Concentration Inequalities : Exercises solutions

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Contents

Chapter 2

Basic Inequalities

Exercise 2.1

Let $\mathbf{M}Z$ be a median of the square integrable variable Z (i.e $\mathbb{P}[Z \geq \mathbf{M}Z] \geq \frac{1}{2}$ and $\mathbb{P}[Z \leq \mathbf{M}Z] \geq \frac{1}{2}$). Show that :

$$|\mathbb{E}[Z] - \mathbf{M}Z| \leq \sqrt{Var(Z)}$$

Solution : We have the following inequalities :

$$\begin{aligned}
|\mathbb{E}[Z] - \mathbf{M}Z| &= |\mathbb{E}[Z - \mathbf{M}Z]| \\
&\leq \mathbb{E}[|Z - \mathbf{M}Z|] \\
&\leq \mathbb{E}[|Z - \mathbb{E}[Z]|] \\
&\leq \sqrt{\mathbb{E}[(Z - \mathbb{E}[Z])^2]}
\end{aligned}$$

Where the second inequality comes from the optimality of the median for the mean absolute error($\mathbf{M}Z \in \arg\min_{c \in \boldsymbol{R}} \mathbb{E}[|Z - c|]$) and the third inequality is Cauchy-Schwarz. ☑

Exercise 2.2

Let X be a random variable with median $\mathbf{M}X$, such that there exist positive constants a and b such that for all $t \geq 0$,

$$\mathbb{P}[|X - \mathbf{M}X| > t] \leq a \exp -\frac{t^2}{b}$$

Show that $|\mathbf{M}X - \mathbb{E}[X]| \leq \min\left(\sqrt{ab}, a\sqrt{b\pi}/2\right)$

Solution : We have the following :

$$\begin{aligned}
|\mathbb{E}[X] - \mathbf{M}X| &= |\mathbb{E}[X - \mathbf{M}X]| \\
&\leq \mathbb{E}[|X - \mathbf{M}X|] \\
&= \int_0^\infty \mathbb{P}[|X - \mathbf{M}X| > t]\, dt \\
&\leq \int_0^\infty a \exp -\frac{t^2}{b} \\
&\leq a \cdot \sqrt{b\pi}/2
\end{aligned}$$

For the second inequality, we will bound the variance and use the first exercise. We know that

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \min_{c \in \boldsymbol{R}} \mathbb{E}\left[(X - c)^2\right]$$

In particular, plugging in the value of the median as c, we get :

$$\begin{aligned}
Var(X) &\leq \mathbb{E}\left[(X - \mathbf{M}X)^2\right] \\
&= \int_0^\infty \mathbb{P}\left[(X - \mathbf{M}X)^2 > t\right] dt \\
&= \int_0^\infty \mathbb{P}\left[|X - \mathbf{M}X| > \sqrt{t}\right] dt \\
&\leq \int_0^\infty a \cdot \exp -\frac{t}{b} dt \\
&\leq ab
\end{aligned}$$

Then using the result of the first exercise we have :

$$|\mathbb{E}[X] - \mathbf{M}X| \leq \sqrt{Var(X)} \leq \sqrt{ab}$$

$\square$

Exercise 2.3 (Chebyshev-Cantelli Inequality)

Prove the following one-sided improvement of Chebyshev's inequality : for any real-valued random variable Y and $t > 0$,

$$\mathbb{P}[Y - \mathbb{E}[Y] \geq t] \leq \frac{Var(Y)}{Var(Y) + t^2}$$

Solution : We start by defining $Z := Y - \mathbb{E}[Y]$ and $v := \mathbb{E}\left[Z^2\right] = Var(Y)$

We are now trying to prove $\mathbb{P}[Z \geq t] \leq \frac{v}{v+t^2}$ We have for any $u \geq 0$

$$\begin{aligned}
\mathbb{P}[Z \geq t] &= \mathbb{P}[Z + u \geq t + u] \\
&\leq \mathbb{P}\left[(Z+u)^2 \geq (t+u)^2\right] \\
&\overset{\text{(Markov)}}{\leq} \frac{\mathbb{E}\left[(Z+u)^2\right]}{(t+u)^2} \\
&\leq \frac{v+u^2}{(t+u)^2}
\end{aligned}$$

We're now going to optimize the value of this fraction over u. Let

$$\begin{aligned}
\varphi : \boldsymbol{R}^+ &\longrightarrow \boldsymbol{R} \\
u &\longmapsto \frac{v+u^2}{(t+u)^2}
\end{aligned}$$

We compute it's derivative :

$$\begin{aligned}
\varphi\prime(u) &= \frac{2u \cdot (t+u)^2 - (v+u^2) \cdot 2(t+u)}{(t+u)^4} \\
&= \frac{(t+u) \cdot (2ut - 2v)}{(t+u)^4}
\end{aligned}$$

In particular, φ reaches its minimum in $u^* := \frac{v}{t}$. We plug that value in the previous inequality :

$$\begin{aligned}
\mathbb{P}[Z \geq t] \leq \varphi(u^*) &= \frac{v + \frac{v}{t}^2}{(t + \frac{v}{t})^2} \\
&= \frac{v \cdot (t^2+v)}{(t^2+v)^2} \\
&= \frac{v}{v+t^2}
\end{aligned}$$

Exercise 2.4

Solution :

Exercise 2.5 (Moments VS. Chernoff Bounds)

Show that moment bounds for tail probabilities are always better than Cramér-Chernoff bounds. More precisely, let Y be a nonnegative random variable and let $t > 0$. The best moment bound for the tail probability $\mathbb{P}[Y \geq t]$ is $\min_q \mathbb{E}[Y^q]\, t^{-q}$ where the minimum is taken over all positive integers. The best Cramér-Chernoff bound is $\inf_{\lambda>0} \mathbb{E}[\exp \lambda(Y-t)]$. Prove that :

$$\min_q \mathbb{E}[Y^q]\, t^{-q} \leq \inf_{\lambda>0} \mathbb{E}[\exp \lambda(Y-t)]$$

Solution : We denote $m := \min_q \mathbb{E}[Y^q]\, t^{-q}$. In particular, we have that for any integer q, $\mathbb{E}[Y^q] \geq m \cdot t^q$. We will now fix a $\lambda > 0$. We have :

$$\begin{aligned}
\mathbb{E}[\exp \lambda(Y-t)] &= e^{-\lambda t}\mathbb{E}\left[\sum_{q=0}^{\infty} \frac{(\lambda Y)^q}{q!}\right] \\
&\overset{\text{(Fubini}-\text{Tonelli)}}{=} e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q \mathbb{E}[Y]^q)}{q!} \\
&\geq e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q \cdot m \cdot t^q)}{q!} \\
&= m \cdot e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q t^q)}{q!} \\
&= m \cdot e^{-\lambda t} \cdot e^{\lambda t} \\
&= m
\end{aligned}$$

$\square$

Exercise 2.6

Solution : $\square$

Exercise 2.7

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Exercise 2.8

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Exercise 2.9

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Exercise 2.10

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Exercise 2.28

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Bibliography