

Prediction Learning and Games – Exercise Solutions

Ludovic Schwartz

Contents

2 Prediction with Expert Advice	2
Exercise 2.	2
Exercise 5. Nonuniform initial weights	3
Exercise 6. Many good experts	4
Exercise 8. The doubling trick	5
Exercise 13.	7
Exercise 17.	7

Chapter 2

Prediction with Expert Advice

Exercise 2.2

Consider a weighted average forecaster based on a potential function

$$\Phi(u) = \psi \left(\sum_{i=1}^N \phi(u_i) \right).$$

Assume further that the quantity $C(r_t)$ appearing in the statement of Theorem 2.1 is bounded by a constant for all values of r_t and that the function $\psi(\phi(u))$ is strictly convex. Show that there exists a nonnegative sequence $\epsilon_n \rightarrow 0$ such that the cumulative regret of the forecaster satisfies, for every n and for every outcome sequence y^n ,

$$\frac{1}{n} \left(\max_{i=1, \dots, N} R_{i,n} \right) \leq \epsilon_n$$

Solution : We start by assuming that $C(r_t) \leq C$ for all values of r_t . We apply theorem 2.1 and we get that :

$$\Phi(R_n) \leq \Phi(0) + \frac{1}{2} \sum_{t=1}^n C(r_t) \leq \Phi(0) + \frac{Cn}{2}$$

Then, since we know that $\psi \circ \phi$ is non decreasing and strictly convex, it must be increasing and as a result, both $\psi \circ \phi$ and ϕ must be invertible and increasing and we have :

$$\psi \left(\phi \left(\max_{i=1, \dots, N} R_{i,n} \right) \right) \leq \psi \left(\max_{i=1, \dots, N} \phi(R_{i,n}) \right) \leq \psi \left(\sum_{i=1}^N \phi(R_{i,n}) \right) = \Phi(R_n)$$

Hence :

$$\max_{i=1,\dots,N} R_{i,n} \leq \phi^{-1}(\psi^{-1}(\Phi(R_n)))$$

$$\frac{1}{n} \left(\max_{i=1,\dots,N} R_{i,n} \right) \leq \underbrace{\frac{(\psi \circ \phi)^{-1} \left(\Phi(0) + \frac{Cn}{2} \right)}{n}}_{\epsilon_n}$$

Now we need to show that $\epsilon_n \rightarrow 0$. That is the same as saying that for a strictly convex increasing function F , we have $\lim_{x \rightarrow \infty} \frac{F^{-1}(x)}{x} = 0$ or equivalently $\lim_{x \rightarrow \infty} \frac{F(x)}{x} = +\infty$. This doesn't seem to be true in general. Indeed, the function $F(x) = \sqrt{x^2 + 1} - 1$ is strictly convex but $F(x) \underset{x \rightarrow \infty}{\sim} x$. The result would be true if we assume that $\psi \circ \phi$ is strongly convex as it would be lower bounded by a positive quadratic.

□

Exercise 2.5 NONUNIFORM INITIAL WEIGHTS

By definition, the weighted average forecaster uses uniform initial weights $w_{i,0} = 1$ for all $i = 1, \dots, N$. However, there is nothing special about this choice, and the analysis of the regret for this forecaster can be carried out using any set of nonnegative numbers for the initial weights.

Consider the exponentially weighted average forecaster run with arbitrary initial weights $w_{1,0}, \dots, w_{N,0} > 0$, defined for all $t = 1, 2, \dots$, by

$$\hat{p}_t = \frac{\sum_{i=1}^N w_{i,t-1} f_{i,t}}{\sum_{j=1}^N w_{j,t-1}}, \quad w_{i,t} = w_{i,t-1} \exp(-\eta \ell(f_{i,t}, y_t)).$$

Under the same conditions as in the statement of Theorem 2.2, show that for every n and for every outcome sequence y^n ,

$$\widehat{L}_n \leq \min_{i=1,\dots,N} \left(L_{i,n} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) + \frac{\log W_0}{\eta} + \frac{\eta}{8} n$$

where $W_0 = w_{1,0} + \dots + w_{N,0}$

Solution : As in the proof of theorem 2.2, we define :

$$\begin{aligned} W_t &= \sum_{i=1}^N w_{i,t} \\ &= \sum_{i=1}^N \exp(-\eta L_{i,t}) w_{i,0} \\ &= \sum_{i=1}^N \exp -\eta \left(L_{i,t} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) \end{aligned}$$

Now as in the proof of theorem 2.2, we analyse $\log \frac{W_t}{W_0}$ and $\log \frac{W_t}{W_{t-1}}$. We have :

$$\begin{aligned} \log \frac{W_n}{W_0} &= \log \left(\sum_{i=1}^N \exp -\eta \left(L_{i,n} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) \right) - \log W_0 \\ &\geq \log \left(\max_{i=1,\dots,N} \exp -\eta \left(L_{i,n} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) \right) \\ &= -\eta \left(\min_{i=1,\dots,N} L_{i,n} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) - \log W_0 \end{aligned}$$

On the other hand, exactly as in the proof of Theorem 2.2, we get for each $t = 1, \dots, n$,

$$\begin{aligned} \log \frac{W_t}{W_{t-1}} &= \log \frac{\sum_{i=1}^N \exp(-\eta \ell(f_{i,t}, y_t)) \exp \left(-\eta \left(L_{i,t-1} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) \right)}{\sum_{i=1}^N \exp \left(-\eta \left(L_{i,t-1} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) \right)} \\ &= \log \frac{\sum_{i=1}^N w_{i,t-1} \exp(-\eta \ell(f_{i,t}, y_t))}{\sum_{i=1}^N w_{i,t-1}} \\ &\leq -\eta \ell(\hat{p}_t, y_t) + \frac{\eta^2}{8} \end{aligned}$$

Where the last inequality is obtained by using Hoeffding's Lemma and the convexity of the loss exactly like in the proof of Theorem 2.2. Summing over $t = 1, \dots, n$, we get

$$\log \frac{W_n}{W_0} \leq -\eta \widehat{L}_n + \frac{\eta^2}{8} n.$$

Putting this together with the previous upper bound and solving for \widehat{L}_n we get :

$$\widehat{L}_n \leq \min_{i=1,\dots,N} \left(L_{i,n} + \frac{1}{\eta} \log \frac{1}{w_{i,0}} \right) + \frac{\log W_0}{\eta} + \frac{\eta}{8} n$$

□

Exercise 2.6 MANY GOOD EXPERTS

Sequences of outcomes on which many experts suffer a small loss are intuitively easier to predict. Adapt the proof of Theorem 2.2 to show that the exponentially weighted forecaster satisfies the following property : for every n , for every outcome sequence y^n and for all $L > 0$,

$$\widehat{L}_n \leq L + \frac{1}{\eta} \log \frac{N}{N_L} + \frac{\eta}{8} n,$$

where N_L is the cardinality of the set $\{1 \leq i \leq N : L_{i,n} \leq L\}$

Solution : We adapt the lower bound of $\log \frac{W_n}{W_0}$. We have :

$$\begin{aligned}
\log \frac{W_n}{W_0} &= \log \left(\sum_{i=1}^N e^{-\eta L_{i,n}} \right) - \log N \\
&\geq \log \left(\sum_{i=1}^N e^{-\eta L_{i,n}} \cdot \mathbb{I}_{L_{i,n} \leq L} \right) - \log N \\
&\geq \log \left(\sum_{i=1}^N e^{-\eta L} \cdot \mathbb{I}_{L_{i,n} \leq L} \right) - \log N \\
&= \log (N_L e^{-\eta L}) - \log N \\
&= -\eta L - \log \frac{N}{N_L}
\end{aligned}$$

We still have the upper bound of $\log \frac{W_n}{W_0}$ from the proof of Theorem 2.2 :

$$\log \frac{W_n}{W_0} \leq -\eta \widehat{L}_n + \frac{\eta^2}{8} n.$$

Putting everything together, we get :

$$\widehat{L}_n \leq L + \frac{1}{\eta} \log \frac{N}{N_L} + \frac{\eta}{8} n,$$

□

Exercise 2.8 THE DOUBLING TRICK

Consider the following forecasting strategy ("Doubling Trick"): time is divided in periods $(2^m, \dots, 2^{m+1} - 1)$, where $m = 0, 1, 2, \dots$. In period $(2^m, \dots, 2^{m+1} - 1)$ the strategy uses the exponentially weighted average forecaster initialized at time 2^m with parameter $\eta_m = \sqrt{8(\log N)/2^m}$. Thus, the weighted average forecaster is reset at each time instance that is an integer power of 2 and is restarted with a new value of η . Using Theorem 2.2 prove that, for any sequence $y_1, y_2, \dots \in \mathcal{Y}$ of outcomes and for any $n \geq 1$, the regret of this forecaster is at most

$$\widehat{L}_n - \min_{i=1, \dots, N} L_{i,n} \leq \frac{\sqrt{2}}{\sqrt{2}-1} \sqrt{\frac{n}{2} \log N}$$

Solution : Let $n \geq 1$, and let M be the only integer such that $2^M \leq n \leq$

2^{M+1} . We have :

$$\begin{aligned}
& \widehat{L}_n - \min_{i=1,\dots,N} L_{i,n} \\
&= \sum_{t=1}^n \ell(\hat{p}_t, y_t) - \min_{i=1,\dots,N} \sum_{t=1}^n f_{i,t} \\
&= \left(\sum_{m=0}^M \sum_{t=2^m}^{2^{m+1} \wedge n} \ell(\hat{p}_t, y_t) \right) - \min_{i=1,\dots,N} \left(\sum_{m=0}^M \sum_{t=2^m}^{2^{m+1} \wedge n} f_{i,t} \right) \\
&\leq \sum_{m=0}^M \left(\sum_{t=2^m}^{2^{m+1} \wedge n} \ell(\hat{p}_t, y_t) \right) - \sum_{m=0}^M \left(\min_{i=1,\dots,N} \sum_{t=2^m}^{2^{m+1} \wedge n} f_{i,t} \right) \\
&= \sum_{m=0}^M \left(\underbrace{\sum_{t=2^m}^{2^{m+1} \wedge n} \ell(\hat{p}_t, y_t) - \min_{i=1,\dots,N} \sum_{t=2^m}^{2^{m+1} \wedge n} f_{i,t}}_{R_{m,n}} \right)
\end{aligned}$$

Where $R_{m,n}$ is the regret suffered by our forecaster during phase m (and up to time n) which has exactly $2^{m+1} \wedge n - 2^m \leq 2^m$ steps. In particular, using the bound of Theorem 2.2, and the choice of η_m we know that

$$\forall m \leq M, R_m \leq \sqrt{(2^m/2) \log N}$$

Finally :

$$\begin{aligned}
& \widehat{L}_n - \min_{i=1,\dots,N} L_{i,n} \\
&\leq \sum_{m=0}^M \sqrt{(2^m/2) \log N} \\
&\leq \sqrt{\frac{\log N}{2}} \cdot \frac{\sqrt{(2^{M+1})} - 1}{\sqrt{2} - 1} \\
&\leq \sqrt{\frac{\log N}{2}} \cdot \frac{\sqrt{(2^{M+1})}}{\sqrt{2} - 1} \\
&\leq \frac{\sqrt{2}}{\sqrt{2} - 1} \sqrt{\frac{n}{2} \log N}
\end{aligned}$$

Where we used $2^{M+1} \leq 2n$ on the last line.

□

Exercise 2.13

Prove Lemma 2.4 ($\forall z \geq \frac{1}{2}, \log(1+z) \geq z - z^2$)

Solution : Let $f(z) = \log(1+z) - z + z^2$. We have that f is differentiable on $[-\frac{1}{2}, +\infty[$. In particular :

$$f'(z) = \frac{1}{1+z} - 1 + 2z = \frac{1}{1+z} \cdot (1 - (1+z) - 2z(1+z)) = \frac{z(2z+1)}{1+z}$$

From this we know that f' is non-positive on $[-\frac{1}{2}, 0]$ and non-negative on $[0, +\infty[$. In particular $\forall z \in [-\frac{1}{2}, +\infty[, f(z) \geq f(0) = 0$. That is to say :

$$\forall z \geq \frac{1}{2}, \log(1+z) \geq z - z^2$$

□

Exercise 2.17

Prove Lemma 2.5. *Hint: Adapt the proof of Theorem 2.3.*

Lemma 2.5. *Let h be a payoff function concave in its first argument. The exponentially weighted average forecaster, run with any nonincreasing sequence η_1, η_2, \dots of parameters satisfies, for any $n \geq 1$ and for any sequence y_1, \dots, y_n of outcomes,*

$$H_n^* - \hat{H}_n \leq \left(\frac{2}{\eta_{n+1}} - \frac{1}{\eta_1} \right) \log N + \sum_{t=1}^n \frac{1}{\eta_t} \log \mathbb{E} \left[e^{\eta_t (X_t - \mathbb{E}[X_t])} \right].$$

Where X_t are independent random variable such that $X_t = h(f_{i,t}, y_t)$ with probability $\frac{w_{i,t-1}}{W_{t-1}}$

Solution : We follow the structure of the proof of Theorem 2.3. We once again define the currently best expert k_t that we will use to lower bound the weight $\log(\frac{w_{k_t,t}}{W_t})$ and $w'_{i,t-1} = \exp \eta_{t-1} H_{i,t-1}$ to denote the weight $w_{i,t-1}$ where the parameter η_t is replaced by η_{t-1} . We write the following :

$$\begin{aligned} & \frac{1}{\eta_t} \log \frac{w_{k_{t-1},t-1}}{W_{t-1}} - \frac{1}{\eta_{t+1}} \log \frac{w_{k_t,t}}{W_t} \\ &= \underbrace{\left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \log \frac{W_t}{w_{k_t,t}}}_{(A)} + \underbrace{\frac{1}{\eta_t} \log \frac{w'_{k_t,t}/W'_t}{w_{k_t,t}/W_t}}_{(B)} + \underbrace{\frac{1}{\eta_t} \log \frac{w_{k_{t-1},t-1}/W_{t-1}}{w'_{k_t,t}/W'_t}}_{(C)} \end{aligned}$$

As in the proof of Theorem 2.3 we have that

$$(A) \leq \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \log N.$$

because the sequence η_t is increasing. For (B), we have :

$$\begin{aligned} (B) &= \frac{1}{\eta_t} \log \frac{w'_{k_t,t}/W'_t}{w_{k_t,t}/W_t} = \frac{1}{\eta_t} \log \frac{\sum_{i=1}^N e^{\eta_{t+1}(H_{i,t}-H_{k_t,t})}}{\sum_{i=1}^N e^{\eta_t(H_{i,t}-H_{k_t,t})}} \\ &\leq \frac{\eta_t - \eta_{t+1}}{\eta_t \eta_{t+1}} \log N = \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \log N \end{aligned}$$

Where we applied Lemma 2.3 with $d_i = H_{k_{t+1},t} - H_t$, $\alpha = \eta_{t+1}$ and $\beta = \eta_t$. As in the proof of theorem 2.3, we have that $d_i \geq 0$ because k_{t+1} is the index of the expert with the highest payoff after the first t rounds and $\sum_{i=1}^N e^{-\eta_{t+1}d_i} \geq 1$ as $d_{k_{t+1}} = 0$.

Now, we split the term (C) :

$$(C) = \frac{1}{\eta_t} \log \frac{w_{k_{t-1},t-1}/W_{t-1}}{w'_{k_t,t}/W'_t} = \frac{1}{\eta_t} \frac{w_{k_{t-1},t-1}}{w'_{k_t,t}} + \frac{1}{\eta_t} \log \frac{W'_t}{W_{t-1}}$$

We treat the two subterms of the right-hand side separately. For the first one, we have :

$$\frac{1}{\eta_t} \log \frac{w_{k_{t-1},t-1}}{w'_{k_t,t}} = \frac{1}{\eta_t} \log \frac{e^{\eta_t H_{k_{t-1},t-1}}}{e^{\eta_t H_{k_t,t}}} = H_{k_{t-1},t-1} - H_{k_t,t}$$

So far, we have not changed anything in the proof of Theorem 2.3, we have just checked that the steps of the proof carry on to the more general signed games setting. Now we will study the second term of the right-hand side of (C) and introduce the variable X_t to replace the Hoeffding's bound that was done in the proof of Theorem 2.3. We have :

$$\frac{1}{\eta_t} \log \frac{W'_t}{W_{t-1}} = \frac{1}{\eta_t} \log \sum_{i=1}^N \frac{w_{i,t-1}}{W_{t-1}} e^{\eta_t h(f_{i,t}, y_t)} = \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t X_t}]$$

by definition of X_t . Now we compute $\mathbb{E} [X_t]$:

$$\begin{aligned} \mathbb{E} [X_t] &= \sum_{i=1}^N \frac{w_{i,t-1}}{W_{t-1}} h(f_{i,t}, y_t) \\ &\leq h \left(\sum_{i=1}^N \frac{w_{i,t-1}}{W_{t-1}} f_{i,t}, y_t \right) \\ &= h(\hat{p}_t, y_t) \end{aligned}$$

Where the inequality comes from the concavity of h in its first argument.

Coming back to the previous bound, we have :

$$\begin{aligned}
\frac{1}{\eta_t} \log \frac{W'_t}{W_{t-1}} &\leq \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t X_t}] \\
&\leq \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t X_t}] + h(\hat{p}_t, y_t) - \mathbb{E} [X_t] \\
&= \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t X_t}] + h(\hat{p}_t, y_t) - \frac{1}{\eta_t} \log e^{\eta_t \mathbb{E}[X_t]} \\
&= \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t (X_t - \mathbb{E}[X_t])}] + h(\hat{p}_t, y_t)
\end{aligned}$$

Putting everything together, we get :

$$\begin{aligned}
&\frac{1}{\eta_t} \log \frac{w_{k_{t-1}, t-1}}{W_{t-1}} - \frac{1}{\eta_{t+1}} \log \frac{w_{k_t, t}}{W_t} \\
&\leq 2 \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \log N + H_{k_{t-1}, t-1} - H_{k_t, t} + \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t (X_t - \mathbb{E}[X_t])}] + h(\hat{p}_t, y_t)
\end{aligned}$$

Or rearranging :

$$\begin{aligned}
&H_{k_{t-1}, t-1} - H_{k_t, t} - h(\hat{p}_t, y_t) \\
&\leq 2 \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \log N + \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t (X_t - \mathbb{E}[X_t])}] - \frac{1}{\eta_t} \log \frac{w_{k_{t-1}, t-1}}{W_{t-1}} + \frac{1}{\eta_{t+1}} \log \frac{w_{k_t, t}}{W_t}
\end{aligned}$$

Summing all of these inequalities for $t = 1, \dots, n$, we get :

$$\begin{aligned}
H_n^* - \hat{H}_n &\leq 2 \left(\frac{1}{\eta_{n+1}} - \frac{1}{\eta_1} \right) \log N + \sum_{t=1}^n \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t (X_t - \mathbb{E}[X_t])}] \\
&\quad + \frac{1}{\eta_{n+1}} \log \frac{w_{k_n, n}}{W_n} - \frac{1}{\eta_1} \log \frac{1}{N} \\
&\leq \left(\frac{2}{\eta_{n+1}} - \frac{1}{\eta_1} \right) \log N + \sum_{t=1}^n \frac{1}{\eta_t} \log \mathbb{E} [e^{\eta_t (X_t - \mathbb{E}[X_t])}]
\end{aligned}$$

Where we used that $\frac{w_{k_n, n}}{W_n} \leq 1$.

□