sum_{m=1}^n \mu(m) f(\lfloor \frac{n}{m} \rfloor)$ .
- **Inclusion-Exclusion:** If  $g(T) = \sum_{S \subseteq T} f(S)$ , then

$$f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} g(T).$$

Corollary:  $b_n = \sum_{k=0}^n \binom{n}{k} a_k \Leftrightarrow a_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} b_k$ .

8.1. **The Twelfold Way.** Putting  $n$  balls into  $k$  boxes.  $p(n, k)$  is # partitions of  $n$  in  $k$  parts, each  $> 0$ .  $p_k(n) = \sum_{i=0}^k p(n, k)$ .

Balls Boxes	same same	distinct same	same distinct	distinct distinct
-	$p_k(n)$	$\sum_{i=0}^k \left\{ \begin{matrix} n \\ i \end{matrix} \right\}$	$\binom{n+k-1}{k-1}$	$k^n$
size $\geq 1$	$p(n, k)$	$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	$\binom{n-1}{k-1}$	$k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$
size $\leq 1$	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n! \binom{k}{n}$

## 9. FORMULAS

- **Legendre symbol:**  $\left(\frac{a}{b}\right) = a^{(b-1)/2} \pmod{b}$ ,  $b$  odd prime.
- **Heron's formula:** A triangle with side lengths  $a, b, c$  has area  $\sqrt{s(s-a)(s-b)(s-c)}$  where  $s = \frac{a+b+c}{2}$ .
- **Shoelace formula:**  $A = \frac{1}{2} |\sum_{i=0}^{n-1} x_i y_{i+1} - x_{i+1} y_i|$ .
- **Pick's theorem:** A polygon on an integer grid strictly containing  $i$  lattice points and having  $b$  lattice points on the boundary has area  $i + \frac{b}{2} - 1$ . (Nothing similar in higher dimensions)
- **Absorption probabilities** A random walk on  $[0, n]$  with probability  $p$  to increase and  $q$  to decrease, starting at  $k$  has at  $n$  absorption probability  $\frac{(q/p)^k - 1}{(q/p)^n - 1}$  if  $q \neq p$ , and  $k/n$  if  $q = p$ .
- **König's theorem:** In any bipartite graph  $G = (L \cup R, E)$ , the number of edges in a maximum matching is equal to the number of

vertices in a minimum vertex cover. Let  $U$  be the set of unmatched vertices in  $L$ , and  $Z$  be the set of vertices that are either in  $U$  or are connected to  $U$  by an alternating path. Then  $K = (L \setminus Z) \cup (R \cap Z)$  is the minimum vertex cover.

- A minimum Steiner tree for  $n$  vertices requires at most  $n - 2$  additional Steiner vertices.
- The number of vertices of a graph is equal to its minimum vertex cover number plus the size of a maximum independent set.
- **Lagrange polynomial** through points  $(x_0, y_0), \dots, (x_k, y_k)$  is

$$L(x) = \sum_{j=0}^k y_j \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}.$$

- **Hook length formula:** If  $\lambda$  is a Young diagram and  $h_\lambda(i, j)$  is the hook-length of cell  $(i, j)$ , then then the number of Young tableaux  $d_\lambda = n! / \prod h_\lambda(i, j)$ .
- #primitive pythagorean triples with hypotenuse  $< n$  approx  $n/(2\pi)$ .
- **Frobenius Number:** largest number which can't be expressed as a linear combination of numbers  $a_1, \dots, a_n$  with non-negative coefficients.  $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$ ,  $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$ .  $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d - 1)$ . An integer  $x > (\max_i a_i)^2$  can be expressed in such a way iff.  $x \mid \text{gcd}(a_1, \dots, a_n)$ .
- **Snell's law:**  $v_2 \sin \theta_1 = v_1 \sin \theta_2$  gives the shortest path between two media.
- **BEST theorem:** The number of Eulerian cycles in a *directed* graph  $G$  is:

$$t_w(G) \prod_{v \in G} (\deg v - 1)!,$$

where  $t_w(G)$  is the number of arborescences ("directed spanning" tree) rooted at  $w$ :  $t_w(G) = \det(q_{ij})_{i,j \neq w}$ , with  $q_{ij} = [i = j] \text{indeg}(i) - \#\{(i, j) \in E\}$ .

9.1. **Burnside's Lemma.** Let a finite group  $G$  act on a set  $X$ . Denote  $X^g = \{x \in X \mid gx = x\}$ . For each  $g$  in  $G$  let  $X^g$  denote the set of elements in  $X$  that are fixed by  $g$ . Then the number of orbits is:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

9.2. **Bézout's identity.** If  $(x, y)$  is a solution to  $ax + by = d$  ( $x, y$  can be found with EGCD), then all solutions are given by

$$(x + k \cdot \text{lcm}(a, b)/a, y - k \cdot \text{lcm}(a, b)/b), \quad k \in \mathbb{Z}$$

## 10. GAME THEORY

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

- **Nim:** Let  $X = \bigoplus_{i=1}^n x_i$ , then  $(x_i)_{i=1}^n$  is a winning position iff  $X \neq 0$ . Find a move by picking  $k$  such that  $x_k > x_k \oplus X$ .
- **Misère Nim:** Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles.

The second player wins  $(a_1, \dots, a_n)$  if 1) there is a pile  $a_i > 1$  and  $\bigoplus_{i=1}^n a_i = 0$  or 2) all  $a_i \leq 1$  and  $\bigoplus_{i=1}^n a_i = 1$ .

- **Staircase Nim:** Stones are moved down a staircase and only removed from the last pile.  $(x_i)_{i=1}^n$  is an  $L$ -position if  $(x_{2i-1})_{i=1}^{n/2}$  is (i.e. only look at odd-numbered piles).
- **Moore's Nim<sub>k</sub>:** The player may remove from at most  $k$  piles (Nim = Nim<sub>1</sub>). Expand the piles in base 2, do a carry-less addition in base  $k+1$  (i.e. the number of ones in each column should be divisible by  $k+1$ ).
- **Dim<sup>+</sup>:** The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is  $k+1$  where  $2^k$  is the largest power of 2 dividing the pile size.
- **Aliquot game:** Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just  $k$ .
- **Nim (at most half):** Write  $n+1 = 2^m y$  with  $m$  maximal, then the Sprague-Grundy function of  $n$  is  $(y-1)/2$ .
- **Lasker's Nim:** Players may alternatively split a pile into two new non-empty piles.  $g(4k+1) = 4k+1$ ,  $g(4k+2) = 4k+2$ ,  $g(4k+3) = 4k+4$ ,  $g(4k+4) = 4k+3$  ( $k \geq 0$ ).
- **Hackenbush on trees:** A tree with stalks  $(x_i)_{i=1}^n$  may be replaced with a single stalk with length  $\bigoplus_{i=1}^n x_i$ .

## 11. DEBUGGING TIPS

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
  - Getting NaN? Make sure `acos` etc. are not getting values out of their range (perhaps `1+eps`).
  - Rounding negative numbers?
  - Outputting in scientific notation?
- Wrong Answer?
  - Read the problem statement again!
  - Are multiple test cases being handled correctly? Try repeating the same test case many times.
  - Integer overflow?
  - Think very carefully about boundaries of all input parameters
  - Try out possible edge cases:
    - \*  $n = 0, n = -1, n = 1, n = 2^{31} - 1$  or  $n = -2^{31}$
    - \* List is empty, or contains a single element
    - \*  $n$  is even,  $n$  is odd
    - \* Graph is empty, or contains a single vertex
    - \* Graph is a multigraph (loops or multiple edges)
    - \* Polygon is concave or non-simple
  - Is initial condition wrong for small cases?
  - Are you sure the algorithm is correct?
  - Explain your solution to someone.
  - Are you using any functions that you don't completely understand? Maybe STL functions?
  - Maybe you (or someone else) should rewrite the solution?
  - Can the input line be empty?
- Run-Time Error?

- Is it actually Memory Limit Exceeded?

### 11.1. Solution Ideas.

- Dynamic Programming
    - Parsing CFGs: CYK Algorithm
    - Drop a parameter, recover from others
    - Swap answer and a parameter
    - When grouping: try splitting in two
    - $2^k$  trick
    - When optimizing
      - \* Convex hull optimization
        - $dp[i] = \min_{j < i} \{dp[j] + b[j] \times a[i]\}$
        - $b[j] \geq b[j + 1]$
        - optionally  $a[i] \leq a[i + 1]$
        - $O(n^2)$  to  $O(n)$
      - \* Divide and conquer optimization
        - $dp[i][j] = \min_{k < j} \{dp[i - 1][k] + C[k][j]\}$
        - $A[i][j] \leq A[i][j + 1]$
        - $O(kn^2)$  to  $O(kn \log n)$
        - sufficient:  $C[a][c] + C[b][d] \leq C[a][d] + C[b][c]$ ,  
 $a \leq b \leq c \leq d$  (QI)
- ```
vvi A; // A[i][j] is voor [i, j]

void divco(ll ls, ll rs, ll lt, ll rt, vi &t,
  ↪ vi &s) { // berekent t_{[lt, rt]}
  if (lt >= rt)
    return;
  ll ms = ls, mt = (lt + rt) / 2;
  t[mt] = -INF;
  rep(i, ls, rs) {
    if (i >= mt) {
      break;
    }
    if (s[i] + A[i][mt] > t[mt]) {
      t[mt] = s[i] + A[i][mt];
      ms = i;
    }
  }
  divco(ls, ms + 1, lt, mt, t, s);
  divco(ms, rs, mt + 1, rt, t, s);
}

* Knuth optimization
  ·  $dp[i][j] = \min_{i < k < j} \{dp[i][k] + dp[k][j] + C[i][j]\}$ 
  ·  $A[i][j - 1] \leq A[i][j] \leq A[i + 1][j]$ 
  ·  $O(n^3)$  to  $O(n^2)$ 
  · sufficient: QI and  $C[b][c] \leq C[a][d]$ ,  $a \leq b \leq c \leq d$ - Greedy
- Randomized
- Optimizations
  - Use bitset (/64)
  - Switch order of loops (cache locality)
- Process queries offline
  - Mo's algorithm
- Square-root decomposition

```

- Precomputation
- Efficient simulation
  - Mo's algorithm
  - Sqrt decomposition
  - Store  $2^k$  jump pointers
- Data structure techniques
  - Sqrt buckets
  - Store  $2^k$  jump pointers
  - $2^k$  merging trick
- Counting
  - Inclusion-exclusion principle
  - Generating functions
- Graphs
  - Can we model the problem as a graph?
  - Can we use any properties of the graph?
  - Strongly connected components
  - Cycles (or odd cycles)
  - Bipartite (no odd cycles)
    - \* Bipartite matching
    - \* Hall's marriage theorem
    - \* Stable Marriage
  - Cut vertex/bridge
  - Biconnected components
  - Degrees of vertices (odd/even)
  - Trees
    - \* Heavy-light decomposition
    - \* Centroid decomposition
    - \* Least common ancestor
    - \* Centers of the tree
  - Eulerian path/circuit
  - Chinese postman problem
  - Topological sort
  - (Min-Cost) Max Flow
  - Min Cut
    - \* Maximum Density Subgraph
  - Huffman Coding
  - Min-Cost Arborescence
  - Steiner Tree
  - Kirchhoff's matrix tree theorem
  - Prüfer sequences
  - Lovász Toggle
  - Look at the DFS tree (which has no cross-edges)
  - Is the graph a DFA or NFA?
    - \* Is it the Synchronizing word problem?
- math
  - Is the function multiplicative?
  - Look for a pattern
  - Permutations
    - \* Consider the cycles of the permutation
  - Functions
    - \* Sum of piecewise-linear functions is a piecewise-linear function

- \* Sum of convex (concave) functions is convex (concave)
- Modular arithmetic
  - \* Chinese Remainder Theorem
  - \* Linear Congruence
- Sieve
- System of linear equations
- Values too big to represent?
  - \* Compute using the logarithm
  - \* Divide everything by some large value
- Linear programming
  - \* Is the dual problem easier to solve?
- Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
  - 2-SAT
  - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ( $\log(n)$ )
- Strings
  - Trie (maybe over something weird, like bits)
  - Suffix array
  - Suffix automaton (+DP?)
  - Aho-Corasick
  - eerTree
  - Work with  $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
  - Lazy propagation
  - Persistent
  - Implicit
  - Segment tree of X
- Geometry
  - Minkowski sum (of convex sets)
  - Rotating calipers
  - Sweep line (horizontally or vertically?)
  - Sweep angle
  - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem
  - Minimize something instead of maximizing



- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)
- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

PRACTICE CONTEST CHECKLIST

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use printf/scanf with long long/long double?
- Are `__int128` and `__float128` available?
- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: `i?python[23]`, `factor`.
- Try submitting with `assert(false)` and `assert(true)`.
- Omitting `return 0;` still works?
- Look for directory with sample test cases.
- Make sure printing works.