

# TCR

## git merge -s octopus solution cup

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```

.bashrc

alias gg='g++ -std=c++17 -Wall -Wconversion
↳ -Wshadow'
alias g='gg -DDEBUG -g -fsanitize=address,undefined'

.vimrc

set nu rnu sw=4 ts=4 sts=4 noet ai hls shcf=-ic
sy on | colo slate

Test script (usage: ./test.sh A/B/..)
g++ -g -Wall -fsanitize=address,undefined
↳ -Wfatal-error -std=c++17 $1.cc || exit
for i in $(ls *.in)
do
    j="$(echo $i | sed 's/\.in$/.ans')
    ./a.out < $i > output
    diff output $j || echo "!!WA on $i!!"
done

template.cc

#include <bits/stdc++.h>
using namespace std;

typedef long long ll;
typedef long double ld;
typedef pair<ll, ll> ii;
typedef vector<ll> vi;
typedef vector<vi> vvi;
typedef vector<ii> vii;

#define x first
#define y second
#define pb push_back
#define eb emplace_back
#define rep(i,a,b) for(auto i=(a); i<(b); ++i)
#define REP(i,n) rep(i,0,n)
#define all(v) (v).begin(), (v).end()
#define rs resize
#define DBG(x) cerr << __LINE__ << ": " << #x << " = " << (x) << endl

const ld PI = acos(-1.0);
template<class T> using min_queue =
    priority_queue<T, vector<T>, greater<T>>;
template<class T> int sz(const T &x) {
    return (int) x.size(); // copy the ampersand(&)!
}

template<class T> ostream& operator<<(ostream &os,
↳ vector<T> &v) {
    os << "\n[";
    for(T &x : v) os << x << ', ';
    return os << "]\n";
}

struct pairhash {
public:
    template<typename T1, typename T2>

```

```

    size_t operator()(const pair<T1, T2> &p) const {
        size_t lhs = hash<T1>()(p.x);
        size_t rhs = hash<T2>()(p.y);
        return lhs ^ (rhs<<9e3779b9+(lhs<<6)+(lhs>>2));
    }
};

void run() {}

signed main() {
    // DON'T MIX "scanf" and "cin"!
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    (cout << fixed).precision(18);
    run();
    return 0;
}

template.py

from sys import *
n,m = [ int(x) for x in
↳ stdin.readline().rstrip().split() ]
stdout.write( str(n*m)+"\n" )

from itertools import *
for (x,y) in product(range(3),repeat=2):
    stdout.write( str(3*x+y)+" " )
stdout.write( "\n" )
for L in combinations(range(4),2):
    stdout.write( str(L)+" " )
stdout.write( "\n" )

from functools import *
y = reduce( lambda x,y: x+y, map( lambda x: x*x,
↳ range(4) ), -3 )
stdout.write( str(y)+"\n" )

from math import *
stdout.write( "{0:.2f}\n".format(pi) )

```

### 0.1. De winnende aanpak.

- Slaap goed & heb een vroeg ritme!
- Drink & eet genoeg voor & tijdens de wedstrijd!
- Houd een lijst bij met info over alle problemen.
- Iedereen (incl. Ludo) moet **ALLE** opgaves goed lezen!
- Analyseer de voorbeeld test cases.
- Houd na 2 uur een pauze en overleg waar iedereen mee bezig is.
- Maak zelf (zwarte) test cases.
- Gebruik ll.

### 0.2. Wrong Answer.

- Print de oplossing om te debuggen!
- Kijk naar wellicht makkelijkere problemen.
- Bedenk zelf test cases met randgevallen!

- Controleer de precisie.
- Controleer op **overflow** (gebruik **OVERALL** ll, ld).  
*Kijk naar overflows in tussenantwoorden bij modulo.*
- Controleer op **typo's**.
- Loop de voorbeeld test case accuraat langs.
- Controleer op off-by-one-errors (in indices of lus-grenzen)?

**Detecting overflow:** This GNU builtin checks for over- and underflow. Result is in res if successful:

```

bool isOverflown =
↳ __builtin_[add|mul|sub]_overflow(a, b, &res);

```

### 1. MATHEMATICS

**XOR sum:**  $\bigoplus_{x=0}^{a-1} x = \{0, a-1, 1, a\}[a \bmod 4]$ .

```

int abs(int x) { return x > 0 ? x : -x; }
int sign(int x) { return (x > 0) - (x < 0); }

```

```

// greatest common divisor
ll gcd(ll a, ll b) { while(b) a%=b, swap(a,b); return a; }
// least common multiple
ll lcm(ll a, ll b) { return a/gcd(a, b)*b; }
ll mod(ll a, ll b) { return (a%b) < 0 ? a+b : a; }

```

```

// ab % m for m <= 4e18 in O(log b)
ll mod_mul(ll a, ll b, ll m) {
    ll r = 0;
    while(b) {
        if (b & 1) r = mod(r+a,m);
        a = mod(a+a,m); b >>= 1;
    }
    return r;
}

```

```

// a^b % m for m <= 2e9 in O(log b)
ll mod_pow(ll a, ll b, ll m) {
    ll r = 1;
    while(b) {
        if (b & 1) r = (r * a) % m; // mod_mul
        a = (a * a) % m; // mod_mul
        b >>= 1;
    }
    return r;
}

```

```

// returns x, y such that ax + by = gcd(a, b)
ll egcd(ll a, ll b, ll &x, ll &y) {
    ll xx = y = 0, yy = x = 1;
    while (b) {
        x -= a / b * xx; swap(x, xx);
        y -= a / b * yy; swap(y, yy);
        a %= b; swap(a, b);
    }
    return a;
}

```

```
// Chinese Remainder Theorem: returns (u, v) s.t.
// x=u (mod v) <=> x=a (mod n) and x=b (mod m)
pair<ll, ll> crt(ll a, ll n, ll b, ll m) {
    ll s, t, d = egcd(n, m, s, t); //n,m<=1e9
    if (mod(a - b, d)) return { 0, -1 };
    return { mod(s*b%m*n + t*a%n*m, n*m)/d, n*m/d };
}

// phi[i] = # { 0 < j <= i | gcd(i, j) = 1 } sieve
vi totient(int N) {
    vi phi(N);
    for (int i = 0; i < N; i++) phi[i] = i;
    for (int i = 2; i < N; i++) if (phi[i] == i)
        for (int j = i; j < N; j+=i) phi[j] -= phi[j]/i;
    return phi;
}

// calculate nCk % p (p prime!)
ll lucas(ll n, ll k, ll p) {
    ll ans = 1;
    while (n) {
        ll np = n % p, kp = k % p;
        if (np < kp) return 0;
        ans = mod(ans * binom(np, kp), p); // (np C kp)
        n /= p; k /= p;
    }
    return ans;
}

// returns if n is prime for n < 3e24 (>2^64)
// but use mul_mod for n > 2e9.
bool millerRabin(ll n){
    if (n < 2 || n % 2 == 0) return n == 2;
    ll d = n - 1, ad, s = 0, r;
    for (; d % 2 == 0; d /= 2) s++;
    for (int a : { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 }) {
        if (n == a) return true;
        if ((ad = mod_pow(a, d, n)) == 1) continue;
        for (r = 0; r < s && ad + 1 != n; r++)
            ad = (ad * ad) % n;
        if (r == s) return false;
    }
    return true;
}
```

**1.1. Primitive Root  $O(\sqrt{m})$ .** Returns a generator of  $\mathbb{F}_m^*$ . If  $m$  not prime, replace  $m-1$  by totient of  $m$ .

```
ll primitive_root(ll m) {
    vector<ll> div;
    for (ll i = 1; i*i < m; i++)
        if ((m-1) % i == 0) {
            if (i < m-1) div.pb(i);
            if ((m-1)/i < m) div.pb((m-1)/i);
        }
    rep(x, 2, m) {
        bool ok = true;
        for (ll d : div) if (mod_pow(x, d, m) == 1)
```

```
        { ok = false; break; }
        if (ok) return x;
    }
    return -1;
}

1.2. Tonelli-Shanks algorithm. Given prime  $p$  and integer
 $1 \leq n < p$ , returns the square root  $r$  of  $n$  modulo  $p$ . There is
also another solution given by  $-r$  modulo  $p$ .
ll legendre(ll a, ll p) {
    if (a % p == 0) return 0;
    return p == 2 || mod_pow(a, (p-1)/2, p) == 1 ? 1 :
        -1;
}
ll tonelli_shanks(ll n, ll p) {
    assert(legendre(n, p) == 1);
    if (p == 2) return 1;
    ll s = 0, q = p-1, z = 2;
    while (~q & 1) s++, q >>= 1;
    if (s == 1) return mod_pow(n, (p+1)/4, p);
    while (legendre(z, p) != -1) z++;
    ll c = mod_pow(z, q, p),
        r = mod_pow(n, (q+1)/2, p),
        t = mod_pow(n, q, p),
        m = s;
    while (t != 1) {
        ll i = 1, ts = (ll)t*t % p;
        while (ts != 1) i++, ts = ((ll)ts * ts) % p;
        ll b = mod_pow(c, 1LL<<(m-i-1), p);
        r = (ll)r * b % p;
        t = (ll)t * b % p * b % p;
        c = (ll)b * b % p;
        m = i;
    }
    return r;
}
```

**1.3. Numeric Integration.** Numeric integration using Simpson's rule.

```
ld numint(ld (*f)(ld), ld a, ld b, ld EPS = 1e-6) {
    ld ba = b - a, m=(a+b)/2;
    return abs(ba) < EPS
        ? ba/8*(f(a)+f(b)+f(a+ba/3)*3+f(b-ba/3)*3)
        : numint(f, a, m, EPS) + numint(f, m, b, EPS);
}
```

**1.4. Fast Hadamard Transform.** Computes XOR-convolutions in  $O(k2^k)$  on  $k$  bits.

For AND-convolution, use  $(x+y, y)$ ,  $(x-y, y)$ .

For OR-convolution, use  $(x, x+y)$ ,  $(x, -x+y)$ .

**Note:** The array size must be a power of 2.

```
void fht(vi &A, bool inv=false, int l, int r) {
    if (l+1 == r) return;
    int k = (r-l)/2;
    if (!inv) fht(A, inv, l, l+k), fht(A, inv, l+k,
        -> r);
```

```
rep(i, l, l+k) {
    int x = A[i], y = A[i+k];
    if (!inv) A[i] = x-y, A[i+k] = x+y;
    else A[i] = (x+y)/2, A[i+k] = (-x+y)/2;
}
if (inv) fht(A, inv, l, l+k), fht(A, inv, l+k, r);
}
```

**1.5. Tridiagonal Matrix Algorithm.** Solves a tridiagonal system of linear equations

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i$$

where  $a_1 = c_n = 0$ . Beware of numerical instability.

```
#define MAXN 5000
ld A[MAXN], B[MAXN], C[MAXN], D[MAXN], X[MAXN];
void solve(int n) {
    C[0] /= B[0]; D[0] /= B[0];
    rep(i, 1, n-1) C[i] /= B[i] - A[i]*C[i-1];
    rep(i, 1, n) D[i] =
        (D[i] - A[i]*D[i-1]) / (B[i] - A[i]*C[i-1]);
    X[n-1] = D[n-1];
    for (int i = n-1; i--;) X[i] = D[i] - C[i]*X[i+1];
}
```

**1.6. Number of Integer Points under Line.** Count the number of integer solutions to  $Ax + By \leq C$ ,  $0 \leq x \leq n$ ,  $0 \leq y$ . In other words, evaluate the sum  $\sum_{x=0}^n \lfloor \frac{C-Ax}{B} + 1 \rfloor$ . To count all solutions, let  $n = \lfloor \frac{C}{A} \rfloor$ . In any case, it must hold that  $C - nA \geq 0$ . Be very careful about overflows.

```
ll floor_sum(ll n, ll a, ll b, ll c) {
    if (c == 0) return 1;
    if (c < 0) return 0;
    if (a % b == 0) return
        -> (n+1)*(c/b+1)-n*(n+1)/2*a/b;
    if (a >= b) return
        -> floor_sum(n, a%b, b, c)-a/b*n*(n+1)/2;
    ll t = (c-a*n+b)/b;
    return floor_sum((c-b*t)/b, b, a, c-b*t)+t*(n+1);
}
```

**1.7. Solving linear recurrences.** Given some brute-forced sequence  $s[0], s[1], \dots, s[2n-1]$ , Berlekamp-Massey finds the shortest possible recurrence relation in  $O(n^2)$ . After that, lin\_rec finds  $s[k]$  in  $O(n^2 \log k)$ .

```
// Given a sequence s[0], ..., s[2n-1] finds the
// smallest linear recurrence
// of size <= n compatible with s.
v1 BerlekampMassey(const v1 &s, ll mod) {
    int n = sz(s), L = 0, m = 0;
    v1 C(n), B(n), T;
    C[0] = B[0] = 1;
    ll b = 1;
    REP(i, n) {
        ++m;
        ll d = s[i] % mod;
```

```

rep(j, 1, L+1) d = (d + C[j] * s[i - j]) % mod;
if (!d) continue;
T = C;
ll coef = d * modpow(b, mod-2, mod) % mod;
rep(j,m,n) C[j] = (C[j] - coef * B[j-m]) % mod;
if (2 * L > i) continue;
L = i + 1 - L;
B = T; b = d; m = 0;
}
C.resize(L + 1);
C.erase(C.begin());
for (auto &x : C) x = (mod - x) % mod;
return C;
}

// Input: A[0,...,n-1], C[0,...,n-1] satisfying
// A[i] = \sum_{j=1}^n C[j-1] A[i-j],
// Outputs A[k]
ll lin_rec(const vl &A, const vl &C, ll k, ll mod){
int n = sz(A);
auto combine = [&](vl a, vl b) {
vl res(sz(a) + sz(b) - 1, 0);
REP(i, sz(a)) REP(j, sz(b))
res[i+j] = (res[i+j] + a[i]*b[j]) % mod;
for (int i = 2*n; i > n; --i) REP(j,n)
res[i-1-j] = (res[i-1-j] + res[i]*C[j]) % mod;
res.resize(n + 1);
return res;
};
vl pol(n + 1, e(pol));
pol[0] = e[1] = 1;
for (++k; k; k /= 2) {
if (k % 2) pol = combine(pol, e);
e = combine(e, e);
}
ll res = 0;
REP(i, n) res = (res + pol[i + 1] * A[i]) % mod;
return res;
}

```

## 1.8. Misc.

1.8.1. *Josephus problem*. Last man standing out of  $n$  if every  $k$ th is killed. Zero-based, and does not kill 0 on first pass.

```

int J(int n, int k) {
if (n == 1 || k == 1) return n-1;
if (n < k) return (J(n-1,k)+k)%n;
int np = n - n/k;
return k*((J(np,k)+np-n%k*np)%np) / (k-1);
}

```

## • Prime numbers:

1031, 32771, 1048583, 8125344, 33554467, 9982451653, 1073741827, 34359738421, 1099511627791, 35184372088891, 1125899906842679, 36028797018963971.

$10^3 + \{-9, -3, 9, 13\}$ ,  $10^6 + \{-17, 3, 33\}$ ,  $10^9 + \{7, 9, 21, 33, 87\}$ .

• **Generating functions:** Ordinary (ogf):  $A(x) := \sum_{n=0}^{\infty} a_n x^n$ .

Calculate product  $c_n = \sum_{k=0}^n a_k b_{n-k}$  with FFT.

Exponential (e.g.f.):  $A(x) := \sum_{n=0}^{\infty} a_n x^n / n!$ ,

$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k} = n! \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}$  (use FFT).

• **General linear recurrences:** If  $a_n = \sum_{k=0}^{n-1} a_k b_{n-k}$ , then  $A(x) = \frac{a_0}{1-B(x)}$ .

• **Inverse polynomial modulo  $x^l$ :** Given  $A(x)$ , find  $B(x)$  such that  $A(x)B(x) = 1 + x^l Q(x)$  for some  $Q(x)$ .

Step 1: Start with  $B_0(x) = 1/a_0$

Step 2:  $B_{k+1}(x) = (-B_k(x)^2 A(x) + 2B_k(x)) \bmod x^{2^{k+1}}$ .

• **Fast subset convolution:** Given array  $a_i$  of size  $2^k$  calculate  $b_i = \sum_{j \& i = i} a_j$ .

```
for (int b = 1; b < (1 << k); b <= 1)
```

```
for (int i = 0; i < (1 << k); i++)
```

```
if (!(i & b)) a[i | b] += a[i];
```

```
// inv: if (!(i & b)) a[i | b] -= a[i];
```

• **Primitive Roots:** It only exists when  $n$  is  $2, 4, p^k, 2p^k$ , where  $p$  odd prime. If  $g$  is a primitive root, all primitive roots are of the form  $g^k$  where  $k, \phi(p)$  are coprime (hence there are  $\phi(\phi(p))$  primitive roots).

• **Maximum number of divisors:**

$\leq N$	$10^3$	$10^6$	$10^9$	$10^{12}$	$10^{18}$
$m$	840	720720	735134400	963761198400	
$\sigma_0(m)$	32	240	1344	6270	103680

For  $n = 10^{18}, m = 897612484786617600$ .

## 2. DATASTRUCTURES

### 2.1. Order tree.

```

#include <bits/extc++.h>
using namespace __gnu_pbds;
template<class TK, class TM> using order_tree =
    tree<TK, TM, greater<TK>, rb_tree_tag,
    tree_order_statistics_node_update>;
template<class TK> using order_set =
    order_tree<TK, null_type>;

```

```

vi s;
order_set<ii> t;
void update( ll k, ll v ) {
t.erase( ii{ s[k], k } );
s[k] = v;
t.insert( ii{ s[k], k } );
}

```

```

signed main() {
ll n = 4;
s.resize(n, 0);
rep(i, 0, n) t.insert(ii{0, i});
}

```

```

update( 2, 3 );
cout << t.find_by_order( 2 )->y << endl;
cout << t.order_of_key( ii{s[3], 3} ) << endl;
}

```

### 2.2. Segment tree $\mathcal{O}(\log n)$ .

#### 2.2.1. Standard segment tree.

```

typedef int S; // or define your own object
const int n = 1 << 20;
S t[2 * n];

// combine must be an associative function!
S combine(S l, S r) { return l+r; } //or max(l,r) etc

```

```

void build() {
for (int i = n; --i; )
t[i] = combine(t[2 * i], t[2 * i + 1]);
}

```

```

// set value v on position i
void update(int i, S v) {
for (t[i+=n] = v; i /= 2; )
t[i] = combine(t[2 * i], t[2 * i + 1]);
}

```

```

// sum on interval [l, r)
S query(int l, int r) {
S resL = 0, resR = 0;
for (l += n, r += n; l < r; l /= 2, r /= 2) {
if (l & 1) resL = combine(resL, t[l++]);
if (r & 1) resR = combine(t[--r], resR);
}
return combine(resL, resR);
}

```

#### 2.2.2. Lazy segment tree.

Be careful: all intervals are right-closed  $[\ell, r]$ .

```

struct node {
int l, r, x, lazy;
node() {}
node(int _l, int _r) : l(_l), r(_r), x(INT_MAX),
    lazy(0) {}
node(int _l, int _r, int _x) : node(_l, _r) {x=_x;}
node(node a, node b) : node(a.l, b.r) {x=min(a.x, b.x);}
void update(int v) { x = v; }
void range_update(int v) { lazy = v; }
void apply() { x += lazy; lazy = 0; }
void push(node &u) { u.lazy += lazy; }
};

```

```

struct segment_tree {
int n;
vector<node> arr;
segment_tree() {}
segment_tree(const vi &a) : n(sz(a)), arr(4*n) {
mk(a, 0, 0, n-1);
}
}

```

```

node mk(const vi &a, int i, int l, int r) {
    int m = (l+r)/2;
    return arr[i] = l > r ? node(l,r) :
        l == r ? node(l,r,a[l]) :
        node(mk(a,2*i+1,l,m),mk(a,2*i+2,m+1,r));
}
node update(int at, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (at < hl || hr < at) return arr[i];
    if (hl == at && at == hr) {
        arr[i].update(v); return arr[i];
    }
    return arr[i] =
        node(update(at,v,2*i+1),update(at,v,2*i+2));
}
node query(int l, int r, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return node(hl,hr);
    if (l <= hl && hr <= r) return arr[i];
    return node(query(l,r,2*i+1),query(l,r,2*i+2));
}
node range_update(int l, int r, ll v, int i=0) {
    propagate(i);
    int hl = arr[i].l, hr = arr[i].r;
    if (r < hl || hr < l) return arr[i];
    if (l <= hl && hr <= r) {
        arr[i].range_update(v);
        propagate(i);
        return arr[i];
    }
    return arr[i] = node(range_update(l,r,v,2*i+1),
        range_update(l,r,v,2*i+2));
}
void propagate(int i) {
    if (arr[i].l < arr[i].r) {
        arr[i].push(arr[2*i+1]);
        arr[i].push(arr[2*i+2]);
    }
    arr[i].apply();
}
};

```

### 2.2.3. Persistent segment tree.

Be careful: all intervals are right-closed  $[\ell, r]$ , including build.

```

int segcnt = 0;
struct segment {
    int l, r, lid, rid, sum;
} S[2000000];

int build(int l, int r) {
    if (l > r) return -1;
    int id = segcnt++;
    S[id].l = l;
    S[id].r = r;
    if (l == r) S[id].lid = -1, S[id].rid = -1;

```

```

    else {
        int m = (l + r) / 2;
        S[id].lid = build(l, m);
        S[id].rid = build(m + 1, r);
    }
    S[id].sum = 0;
    return id;
}
int update(int idx, int v, int id) {
    if (id == -1) return -1;
    if (idx < S[id].l || idx > S[id].r) return id;
    int nid = segcnt++;
    S[nid].l = S[id].l;
    S[nid].r = S[id].r;
    S[nid].lid = update(idx, v, S[id].lid);
    S[nid].rid = update(idx, v, S[id].rid);
    S[nid].sum = S[id].sum + v;
    return nid;
}
int query(int id, int l, int r) {
    if (r < S[id].l || S[id].r < l) return 0;
    if (l <= S[id].l && S[id].r <= r) return S[id].sum;
    return query(S[id].lid, l, r) + query(S[id].rid, l, r);
}

2.3. Binary Indexed Tree  $\mathcal{O}(\log n)$ . Use one-based indices ( $i > 0$ )!
struct BIT {
    int n; vi A;
    BIT(int _n) : n(_n), A(_n+1, 0) {}
    BIT(vi &v) : n(sz(v)), A(1) {
        for (auto x:v)
            for (int i=1, j; j=i&-1, i<=n; i++)
                if (i+j <= n) A[i+j] += A[i];
    }
    void update(int i, ll v) { // a[i] += v
        while (i <= n) A[i] += v, i += i&-1;
    }
    ll query(int i) { // sum_{j<=i} a[j]
        ll v = 0;
        while (i) v += A[i], i -= i&-1;
        return v;
    }
};

struct rangeBIT {
    int n; BIT b1, b2;
    rangeBIT(int _n) : n(_n), b1(_n), b2(_n+1) {}
    rangeBIT(vi &v) : n(sz(v)), b1(v), b2(sz(v)+1) {}
    void pupdate(int i, ll v) { b1.update(i, v); }
    void rupdate(int i, int j, ll v) { // a[i..j] += v
        b2.update(i, v);
        b2.update(j+1, -v);
        b1.update(j+1, v*j);
        b1.update(i, (1-i)*v);
    }
    ll query(int i) { return b1.query(i) + b2.query(i)*i; }
};

```

### 2.4. Disjoint-Set / Union-Find $\mathcal{O}(\alpha(n))$ .

```

struct dsu {
    vi par, rnk;
    dsu(int n) : par(n, -1), rnk(n, 0) {}
    int find(int i) { return
        par[i] < 0 ? i : par[i] = find(par[i]); }
    void unite(int a, int b) {
        if ((a = find(a)) == (b = find(b))) return;
        if (rnk[a] < rnk[b]) swap(a, b);
        if (rnk[a] == rnk[b]) rnk[a]++;
        par[a] += par[b]; par[b] = a;
    }
};

```

### 2.5. Cartesian tree.

```

struct node {
    int x, y, sz;
    node *l, *r;
    node(int _x, int _y)
        : x(_x), y(_y), sz(1), l(NULL), r(NULL) {}
};
int tsize(node* t) { return t ? t->sz : 0; }
void augment(node* t) {
    t->sz = 1 + tsize(t->l) + tsize(t->r);
}
pair<node*, node*> split(node* t, int x) {
    if (!t) return make_pair((node*)NULL, (node*)NULL);
    if (t->x < x) {
        pair<node*, node*> res = split(t->r, x);
        t->r = res.x; augment(t);
        return make_pair(t, res.y);
    }
    pair<node*, node*> res = split(t->l, x);
    t->l = res.y; augment(t);
    return make_pair(res.x, t);
}
node* merge(node* l, node* r) {
    if (!l) return r; if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r); augment(l); return l;
    }
    r->l = merge(l, r->l); augment(r); return r;
}
node* find(node* t, int x) {
    while (t) {
        if (x < t->x) t = t->l;
        else if (t->x < x) t = t->r;
        else return t;
    }
    return NULL;
}
node* insert(node* t, int x, int y) {
    if (find(t, x) != NULL) return t;
    pair<node*, node*> res = split(t, x);
    return merge(res.x, merge(new node(x, y), res.y));
}
node* erase(node* t, int x) {
    if (!t) return NULL;
    if (t->x < x) t->r = erase(t->r, x);
    else if (x < t->x) t->l = erase(t->l, x);
    else {node* old=t; t=merge(t->l, t->r); delete old;}
    if (t) augment(t); return t;
}

```

```

int kth(node *t, int k) {
    if (k < tsize(t->l)) return kth(t->l, k);
    else if (k == tsize(t->l)) return t->x;
    else return kth(t->r, k - tsize(t->l) - 1);
}

2.6. Heap. An implementation of a binary heap.
#define RESIZE
#define SWP(x,y) tmp = x, x = y, y = tmp
struct int_less {
    int_less() {}
    bool operator () (const int &a, const int &b) {
        return a < b;
    }
};
template <class Compare = int_less> struct heap {
    int cap, len, *q, *loc, tmp;
    Compare _cmp;
    inline bool cmp(int i, int j) {
        return _cmp(q[i], q[j]);
    }
    inline void swp(int i, int j) {
        SWP(q[i], q[j]), SWP(loc[q[i]], loc[q[j]]);
    }
    void swim(int i) {
        while (i > 0) {
            int p = (i - 1) / 2;
            if (!cmp(i, p)) break;
            swp(i, p), i = p;
        }
    }
    void sink(int i) {
        while (true) {
            int l = 2*i + 1, r = l + 1;
            if (l >= len) break;
            int m = r >= len || cmp(l, r) ? l : r;
            if (!cmp(m, i)) break;
            swp(m, i), i = m;
        }
    }
    heap(int C=128) : len(0), cap(C), _cmp(Compare())
    ↪ {
        q = new int[C]; loc = new int[C];
        memset(loc, 255, cap << 2);
    }
    ~heap() {
        delete[] q; delete[] loc;
    }
    void push(int n, bool fix = true) {
        if (cap == len || n >= cap) {
            #ifdef RESIZE
            int newcap = 2 * cap;
            while (n >= newcap) newcap *= 2;
            int *newq = new int[newcap], *newloc = new
            ↪ int[newcap];
            REP(i, cap) newq[i] = q[i], newloc[i] = loc[i];
            memset(newloc+cap, 255, (newcap-cap) << 2);
            delete[] q, delete[] loc;

```

```

        loc = newloc, q = newq, cap = newcap;
        #else
        assert(false);
        #endif
    }
    assert(loc[n] == -1);
    loc[n] = len, q[len++] = n;
    if (fix) swim(len-1);
}
void pop(bool fix = true) {
    assert(len > 0);
    loc[q[0]] = -1, q[0] = q[--len], loc[q[0]] = 0;
    if (fix) sink(0);
}
int top() { assert(len > 0); return q[0]; }
void heapify() {
    for (int i = len - 1; i > 0; i--)
        if (cmp(i, (i-1)/2)) swp(i, (i-1)/2);
}
void update_key(int n) {
    assert(loc[n] != -1); swim(loc[n]); sink(loc[n]);
}
bool empty() { return len == 0; }
int size() { return len; }
void clear() {
    len = 0; memset(loc, 255, cap << 2);
}
};

```

2.7. **Dancing Links.** An implementation of Donald Knuth's Dancing Links data structure. A linked list supporting deletion and restoration of elements.

```

template <class T>
struct dancing_links {
    struct node {
        T item;
        node *l, *r;
        node(const T &_item, node *_l=NULL, node *_r=NULL)
        : item(_item), l(_l), r(_r) {
            if (l) l->r = this;
            if (r) r->l = this; } };
    node *front, *back;
    dancing_links() { front = back = NULL; }
    node *push_back(const T &item) {
        back = new node(item, back, NULL);
        if (!front) front = back;
        return back; }
    node *push_front(const T &item) {
        front = new node(item, NULL, front);
        if (!back) back = front;
        return front; }
    void erase(node *n) {
        if (!n->l) front = n->r; else n->l->r = n->r;
        if (!n->r) back = n->l; else n->r->l = n->l; }
    void restore(node *n) {
        if (!n->l) front = n; else n->l->r = n;
        if (!n->r) back = n; else n->r->l = n; } };

```

2.8. **Misof Tree.** A simple tree data structure for inserting, erasing, and querying the  $n$ th largest element.

```

const int BITS = 15;
struct misof_tree {
    int cnt[BITS][1<<BITS];
    misof_tree() { memset(cnt, 0, sizeof(cnt)); }
    void insert(int x) {
        for (int i=0; i<BITS; cnt[i++][x]++, x >>= 1); }
    void erase(int x) {
        for (int i=0; i<BITS; cnt[i++][x]--, x >>= 1); }
    int nth(int n) {
        int res = 0;
        for (int i = BITS-1; i >= 0; i--)
            if (cnt[i][res <= 1] <= n)
                n -= cnt[i][res], res |= 1;
        return res;
    }
};

```

2.9. **k-d Tree.** A  $k$ -dimensional tree supporting fast construction, adding points, and nearest neighbor queries. NOTE: Not completely stable, occasionally segfaults.

```

#define INC(c) ((c) == K - 1 ? 0 : (c) + 1)
template <int K> struct kd_tree {
    struct pt {
        double coord[K];
        pt() {}
        pt(double c[K]) { REP(i,K) coord[i] = c[i]; }
        double dist(const pt &other) const {
            double sum = 0.0;
            REP(i,K) sum +=
            ↪ pow(coord[i]-other.coord[i], 2);
            return sqrt(sum); } };
    struct cmp {
        int c;
        cmp(int _c) : c(_c) {}
        bool operator () (const pt &a, const pt &b) {
            for (int i = 0, cc; i <= K; i++) {
                cc = i == 0 ? c : i - 1;
                if (abs(a.coord[cc] - b.coord[cc]) > EPS)
                    return a.coord[cc] < b.coord[cc];
            }
            return false; } };
    struct bb {
        pt from, to;
        bb(pt _from, pt _to) : from(_from), to(_to) {}
        double dist(const pt &p) {
            double sum = 0.0;
            REP(i,K) {
                if (p.coord[i] < from.coord[i])
                    sum += pow(from.coord[i] - p.coord[i],
                    ↪ 2.0);
                else if (p.coord[i] > to.coord[i])
                    sum += pow(p.coord[i] - to.coord[i], 2.0);
            }
        }
    };
};

```



```

    return sqrt(sum); }
    bb bound(double l, int c, bool left) {
        pt nf(from.coord), nt(to.coord);
        if (left) nt.coord[c] = min(nt.coord[c], l);
        else nf.coord[c] = max(nf.coord[c], l);
        return bb(nf, nt); } };
struct node {
    pt p; node *l, *r;
    node(pt _p, node *_l, node *_r)
        : p(_p), l(_l), r(_r) { } };
node *root;

// kd_tree() : root(NULL) { }
kd_tree(vector<pt> pts) {
    root = construct(pts, 0, size(pts) - 1, 0); }
node* construct(vector<pt> &pts, int fr, int to,
    ↪ int c) {
    if (fr > to) return NULL;
    int mid = fr + (to-fr) / 2;
    nth_element(pts.begin() + fr, pts.begin() + mid,
        pts.begin() + to + 1, cmp(c));
    return new node(pts[mid],
        construct(pts, fr, mid - 1, INC(c)),
        construct(pts, mid + 1, to, INC(c))); }
bool contains(const pt &p) { return
    ↪ _con(p, root, 0); }
bool _con(const pt &p, node *n, int c) {
    if (!n) return false;
    if (cmp(c)(p, n->p)) return _con(p, n->l, INC(c));
    if (cmp(c)(n->p, p)) return _con(p, n->r, INC(c));
    return true; }
void insert(const pt &p) { _ins(p, root, 0); }
void _ins(const pt &p, node *n, int c) {
    if (!n) n = new node(p, NULL, NULL);
    else if (cmp(c)(p, n->p)) _ins(p, n->l, INC(c));
    else if (cmp(c)(n->p, p)) _ins(p, n->r, INC(c));
}
void clear() { _clr(root); root = NULL; }
void _clr(node *n) {
    if (n) _clr(n->l), _clr(n->r), delete n; }
pt nearest_neighbour(const pt &p, bool same=true)
    ↪ {
    assert(root);
    double mn = INFINITY, cs[K];
    REP(i, K) cs[i] = -INFINITY;
    pt from(cs);
    REP(i, K) cs[i] = INFINITY;
    pt to(cs);
    return _nn(p, root, bb(from, to), mn, 0,
        ↪ same).x;
}
pair<pt, bool> _nn(const pt &p, node *n, bb b,
    double &mn, int c, bool same) {
    if (!n || b.dist(p) > mn)
        return make_pair(pt(), false);
    bool found = same || p.dist(n->p) > EPS,
        l1 = true, l2 = false;

```

```

    pt resp = n->p;
    if (found) mn = min(mn, p.dist(resp));
    node *n1 = n->l, *n2 = n->r;
    REP(i, 2) {
        if (i == 1 || cmp(c)(n->p, p))
            swap(n1, n2), swap(l1, l2);
        auto res = _nn(p, n1, b.bound(n->p.coord[c],
            ↪ c, l1), mn, INC(c), same);
        if (res.y && (!found || p.dist(res.x) <
            ↪ p.dist(resp)))
            resp = res.x, found = true;
    }
    return make_pair(resp, found); } };

```

2.10. **Sqrt Decomposition.** Design principle that supports many operations in amortized  $\sqrt{n}$  per operation.

```

struct segment {
    vi arr;
    segment(vi _arr) : arr(_arr) { } };
vector<segment> T;
int K;
void rebuild() {
    int cnt = 0;
    rep(i, 0, size(T))
        cnt += size(T[i].arr);
    K = static_cast<int>(ceil(sqrt(cnt)) + 1e-9);
    vi arr(cnt);
    for (int i = 0, at = 0; i < size(T); i++)
        rep(j, 0, size(T[i].arr))
            arr[at++] = T[i].arr[j];
    T.clear();
    for (int i = 0; i < cnt; i += K)
        T.push_back(segment(vi(arr.begin() + i,
            ↪ arr.begin() + min(i + K, cnt)))); }
int split(int at) {
    int i = 0;
    while (i < size(T) && at >= size(T[i].arr))
        at -= size(T[i].arr), i++;
    if (i >= size(T)) return size(T);
    if (at == 0) return i;
    T.insert(T.begin() + i + 1,
        segment(vi(T[i].arr.begin() + at,
            ↪ T[i].arr.end())));
    T[i] = segment(vi(T[i].arr.begin(),
        ↪ T[i].arr.begin() + at));
    return i + 1; }
void insert(int at, int v) {
    vi arr; arr.push_back(v);
    T.insert(T.begin() + split(at), segment(arr)); }
void erase(int at) {
    int i = split(at); split(at + 1);
    T.erase(T.begin() + i); }

```

2.11. **Monotonic Queue.** A queue that supports querying for the minimum element. Useful for sliding window algorithms.

```

struct min_stack {
    stack<int> S, M;
    void push(int x) {
        S.push(x);
        M.push(M.empty() ? x : min(M.top(), x)); }
    int top() { return S.top(); }
    int mn() { return M.top(); }
    void pop() { S.pop(); M.pop(); }
    bool empty() { return S.empty(); } };
struct min_queue {
    min_stack inp, outp;
    void push(int x) { inp.push(x); }
    void fix() {
        if (outp.empty()) while (!inp.empty())
            outp.push(inp.top()), inp.pop(); }
    int top() { fix(); return outp.top(); }
    int mn() {
        if (inp.empty()) return outp.mn();
        if (outp.empty()) return inp.mn();
        return min(inp.mn(), outp.mn()); }
    void pop() { fix(); outp.pop(); }
    bool empty() { return inp.empty() && outp.empty(); }
};

```

2.12. **Line container à la ‘Convex Hull Trick’**  $\mathcal{O}(n \log n)$ . Container where you can add lines of the form  $y_i(x) = k_i x + m_i$  and query  $\max_i y_i(x)$ .

```

bool Q;
struct Line {
    mutable ll k, m, p;
    bool operator<(const Line& o) const {
        return Q ? p < o.p : k < o.k;
    }
};
struct LineContainer : multiset<Line> {
    // (for doubles, use inf = 1/.0, div(a,b) = a/b)
    const ll inf = LLONG_MAX;
    ll div(ll a, ll b) { // floored division
        return a / b - ((a ^ b) < 0 && a % b); }
    bool isect(iterator x, iterator y) {
        if (y == end()) { x->p = inf; return false; }
        if (x->k == y->k)
            x->p = x->m > y->m ? inf : -inf;
        else
            x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    }
    void add(ll k, ll m) {
        auto z = insert({k, m, 0}), y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y))
            isect(x, y = erase(y));
        while ((y = x) != begin() && (--x)->p >= y->p)
            isect(x, erase(y));
    }
    ll query(ll x) {
        assert(!empty());
    }
};

```

```

    Q=1; auto l = *lower_bound({0,0,x}); Q=0;
    return l.k * x + l.m;
}
};

```

### 2.13. Sparse Table $O(\log n)$ per query.

```

struct sparse_table {
    vvi m;
    sparse_table(vi arr) {
        m.pb(arr);
        for (int k=0; (1<<(+k)) <= sz(arr); ) {
            int w = (1<<k), hw = w/2;
            m.pb(vi(sz(arr) - w + 1);
            for (int i = 0; i+w <= sz(arr); i++) {
                m[k][i] = min(m[k-1][i], m[k-1][i+hw]);
            }
        }
        int query(int l, int r) { // query min in [l,r]
            int k = 31 - __builtin_clz(r-l); // k = 0;
            // while (1<<(k+1) <= r-l+1) k++;
            return min(m[k][l], m[k][r-(1<<k)+1]);
        }
    }
};

```

## 3. GRAPH ALGORITHMS

### 3.1. Shortest path.

#### 3.1.1. Dijkstra $O(|E| \log |V|)$ .

```

const ll INFTY = -1;
vi dijkstra( vector<vii> G, ll s ) {
    vi d( G.size(), INFTY );
    priority_queue<ii,vector<ii>,greater<ii>> Q;
    Q.emplace(0,s);
    while(!Q.empty()){
        ll c = Q.top().x, a = Q.top().y;
        Q.pop();
        if(d[a] != INFTY) continue;
        d[a] = c;
        for(ii e : G[a])
            Q.emplace(d[a] + e.y, e.x);
    }
    return d;
}

```

3.1.2. *Floyd-Warshall*  $O(V^3)$ . Be careful with negative edges! Note:  $|d[i][j]|$  can grow exponentially, and  $INFTY + \text{negative} < INFTY$ .

```

const ll INF = 1LL << 61;
void floyd_warshall( vvi& d ) {
    ll n = d.size();
    REP(i,n) REP(j,n) REP(k,n)
        if(d[j][i] < INF && d[i][k] < INF) // neg edges!
            d[j][k] = max(-INF,
                min(d[j][k], d[j][i] + d[i][k]));
}

```

```

vvi d(n,vi(n,INF));
REP(i,n) d[i][i] = 0;

```

3.1.3. *Bellman Ford*  $O(VE)$ . This is only useful if there are edges with weight  $w_{ij} < 0$  in the graph.

```

const ll INF = 1LL << 61;
// G[u] = { (v,w) | edge u->v, cost w }
vi bellman_ford(vector<vii> G, ll s) {
    ll n = G.size();
    vi d(n, INF); d[s] = 0;
    REP(loops, n) REP(u, n) if(d[u] != INF)
        for(ii e : G[u]) if(d[u] + e.y < d[e.x])
            d[e.x] = d[u] + e.y;
    // detect paths of -INF length
    for( ll change = 1; change--;)
        REP(u, n) if(d[u] != INF)
            for(ii e : G[u]) if(d[u] + e.y < d[e.x])
                d[e.x] = -INF, change = 1;
    return d;
}

```

#### 3.1.4. IDA\* algorithm.

```

int n, cur[100], pos;
int calch() {
    int h = 0;
    rep(i,0,n) if (cur[i] != 0) h += abs(i - cur[i]);
    return h;
}
int dfs(int d, int g, int prev) {
    int h = calch();
    if (g + h > d) return g + h;
    if (h == 0) return 0;
    int mn = INT_MAX;
    rep(di,-2,3) {
        if (di == 0) continue;
        int nxt = pos + di;
        if (nxt == prev) continue;
        if (0 <= nxt && nxt < n) {
            swap(cur[pos], cur[nxt]);
            swap(pos,nxt);
            mn = min(mn, dfs(d, g+1, nxt));
            swap(pos,nxt);
            swap(cur[pos], cur[nxt]);
        }
        if (mn == 0) break;
    }
    return mn;
}
int idastar() {
    rep(i,0,n) if (cur[i] == 0) pos = i;
    int d = calch();
    while (true) {
        int nd = dfs(d, 0, -1);
        if (nd == 0 || nd == INT_MAX) return d;
        d = nd;
    }
}

```

### 3.2. Maximum Matching.

**Matching:** A set of edges without common vertices (*Maximum is the largest such set, maximal is a set which*

*you cannot add more edges to without breaking the property*).

**Minimum Vertex Cover:** A set of vertices such that each edge in the graph is incident to at least one vertex of the set.

**Minimum Edge Cover:** A set of edges such that every vertex is incident to at least one edge of the set.

**Maximum Independent Set:** A set of vertices in a graph such that no two of them are adjacent.

Minimum edge cover  $\iff$  Maximum independent set.

**König's theorem:** In any bipartite graph  $G = (L \cup R, E)$ , the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover. Let  $U$  be the set of unmatched vertices in  $L$ , and  $Z$  be the set of vertices that are either in  $U$  or are connected to  $U$  by an alternating path. Then  $K = (L \setminus Z) \cup (R \cap Z)$  is the minimum vertex cover.

In any bipartite graph,

$$\text{maxmatch} = \text{MVC} = V - \text{MIS}.$$

See 3.2.3.

#### 3.2.1. Standard bipartite matching $O(nm)$ .

```
const int sizeL = 1e4, sizeR = 1e4;
```

```

bool vis[sizeR];
int par[sizeR]; // par : R -> L
vi adj[sizeL]; // adj : L -> (N -> R)

bool match(int u) {
    for (int v : adj[u]) {
        if (vis[v]) continue; vis[v] = true;
        if (par[v] == -1 || match(par[v])) {
            par[v] = u;
            return true;
        }
    }
    return false;
}

```

```

// perfect matching iff ret == sizeL == sizeR
int maxmatch() {
    fill_n(par, sizeR, -1); int ret = 0;
    for (int i = 0; i < sizeL; i++) {
        fill_n(vis, sizeR, false);
        ret += match(i);
    }
    return ret;
}

```



3.2.2. Hopcroft-Karp bipartite matching  $\mathcal{O}(E\sqrt{V})$ .

```
const ll INFTY = (1LL<<61LL);

struct bi_graph {
    ll n, m;
    vvi adj;
    vi L, R, d;
    queue<ll> q;
    bi_graph( ll _n, ll _m ) : n(_n), m(_m),
        adj(n), L(n,-1), R(m,n), d(n+1) {}
    ll add_edge( ll a, ll b ) { adj[a].pb(b); }
    ll bfs() {
        rep(v,0,n)
            if( L[v] == -1 ) d[v] = 0, q.push(v);
        else d[v] = INFTY;
        d[n] = INFTY;
        while( !q.empty() ) {
            ll v = q.front(); q.pop();
            if( d[v] < d[n] )
                for( ll u : adj[v] ) if( d[R[u]] == INFTY )
                    d[R[u]] = d[v]+1, q.push(R[u]);
        }
        return d[n] != INFTY;
    }
    ll dfs( ll v ) {
        if( v == n ) return true;
        for( ll u : adj[v] )
            if( d[R[u]] == d[v] + 1 and dfs(R[u]) ) {
                R[u] = v; L[v] = u;
                return true;
            }
        d[v] = INFTY;
        return false;
    }
    ll maximum_matching() {
        ll s = 0;
        while( bfs() ) rep(i,0,n)
            s += L[i] == -1 && dfs( i );
        return s;
    }
};
```

## 3.2.3. Minimum Vertex Cover in Bipartite Graphs.

```
#include "hopcroft_karp.cpp"
vi alt;
void dfs( bi_graph &G, ll v ) {
    alt[v] = 1;
    for( ll u : G.adj[v] ) {
        alt[u+G.n] = 1;
        if( G.R[u] != G.n && !alt[G.R[u]] )
            dfs(G,G.R[u]);
    }
}
vi mvc_bipartite( bi_graph &G ) {
    vi res; G.maximum_matching();
    alt.assign( G.n + G.m, 0 );
    rep(i,0,G.n) if( G.L[i] == -1 ) dfs(G,i);
    rep(i,0,G.n) if( !alt[i] ) res.pb(i);
    rep(i,0,G.n) if( alt[G.n+i] ) res.pb(G.n+i);
}
```

```
return res;
}
```

3.2.4. *Stable marriage*. With  $n$  men,  $m \geq n$  women,  $n$  preference lists of women for each men, and for every woman  $j$  a preference of men defined by  $\text{pref}[][j]$  (lower is better) find for every man a women such that no pair of a men and a woman want to run off together.

```
// n = aantal mannen, m = aantal vrouwen
// voor een man i, is order[i] de prefere
vi stable(int n, int m, vvi order, vvi pref) {
    queue<int> q;
    REP(i, n) q.push(i);
    vi mas(m,-1), mak(n,-1), p(n,0);
    while (!q.empty()) {
        int k = q.front();
        q.pop();
        int s = order[k][p[k]], k2 = mas[s];
        if (mas[s] == -1) {
            mas[s] = k;
            mak[k] = s;
        } else if (pref[k][s] < pref[k2][s]) {
            mas[s] = k;
            mak[k] = s;
            mak[k2] = -1;
            q.push(k2);
        } else {
            q.push(k);
        }
        p[k]++;
    }
    return mak;
}
```

3.3. Cycle Detection  $\mathcal{O}(V + E)$ .

vvi adj; // assumes a bidirected graph

```
bool cycle_detection() {
    stack<int> s; vector<bool> vis(MAXN, false);
    vi par(MAXN, -1); s.push(0);
    vis[0] = true;
    while (!s.empty()) {
        int cur = s.top(); s.pop();
        for (int i : adj[cur]) {
            if (vis[i] && par[cur] != i) return true;
            s.push(i); par[i] = cur; vis[i] = true;
        }
    }
    return false;
}
```

## 3.4. Depth first searches.

3.4.1. Topological Sort  $\mathcal{O}(V + E)$ .

```
vi topo(vvi &adj) { // requires C++14
    int n=sz(adj); vector<bool> vis(n,0); vi ans;
    auto dfs = [&](int v, const auto& f)->void {
        vis[v] = true;
```

```
for (int w : adj[v]) if (!vis[w]) f(w, f);
    ans.pb(v);
};
REP(i, n) if (!vis[i]) dfs(i, dfs);
reverse(all(ans));
return ans;
}
```

3.4.2. Cut Points and Bridges  $\mathcal{O}(V + E)$ .

```
const int MAXN = 5000;
int low[MAXN], num[MAXN], curnum;

void dfs(const vvi &adj, vi &cp, vii &bri, int u, int
    p) {
    low[u] = num[u] = curnum++;
    int cnt = 0; bool found = false;
    REP(i, sz(adj[u])) {
        int v = adj[u][i];
        if (num[v] == -1) {
            dfs(adj, cp, bri, v, u);
            low[u] = min(low[u], low[v]);
            cnt++;
            found = found || low[v] >= num[u];
            if (low[v] > num[u]) bri.pb(u, v);
        } else if (p != v) low[u] = min(low[u], num[v]);
    }
    if (found && (p != -1 || cnt > 1)) cp.pb(u);
}
```

```
pair<vi,vii> cut_points_and_bridges(const vvi &adj)
    {
        int n = size(adj);
        vi cp; vii bri;
        memset(num, -1, n << 2);
        curnum = 0;
        REP(i,n) if(num[i] == -1) dfs(adj, cp, bri, i,
            -1);
        return make_pair(cp, bri);
    }
```

3.4.3. Strongly Connected Components  $\mathcal{O}(V + E)$ .

```
vvi adj, comps;
vi tidx, lnk, cnr, st;
vector<bool> vis;
int age, ncomps;

void tarjan(int v) {
    tidx[v] = lnk[v] = ++age; vis[v] = true; st.pb(v);
    for (int w : adj[v]) {
        if (!tidx[w])
            tarjan(w), lnk[v] = min(lnk[v], lnk[w]);
        else if (vis[w]) lnk[v] = min(lnk[v], tidx[w]);
    }
    if (lnk[v] != tidx[v]) return;
    comps.pb(vi());
    int w;
```

```
do {
    vis[w = st.back()] = false; cnr[w] = ncomps;
    comps.back().pb(w);
    st.pop_back();
} while (w != v);
ncomps++;
}
```

```
void findSCC(int n) {
    age = ncomps = 0;
    vis.assign(n, false);
    tidx.assign(n, 0);
    lnk.resize(n); cnr.resize(n); comps.clear();
    for (int i = 0; i < n; i++)
        if (tidx[i] == 0) tarjan(i);
}
```

3.4.4. 2-SAT  $\mathcal{O}(V + E)$ . Include findSCC.

```
void init2sat(int n) { adj.assign(2 * n, vi()); }
```

```
// (var xl = vl) ==> (var xr = vr)
void imply(int xl, bool vl, int xr, bool vr) {
    adj[2 * xl + vl].pb(2 * xr + vr);
    adj[2 * xr + !vr].pb(2 * xl + !vl);
}
```

```
void satOr(int xl, bool vl, int xr, bool vr) {
    imply(xl, !vl, xr, vr);
}
```

```
void satConst(int x, bool v) { imply(x, !v, x, v); }
void satIff(int xl, bool vl, int xr, bool vr) {
    imply(xl, vl, xr, vr); imply(xr, vr, xl, vl);
}
```

```
bool solve2sat(int n, vector<bool> &sol) {
    findSCC(2 * n);
    for (int i = 0; i < n; i++)
        if (cnr[2 * i] == cnr[2 * i + 1]) return false;
    vector<bool> seen(n, false); sol.assign(n, false);
    for (vi &comp : comps) {
        for (int v : comp) {
            if (seen[v / 2]) continue;
            seen[v / 2] = true;
            sol[v / 2] = v & 1;
        }
    }
    return true;
}
```

3.4.5. Dominator graph.

```
const int N = 1234567;
```

```
vi g[N], g_rev[N], bucket[N];
int pos[N], cnt, order[N], parent[N], sdom[N], p[N],
    ↪ best[N], idom[N], link[N];
```

```
void dfs(int v) {
    pos[v] = cnt;
    order[cnt++] = v;
```

```
for (int u : g[v]) {
    if (pos[u] == -1) {
        parent[u] = v;
        dfs(u);
    }
}
```

```
int find_best(int x) {
    if (p[x] == x) return best[x];
    int u = find_best(p[x]);
    if (pos[sdom[u]] < pos[sdom[best[x]]])
        best[x] = u;
    p[x] = p[p[x]];
    return best[x];
}
```

```
void dominators(int n, int root) {
    fill_n(pos, n, -1);
    cnt = 0;
    dfs(root);
    for (int i = 0; i < n; i++)
        for (int u : g[i]) g_rev[u].push_back(i);
    for (int i = 0; i < n; i++)
        p[i] = best[i] = sdom[i] = i;
    for (int it = cnt - 1; it >= 1; it--) {
        int w = order[it];
        for (int u : g_rev[w]) {
            int t = find_best(u);
            if (pos[sdom[t]] < pos[sdom[w]])
                sdom[w] = sdom[t];
        }
        bucket[sdom[w]].push_back(w);
        idom[w] = sdom[w];
        for (int u : bucket[parent[w]])
            link[u] = find_best(u);
        bucket[parent[w]].clear();
        p[w] = parent[w];
    }
    for (int it = 1; it < cnt; it++) {
        int w = order[it];
        idom[w] = idom[link[w]];
    }
}
```

3.5. Min Cut / Max Flow.

3.5.1. Dinic's Algorithm  $\mathcal{O}(V^2E)$ .

```
struct Edge { int t; ll c, f; };
struct Dinic {
    vi H, P; vvi E;
    vector<Edge> G;
    Dinic(int n) : H(n), P(n), E(n) {}
```

```
void addEdge(int u, int v, ll c) {
    E[u].pb(G.size()); G.pb({v, c, 0LL});
    E[v].pb(G.size()); G.pb({u, 0LL, 0LL});
}
```

```
ll dfs(int t, int v, ll f) {
    if (v == t || !f) return f;
    for ( ; P[v] < (int) E[v].size(); P[v]++) {
        int e = E[v][P[v]], w = G[e].t;
        if (H[w] != H[v] + 1) continue;
        ll df = dfs(t, w, min(f, G[e].c - G[e].f));
        if (df > 0) {
            G[e].f += df, G[e ^ 1].f -= df;
            return df;
        }
    }
    return 0;
}
ll maxflow(int s, int t, ll f = 0) {
    while (1) {
        fill(all(H), 0); H[s] = 1;
        queue<int> q; q.push(s);
        while (!q.empty()) {
            int v = q.front(); q.pop();
            for (int w : E[v])
                if (G[w].f < G[w].c && !H[G[w].t])
                    H[G[w].t] = H[v] + 1, q.push(G[w].t);
        }
        if (!H[t]) return f;
        fill(all(P), 0);
        while (ll df = dfs(t, s, LLONG_MAX)) f += df;
    }
};
```

3.5.2. Min-cost max-flow  $\mathcal{O}(n^2m^2)$ . Find the cheapest possible way of sending a certain amount of flow through a flow network.

```
const int maxn = 300;
```

```
struct edge { ll x, y, f, c, w; };
ll V, par[maxn], D[maxn]; vector<edge> g;
inline void addEdge(int u, int v, ll c, ll w) {
    g.pb({u, v, 0, c, w});
    g.pb({v, u, 0, 0, -w});
}
```

```
void sp(int s, int t) {
    fill_n(D, V, LLONG_MAX); D[s] = 0;
    for (int ng = g.size(), _ = V; _--;) {
        bool ok = false;
        for (int i = 0; i < ng; i++)
            if (D[g[i].x] != LLONG_MAX && g[i].f < g[i].c
                ↪ && D[g[i].x] + g[i].w < D[g[i].y]) {
                D[g[i].y] = D[g[i].x] + g[i].w;
                par[g[i].y] = i; ok = true;
            }
        if (!ok) break;
    }
}
```

```
void minCostMaxFlow(int s, int t, ll &c, ll &f) {
    for (c = f = 0; sp(s, t), D[t] < LLONG_MAX; ) {
```

```

ll df = LLONG_MAX, dc = 0;
for (int v = t, e; e = par[v], v != s; v =
    ↪ g[e].x) df = min(df, g[e].c - g[e].f);
for (int v = t, e; e = par[v], v != s; v =
    ↪ g[e].x) g[e].f += df, g[e^1].f -= df, dc +=
    ↪ g[e].w;
f += df; c += dc * df;
}
}

```

3.5.3. *Gomory-Hu Tree - All Pairs Maximum Flow.* An implementation of the Gomory-Hu Tree. The spanning tree is constructed using Gusfield's algorithm in  $O(|V|^2)$  plus  $|V|-1$  times the time it takes to calculate the maximum flow. If Dinic's algorithm is used to calculate the max flow, the running time is  $O(|V|^3|E|)$ . NOTE: Not sure if it works correctly with disconnected graphs.

```

#include "dinic.cpp"
bool same[MAXV];
pair<vii, vvi> construct_gh_tree(flow_network &g) {
    int n = g.n, v;
    vii par(n, ii(0, 0)); vvi cap(n, vi(n, -1));
    rep(s, 1, n) {
        int l = 0, r = 0;
        par[s].second = g.max_flow(s, par[s].first,
            ↪ false);
        memset(d, 0, n * sizeof(int));
        memset(same, 0, n * sizeof(bool));
        d[q[r++] = s] = 1;
        while (l < r) {
            same[v = q[l++]] = true;
            for (int i = g.head[v]; i != -1; i =
                ↪ g.e[i].nxt)
                if (g.e[i].cap > 0 && d[g.e[i].v] == 0)
                    d[q[r++] = g.e[i].v] = 1;
            rep(i, s+1, n)
                if (par[i].first == par[s].first && same[i])
                    par[i].first = s;
            g.reset();
        }
        rep(i, 0, n) {
            int mn = INT_MAX, cur = i;
            while (true) {
                cap[cur][i] = mn;
                if (cur == 0) break;
                mn = min(mn, par[cur].second), cur =
                    ↪ par[cur].first;
            }
            return make_pair(par, cap);
        }
    }
    int compute_max_flow(int s, int t, const pair<vii,
        ↪ vvi> &gh) {
        int cur = INT_MAX, at = s;
        while (gh.second[at][t] == -1)
            cur = min(cur, gh.first[at].second),
            at = gh.first[at].first;
        return min(cur, gh.second[at][t]);
    }
}

```

3.6. Minimal Spanning Tree  $O(E \log V)$ .

```

struct edge { int x, y; ll w; };
ll kruskal(int n, vector<edge> edges) {
    dsu D(n);
    sort(all(edges), [] (edge a, edge b) -> bool {
        return a.w < b.w; });
    ll ret = 0;
    for (edge e : edges)
        if (D.find(e.x) != D.find(e.y))
            ret += e.w, D.unite(e.x, e.y);
    return ret;
}

```

3.7. **Euler Path  $O(V + E)$  hopefully.** Finds an Euler Path (or circuit) in a *directed* graph iff one exists.

```

const int MAXV = 1000, MAXE = 5000;
vi adj[MAXV];
int n, m, indeg[MAXV], outdeg[MAXV], res[MAXE + 1];
ii start_end() {
    int start = -1, end = -1, any = 0, c = 0;
    REP(i, n) {
        if (outdeg[i] > 0) any = i;
        if (indeg[i] + 1 == outdeg[i]) start = i, c++;
        else if (indeg[i] == outdeg[i] + 1) end = i, c++;
        else if (indeg[i] != outdeg[i]) return ii(-1, -1);
    }
    if ((start == -1) != (end == -1) || (c != 2 && c))
        return ii(-1, -1);
    if (start == -1) start = end = any;
    return ii(start, end);
}
bool euler_path() {
    ii se = start_end();
    int cur = se.first, at = m + 1;
    if (cur == -1) return false;
    stack<int> s;
    while (true) {
        if (outdeg[cur] == 0) {
            res[--at] = cur;
            if (s.empty()) break;
            cur = s.top(); s.pop();
        } else s.push(cur), cur =
            ↪ adj[cur][--outdeg[cur]];
    }
    return at == 0;
}

```

Finds an Euler *cycle* in a *undirected* graph:

```

const int MAXV = 1000;
multiset<int> adj[MAXV];
list<int> L;
list<int>::iterator euler(int at, int to,
    list<int>::iterator it) {
    if (at == to) return it;
    L.insert(it, at), --it;
    while (!adj[at].empty()) {
        int nxt = *adj[at].begin();
        adj[at].erase(adj[at].find(nxt));
        adj[nxt].erase(adj[nxt].find(at));
        if (to == -1) {

```

```

            it = euler(nxt, at, it);
            L.insert(it, at);
            --it;
        } else {
            it = euler(nxt, to, it);
            to = -1;
        }
    }
    return it;
}
// usage: euler(0, -1, L.begin());

```

3.8. **Heavy-Light Decomposition.**

```

#include "../data-structures/segment_tree.cpp"
const int ID = 0;
int f(int a, int b) { return a + b; }
struct HLD {
    int n, curhead, curloc;
    vi sz, head, parent, loc;
    vvi adj; segment_tree values;
    HLD(int _n) : n(_n), sz(n, 1), head(n),
        parent(n, -1), loc(n), adj(n) {
        vector<ll> tmp(n, ID); values =
            ↪ segment_tree(tmp);
    }
    void add_edge(int u, int v) {
        adj[u].push_back(v); adj[v].push_back(u);
    }
    void update_cost(int u, int v, int c) {
        if (parent[v] == u) swap(u, v); assert(parent[u]
            ↪ == v);
        values.update(loc[u], c);
    }
    int csz(int u) {
        rep(i, 0, size(adj[u])) if (adj[u][i] !=
            ↪ parent[u])
            sz[u] += csz(adj[parent[adj[u][i]] = u][i]);
        return sz[u];
    }
    void part(int u) {
        head[u] = curhead; loc[u] = curloc++;
        int best = -1;
        rep(i, 0, size(adj[u]))
            if (adj[u][i] != parent[u] &&
                (best == -1 || sz[adj[u][i]] > sz[best]))
                best = adj[u][i];
        if (best != -1) part(best);
        rep(i, 0, size(adj[u]))
            if (adj[u][i] != parent[u] && adj[u][i] !=
                ↪ best)
                part(curhead = adj[u][i]);
    }
    void build(int r = 0) {
        curloc = 0, csz(curhead = r), part(r);
    }
    int lca(int u, int v) {
        vi uat, vat; int res = -1;
        while (u != -1) uat.push_back(u), u =
            ↪ parent[head[u]];
        while (v != -1) vat.push_back(v), v =
            ↪ parent[head[v]];
        u = size(uat) - 1, v = size(vat) - 1;
        while (u >= 0 && v >= 0 && head[uat[u]] ==
            ↪ head[vat[v]])

```

```

    res = (loc[uat[u]] < loc[vat[v]] ? uat[u] :
    ↪ vat[v]),
    u--, v--;
    return res; }
int query_upto(int u, int v) { int res = ID;
while (head[u] != head[v])
    res = f(res, values.query(loc[head[u]],
    ↪ loc[u]).x),
    u = parent[head[u]];
return f(res, values.query(loc[v] + 1,
    ↪ loc[u]).x); }
int query(int u, int v) { int l = lca(u, v);
return f(query_upto(u, l), query_upto(v, l)); }
↪ };

```

### 3.9. Centroid Decomposition.

```

#define MAXV 100100
#define LGMAXV 20
int jmp[MAXV][LGMAXV],
    path[MAXV][LGMAXV],
    sz[MAXV], seph[MAXV],
    shortest[MAXV];
struct centroid_decomposition {
    int n; vvi adj;
    centroid_decomposition(int _n) : n(_n), adj(n) {}
    void add_edge(int a, int b) {
        adj[a].push_back(b); adj[b].push_back(a); }
    int dfs(int u, int p) {
        sz[u] = 1;
        rep(i, 0, size(adj[u]))
            if (adj[u][i] != p) sz[u] += dfs(adj[u][i],
            ↪ u);
        return sz[u]; }
    void makepaths(int sep, int u, int p, int len) {
        jmp[u][seph[sep]] = sep, path[u][seph[sep]] =
        ↪ len;
        int bad = -1;
        rep(i, 0, size(adj[u])) {
            if (adj[u][i] == p) bad = i;
            else makepaths(sep, adj[u][i], u, len + 1);
        }
        if (p == sep)
            swap(adj[u][bad], adj[u].back()),
            ↪ adj[u].pop_back(); }
    void separate(int h=0, int u=0) {
        dfs(u, -1); int sep = u;
        down: iter(nxt, adj[sep])
            if (sz[*nxt] < sz[sep] && sz[*nxt] > sz[u]/2)
            ↪ {
                sep = *nxt; goto down; }
        seph[sep] = h, makepaths(sep, sep, -1, 0);
        rep(i, 0, size(adj[sep])) separate(h+1,
        ↪ adj[sep][i]); }
    void paint(int u) {
        rep(h, 0, seph[u]+1)
            shortest[jmp[u][h]] = min(shortest[jmp[u][h]],
            ↪ path[u][h]); }

```

```

int closest(int u) {
    int mn = INT_MAX/2;
    rep(h, 0, seph[u]+1)
        mn = min(mn, path[u][h] +
        ↪ shortest[jmp[u][h]]);
    return mn; } };

```

### 3.10. Least Common Ancestors, Binary Jumping.

```

const int LOGSZ = 20, SZ = 1 << LOGSZ;
int P[SZ], BP[SZ][LOGSZ];

void initLCA() { // assert P[root] == root
    rep(i, 0, SZ) BP[i][0] = P[i];
    rep(j, 1, LOGSZ) rep(i, 0, SZ)
        BP[i][j] = BP[BP[i][j-1]][j-1];
}

int LCA(int a, int b) {
    if (H[a] > H[b]) swap(a, b);
    int dh = H[b] - H[a], j = 0;
    rep(i, 0, LOGSZ) if (dh & (1 << i)) b = BP[b][i];
    while (BP[a][j] != BP[b][j]) j++;
    while (--j >= 0) if (BP[a][j] != BP[b][j])
        a = BP[a][j], b = BP[b][j];
    return a == b ? a : P[a];
}

```

### 3.11. Miscellaneous.

3.11.1. *Misra-Gries  $D+1$ -edge coloring.* Finds a  $\max_i \deg(i) + 1$ -edge coloring where there all incident edges have distinct colors. Finding a  $D$ -edge coloring is NP-hard.

```

struct Edge { int to, col, rev; };

struct MisraGries {
    int N, K=0; vvi F;
    vector<vector<Edge>> G;

    MisraGries(int n) : N(n), G(n) {}
    // add an undirected edge, NO DUPLICATES ALLOWED
    void addEdge(int u, int v) {
        G[u].pb({v, -1, (int) G[v].size()});
        G[v].pb({u, -1, (int) G[u].size()-1});
    }

    void color(int v, int i) {
        vi fan = { i };
        vector<bool> used(G[v].size());
        used[i] = true;
        for (int j = 0; j < (int) G[v].size(); j++)
            if (!used[j] && G[v][j].col >= 0 &&
            ↪ F[G[v][fan.back()]].to[G[v][j].col] < 0)
                used[j] = true, fan.pb(j), j = -1;
        int c = 0; while (F[v][c] >= 0) c++;
        int d = 0; while (F[G[v][fan.back()]].to[d] >=
        ↪ 0) d++;
        int w = v, a = d, k = 0, ccol;

```

```

while (true) {
    swap(F[w][c], F[w][d]);
    if (F[w][c] >= 0) G[w][F[w][c]].col = c;
    if (F[w][d] >= 0) G[w][F[w][d]].col = d;
    if (F[w][a^c^d] < 0) break;
    w = G[w][F[w][a]].to;
}
do {
    Edge &e = G[v][fan[k]];
    ccol = F[e.to][d] < 0 ? d :
    ↪ G[v][fan[k+1]].col;
    if (e.col >= 0) F[e.to][e.col] = -1;
    F[e.to][ccol] = e.rev;
    F[v][ccol] = fan[k];
    e.col = G[e.to][e.rev].col = ccol;
    k++;
} while (ccol != d);
}
// finds a K-edge-coloring
void color() {
    REP(v, N) K = max(K, (int) G[v].size() + 1);
    F = vvi(N, vi(K, -1));
    REP(v, N) for (int i = G[v].size(); i--;)
        if (G[v][i].col < 0) color(v, i);
}
};

```

3.11.2. *Minimum Mean Weight Cycle.* Given a strongly connected directed graph, finds the cycle of minimum mean weight. If you have a graph that is not strongly connected, run this on each strongly connected component.

```

double
↪ min_mean_cycle(vector<vector<pair<int, double>>>
↪ adj) {
    int n = size(adj); double mn = INFINITY;
    vector<vector<double> > arr(n+1, vector<double>(n,
    ↪ mn));
    arr[0][0] = 0;
    rep(k, 1, n+1) rep(j, 0, n) iter(it, adj[j])
        arr[k][it->first] = min(arr[k][it->first],
        ↪ it->second +
        ↪ arr[k-1][j]);

    rep(k, 0, n) {
        double mx = -INFINITY;
        rep(i, 0, n) mx = max(mx,
        ↪ (arr[n][i]-arr[k][i])/(n-k));
        mn = min(mn, mx); }
    return mn; }

```

3.11.3. *Minimum Arborescence.* Given a weighted directed graph, finds a subset of edges of minimum total weight so that there is a unique path from the root  $r$  to each vertex. Returns a vector of size  $n$ , where the  $i$ th element is the edge for the  $i$ th vertex. The answer for the root is undefined!

$\mathcal{O}(EV)$  runtime and  $\mathcal{O}(E)$  memory:

```
#include "../datastructures/union_find.cpp"
struct arborescence {
    int n; union_find uf;
    vector<vector<pair<ii,int>>> adj;
    arborescence(int _n) : n(_n), uf(n), adj(n) {}
    void add_edge(int a, int b, int c) {
        adj[b].eb(ii(a,b),c);
    }
    vii find_min(int r) {
        vi vis(n,-1), mn(n,INT_MAX); vii par(n);
        REP(i, n) {
            if (uf.find(i) != i) continue;
            int at = i;
            while (at != r && vis[at] == -1) {
                vis[at] = i;
                for (auto it : adj[at])
                    if (it.y < mn[at] && uf.find(it.x.x) !=
                        ↪ at)
                        mn[at] = it.y, par[at] = it.x;
                if (par[at] == ii(0,0)) return vii();
                at = uf.find(par[at].x);
            }
            if (at == r || vis[at] != i) continue;
            union_find tmp = uf;
            vi seq;
            do seq.pb(at), at = uf.find(par[at].x);
            while (at != seq.front());
            int c = uf.find(seq[0]);
            for (auto it : seq) uf.unite(it, c);
            for (auto &jt : adj[c]) jt.y -= mn[c];
            for (auto it : seq) {
                if (it == c) continue;
                for (auto jt : adj[it])
                    adj[c].eb(jt.x, jt.y - mn[it]);
                adj[it].clear();
            }
            vii rest = find_min(r);
            if (rest.empty()) return rest;
            ii use = rest[c];
            rest[at = tmp.find(use.y)] = use;
            for (int it : seq) if (it != at)
                rest[it] = par[it];
            return rest;
        }
        return par; } };

O(V2 log V) runtime and O(E) memory:

const int oo = 0x3f3f3f3f, MAXN = 4024;

// N = #V, R = root
int N, R;
// for each node a list of pairs (predecessor,
↪ cost):
vector<pii> g[MAXN];
int pred[MAXN], label[MAXN], node[MAXN],
↪ helper[MAXN];

int get_node(int n) {
    return node[n] == n ? n :
```

```
(node[n] = get_node(node[n]));
}

int update_node(int n) {
    int m = oo;
    for (auto ed : g[n]) m = min(m, ed.y);
    REP(j, sz(g[n])) {
        g[n][j].y -= m;
        if (g[n][j].y == 0)
            pred[n] = g[n][j].x;
    }
    return m;
}

ll cycle(vi &active, int n, int &cend) {
    n = get_node(n);
    if (label[n] == 1) return false;
    if (label[n] == 2) { cend = n; return 0; }

    active.pb(n);
    label[n] = 2;
    auto res = cycle(active, pred[n], cend);
    if (cend == n) {
        int F = find(all(active), n)-active.begin();
        vi todo(active.begin() + F, active.end());
        active.resize(F);
        vii> newg;
        for (auto i: todo) node[i] = n;
        for (auto i: todo) for (auto &ed : g[i])
            helper[ed.x = get_node(ed.x)] = ed.y;
        for (auto i: todo) for (auto &ed : g[i])
            helper[ed.x] = min(ed.y, helper[ed.x]);
        for (auto i: todo) for (auto &ed : g[i]) {
            if (helper[ed.x] != oo && ed.x != n) {
                newg.eb(ed.x, helper[ed.x]);
                helper[ed.x] = oo;
            }
        }
        g[n] = newg;
        res += update_node(n);
        label[n] = 0;
        cend = -1;
        return cycle(active, n, cend) + res;
    }
    if (cend == -1) {
        active.pop_back();
        label[n] = 1;
    }
    return res;
}

// Calculates value of minimal arborescence from R,
// assuming it exists.
// NOTE: N, R must be initialized at this point!!!
// Algo changes g!!
ll min_arbor() {
    ll res = 0;
    REP(i, N) {
```

```
    node[i] = i;
    if (i != R) res += update_node(i);
}
REP(i, N) label[i] = (i==R);
REP(i, N) {
    if (label[i] == 1 || get_node(i) != i)
        continue;
    vi active;
    int cend = -1;
    res += cycle(active, i, cend);
}
return res;
}
```

3.11.4. *Maximum Density Subgraph*. Given (weighted) undirected graph  $G$ . Binary search density. If  $g$  is current density, construct flow network:  $(S, u, m)$ ,  $(u, T, m + 2g - d_u)$ ,  $(u, v, 1)$ , where  $m$  is a large constant (larger than sum of edge weights). Run floating-point max-flow. If minimum cut has empty  $S$ -component, then maximum density is smaller than  $g$ , otherwise it's larger. Distance between valid densities is at least  $1/(n(n-1))$ . Edge case when density is 0. This also works for weighted graphs by replacing  $d_u$  by the weighted degree, and doing more iterations (if weights are not integers).

3.11.5. *Maximum-Weight Closure*. Given a vertex-weighted directed graph  $G$ . Turn the graph into a flow network, adding weight  $\infty$  to each edge. Add vertices  $S, T$ . For each vertex  $v$  of weight  $w$ , add edge  $(S, v, w)$  if  $w \geq 0$ , or edge  $(v, T, -w)$  if  $w < 0$ . Sum of positive weights minus minimum  $S - T$  cut is the answer. Vertices reachable from  $S$  are in the closure. The maximum-weight closure is the same as the complement of the minimum-weight closure on the graph with edges reversed.

3.11.6. *Maximum Weighted Independent Set in a Bipartite Graph*. This is the same as the minimum weighted vertex cover. Solve this by constructing a flow network with edges  $(S, u, w(u))$  for  $u \in L$ ,  $(v, T, w(v))$  for  $v \in R$  and  $(u, v, \infty)$  for  $(u, v) \in E$ . The minimum  $S, T$ -cut is the answer. Vertices adjacent to a cut edge are in the vertex cover.

3.11.7. *Synchronizing word problem*. A DFA has a synchronizing word (an input sequence that moves all states to the same state) iff. each pair of states has a synchronizing word. That can be checked using reverse DFS over pairs of states. Finding the shortest synchronizing word is NP-complete.

#### 4. STRING ALGORITHMS

##### 4.1. Trie.

```

const int SIGMA = 26;

struct trie {
    bool word; trie **adj;

    trie() : word(false), adj(new trie*[SIGMA]) {
        for (int i = 0; i < SIGMA; i++) adj[i] = NULL;
    }

    void addWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) cur->adj[i] = new trie();
            cur = cur->adj[i];
        }
        cur->word = true;
    }

    bool isWord(const string &str) {
        trie *cur = this;
        for (char ch : str) {
            int i = ch - 'a';
            if (!cur->adj[i]) return false;
            cur = cur->adj[i];
        }
        return cur->word;
    }
};

```

#### 4.2. Z-algorithm $\mathcal{O}(n)$ .

//  $z[i]$  = length of longest substring starting from  $\hookrightarrow s[i]$  which is also a prefix of  $s$ .

```

vi z_function(const string &s) {
    int n = (int) s.length();
    vi z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);
        while (i + z[i] < n && s[z[i]] == s[i + z[i]])
             $\hookrightarrow$  ++z[i];
        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    }
    return z;
}

```

#### 4.3. Suffix array $\mathcal{O}(n \log n)$ . Lexicographically sorts the cyclic shifts of $S$ where $p[0]$ is the index of the smallest string, etc.

```

vi sort_cyclic_shifts(const string &s) {
    const int alphabet = 256, n = sz(s);

    vi p(n), c(n), cnt(max(alphabet, n), 0);
    REP(i, n) cnt[s[i]]++;
    partial_sum(all(cnt), cnt.begin());
    REP(i, n) p[--cnt[s[i]]] = i;
    c[p[0]] = 0;
    int cl = 1;

```

```

    rep(i, 1, n) {
        if (s[p[i]] != s[p[i-1]]) cl++;
        c[p[i]] = cl - 1;
    }

    vi pn(n), cn(n);
    for (int h = 0, l = 1; l < n; l*=2, ++h) {
        REP(i, n) {
            pn[i] = p[i] - (1 << h);
            if (pn[i] < 0) pn[i] += n;
        }
        fill(cnt.begin(), cnt.begin() + cl, 0);
        REP(i, n) cnt[c[pn[i]]]++;
        rep(i, 1, cl) cnt[i] += cnt[i-1];
        for (int i = n-1; i >= 0; i--)
            p[--cnt[c[pn[i]]]] = pn[i];
        cn[p[0]] = 0;
        cl = 1;
        rep(i, 1, n) {
            if (c[p[i]] != c[p[i-1]] || c[(p[i]+l)%n]
                != c[(p[i-1]+l)%n]) cl++;
            cn[p[i]] = cl - 1;
        }
        c.swap(cn);
    }
    return p;
}

```

```

vi suffix_array(string s) {
    s += '\0';
    vi v = sort_cyclic_shifts(s);
    v.erase(v.begin());
    return v;
}

```

#### 4.4. Longest Common Subsequence $\mathcal{O}(n^2)$ . SUBSTRING: consecutive characters!!!

```

int dp[STR_SIZE][STR_SIZE]; // DP problem

int lcs(const string &w1, const string &w2) {
    int n1 = w1.size(), n2 = w2.size();
    for (int i = 0; i < n1; i++) {
        for (int j = 0; j < n2; j++) {
            if (i == 0 || j == 0) dp[i][j] = 0;
            else if (w1[i-1] == w2[j-1])
                dp[i][j] = dp[i-1][j-1] + 1;
            else dp[i][j] = max(dp[i-1][j], dp[i][j-1]);
        }
    }
    return dp[n1][n2];
}

```

```

// backtrace
string getLCS(const string &w1, const string &w2) {
    int i = w1.size(), j = w2.size(); string ret = "";
    while (i > 0 && j > 0) {
        if (w1[i-1] == w2[j-1]) ret += w1[--i], j--;
        else if (dp[i][j-1] > dp[i-1][j]) j--;

```

```

        else i--;
    }
    reverse(ret.begin(), ret.end());
    return ret;
}

```

#### 4.5. Levenshtein Distance $\mathcal{O}(n^2)$ . Minimal number of insertions, removals and edits required to transform one string in the other.

```

int dp[MAX_SIZE][MAX_SIZE]; // DP problem

int levDist(const string &w1, const string &w2) {
    int n1 = sz(w1)+1, n2 = sz(w2)+1;
    REP(i, n1) dp[i][0] = i; // removal
    REP(j, n2) dp[0][j] = j; // insertion
    rep(i, 1, n1) rep(j, 1, n2)
        dp[i][j] = min(
            1 + min(dp[i-1][j], dp[i][j-1]),
            dp[i-1][j-1] + (w1[i-1] != w2[j-1])
        );
    return dp[n1][n2];
}

```

#### 4.6. Knuth-Morris-Pratt algorithm $\mathcal{O}(N + M)$ .

```

int kmp(const string &word, const string &text) {
    int n = word.size();
    vi T(n + 1, 0);
    for (int i = 1, j = 0; i < n; ) {
        if (word[i] == word[j]) T[++i] = ++j; // match
        else if (j > 0) j = T[j]; // fallback
        else i++; // no match, keep zero
    }
    int matches = 0;
    for (int i = 0, j = 0; i < text.size(); ) {
        if (text[i] == word[j]) {
            i++;
            if (++j == n) // match at interval [i - n, i)
                matches++, j = T[j];
        } else if (j > 0) j = T[j];
        else i++;
    }
    return matches;
}

```

#### 4.7. Aho-Corasick Algorithm $\mathcal{O}(N + \sum_{i=1}^m |S_i|)$ . Dictionary substring matching as automaton. All given $P$ must be unique!

```

const int MAXP = 100, MAXLEN = 200, SIGMA = 26,
 $\hookrightarrow$  MAXTRIE = MAXP * MAXLEN;

int nP;
string P[MAXP], S;

int pnr[MAXTRIE], to[MAXTRIE][SIGMA],
 $\hookrightarrow$  sLink[MAXTRIE], dLink[MAXTRIE], nnodes;

```



```

void ahoCorasick() {
    fill_n(pnr, MAXTRIE, -1);
    for (int i = 0; i < MAXTRIE; i++) fill_n(to[i],
        ↳ SIGMA, 0);
    fill_n(sLink, MAXTRIE, 0); fill_n(dLink, MAXTRIE,
        ↳ 0);
    nnodes = 1;
    // STEP 1: MAKE A TREE
    for (int i = 0; i < nP; i++) {
        int cur = 0;
        for (char c : P[i]) {
            int i = c - 'a';
            if (to[cur][i] == 0) to[cur][i] = nnodes++;
            cur = to[cur][i];
        }
        pnr[cur] = i;
    }
    // STEP 2: CREATE SUFFIX_LINKS AND DICT_LINKS
    queue<int> q; q.push(0);
    while (!q.empty()) {
        int cur = q.front(); q.pop();
        for (int c = 0; c < SIGMA; c++) {
            if (to[cur][c]) {
                int sl = sLink[to[cur][c]] = cur == 0 ? 0 :
                    ↳ to[sLink[cur]][c];
                // if all strings have equal length, remove
                ↳ this:
                dLink[to[cur][c]] = pnr[sl] >= 0 ? sl :
                    ↳ dLink[sl];
                q.push(to[cur][c]);
            } else to[cur][c] = to[sLink[cur]][c];
        }
    }
    // STEP 3: TRAVERSE S
    for (int cur = 0, i = 0, n = S.size(); i < n; i++)
        ↳ {
            cur = to[cur][S[i] - 'a'];
            for (int hit = pnr[cur] >= 0 ? cur : dLink[cur];
                ↳ hit; hit = dLink[hit]) {
                cerr << P[pnr[hit]] << " found at [" << (i + 1)
                    ↳ << - P[pnr[hit]].size()) << ", " << i << "]"
                    ↳ << endl;
            }
        }
}

```

4.8. **eerTree**. Constructs an eerTree in  $O(n)$ , one character at a time.

```

#define MAXN 100100
#define SIGMA 26
#define BASE 'a'
char *s = new char[MAXN];
struct state {
    int len, link, to[SIGMA];
} *st = new state[MAXN+2];
struct eertree {
    int last, sz, n;

```

```

eertree() : last(1), sz(2), n(0) {
    st[0].len = st[0].link = -1;
    st[1].len = st[1].link = 0; }
int extend() {
    char c = s[n++]; int p = last;
    while (n - st[p].len - 2 < 0 || c != s[n -
        ↳ st[p].len - 2])
        p = st[p].link;
    if (!st[p].to[c-BASE]) {
        int q = last = sz++;
        st[p].to[c-BASE] = q;
        st[q].len = st[p].len + 2;
        do { p = st[p].link;
            } while (p != -1 && (n < st[p].len + 2 ||
                ↳ c != s[n - st[p].len - 2]));
        if (p == -1) st[q].link = 1;
        else st[q].link = st[p].to[c-BASE];
        return 1; }
    last = st[p].to[c-BASE];
    return 0; } }

```

4.9. **Suffix Automaton**. Minimum automata that accepts all suffixes of a string with  $O(n)$  construction. The automata itself is a DAG therefore suitable for DP, examples are counting unique substrings, occurrences of substrings and suffix.

```

// TODO: Add longest common subsring
const int MAXL = 100000;
struct suffix_automaton {
    vi len, link, occur, cnt;
    vector<map<char,int>> next;
    vector<bool> isclone;
    ll *occuratleast;
    int sz, last;
    string s;
    suffix_automaton() : len(MAXL*2), link(MAXL*2),
        occur(MAXL*2), next(MAXL*2), isclone(MAXL*2) {
        clear(); }
    void clear() { sz = 1; last = len[0] = 0; link[0]
        ↳ = -1;
        next[0].clear(); isclone[0] =
            ↳ false; }
    bool issubstr(string other) {
        for (int i = 0, cur = 0; i < size(other); ++i) {
            if (cur == -1) return false; cur =
                ↳ next[cur][other[i]]; }
        return true; }
    void extend(char c) { int cur = sz++; len[cur] =
        ↳ len[last]+1;
        next[cur].clear(); isclone[cur] = false; int p =
            ↳ last;
        for (; p != -1 && !next[p].count(c); p = link[p])
            next[p][c] = cur;
        if (p == -1) { link[cur] = 0; }
        else { int q = next[p][c];
            if (len[p] + 1 == len[q]) { link[cur] = q; }

```

```

        else { int clone = sz++; isclone[clone] =
            ↳ true;
            len[clone] = len[p] + 1;
            link[clone] = link[q]; next[clone] =
                ↳ next[q];
            for (; p != -1 && next[p].count(c) &&
                ↳ next[p][c] == q;
                p = link[p]) {
                next[p][c] = clone; }
            link[q] = link[cur] = clone;
        } } last = cur; }
    void count() {
        cnt=vi(sz, -1); stack<ii> S; S.push(ii(0,0));
        map<char,int>::iterator i;
        while (!S.empty()) {
            ii cur = S.top(); S.pop();
            if (cur.y) {
                for (i = next[cur.x].begin();
                    i != next[cur.x].end(); ++i) {
                    cnt[cur.x] += cnt[*i].y; } }
            else if (cnt[cur.x] == -1) {
                cnt[cur.x] = 1; S.push(ii(cur.x, 1));
                for (i = next[cur.x].begin();
                    i != next[cur.x].end(); ++i) {
                    S.push(ii((*i).y, 0)); } } } }
        string lexicok(ll k) {
            int st=0; string res; map<char,int>::iterator i;
            while (k) {
                for (i = next[st].begin(); i != next[st].end();
                    ↳ ++i) {
                    if (k <= cnt[*i].y) { st = (*i).y;
                        res.push_back((*i).x); k--; break;
                    } else { k -= cnt[*i].y; } } }
            return res; }
    void countoccur() {
        REP(i, sz) occur[i] = 1 - isclone[i];
        vii states(sz);
        REP(i, sz) states[i] = ii(len[i], i);
        sort(states.begin(), states.end());
        for (int i = size(states)-1; i >= 0; --i) {
            int v = states[i].y;
            if (link[v] != -1)
                occur[link[v]] += occur[v]; } } }

```

4.10. **Hashing**. Modulus should be a large prime. Can also use multiple instances with different moduli to minimize chance of collision.

```

struct hasher {
    int b = 311, m; vi h, p;
    hasher(string s, int _m) :
        m(_m), h(sz(s)+1), p(sz(s)+1) {
        p[0] = 1; h[0] = 0;
        REP(i, sz(s)) p[i+1] = (ll)p[i] * b % m;
        REP(i, sz(s)) h[i+1] = ((ll)h[i] * b + s[i]) % m;
    }
    int hash(int l, int r) {
        return (h[r+1] + m - (ll)h[l]*p[r-l+1] % m) % m;
    }

```

```

}
};

```

## 5. GEOMETRY

```

const ld EPS = 1e-7, PI = acos(-1.0);
typedef ld NUM; // EITHER ld OR ll
typedef pair<NUM, NUM> pt;

```

```

pt operator+(pt p, pt q) { return {p.x+q.x, p.y+q.y}; }
pt operator-(pt p, pt q) { return {p.x-q.x, p.y-q.y}; }
pt operator*(pt p, NUM n) { return {p.x*n, p.y*n}; }

```

```

pt& operator+=(pt &p, pt q) { return p = p+q; }
pt& operator-=(pt &p, pt q) { return p = p-q; }

```

```

NUM operator*(pt p, pt q) { return p.x*q.x+p.y*q.y; }
NUM operator^(pt p, pt q) { return p.x*q.y-p.y*q.x; }

```

```

// square distance from p to line ab
ld distPtLineSq(pt p, pt a, pt b) {
    p -= a; b -= a;
    return ld(p^b) * (p^b) / (b*b);
}

```

```

// square distance from p to linesegment ab
ld distPtSegmentSq(pt p, pt a, pt b) {
    p -= a; b -= a;
    NUM dot = p*b, len = b*b;
    if (dot <= 0) return p*p;
    if (dot >= len) return (p-b)*(p-b);
    return p*p - ld(dot)*dot/len;
}

```

```

// Test if p is on line segment ab
bool segmentHasPoint(pt p, pt a, pt b) {
    pt u = p-a, v = p-b;
    return abs(u^v) < EPS && u*v <= 0;
}

```

```

// projects p onto the line ab
pair<ld,ld> proj(pt p, pt a, pt b) {
    p -= a; b -= a;
    return a + b*(ld(b*p) / (b*b));
}

```

```

bool col(pt a, pt b, pt c) {
    return abs((a-b) ^ (a-c)) < EPS;
}

```

```

// true => 1 intersection, false => parallel or same
bool linesIntersect(pt a, pt b, pt c, pt d) {
    return abs((a-b) ^ (c-d)) > EPS;
}

```

```

pair<ld,ld> lineLineIntersection(pt a, pt b, pt c,
    pt d) {
    ld det = (a-b) ^ (c-d);
    assert(abs(det) > EPS);
}

```

```

return ((c-d)*(a^b) - (a-b)*(c^d)) *
    (1/det);
}

```

```

// dp, dq are directions from p, q
// intersection at p + t_i dp, for 0 <= i < return
// value
int segmentIntersection(pt p, pt dp, pt q, pt dq,
    pt &A, pt &B) {
    if (abs(dp * dp) < EPS)
        swap(p, q), swap(dp, dq); // dq=0
    if (abs(dp * dp) < EPS) {
        A = p; // dp = dq = 0
        return p == q;
    }
}

```

```

pt dpq = q-p;
NUM c = dp^dq, c0 = dp^dp, c1 = dp^dq;
if (abs(c) < EPS) { // parallel, dp > 0, dq >= 0
    if (abs(c0) > EPS) return 0; // not collinear
    NUM v0 = dpq*dp, v1 = v0 + dq*dp, dp2 = dp*dp;
    if (v1 < v0) swap(v0, v1);
}

```

```

v0 = max(v0, NUM(0));
v1 = min(v1, dp2);

A = p + dp * (ld(v0) / dp2);
B = p + dp * (ld(v1) / dp2);

return (v0 <= v1) + (v0 < v1);
}

```

```

if (c < 0) {
    c = -c; c0 = -c0; c1 = -c1;
}

```

```

A = p + dp * (ld(c1)/c);
return 0 <= min(c0,c1) && max(c0,c1) <= c;
}

```

```

// Returns TWICE the area of a polygon (for
// integers)

```

```

NUM polygonTwiceArea(const vector<pt> &p) {
    NUM area = 0;
    for (int n = sz(p), i=0, j=n-1; i<n; j = i++)
        area += p[i] ^ p[j];
    return abs(area); // area < 0 <=> p ccw
}

```

```

bool insidePolygon(const vector<pt> &pts, pt p, bool
    strict = true) {
    int n = 0;
    for (int N = sz(pts), i = 0, j = N - 1; i < N; j =
        i++) {
        // if p is on edge of polygon
        if (segmentHasPoint(p, pts[i], pts[j])) return
            !strict;
    }
}

```

```

// or: if(distPtSegmentSq(p, pts[i], pts[j]) <=
    EPS) return !strict;

```

```

// increment n if segment intersects line from p
n += (max(pts[i].y, pts[j].y) > p.y &&
    min(pts[i].y, pts[j].y) <= p.y &&
    ((pts[j] - pts[i])^(p-pts[i])) > 0) ==
    (pts[i].y <= p.y));
}
return n & 1; // inside if odd number of
    intersections
}

```

## 5.1. Convex Hull $\mathcal{O}(n \log n)$ .

```

// the convex hull consists of: { pts[ret[0]],
    pts[ret[1]], ... pts[ret.back()] }
vi convexHull(const vector<pt> &pts) {
    if (pts.empty()) return vi();
    vi ret, ord;
    int n = pts.size(), st = min_element(all(pts)) -
        pts.begin();
    rep(i, 0, n)
        if (pts[i] != pts[st]) ord.pb(i);
    sort(all(ord), [&pts, &st] (int a, int b) {
        pt p = pts[a] - pts[st], q = pts[b] - pts[st];
        return (p ^ q) != 0 ? (p ^ q) > 0 : lenSq(p) <
            lenSq(q);
    });
    ord.pb(st); ret.pb(st);
    for (int i : ord) {
        // use '>' to include ALL points on the
            hull-line
        for (int s = ret.size() - 1; s > 0 &&
            ((pts[ret[s-1]] - pts[ret[s]]) ^ (pts[i] -
                pts[ret[s]])) >= 0; s--)
            ret.pop_back();
        ret.pb(i);
    }
    ret.pop_back();
    return ret;
}

```

## 5.2. Rotating Calipers $\mathcal{O}(n)$ . Finds the longest distance between two points in a convex hull.

```

NUM rotatingCalipers(vector<pt> &hull) {
    int n = hull.size(), a = 0, b = 1;
    if (n <= 1) return 0.0;
    while (((hull[1] - hull[0]) ^ (hull[(b + 1) % n] -
        hull[b])) > 0) b++;
    NUM ret = 0.0;
    while (a < n) {
        ret = max(ret, lenSq(hull[a], hull[b]));
        if (((hull[(a + 1) % n] - hull[a % n]) ^
            (hull[(b + 1) % n] - hull[b])) <= 0) a++;
        else if (++b == n) b = 0;
    }
}

```

```
    return ret;
}
```

### 5.3. Closest points $\mathcal{O}(n \log n)$ .

```
int n; pt pts[maxn];
```

```
struct byY {
    bool operator()(int a, int b) const { return
        ↪ pts[a].y < pts[b].y; }
};
```

```
inline NUM dist(ii p) { return hypot(pts[p.x].x -
    ↪ pts[p.y].x, pts[p.x].y - pts[p.y].y); }
```

```
ii minpt(ii p1, ii p2) { return dist(p1) < dist(p2)
    ↪ ? p1 : p2; }
```

```
// closest pts (by index) inside pts[l ... r], with
    ↪ sorted y values in ys
```

```
ii closest(int l, int r, vi &ys) {
    if (r - l == 2) { // don't assume 1 here.
        ys = { l, l + 1 };
        return ii(l, l + 1);
    } else if (r - l == 3) { // brute-force
        ys = { l, l + 1, l + 2 };
        sort(all(ys), byY());
        return minpt(ii(l, l + 1), minpt(ii(l, l + 2),
            ↪ ii(l + 1, l + 2)));
    }
    int m = (l + r) / 2; vi yl, yr;
    ii delta = minpt(closest(l, m, yl), closest(m, r,
        ↪ yr));
    NUM ddelta = dist(delta), xm = .5 * (pts[m-1].x +
        ↪ pts[m].x);
    merge(all(yl), all(yr), back_inserter(ys), byY());
    deque<int> q;
    for (int i : ys) if (abs(pts[i].x - xm) <= ddelta)
        ↪ {
            for (int j : q) delta = minpt(delta, ii(i, j));
            q.pb(i);
            if (q.size() > 8) q.pop_front(); // magic from
                ↪ Introduction to Algorithms.
        }
    return delta;
}
```

5.4. **Great-Circle Distance.** Computes the distance between two points (given as latitude/longitude coordinates) on a sphere of radius  $r$ .

```
ld gc_distance(ld pLat, ld pLong, ld qLat, ld qLong,
    ↪ ld r) {
    pLat *= pi / 180; pLong *= pi / 180;
    qLat *= pi / 180; qLong *= pi / 180;
    return r * acos(cos(pLat)*cos(qLat)*cos(pLong -
        ↪ qLong) + sin(pLat)*sin(qLat)); }
```

### 5.5. 3D Primitives.

```
#define P(p) const point3d &p
#define L(p0, p1) P(p0), P(p1)
#define PL(p0, p1, p2) P(p0), P(p1), P(p2)
struct point3d {
    double x, y, z;
    point3d() : x(0), y(0), z(0) {}
    point3d(double _x, double _y, double _z)
        : x(_x), y(_y), z(_z) {}
    point3d operator+(P(p)) const {
        return point3d(x + p.x, y + p.y, z + p.z); }
    point3d operator-(P(p)) const {
        return point3d(x - p.x, y - p.y, z - p.z); }
    point3d operator-() const {
        return point3d(-x, -y, -z); }
    point3d operator*(double k) const {
        return point3d(x * k, y * k, z * k); }
    point3d operator/(double k) const {
        return point3d(x / k, y / k, z / k); }
    double operator%(P(p)) const {
        return x * p.x + y * p.y + z * p.z; }
    point3d operator*(P(p)) const {
        return point3d(y*p.z - z*p.y,
            ↪ z*p.x - x*p.z, x*p.y - y*p.x); }

    double length() const {
        return sqrt(*this % *this); }
    double distTo(P(p)) const {
        return (*this - p).length(); }
    double distTo(P(A), P(B)) const {
        // A and B must be two different points
        return ((*this - A) * (*this - B)).length() /
            ↪ A.distTo(B); }
    point3d normalize(double k = 1) const {
        // length() must not return 0
        return (*this) * (k / length()); }
    point3d getProjection(P(A), P(B)) const {
        point3d v = B - A;
        return A + v.normalize((v % (*this - A)) /
            ↪ v.length()); }
    point3d rotate(P(normal)) const {
        //normal must have length 1 and be orthogonal to
        ↪ the vector
        return (*this) * normal; }
    point3d rotate(double alpha, P(normal)) const {
        return (*this) * cos(alpha) + rotate(normal) *
            ↪ sin(alpha); }
    point3d rotatePoint(P(O), P(axe), double alpha)
        ↪ const {
        point3d Z = axe.normalize(axe % (*this - O));
        return O + Z + (*this - O - Z).rotate(alpha, O);
        ↪ }
    bool isZero() const {
        return abs(x) < EPS && abs(y) < EPS && abs(z) <
            ↪ EPS; }
    bool isOnLine(L(A, B)) const {
        return ((A - *this) * (B - *this)).isZero(); }
    bool isInSegment(L(A, B)) const {
```

```
    return isOnLine(A, B) && ((A - *this) % (B -
        ↪ *this)) < EPS; }
    bool isInSegmentStrictly(L(A, B)) const {
        return isOnLine(A, B) && ((A - *this) % (B -
            ↪ *this)) < -EPS; }
    double getAngle() const {
        return atan2(y, x); }
    double getAngle(P(u)) const {
        return atan2((*this * u).length(), *this % u); }
    bool isOnPlane(PL(A, B, C)) const {
        return
            ↪ abs((A - *this) * (B - *this) % (C - *this)) <
                ↪ EPS; } };
int line_line_intersect(L(A, B), L(C, D), point3d
    ↪ &O) {
    if (abs((B - A) * (C - A) % (D - A)) > EPS) return
        ↪ 0;
    if (((A - B) * (C - D)).length() < EPS)
        return A.isOnLine(C, D) ? 2 : 0;
    point3d normal = ((A - B) * (C - D)).normalize();
    double s1 = (C - A) * (D - A) % normal;
    O = A + ((B - A) / (s1 + ((D - B) * (C - B) %
        ↪ normal))) * s1;
    return 1; }
int line_plane_intersect(L(A, B), PL(C, D, E),
    ↪ point3d &O) {
    double V1 = (C - A) * (D - A) % (E - A);
    double V2 = (D - B) * (C - B) % (E - B);
    if (abs(V1 + V2) < EPS)
        return A.isOnPlane(C, D, E) ? 2 : 0;
    O = A + ((B - A) / (V1 + V2)) * V1;
    return 1; }
bool plane_plane_intersect(P(A), P(nA), P(B), P(nB),
    point3d &P, point3d &Q) {
    point3d n = nA * nB;
    if (n.isZero()) return false;
    point3d v = n * nA;
    P = A + (n * nA) * ((B - A) % nB / (v % nB));
    Q = P + n;
    return true; }
```

### 5.6. Polygon Centroid.

$$C_x = \frac{1}{6A} \sum_{i=0}^{n-1} (x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$C_y = \frac{1}{6A} \sum_{i=0}^{n-1} (y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)$$

$$A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$$

**5.7. Rectilinear Minimum Spanning Tree.** Given a set of  $n$  points in the plane, and the aim is to find a minimum spanning tree connecting these  $n$  points, assuming the Manhattan distance is used. The function candidates returns at most  $4n$  edges that are a superset of the edges in a minimum spanning tree, and then one can use Kruskal's algorithm.

```
#define MAXN 100100
struct RMST {
    struct point {
        int i; ll x, y;
        point() : i(-1) {}
        ll d1() { return x + y; }
        ll d2() { return x - y; }
        ll dist(point other) {
            return abs(x - other.x) + abs(y - other.y); }
        bool operator < (const point &other) const {
            return y==other.y ? x > other.x : y < other.y; }
    }
    best[MAXN], A[MAXN], tmp[MAXN];
    int n;
    RMST() : n(0) {}
    void add_point(int x, int y) {
        A[n].i = n; x = x, A[n].y = y; }
    void rec(int l, int r) {
        if (l >= r) return;
        int m = (l+r)/2;
        rec(l,m), rec(m+1,r);
        point bst;
        for(int i=l, j=m+1, k=l; i <= m || j <= r; k++){
            if(j>r || (i <= m && A[i].d1() < A[j].d1())){
                tmp[k] = A[i++];
                if (bst.i != -1 && (best[tmp[k].i].i == -1
                    || best[tmp[k].i].d2() < bst.d2()))
                    best[tmp[k].i] = bst;
            } else {
                tmp[k] = A[j++];
                if (bst.i == -1 || bst.d2() < tmp[k].d2())
                    bst = tmp[k]; }
            rep(i,l,r+1) A[i] = tmp[i]; }
        vector<pair<ll,ii>> candidates() {
            vector<pair<ll, ii>> es;
            REP(p, 2) {
                REP(q, 2) {
                    sort(A, A+n);
                    REP(i, n) best[i].i = -1;
                    rec(0, n-1);
                    REP(i, n) {
                        if(best[A[i].i].i != -1)
                            es.pb({A[i].dist(best[A[i].i]),
                                {A[i].i, best[A[i].i].i}});
                        swap(A[i].x, A[i].y);
                        A[i].x *= -1, A[i].y *= -1; } }
                    REP(i, n) A[i].x *= -1; }
            return es; } };
```

## 5.8. Points and lines (CP3).

```
const ld EPS = 1e-9;

ld DEG_to_RAD(ld d) { return d*PI/180.0; }
ld RAD_to_DEG(ld r) { return r*180.0/PI; }

struct point { ld x, y;
    point() { x = y = 0.0; }
    point(ld _x, ld _y) : x(_x), y(_y) {}
    // useful for sorting
    bool operator < (point other) const {
        if (fabs(x - other.x) > EPS)
            return x < other.x;
        return y < other.y; }
    // use EPS (1e-9) when testing for equality
    bool operator == (point other) const {
        return fabs(x-other.x)<EPS &&
            ↪ fabs(y-other.y)<EPS; }
};

ld dist(point p1, point p2) {
    // hypot(dx, dy) returns sqrt(dx * dx + dy * dy)
    return hypot(p1.x - p2.x, p1.y - p2.y); }

// rotate p by rad RADIANS CCW w.r.t origin (0, 0)
point rotate(point p, ld rad) {
    return point(p.x*cos(rad) - p.y*sin(rad),
        p.x*sin(rad) + p.y*cos(rad)); }

// lines are (x,y) s.t. ax + by = c. AND b=0,1.
struct line { ld a, b, c; };

// gives line through p1, p2
line pointsToLine(point p1, point p2) {
    if (fabs(p1.x - p2.x) < EPS) // vertical line
        return { 1.0, 0.0, -p1.x };
    else return {
        -(ld)(p1.y - p2.y) / (p1.x - p2.x),
        1.0,
        -(ld)(1.a * p1.x) - p1.y;
    }; }

bool areParallel(line l1, line l2) {
    return fabs(l1.a-l2.a)<EPS && fabs(l1.b-l2.b)<EPS; }

bool areSame(line l1, line l2) {
    return areParallel(l1,l2) && fabs(l1.c-l2.c)<EPS; }

// returns true (+ intersection) if l1,l2 intersect
bool areIntersect(line l1, line l2, point &p) {
    // solve two equations:
    p.x = (l2.b * l1.c - l1.b * l2.c)
        / (l2.a * l1.b - l1.a * l2.b); }
```

```
// special case: test for vertical line:
if (fabs(l1.b) > EPS) p.y = -(l1.a * p.x + l1.c);
else p.y = -(l2.a * p.x + l2.c);
return true;
}

// name: `vec' is different from STL vector
struct vec { ld x, y;
    vec(ld _x, ld _y) : x(_x), y(_y) {} };
// convert 2 points to vector a->b
vec toVec(point a, point b) {
    return vec(b.x - a.x, b.y - a.y); }
vec scale(vec v, ld s) { return vec(v.x*s, v.y*s); }
// translate p according to v
point translate(point p, vec v) {
    return point(p.x + v.x, p.y + v.y); }

// convert point and gradient/slope to line
void pointSlopeToLine(point p, ld m, line &l) {
    l.a = -m; // always -m
    l.b = 1; // always 1
    l.c = -((l.a * p.x) + (l.b * p.y)); }

void closestPoint(line l, point p, point &ans) {
    if (fabs(l.b) < EPS) { // case 1: vertical line
        ans.x = -(l.c); ans.y = p.y; return; }

    if (fabs(l.a) < EPS) { // case 2: horizontal line
        ans.x = p.x; ans.y = -(l.c); return; }
    // normal line:
    line perpendicular;
    pointSlopeToLine(p, 1 / l.a, perpendicular);
    // intersect line l with this perpendicular line
    // the intersection point is the closest point
    areIntersect(l, perpendicular, ans); }

// returns the reflection of point on a line
void reflectionPoint(line l, point p, point &ans) {
    point b;
    closestPoint(l, p, b); // similar to distToLine
    return point(2*b.x - p.x, 2*b.y - p.y); }

ld dot(vec a, vec b) { return a.x*b.x + a.y*b.y; }
ld cross(vec a,vec b){ return a.x*b.y - a.y*b.x; }
ld norm_sq(vec v) { return v.x*v.x + v.y*v.y; }

// returns the distance from p to the line defined
// by points a and b (a != b), closest point in c.
ld distToLine(point p, point a, point b, point &c) {
    // formula: c = a + u * ab
    vec ap = toVec(a, p), ab = toVec(a, b);
    ld u = dot(ap, ab) / norm_sq(ab);
    c = translate(a, scale(ab, u));
    return dist(p, c); }

// returns the distance from p to the line segment
// ab defined by points a and b (still OK if a == b)
// the closest point is stored in c byref.
```

```

ld distToLineSegment(point p, point a, point b,
↳ point &c) {
    vec ap = toVec(a, p), ab = toVec(a, b);
    ld u = dot(ap, ab) / norm_sq(ab);
    if (u < 0.0) { c = point(a.x, a.y);
        return dist(p, a); } // closer to a
    if (u > 1.0) { c = point(b.x, b.y);
        return dist(p, b); } // closer to b
    // otherwise closest is perp to line:
    return distToLine(p, a, b, c); }

// returns angle aob in rad
ld angle(point a, point o, point b) {
    vec oa = toVec(o, a), ob = toVec(o, b);
    return acos(dot(oa, ob)
        / sqrt(norm_sq(oa) * norm_sq(ob)));
}

// note: to accept collinear points, change '> 0'
// returns true if r is on the left side of line pq
bool ccw(point p, point q, point r) {
    return cross(toVec(p, q), toVec(p, r)) > 0; }

// returns true if r is on the same line as line pq
bool collinear(point p, point q, point r) {
    return fabs(cross(toVec(p, q), toVec(p, r))) < EPS;
}

5.9. Polygon (CP3). Polygons have  $P_0 = P_{n-1}$  here.
typedef vector<point> poly;

// returns the perimeter: sum of Euclidean distances
// of consecutive line segments (polygon edges)
ld perimeter(const poly &P) {
    ld result = 0.0;
    REP(i, sz(P)-1) // remember that P[0] = P[n-1]
        result += dist(P[i], P[i+1]);
    return result; }

// returns the area, which is half the determinant
ld area(const poly &P) {
    ld result = 0.0;
    REP(i, sz(P)-1)
        result += P[i].x * P[i+1].y - P[i+1].x * P[i].y;
    return result;
}

// returns true if we always make the same turn
// throughout the polygon
bool isConvex(const poly &P) {
    int n = sz(P);
    if (n <= 3) return false; // point=2; line=3
    bool isLeft = ccw(P[0], P[1], P[2]);
    rep(i, n-2) if (ccw(P[i], P[i+1],
        P[(i+2) == n ? 1 : i+2]) != isLeft)
        return false; // different sign -> concave
    return true; } // convex

```

```

// returns true if pt is in polygon P
bool inPolygon(point pt, const poly &P) {
    if (sz(P) == 0) return false;
    ld sum = 0; // Assume P[0] == P[n-1]
    REP(i, sz(P)-1) {
        if (ccw(pt, P[i], P[i+1]))
            sum += angle(P[i], pt, P[i+1]);
        else sum -= angle(P[i], pt, P[i+1]); }
    return fabs(fabs(sum) - 2*PI) < EPS;
}

// line segment p-q intersect with line A-B.
point lineIntersectSeg(point p, point q,
    point A, point B) {
    ld a = B.y - A.y;
    ld b = A.x - B.x;
    ld c = B.x * A.y - A.x * B.y;
    ld u = fabs(a * p.x + b * p.y + c);
    ld v = fabs(a * q.x + b * q.y + c);
    return point((p.x*v + q.x*u) / (u+v),
        (p.y*v + q.y*u) / (u+v)); }

// cuts polygon Q along the line formed by a -> b
// (note: Q[0] == Q[n-1] is assumed)
poly cutPolygon(point a, point b, const poly &Q) {
    poly P;
    REP(i, sz(Q)) {
        ld left1 = cross(toVec(a, b), toVec(a, Q[i]));
        ld left2 = 0;
        if (i != sz(Q)-1)
            left2 = cross(toVec(a, b), toVec(a, Q[i+1]));
        if (left1 > -EPS)
            P.pb(Q[i]); // Q[i] is left of ab
        if (left1 * left2 < -EPS)
            // edge Q[i]-Q[i+1] crosses line ab
            P.pb(lineIntersectSeg(Q[i], Q[i+1], a, b));
    }
    if (!P.empty() && !(P.back() == P.front()))
        P.pb(P.front()); // make P[0] == P[n-1]
    return P; }

point pivot; // sorts points by angle around pivot
bool angleCmp(point a, point b) {
    if (collinear(pivot, a, b)) // special case
        return dist(pivot, a) < dist(pivot, b);
    ld dlx = a.x - pivot.x, dly = a.y - pivot.y;
    ld d2x = b.x - pivot.x, d2y = b.y - pivot.y;
    return (atan2(dly, dlx) - atan2(d2y, d2x)) < 0;
}

poly CH(poly P) { // no order of P assumed!
    int i, j, n = sz(P)
    if (n <= 3) {
        // safeguard from corner case
        if (!(P[0] == P[n-1])) P.pb(P[0]);
        return P; // special case, the CH is P itself
    }

```

```

// P0 = point with lowest Y (if tie rightmost X)
int P0 = 0;
rep(i, 1, n) if (P[i].y < P[P0].y
    || (P[i].y == P[P0].y && P[i].x > P[P0].x))
    P0 = i;
// swap P[P0] with P[0]:
point temp = P[0]; P[0] = P[P0]; P[P0] = temp;

// second, sort points by angle w.r.t. pivot P0
pivot = P[0];
sort(++P.begin(), P.end(), angleCmp); // keep P[0]

// third, the ccw tests
poly S = { P[n-1], P[0], P[1] }; // initial S
i = 2; // then, we check the rest
while (i < n) { // required: N must be >= 3
    j = sz(S) - 1;
    if (ccw(S[j-1], S[j], P[i]))
        S.pb(P[i++]); // left turn, accept
    else // pop top of S when right turn
        S.pop_back();
}
return S;
}

```

### 5.10. Triangle (CP3).

```

ld perimeter(point a, point b, point c) {
    return dist(a, b) + dist(b, c) + dist(c, a); }

ld area(ld ab, ld bc, ld ca) {
    // Heron's formula
    ld s = 0.5 * (ab+bc+ca);
    return sqrt(s)*sqrt(s-ab)*sqrt(s-bc)*sqrt(s-ca);
}

ld area(point a, point b, point c) {
    return area(dist(a, b), dist(b, c), dist(c, a));
}

ld rInCircle(ld ab, ld bc, ld ca) {
    return area(ab, bc, ca) * 2.0 / (ab+bc+ca);
}

ld rInCircle(point a, point b, point c) {
    return rInCircle(dist(a, b), dist(b, c), dist(c, a));
}

// assumption: the required points/lines functions
// have been written.
// Returns if there is an inCircle center
// if it returns TRUE, ctr will be the inCircle
// center and r is the same as rInCircle
int inCircle(point p1, point p2, point p3, point
↳ &ctr, ld &r) {
    r = rInCircle(p1, p2, p3);
    if (fabs(r) < EPS) return false;

    line l1, l2; // compute these two angle bisectors
    ld ratio = dist(p1, p2) / dist(p1, p3);

```



```

    point p = translate(p2,
        scale(toVec(p2, p3), ratio / (1 + ratio)));
    pointsToLine(p1, p, l1);

    ratio = dist(p2, p1) / dist(p2, p3);
    p = translate(p1,
        scale(toVec(p1, p3), ratio / (1 + ratio)));
    pointsToLine(p2, p, l2);
    // get their intersection point:
    areIntersect(l1, l2, ctr);
    return true;
}

ld rCircumCircle(ld ab, ld bc, ld ca) {
    return ab * bc * ca / (4.0 * area(ab, bc, ca)); }

ld rCircumCircle(point a, point b, point c) {
    return rCircumCircle(
        dist(a,b), dist(b,c), dist(c,a));
}

// assumption: the required points/lines functions
// have been written.
// Returns 1 iff there is a circumCenter center
// if this function returns 1, ctr will be the
// circumCircle center and r = rCircumCircle
bool circumCircle(point p1, point p2, point p3,
    ⇐ point &ctr, ld &r){
    ld a = p2.x - p1.x, b = p2.y - p1.y;
    ld c = p3.x - p1.x, d = p3.y - p1.y;
    ld e = a * (p1.x + p2.x) + b * (p1.y + p2.y);
    ld f = c * (p1.x + p3.x) + d * (p1.y + p3.y);
    ld g = 2.0 * (a * (p3.y - p2.y) - b * (p3.x - p2.x));
    if (fabs(g) < EPS) return false;

    ctr.x = (d*e - b*f) / g;
    ctr.y = (a*f - c*e) / g;
    r = dist(p1, ctr); // r = dist(center, p_i)
    return true;
}

// returns if pt d is inside the circumCircle
// defined by a,b,c
bool inCircumCircle(point a, point b,
    point c, point d) {
    vec va=toVec(a,d), vb=toVec(b,d), vc=toVec(c,d);
    return 0 <
        va.x * vb.y * (vc.x*vc.x + vc.y*vc.y) +
        va.y * (vb.x*vb.x + vb.y*vb.y) * vc.x +
        (va.x*va.x + va.y*va.y) * vb.x * vc.y -
        (va.x*va.x + va.y*va.y) * vb.y * vc.x -
        va.y * vb.x * (vc.x*vc.x + vc.y*vc.y) -
        va.x * (vb.x*vb.x + vb.y*vb.y) * vc.y;
}

bool canFormTriangle(ld a, ld b, ld c) {
    return a+b > c && a+c > b && b+c > a; }

```

### 5.11. Circle (CP3).

```

int insideCircle(point_i p, point_i c, int r) { //
    ⇐ all integer version
    int dx = p.x - c.x, dy = p.y - c.y;
    int Euc = dx * dx + dy * dy, rSq = r * r;
    ⇐ // all integer
    return Euc < rSq ? 0 : Euc == rSq ? 1 : 2; }
    ⇐ //inside/border/outside

bool circle2PtsRad(point p1, point p2, double r,
    ⇐ point &c) {
    double d2 = (p1.x - p2.x) * (p1.x - p2.x) +
        (p1.y - p2.y) * (p1.y - p2.y);
    double det = r * r / d2 - 0.25;
    if (det < 0.0) return false;
    double h = sqrt(det);
    c.x = (p1.x + p2.x) * 0.5 + (p1.y - p2.y) * h;
    c.y = (p1.y + p2.y) * 0.5 + (p2.x - p1.x) * h;
    return true; } // to get the other center,
    ⇐ reverse p1 and p2

```

5.12. **Formulas.** Let  $a = (a_x, a_y)$  and  $b = (b_x, b_y)$  be two-dimensional vectors.

- $a \cdot b = |a||b| \cos \theta$ , where  $\theta$  is the angle between  $a$  and  $b$ .
- $a \times b = |a||b| \sin \theta$ , where  $\theta$  is the signed angle between  $a$  and  $b$ .
- $a \times b$  is equal to the area of the parallelogram with two of its sides formed by  $a$  and  $b$ . Half of that is the area of the triangle formed by  $a$  and  $b$ .
- **Euler's formula:**  $V - E + F = 2$
- Side lengths  $a, b, c$  can form a triangle iff.  $a + b > c$ ,  $b + c > a$  and  $a + c > b$ .
- Sum of internal angles of a regular convex  $n$ -gon is  $(n - 2)\pi$ .
- **Law of sines:**  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- **Law of cosines:**  $b^2 = a^2 + c^2 - 2ac \cos B$
- Internal tangents of circles  $(c_1, r_1), (c_2, r_2)$  intersect at  $(c_1 r_2 + c_2 r_1) / (r_1 + r_2)$ , external intersect at  $(c_1 r_2 - c_2 r_1) / (r_1 + r_2)$ .

## 6. MISCELLANEOUS

6.1. **Binary search**  $\mathcal{O}(\log(hi - lo))$ .

```

bool test(int n);

int search(int lo, int hi) {
    assert(test(lo) && !test(hi)); // BE CERTAIN
    while (hi - lo > 1) {
        int m = (lo + hi) / 2;
        (test(m) ? lo : hi) = m;
    }
    // assert(test(lo) && !test(hi));
    return lo;
}

```

6.2. **Fast Fourier Transform**  $\mathcal{O}(n \log n)$ . Given two polynomials  $A(x) = a_0 + a_1x + \dots + a_{n/2}x^{n/2}$  and  $B(x) = b_0 + b_1x + \dots + b_{n/2}x^{n/2}$ , FFT calculates all coefficients of  $C(x) = A(x) \cdot B(x) = c_0 + c_1x + \dots + c_nx^n$ , with  $c_i = \sum_{j=0}^i a_j b_{i-j}$ .

```

typedef complex<double> cpx;
const int LOGN = 19, MAXN = 1 << LOGN;

int rev[MAXN];
cpx rt[MAXN], a[MAXN] = {}, b[MAXN] = {};

void fft(cpx *A) {
    REP(i, MAXN) if (i < rev[i]) swap(A[i],
        ⇐ A[rev[i]]);
    for (int k = 1; k < MAXN; k *= 2)
        for (int i = 0; i < MAXN; i += 2*k) REP(j, k) {
            cpx t = rt[j + k] * A[i + j + k];
            A[i + j + k] = A[i + j] - t;
            A[i + j] += t;
        }
}

void multiply() { // a = convolution of a * b
    rev[0] = 0; rt[1] = cpx(1, 0);
    REP(i, MAXN) rev[i] = (rev[i/2] | (i&1)<<LOGN)/2;
    for (int k = 2; k < MAXN; k *= 2) {
        cpx z(cos(PI/k), sin(PI/k));
        rep(i, k/2, k) rt[2*i] = rt[i], rt[2*i+1] = rt[i]*z;
    }
    fft(a); fft(b);
    REP(i, MAXN) a[i] *= b[i] / (double)MAXN;
    reverse(a+1, a+MAXN); fft(a);
}

```

6.3. **Minimum Assignment (Hungarian Algorithm)**  $\mathcal{O}(n^3)$ .

```

int a[MAXN + 1][MAXM + 1]; // matrix, 1-based
int minimum_assignment(int n, int m) { // n rows, m
    ⇐ columns
    vi u(n + 1), v(m + 1), p(m + 1), way(m + 1);
    for (int i = 1; i <= n; i++) {
        p[0] = i;
        int j0 = 0;
        vi minv(m + 1, INT_MAX);
        vector<char> used(m + 1, false);
        do {
            used[j0] = true;
            int i0 = p[j0], delta = INT_MAX, j1;
            for (int j = 1; j <= m; j++)
                if (!used[j]) {
                    int cur = a[i0][j] - u[i0] - v[j];
                    if (cur < minv[j]) minv[j] = cur, way[j] =
                        ⇐ j0;
                    if (minv[j] < delta) delta = minv[j], j1 =
                        ⇐ j;
                }
        }
    }
}

```



```

    for (int j = 0; j <= m; j++) {
        if (used[j]) u[p[j]] += delta, v[j] -= delta;
        else minv[j] -= delta;
    }
    j0 = j1;
    while (p[j0] != 0);
    do {
        int j1 = way[j0]; p[j0] = p[j1]; j0 = j1;
    } while (j0);
    // column j is assigned to row p[j]
    return -v[0];
}

```

#### 6.4. Partial linear equation solver $\mathcal{O}(N^3)$ .

```

typedef double NUM;
const int MAXROWS = 200, MAXCOLS = 200;
const NUM EPS = 1e-5;

// F2: bitset<MAXCOLS+1> mat[MAXROWS];
// bitset<MAXROWS> vals;
NUM mat[MAXROWS][MAXCOLS + 1], vals[MAXCOLS]; bool
// hasval[MAXCOLS];
bool is0(NUM a) { return -EPS < a && a < EPS; }

// finds x such that Ax = b
// A_ij is mat[i][j], b_i is mat[i][m]
int solvemmat(int n, int m) {
    // F2: vals.reset();
    int pr = 0, pc = 0;
    while (pc < m) {
        int r = pr, c;
        while (r < n && is0(mat[r][pc])) r++;
        if (r == n) { pc++; continue; }

        // F2: mat[pr] ^= mat[r]; mat[r] ^= mat[pr];
        // mat[pr] ^= mat[r];
        for (c = 0; c <= m; c++) swap(mat[pr][c],
        // mat[r][c]);

        r = pr++; c = pc++;
        // F2: vals.set(pc, mat[pr][m]);
        NUM div = mat[r][c];
        for (int col = c; col <= m; col++) mat[r][col]
        // /= div;
        REP(row, n) {
            if (row == r) continue;
            // F2: if (mat[row].test(c)) mat[row] ^=
            // mat[r];
            NUM times = -mat[row][c];
            for (int col = c; col <= m; col++)
                mat[row][col] += times * mat[r][col];
        }
    } // now mat is in RREF

    for (int r = pr; r < n; r++)
        if (!is0(mat[r][m])) return 0;
}

```

```

// F2: return 1;
fill_n(hasval, n, false);
for (int col = 0, row; col < m; col++) {
    hasval[col] = !is0(mat[row][col]);
    if (!hasval[col]) continue;
    for (int c = col + 1; c < m; c++) {
        if (!is0(mat[row][c])) hasval[col] = false;
    }
    if (hasval[col]) vals[col] = mat[row][m];
    row++;
}
REP(i, n) if (!hasval[i]) return 2;
return 1;
}

```

#### 6.5. Cycle-Finding.

```

ii find_cycle(int x0, int (*f)(int)) {
    int t = f(x0), h = f(t), mu = 0, lam = 1;
    while (t != h) t = f(t), h = f(f(h));
    h = x0;
    while (t != h) t = f(t), h = f(h), mu++;
    h = f(t);
    while (t != h) h = f(h), lam++;
    return ii(mu, lam); }

```

#### 6.6. Longest Increasing Subsequence.

```

vi lis(vi arr) {
    vi seq, back(size(arr)), ans;
    rep(i, 0, size(arr)) {
        int res = 0, lo = 1, hi = size(seq);
        while (lo <= hi) {
            int mid = (lo+hi)/2;
            if (arr[seq[mid-1]] < arr[i]) res = mid, lo =
            // mid + 1;
            else hi = mid - 1; }
        if (res < size(seq)) seq[res] = i;
        else seq.push_back(i);
        back[i] = res == 0 ? -1 : seq[res-1]; }
    int at = seq.back();
    while (at != -1) ans.push_back(at), at = back[at];
    reverse(ans.begin(), ans.end());
    return ans; }

```

#### 6.7. Dates.

```

int intToDay(int jd) { return jd % 7; }
int dateToInt(int y, int m, int d) {
    return 1461 * (y + 4800 + (m - 14) / 12) / 4 +
    367 * (m - 2 - (m - 14) / 12 * 12) / 12 -
    3 * ((y + 4900 + (m - 14) / 12) / 100) / 4 +
    d - 32075; }
void intToDate(int jd, int &y, int &m, int &d) {
    int x, n, i, j;
    x = jd + 68569;
    n = 4 * x / 146097;
    x -= (146097 * n + 3) / 4;
    i = (4000 * (x + 1)) / 1461001;
    x -= 1461 * i / 4 - 31;
    j = 80 * x / 2447;
}

```

```

d = x - 2447 * j / 80;
x = j / 11;
m = j + 2 - 12 * x;
y = 100 * (n - 49) + i + x; }

```

#### 6.8. Simplex.

```

typedef vector<ld> VD;
typedef vector<VD> VVD;
const ld EPS = 1e-9;
struct LPSolver {
    int m, n; vi B, N; VVD D;
    LPSolver(const VVD &A, const VD &b, const VD &c) :
        m(b.size()), n(c.size()),
        N(n + 1), B(m), D(m + 2, VD(n + 2)) {
        REP(i, m) REP(j, n) D[i][j] = A[i][j];
        REP(i, m) { B[i] = n + i; D[i][n] = -1;
        D[i][n + 1] = b[i]; }
        REP(j, n) N[j] = j, D[m][j] = -c[j];
        N[n] = -1; D[m + 1][n] = 1;
    }
    void Pivot(int r, int s) {
        double inv = 1.0 / D[r][s];
        REP(i, m+2) if (i != r) REP(j, n+2) if (j != s)
            D[i][j] -= D[r][j] * D[i][s] * inv;
        REP(j, n+2) if (j != s) D[r][j] *= inv;
        REP(i, m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]); }
    bool Simplex(int phase) {
        int x = phase == 1 ? m + 1 : m;
        while (true) {
            int s = -1;
            for (int j = 0; j <= n; j++) {
                if (phase == 2 && N[j] == -1) continue;
                if (s == -1 || D[x][j] < D[x][s] ||
                D[x][j] == D[x][s] && N[j] < N[s]) s = j; }
            if (D[x][s] > -EPS) return true;
            int r = -1;
            REP(i, m) {
                if (D[i][s] < EPS) continue;
                if (r == -1 || D[i][n + 1] / D[i][s] < D[r][n +
                // 1] /
                D[r][s] || (D[i][n + 1] / D[i][s]) ==
                // (D[r][n + 1] /
                D[r][s]) && B[i] < B[r]) r = i; }
            if (r == -1) return false;
            Pivot(r, s); } }
    ld Solve(VD &x) {
        int r = 0;
        rep(i, 1, m) if (D[i][n+1] < D[r][n+1]) r = i;
        if (D[r][n + 1] < -EPS) {
            Pivot(r, n);
            if (!Simplex(1) || D[m + 1][n + 1] < -EPS)
                return -numeric_limits<ld>::infinity();
            REP(i, m) if (B[i] == -1) {
                int s = -1;

```

```

for (int j = 0; j <= n; j++)
  if (s == -1 || D[i][j] < D[i][s] ||
      D[i][j] == D[i][s] && N[j] < N[s])
    s = j;
Pivot(i, s); }
if (!Simplex(2)) return
  ↪ numeric_limits<ld>::infinity();
x = VD(n);
for (int i = 0; i < m; i++) if (B[i] < n)
  x[B[i]] = D[i][n + 1];
return D[m][n + 1]; } }
// 2-phase simplex solves linear system:
// maximize c^T x
// subject to Ax <= b, x >= 0
// INPUT: A -- an m x n matrix
// b -- an m-dimensional vector
// c -- an n-dimensional vector
// x -- optimal solution (by reference)
// OUTPUT: c^T x (inf. if unbounded above, nan if
  ↪ infeasible)
// *** Example ***
// const int m = 4, n = 3;
// ld _A[m][n] = {{6,-1,0}, {-1,-5,0},
// {1,5,1}, {-1,-5,-1}};
// ld _b[m] = {10,-4,5,-5}, _c[n] = {1,-1,0};
// VVD A(m);
// VD b(_b, _b + m), c(_c, _c + n), x;
// REP(i, m) A[i] = VD(_A[i], _A[i] + n);
// LPSolver solver(A, b, c);
// ld value = solver.Solve(x);
// cerr << "VALUE: " << value << endl; // 1.29032
// cerr << "SOLUTION:"; // 1.74194 0.451613 1
// REP(i, sz(x)) cerr << " " << x[i];
// cerr << endl;

```

## 7. COMBINATORICS

- Catalan numbers (valid bracket seq's of length  $2n$ ):  
 $C_0 = 1, C_n = \frac{1}{n+1} \binom{2n}{n} = \sum_{i=0}^{n-1} C_i C_{n-i-1}$ .
- Stirling 1<sup>th</sup> kind ( $\#\pi \in \mathfrak{S}_n$  with exactly  $k$  cycles):  
 $[n] = [0] = \delta_{0n}, [n]_k = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ .
- Stirling 2<sup>nd</sup> kind ( $k$ -partitions of  $[n]$ ):  
 $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1, \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\}$ .
- Bell numbers (partitions of  $[n]$ ):  
 $B_0 = 1, B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k} = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ .
- Euler ( $\#\pi \in \mathfrak{S}_n$  with exactly  $k$  ascents):  
 $\left\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\rangle = 1, \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (n-k) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle$ .
- Euler 2<sup>nd</sup> order (nr perms of  $1, 1, 2, 2, \dots, n, n$  with exactly  $k$  ascents):  
 $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle = (k+1) \left\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\rangle + (2n-k-1) \left\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\rangle$ .
- Rooted trees:  $n^{n-1}$ , unrooted:  $n^{n-2}$ .

- Forests of  $k$  rooted trees:  $\binom{n}{k} k \cdot n^{n-k-1}$ .
- $1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}, \quad 1^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{i=1}^n \binom{n}{i} F_i = F_{2n}, \quad \sum_i \binom{n-i}{i} = F_{n+1}$
- $\sum_{k=0}^n \binom{n}{k} = \binom{n+1}{n+1}, \quad x^k = \sum_{i=0}^k i! \left\{ \begin{smallmatrix} k \\ i \end{smallmatrix} \right\} \binom{x+i}{k}$
- $a \equiv b \pmod{x, y} \Leftrightarrow a \equiv b \pmod{\text{lcm}(x, y)}$ .
- $ac \equiv bc \pmod{m} \Leftrightarrow a \equiv b \pmod{m/\text{gcd}(c, m)}$ .
- $\text{gcd}(n^a - 1, n^b - 1) = \text{gcd}(a, b) - 1$ .
- **Möbius inversion formula:** If  $f(n) = \sum_{d|n} g(d)$ , then  $g(n) = \sum_{d|n} \mu(d) f(n/d)$ . If  $f(n) = \sum_{m=1}^n g(\lfloor n/m \rfloor)$ , then  $g(n) = \sum_{m=1}^n \mu(m) f(\lfloor \frac{n}{m} \rfloor)$ .
- **Inclusion-Exclusion:** If  $g(T) = \sum_{S \subseteq T} f(S)$ , then

$$f(T) = \sum_{S \subseteq T} (-1)^{|T \setminus S|} g(T).$$

Corollary:  $b_n = \sum_{k=0}^n \binom{n}{k} a_k \Leftrightarrow a_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} b_k$ .

- **The Twelfold Way:** Putting  $n$  balls into  $k$  boxes.  $p(n, k)$  is # partitions of  $n$  in  $k$  parts, each  $> 0$ .  $p_k(n) = \sum_{i=0}^k p(n, k)$ .

Balls Boxes	same same	distinct same	same distinct	distinct distinct
-	$p_k(n)$	$\sum_{i=0}^k \left\{ \begin{smallmatrix} n \\ i \end{smallmatrix} \right\}$	$\binom{n+k-1}{k-1}$	$k^n$
size $\geq 1$	$p(n, k)$	$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$	$\binom{n-1}{k-1}$	$k! \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$
size $\leq 1$	$[n \leq k]$	$[n \leq k]$	$\binom{k}{n}$	$n! \binom{k}{n}$

## 8. FORMULAS

- **Legendre symbol:**  $\left( \frac{a}{b} \right) = a^{(b-1)/2} \pmod{b}$ ,  $b$  odd prime.
- **Heron's formula:** A triangle with side lengths  $a, b, c$  has area  $\sqrt{s(s-a)(s-b)(s-c)}$  where  $s = \frac{a+b+c}{2}$ .
- **Shoelace formula:**  $A = \frac{1}{2} \left| \sum_{i=0}^{n-1} x_i y_{i+1} - x_{i+1} y_i \right|$ .
- **Pick's theorem:** A polygon on an integer grid strictly containing  $i$  lattice points and having  $b$  lattice points on the boundary has area  $i + \frac{b}{2} - 1$ . (Nothing similar in higher dimensions)
- **Absorption probabilities** A random walk on  $[0, n]$  with probability  $p$  to increase and  $q$  to decrease, starting at  $k$  has at  $n$  absorption probability  $\frac{(q/p)^{k-1}}{(q/p)^{n-1}}$  if  $q \neq p$ , and  $k/n$  if  $q = p$ .
- A minimum Steiner tree for  $n$  vertices requires at most  $n-2$  additional Steiner vertices.
- **Lagrange polynomial** through points  $(x_0, y_0), \dots, (x_k, y_k)$  is

$$L(x) = \sum_{j=0}^k y_j \prod_{\substack{0 \leq m \leq k \\ m \neq j}} \frac{x - x_m}{x_j - x_m}.$$

- **Hook length formula:** If  $\lambda$  is a Young diagram and  $h_\lambda(i, j)$  is the hook-length of cell  $(i, j)$ , then then the number of Young tableaux  $d_\lambda = n! / \prod h_\lambda(i, j)$ .
- #primitive pythagorean triples with hypotenuse  $< n$  approx  $n/(2\pi)$ .
- **Frobenius Number:** largest number which can't be expressed as a linear combination of numbers  $a_1, \dots, a_n$  with non-negative coefficients.  $g(a_1, a_2) = a_1 a_2 - a_1 - a_2$ ,  $N(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$ .  $g(d \cdot a_1, d \cdot a_2, a_3) = d \cdot g(a_1, a_2, a_3) + a_3(d-1)$ . An integer  $x > (\max_i a_i)^2$  can be expressed in such a way iff.  $x \mid \text{gcd}(a_1, \dots, a_n)$ .
- **Snell's law:**  $v_2 \sin \theta_1 = v_1 \sin \theta_2$  gives the shortest path between two media.
- **BEST theorem:** The number of Eulerian cycles in a directed graph  $G$  is:

$$t_w(G) \prod_{v \in G} (\deg v - 1)!,$$

where  $t_w(G)$  is the number of arborescences ("directed spanning" tree) rooted at  $w$ :  $t_w(G) = \det(q_{ij})_{i,j \neq w}$ , with  $q_{ij} = [i=j] \text{indeg}(i) - \#\{(i, j) \in E\}$ .

- **Burnside's Lemma:** Let a finite group  $G$  act on a set  $X$ . Denote  $X^g = \{x \in X \mid gx = x\}$ . For each  $g$  in  $G$  let  $X^g$  denote the set of elements in  $X$  that are fixed by  $g$ . Then the number of orbits is:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

- **Bézout's identity:** If  $(x, y)$  is a solution to  $ax + by = d$  ( $x, y$  can be found with EGCD), then all solutions are given by

$$(x + k \cdot \text{lcm}(a, b)/a, y - k \cdot \text{lcm}(a, b)/b), \quad k \in \mathbb{Z}$$

## 9. GAME THEORY

A game can be reduced to Nim if it is a finite impartial game. Nim and its variants include:

- **Nim:** Let  $X = \bigoplus_{i=1}^n x_i$ , then  $(x_i)_{i=1}^n$  is a winning position iff  $X \neq 0$ . Find a move by picking  $k$  such that  $x_k > x_k \oplus X$ .
- **Misère Nim:** Regular Nim, except that the last player to move *loses*. Play regular Nim until there is only one pile of size larger than 1, reduce it to 0 or 1 such that there is an odd number of piles. The second player wins  $(a_1, \dots, a_n)$  if 1) there is a pile  $a_i > 1$  and  $\bigoplus_{i=1}^n a_i = 0$  or 2) all  $a_i \leq 1$  and  $\bigoplus_{i=1}^n a_i = 1$ .
- **Staircase Nim:** Stones are moved down a staircase and only removed from the last pile.  $(x_i)_{i=1}^n$  is an  $L$ -position if  $(x_{2i-1})_{i=1}^{n/2}$  is (i.e. only look at odd-numbered piles).

- **Moore's Nim<sub>k</sub>**: The player may remove from at most  $k$  piles (Nim = Nim<sub>1</sub>). Expand the piles in base 2, do a carry-less addition in base  $k+1$  (i.e. the number of ones in each column should be divisible by  $k+1$ ).
- **Dim<sup>+</sup>**: The number of removed stones must be a divisor of the pile size. The Sprague-Grundy function is  $k+1$  where  $2^k$  is the largest power of 2 dividing the pile size.
- **Aliquot game**: Same as above, except the divisor should be proper (hence 1 is also a terminal state, but watch out for size 0 piles). Now the Sprague-Grundy function is just  $k$ .
- **Nim (at most half)**: Write  $n+1 = 2^m y$  with  $m$  maximal, then the Sprague-Grundy function of  $n$  is  $(y-1)/2$ .
- **Lasker's Nim**: Players may alternatively split a pile into two new non-empty piles.  $g(4k+1) = 4k+1$ ,  $g(4k+2) = 4k+2$ ,  $g(4k+3) = 4k+4$ ,  $g(4k+4) = 4k+3$  ( $k \geq 0$ ).
- **Hackenbush on trees**: A tree with stalks  $(x_i)_{i=1}^n$  may be replaced with a single stalk with length  $\bigoplus_{i=1}^n x_i$ .

## 10. SCHEDULING THEORY

Let  $p_j$  be the time task  $j$  takes on a machine,  $d_j$  the deadline,  $C_j$  the time it is completed,  $L_j = C_j - d_j$  the lateness,  $T_j = \max(L_j, 0)$  the tardiness,  $U_j = 1$  iff  $T_j > 0$  and else 0.

- One machine, minimise  $L_{\max}$ : do the tasks in increasing deadline
- One machine, minimise  $\sum_j w_j C_j$ : do the task increasing in  $p_j/w_j$
- One machine, minimise  $\sum_{j=1}^n C_j$  under the condition that all tasks can be done on time:
  - (1) Initialise  $k = n, \tau = \sum_j p_j, J = [n]$
  - (2) Take  $i_k \in J$  with  $d_{i_k} \geq \tau$  and  $p_{i_k} \geq p_\ell$  for  $\ell \in J$  with  $d_\ell \geq \tau$
  - (3)  $\tau := \tau - p_{i_k}, k := k - 1, J := J - \{i_k\}$ . If  $k \neq 0$ , go to step 2.
  - (4) The optimale schedule is  $i_1, \dots, i_n$ .
- One machine, minimise  $\sum_j U_j$ . Add all tasks in order of increasing deadline; if adding a task makes it contrary with its deadline, remove the processed task with the highest processing time.
- Two machines (all tasks have to be done on both machines, in any order), minimise  $C_{\max}$ : a greedy algorithm, when a machine is free it picks a task that hasn't been done yet on either machine and has longest processing time on the other machine.
- Two machines (all tasks have to be done first on machine 1, then machine 2), minimise  $C_{\max}$ . There is an optimal schedule with on both machines the same order of tasks. Take

$X = \{j : p_{1j} \leq p_{2j}\}$  and  $Y$  the complement. Sort  $X$  increasing in  $p_{1j}$  and  $Y$  decreasing in  $p_{2j}$ . Then  $X, Y$  is an optimal schedule.

- Two machines (all tasks have to be done first on machine 1, then on 2, or vice versa), minimise  $C_{\max}$ : let  $J_{12}$  be the tasks that have to be done first on machine 1, then on 2 and similar  $J_{21}$ . Use the above algorithm to find  $S_{12}, S_{21}$  optimal for  $J_{12}, J_{21}$ . Then optimal is  $S_{12}, S_{21}$  for M1 and  $S_{21}, S_{12}$  for M2. (If there are tasks that have to be done on only one machine, do them in the middle.)

## 11. JAVA ESSENTIALS

### 11.1. Round to n decimals.

```
DecimalFormatSymbols dfs = new
    DecimalFormatSymbols();
dfs.setDecimalSeparator('.');
DecimalFormat df = new DecimalFormat("#0.00", dfs);
double x = 12.5093;
System.out.println(df.format(x));
```

### 11.2. Example usage BufferedReader.

```
BufferedReader br = new BufferedReader(new
    InputStreamReader(System.in));
String line = br.readLine();
String splittedLine = br.readLine().split(" ");
int N = Integer.parseInt(splittedLine[0]);
```

### 11.3. Example usage sort().

```
class ExampleComparator implements
    Comparator<Integer> {
    public int compare(Integer n, Integer m) {
        if (n < m) return -1;
        else if (n > m) return 1;
        else return 0;
    }
}

// In some other function:
Collections.reverse(arr);
Collections.sort(arr);
Collections.sort(arr, new ExampleComparator());
```

```
ArrayList<String> stringArr = new ArrayList<>();
stringArr.add("a"); stringArr.add("b");
    stringArr.add("c");
Collections.sort(stringArr); // yields [C, a, b]
Collections.sort(stringArr,
    String.CASE_INSENSITIVE_ORDER); // yields [a, b,
    C]

int[] arr2 = new int[3];
arr2[0] = 0; arr2[1] = 2; arr2[2] = 1;
Arrays.sort(arr2); // yields [0,1,2]
```

### 11.4. Shortest path (Dijkstra).

```
// Running time is O((E + V) log V)
class Node {
    ArrayList<Edge> adj;
    int dist; // initially Integer.MAX_VALUE (must
        ↪ initialize!)
    Node parent;
}

class NodeDist implements Comparable<NodeDist> {
    int i, d; // node index and distance
    NodeDist(int index, int dist) {...};
    public int compareTo(NodeDist other) {
        return (d - other.d);
    }
}

void dijkstra(int source) { // can also be done for
    ↪ multiple sources...
    PriorityQueue<NodeDist> Q = new
        PriorityQueue<NodeDist>();
    V[source].dist = 0; V[source].parent = null;
    Q.add(new NodeDist(source, 0));
    while (!Q.isEmpty()) {
        NodeDist nd = Q.poll();
        int k = nd.i;
        int d = nd.d;
        if (V[k].dist < d) continue;
        for (Edge e: V[k].adj) {
            int newDist = d + e.weight; // e.weight
            ↪ cannot be MAX_VALUE!
            if (newDist < V[e.target].dist) {
                V[e.target].dist = newDist;
                V[e.target].parent = V[k];
                Q.add(new NodeDist(e.target,
                    ↪ newDist));
            }
        }
    }
    // Nodes contain distance and parent info
}
```

## 12. DEBUGGING TIPS

- Stack overflow? Recursive DFS on tree that is actually a long path?
- Floating-point numbers
  - Getting NaN? Make sure acos etc. are not getting values out of their range (perhaps 1+eps).
  - Rounding negative numbers?
  - Outputting in scientific notation?
- Wrong Answer?
  - Read the problem statement again!

- Are multiple test cases being handled correctly? Try repeating the same test case many times.
- Integer overflow?
- Think very carefully about boundaries of all input parameters
- Try out possible edge cases:
  - \*  $n = 0, n = -1, n = 1, n = 2^{31} - 1$  or  $n = -2^{31}$
  - \* List is empty, or contains a single element
  - \*  $n$  is even,  $n$  is odd
  - \* Graph is empty, or contains a single vertex
  - \* Graph is a multigraph (loops or multiple edges)
  - \* Polygon is concave or non-simple
- Is initial condition wrong for small cases?
- Are you sure the algorithm is correct?
- Explain your solution to someone.
- Are you using any functions that you don't completely understand? Maybe STL functions?
- Maybe you (or someone else) should rewrite the solution?
- Can the input line be empty?
- Run-Time Error?
  - Is it actually Memory Limit Exceeded?

### 12.1. Solution Ideas.

- Dynamic Programming
    - Parsing CFGs: CYK Algorithm
    - Drop a parameter, recover from others
    - Swap answer and a parameter
    - When grouping: try splitting in two
    - $2^k$  trick
    - When optimizing
      - \* Convex hull optimization
        - $dp[i] = \min_{j < i} \{dp[j] + b[j] \times a[i]\}$
        - $b[j] \geq b[j + 1]$
        - optionally  $a[i] \leq a[i + 1]$
        - $O(n^2)$  to  $O(n)$
      - \* Divide and conquer optimization
        - $dp[i][j] = \min_{k < j} \{dp[i - 1][k] + C[k][j]\}$
        - $A[i][j] \leq A[i][j + 1]$
        - $O(kn^2)$  to  $O(kn \log n)$
        - sufficient:  $C[a][c] + C[b][d] \leq C[a][d] + C[b][c], a \leq b \leq c \leq d$  (QI)
- ```
vvi A; // A[i][j] is voor [i,j]

void divco(ll ls,ll rs,ll lt,ll rt,vi &t, vi
↳ &s){ // berekent t[_]{[lt,rt)}
    if(lt >= rt)
        return;
    ll ms = ls, mt = (lt + rt)/2;
```

```
t[mt] = -INF;
rep(i,ls,rs){
    if(i >= mt){
        break;
    }
    if(s[i] + A[i][mt] > t[mt]){
        t[mt] = s[i] + A[i][mt];
        ms = i;
    }
}
divco(ls,ms+1,lt,mt,t,s);
divco(ms,rs,mt+1,rt,t,s);
}
```

- Knuth optimization
  - $dp[i][j] = \min_{i < k < j} \{dp[i][k] + dp[k][j] + C[i][j]\}$
  - $A[i][j - 1] \leq A[i][j] \leq A[i + 1][j]$
  - $O(n^3)$  to  $O(n^2)$
  - sufficient: QI and  $C[b][c] \leq C[a][d], a \leq b \leq c \leq d$

- Greedy
- Randomized
- Optimizations
  - Use bitset (/64)
  - Switch order of loops (cache locality)
- Process queries offline
  - Mo's algorithm
- Square-root decomposition
- Precomputation
- Efficient simulation
  - Mo's algorithm
  - Sqrt decomposition
  - Store  $2^k$  jump pointers
- Data structure techniques
  - Sqrt buckets
  - Store  $2^k$  jump pointers
  - $2^k$  merging trick
- Counting
  - Inclusion-exclusion principle
  - Generating functions
- Graphs
  - Can we model the problem as a graph?
  - Can we use any properties of the graph?
  - Strongly connected components
  - Cycles (or odd cycles)
  - Bipartite (no odd cycles)
    - \* Bipartite matching
    - \* Hall's marriage theorem
    - \* Stable Marriage
  - Cut vertex/bridge
  - Biconnected components
  - Degrees of vertices (odd/even)

- Trees
  - \* Heavy-light decomposition
  - \* Centroid decomposition
  - \* Least common ancestor
  - \* Centers of the tree
- Eulerian path/circuit
- Chinese postman problem
- Topological sort
- (Min-Cost) Max Flow
- Min Cut
  - \* Maximum Density Subgraph
- Huffman Coding
- Min-Cost Arborescence
- Steiner Tree
- Kirchoff's matrix tree theorem
- Prüfer sequences
- Lovász Toggle
- Look at the DFS tree (which has no cross-edges)
- Is the graph a DFA or NFA?
  - \* Is it the Synchronizing word problem?
- math
  - Is the function multiplicative?
  - Look for a pattern
  - Permutations
    - \* Consider the cycles of the permutation
  - Functions
    - \* Sum of piecewise-linear functions is a piecewise-linear function
    - \* Sum of convex (concave) functions is convex (concave)
  - Modular arithmetic
    - \* Chinese Remainder Theorem
    - \* Linear Congruence
  - Sieve
  - System of linear equations
  - Values too big to represent?
    - \* Compute using the logarithm
    - \* Divide everything by some large value
  - Linear programming
    - \* Is the dual problem easier to solve?
  - Can the problem be modeled as a different combinatorial problem? Does that simplify calculations?
- Logic
  - 2-SAT
  - XOR-SAT (Gauss elimination or Bipartite matching)
- Meet in the middle
- Only work with the smaller half ( $\log(n)$ )
- Strings

- Trie (maybe over something weird, like bits)
- Suffix array
- Suffix automaton (+DP?)
- Aho-Corasick
- `eerTree`
- Work with  $S + S$
- Hashing
- Euler tour, tree to array
- Segment trees
  - Lazy propagation
  - Persistent
  - Implicit
  - Segment tree of X
- Geometry
  - Minkowski sum (of convex sets)
  - Rotating calipers
  - Sweep line (horizontally or vertically?)
  - Sweep angle
  - Convex hull
- Fix a parameter (possibly the answer).
- Are there few distinct values?
- Binary search
- Sliding Window (+ Monotonic Queue)
- Computing a Convolution? Fast Fourier Transform
- Computing a 2D Convolution? FFT on each row, and then on each column
- Exact Cover (+ Algorithm X)
- Cycle-Finding
- What is the smallest set of values that identify the solution? The cycle structure of the permutation? The powers of primes in the factorization?
- Look at the complement problem
  - Minimize something instead of maximizing
- Immediately enforce necessary conditions. (All values greater than 0? Initialize them all to 1)
- Add large constant to negative numbers to make them positive
- Counting/Bucket sort

- Does MLE give RTE or MLE as a verdict? What about stack overflow?
- What is `RAND_MAX`?
- How does the judge handle extra spaces (or missing newlines) in the output?
- Look at documentation for programming languages.
- Try different programming languages: C++, Java and Python.
- Try the submit script.
- Try local programs: `i?python[23]`, `factor`.
- Try submitting with `assert(false)` and `assert(true)`.
- Omitting `return 0;` still works?
- Look for directory with sample test cases.
- Make sure printing works.

## PRACTICE CONTEST CHECKLIST

- How many operations per second? Compare to local machine.
- What is the stack size?
- How to use `printf/scanf` with `long long/long double`?
- Are `__int128` and `__float128` available?