Concentration Inequalities : Exercises solutions

Ludovic Schwartz

Contents

Chapter 2

Basic Inequalities

Exercise 2.1

Let $\mathbf{M}Z$ be a median of the square integrable variable Z (i.e $\mathbb{P}[Z \geq \mathbf{M}Z] \geq \frac{1}{2}$ and $\mathbb{P}[Z \leq \mathbf{M}Z] \geq \frac{1}{2}$). Show that :

$$|\mathbb{E}[Z] - \mathbf{M}Z| \leq \sqrt{Var(Z)}$$

Solution : We have the following inequalities :

$$\begin{aligned}
|\mathbb{E}[Z] - \mathbf{M}Z| &= |\mathbb{E}[Z - \mathbf{M}Z]| \\
&\leq \mathbb{E}[|Z - \mathbf{M}Z|] \\
&\leq \mathbb{E}[|Z - \mathbb{E}[Z]|] \\
&\leq \sqrt{\mathbb{E}[(Z - \mathbb{E}[Z])^2]}
\end{aligned}$$

Where the second inequality comes from the optimality of the median for the mean absolute error($\mathbf{M}Z \in \arg\min_{c \in \boldsymbol{R}} \mathbb{E}[|Z - c|]$) and the third inequality is Cauchy-Schwarz. $\square$

Exercise 2.2

Let X be a random variable with median $\mathbf{M}X$, such that there exist positive constants a and b such that for all $t \geq 0$,

$$\mathbb{P}[|X - \mathbf{M}X| > t] \leq a \exp -\frac{t^2}{b}$$

Show that $|\mathbf{M}X - \mathbb{E}[X]| \leq \min\left(\sqrt{ab}, a\sqrt{b\pi}/2\right)$

Solution : We have the following :

$$\begin{aligned}|\mathbb{E}[X] - \mathbf{M}X| &= |\mathbb{E}[X - \mathbf{M}X]| \\ &\leq \mathbb{E}[|X - \mathbf{M}X|] \\ &= \int_0^\infty \mathbb{P}[|X - \mathbf{M}X| > t]\, dt \\ &\leq \int_0^\infty a \exp -\frac{t^2}{b} \\ &\leq a \cdot \sqrt{b\pi}/2\end{aligned}$$

For the second inequality, we will bound the variance and use the first exercise. We know that

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \min_{c \in \boldsymbol{R}} \mathbb{E}\left[(X - c)^2\right]$$

In particular, plugging in the value of the median as c, we get :

$$\begin{aligned}Var(X) &\leq \mathbb{E}\left[(X - \mathbf{M}X)^2\right] \\ &= \int_0^\infty \mathbb{P}\left[(X - \mathbf{M}X)^2 > t\right] dt \\ &= \int_0^\infty \mathbb{P}\left[|X - \mathbf{M}X| > \sqrt{t}\right] dt \\ &\leq \int_0^\infty a \cdot \exp -\frac{t}{b} dt \\ &\leq ab\end{aligned}$$

Then using the result of the first exercise we have :

$$|\mathbb{E}[X] - \mathbf{M}X| \leq \sqrt{Var(X)} \leq \sqrt{ab}$$

$\square$

Exercise 2.3 (Chebyshev-Cantelli Inequality)

Prove the following one-sided improvement of Chebyshev's inequality : for any real-valued random variable Y and $t > 0$,

$$\mathbb{P}[Y - \mathbb{E}[Y] \geq t] \leq \frac{Var(Y)}{Var(Y) + t^2}$$

Solution : We start by defining $Z := Y - \mathbb{E}[Y]$ and $v := \mathbb{E}\left[Z^2\right] = Var(Y)$

We are now trying to prove $\mathbb{P}[Z \geq t] \leq \frac{v}{v+t^2}$ We have for any $u \geq 0$

$$\begin{aligned}
\mathbb{P}[Z \geq t] &= \mathbb{P}[Z+u \geq t+u] \\
&\leq \mathbb{P}\left[(Z+u)^2 \geq (t+u)^2\right] \\
&\overset{\text{(Markov)}}{\leq} \frac{\mathbb{E}\left[(Z+u)^2\right]}{(t+u)^2} \\
&\leq \frac{v+u^2}{(t+u)^2}
\end{aligned}$$

We're now going to optimize the value of this fraction over u. Let

$$\begin{aligned}
\varphi : \boldsymbol{R}^+ &\longrightarrow \boldsymbol{R} \\
u &\longmapsto \frac{v+u^2}{(t+u)^2}
\end{aligned}$$

We compute it's derivative :

$$\begin{aligned}
\varphi\prime(u) &= \frac{2u \cdot (t+u)^2 - (v+u^2) \cdot 2(t+u)}{(t+u)^4} \\
&= \frac{(t+u) \cdot (2ut - 2v)}{(t+u)^4}
\end{aligned}$$

In particular, φ reaches its minimum in $u^* := \frac{v}{t}$. We plug that value in the previous inequality :

$$\begin{aligned}
\mathbb{P}[Z \geq t] \leq \varphi(u^*) &= \frac{v + \frac{v}{t}^2}{(t+\frac{v}{t})^2} \\
&= \frac{v \cdot (t^2+v)}{(t^2+v)^2} \\
&= \frac{v}{v+t^2}
\end{aligned}$$

□

Exercise 2.4

Solution : □

Exercise 2.5 (Moments VS. Chernoff Bounds)

Show that moment bounds for tail probabilities are always better than Cramér-Chernoff bounds. More precisely, let Y be a nonnegative random variable and let $t > 0$. The best moment bound for the tail probability $\mathbb{P}[Y \geq t]$ is $\min_q \mathbb{E}[Y^q]\, t^{-q}$ where the minimum is taken over all positive integers. The best Cramér-Chernoff bound is $\inf_{\lambda>0} \mathbb{E}[\exp \lambda(Y-t)]$. Prove that :

$$\min_q \mathbb{E}[Y^q]\, t^{-q} \leq \inf_{\lambda>0} \mathbb{E}[\exp \lambda(Y-t)]$$

Solution : We denote $m := \min_q \mathbb{E}[Y^q]\, t^{-q}$. In particular, we have that for any integer q, $\mathbb{E}[Y^q] \geq m \cdot t^q$. We will now fix a $\lambda > 0$. We have :

$$\begin{aligned}
\mathbb{E}[\exp \lambda(Y-t)] &= e^{-\lambda t}\mathbb{E}\left[\sum_{q=0}^{\infty} \frac{(\lambda Y)^q}{q!}\right] \\
&\overset{\text{(Fubini–Tonelli)}}{=} e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q \mathbb{E}[Y]^q)}{q!} \\
&\geq e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q \cdot m \cdot t^q)}{q!} \\
&= m \cdot e^{-\lambda t} \sum_{q=0}^{\infty} \frac{(\lambda^q t^q)}{q!} \\
&= m \cdot e^{-\lambda t} \cdot e^{\lambda t} \\
&= m
\end{aligned}$$

$\square$

Exercise 2.6

Let Z be a real-valued random variable. Show that the set of positive numbers $S = \{\lambda > 0 : \mathbb{E}\left[e^{\lambda Z}\right] < \infty\}$ is either empty or an interval with left end point equal to 0. Let $b = \sup S$. Show that $\varphi_Z(\lambda) = \log \mathbb{E}\left[e^{\lambda Z}\right]$ is convex and infinitely many times differentiable on $I = (0, b)$ with $\varphi'_Z(0) = \varphi_Z(0) = 0$ and the Cramér transform of Z equals $\varphi^*_Z = \sup_{\lambda \in I}(\lambda t - \varphi_Z(\lambda))$

Solution : We start with the first claim, we are going to prove that:

$$\text{For any } 0 < \lambda < \mu, \quad \mu \in S \implies \lambda \in S$$

This will prove that S is either empty or an interval with left end point equal to 0. Let $0 < \lambda < \mu$ with $\mu \in S$. We have :

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda Z}\right] &= \mathbb{E}\left[e^{\lambda Z}\mathbb{I}_{Z\leq 0} + e^{\lambda Z}\mathbb{I}_{Z>0}\right] \\
&= \mathbb{E}[1 \cdot \mathbb{I}_{Z\leq 0}] + \mathbb{E}\left[e^{\mu Z}\right] \\
&< \infty
\end{aligned}$$

Now for the differentiability, let $0 < \alpha < \beta < b$, $\lambda \in (\alpha, \beta)$ and $k \in \mathbb{N}$. We have :

$$\frac{d^k}{d\lambda} e^{\lambda Z} = Z^k e^{\lambda Z}$$

To prove the differentiability of φ on (α, β), it is enough to provide a domination of $\frac{d^k}{d\lambda}$ on (α, β). We have :

$$\begin{aligned} |Z^k e^{\lambda Z}| &\leq |Z|^k e^{\lambda Z} \mathbb{I}_{Z \leq 0} + |Z|^k e^{\lambda Z} \mathbb{I}_{Z > 0} \\ &\leq |Z|^k e^{\alpha Z} \mathbb{I}_{Z \leq 0} + (|Z|^k \cdot e^{-\frac{b-\beta}{2} Z})(e^{\frac{b+\beta}{2} Z}) \mathbb{I}_{Z > 0} \end{aligned}$$

The left term is integrable because it is bounded and the right term is integrable as the product of a bounded function and an integrable function. Now since $\mathbb{E}\left[e^{\lambda Z} > 0\right]$ on S, we have proven that φ is infinitely differentiable on I.

To prove convexity, we use the previous result and compute the 2nd order derivative of φ. For any $\lambda \in I$:

$$\begin{aligned} \varphi'(\lambda) &= \frac{\mathbb{E}\left[Z e^{\lambda Z}\right]}{\mathbb{E}\left[e^{\lambda Z}\right]} \\ \varphi'' &= \frac{\mathbb{E}\left[Z^2 e^{\lambda Z}\right] \cdot \mathbb{E}\left[e^{\lambda Z}\right] - \mathbb{E}\left[Z e^{\lambda Z}\right]^2}{\mathbb{E}\left[e^{\lambda Z}\right]^2} \overset{\text{(C.S)}}{\geq} 0 \end{aligned}$$

Where the second inequality comes from Cauchy-Schwarz Inequality. This proves the convexity of φ.

To finish the exercise, it remains to study the case in which the expectation of Z is 0. To prove that φ is continuously differentiable on $[0, b)$, we simply need to verify that its derivative has a right limit in 0. It is clear that $Z e^{\lambda Z} \underset{\lambda \to 0}{\to} Z$. Then :

$$\begin{aligned} \mathbb{E}\left[Z e^{\lambda Z}\right] &= \mathbb{E}\left[Z e^{\lambda Z} \mathbb{I}_{Z \leq 0}\right] + \mathbb{E}\left[Z e^{\lambda Z} \mathbb{I}_{Z > 0}\right] \\ &\underset{\lambda \to 0}{\to} \mathbb{E}\left[Z \mathbb{I}_{Z \leq 0}\right] + \mathbb{E}\left[Z \mathbb{I}_{Z > 0}\right] = \mathbb{E}\left[Z\right] = 0 \end{aligned}$$

Where the left expectation converges because of dominated convergence and the right one because of monotonic convergence. We have proven continous differntiability of φ on $[0, b)$ and that $\varphi(0) = \varphi'(0) = 0$. $\square$

Exercise 2.7

Prove that if Z is a centered normal random variable with variance σ^2, then

$$\sup_{t>0} \left(\mathbb{P}\left[Z \geq t\right] \exp \frac{t^2}{2\sigma^2} \right) = \frac{1}{2}$$

Solution : Without loss of generality, we assume that $\sigma = 1$. Let $t > 0$, we have

$$\begin{aligned}
\mathbb{P}\left[Z \geq t\right] &= \int_t^{+\infty} \frac{1}{\sqrt{2\pi}} \exp - \frac{u^2}{2}\, du \\
&= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp - \frac{(u+t)^2}{2} du \\
&= \exp - \frac{t^2}{2} \cdot \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du \\
\mathbb{P}\left[Z \geq t\right] \cdot \exp\left(\frac{t^2}{2}\right) &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du
\end{aligned}$$

Then we notice that $\exp(-ut) \underset{t\to\infty}{\rightarrow} 1$ pointwise. More importantly it converges in an increasing manner and since everything inside the integral is nonnegative, we can use the monotone convergence theorem to conclude that :

$$\begin{aligned}
\sup_{t>0} \mathbb{P}\left[Z \geq t\right] \cdot \exp\left(\frac{t^2}{2}\right) &= \sup_{t>0} \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du \\
&= \lim_{t\to\infty} \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du \\
&= \int_0^{+\infty} \lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-ut\right) \exp\left(-\frac{u^2}{2}\right) du \\
&= \int_0^{+\infty} \frac{1}{\sqrt{2\pi}} \exp - \frac{u^2}{2} du = \frac{1}{2}
\end{aligned}$$

$\square$

Bibliography