Proofs and Computations

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Contents

1 Logic

1.1 Natural Deduction

Negation is defined by

$$\neg A := (A \to \bot)$$

Definition 1.1 (Gentzen)**. Subformulas** of A are defined by

1. A is a subformula of A
2. if $B \circ C$ is a subformula of A then so are B, C for $\circ = \to, \wedge, \vee$
3. if $\forall_x B(x)$ or $\exists_x B(x)$ is a subformula of A, then so is $B(r)$

Definition 1.2. The notions of **positive**, **negative**, **strictly positive** subformula are defined in a similar style

1. A is a positive and a strictly positive subformula of itself
2. if $B \wedge C$ or $B \vee C$ is a positive (negative, strictly positive) subformula of A, then so are B, C
3. if $\forall_x B(x)$ or $\exists_x B(x)$ is a positive (negative, strictly positive) subformula of A, then so is $B(r)$

4. if $B \to C$ is a positive (negative) subformula of A, then B is a negative (positive)subformula of A, and C is a positive (negative)subformula of A

5. if $B \to C$ is a strictly subformula of A, then so is C

A strictly positive subformula of A is also called a **strictly positive part** (**s.p.p.**) of A

$$\dfrac{\begin{matrix}[u : A]\\ \vdots\, \mathrm{D}\\ B\end{matrix}}{A \to B}\to^+ u \qquad \dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ A \to B\end{matrix} \quad \begin{matrix}\vdots\, \mathrm{N}\\ A\end{matrix}}{B}\to^-$$

The rule $\forall^+ x$ with conclusion $\forall_x A$ is subject to the following **(eigen-)variable condition**: the derivation M of the premise A should not contain any open assumption having x as a free variable

$$\dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ A\end{matrix}}{\forall_x A}\forall^+ x \qquad \dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ \forall_x A(x)\end{matrix} \quad r}{A(r)}\forall^-$$

$$\dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ A\end{matrix}}{A \vee B}\vee^+_0 \qquad \dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ B\end{matrix}}{A \vee B}\vee^+_1 \qquad \dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ A \vee B\end{matrix} \quad \begin{matrix}[u : A]\\ \vdots\, \mathrm{N}\\ C\end{matrix} \quad \begin{matrix}[v : B]\\ \vdots\, \mathrm{K}\\ C\end{matrix}}{C}\vee^- u, v$$

$$\dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ A\end{matrix} \quad \begin{matrix}\vdots\, \mathrm{N}\\ B\end{matrix}}{A \wedge B}\wedge^+ \qquad \dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ A \wedge B\end{matrix} \quad \begin{matrix}[u : A] \quad [v : B]\\ \vdots\, \mathrm{N}\\ C\end{matrix}}{C}\wedge^- u, v$$

$$\dfrac{r \quad \begin{matrix}\vdots\, \mathrm{M}\\ A(r)\end{matrix}}{\exists_x A(x)}\exists^+ \qquad \dfrac{\begin{matrix}\vdots\, \mathrm{M}\\ \exists_x A\end{matrix} \quad \begin{matrix}[u : A]\\ \vdots\, \mathrm{N}\\ B\end{matrix}}{B}\exists^- x, u(\text{var.cond.})$$

Rule $\exists^- x, u$ is subject to an **(eigen-)variable condition**: in the derivation N the variable x

1. should not occur free in the formula of any open assumption other than $u : A$
2. should not occur free in B

For each of the connectives $\wedge, \vee, \exists$ the rules and the following axioms are equivalent over minimal logic

$$\exists^+ : A \to \exists_x A, \quad \exists^- : \exists_x A \to \forall_x (A \to B) \to B (x \notin FV(B))$$

Lemma 1.3. *The following are derivable*

$$\begin{aligned}
&(A \wedge B \to C) \leftrightarrow (A \to B \to C) \\
&(A \to B \wedge C) \leftrightarrow (A \to B) \wedge (A \to C) \\
&(A \vee B \to C) \leftrightarrow (A \to C) \wedge (B \to C) \\
&(A \to B) \vee (A \to C) \to (A \to B \vee C) \\
&\exists_x (A \to B) \to (\forall_x A \to B) \quad \textit{if } x \notin FV(B) \\
&\forall_x (A \to B) \leftrightarrow (A \to \forall_x B) \quad \textit{if } x \notin FV(A) \\
&\forall_x (A \to B) \leftrightarrow (\exists_x A \to B) \quad \textit{if } x \notin FV(B) \\
&\exists_x (A \to B) \to (A \to \exists_x B) \quad \textit{if } x \notin FV(A)
\end{aligned}$$

Proof.

$$\dfrac{\dfrac{u : \exists_x (A \to B) \qquad \dfrac{x \qquad \dfrac{w : A \to B \qquad v : A}{B}}{\exists_x B}}{\dfrac{\exists_x B}{A \to \exists_x B} \to^+ v} \exists^- x, w}{\exists_x (A \to B) to A \to \exists_x B} \to^+ u$$

□

weak disjuction and **weak existence**

$$A \tilde{\vee} B := \neg A \to \neg B \to \bot, \quad \tilde{\exists}_x A := \neg \forall_x \neg A$$

These weak variants are no stronger than the proper ones

$$A \vee B \to A \tilde{\vee} B, \quad \exists_x A \to \tilde{\exists}_x A$$

by putting $C := \bot$ in $\vee^-$ and $B := \bot$ in $\exists^-$

Moreover

$$\tilde{\exists}_{x_1,\dots,x_n} A := \forall_{x_1,\dots,x_n}(A \to \bot) \to \bot$$
$$\tilde{\exists}_{x_1,\dots,x_n}(A_1 \tilde{\wedge} \cdots \tilde{\wedge} A_m) := \forall_{x_1,\dots,x_n}(A_1 \to \cdots \to A_m \to \bot) \to \bot$$

In the previous contexts falsity $\bot$ plays no role. We may change this and require **ex-falso-quodlibet** axioms of the form

$$\forall_{\vec{x}}(\bot \to R\vec{x})$$

with R a relation symbol distinct from $\bot$. Let Efq denote the set of all such axioms. A formula A is called **intuitionistically derivable**, written $\vdash_i A$ if $\text{Efq} \vdash A$. We write $\Gamma \vdash_i B$ for $\Gamma \cup \text{Efq} \vdash B$

We may even go further and require **stability** axioms, of the form

$$\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x})$$

with R again a relation distinct from $\bot$. Let Stab denote the set of all these axioms. A formula A is called **classically derivable**, written $\vdash_c A$, if $\text{Stab} \vdash A$. We write $\Gamma \vdash_c B$ for $\Gamma \cup \text{Stab} \vdash B$

Theorem 1.4 (Stability, or principle of indirect proof)**.**

1. *$\vdash (\neg\neg A \to A) \to (\neg\neg B \to B) \to \neg\neg(A \wedge B) \to A \wedge B$*
2. *$\vdash (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B$*
3. *$\vdash (\neg\neg A \to A) \to \neg\neg\forall_x A \to A$*
4. *$\vdash_c \neg\neg A \to A$ for every formula A without $\vee, \exists$*

Proof.

1. $(\neg\neg A \to A) \to \neg\neg A \to A$
2.

$$\begin{prooftree}
\AxiomC{$u : \neg\neg B \to B$}
\AxiomC{$v : \neg\neg(A \to B)$}
\AxiomC{$u_1 : \neg B$}
\AxiomC{$u_2 : A \to B$}
\AxiomC{$w : A$}
\BinaryInfC{B}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_2$}
\UnaryInfC{$\neg(A \to B)$}
\BinaryInfC{$\bot$}
\RightLabel{$\to^+ u_1$}
\UnaryInfC{$\neg\neg B$}
\BinaryInfC{B}
\end{prooftree}$$

3.

$$
\dfrac{u : \neg\neg A \to A \qquad \dfrac{\dfrac{v : \neg\neg\forall_x A \qquad \dfrac{\dfrac{u_1 : \neg A \qquad \dfrac{u_2 : \forall_x A \qquad x}{A}}{\bot}}{\neg\forall_x A}\to^+ u_2}{\bot}}{\neg\neg A}\to^+ u_1}{A}
$$

4. Induction on A. The case $R\vec{t}$ with R distinct from $\bot$ is given by Stab. In the case $\bot$ the desired derivation is

$$
\dfrac{v : (\bot \to \bot) \to \bot \qquad \dfrac{u : \bot}{\bot \to \bot}\to^+ u}{\bot}
$$

In the case $A \wedge B, A \to B$ and $\forall_x A$ use 1,2,3 respectively

$\square$

Lemma 1.5. *The following are derivable*

$$
\begin{aligned}
(\tilde{\exists}_x A \to B) &\to \forall_x (A \to B) && \text{if } x \notin FV(B) \\
(\neg\neg B \to B) \to \forall_x (A \to B) &\to \tilde{\exists}_x A \to B && \text{if } x \notin FV(B) \\
(\bot \to B[x := c]) \to (A \to \tilde{\exists}_x B) &\to \tilde{\exists}_x (A \to B) && \text{if } x \notin FV(A) \\
\tilde{\exists}_x (A \to B) &\to A \to \tilde{\exists}_x B && \text{if } x \notin FV(A)
\end{aligned}
$$

The last two items can also be seen as simplifying a weakly existentially quantified implication whose premise doesn't contain the quantified variable. In case the conclusion does not contain the quantified variable we have align

$$
\begin{aligned}
(\neg\neg B \to B) \to \tilde{\exists}_x (A \to B) \to \forall_x A \to B && \text{if } x \notin FV(A) \\
\forall_x (\neg\neg A \to A) \to (\forall_x A \to B) \to \tilde{\exists}_x (A \to B) && \text{if } x \notin FV(A)
\end{aligned}
$$

Proof. 3. Writing B_0 for $B[x := c]$ we have

$$\dfrac{\dfrac{\forall_x\neg(A\to B)\quad c}{\neg(A\to B_0)} \qquad \dfrac{\dfrac{\bot\to B_0 \qquad \dfrac{\dfrac{A\to\tilde{\exists}_x B \quad u_2 : A}{\tilde{\exists}_x B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x\neg(A\to B)\quad x}{\neg(A\to B)} \quad \dfrac{u_1 : B}{A\to B}}{\bot}}{\neg B}\to^+ u}{\forall_x\neg B}}{\bot}}{B_0}}{A\to B_0}\to^+ u_2}{\bot}$$

4.

$$\dfrac{\tilde{\exists}_x(A\to B) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x\neg B \quad x}{\neg B}\quad \dfrac{u_1 : A\to B \quad A}{B}}{\bot}}{\neg(A\to B)}\to^+ u_1}{\forall_x\neg(A\to B)}}{\bot}$$

□

An immediate consequence of 6 is the classical derivability of the ”drinker formula” $\tilde{\exists}_x(Px \to \forall_x Px)$ to be read ”in every non-empty bar there is a person s.t. if this person drinks, then everybody drinks”

Corollary 1.6.

$$\vdash_c (\tilde{\exists}_x A \to B) \leftrightarrow \forall_x(A\to B) \quad \textit{if } x \notin FV(B) \textit{ and } B \textit{ without } \forall, \exists$$
$$\vdash_i (A \to \tilde{\exists}_x B) \leftrightarrow \tilde{\exists}_x(A\to B) \quad \textit{if } x \notin FV(A)$$
$$\vdash_c \tilde{\exists}_x(A \to B) \leftrightarrow (\forall_x A\to B) \quad \textit{if } x \notin FV(B) \textit{ and } A, B \textit{ without } \forall, \exists$$

Lemma 1.7. *The following are derivable*

$$(A\tilde{\vee}B \to C) \to (A\to C)\wedge(B\to C)$$
$$(\neg\neg C\to C)\to(A\to C)\to(B\to C)\to A\tilde{\vee}B\to C$$
$$(\bot\to B)\to(A\to B\tilde{\vee}C)\to(A\to B)\tilde{\vee}(A\to C)$$
$$(A\to B)\tilde{\vee}(A\to C)\to A\to B\tilde{\vee}C$$
$$(\neg\neg C\to C)\to(A\to C)\tilde{\vee}(B\to C)\to A\to B\to C$$
$$(\bot\to C)\to(A\to B\to C)\to(A\to C)\tilde{\vee}(B\to C)$$

Proof. 6.

$$\dfrac{\neg(B \to C) \qquad \dfrac{\dfrac{\bot \to C \qquad \dfrac{\neg(A \to C) \qquad \dfrac{\dfrac{\dfrac{A \to B \to C \qquad u_1 : A}{B \to C} \qquad u_2 : B}{C}}{A \to C}\ \to^+ u_1}{\bot}}{C}}{B \to C}\ \to^+ u_2}{\bot}$$

□

Corollary 1.8.

$$\vdash_c (A \tilde{\vee} B \to C) \leftrightarrow (A \to C) \wedge (B \to C) \quad \textit{for } C \textit{ without } \forall, \exists$$
$$\vdash_i (A \to B \tilde{\vee} C) \leftrightarrow (A \to B) \tilde{\vee} (A \to C)$$
$$\vdash_c (A \to C) \tilde{\vee} (B \to C) \leftrightarrow (A \to B \to C) \quad \textit{for } C \textit{ without } \forall, \exists$$

Remark. It is easy to see that weak disjuction and the weak existential quantifier satisfy the same axioms as the strong variants, if one restricts the conclusion of the elimination axioms to formulas without $\forall, \exists$. In fact we have

$$\vdash A \to A \tilde{\vee} B, \quad \vdash B \to A \tilde{\vee} B$$
$$\vdash_c A \tilde{\vee} B \to (A \to C) \to (B \to C) \to C \quad (C \text{ without } \forall, \exists)$$
$$\vdash A \to \tilde{\vee}_x A$$
$$\vdash_c \tilde{\exists}_x A \to \forall_x (A \to B) \to B \quad (x \notin FV(B), B \text{ without } \forall, \exists)$$

Proof. 2.

$$\dfrac{\neg\neg C \to C \qquad \dfrac{\dfrac{\dfrac{\neg A \to \neg B \to \bot \qquad \dfrac{\dfrac{u_1 : \neg C \qquad \dfrac{A \to C \qquad u_2 : A}{C}}{\bot}}{\neg A}\ \to^+ u_2}{\neg B \to \bot} \qquad \dfrac{\dfrac{u_1 : \neg C \qquad \dfrac{B \to C \qquad u_3 : B}{C}}{\bot}}{\neg B}\ \to^+ u_3}{\bot}}{\neg\neg C}\ \to^+ u_1}{C}$$

□

A is derivable in classical logic iff its translation A^g is derivable in minimal logic

Definition 1.9 (Gödel-Gentzen translation A^g)**.**

$$
\begin{aligned}
(R\vec{t})^g &:= \neg\neg R\vec{t} \qquad \text{for } R \text{ distinct from } \bot \\
\bot^g &:= \bot \\
(A \vee B)^g &:= A^g \tilde{\vee} B^g \\
(\exists_x A)^g &:= \tilde{\exists}_x A^g \\
(A \circ B)^g &:= A^g \circ B^g \qquad \text{for } \circ = \to, \wedge \\
(\forall_x A)^g &:= \forall_x A^g
\end{aligned}
$$

Lemma 1.10. $\vdash \neg\neg A^g \to A^g$

Proof. Induction on A

Case $R\vec{t}$ with R distinct from $\bot$. We must show $\neg\neg\neg\neg R\vec{t} \to \neg\neg R\vec{t}$, which is a special case of $\vdash \neg\neg\neg B \to \neg B$

Case $\bot$. Use $\vdash \neg\neg\bot \to \bot$

Case $A \vee B$. $\vdash \neg\neg(A^g \tilde{\vee} B^g) \to A^g \tilde{\vee} B^g$ is a special case of $\vdash \neg\neg(\neg C \to \neg D \to \bot) \to \neg C \to \neg D \to \bot$

$$
\dfrac{\neg\neg(\neg C \to \neg D \to \bot) \qquad \dfrac{\dfrac{\dfrac{\dfrac{u_1 : \neg C \to \neg D \to \bot \qquad \neg C}{\neg D \to \bot} \qquad \neg D}{\bot}}{\neg(\neg C \to \neg D \to \bot)} \to^+ u_1}{}}{\bot}
$$

Case $\exists_x A$. We need to show $\vdash \neg\neg\tilde{\exists}_x A^g \to \tilde{\exists}_x A^g$, and this is a special case of $\vdash \neg\neg\neg B \to \neg B$ $\square$

Theorem 1.11. *1.* $\Gamma \vdash_c A$ *implies* $\Gamma^g \vdash A^g$

2. $\Gamma^g \vdash A^g$ *implies* $\Gamma \vdash_c A$ *for* Γ, A *without* $\vee, \exists$