Prediction Learning and Games – Exercise Solutions

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Chapter 2

Prediction with Expert Advice

Exercise 2.2

Consider a weighted average forecaster based on a potential function

$$\Phi(u) = \psi\left(\sum_{i=1}^{N} \phi(u_i)\right).$$

Assume further that the quantity $C(r_t)$ appearing in the statement of Theorem 2.1 is bounded by a constant for all values of r_t and that the function $\psi(\phi(u))$ is strictly convex. Show that there exists a nonnegative sequnce $\epsilon_n \to 0$ such that the cumulative regret of the forecaster satisfies, for every n and for every outcome sequence y^n,

$$\frac{1}{n}\left(\max_{i=1,\ldots N} R_{i,n}\right) \leq \epsilon_n$$

Solution : We start by assuming that $C(r_t) \leq C$ for all values of r_t. We apply theorem 2.1 and we get that :

$$\Phi(R_n) \leq \Phi(0) + \frac{1}{2}\sum_{t=1}^{n} C(r_t) \leq \Phi(0) + \frac{Cn}{2}$$

Then, since we know that $\psi \circ \phi$ is non decreasing and strictly convex, it must be increasing and as a result, both $\psi \circ \phi$ and ϕ must be invertible and increasing and we have :

$$\psi\left(\phi\left(\max_{i=1,\ldots,N} R_{i,n}\right)\right) \leq \psi\left(\max_{i=1,\ldots,N} \phi(R_{i,n})\right) \leq \psi\left(\sum_{i=1}^{N} \phi(R_{i,n})\right) = \Phi(R_n)$$

Hence :

$$\max_{i=1,\ldots,N} R_{i,n} \leq \phi^{-1}(\psi^{-1}(\Phi(R_n)))$$
$$\frac{1}{n}\left(\max_{i=1,\ldots,N} R_{i,n}\right) \leq \underbrace{\frac{(\psi\circ\phi)^{-1}\left(\Phi(0)+\frac{Cn}{2}\right)}{n}}_{\epsilon_n}$$

Now we need to show that $\epsilon_n \to 0$. That is the same as saying that for a stricly convex increasing function F, we have $\lim_{x\to\infty} \frac{F^{-1}(x)}{x} = 0$ or equivalenty $\lim_{x\to\infty} \frac{F(x)}{x} = +\infty$. This doesn't seem to be true in general. Indeed, the function $F(x) = \sqrt{x^2+1} - 1$ is stricly convex but $F(x) \underset{x\to\infty}{\sim} x$. The result would be true if we assume that $\psi\circ\phi$ is strongly convex as it would be lower bounded by a positive quadratic.

$\square$