Concentration Inequalities : Exercises solutions

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Chapter 2

Basic Inequalities

Exercise 2.1

Let $\mathbf{M}Z$ be a median of the square integrable variable Z (i.e $\mathbb{P}\left[Z \geq \mathbf{M}Z\right] \geq \frac{1}{2}$ and $\mathbb{P}\left[Z \leq \mathbf{M}Z\right] \geq \frac{1}{2}$). Show that :

$$|\mathbb{E}\left[Z\right] - \mathbf{M}Z| \leq \sqrt{Var(Z)}$$

Solution : We have the following inequalities :

$$
\begin{aligned}
|\mathbb{E}\left[Z\right] - \mathbf{M}Z| &= |\mathbb{E}\left[Z - \mathbf{M}Z\right]| \\
&\leq \mathbb{E}\left[|Z - \mathbf{M}Z|\right] \\
&\leq \mathbb{E}\left[|Z - \mathbb{E}\left[Z\right]|\right] \\
&\leq \sqrt{\mathbb{E}\left[(Z - \mathbb{E}\left[Z\right])^2\right]}
\end{aligned}
$$

Where the second inequality comes from the optimality of the median for the mean absolute error($\mathbf{M}Z \in \arg\min_{c \in \boldsymbol{R}} \mathbb{E}\left[|Z - c|\right]$) and the third inequality is Cauchy-Schwarz. $\square$

Exercise 2.2

Let X be a random variable with median $\mathbf{M}X$, such that there exist positive constants a and b such that for all $t \geq 0$,

$$\mathbb{P}\left[|X - \mathbf{M}X| > t\right] \leq a \exp -\frac{t^2}{b}$$

Show that $|\mathbf{M}X - \mathbb{E}\left[X\right]| \leq \min\left(\sqrt{ab}, a\sqrt{b\pi}/2\right)$

Solution : We have the following :

$$|\mathbb{E}\left[X\right] - \mathbf{M}X| = |\mathbb{E}\left[X - \mathbf{M}X\right]|$$
$$\leq \mathbb{E}\left[|X - \mathbf{M}X|\right]$$
$$= \int_0^\infty \mathbb{P}\left[|X - \mathbf{M}X| > t\right] dt$$
$$\leq \int_0^\infty a \exp -\frac{t^2}{b}$$
$$\leq a \cdot \sqrt{b\pi}/2$$

For the second inequality, we will bound the variance and use the first exercise. We know that

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right] = \min_{c \in \mathbf{R}} \mathbb{E}\left[(X - c)^2\right]$$

In particular, plugging in the value of the median as c, we get :

$$Var(X) \leq \mathbb{E}\left[(X - \mathbf{M}X)^2\right]$$
$$= \int_0^\infty \mathbb{P}\left[(X - \mathbf{M}X)^2 > t\right] dt$$
$$= \int_0^\infty \mathbb{P}\left[|X - \mathbf{M}X| > \sqrt{t}\right] dt$$
$$\leq \int_0^\infty a \cdot \exp -\frac{t}{b} dt$$
$$\leq ab$$

Then using the result of the first exercise we have :

$$|\mathbb{E}\left[X\right] - \mathbf{M}X| \leq \sqrt{Var(X)} \leq \sqrt{ab}$$

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Exercise 2.3 (CHEBYSHEV-CANTELLI INEQUALITY)

Prove the following one-sided improvement of Chebyshev's inequality : for any real-valued random variable Y and $t > 0$,

$$\mathbb{P}\left[Y - \mathbb{E}\left[Y\right] \geq t\right] \leq \frac{Var(Y)}{Var(Y) + t^2}$$

Solution : We start by defining $Z := Y - \mathbb{E}\left[Y\right]$ and $v := \mathbb{E}\left[Z^2\right] = Var(Y)$

We are now trying to prove $\mathbb{P}[Z \geq t] \leq \frac{v}{v+t^2}$ We have for any $u \geq 0$

$$\mathbb{P}[Z \geq t] = \mathbb{P}[Z + u \geq t + u]$$
$$\leq \mathbb{P}\left[(Z+u)^2 \geq (t+u)^2\right]$$
$$\overset{\text{(Markov)}}{\leq} \frac{\mathbb{E}\left[(Z+u)^2\right]}{(t+u)^2}$$
$$\leq \frac{v+u^2}{(t+u)^2}$$

We're now going to optimize the value of this fraction over u. Let

$$\varphi : \mathbf{R}^+ \longrightarrow \mathbf{R}$$
$$u \longmapsto \frac{v+u^2}{(t+u)^2}$$

We compute it's derivative :

$$\varphi\prime(u) = \frac{2u \cdot (t+u)^2 - (v+u^2) \cdot 2(t+u)}{(t+u)^4}$$
$$= \frac{(t+u) \cdot (2ut - 2v)}{(t+u)^4}$$

In particular, φ reaches its minimum in $u^* := \frac{v}{t}$. We plug that value in the previous inequality :

$$\mathbb{P}[Z \geq t] \leq \varphi(u^*) = \frac{v + \frac{v}{t}^2}{(t + \frac{v}{t})^2}$$
$$= \frac{v \cdot (t^2 + v)}{(t^2 + v)^2}$$
$$= \frac{v}{v + t^2}$$

$$\boxed{}$$

Exercise 2.4 (Paley-Zigmund Inequality)

Show that if Y is a nonnegative random variable, then for any $a \in (0,1)$,

$$\mathbb{P}[Y - \mathbb{E}[Y]] \geq (1-a)^2 \frac{(\mathbb{E}[Y])^2}{\mathbb{E}[Y^2]}$$

Solution : We have :

$$\mathbb{E}[Y] = \mathbb{E}\left[Y\mathbb{I}_{Y<a\mathbb{E}[Y]}\right] + \mathbb{E}\left[Y\mathbb{I}_{Y\geq a\mathbb{E}[Y]}\right]$$

$$\leq \mathbb{E}\left[a\mathbb{E}[Y]\,\mathbb{I}_{Y<a\mathbb{E}[Y]}\right] + \sqrt{\mathbb{E}[Y^2]\cdot\mathbb{E}\left[\mathbb{I}^2_{Y\geq a\mathbb{E}[Y]}\right]}$$

$$\leq a\mathbb{E}[Y] + \sqrt{\mathbb{E}[Y^2]\cdot\mathbb{E}\left[\mathbb{I}_{Y\geq a\mathbb{E}[Y]}\right]}$$

$$(1-a)\mathbb{E}[Y] \leq \sqrt{\mathbb{E}[Y^2]\cdot\mathbb{P}[Y\geq a\mathbb{E}[Y]]}$$

Where the first inequality comes from Cauchy Schwarz on the right term. Taking the square and rearranging, we get the claimed result.

$\square$

Exercise 2.5 (MOMENTS VS. CHERNOFF BOUNDS)

Show that moment bounds for tail probabilities are always better than Cramér-Chernoff bounds. More precisely, let Y be a nonnegative random variable and let $t > 0$. The best moment bound for the tail probability $\mathbb{P}[Y \geq t]$ is $\min_q \mathbb{E}[Y^q]\,t^{-q}$ where the minimum is taken over all positive integers. The best Cramér-Chernoff bound is $\inf_{\lambda>0}\mathbb{E}[\exp\lambda(Y-t)]$. Prove that :

$$\min_q \mathbb{E}[Y^q]\,t^{-q} \leq \inf_{\lambda>0}\mathbb{E}[\exp\lambda(Y-t)]$$

Solution : We denote $m := \min_q \mathbb{E}[Y^q]\,t^{-q}$. In particular, we have that for any integer q, $\mathbb{E}[Y^q] \geq m\cdot t^q$. We will now fix a $\lambda > 0$. We have :

$$\mathbb{E}[\exp\lambda(Y-t)] = e^{-\lambda t}\mathbb{E}\left[\sum_{q=0}^{\infty}\frac{(\lambda Y)^q}{q!}\right]$$

$$\overset{\text{(Fubini-Tonelli)}}{=} e^{-\lambda t}\sum_{q=0}^{\infty}\frac{(\lambda^q\mathbb{E}[Y]^q)}{q!}$$

$$\geq e^{-\lambda t}\sum_{q=0}^{\infty}\frac{(\lambda^q\cdot m\cdot t^q)}{q!}$$

$$= m\cdot e^{-\lambda t}\sum_{q=0}^{\infty}\frac{(\lambda^q t^q)}{q!}$$

$$= m\cdot e^{-\lambda t}\cdot e^{\lambda t}$$

$$= m$$

$\square$

Exercise 2.6

Let Z be a real-valued random variable. Show that the set of positive numbers $S = \{\lambda > 0 : \mathbb{E}\left[e^{\lambda Z}\right] < \infty\}$ is either empty or an interval with left end point equal to 0. Let $b = \sup S$. Show that $\varphi_Z(\lambda) = \log \mathbb{E}\left[e^{\lambda Z}\right]$ is convex and infinitely many times differentiable on $I = (0, b)$ with $\varphi'_Z(0) = \varphi_Z(0) = 0$ and the Cramér transform of Z equals $\varphi^*_Z = \sup_{\lambda \in I}(\lambda t - \varphi_Z(\lambda))$

Solution : We start with the first claim, we are going to prove that:

$$\text{For any } 0 < \lambda < \mu, \quad \mu \in S \implies \lambda \in S$$

This will prove that S is either empty or an interval with left end point equal to 0. Let $0 < \lambda < \mu$ with $\mu \in S$. We have :

$$\mathbb{E}\left[e^{\lambda Z}\right] = \mathbb{E}\left[e^{\lambda Z}\mathbb{I}_{Z \leq 0} + e^{\lambda Z}\mathbb{I}_{Z > 0}\right]$$
$$= \mathbb{E}\left[1 \cdot \mathbb{I}_{Z \leq 0}\right] + \mathbb{E}\left[e^{\mu Z}\right]$$
$$< \infty$$

Now for the differentiability, let $0 < \alpha < \beta < b$, $\lambda \in (\alpha, \beta)$ and $k \in \mathbb{N}$. We have :

$$\frac{d^k}{d\lambda^k}e^{\lambda Z} = Z^k e^{\lambda Z}$$

To prove the differentiability of φ on (α, β), it is enough to provide a domination of $\frac{d^k}{d\lambda^k}$ on (α, β). We have :

$$|Z^k e^{\lambda Z}| \leq |Z|^k e^{\lambda Z}\mathbb{I}_{Z \leq 0} + |Z|^k e^{\lambda Z}\mathbb{I}_{Z > 0}$$
$$\leq |Z|^k e^{\alpha Z}\mathbb{I}_{Z \leq 0} + (|Z|^k \cdot e^{-\frac{b-\beta}{2}Z})(e^{\frac{b+\beta}{2}Z})\mathbb{I}_{Z > 0}$$

The left term is integrable because it is bounded and the right term is integrable as the product of a bounded function and an integrable function. Now since $\mathbb{E}\left[e^{\lambda Z} > 0\right]$ on S, we have proven that φ is infinitely differentiable on I.

To prove convexity, we use the previous result and compute the 2nd order derivative of φ. For any $\lambda \in I$:

$$\varphi'(\lambda) = \frac{\mathbb{E}\left[Ze^{\lambda Z}\right]}{\mathbb{E}\left[e^{\lambda Z}\right]}$$
$$\varphi'' = \frac{\mathbb{E}\left[Z^2 e^{\lambda Z}\right] \cdot \mathbb{E}\left[e^{\lambda Z}\right] - \mathbb{E}\left[Ze^{\lambda Z}\right]^2}{\mathbb{E}\left[e^{\lambda Z}\right]^2} \overset{(C.S)}{\geq} 0$$

Where the second inequality comes from Cauchy-Schwarz Inequality. This proves the convexity of φ.

To finish the exercise, it remains to study the case in which the expectation of Z is 0. To prove that φ is continuously differentiable on $[0, b)$, we simply need to verify that its derivative has a right limit in 0. It is clear that $Ze^{\lambda Z} \underset{\lambda \to 0}{\to} Z$. Then :

$$\mathbb{E}\left[Ze^{\lambda Z}\right] = \mathbb{E}\left[Ze^{\lambda Z}\mathbb{I}_{Z \leq 0}\right] + \mathbb{E}\left[Ze^{\lambda Z}\mathbb{I}_{Z > 0}\right]$$
$$\underset{\lambda \to 0}{\to} \mathbb{E}\left[Z\mathbb{I}_{Z \leq 0}\right] + \mathbb{E}\left[Z\mathbb{I}_{Z > 0}\right] = \mathbb{E}\left[Z\right] = 0$$

Where the left expectation converges because of dominated convergence and the right one because of monotonic convergence. We have proven continous differntiability of φ on $[0, b)$ and that $\varphi(0) = \varphi'(0) = 0$. $\qquad \boxdot$

Exercise 2.7

Prove that if Z is a centered normal random variable with variance σ^2, then
$$\sup_{t > 0}\left(\mathbb{P}\left[Z \geq t\right]\exp\frac{t^2}{2\sigma^2}\right) = \frac{1}{2}$$

Solution : Without loss of generality, we assume that $\sigma = 1$. Let $t > 0$, we have

$$\mathbb{P}\left[Z \geq t\right] = \int_t^{+\infty}\frac{1}{\sqrt{2\pi}}\exp-\frac{u^2}{2}\,du$$
$$= \int_0^{+\infty}\frac{1}{\sqrt{2\pi}}\exp-\frac{(u+t)^2}{2}du$$
$$= \exp-\frac{t^2}{2}\cdot\int_0^\infty\frac{1}{\sqrt{2\pi}}\exp\left(-ut\right)\exp\left(-\frac{u^2}{2}\right)du$$
$$\mathbb{P}\left[Z \geq t\right]\cdot\exp\left(\frac{t^2}{2}\right) = \int_0^\infty\frac{1}{\sqrt{2\pi}}\exp\left(-ut\right)\exp\left(-\frac{u^2}{2}\right)du$$

Then we notice that $\exp(-ut) \underset{t \to \infty}{\to} 1$ pointwise. More importantly it converges in an increasing manner and since everything inside the integral is nonnegative, we can use the monotone convergence theorem to conclude

that :

$$\sup_{t>0} \mathbb{P}\left[Z \geq t\right] \cdot \exp\left(\frac{t^2}{2}\right) = \sup_{t>0} \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du$$

$$= \lim_{t\to\infty} \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du$$

$$= \int_0^{+\infty} \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du$$

$$= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp-\frac{u^2}{2} du = \frac{1}{2}$$

□

Exercise 2.9 (SUB-GAUSSIAN LOWER TAIL FOR NONNEGATIVE RANDOM

VARIABLES)

Let X be a nonnegative random variable with finite second moment. Show that for any $\lambda > 0$, $\mathbb{E}\left[e^{-\lambda(X-\mathbb{E}[X])}\right] \leq e^{\lambda^2 \mathbb{E}[X^2]/2}$. In particular, if $X_1, \ldots, X_n$ are independent nonnegative random variables, then for any t¿0,

$$\mathbb{P}\left[S \leq -t\right] \leq \exp\left(\frac{-t^2}{v}\right)$$

where $S = \sum_{i=1}^n (X_i - \mathbb{E}\left[X_i\right])$ and $v = \sum_{i=1}^n \mathbb{E}\left[X_i^2\right]$

Solution : We have the following elementary inequalities :

$$\forall x \geq 0, \ \exp-x \leq 1 - x + \frac{x^2}{2}$$

$$\forall x \in \mathbb{R}, \ 1 + x \leq \exp x$$

Using this we get :

$$\mathbb{E}\left[e^{-\lambda(X-\mathbb{E}[X])}\right] = e^{\lambda\mathbb{E}[X]} \cdot \mathbb{E}\left[e^{-\lambda X}\right]$$

$$\leq e^{\lambda\mathbb{E}[X]} \cdot \mathbb{E}\left[1 - \lambda X + \frac{\lambda^2 X^2}{2}\right]$$

$$\leq e^{\lambda\mathbb{E}[X]} \cdot \left(1 + \left(-\lambda\mathbb{E}\left[X\right] + \frac{\lambda^2}{2}\mathbb{E}\left[X^2\right]\right)\right)$$

$$\leq e^{\lambda\mathbb{E}[X]} \cdot e^{-\lambda X + \frac{\lambda^2}{2}\mathbb{E}[X^2]}$$

$$= e^{\lambda^2\mathbb{E}[X^2]/2}$$

□

Bibliography