Category Theory

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1 Categories

1.1 Examples of categories

Definition 1.1. A functor

$$F: \mathbf{C} \to \mathbf{C}$$

between categories \boldsymbol{C} and \boldsymbol{D} is a mapping of objects to objects and arrows to arrows, in such a way that

- 1. $F(f:A \rightarrow B) = F(f):F(A) \rightarrow F(B)$
- 2. $F(1_A) = 1_{F(A)}$
- 3. $F(g \circ f) = F(g) \circ F(f)$

1.2 Free categories

The "Kleene closure" of *A* is defined to be the set

$$A^* = \{ words over A \}$$

Also

$$i:A\to A^*$$

defined by i(a) = a and called the "intersection of generators"

A monoid M is **freely generated** by a subset A of M if the following conditions hold:

- 1. Every element $m \in M$ can be written as a product of elements of A
- 2. No "nontrivial" relations hold in M, that is, if $a_1 \dots a_j = a_1' \dots a_k'$, then this is required by the axioms for monoids

Every monoid N has an underlying set |N|, and every monoid homomorphism $f:N\to M$ has an underlying function $|f|:|N|\to |M|$. The free monoid M(A) on a set A is by definition "the" monoid with the following UMP

Universal Mapping Property of M(A)

There is a function $i:A\to |M(A)|$, and given any monoid N and any function $f:A\to |N|$, there is a **unique** monoid homomorphism $\overline{f}:M(A)\to N$ s.t. $\left|\overline{f}\right|\circ i=f$

in Mon

$$M(A) \xrightarrow{\overline{f}} N$$

in Sets

$$|M(A)| \xrightarrow{|\overline{f}|} |N|$$

$$\downarrow \uparrow \qquad \qquad \downarrow \uparrow$$

$$A$$

Proposition 1.2. A^* has the UMP of the free monoid on A

Proof. Given
$$f:A \to |N|$$
, define $\overline{f}:A^* \to N$ by

$$\begin{split} \overline{f}(-) &= u_N, \quad \text{the unit of } N \\ \overline{f}(a_1 \dots a_i) &= f(a_1) \cdot_N \dots \cdot_N f(a_i) \end{split}$$

2 Abstract structures

2.1 Initial and terminal objects

Example 2.1. A **Boolean algebra** is a poset B equipped with distinguished elements 0,1, binary operations $a \lor b$ of join and $a \land b$ of meet, and a unary operation $\neg b$ of complementation. These are required to satisfy the conditions

$$0 \le a$$

$$a \le 1$$

$$a \le c \quad \text{and} \quad b \le c \quad \text{iff} \quad a \lor b \le c$$

$$c \le a \quad \text{and} \quad c \le b \quad \text{iff} \quad c \le a \land b$$

$$a \le \neg b \quad \text{iff} \quad a \land b = 0$$

 $\mathbf{2} = \{0,1\}$ is an initial elements of **BA**. **BA** has as arrows the Boolean homomorphisms that $h(0) = 0, h(a \lor b) = h(a) \lor h(b)$, etc.

2.2 Products

Definition 2.1. In any category C, a **product diagram** for the objects A and B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP: Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique $u: X \to P$ making the diagram

$$A \xleftarrow{x_1} \downarrow u \qquad x_2 \\ \downarrow u \qquad X \\ \downarrow u \qquad X_2 \\ A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

2.3 Categories with products

Let **C** be a category that has a product diagram for every pair of objects. Suppose we have objects and arrows

with indicated products. Then we write

$$f\times f':A\times A'\to B\times B$$

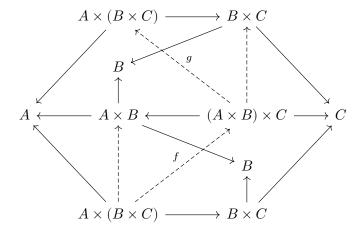
for $f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle$

$$\begin{array}{ccccc} A \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\ f \downarrow & & \downarrow f \times f' & \downarrow f' \\ B \xleftarrow{q_1} & B \times B' & \xrightarrow{q_2} & B' \end{array}$$

In this way, if we choose a product for each pair of objects, we get a functor

$$(g\circ f)\times (g'\circ f')=(f\times f')\circ (g\times g')$$
 To prove
$$(A\times B)\times C\cong A\times (B\times C)$$

Consider



Given no objects, there is an object 1 with no maps, and give nany other object X and no maps, there is a unique arrow

$$!: X \to 1$$

Definition 2.2. A category **C** is said to **have all finite products**, if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category **C** has all (small) products if every set of objects in **C** has a product

2.4 Hom-sets

In this section, we assume that all categories are locally small Given any objects A and B in category C, we write

$$\operatorname{Hom}(A,B) = \{ f \in \mathbf{C} \mid f : A \to B \}$$

and call such a set of arrows a Hom-set

Note that any arrow $g: B \to B'$ in **C** induces a function

$$\operatorname{Hom}(A,g):\operatorname{Hom}(A,B)\to\operatorname{Hom}(A,B')$$

$$(f:A\to B)\mapsto (g\circ f:A\to B\to B')$$

Let's show that this determines a functor

$$\operatorname{Hom}(A,-):\mathbf{C}\to\operatorname{\mathbf{Sets}}$$

called the (covariant) **representable functor** of *A*. We need to show that

$$\operatorname{Hom}(A,1_X)=1_{\operatorname{Hom}(A,X)}$$

and that

$$\operatorname{Hom}(A, g \circ f) = \operatorname{Hom}(A, g) \circ \operatorname{Hom}(A, f)$$

For any object P, a pair of arrows $p_1:P\to A$ and $p_2:P\to B$ determine an element (p_1,p_2) of the set

$$\operatorname{Hom}(P,A) \times \operatorname{Hom}(P,B)$$

Now given any arrow

$$x: X \to P$$

composing with p_1 and p_2 gives a pair of arrows $x_1=p_1\circ x:X\to A$ and $x_2=p_2\circ x:X\to B$

In this way, we have a function

 $\theta_X=(\mathrm{Hom}(X,p_1),\mathrm{Hom}(X,p_2)):\mathrm{Hom}(X,P)\to\mathrm{Hom}(X,A)\times\mathrm{Hom}(X,B)$ defined by

$$\theta_X(x) = (x_1, x_2)$$

Proposition 2.3. A diagram of the form

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

is a product for A and B iff for every object X, the canonical function θ_X is an isomorphism

$$\theta_X : \operatorname{Hom}(X, P) \cong \operatorname{Hom}(X, A) \times \operatorname{Hom}(X, B)$$

Proof. Note that we are talking about isomorphism on the set \Box

Definition 2.4. Let C, D be categories with binary products. A functor $F : C \to D$ is said to **preserve binary products** if it takes every product diagram

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

to a product diagram

$$FA \leftarrow_{Fp_1} F(A \times B) \xrightarrow{Fp_2} FB$$

F preserves products just if

$$F(A \times B) \cong FA \times FB$$

Corollary 2.5. For any object X in a category C with products, the (covariant) representable functor

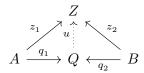
$$\operatorname{Hom}_{\mathcal{C}}(X,-): \mathcal{C} \to \operatorname{Sets}$$

preserves products

3 Duality

3.1 Coproducts

Definition 3.1. A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A,B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u:Q \to Z$ with $u \circ q_i = z_i$



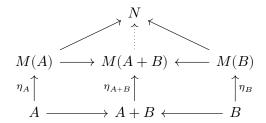
written as A + B

In **Sets**, every finite set A is a coproduct

$$A \cong 1 + 1 + \dots + 1$$
 (*n*-times)

Example 3.1. If M(A) and M(B) are free monoids on sets A and B, then in **Mon** we can construct their coproduct as

$$M(A) + M(B) \cong M(A+B)$$



Here we are working in two different categories. Half below is in **Sets**, the other is **Mon**

Product of two powerset Boolean algebras $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is also a powerset

$$\mathcal{P}(A) \times \mathcal{P}(B) \cong \mathcal{P}(A+B)$$

Example 3.2. Two monoids M(|A| + |B|) is strings over the disjoint union |A| + |B| of the underlying sets. That is, the elements of A and B and the

equivalence relation $v \sim w$ is the least one containing all instances of the following equations

$$\begin{split} (\dots xu_Ay\dots) &= (\dots xy\dots) \\ (\dots xu_By\dots) &= (\dots xy\dots) \\ (\dots aa'\dots) &= (\dots a\cdot_A a'\dots) \\ (\dots bb'\dots) &= (\dots b\cdot_B b'\dots) \end{split}$$

The unit is the equivalence class [-] of the empty word. Multiplication is

$$[x\ldots y]\cdot [x'\ldots y'] = [x\ldots yx'\ldots y']$$

The coproduct injections $i_A:A\to A+B$ and $i_B:B\to A+B$ are

$$i_A(a) = [a], \quad i_B(b) = [b]$$

Given any homomorphisms $f:A\to M$ and $g:B\to M$ into a monoid, the unique homomorphism

$$[f,g]:A+B\to M$$

is defined by

$$|A| + |B| \xrightarrow{[|f|+|g|]} |M|$$

$$M(|A| + |B|) \xrightarrow{[f,g]'} M$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow$$