

Concentration Inequalities : Exercises solutions

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Chapter 2

Basic Inequalities

Exercise 2.1

Let $\mathbf{M}Z$ be a median of the square integrable variable Z (i.e $\mathbb{P}[Z \geq \mathbf{M}Z] \geq \frac{1}{2}$ and $\mathbb{P}[Z \leq \mathbf{M}Z] \geq \frac{1}{2}$). Show that :

$$|\mathbb{E}[Z] - \mathbf{M}Z| \leq \sqrt{\text{Var}(Z)}$$

Solution : We have the following inequalities :

$$\begin{aligned} |\mathbb{E}[Z] - \mathbf{M}Z| &= |\mathbb{E}[Z - \mathbf{M}Z]| \\ &\leq \mathbb{E}[|Z - \mathbf{M}Z|] \\ &\leq \mathbb{E}[|Z - \mathbb{E}[Z]|] \\ &\leq \sqrt{\mathbb{E}[(Z - \mathbb{E}[Z])^2]} \end{aligned}$$

Where the second inequality comes from the optimality of the median for the mean absolute error ($\mathbf{M}Z \in \arg \min_{c \in \mathbf{R}} \mathbb{E}[|Z - c|]$) and the third inequality is Cauchy-Schwarz. \square

Exercise 2.2

Let X be a random variable with median $\mathbf{M}X$, such that there exist positive constants a and b such that for all $t \geq 0$,

$$\mathbb{P}[|X - \mathbf{M}X| > t] \leq a \exp -\frac{t^2}{b}$$

Show that $|\mathbf{M}X - \mathbb{E}[X]| \leq \min\left(\sqrt{ab}, a\sqrt{b\pi}/2\right)$

Solution : We have the following :

$$\begin{aligned}
|\mathbb{E}[X] - \mathbf{M}X| &= |\mathbb{E}[X - \mathbf{M}X]| \\
&\leq \mathbb{E}[|X - \mathbf{M}X|] \\
&= \int_0^\infty \mathbb{P}[|X - \mathbf{M}X| > t] dt \\
&\leq \int_0^\infty a \exp -\frac{t^2}{b} \\
&\leq a \cdot \sqrt{b\pi}/2
\end{aligned}$$

For the second inequality, we will bound the variance and use the first exercise. We know that

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \min_{c \in \mathbf{R}} \mathbb{E}[(X - c)^2]$$

In particular, plugging in the value of the median as c , we get :

$$\begin{aligned}
Var(X) &\leq \mathbb{E}[(X - \mathbf{M}X)^2] \\
&= \int_0^\infty \mathbb{P}[(X - \mathbf{M}X)^2 > t] dt \\
&= \int_0^\infty \mathbb{P}[|X - \mathbf{M}X| > \sqrt{t}] dt \\
&\leq \int_0^\infty a \cdot \exp -\frac{t}{b} dt \\
&\leq ab
\end{aligned}$$

Then using the result of the first exercise we have :

$$|\mathbb{E}[X] - \mathbf{M}X| \leq \sqrt{Var(X)} \leq \sqrt{ab}$$

□

Exercise 2.3 (CHEBYSHEV-CANTELLI INEQUALITY)

Prove the following one-sided improvement of Chebyshev's inequality : for any real-valued random variable Y and $t > 0$,

$$\mathbb{P}[Y - \mathbb{E}[Y] \geq t] \leq \frac{Var(Y)}{Var(Y) + t^2}$$

Solution : We start by defining $Z := Y - \mathbb{E}[Y]$ and $v := \mathbb{E}[Z^2] = Var(Y)$

We are now trying to prove $\mathbb{P}[Z \geq t] \leq \frac{v}{v+t^2}$. We have for any $u \geq 0$

$$\begin{aligned}\mathbb{P}[Z \geq t] &= \mathbb{P}[Z + u \geq t + u] \\ &\leq \mathbb{P}[(Z + u)^2 \geq (t + u)^2] \\ &\stackrel{(\text{Markov})}{\leq} \frac{\mathbb{E}[(Z + u)^2]}{(t + u)^2} \\ &\leq \frac{v + u^2}{(t + u)^2}\end{aligned}$$

We're now going to optimize the value of this fraction over u . Let

$$\begin{aligned}\varphi : \mathbf{R}^+ &\longrightarrow \mathbf{R} \\ u &\longmapsto \frac{v + u^2}{(t + u)^2}\end{aligned}$$

We compute its derivative :

$$\begin{aligned}\varphi'(u) &= \frac{2u \cdot (t + u)^2 - (v + u^2) \cdot 2(t + u)}{(t + u)^4} \\ &= \frac{(t + u) \cdot (2ut - 2v)}{(t + u)^4}\end{aligned}$$

In particular, φ reaches its minimum in $u^* := \frac{v}{t}$. We plug that value in the previous inequality :

$$\begin{aligned}\mathbb{P}[Z \geq t] \leq \varphi(u^*) &= \frac{v + \frac{v^2}{t}}{(t + \frac{v}{t})^2} \\ &= \frac{v \cdot (t^2 + v)}{(t^2 + v)^2} \\ &= \frac{v}{v + t^2}\end{aligned}$$

□

Exercise 2.4

Solution :

□

Exercise 2.5 (MOMENTS VS. CHERNOFF BOUNDS)

Show that moment bounds for tail probabilities are always better than Cramér-Chernoff bounds. More precisely, let Y be a nonnegative random

variable and let $t > 0$. The best moment bound for the tail probability $\mathbb{P}[Y \geq t]$ is $\min_q \mathbb{E}[Y^q] t^{-q}$ where the minimum is taken over all positive integers. The best Cramér-Chernoff bound is $\inf_{\lambda > 0} \mathbb{E}[\exp \lambda(Y - t)]$. Prove that :

$$\min_q \mathbb{E}[Y^q] t^{-q} \leq \inf_{\lambda > 0} \mathbb{E}[\exp \lambda(Y - t)]$$

Solution : We denote $m := \min_q \mathbb{E}[Y^q] t^{-q}$. In particular, we have that for any integer q , $\mathbb{E}[Y^q] \geq m \cdot t^q$. We will now fix a $\lambda > 0$. We have :

$$\begin{aligned} \mathbb{E}[\exp \lambda(Y - t)] &= e^{-\lambda t} \mathbb{E} \left[\sum_{q=0}^{\infty} \frac{(\lambda Y)^q}{q!} \right] \\ &\stackrel{\text{(Fubini-Tonelli)}}{=} e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q \mathbb{E}[Y^q])}{q!} \\ &\geq e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q \cdot m \cdot t^q)}{q!} \\ &= m \cdot e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q t^q)}{q!} \\ &= m \cdot e^{-\lambda t} \cdot e^{\lambda t} \\ &= m \end{aligned}$$

□

Exercise 2.6

Let Z be a real-valued random variable. Show that the set of positive numbers $S = \{\lambda > 0 : \mathbb{E}[e^{\lambda Z}] < \infty\}$ is either empty or an interval with left end point equal to 0. Let $b = \sup S$. Show that $\varphi_Z(\lambda) = \log \mathbb{E}[e^{\lambda Z}]$ is convex and infinitely many times differentiable on $I = (0, b)$ with $\varphi'_Z(0) = \varphi_Z(0) = 0$ and the Cramér transform of Z equals $\varphi_Z^* = \sup_{\lambda \in I} (\lambda t - \varphi_Z(\lambda))$

Solution : We start with the first claim, we are going to prove that:

$$\text{For any } 0 < \lambda < \mu, \quad \mu \in S \implies \lambda \in S$$

This will prove that S is either empty or an interval with left end point equal to 0. Let $0 < \lambda < \mu$ with $\mu \in S$. We have :

$$\begin{aligned} \mathbb{E}[e^{\lambda Z}] &= \mathbb{E}[e^{\lambda Z} \mathbb{I}_{Z \leq 0} + e^{\lambda Z} \mathbb{I}_{Z > 0}] \\ &= \mathbb{E}[1 \cdot \mathbb{I}_{Z \leq 0}] + \mathbb{E}[e^{\mu Z}] \\ &< \infty \end{aligned}$$

Now for the differentiability, let $0 < \alpha < \beta < b$, $\lambda \in (\alpha, \beta)$ and $k \in \mathbb{N}$. We have :

$$\frac{d^k}{d\lambda} e^{\lambda Z} = Z^k e^{\lambda Z}$$

To prove the differentiability of φ on (α, β) , it is enough to provide a domination of $\frac{d^k}{d\lambda}$ on (α, β) . We have :

$$\begin{aligned} |Z^k e^{\lambda Z}| &\leq |Z|^k e^{\lambda Z} \mathbb{I}_{Z \leq 0} + |Z|^k e^{\lambda Z} \mathbb{I}_{Z > 0} \\ &\leq |Z|^k e^{\alpha Z} \mathbb{I}_{Z \leq 0} + (|Z|^k \cdot e^{-\frac{b-\beta}{2} Z}) (e^{\frac{b+\beta}{2} Z}) \mathbb{I}_{Z > 0} \end{aligned}$$

The left term is integrable because it is bounded and the right term is integrable as the product of a bounded function and an integrable function. Now since $\mathbb{E}[e^{\lambda Z} > 0]$ on S , we have proven that φ is infinitely differentiable on I .

To prove convexity, we use the previous result and compute the 2nd order derivative of φ . For any $\lambda \in I$:

$$\begin{aligned} \varphi'(\lambda) &= \frac{\mathbb{E}[Z e^{\lambda Z}]}{\mathbb{E}[e^{\lambda Z}]} \\ \varphi'' &= \frac{\mathbb{E}[Z^2 e^{\lambda Z}] \cdot \mathbb{E}[e^{\lambda Z}] - \mathbb{E}[Z e^{\lambda Z}]^2}{\mathbb{E}[e^{\lambda Z}]^2} \stackrel{(C.S)}{\geq} 0 \end{aligned}$$

Where the second inequality comes from Cauchy-Schwarz Inequality. This proves the convexity of φ .

To finish the exercise, it remains to study the case in which the expectation of Z is 0. To prove that φ is continuously differentiable on $[0, b)$, we simply need to verify that its derivative has a right limit in 0. It is clear that $Z e^{\lambda Z} \xrightarrow{\lambda \rightarrow 0} Z$. Then :

$$\begin{aligned} \mathbb{E}[Z e^{\lambda Z}] &= \mathbb{E}[Z e^{\lambda Z} \mathbb{I}_{Z \leq 0}] + \mathbb{E}[Z e^{\lambda Z} \mathbb{I}_{Z > 0}] \\ &\xrightarrow{\lambda \rightarrow 0} \mathbb{E}[Z \mathbb{I}_{Z \leq 0}] + \mathbb{E}[Z \mathbb{I}_{Z > 0}] = \mathbb{E}[Z] = 0 \end{aligned}$$

Where the left expectation converges because of dominated convergence and the right one because of monotonic convergence. We have proven continuous differentiability of φ on $[0, b)$ and that $\varphi(0) = \varphi'(0) = 0$. \square

Exercise 2.7

Solution :

\square

Exercise 2.8

Solution :



Exercise 2.9

Solution :



Exercise 2.10

Solution :



Exercise 2.11

Solution :



Exercise 2.12

Solution :



Exercise 2.13

Solution :



Exercise 2.14

Solution :



Exercise 2.15

Solution :



Exercise 2.16

Solution :



Exercise 2.17

Solution :



Exercise 2.18

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Exercise 2.19

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Exercise 2.20

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Exercise 2.21

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Exercise 2.22

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Exercise 2.23

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Exercise 2.24

Solution :



Exercise 2.25

Solution :



Exercise 2.26

Solution :



Exercise 2.27

Solution :



Exercise 2.28

Solution :



Bibliography