Modern intro to Online Learning – Exercise Solutions

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Chapter 1

What is online learning ?

Exercise 1.1

Extend the previous algorithm and analysis to the case when the adversary selects a vector $y_t \in \mathbb{R}^d$ such that $\|y\|_2 \leq 1$, the algorithm guesses a vector $x_t \in \mathbb{R}^d$, and the loss function is $\|x_t - y_t\|_2^2$. Show an upper bound to the regret logarithmic in T and that does not depend on d. Among the other things, you will probably need the Cauchy-Schwarz inequality : $\langle x, y \rangle \leq \|x\|_2 \|y\|_2$.

Solution : The natural extension of the previous algorithm is to pick the x_t which minimizes the cumulated loss up to time $t-1$. We also define x_t^* the optimal comparator at times t :

$$x_t = x_{t-1}^* = \operatorname*{arg\,min}_{x \in \mathbb{R}} \sum_{s=1}^{t-1} \|x - y_t\|^2$$

We can once again explicitely compute the value of x_t. Indeed, if we define $F_t(x) := \sum_{s=1}^{t} \|x - y_t\|^2$, F_t is a strictly convex function and reaches its minimum where its gradient vanishes. A simple computation then gives $\nabla F_t(x) = 0 \leftrightarrow \sum_{s=1}^{t} 2(x - y_s) = 0 \leftrightarrow x = \frac{1}{t} \sum_{s=1}^{t} y_s$ In particular, we have again $x_t = \frac{1}{t-1} \sum_{s=1}^{t-1} y_s$ and we can notice that $\|x_t\| \leq 1$ at all time.

Now, by lemma 1.2 with the loss $\ell_t(x) = \|x - y_t\|^2$, we have :

$$\forall T, \sum_{t=1}^{T} \|x_T^* - y_t\|^2 \geq \sum_{s=1}^{T} \|x_t^* - y_t\|^2$$

We can now complete the proof :

$$
\begin{aligned}
R_T &= \sum_{t=1}^{T} \|x_t - y_t\|^2 - \min_{x \in \mathbb{R}} \sum_{t=1}^{T} \|x - y_t\|^2 \\
&= \sum_{t=1}^{T} \left\|x_{t-1}^* - y_t\right\|^2 - \sum_{t=1}^{T} \|x_T - y_t\|^2 \\
&\leq \sum_{t=1}^{T} \left\|x_{t-1}^* - y_t\right\|^2 - \sum_{t=1}^{T} \|x_t^* - y_t\|^2 \\
&= \sum_{t=1}^{T} \langle x_{t-1}^* + x_t^* - 2y_t, x_{t-1}^* - x_t^* \rangle \\
&\overset{\text{(C.S)}}{\leq} \sum_{t=1}^{T} \left\|x_{t-1}^* + x_t^* - 2y_t\right\| \cdot \left\|x_{t-1}^* - x_t^*\right\| \\
&\leq \sum_{t=1}^{T} 4 \left\|x_{t-1}^* - x_t^*\right\|
\end{aligned}
$$

Where the first inequality uses lemma 1.2, the second one uses Cauchy-Schwarz and the third uses that $\forall t,\ \|x_t^*\| \leq 1 \quad \text{and} \quad \|y_t\| \leq 1$. Now we notice that

$$
\begin{aligned}
\left\|x_{t-1}^* - x_t\right\| &= \left\| \frac{1}{t-1} \sum_{s=1}^{t-1} y_s - \frac{1}{t} \sum_{s=1}^{t} y_s \right\| \\
&= \left\| \frac{1}{t(t-1)} \sum_{s=1}^{t-1} y_s + \frac{1}{t} y_t \right\| \\
&\leq \frac{1}{t(t-1)} \sum_{s=1}^{t-1} \|y_s\| + \frac{1}{t} \|y_t\| \\
&\leq \frac{2}{t}
\end{aligned}
$$

Now we plug everything together :

$$
R_T \leq 4 \cdot \sum_{t=1}^{T} \left\|x_{t-1}^* - x_t^*\right\| \leq 8 \cdot \sum_{t=1}^{T} \frac{1}{t} \leq 8 + 8 \log T
$$

$\square$

Chapter 2

Online Subgradient Descent

Exercise 2.1

Prove that $\sum_{t=1}^{T} \frac{1}{\sqrt{t}} \leq 2\sqrt{T} - 1$

Solution : We have :

$$\begin{aligned}
\sum_{t=1}^{T} \frac{1}{\sqrt{t}} &= 1 + \sum_{t=2}^{T} \frac{1}{\sqrt{t}} \\
&= 1 + \sum_{t=2}^{T} \int_{t-1}^{t} \frac{1}{\sqrt{t}} du \\
&\leq 1 + \sum_{t=2}^{T} \int_{t-1}^{t} \frac{1}{\sqrt{u}} du \\
&\leq 1 + \int_{1}^{T} \frac{1}{\sqrt{u}} du \\
&= 1 + 2\sqrt{T} - 2 = 2\sqrt{T} - 1
\end{aligned}$$

$\square$

Exercise 2.2

Using the inequality in the previous exercise, prove that a learning rate $\propto \frac{1}{\sqrt{t}}$ gives rise to a regret only a constant multiplicative factor worse than the one in (2.1)($R_T \leq DL\sqrt{T}$).

Solution : We start at the result of theorem 2.13 :

$$R_T \leq \frac{D^2}{2\eta_T} + \sum_{t=1}^{T} \frac{\eta_t}{2} \|g_t\|^2$$

Then we bound for any t, $\|g_t\|^2 \leq L^2$ and set $\eta_t = \alpha \frac{1}{\sqrt{t}}$ with $\alpha > 0$ to be determined later. We have

$$\begin{aligned}
R_T &\leq \frac{D^2\sqrt{T}}{2\alpha} + \sum_{t=1}^{T} \frac{\alpha}{2\sqrt{t}} L^2 \\
&= \frac{D^2\sqrt{T}}{2\alpha} + \frac{\alpha L^2}{2} \sum_{t=1}^{T} \frac{1}{\sqrt{t}} \\
&\leq \frac{D^2\sqrt{T}}{2\alpha} + \alpha L^2 \sqrt{T} \\
&= \sqrt{T}\left(\frac{D^2}{2\alpha} + \alpha L^2\right) \\
&= DL\sqrt{2T}
\end{aligned}$$

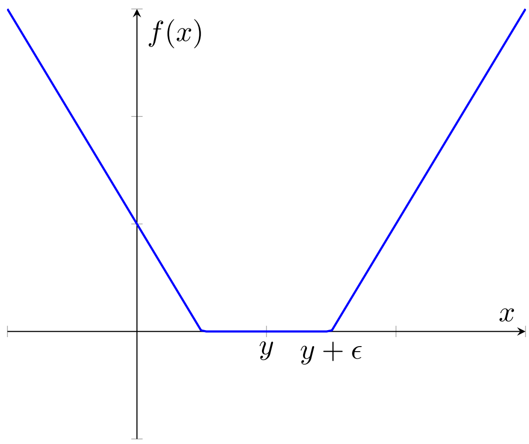
Where the third line uses the result of the previous exercise and the last lines uses the choice $\alpha = \sqrt{\frac{D^2}{2L^2}}$. We remark that we are only a factor $\sqrt{2}$ worse than the bound obtained with a fixed η.

$\square$

Exercise 2.3

Calculate the subdifferential set of the ϵ-insensitive loss :
$f(x) = \max(|x - y| - \epsilon, 0)$

Solution : We start by a drawing of the function :



Then the subdifferential is :

$$\partial f(x) = \begin{cases} \{0\} & \text{if } x \in]y-\epsilon, y+\epsilon[\\ \{-1\} & \text{if } x \in]-\infty, y-\epsilon[\\ \{1\} & \text{if } x \in]y+\epsilon, \infty[\\ [0,1] & \text{if } x = y+\epsilon \\ [-1,1] & \text{if } x = y-\epsilon \end{cases}$$