Bandit Convex Optimisation – Exercise Solutions

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Contents

Chapter 3

Mathematical tools

Exercise 3.2

Prove the following statements:

(a) Given convex bodies $K \subset J$, polarity reverses inclusion: $J^\circ \subset K^\circ$.

(b) The polar body of a non-empty ball $K = \mathbb{B}_r^d$ is $K^\circ = \mathbb{B}_{1/r}^d$.

(c) For symmetric convex body K, $||\cdot||_K := \pi_K(\cdot)$ is a norm.

(d) For symmetric convex body K, the dual of $||\cdot||_K$ is $||\cdot||_{K^\circ}$ where:

$$||u||_K = \max\{\langle u, x\rangle : ||x||_K \leq 1, x \in \mathbb{R}^d\}.$$

Solution : (a) For sake of concreteness we recall that $K^\circ = \{u \in \mathbb{R}^d : h_K(u) \leq 1\}$ where $h_K(u) = \sup_{x \in K}\langle u, x\rangle$. Now remark that $K \subset J$ implies that $h_K(u) \leq h_J(u)$ for all $u \in \mathbb{R}^d$, in particular let $u \in J^\circ$ then the following holds:

$$h_K(u) \leq h_J(u) \leq 1$$

thus $u \in J^\circ$ which concludes.

(b) We will show the statement by double inclusion. If $u \in K = \mathbb{B}_r^d$ then by Cauchy-Scwharz $\langle u, x\rangle \leq ||u||||x||$ for all $x \in K$ so in particular $h_K(u) \leq ||u||||x|| \leq r \times 1/r = 1$ and therefore $u \in K^\circ$. Conversely let $u \in K^\circ$ and assume $u \neq 0$ (we can discard the case $u = 0$ because the stament is trivial in that case) then $\langle u, x\rangle \leq 1$ for all $x \in \mathbb{B}_r^d$ in particular it holds for $x^0 = r\frac{u}{||u||}$. therefore $\langle u, x^0\rangle = r||u|| \leq 1$ which implies that $||u|| \leq 1/r$ thus $K^\circ = \mathbb{B}_{1/r}^d$ which concludes.

(c) We assume K is symmetric, *i.e* $K = -K$, and recall that $\pi_K(x) = \inf\{t > 0 : x \in tK\}$.

Positive definiteness: Suppose $\pi_K(x) = 0$ and $x \neq 0$ then because K is a convex body there exist $\epsilon > 0$ such that $\mathbb{B}_\epsilon^d \subset K$ and therefore since $x \in \mathbb{B}_{||x||}^d$ setting $t = \frac{||x||}{\epsilon}$ ensures that $x \in tK$ which contradicts that $\pi_K(x) = 0$. So necessarily $x = 0$.

Homogenity: By definition $\pi_K(\lambda x) = \inf\{t > 0 : \lambda x \in tK\} = |\lambda| \inf\{t > 0 : x \in tK\} = |\lambda|\pi_K(x)$. In the second inequality we used the fact that if $\lambda x \in K$, $-\lambda x \in K$.

Triangle Inequality: $\pi_K(x + y) = \inf\{t > 0 : x + y \in tK\} \leq \inf\{t > 0 : x \in tK\} + \inf\{t > 0 : y \in tK\} = \pi_K(x) + \pi_K(y)$.

$\square$

Exercise 3.5

Suppose that $f \in \mathcal{F}_{sm,sc}$ and f is twice differentiable. Show that

$$\alpha\mathbf{1} \preceq f''(x) \preceq \beta\mathbf{1} \text{ for all } x \in int(K).$$

Solution : Since $f \in \mathcal{F}_{sm,sc}$, it holds that $g : x \mapsto f(x) - \frac{\alpha}{2}||x||^2$ is convex and $h : x \mapsto f(x) - \frac{\beta}{2}||x||^2$ is concave. Further more f is twice differentiable which implies that g and h are also twice differentiable and it follows that:

$$g''(x) \succeq 0 \text{ and } h''(x) \preceq 0 \text{ for all } x \in int(K).$$

Remarking that $g''(x) = f''(x) - \alpha \mathbf{1}$ and $h''(x) = f''(x) - \beta \mathbf{1}$ yields the statement. ☐

Exercise 3.9

Suppose that A is convex and f has directional derivatives and $Df(x)[\eta] \leq L$ for all $x \in A$ and $\eta \in \mathbb{S}_1^{d-1}$. Show that $\text{lip}_A(f) \leq L$.

Solution : We remark that if A is open and f is differentiable on A, the result is a direct consequence of the mean value theorem. Indeed, let $x \neq y \in A$ and define

$$\begin{aligned} g : [0,1] &\longrightarrow \mathbb{R} \\ t &\longmapsto f((1-t)x + ty). \end{aligned}$$

g is then differentiable on [0,1] and by the mean value theorem, we can find a real number $c \in]0,1[$ such that $g(1) - g(0) = g'(c)$. Moreover, by the chain rule we have

$$\begin{aligned} g'(c) &= \nabla f((1-c)x + cy) \cdot (y - x) \\ &= Df((1-c)x + cy)[x - y] \\ &= \|x - y\| \, Df((1-c)x + cy) \left[\frac{x-y}{\|x-y\|} \right] \\ &\leq \|x - y\| \, L. \end{aligned}$$

where we use the fact that $\frac{x-y}{\|x-y\|} \in \mathbb{S}_1^{d-1}$. Then we have

$$f(y) - f(x) = g(1) - g(0) = g'(c) \leq \|x - y\| \, L.$$

By symmetry of x and y, we have proven $|f(x) - f(y)| \leq L \|x - y\|$.

We now look at the case where we don't assume that f is differentiable but just has directional derivatives. As before, we will show that $g(1) - g(0) \leq \|x - y\| \, L$. We let $M = \|x - y\| \, L$, $\epsilon > 0$ and $h(x) = g(x) - (M + \epsilon)x$. We will show that h is decreasing. The hypothesis that f has directional derivatives translates into g having left and right derivatives. Moreover for any $t \in [0,1[$

$$g'^{+}(t) = \lim_{h \to 0^+} \frac{g(t+h) - g(t)}{h} = \|x - y\| \, Df((1-t)x + ty) \left[\frac{y-x}{\|y-x\|} \right] \leq \|x - y\| \, L = M.$$

As a consequence of this $h'^{+}(x) = g'^{+}(x) - (M + \epsilon) \leq -\epsilon < 0$. We will now show that h is decreasing on $[0,1]$ by contradiction.

Assume that there are $0 \leq u < v < 1$ such that $h(u) < h(v)$. We let $w = inf(\{z \in [u,1], h(u) < h(z)\})$. Since $h'^{+}(u) < 0$, there is an $\alpha > 0$ such that for any $z \in [u, u+\alpha], h(z) \leq h(u)$. This proves that $w > u$. Also, because h is continuous(Because it has left and right derivatives), we must have $h(w) = h(u)$.

Now since $g'^{+}(w) < 0$, there is once again an $\alpha > 0$ such that for any $z \in [w, w+\alpha], h(z) \leq h(w)$. Now by the definition of w, we can find an element $z \in]w, w+\alpha[$ such that $h(z) < h(u) = h(w)$, this is a contradiction.

Finally we have proven that h is decreasing on [0,1[(and on [0,1] by continuity). This means that

$$0 \leq h(0) - h(1) = g(0) - g(1) + (M + \epsilon).$$

Rearranging we get

$$f(y) - f(x) \leq M + \epsilon.$$

Since ϵ was arbitrary, this means that $f(y) - f(x) \leq M$ and finally by symmetry

$$|f(y) - f(x)| \leq M = L \|x - y\| .$$

$\square$

Exercise 3.17

Prove Proposition 3.16. We suggest you start by assuming $\mathrm{dom}(f)$ has non-empty interior. In case $\mathrm{dom}(f)$ has no interior you should first extend f to the affine hull of the relative interior and then extend the extension to the whole space. You may find it useful to use the fact that for $y \in \mathrm{int}(\mathrm{dom}(f))$, $Df(y)[h] = \sup_{g \in \partial f(y) \langle g,h \rangle}$.

Exercise 3.19

Give an example showing that the constant mR in the last term of the definition of $\bar{f}$ cannot be reduced.

Exercise 3.22

Suppose that $f : \mathbb{R}^d \to \mathbb{R} \bigcup \{\infty\}$ is convex and $\mathrm{lip}(f) \leq \infty$ and let $g = f * \phi_\epsilon$ with $*$ the convolution, which is defined on $\mathrm{dom}(g) = \{x \in \mathbb{R}^d : x + \mathbb{B}_\epsilon^d \subset \mathrm{cl}(\mathrm{dom}(f))\}$. Prove the following statements:

(a) g is twice differentiable on $\mathrm{int}(\mathrm{dom}(g))$.

(b) $\mathrm{lip}(g) \leq \mathrm{lip}(f)$.

(c) g is smooth: $||g''(x)|| \leq \frac{(d+1)(d+6)\mathrm{lip}(f)}{\epsilon}, \forall x \in \mathrm{int}(\mathrm{dom}(g))$.

(d) $\max_{x \in \mathrm{dom}((g))} |f(x) - g(x)| \leq \epsilon \mathrm{lip}(f)$.