Dynamical Systems 1
Course 01617
E20

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Solutions to Week 6

Test questions

(1) Non-hyberbolic (in particular, saddle can be ruled out since there is no neighbourhood of zero with a 1-D stable and a 1-D unstable manifold).

Exercise 1

(a)

From the set of differential equations we have:

$$\begin{aligned} \dot{x}_2 &= 2x_2 \Rightarrow \\ x_2(t) &= x_2(0)e^{2t} \end{aligned}$$

thus the equilibrium point is unstable.

(b)

The equilibria of the system:

$$\begin{aligned} \dot{x} &= y - x^2 + 2 \\ \dot{y} &= 2y^2 - 2xy, \end{aligned} \tag{1}$$

are:

$$p_{1,2} = \left(\pm\sqrt{2}, 0\right), \quad p_3 = (-1, -1), \quad p_4 = (2, 2).$$

The Jacobian matrix of system (1) is:

$$Df(x, y) = \begin{pmatrix} -2x & 1 \\ -2y & 4y - 2x \end{pmatrix}.$$

In order to investigate the stability of the four equilibria, we use the theorem in p.25 ($\delta = \det(Df)$, $\tau = \operatorname{tr}(Df)$).

- p_1 : We have:

$$Df(p_1) = \begin{pmatrix} -2\sqrt{2} & 1 \\ 0 & -2\sqrt{2} \end{pmatrix}.$$

Since $\delta_1 > 0$, p_1 is a (stable) node.

- p_2 : We have:

$$Df(p_2) = \begin{pmatrix} 2\sqrt{2} & 1 \\ 0 & 2\sqrt{2} \end{pmatrix}.$$

Since $\delta_2 > 0$, p_2 is a (unstable) node.

- p_3 : We have:

$$Df(p_3) = \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}.$$

Since $\delta_3 = -6$, $\tau_3 = --$ we have p_3 is an unstable saddle.

- p_4 : We have:

$$Df(p_4) = \begin{pmatrix} -4 & 1 \\ -4 & 4 \end{pmatrix}.$$

Since $\delta_4 = -12$, $\tau_4 = 0$ we have p_4 is an unstable saddle.

(c)

The equilibria of the system:

$$\begin{aligned} \dot{x} &= -4x - 1y + 4 \\ \dot{y} &= xy, \end{aligned} \tag{2}$$

are:

$$p_1 = (0, 2), \quad p_2 = (1, 0).$$

The Jacobian matrix of system (2) is:

$$Df(x, y) = \begin{pmatrix} -4 & -2 \\ y & x \end{pmatrix}.$$

As in (b), In order to investigate the stability of the four equilibria, we use the theorem in p.25 ($\delta = \det(Df)$, $\tau = \text{tr}(Df)$).

- p_1 : We have:

$$Df(p_1) = \begin{pmatrix} -4 & -2 \\ 2 & 0 \end{pmatrix}.$$

Since $\delta_1 = 4$, $\tau_1 = -4$ we have $\tau_1^2 - 4\delta_1 = 0$ and p_1 is a node (stable, as $\tau < 0$)

- p_2 : We have:

$$Df(p_2) = \begin{pmatrix} -4 & -2 \\ 0 & 1 \end{pmatrix}.$$

Since $\delta_2 < 0$, p_2 is a saddle.

Exercise 2

(a) Since $\dot{V}(\mathbf{x}) < 0$ for $\mathbf{x} \neq 0$ the equilibrium point $(0, 0)$ is asymptotically stable.

(b) $\dot{V}(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ and this shows that $(0, 0)$ is unstable.

(c) $\dot{V}(\mathbf{x}) = 0$ for $\mathbf{x} \neq 0$ so trajectories move on level curves of $\dot{V}(\mathbf{x})$, so $(0, 0)$ is stable (but not asymptotically zero). In this case the level curves are circles.

Exercise 3

We whish to check the equation

$$H \circ \phi_t(x) = \psi_t \circ H(x)$$

or, equivalently,

$$H\left(\phi_t(x)\right) = \psi_t\left(H(x)\right)$$

Since $H(x) = y$, and since $\psi_t(y) = y_0 + t$ (the equation for y is easy to solve), we need to check that

$$H(x(t)) = H(x_0) + t$$

Differentating both sides with respect to time, we find that we would need to have

$$H'(x)\dot{x} = 1$$

Working out H' and using the dynamical equation ofr $\dot{x}$, we find that the equation is satisfied. If two C^1 functions have the same derivative, they differ by at most a constant. But we have already seen that the two sides of the euqation match for $t = 0$. So H is indeed a conjugation homeomorphism.

Exercise 4

(a)

Consider $x \in \mathbb{R}^2$, suppose that **x=0** is hyperbolic and letλ_1, λ_2 be its eigenvalues. It is not possible that both eigenvalues have positive real part, thus the possible cases are:

1. Re$\{\lambda_i\} < 0$, $i = 1, 2$
2. $\lambda_1 < 0 < \lambda_2$.

In the first case, the equilibrium point is either a stable node or a stable focus, thus asymptotically stable. In the second case we have a contradiction, since $W^s_{loc}(0)$ is 1D and therefore $W^s(0)$ would be 1D, while by assumption $W^s(0)$ is 2D. Thus the only possible case is the first one.

(b)

Consider the system:

$$\begin{aligned} \dot{x} &= x^2\left(y - x\right) + y^5 \\ \dot{y} &= y^2\left(y - 2x\right), \end{aligned} \tag{3}$$

We have $\dot{y} = 0$ for $y = 0$ or $y = 2x$, and $\dot{x} = 0$ for $x = 0$ or $y = x$. Thus the only equilibrium point is:

$$p = (0, 0).$$

(c)

The Jacobian matrix of system (3) is:

$$Df(x, y) = \begin{pmatrix} 2xy - 3x^2 & x^2 + 5y^4 \\ 3y^2 - 2x & 4y - 2x - 2y^2 \end{pmatrix}.$$

and:

$$Df(p) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

thus p is not hyperbolic.

(d)

For $y = 0$ (x-axis) we have:

$$\dot{y} = 0,$$

thus the flow evolves on the x-axis, with dynamics given by:

$$\dot{x} = -x^3$$

(e)

Consider the system:

$$\begin{aligned} \dot{x} &= \frac{x^2\,(y-x)+y^5}{r^2} \\ \dot{y} &= \frac{y^2\,(y-2x)}{r^2}, \end{aligned} \tag{4}$$

Let $\tau = r^2 t$. Then:

$$\begin{aligned} x' &= \frac{\mathsf{d}x}{\mathsf{d}\tau} = \frac{\mathsf{d}x}{\mathsf{d}t}\frac{\mathsf{d}t}{\mathsf{d}\tau} = \frac{1}{r^2}\dot{x} \\ y' &= \frac{\mathsf{d}y}{\mathsf{d}\tau} = \frac{\mathsf{d}y}{\mathsf{d}t}\frac{\mathsf{d}t}{\mathsf{d}\tau} = \frac{1}{r^2}\dot{y} \end{aligned}$$

and systems (3) and (4) are topologically equivalent.