Game Theory

Lecture 7: Dynamic games

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1. This week's topic: Dynamic games

Also called multi-stage games, or games played over time

Problem #1: When such a game is written in strategic form, important information regarding the timing of the moves may be lost

Solution: Representation via **extensive form** ('game tree plus info sets')
Equilibrium concepts stronger than Nash's are appropriate

- common theme 'sequential rationality' (a.k.a. 'perfection')

Central themes in the study of dynamic games:

- credibility (=sequential rationality) of threats
- what to think following an unexpected move by opponent?

Models in which players manipulate each others' information over time (signalling, signal-jamming, reputation-building...) are prevalent in economics

Ship-burning

Muslim conquest of the Iberian peninsula, 711 AD:

After landing his army, commander Ṭāriq ibn Ziyād ordered ships to be burned, making retreat impossible.

Why?

Bonus questions:

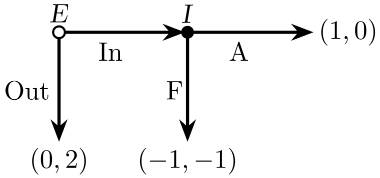
- which kingdom was being conquered? what was its capital?
- which British Overseas Territory is named after Ṭāriq?

The ‘entry deterrence’ game

A monopolist (player 2, called I) faces a potential entrant (E). If the entrant stays out then the entrant receives 0 and the incumbent gets 2 units of utility.

If the entrant enters then the incumbent can either fight or acquiesce. The former yields $(-1, -1)$, the latter $(1, 0)$ for E and I, respectively.

We may represent the moves and payoffs as a **game tree:**



Payoffs are written in the order (E, I).

Strategic form and Nash equilibria

Strategies:

- Player E has two strategies: In and Out.
- Player I's strategy is a plan of what to do when called to act. Two strategies: F and A.

Entry deterrence in strategic form

E is player 1 choosing row.

	F	A
In	$-1, -1$	$\underline{1}, \underline{0}$
Out	$\underline{0}, \underline{2}$	$0, \underline{2}$

Best-response payoffs underlined. Two pure-strategy Nash equilibria: $(\mathrm{In}, \mathrm{A})$ and $(\mathrm{Out}, \mathrm{F})$. All mixed Nash equilibria: (Out, pF+$(1-p)$A) with $p \geq 1/2$.

Do you find the Nash equilibria involving Out by the entrant and a sufficiently likely threat of F by the incumbent sensible game-play predictions when players are rational?

Credibility of threats

I (incumbent) would love to be able to *commit* to play F in order to keep E out.
But fighting is *not sequentially rational:* Once E is in, playing F is not rational for I.

(In macro, sequential rationality is called 'time-consistency'.)

That is, if E plays In then I's best response is A.
Anticipating this (i.e., believing I is rational), E is better off with In than Out.

$\Longrightarrow$ The expected outcome is In, followed by A, for payoffs (1, 0).

This procedure is called **backward induction**; results in a **subgame-perfect equilibrium (SPE or SPNE)**, a refinement of Nash equilibrium, in perfect-information dynamic games (Due to Reinhard Selten, Nobel Prize in 1994.)

More precise definitions, as well as a discussion of refinements of Nash equilibrium in extensive form games, are coming next.

2. Extensive-form games

An **extensive-form game** is defined by the following:

- **Players**, set N. Note: *Nature* can be one of the players.
- **Histories**, set H. A history h is a sequence of moves up to a given point in time. History $\emptyset \in H$ represents the start of the game.
- For all $h \in H$ define the set of **available actions**, $A(h)$. Terminal histories $Z \subseteq H$ are such that $A(z) = \emptyset$ for all $z \in Z$.
- **Player assignment** function, $P(h) \in N$: who moves at $h \in H \setminus Z$. If $P(h) = \textit{Nature}$ then there is a probability distribution over $A(h)$. $P(z) = \varnothing$ for $z \in Z$.
- **Information sets**, $\mathcal{I}$: a collection of disjoint subsets of $H \setminus Z$ whose union is $H \setminus Z$ (i.e., a *partition* of $H \setminus Z$). For $h, h' \in I \in \mathcal{I}$, $P(h) = P(h')$ and $A(h) = A(h')$. Interpretation: if $h, h' \in I \in \mathcal{I}$, then player $P(h)$ cannot distinguish h and h'.
- **Payoffs**, $\{u_i(z)\}_{i \in N}$, at every terminal history $z \in Z$.

Representation and conversion to strategic form

1) Lossless graphical representation: 'Game tree + information sets'.
   - Directed graph with a single initial node; edges represent moves.
   - Probabilities on edges representing Nature (chance) moves.
   - Nodes that the deciding player cannot distinguish (which are in the same information set) are connected by a dashed line.

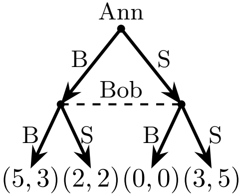
2) Lossy conversion: Strategic (also called 'normal') form.
   - A strategy is a player's **complete plan of action**, listing a move at every info set
   - A strategy profile (one strategy from each player) determines an outcome (in fact, more)
   - Different extensive-form games may have same normal form

Observation: The number of a player's strategies equals the product of the numbers of actions available at each of her information sets.

Simultaneous vs. sequential moves

Ann (P1, row) and Bob (P2, column) play Battle of the Sexes.

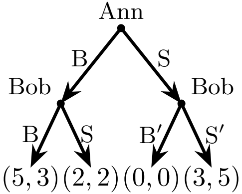
	B	S
B	$\underline{5}$, $\underline{3}$	2, 2
S	0, 0	$\underline{3}$, $\underline{5}$



Ann first, observed by Bob (**singleton info sets** in extensive form!):

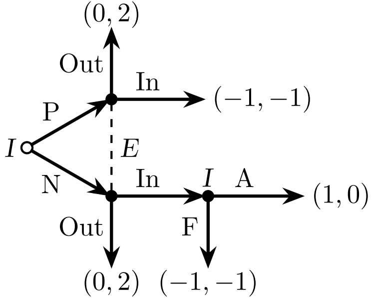
	BB′	BS′	SB′	SS′
B	$\underline{5}, \underline{3}$	$\underline{5}, \underline{3}$	$\underline{2}, 2$	$2, 2$
S	$\underline{0}, \underline{0}$	$3, \underline{5}$	$0, 0$	$\underline{3}, \underline{5}$

Bob's strategy XY means
'X after Ann's B and Y after Ann's S'



Reduced strategic form

Example: I can secretly take a 'poison pill' (triggering F in case E enters) before the entry deterrence game is played. As I has two nodes with two actions at each, I has 4 strategies.



'Poison pill' in entry deterrence
(E is player 1 choosing row.)

	PA	PF	NA	NF
In	$-1,-1$	$-1,-1$	$\underline{1}, \underline{0}$	$-1,-1$
Out	$\underline{0}, \underline{2}$	$\underline{0}, \underline{2}$	$0, \underline{2}$	$\underline{0}, \underline{2}$

In the **reduced strategic form** a single strategy P can stand in for two equivalent strategies, PA and PF, with no further loss of information. We will use the reduced strategic form.

3. Solution concepts for extensive-form games

What we know, so far:

(1) Every extensive-form game can be converted into its (reduced) strategic form. Information about dynamics may be lost, though.

(2) Every finite, normal-form game has a Nash equilibrium (pure or mixed).

(3) A Nash equilibrium may fail to be sequentially rational: strategies may fail to best-respond in some out-of-equilibrium continuation.

Refinements of Nash equilibrium in dynamic games:

(1) Subgame-perfect equilibrium (SPE)

(2) Perfect Bayesian equilibrium (PBE)

(3) Sequential equilibrium (SE)

(4) In pursuit of the 'holy grail', going beyond PBE/SE: trembling-hand perfect equilibrium, proper equilibrium, PBE/SE with forward-induction refinements (e.g the Intuitive Criterion: next lecture), . . . , stability.

Subgame-perfect equilibrium

A **subgame** is a continuation of the game (sub-tree) after a specific history (node), such that no information set is 'broken up' (no info set with nodes both inside and outside a subgame)

- see e.g. MWG, Definition 9.B.1, for precise notation
- Every game is a subgame of itself. Other subgames are called *proper subgames.*

A **subgame-perfect equilibrium** (SPE or SPNE) of a game is a strategy profile that is Nash equilibrium in every subgame.

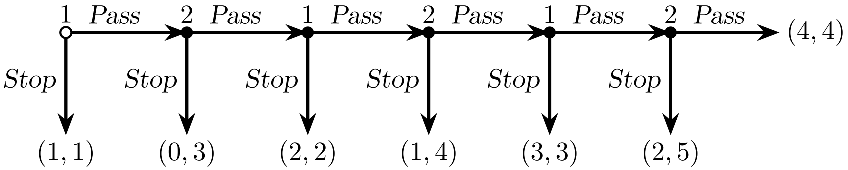
Theorem (H. Kuhn, 1953): Every finite game in which players have perfect recall (they do not forget their <u>own</u> prior moves; not the same as perfect info) has a SPE.

Example: Chess. (Homework: find the equilibrium strategies ☺.)

Simultaneous-move games have no proper subgames, hence in such games all Nash are SPE.

The centipede game

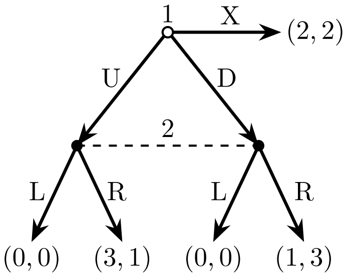
Sequential Prisoner's Dilemma, by Bob Rosenthal (1981): Each player can *Pass* up to 50 times, or *Stop* to end the game. *Pass* decreases one's own payoff by 1 and increases the opponent's by 2.



(Short version: 3 passes each.) By backward induction, the unique SPE is *Stop* at every node. In centipede experiments most subjects *Pass* at least initially: a 'trust bubble' forms.

Palacios-Huerta & Volij (2009), 'Field centipedes', *AER 99*(4), 1619–35: chess grandmasters stop right away; other subjects do not… unless they are told they are playing chess players. More subtle than 'altruism' or 'people are unsophisticated, do not play equilibrium'.

A valid problem with SPE



	L	R
U	0, 0	$\underline{3}$, $\underline{1}$
D	0, 0	1, $\underline{3}$
X	$\underline{2}$, $\underline{2}$	2, $\underline{2}$

Nash equilibria: (U, R) and $\{(\mathrm{X}, p\mathrm{L} + (1-p)\mathrm{R}) : p \geq \frac{1}{3}\}$

No proper subgame to check; we cannot do backward induction as originally defined. Hence all Nash equilibria are SPE. But are they all reasonable?

(X, L) is Nash: If P1 plays X then P2's move does not matter, hence P2 may play L. But at P2's information set, **no matter what he thinks P1 has done to get there**, playing L is not rational: P2 is strictly better off with R at both nodes.

Resolution: Perfect Bayesian equilibrium

Beliefs: probability distribution over nodes in an information set.

Assessment: (σ, μ) where σ is a strategy profile and μ specifies beliefs at each info set.

Perfect Bayesian equilibrium (PBE): an assessment (σ, μ) such that σ is an SPE and

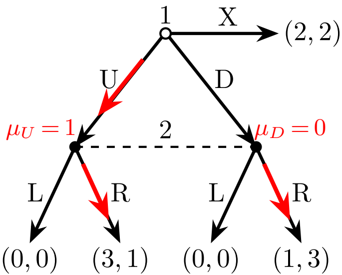
(1) *Sequential rationality:* Each player chooses optimally given her beliefs at each information set and the others' equilibrium strategies.

(2) *Bayesian beliefs:* Beliefs are computed based on equilibrium strategies via Bayes's rule whenever possible. 'Whenever possible': on the equilibrium path, and more. If ≥ 3 players: sensible additional restrictions ('you can't signal what you don't know').

Theorem: Every finite extensive-form game with perfect recall has a PBE.
(Proved by D. Fudenberg & J. Tirole, 1991, for a large class of games; more results since.)

In the example the unique PBE is (U, R) with P2's beliefs set at $\mu_U = 1$. (Next slide.)

Unique PBE in the example



	L	R
U	0, 0	$\underline{3}$, $\underline{1}$
D	0, 0	1, $\underline{3}$
X	$\underline{2}$, $\underline{2}$	2, $\underline{2}$

In the game above the unique PBE is (U, R) with $\mu_U = 1$.

That is, player 1 (she) plays U and player 2 (he) believes he is at the left-hand side node if called upon to play. Why PBE:

- R is rational for player 2 given his belief $\mu_U = 1$, U is rational for player 1 anticipating R
- Player 2's beliefs are Bayesian: $\mu_U = 1$ as player 1 plays U

Related concept: Sequential equilibrium

Assessment (σ, μ) is **consistent** if there is a sequence $((\sigma^m, \mu^m))_{m=1}^{\infty}$ converging to (σ, μ) such that σ^m is a fully mixed strategy profile and μ^m is computed from σ^m via Bayes's rule

An assessment (σ, μ) is a **sequential equilibrium** (SE) if it is sequentially rational (as in the definition of PBE) as well as consistent.

Theorem (D. Kreps & R. Wilson, 1982): Every finite game with perfect recall has a sequential equilibrium. All SE are PBE, and all PBE are SPE.

The course notes (Section 3.4) discuss Selten's TPE (trembling-hand perfect eq'm) as well. In almost all games TPE, PBE, and SE outcomes coincide; PBE is often easiest to find.

*Bayes's rule: Updating beliefs conditional on new info

Review Bayes's rule using this medical application:

- The prior probability of a disease is 0.8%, i.e., 0.008.
- If the disease is present then the test is positive with 90% chance.
- The test also gives a false positive with 7% probability.
- How likely is that the patient is ill if the test is positive?

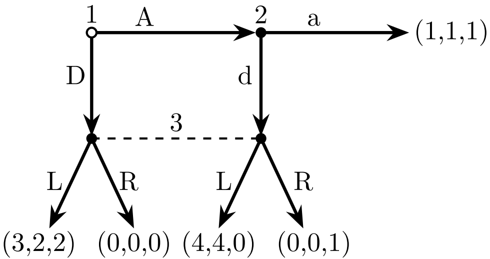
Gigerenzer (2002) reports, 95 of 100 US doctors guessed $\approx$75%.

Correctly: 9%. Use a 'natural frequencies' approach:

- Imagine 1000 patients, 8 sick, 992 healthy, test them all.
- Out of 8 sick patients, $90\% \times 8 \approx 7$ test positive;
- out of 992 healthy patients, $7\% \times 992 \approx 70$ test positive.
- Of the total 77 with positive tests, $7/77 \approx 9\%$ are ill.

Understanding Bayes's rule is a great competitive advantage in life.

Example: Selten's 'horse' (Reinhard Selten, 1975)



	a	d
A	1,1,$\underline{1}$	$\underline{4}$,$\underline{4}$,0
D	$\underline{3}$,$\underline{2}$,$\underline{2}$	3,$\underline{2}$,$\underline{2}$

L

	a	d
A	$\underline{1}$,$\underline{1}$,$\underline{1}$	$\underline{0}$,0,$\underline{1}$
D	0,$\underline{0}$,0	$\underline{0}$,$\underline{0}$,0

R

P1 picks row, P2 picks column, P3 picks matrix. Pure Nash/SPE: $(\mathrm{D}, \mathrm{a}, \mathrm{L})$ and $(\mathrm{A}, \mathrm{a}, \mathrm{R})$.
All: $\{$ $(\mathrm{D}, p\mathrm{a}$+(1-p)$\mathrm{d}, \mathrm{L})$ with $p \geq \frac{1}{3}$ $\}$ and $\{$ $(\mathrm{A}, \mathrm{a}, q\mathrm{L}$+(1-$q$)$\mathrm{R})$ with $q \leq \frac{1}{4}$ $\}$.

PBE/SE selects unique outcome in Selten's horse

Fact 1: $(\mathrm{D}, \mathrm{a}, \mathrm{L})$ is not part of any PBE/SE assessment

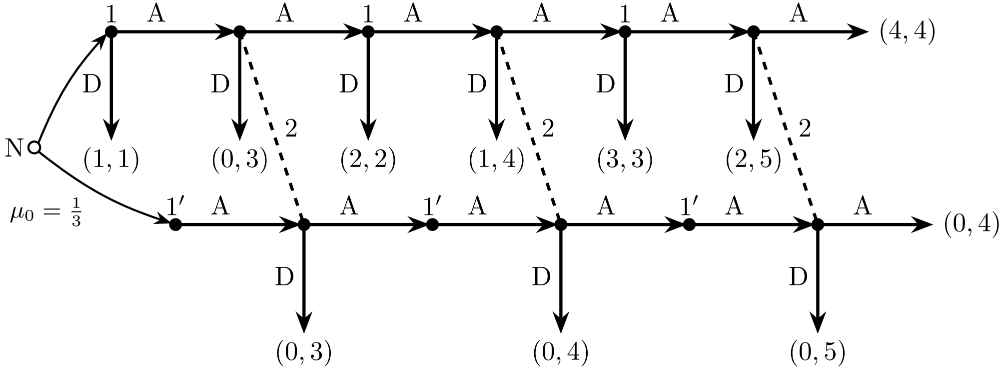
- If P2 gets to move then by picking d she induces $(\mathrm{A}, \mathrm{d}, \mathrm{L})$ which gives her 4 instead of 1
- Hence in PBE/SE where P3 plays L, action 'a' is not sequentially rational for P2

Fact 2: $(\mathrm{A}, \mathrm{a}, \mathrm{R})$ is PBE/SE with player 3's beliefs set $\mu_D \leq 1/3$.

- If P3 holds belief $\mu_D \leq 1/3$ then P3's expected payoff from L, $\mu_D \cdot 2 + (1 - \mu_D) \cdot 0$, is less than that from R, $\mu_D \cdot 0 + (1 - \mu_D) \cdot 1$
- As P3 chooses R, both P1 and P2 are better off going across

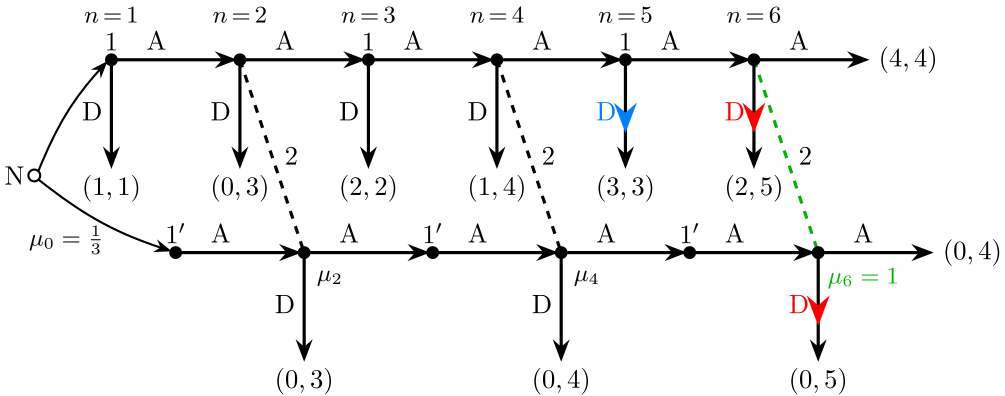
There are also PBE with $(\mathrm{A}, \mathrm{a}, q\mathrm{L} + (1 - q)\mathrm{R})$, $\mu_D = 1/3$, $q \leq 1/4$.
'Off-path mixing' by P3; outcome-equivalent to PBE $(\mathrm{A}, \mathrm{a}, \mathrm{R})$.

4. Application: Centipede with ‘doubt’ about P1’s rationality



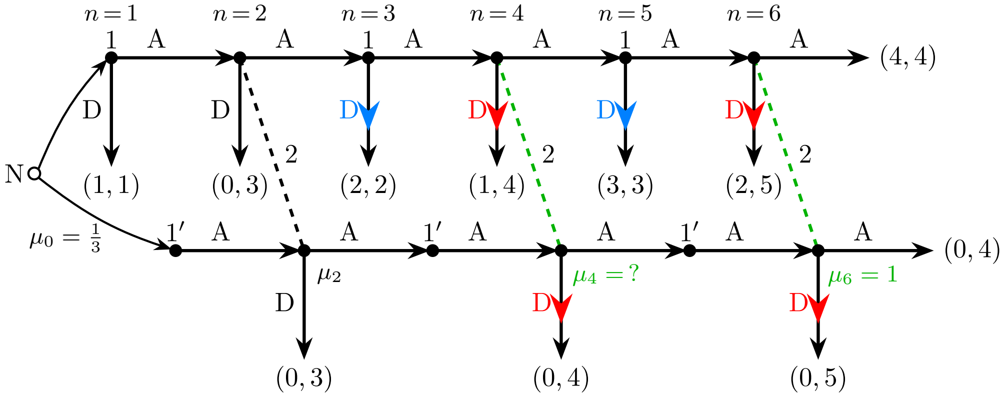
In a shortened version of Rosenthal’s (1981) centipede game with 3 passes for each player, P2 (she) believes there is a $\mu_0 = \frac{1}{3}$ chance that P1 (he) is irrational (a ‘commitment/crazy type’) playing A at every node. P2’s prior is commonly known. P1 knows whether he is rational or not.

The endgame



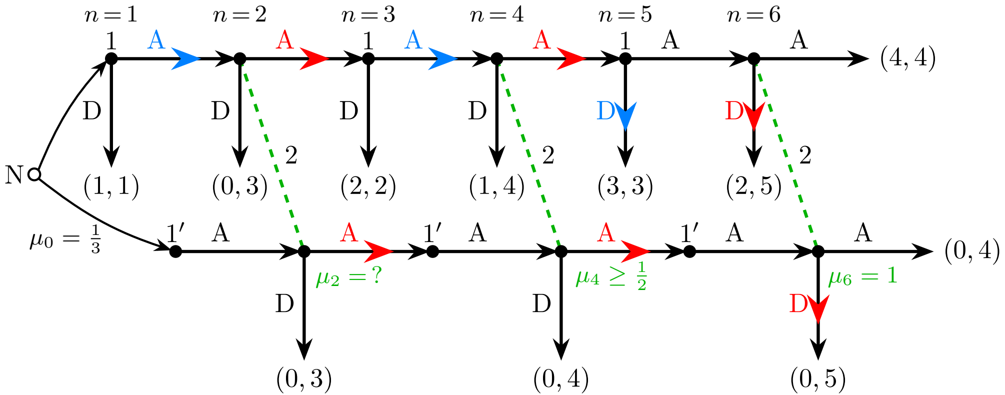
At node (info set) $n = 6$, it is optimal for P2 to play D irrespective of her beliefs. Hence at $n = 5$, rational P1 plays D as well. At $n = 6$, P2 infers that P1 is crazy, if he has played A at $n = 5$. Therefore, in any PBE, P2's belief at node 6 that P1 is crazy is $\mu_6 = 1$.

Why P2 won't play pure action D at $n = 4$ in any PBE



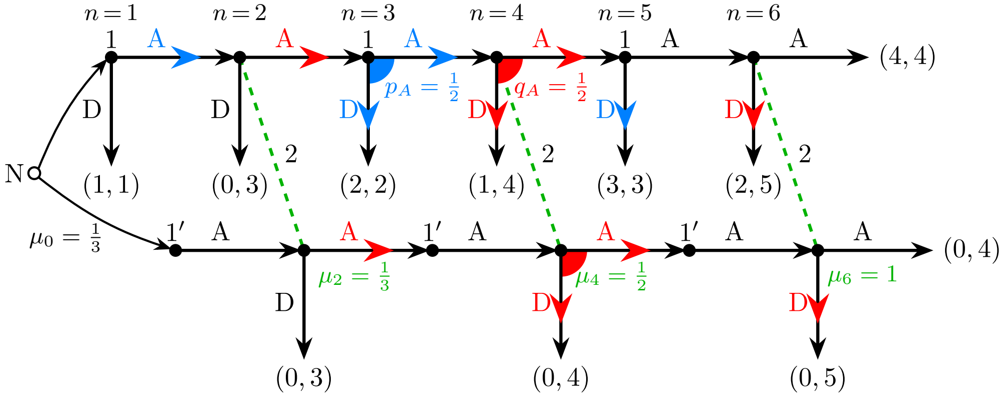
P2 plays pure D at $n = 4$ only if she weakly prefers D to A $\Leftrightarrow 4 \geq \mu_4 5 + (1 - \mu_4)3$
$\Leftrightarrow \mu_4 \leq \frac{1}{2}$. If P2 plays pure D at $n = 4$ then rational P1 plays D at $n = 3$ as well.
Then P2 gets to make a move at $n = 4$ only if P1 is crazy, hence $\mu_4 = 1$. Contradiction!

Why P2 won't play pure action A at $n = 4$ in any PBE



P2 plays A at $n = 4$ only if $\mu_4 \geq \frac{1}{2}$. If she does then rational P1 also plays A at $n = 3$. Both types of P1 play A at $n = 3$, hence Bayes's rule implies $\mu_2 \equiv \mu_4 \geq \frac{1}{2}$, and so P2 picks A at $n = 2$ as well. Then rational P1 plays A at $n = 1$, and so $\mu_2 = \mu_0 = \frac{1}{3}$. Contradiction!

Verify the following PBE (there is no other)



At $n=1$, rational P1 plays A, hence $\mu_2 = \mu_0 = \frac{1}{3}$. At $n=2$, P2 also plays A.
At $n=3$, rational P1 mixes 50–50%, this makes P2 update her beliefs to $\mu_4 = \frac{1}{2}$.
At $n=4$, P2 also mixes, with 50–50% as well. Beyond that point both players play D.

Steps for checking the solution

In any PBE, we must make P2's beliefs at $n = 4$ equal $\mu_4 = \frac{1}{2}$, so she is willing to mix.

To that end, let rational P1 play A with probability $p_A = \frac{1}{2}$ at $n = 3$.

Then Bayes's rule yields $\mu_4 = \frac{\mu_2}{\mu_2 + (1-\mu_2)p_A} = \frac{\frac{1}{3}}{\frac{1}{3}\times 1 + \frac{2}{3}\times\frac{1}{2}} = \frac{1}{2}$, as desired.

Final step: At $n = 2$, with beliefs $\mu_2 = \frac{1}{3}$, P2 prefers A, because

$$\frac{2}{3}\left[\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 3\right)\right] + \frac{1}{3}\left(\frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 5\right) = 3.\dot{3} > 3.$$

Interpretation of the PBE/SE of the game: Initially, rational P1 plays A to make P2 believe he is likely to be crazy. P2 rationally goes along, but the players 'burst the bubble' before the end. More detail in course notes and especially Kreps's textbook, chapter 14.6 (pp. 536–43)

Take-away from 'centipede with doubt'

In the PBE of the centipede game with doubt, both players play A in early periods

- true with 100 nodes as well: if $\frac{1}{4} < \mu_0 < \frac{1}{2}$ then both play A at all but last four nodes
- if μ_0 is smaller then the 'bubble-bursting' (mixing) stage before the endgame is longer
- the PBE is unique (in fact: unique Nash!); we did not fully prove this, but it can be done

Play A (or Pass) not because you are irrational, but because you cannot be sure whether the other is rational, or whether the other knows that you are rational, and so on.

- this idea, due to Kreps, Milgrom, Roberts, Wilson (1982), explains **rational bubbles**

Slightly irrelevant trivia concerning centipedes:

- a house centipede has 30 legs, other species as many as 354
- centipedes always have an *odd number of pairs* of legs, never exactly 100

Take-away from today

Represent dynamic games in extensive form, and use stronger equilibrium concepts than Nash such as SPE or PBE/SE as applicable, because Nash equilibria may involve incredible (=sequentially irrational) threats.

When converting an extensive-form game to reduced normal form, remember that strategies are complete contingent plans. (But feel free to remove truly duplicate strategies!)

PBE and SE specify assessments, which comprise a strategy profile *and* (Bayesian or 'consistent') beliefs at every information set

These concepts place few restrictions on off-equilibrium-path beliefs. More ideas to come.