Modern intro to Online Learning – Exercise Solutions

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Chapter 1

What is online learning ?

Exercise 1.1

Extend the previous algorithm and analysis to the case when the adversary selects a vector $y_t \in \mathbb{R}^d$ such that $\|y\|_2 \leq 1$, the algorithm guesses a vector $x_t \in \mathbb{R}^d$, and the loss function is $\|x_t - y_t\|_2^2$. Show an upper bound to the regret logarithmic in T and that does not depend on d. Among the other things, you will probably need the Cauchy-Schwarz inequality : $\langle x, y \rangle \leq \|x\|_2 \|y\|_2$.

Solution : The natural extension of the previous algorithm is to pick the x_t which minimizes the cumulated loss up to time $t-1$. We also define x_t^* the optimal comparator at times t :

$$x_t = x_{t-1}^* = \underset{x \in \mathbb{R}}{\arg\min} \sum_{s=1}^{t-1} \|x - y_t\|^2$$

We can once again explicitely compute the value of x_t. Indeed, if we define $F_t(x) := \sum_{s=1}^{t} \|x - y_t\|^2$, F_t is a strictly convex function and reaches its minimum where its gradient vanishes. A simple computation then gives $\nabla F_t(x) = 0 \leftrightarrow \sum_{s=1}^{t} 2(x - y_s) = 0 \leftrightarrow x = \frac{1}{t} \sum_{s=1}^{t} y_s$ In particular, we have again $x_t = \frac{1}{t-1} \sum_{s=1}^{t-1} y_s$ and we can notice that $\|x_t\| \leq 1$ at all time.

Now, by lemma 1.2 with the loss $\ell_t(x) = \|x - y_t\|^2$, we have :

$$\forall T, \sum_{t=1}^{T} \|x_T^* - y_t\|^2 \geq \sum_{s=1}^{T} \|x_t^* - y_t\|^2$$

We can now complete the proof :

$$\begin{aligned}
R_T &= \sum_{t=1}^{T} \|x_t - y_t\|^2 - \min_{x\in\mathbb{R}} \sum_{t=1}^{T} \|x - y_t\|^2 \\
&= \sum_{t=1}^{T} \left\|x_{t-1}^* - y_t\right\|^2 - \sum_{t=1}^{T} \|x_T - y_t\|^2 \\
&\leq \sum_{t=1}^{T} \left\|x_{t-1}^* - y_t\right\|^2 - \sum_{t=1}^{T} \|x_t^* - y_t\|^2 \\
&= \sum_{t=1}^{T} \langle x_{t-1}^* + x_t^* - 2y_t, x_{t-1}^* - x_t^* \rangle \\
&\overset{\text{(C.S)}}{\leq} \sum_{t=1}^{T} \left\|x_{t-1}^* + x_t^* - 2y_t\right\| \cdot \left\|x_{t-1}^* - x_t^*\right\| \\
&\leq \sum_{t=1}^{T} 4 \left\|x_{t-1}^* - x_t^*\right\|
\end{aligned}$$

Where the first inequality uses lemma 1.2, the second one uses Cauchy-Schwarz and the third uses that $\forall t$, $\|x_t^*\| \leq 1 \quad \text{and} \quad \|y_t\| \leq 1$. Now we notice that

$$\begin{aligned}
\left\|x_{t-1}^* - x_t\right\| &= \left\| \frac{1}{t-1} \sum_{s=1}^{t-1} y_s - \frac{1}{t} \sum_{s=1}^{t} y_s \right\| \\
&= \left\| \frac{1}{t(t-1)} \sum_{s=1}^{t-1} y_s + \frac{1}{t} y_t \right\| \\
&\leq \frac{1}{t(t-1)} \sum_{s=1}^{t-1} \|y_s\| + \frac{1}{t} \|y_t\| \\
&\leq \frac{2}{t}
\end{aligned}$$

Now we plug everything together :

$$R_T \leq 4 \cdot \sum_{t=1}^{T} \left\|x_{t-1}^* - x_t^*\right\| \leq 8 \cdot \sum_{t=1}^{T} \frac{1}{t} \leq 8 + 8 \log T$$

$\square$

Chapter 2

Online Subgradient Descent

Exercise 2.1

Prove that $\sum_{t=1}^{T} \frac{1}{\sqrt{t}} \leq 2\sqrt{T} - 1$

Solution : We have :

$$\begin{aligned}
\sum_{t=1}^{T} \frac{1}{\sqrt{t}} &= 1 + \sum_{t=2}^{T} \frac{1}{\sqrt{t}} \\
&= 1 + \sum_{t=2}^{T} \int_{t-1}^{t} \frac{1}{\sqrt{t}} du \\
&\leq 1 + \sum_{t=2}^{T} \int_{t-1}^{t} \frac{1}{\sqrt{u}} du \\
&\leq 1 + \int_{1}^{T} \frac{1}{\sqrt{u}} du \\
&= 1 + 2\sqrt{T} - 2 = 2\sqrt{T} - 1
\end{aligned}$$

Exercise 2.2

Using the inequality in the previous exercise, prove that a learning rate $\propto \frac{1}{\sqrt{t}}$ gives rise to a regret only a constant multiplicative factor worse than the one in (2.1)($R_T \leq DL\sqrt{T}$).

Solution : We start at the result of theorem 2.13 :

$$R_T \leq \frac{D^2}{2\eta_T} + \sum_{t=1}^{T} \frac{\eta_t}{2} \|g_t\|^2$$

Then we bound for any t, $\|g_t\|^2 \leq L^2$ and set $\eta_t = \alpha \frac{1}{\sqrt{t}}$ with $\alpha > 0$ to be determined later. We have

$$\begin{aligned} R_T &\leq \frac{D^2\sqrt{T}}{2\alpha} + \sum_{t=1}^{T} \frac{\alpha}{2\sqrt{t}} L^2 \\ &= \frac{D^2\sqrt{T}}{2\alpha} + \frac{\alpha L^2}{2} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} \\ &\leq \frac{D^2\sqrt{T}}{2\alpha} + \alpha L^2 \sqrt{T} \\ &= \sqrt{T}\left(\frac{D^2}{2\alpha} + \alpha L^2\right) \\ &= DL\sqrt{2T} \end{aligned}$$

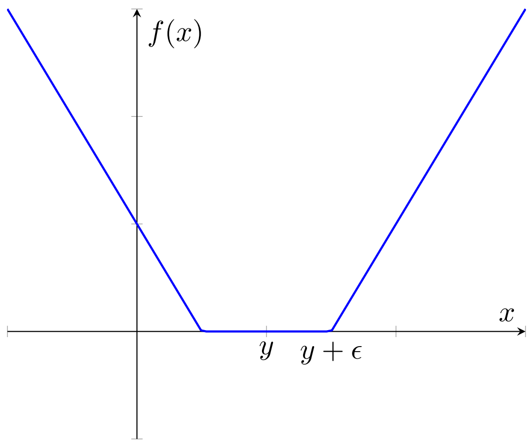
Where the third line uses the result of the previous exercise and the last lines uses the choice $\alpha = \sqrt{\frac{D^2}{2L^2}}$. We remark that we are only a factor $\sqrt{2}$ worse than the bound obtained with a fixed η.

$\square$

Exercise 2.3

Calculate the subdifferential set of the ϵ-insensitive loss :
$f(x) = \max(|x - y| - \epsilon, 0)$

Solution : We start by a drawing of the function :



Then the subdifferential is :

$$\partial f(x) = \begin{cases} \{0\} & \text{if } x \in]y-\epsilon, y+\epsilon[\\ \{-1\} & \text{if } x \in]-\infty, y-\epsilon[\\ \{1\} & \text{if } x \in]y+\epsilon, \infty[\\ [0,1] & \text{if } x = y+\epsilon \\ [-1,1] & \text{if } x = y-\epsilon \end{cases}$$

$\square$

Exercise 2.4

Using the definition of subgradient, find the subdifferential set of $f(x) = \|x\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$, $x \in \mathbb{R}^d$

Solution : We start by treating the case $x \neq 0$. Then f is actually differentiable in x and we have :

$$\frac{\partial f}{\partial x_i}(x) = \frac{x_i}{\sqrt{\sum_{i=1}^d x_i^2}} = \frac{x_i}{\|x\|_2}$$

In that case, we simply have :

$$\partial f(x) = \{\nabla f(x)\} = \left\{ \frac{x}{\|x\|_2} \right\}$$

For the case $x = 0$ we use the definition of a subgradient, g is a subgradient of f at the point $x = 0$ if and only if

$$\forall y \in \mathbb{R}^d,\ \|y\|_2 \geq \|0\|_2 + \langle g, y \rangle = \langle g, y \rangle$$

By Cauchy Schwarz Inequality, we have :

$$\forall g, y \in \mathbb{R}^d,\ \langle g, y \rangle \leq \|g\|_2 \cdot \|y\|_2$$

So any g of norm smaller than 1 will be a subgradient of f at $x = 0$. Conversely, if $\|g\|_2 > 1$, then we have for $y = g$, $\langle g, y \rangle = \langle y, y \rangle = \|y\|_2^2 > \|y\|_2$ and g will not be a subgradient of f at $x = 0$. To conclude, we have that :

$$\partial f(x) = \begin{cases} \{\frac{x}{\|x\|_2}\} & \text{if } x \neq 0 \\ \{g, \|g\|_2 \leq 1\} & \text{if } x = 0 \end{cases}$$

$\square$

Exercise 2.5

Consider Projected Online Subgradient Descent for the example 2.10 on the failure of Follow-the-Leader: Can we use it on that problem ? Would it guarantee sublinear regret ? How would the behaviour of the algorithm differ from FTL ?

Solution : The setting of Example 2.10 is the following : Let $V = [-1, 1]$ and consider the sequence of losses $\ell_t(x) = z_t x + i_v(x)$, where

$$z_t = \begin{cases} -0.5 & \text{if } t = 1 \\ (-1)^t & \text{if } t > 1 \end{cases}$$

We can apply Projected Online Subgradient Descent in that problem. In particular, since we only do prediction inside of V, we can consider the loss function $\ell_t(x) = z_t x$. We have :

$$\nabla \ell_t(x) = z_t$$

Now we can verify that :

$$\forall x \in V, \|\nabla \ell_t(x)\| \leq 1 := L$$

and that :

$$D := \sup_{x,y \in V} \|x - y\| = 2 < \infty$$

Theorem 2.13 applies and we get a bound of order $DL\sqrt{T} = 2\sqrt{T}$, that is to say, sublinear regret. The main difference with Follow the Leader is that Projected Online Subgradient Decent will pick points in the interior of V and do small updates on them while Follow the leader picks extremal points and moves a lot between each prediction.

$\square$

Chapter 4

Beyond $\sqrt{T}$ Regret

Exercise 4.1

Prove that OSD in Example 4.11 with $x_1 = 0$ is exactly the Follow-the-Leader strategy for that particular problem.

Solution : In example 4.11, we have $\ell_t(x) = (x - y_t)^2$, $\eta_t = \frac{1}{2t}$. We can explicitely compute the predictions of OSD :

$$x_1 = 0$$

and

$$\begin{aligned}
x_{t+1} &= x_t - \eta_t \nabla \ell_t(x_t) \\
&= x_t - \frac{1}{t}(x_t - y_t) \\
&= \frac{t-1}{t} x_t + \frac{1}{t}
\end{aligned}$$

An elementary induction then gives $x_{t+1} = \frac{1}{t}\sum_{s=1}^{t} y_s$ which is exactly the strategy of Follow-the-Leader on that problem. $\square$

Exercise 4.4

Prove that the dual norm of $\|\cdot\|_p$ is $\|\cdot\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q \geq 1$

Solution : Let $p, q \geq 1$. By absolute homogeneity of the norms, it suffices to prove that $\forall y \in \mathbb{R}^d$, $\|y\|_q = 1 \implies \|y\|_p^* = 1$.

Let $y \in \mathbb{R}^d$ such that $\|y\|_q = 1$, we have

$$\|y\|_p^* = \max_{\|x\|_p = 1} \langle x, y \rangle \leq \|x\|_p \|y\|_q = 1$$

by Hölder's inequality with equality for some value of x. For completeness, we reprove the equality case of Hölder's inequality here. If $1 < p < \infty$, we define $x_i = sgn(y_i) \cdot |y_i|^{\frac{q}{p}}$ (With the convention $sgn(0) = 0$) We have :

$$\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{d} |y_i|^{p \cdot \frac{q}{p}}\right)^{\frac{1}{p}} = \|y\|_q^{\frac{q}{p}} = 1$$

And

$$\langle x, y\rangle = \sum_{i=1}^{d} x_i \cdot y_i = \sum_{i=1}^{d} |y_i|^{1+\frac{q}{p}} = \sum_{i=1}^{d} |y_i|^q = \|y\|_q^q = 1$$

If $p = 1$, let $i^* \in \arg\max_i |y_i|$ and $x_i = sgn(y_{i^*})\mathbb{I}_{i=i^*}$. We have $\|x\|_1 = 1$, and $\langle x, y\rangle = |y_{i^*}| = \|y\|_\infty = 1$

If $p = \infty$, let $x_i = sgn(y_i)$(with the convention $sgn(0) = 0$). We have $\|x\|_\infty = 1$ and $\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i = \sum_{i=1}^{d} |y_i| = \|y\|_1 = 1$

$\square$