An Introduction To Algebraic Topology

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July 13, 2021

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1 Introduction

1.1 Notation

$\mathbf{I} = [0, 1]$.

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

S^n is called the n**-sphere**. $S^n \subset \mathbb{R}^{n+1}$ (S^1 is the circle); 0-sphere S^0 consists of the two points $\{-1, 1\}$ and hence is a discrete two-point space. We may regard S^n as the **equator** of S^{n+1}

$$S^n = \mathbb{R}^{n+1} \cap S^{n+1} = \{(x_1, \dots, x_{n+2}) \in S^{n+1} : x_{n+2} = 0\}$$

The **north pole** is $(0, 0, \dots, 0, 1) \in S^n$; the **south pole** is $(0, 0, \dots, 0, -1)$. The **antipode** of $x = (x_1, \dots, x_{n+1}) \in S^n$ is the other endpoint of the diameter having one endpoint x; thus the antipode of x is $-x = (-x_1, \dots, -x_{n+1})$, for the distance from $-x$ to x is 2.

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

D^n is called the n**-disk** (or n**-ball**). Observe that $S^{n-1} \subset D^n \subset \mathbb{R}^n$; indeed S^{n-1} is the boundary of D^n in $\mathbb{R}^n$

$$\Delta^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \text{ each } x_i \geq 0 \text{ and } \sum x_i = 1\}$$

Δ^n is called the **standard** n**-simplex**. Δ^0 is a point, Δ^1 is a closed interval, Δ^2 is a triangle (with interior), Δ^3 is a (solid) tetrahedron, and so on.

There is a standard homeomorphism from S^n - {north pole} to $\mathbb{R}^n$, called **stereographic projection**. Denote the north pole by N, and define $\sigma : S^n - \{N\} \to \mathbb{R}^n$ to be the intersection of $\mathbb{R}^n$ and the line joining x and N. Points on the latter line have the form $tx + (1-t)N$, hence they have coordinates $(tx_1, \dots, tx_n, tx_{n+1} + (1-t))$. The last coordinate is zero for $t = (1 - x_{n+1})^{-1}$; hence

$$\sigma(x) = (tx_1, \dots, tx_n)$$

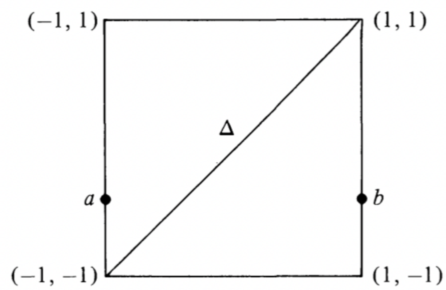
where $t = (1 - x_{n+1})^{-1}$. It is now routine to check that σ is indeed a homeomorphism. Note that $\sigma(x) = x$ iff x lies on the equator S^{n-1}

1.2 Brouwer Fixed Point Theorem

Theorem 1.1. *Every continuous $f : D^1 \to D^1$ has a fixed point*

Proof. Let $f(-1) = a$ and $f(1) = b$.If either $f(-1) = -1$ or $f(1) = 1$, we are done. Therefore we may assume that $f(-1) = a > -1$ and that $f(1) = b < 1$ as drawn. If G is the graph of f and Δ is the graph of the identity function, then we must prove that $G \cap \Delta \neq \emptyset$. The idea is to use a connectness argument to show that every path in $D^1 \times D^1$ from a to b must cross Δ.

Since f is continuous, $G = \{(x, f(x)) : x \in D^1\}$ is connected (continuous image of connected space is connected). Define $A = \{(x, f(x)) : f(x) > x\}$ and $B = \{(x, f(x)) : f(x) < x\}$. Note that $a \in A$ and $b \in B$, so that $A \neq \emptyset$ and $B \neq \emptyset$. If $G \cap \Delta = \emptyset$, then G is the disjoint union of A and B. $\square$



Definition 1.2. A subspace X of a topological space Y is a **retract** of Y if there is a continuous map $r : Y \to X$ with $r(x) = x$ for all $x \in X$; such a map r is called a **retraction**

Theorem 1.3 (Brouwer fixed point theorem)**.** *If $f : D^n \to D^n$ is continuous, then there exists $x \in D^n$ with $f(x) = x$*

2 Categories and Functors

Definition 2.1. A category $\mathcal{C}$ consists of three ingredients: a class of **objects**, obj $\mathcal{C}$; sets of **morphisms** $\mathrm{Hom}(A, B)$, one for every ordered pair $A, B \in$ obj $\mathcal{C}$; **composition** $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, denoted by $(f, g) \to g \circ f$, for every $A, B, C \in$ obj $\mathcal{C}$ satisfying the following axioms

1. the family of $\mathrm{Hom}(A, B)$'s is pairwise disjoint

2. composition is associative when defined

3. for each $A \in$ obj $\mathcal{C}$ there exists an identity $1_A \in \mathrm{Hom}(A, A)$ satisfying $1_A \circ f = f$ for every $f \in \mathrm{Hom}(B, A)$, all $B \in$ obj $\mathcal{C}$ and $g \circ 1_A = g$ for every $g \in \mathrm{Hom}(A, C)$, all C $\in$ obj $\mathcal{C}$

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{A}$ be categories with obj $\mathcal{C} \subset$ obj $\mathcal{A}$. If $A, B \in$ obj $\mathcal{C}$, let's denote the two possible Hom sets by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{A}}(A, B)$. Then $\mathcal{C}$ is a **subcategory** of $\mathcal{A}$ if $\mathrm{Hom}_{\mathcal{C}}(A, B) \subset \mathrm{Hom}_{\mathcal{A}}(A, B)$ for all $A, B \in$ obj $\mathcal{C}$ and if composition in $\mathcal{C}$ is the same as composition in $\mathcal{A}$

Example 2.1. $\mathcal{C} = \mathbf{Top}^2$. here obj $\mathcal{C}$ consists of all ordered pairs (X, A) where X is a topological space and A is a subspace of X. A morphism $f : (X, A) \to (Y, B)$ is an ordered pair (f, f') where $f : X \to Y$ is continuous and $fi = jf'$ (where i and j are inclusions)

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ B \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

and composition is coordinatewise. $\mathbf{Top}^2$ is called the category of **pairs** (of topological spaces)

Example 2.2. $\mathcal{C} = \mathbf{Top}_*$. Here obj $\mathcal{C}$ consists of all ordered pairs (X, x_0) where X is a topological space and x_0 is a point of X. $\mathbf{Top}_*$ is a subcategory of $\mathbf{Top}^2$ and it is called the category of **pointed spaces**; x_0 is called the **basepoint** of (X, x_0) and morphisms are called **pointed maps** (or **basepoint preserving maps**). The category $\mathbf{Sets}_*$ of pointed sets is defined similarly

Exercise 2.0.1. Let $f \in \mathrm{Hom}(A, B)$ be a morphism in a category $\mathcal{C}$. If f has a left inverse g ($g \in \mathrm{Hom}(B, A)\backslash$ and $g \circ f = 1_A$) and a right inverse h ($h \in \mathrm{Hom}(B, A)$ and $f \circ h = 1_B$), then $g = h$

Exercise 2.0.2. A set X is called **quasi-ordered** (or **pre-ordered**) if X has a transitive and reflexive relation $\leq$ (such a set is partially ordered if $\leq$ is antisymmetric). Prove that the following construction gives a category $\mathcal{C}$. Define obj $\mathcal{C} = X$, if $x, y \in X$ and $x \not\leq y$, define $\mathrm{Hom}(x, y) = \emptyset$; if $x \leq y$, define $\mathrm{Hom}(x, y)$ to be a set with exactly one element, denoted by i^x_y; if $x \leq y \leq z$ define composition by $i^y_z \circ i^x_y = i^x_z$

Exercise 2.0.3. Let G be a **monoid**, that is, a semigroup with 1. Show that the following gives a category $\mathcal{C}$. Let obj $\mathcal{C}$ have exactly one element, denoted by *; define $\mathrm{Hom}(*, *) = G$ and define composition $G \times G \to G$ as the given multiplication in G

Definition 2.3. A **diagram** in a category $\mathcal{C}$ is a directed graph whose vertices are labeled by objects of $\mathcal{C}$ and whose directed edges are labeled by morphisms in $\mathcal{C}$. A **commutative diagram** in $\mathcal{C}$ is a diagram in which, for each pair of vertices, every two paths (composites) between them are equal as morphisms.

Exercise 2.0.4. Given a category $\mathcal{C}$, shows that the following construction gives a category $\mathcal{M}$. First, an object of $\mathcal{M}$ is a morphism of $\mathcal{C}$. Next, if $f, g \in$ obj $\mathcal{M}$, say $f : A \to B$ and $g : C \to D$, then a morphism in $\mathcal{M}$ is an ordered pair (h, k) of morphisms in $\mathcal{C}$ s.t. the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ C & \xrightarrow[g]{} & D \end{array}$$

commutes. Define composition coordinatewise

$$(h', k') \circ (h, k) = (h' \circ h, k' \circ k)$$

Definition 2.4. A **congruence** on a category $\mathcal{C}$ is an equivalence relation $\sim$ on the class $\bigcup_{(A,B)} \mathrm{Hom}(A, B)$ of all morphisms in $\mathcal{C}$ s.t.

1. $f \in \mathrm{Hom}(A, B)$ and $f \sim f'$ implies $f' \in \mathrm{Hom}(A, B)$
2. $f \sim f'$, $g \sim g'$ and the composite $g \circ f$ exists imply that

$$g \circ f \sim g' \circ f'$$

Theorem 2.5. *Let $\mathcal{C}$ be a category with congruence $\sim$ and let $[f]$ denote the equivalence class of a morphism f. Define $\mathcal{C}'$ as follows*

$$\begin{aligned} \mathrm{obj}\, \mathcal{C}' &= \mathrm{obj}\, \mathcal{C} \\ \mathrm{Hom}_{\mathcal{C}'}(A, B) &= \{[f] : f \in \mathrm{Hom}_{\mathcal{C}}(A, B)\} \\ [g] \circ [f] &= [g \circ f] \end{aligned}$$

Then $\mathcal{C}'$ is a category

Proof. Property 1 in the definition of congruence shows that $\sim$ partitions each set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and this implies that $\mathrm{Hom}_{\mathcal{C}'}(A, B)$ is a set; moreover, the family of these sets is pairwise disjoint. Property 2 in the definition of congruence shows that composition in $\mathcal{C}'$ is well. $\mathcal{C}'$ is associative and $[1_A]$ is the identity is not hard ☐

The category $\mathcal{C}'$ is called a **quotient category** of $\mathcal{C}$; one usually denotes $\mathrm{Hom}_{\mathcal{C}'}(A, B)$ by $[A, B]$

Exercise 2.0.5. Let G be a group and let $\mathcal{C}$ be the one-object category it defines: $\mathrm{obj}\, \mathcal{C} = \{*\}$, $\mathrm{Hom}(*, *) = G$ and composition is the group operation. If H is a normal subgroup of G, define $x \sim y$ to mean $xy^{-1} \in H$. Show that $\sim$ is a congruence on $\mathcal{C}$ and that $[*, *] = G/H$ in the corresponding quotient category

Definition 2.6. If $\mathcal{A}$ and $\mathcal{C}$ are categories, a **functor** $T : \mathcal{A} \to \mathcal{C}$ is a function, that is,

1. $A \in \operatorname{obj} \mathcal{A}$ implies $TA \in \operatorname{obj} \mathcal{C}$
2. if $f : A \to A'$ is a morphism in $\mathcal{A}$, then $Tf : TA \to TA'$ is a morphism in $\mathcal{C}$
3. if f, g are morphisms in $\mathcal{A}$ for which $g \circ f$ is defined, then

$$T(g \circ f) = (Tg) \circ (Tf)$$

4. $T(1_A) = 1_{TA}$ for every $A \in \operatorname{obj} \mathcal{A}$

Example 2.3. If $\mathcal{C}$ is a category, the **identity functor** $J : \mathcal{C} \to \mathcal{C}$ is defined by $JA = A$ for every object A and $Jf = f$ for every morphism

Example 2.4. If M is a fixed topological space, Then T_m : **Top** $\to$ **Top** is a functor, where $T_M(X) = X \times M$ and if $f : X \to Y$ is continuous, then $T_M(f) : X \times M \to Y \times M$ is defined by $(x, m) \mapsto (f(x), m)$

Example 2.5. Fix an object A in category $\mathcal{C}$. Then $\operatorname{Hom}(A, -) : \mathcal{C} \to$ **Sets** is a functor assigning to each object B the set $\operatorname{Hom}(A, B)$ and to each morphism $f : B \to B'$ the **induced map** $\operatorname{Hom}(A,f) : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B')$ defined by $g \mapsto f \circ g$. One usually denotes the induced map $\operatorname{Hom}(A,f)$ by f_*

Functors as just defined are also called **covariant functors** to distinguish them from **contravariant functors** that reverse the direction of arrows. Thus the functor of the example is sometimes called a **covariant Hom functor**.

Definition 2.7. if $\mathcal{A}$ and $\mathcal{C}$ are categories, a **contravariant functor** $S : \mathcal{A} \to \mathcal{C}$ is a function that

1. $A \in \operatorname{obj} \mathcal{A}$ implies $SA \in \operatorname{obj} \mathcal{C}$
2. if $f : A \to A'$ is a morphism in $\mathcal{C}$, then $Sf : SA' \to SA$ is a morphism in $\mathcal{C}$
3. if f, g are morphisms in $\mathcal{A}$ for which $g \circ f$ is defined, then

$$S(g \circ f) = S(f) \circ S(g)$$

4. $S(1_A) = 1_{SA}$ for every $A \in \operatorname{obj} \mathcal{A}$

Example 2.6. Fix an object B in a category $\mathcal{C}$. Then $\mathrm{Hom}(-, B) : \mathcal{C} \to$ **Sets** is a contravariant functor assigning to each object A the set $\mathrm{Hom}(A, B)$ and to each morphism $g : A \to A'$ the **induced map** $\mathrm{Hom}(g, B) : \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B)$ defined by $h \mapsto h \circ g$. One usually denotes the induced map $\mathrm{Hom}(g, B)$ by g^*; $\mathrm{Hom}(-, B)$ is called a **contravariant Hom functor**

Definition 2.8. An **equivalence** in a category $\mathcal{C}$ is a morphism $f : A \to B$ for which there exists a morphism $g : B \to A$ with $f \circ g = 1_B$ and $g \circ f = 1_A$

Theorem 2.9. *If $\mathcal{A}$ and $\mathcal{C}$ are categories and $T : \mathcal{A} \to \mathcal{C}$ is a functor of either variance, then f an equivalence in $\mathcal{A}$ implies that Tf is an equivalence in $\mathcal{C}$*

Exercise 2.0.6. Let $\mathcal{C}$ and $\mathcal{A}$ be categories, let $\sim$ be a congruence on $\mathcal{C}$. If $T : \mathcal{C} \to \mathcal{A}$ is a functor with $T(f) = T(g)$ whenever $f \sim g$, then T defines a functor $T' : \mathcal{C}' \to \mathcal{A}$ (where $\mathcal{C}'$ is the quotient category) by $T'(X) = T(X)$ for every object X and $T'([f]) = T(f)$ for every morphism f.

Exercise 2.0.7.
1. if X is a topological space, show that $C(X)$, the set of all continuous real-valued functions on X, is a commutative ring with 1 under pointwise operations

$$f + g : x \mapsto f(x) + g(x) \quad \text{and} \quad f \cdot g \mapsto f(x)g(x)$$

   for all $x \in X$

2. show that $X \mapsto C(X)$ gives a (contravariant) functor **Top** $\to$ **Rings**

Proof.
2. From exercise 2.0.4

$\square$

3 Some Basic Topological Notions

3.1 Homotopy

Definition 3.1. If X and Y are spaces and if f_0, f_1 are continuous maps from X to Y, then f_0 is **homotopic** to f_1, denoted by $f_0 \simeq f_1$ if there is a continuous map $F : X \times \mathbf{I} \to Y$ with

$$F(x, 0) = f_0(x) \quad \text{and} \quad F(x, 1) = f_1(x) \quad \text{for all } x \in X$$

Such a map F is called a **homotopy**, written as $F : f_0 \simeq f_1$

If $f_t : X \to Y$ is defined by $f_t(x) = F(x, t)$, then a homotopy F gives a one-parameter family of continuous maps deforming f_0 into f_1

Lemma 3.2 (Gluing lemma)**.** *Assume that a space X is a finite union of closed subsets $X = \bigcup_{i=1}^{n} X_i$. If, for some space Y, there are continuous maps $f_i : X_i \to Y$ that agree on overlaps ($f_i|X_i \cap X_j = f_j|X_i \cap X_j$ for all i, j), then there exists a unique continuous $f : X \to Y$ with $f|X_i = f_i$ for all i*

Proof. If C is closed in Y, then

$$\begin{aligned} f^{-1}(C) &= X \cap f^{-1}(C) = (\bigcup X_i) \cap f^{-1}(C) \\ &= \bigcup (X_i \cap f^{-1}(C)) \\ &= \bigcup (X_i \cap f_i^{-1}(C)) = \bigcup f_i^{-1}(C) \end{aligned}$$

Since each f_i is continuous, $f_i^{-1}(C)$ is closed in X_i. Since X_i is closed in X, $f_i^{-1}(C)$ is closed in X, therefore $f^{-1}(C)$ is closed in X and f is continuous □

Lemma 3.3 (Gluing Lemma)**.** *Assume that a space X has a (possibly infinite) open cover $X = \bigcup X_i$. If for some space Y, there are continuous maps $f_i : X_i \to Y$ that agree on overlaps, then there exists a unique continuous $f : X \to Y$ with $f|X_i = f_i$ for all i*

Theorem 3.4. *Homotopy is an equivalence relation on the set of all continuous maps $X \to Y$*

Proof. Reflexivity. If $f : X \to Y$, define $F : X \times \mathbf{I} \to Y$ by $F(x, t) = f(x)$ for all $x \in X$ and $t \in \mathbf{I}$; clearly $F : f \simeq f$

Symmetry: Assume that $f \simeq g$, so there is a continuous $F : X \times \mathbf{I} \to Y$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. Define $G : X \times \mathbf{I} \to Y$ by $G(x, t) = F(x, 1 - t)$, and note that $G : g \simeq f$.

Transivity: assume that $F : f \simeq g$ and $G : g \simeq h$. Define $H : X \times \mathbf{I} \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t) & 0 \leq t \leq 1/2 \\ G(x, 2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

Because these functions agree on the overlap $\{(x, 1/2) : x \in X\}$, the gluing lemma shows that H is continuous. Therefore $H : f \simeq h$ □

Definition 3.5. If $f : X \to Y$ is continuous, its **homotopy class** is the equivalence class

$$[f] = \{\text{continuous } g : X \to Y : g \simeq f\}$$

The family of all such homotopy classes is denoted by $[X, Y]$

Theorem 3.6. *Let $f_i : X \to Y$ and $g_i : Y \to Z$, for $i = 0, 1$, be continuous. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$; that is, $[g_0 \circ f_0] = [g_1 \circ f_1]$*

Proof. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be homotopies. First, we show that

$$g_0 \circ f_0 \simeq g_1 \circ f_0$$

Define $H : X \times \mathbf{I} \to Z$ by $H(x, t) = G(f_0(x), t)$. Clearly, H is continuous; moreover, $H(x, 0) = G(f_0(x), 0) = g_0(f_0(x))$ and $H(x, 1) = G(f_0(x), 1) = g_1(f_0(x))$. Now observe that

$$K : g_1 \circ f_0 \sim g_1 \circ f_1$$

where $K : X \times \mathbf{I} \to Z$ is the composite $g_1 \circ F$. Now use the transitivity of the homotopy relation, we have $g_0 \circ f_0 \simeq g_1 \circ f_1$ □

Corollary 3.7. *Homotopy is a congruence on the category **Top**.*

It follows from Theorem 2.5 that there is a quotient category whose objects are topological spaces X, whose Hom sets are $\mathrm{Hom}(X, Y) = [X, Y]$ and whose composition is $[g] \circ [f] = [g \circ f]$

Definition 3.8. The quotient category just described is called the **homotopy category**, and it is denoted by **hTop**

All the functors $T : \mathbf{Top} \to \mathcal{A}$ that we shall construct, where $\mathcal{A}$ is some "algebraic" category (e.g. **Ab**, **Groups**, **Rings**) will have the property that $f \simeq g$ implies $T(f) = T(g)$. This fact, aside from a natural wish to identify homotopic maps, makes homotopy valuable, *beacause it guarantees that the algebraic problem in $\mathcal{A}$ arising from a topological problem via T is simpler than the original problem*

Definition 3.9. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Two spaces X and Y have the **same homotopy type** if there is a homotopy equivalence $f : X \to Y$

f is a homotopy equivalence iff $[f] \in [X, Y]$ is an equivalence in **hTop**. ()

Definition 3.10. Let X and Y be spaces, and let $y_0 \in Y$. The **constant map** at y_0 is the function $c : X \to Y$ with $c(x) = y_0$ for all $x \in X$. A continuous map $f : X \to Y$ is **nullhomotopic** if there is a constant map $c : X \to Y$ with $f \simeq c$

Theorem 3.11. *Let $\mathbb{C}$ denote the complex numbers, let $\Sigma_\rho \subset \mathbb{C} \approx \mathbb{R}^2$ denote the circle with center at the origin 0 and radius ρ, and let $f^n_\rho : \Sigma_\rho \to \mathbb{C} - \{0\}$ denote the restriction to Σ^ρ of $z \mapsto z^n$. If none of the maps f^n_ρ is nullhomotopic ($n \geq 1$ and $\rho > 0$) then the fundamental theorem of algebra is true (i.e., every nonconstant complex polynomial has a complex root)*

Proof. Consider the polynomial with complex coefficients

$$g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

Choose $\rho > \max\{1, \sum_{i=1}^{n-1}|a_i|\}$ and define $F : \Sigma_\rho \times \mathbf{I} \to \mathbb{C}$

$$F(z,t) = z^n + \sum_{i=0}^{n-1}(1-t)a_i z^i$$

It's obvious that $F : g|\Sigma_\rho \simeq f_\rho^n$ if we can show that the image of F is contained in $\mathbb{C} - \{0\}$. that is, $F(z,t) \neq 0$. If, on the contrary, $F(z,t) = 0$ for some $t \in \mathbf{I}$ and some z with $|z| = \rho$, then $z^n = -\sum_{i=0}^{n-1}(1-t)a_i z^i$. The triangle inequality gives

$$\rho^n \leq \sum_{i=0}^{n-1}(1-t)|a_i|\rho^i \leq \sum_{i=0}^{n-1}|a_i|\rho^i \leq \left(\sum_{i=0}^{n-1}|a_i|\right)\rho^{n-1}$$

for $\rho > 1$ implies that $\rho^i \leq \rho^{n-1}$. Canceling ρ^{n-1} gives $\rho \leq \sum_{i=0}^{n-1}|a_i|$, a contradiction.

Assume now that g has no complex roots. Define $G : \Sigma_\rho \times \mathbf{I} \to \mathbb{C} - \{0\}$ by $G(z,t) = g((1-t)z)$. (Since g has no roots, the values of G do lie in $\mathbb{C} - \{0\}$) Visibly, $G : g|\Sigma_\rho \simeq k$, where k is the constant function at a_0. Therefore $g|\Sigma_\rho$ is nullhomotopic and by transitivity, f_ρ^n is nullhomotopic, contradicting the hypothesis. ☐

A common problem involves extending a map $f : X \to Z$ to a larger space Y; the picture is

$$\begin{tikzcd} Y \arrow[dr, dashed, "g"] & \\ X \arrow[u, hook] \arrow[r, "f"'] & Z \end{tikzcd}$$

Homotopy itself raises such a problem: if $f_0, f_1 : X \to Z$ then $f_0 \simeq f_1$ if we can extend $f_0 \cup f_1 : X \times \{0\} \cup X \times \{1\} \to Z$ to all of $X \times \mathbf{I}$

Theorem 3.12. *Let $f : S^n \to Y$ be a continuous map into some space Y. TFAE*

1. *f is nullhomotopic*
2. *f can be extended to a continuous map $D^{n+1} \to Y$*
3. *if $x_0 \in S^n$ and $k : S^n \to Y$ is the constant map at $f(x_0)$, then there is a homotopy $F : f \simeq k$ with $F(x_0, t) = f(x_0)$ for all $t \in \mathbf{I}$*

Proof. $1 \to 2$. Assume that $F : f \simeq c$, where $c(x) = y_0$ for all $x \in S^n$. Define $g : D^{n+1} \to Y$ by

$$g(x) = \begin{cases} y_0 & 0 \le \|x\| \le 1/2 \\ F(x/\|x\|, 2 - 2\|x\|) & 1/2 \le \|x\| \le 1 \end{cases}$$

if $x \neq 0$, then $x/\|x\| \in S^n$; if $1/2 \le \|x\| \le 1$ then $2-2\|x\| \in \mathbf{I}$; if $\|x\| = 1/2$ then $2 - 2\|x\| = 1$ and $F(x/\|x\|, 1) = c(x/\|x\|) = y_0$. The gluing lemma shows that g is continuous. Finally g does extend f: if $x \in S^n$, then $\|x\| = 1$ and $g(x) = F(x, 0) = f(x)$.

$2 \to 3$. Assume that $g : D^{n+1} \to Y$ extends f. Define $F : S^n \times \mathbf{I} \to Y$ by $F(x,t) = g((1-t)x + tx_0)$; note that $(1-t)x + tx_0 \in D^{n+1}$. Visibly F is continuous. Now $F(x, 0) = g(x) = f(x)$ while $F(x, 1) = g(x_0) = f(x_0)$ for all $x \in S^n$; hence $F : f \simeq k$ where $k : S^n \to Y$ is the constant map at $f(x_0)$. Finally, $F(x_0, t) = g(x_0) = f(x_0)$ for all $t \in \mathbf{I}$

$3 \to 1$ obvious $\square$

3.2 Convexity, Contractibility, and Cones

Definition 3.13. A subset X of $\mathbb{R}^m$ is **convex** if for each pair of points $x, y \in X$ the line segment joining x and y is contained in X. In other words, if $x, y \in X$, then $tx + (1-t)y \in X$ for all $t \in \mathbf{I}$

Definition 3.14. A space X is **contractible** if 1_X is nullhomotopic

Theorem 3.15. *Every convex set X is contractible*

Proof. Choose $x_0 \in X$, and define $c : X \to X$ by $c(x) = x_0$ for all $x \in X$. Define $F : X \times \mathbf{I} \to X$ by $F(x,t) = tx_0 + (1-t)x$. Hence $F : 1_X \simeq c$. $\square$

A hemisphere is contractible but not convex, so that the converse of Theorem 3.15 is not true

Exercise 3.2.1. Let $R : S^1 \to S^1$ be rotation by α radians. Prove that $R \simeq 1_S$. Conclude that every continuous map $f : S^1 \to S^1$ is homotopic to a continuous map $g : S^1 \to S^1$ with $g(1) = 1$ (where $1 = e^{2\pi i 0} \in S^1$)

Proof. Let $F : S^1 \times \mathbf{I} \to S^1$ be

$$F((\cos\theta, \sin\theta), t) = (\cos(\theta + \alpha(1-t)), \sin(\theta + \alpha(1-t)))$$

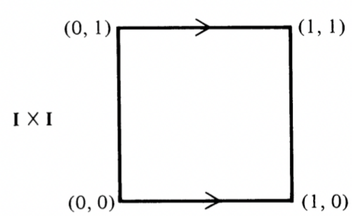
$\square$

Exercise 3.2.2. Let $X = \{0\} \cup \{1, 1/2, 1/3, \dots, 1/n, \dots\}$ and let Y be a countable discrete space. Show that X and Y do not have the same homotopy type.

Definition 3.16. Let X be a topological space and let $X' = \{X_j : j \in J\}$ be a partition of X. The **natural map** $\nu : X \to X'$ is defined by $\nu(x) = X_j$ where $x \in X_j$. The **quotient topology** on X' is the family of all subsets U' of X' for which $\nu^{-1}(U')$ is open in X

$\nu : X \to X'$ is continuous when X' has the quotient topology. There are two special cases that we wish to mention. If A is a subset of X, then we write X/A for X', where the partition of X consists of A together with all the one-point subsets of $X - A$. The second special case arises from an equivalence relation $\sim$ on X. In this case, the partition consists of the equivalence classes, the natural map is given by $\nu : x \mapsto [x]$, and the quotient space is denoted by $X/\sim$.

Example 3.1. Let $X = \mathbf{I} \times \mathbf{I}$ and define $(x, 0) \sim (x, 1)$ for every $x \in \mathbf{I}$. Then



$X/\sim$ is homeomorphic to the cylinder $S^1 \times \mathbf{I}$. Furthermore, suppose we define a second equivalence relation on $\mathbf{I} \times \mathbf{I}$ by $(x, 0) \sim (x, 1)$ for all $x \in \mathbf{I}$ and $(0, y) \sim (1, y)$ for all $y \in \mathbf{I}$. Then $\mathbf{I} \times \mathbf{I} / \sim$ is the **torus** $S^1 \times S^1$

Example 3.2. If $h : X \to Y$ is a function, then **ker *h*** is the equivalence relation on X defined by $x \sim x'$ if $h(x) = h(x')$. The corresponding quotient space is denoted by $X/\ker h$. Note that, given $h : X \to Y$ there always exists an injection $\varphi : X/\ker h \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "\nu"'] & & Y \\ & X/\ker h \arrow[ur, "\varphi"'] & \end{tikzcd}$$

namely, $\varphi([x]) = h(x)$

Definition 3.17. A continuous surjection $f : X \to Y$ is an **identification** if a subset U of Y is open iff $f^{-1}(U)$ is open in X

Example 3.3. If $\sim$ is an equivalence relation on $X$and $X/\sim$ is given the quotient topology, then the natural map $\nu : X \to X/\sim$ is an identification

Example 3.4. If $f : X \to Y$ is a continuous surjective map having a **section** (i.e., there is a continuous $s : Y \to X$ with $fs = 1_Y$), then f is an identification

Theorem 3.18. *Let $f : X \to Y$ be a continuous surjection. Then f is an identification iff for all spaces Z and all functions $g : Y \to Z$, one has g continuous iff gf is continuous*

$$\begin{tikzcd} X \arrow[rr, "gf"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Assume f is an identification. If g is continuous, then gf is continuous. Conversely, if f is continuous and let V be open in Z. Then $f^{-1}g^{-1}(V)$ is open in X; since f is an identification, $g^{-1}(V)$ is open in Y

Assume the condition. Let $Z/\ker f$, let $\nu : X \to X/\ker f$ be the natural map and let $\varphi : X/\ker f \to Y$ be the injection of Example 3.2. Note that φ is surjective because f is. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\nu"] \arrow[dr, "f"'] & & X/\ker f \\ & Y \arrow[ur, "\varphi^{-1}"'] & \end{tikzcd}$$

Then $\varphi^{-1}f = \nu$ is continuous implies that φ^{-1} is continuous, by hypothesis. Also φ is continuous because ν is an identification. We conclude that φ is a homeomorphism $\square$

Definition 3.19. Let $f : X \to Y$ be a function and let $y \in Y$. Then $f^{-1}(y)$ is called the **fiber** over y

Corollary 3.20. *Let $f : X \to Y$ be an identification and, for some space Z, let $h : X \to Z$ be a continuous function that is constant on each fiber of f. Then $hf^{-1} : Y \to Z$ is continuous*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "hf^{-1}"'] & \end{tikzcd}$$

Moreover, hf^{-1} is an open $($closed$)$ map iff $h(U)$ is open $($closed$)$ in Z whenever U is an open $($closed $)$ set in X of the form $U = f^{-1}f(U)$

Proof. h is constant on each fiber of f implies that hf^{-1} is well-defined. hf^{-1} is continuous because $(hf^{-1})(f) = h$ is continuous, and Theorem 3.18 applies. Finally if V is open in Y, then $f^{-1}(V)$ is an open set of the stated form $f^{-1}(V) = f^{-1}f(f^{-1}(V))$ □

Corollary 3.21. *Let X and Z be spaces and let $h : X \to Z$ be an identification. Then the map $\varphi : X/\ker h \to Z$ defined by $[x] \mapsto h(x)$ is a homeomorphism*

Proof. φ is a bijection. φ is continuous by Corollary 3.20. The $\nu : X \to X/\ker h$ be the natural map. Let U open in $X/\ker h$. Then $h^{-1}\varphi(U) = \nu^{-1}(U)$ is an open set in X, because ν is continuous and hence $\varphi(U)$ is open, because h is an identification

Weird □

Exercise 3.2.3. Let $f : X \to Y$ be an identification and let $g : Y \to Z$ be a continuous surjection. Then g is an identification iff gf is an identification

Exercise 3.2.4. Let X and Y be spaces with equivalence relations $\sim$ and $\square$, respectively, and let $f : X \to Y$ be a continuous map preserving the relations (if $x \sim x'$ then $f(x) \square f'(x)$). Prove that the induced map $\bar{f} : X/ \sim \to Y/\square$ is continuous; moreover, if f is an identification then so is $\bar{f}$

Proof. Consider

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\nu_1"'] & Y \arrow[d, "\nu_2"] \\ X/\sim \arrow[r, "\bar{f}"'] & Y/\square \end{tikzcd}$$

Visibly, the diagram commutes □

Definition 3.22. If X is a space, define an equivalence relation on $X \times \mathbf{I}$ by $(x, t) \sim (x', t')$ if $t = t' = 1$. Denote the equivalence class of (x, t) by $[x, t]$. The **cone** over X, denoted by CX, is the equivalence space $X \times \mathbf{I} / \sim$

One may regard CX as the quotient space $X \times \mathbf{I} / X \times \{1\}$. The identified point $[x, 1]$ is called the **vertex**

Example 3.5. For spaces X and Y, every continuous map $f : X \times \mathbf{I} \to Y$ with $f(x, 1) = y_0$, say, for all $x \in X$, induces a continuous map $\bar{f} : CX \to Y$, namely, $\bar{f} : [x, t] \to f(x, t)$. In particular, let $f : S^n \times \mathbf{I} \to D^{n+1}$ be the map $(u, t) \mapsto (1 - t)u$; since $f(u, 1) = 0$ for all $u \in S^n$, there is a continuous map $\bar{f} : CS^n \to D^{n+1}$ with $[u, t] \mapsto (1 - t)u$. Check: $\bar{f}$ is a homeomorphism.

Exercise 3.2.5. For fixed t with $0 \leq t < 1$, prove that $x \mapsto [x, t]$ defines a homeomorphism from a space X to a subspace of CX

Theorem 3.23. *For every space X, the cone CX is contractible*

Proof. Define $F : CX \times \mathbf{I} \to CX$ by $F([x, t], s) = [x, (1 - s)t + s]$ □

Theorem 3.24. *A space X has the same homotopy type as a point iff X is contractible*

Proof. Let $\{a\}$ be a one-point space, and assume that X and $\{a\}$ have the same homotopy type. There are thus maps $f : X \to \{a\}$ and $g : \{a\} \to X$ (with $g(a) = x_0 \in X$) with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_{\{a\}}$ (actually $f \circ g = 1_{\{a\}}$). But $gf(x) = g(a) = x_0$ for all $x \in X$, so that $g \circ f$ is constant. Therefore 1_X is nullhomotopic and X is contractible

Assume that $1_X \simeq k$ where $k(x) \equiv x_0 \in X$. Define $f : X \to \{x_0\}$ as the constant map at x_0 and define $g : \{x_0\} \to X$ by $g(x_0) = x_0$. $f \circ g = 1_{\{x_0\}}$ and $g \circ f = k \simeq 1_X$, by hypothesis. □

This theorem suggests that contractible spaces may behave as singletons

Theorem 3.25. *If Y is contractible, then any two maps $X \to Y$ are homotopic* (*indeed they are nullhomotopic*)

Proof. Assume that $1_Y \simeq k$, where there is $y_0 \in y$ with $k(y) = y_0$ for all $y \in Y$. Define $g : X \to Y$ as the constant map $g(x) = y_0$ for all $x \in X$. If $f : X \to Y$ is any continuous map, we claim that $f \simeq g$. Consider the diagram

$$X \longrightarrow Y \underset{1_Y}{\overset{k}{\rightrightarrows}} Y$$

Since $1_Y \simeq k$, Theorem 3.6 gives $f = 1_Y \circ f \simeq k \circ f = g$

Since homotopy relation is an equivalence, any two maps $X \to Y$ are homotopic □

Definition 3.26. A **path** in X is a continuous map $f : \mathbf{I} \to X$. if $f(0) = a$ and $f(1) = b$, one says that f is a path **from** a **to** b

Definition 3.27. A space X is **path connected** if, for every $a, b \in X$, there exists a path in X from a to b

Theorem 3.28. *If X is path connected, then X is connected*

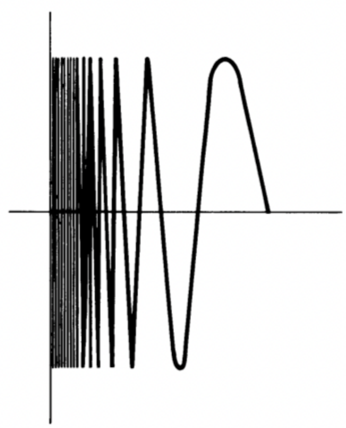
Proof. Suppose $X = A \coprod B$, where A and B are nonempty open subsets of X. Choose $a \in A$ and $b \in B$ and let $f : \mathbf{I} \to X$ be a path from a to b. Now $f(\mathbf{I})$ is connected, yet

$$f(\mathbf{I}) = (A \cap f(\mathbf{I})) \cup (B \cap f(\mathbf{I}))$$

displays $f(\mathbf{I})$ as disconnected, a contradiction $\square$

The converse of Theorem 3.28 is false

Example 3.6. The $\mathbf{sin(1/x)}$ **space** X is the subspace $X = A \cup G$ of $\mathbb{R}^2$, where $A = \{(0, y) : -1 \leq y \leq 1\}$ and $G = \{(x, \sin(1/x)) : 0 < x \leq 1/2\pi\}$



$\bar{G} = X$. To show $\bar{G} \subseteq X$ we prove that X is closed. Let $\{(x_n, y_n)\}$ be a sequence in X with limit $(x, y) \in \mathbb{R}^2$. We must prove $(x, y) \in X$. If $x = 0$ then $(x, y) = (0, y) \in X$. If $x > 1$, then upon dropping the first few terms of the sequence we can assume $x_n > 0$ for all n. Then $(x_n, y_n) \in G$. Since the function $t \mapsto \sin(1/t)$ on $(0, \infty)$ is continuous, from the condition $x_n \to x$ we conclude

$$y = \lim y_n = \lim \sin(1/x_n) = \sin(1/x)$$

Then as G is connected, $\bar{G}$ is connected

Exercise 3.2.6. Show that the $\sin(1/x)$ space X is not path connected

Proof. Assume that $f : \mathbf{I} \to X$ is a path from $(0,0) \to (1/2\pi, 0)$. If $t_0 = \sup\{t \in \mathbf{I} : f(t) \in A\}$, then $a = f(t_0) \in A$ and $f(s) \notin A$ for all $s > t_0$. One may thus assume that there is a path $g : \mathbf{I} \to X$ with $g(0) \in A$ and with $g(t) \in G$ for all $t > 0$.

From StackExchange.

If $f = (f_1, f_2) : [0,1] \to X \subseteq \mathbb{R}^2$. is a path with $f(0) = (0,0)$,then $f(t) = (0,0)$ for all t

Suppose that $f(t)$ is not always $(0,0)$. Removing an initial part of the interval and then rescaling if necessary, assuming that $0 = \sup\{t : f([0,t] = \{(0,0)\})\}$. By continuity of f_2, there is a $\delta > 0$ s.t. $|f_2(t)| < 1$ for all $t < \delta$. Take $0 < t_0 < \delta$ with $f_1(t_0) > 0$. By continuity of f_1 and the intermediate value theorem, $[0, f_1(t_0)]$ is in the image of f_1 restricted to $[0, t_0]$. Since $f_2(t) = \sin(1/f_1(t))$ for all t with $f_1(t) \neq 0$. It follows that $[-1,1]$ is in the image of f_2 restricted to $[0, t_0]$, this contradicts $t_0 < \delta$. $\square$

Exercise 3.2.7.

1. A space X is path connected iff every two constant maps $X \to X$ are homotopic
2. If X is contractible and Y is path connected, then any two continuous maps $X \to Y$ are homotopic (and each is nullhomotopic)

Proof.

1. $\Rightarrow$. Take two constants as initial point and end point.

   $\Leftarrow$. Same
2. Suppose $k : X \to X$ is a constant map.

$$X \underset{k}{\overset{1_X}{\rightrightarrows}} X \xrightarrow{f} Y$$

Then $f \circ 1_X \simeq f \circ k$, and $f \circ k$ is homotopic to any constant map. So f is homotopic to every constant map.

$\square$

Exercise 3.2.8. If X and Y are path connected, then $X \times Y$ is path connected

Proof. A function $f \times g : \mathbf{I} \to X \times Y$ $\square$

Exercise 3.2.9. If $f : X \to Y$ is continuous and X is path connected, then $f(X)$ is path connected

Theorem 3.29. *If X is a space, then the binary relation $\sim$ on X defined by "$a \sim b$ if there is a path in X from a to b" is an equivalence relation*

Proof. *Transivity*: if f is a path from a to b and g is a path from b to c, define $h : \mathbf{I} \to X$ by

$$h(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

This is continuous by gluing lemma □

Definition 3.30. The equivalence classes of X under the relation $\sim$ in Theorem 3.29 are called the **path components** of X

Exercise 3.2.10. The path components of a space X are maximal path connected subspaces; moreover, every path connected subset of X is contained in a unique path component of X

Definition 3.31. Define $\pi_0(X)$ to be the set of path components of X. If $f : X \to Y$, define $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ to be the function taking a path component C of X to the (unique) path component of Y containing $f(C)$ (Exercise 3.2.9 and 3.2.10)

Theorem 3.32. *$\pi_0 : \textbf{\textit{Top}} \to \textbf{\textit{Sets}}$ is a functor. Moreover, if $f \simeq g$, then $\pi_0(f) = \pi_0(g)$*

Proof. If $F : f \simeq g$, where $f, g : X \to Y$. If C is a path component of X, then $C \times \mathbf{I}$ is path connected (Exercise 3.2.8), hence $F(C \times \mathbf{I})$ is path connected (Exercise 3.2.9). Now

$$f(C) = F(C \times \{0\}) \subset F(C \times \mathbf{I})$$

and

$$g(C) = F(C \times \{1\}) \subset F(C \times \mathbf{I})$$

the unique path component of Y containing $F(C \times \mathbf{I})$ thus contains both $f(C)$ and $g(C)$. This says that $\pi_0(f) = \pi_0(g)$ □

Corollary 3.33. *If X and Y have the same homotopy type, then they have the same number of path components*

Proof. Assume that $f : X \to Y$ and $g : Y \to X$ are continuous with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Then $\pi_0(g \circ f) = \pi_0(1_X)$ and $\pi_0(f \circ g) = \pi_0(1_Y)$ by Theorem 3.32. Since π_0 is a functor, it follows that $\pi_0(f)$ is a bijection □

Definition 3.34. A space X is **locally path connected** if, for each $x \in X$ and every open neighborhood U of x, there is an open V with $x \in V \subset U$ s.t. any two points in V can be joined by a path in U

Example 3.7. Let X be the subspace of $\mathbb{R}^2$ obtained from the sin(1/x) space by adjoining a curve from $0, 1$ to $(\frac{1}{2\pi}, 0)$. X is path connected but not locally path connected

Theorem 3.35. *A space X is locally path connected iff path components of open subsets are open. In particular, if X is locally path connected, then its path components are open.*

Proof. Assume that X is locally path connected and that U is an open subset of X. Let C be a path component of U and let $x \in C$. There is an open V with $x \in V \subset U$ s.t. every point of V can be joined to x by a path in U. Hence $V \subset C$. Therefore C is open $(C = \bigcup V_x)$.

Conversely, let U be an open set in X, let $x \in U$ and let V be the path component of x in U. By hypothesis, V is open. Therefore X is locally path connected □

Corollary 3.36. *X is locally path connected iff for each $x \in X$ and each open neighborhood U of x ,there is an open path connected V with $x \in V \subset U$.*

Corollary 3.37. *If X is locally path connected, then the components of every open set coincide with its path components. In particular, the components of X coincide with the path components of X*

guess in here, component means open set

Proof. Let C be a component of an open set U in X, and let $\{A_j : j \in J\}$ be the path components of C; then C is the disjoint union of A_j; By Theorem 3.35 each A_j is open in C, hence each A_j is closed in C. Were there more than one A_j, then C would be disconnected □

Corollary 3.38. *If X is connected and locally path connected, then X is path connected*

Proof. Since X is connected, X has only one component; since X is locally path connected, this component is a path component □

Definition 3.39. Let A be a subspace of X and let $i : A \hookrightarrow X$ be the inclusion. Then A is a **deformation retract** of X if there is a continuous $r : X \to A$ s.t. $r \circ i = 1_A$ and $i \circ r \simeq 1_X$

Every deformation retract is a retract. One can repharse the definition as follows: there is a continuous $F : X \times \mathbf{I} \to X$ s.t. $F(x, 0) = x$ for all $x \in X$, $F(x, 1) \in A$ for all $x \in X$, and $F(a, 1) = a$ for all $a \in A$ (in this formulation we have $r(x) = F(x, 1)$).

Theorem 3.40. *If A is a deformation retract of X, then A and X have the same homotopy type.*

Corollary 3.41. *S^1 is a deformation retract of $\mathbf{C}-\{0\}$ and so these spaces have the same homotopy type.*

Proof. Write each nonzero complex number z in polar coordinates

$$z = \rho e^{i\theta}, \quad \rho > 0, 0 \leq \theta < 2\pi$$

Define $F : (\mathbf{C}-\{0\}) \times \mathbf{I} \to \mathbf{C}-\{0\}$ by

$$F(\rho e^{i\theta}, t) = [(1-t)\rho + t]e^{i\theta}$$

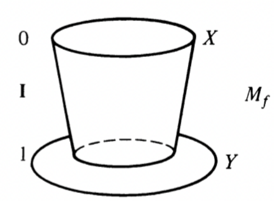
□

Exercise 3.2.11. For $n \geq 1$, show that S^n is a deformation retract of $\mathbb{R}^{n+1}-\{0\}$

Definition 3.42. Let $f : X \to Y$ be continuous and define [1]

$$M_f = ((X \times \mathbf{I}) \coprod Y)/\sim$$

where $(x, t) \sim y$ if $y = f(x)$ and $t = 1$. Denote the class of (x, t) in M_f by $[x, t]$ and the class of y in M_f by $[y]$ (so that $[x, 1] = [f(x)]$). The space M_f is called the **mapping cylinder** of f



4 Simplexes

4.1 Affine Spaces

Definition 4.1. A subset A of euclidean space is called **affine** if for every pair of distinct points $x, x' \in A$, the line determined by x, x' is contained in A

[1] definition

Observe that affine subsets are convex

Theorem 4.2. *If $\{X_j : j \in J\}$ is a family of convex (or affine) subsets of $\mathbb{R}^n$, then $\bigcup X_j$ is also convex (or affine)*

It thus makes sense to speak of the **convex (affine) set** in $\mathbb{R}^n$ **spanned** by a subset X of $\mathbb{R}^n$ (also called the **convex hull** of X), namely, the intersection of all convex (affine) subsets of $\mathbb{R}^n$ containing X. We denote the convex set spanned by X by $[X]$

Definition 4.3. An **affine combination** of points $p_0, p_1, \dots, p_m$ in $\mathbb{R}^n$ is a point with

$$x = t_0 p_0 + t_1 p_1 + \cdots + t_m p_m$$

where $\sum_{i=0}^{m} t_i = 1$. A **convex combination** is an affine combination for which $t_i \geq 0$ for all i

Theorem 4.4. *If $p_0, p_1, \dots, p_m \in \mathbb{R}^n$, then $[p_0, p_1, \dots, p_m]$ is the set of all convex combinations of $p_0, p_1, \dots, p_m$*

Proof. Let S denote the set of all convex combinations

$[p_0, p_1, \dots, p_m] \subset S$: it suffices to show that S is a convex set containing $\{p_0, \dots, p_m\}$. First, if we set $t_j = 1$ and the other $t_i = 0$, then we see that $p_j \in S$ for every j. Second, let $\alpha = \sum a_i p_i$ and $\beta = \sum b_i p_i \in S$, where $a_i, b_i \geq 0$ and $\sum a_i = 1 = \sum b_i$. We claim that $t\alpha + (1-t)\beta \in S$ for $t \in \mathbf{I}$.

$S \subset [p_0, \dots, p_m]$: if X is any convex set containing $\{p_0, \dots, p_m\}$, we show that $S \subset X$ by induction on $m \geq 0$. If $m = 0$, then $S = \{p_0\}$ and we are done. Let $m > 0$. If $t_i \geq 0$ and $\sum t_i = 1$. assume that $t_0 \neq 0$; by induction

$$q = \left(\frac{t_1}{1-t_0}\right) p_1 + \cdots + \left(\frac{t_m}{1-t_0}\right) p_m \in X$$

and so

$$p = t_0 p_0 + (1-t_0) q \in X$$

because X is convex □

Corollary 4.5. *The affine set spanned by $\{p_0, p_1, \dots, p_m\} \subset \mathbb{R}^n$ consists of all affine combinations of these points*

Definition 4.6. An ordered set of points $\{p_0, p_1, \dots, p_m\} \subset \mathbb{R}^n$ is **affine independent** if $\{p_1 - p_0, p_2 - p_0, \dots, p_m - p_0\}$ is a linearly independent subset of the real vector space $\mathbb{R}^n$

Theorem 4.7. *The following conditions on an ordered set of points* $\{p_0, p_1, \ldots, p_m\}$ *in* $\mathbb{R}^n$ *are equivalent*

1. $\{p_0, p_1, \ldots, p_m\}$ *is affine independent*
2. *if* $\{s_0, s_1, \ldots, s_m\} \subset \mathbb{R}$ *satisfies* $\sum_{i=0}^m s_i p_i = 0$ *and* $\sum_{i=0}^m s_i = 0$, *then* $s_0 = s_1 = \cdots = s_m = 0$
3. *each* $x \in A$, *the affine set spanned by* $\{p_0, p_1, \ldots, p_m\}$ *has a unique expression as an affine combination*

$$x = \sum_{i=0}^m t_i p_i \quad \text{and} \quad \sum_{i=0}^m t_i = 1$$

Proof. $1 \to 2$. Assume that $\sum s_i = 0$ and that $\sum s_i p_i = 0$. Then

$$\sum_{i=0}^m s_i p_i = \sum_{i=0}^m s_i p_i - \left(\sum_{i=0}^m s_i\right) = \sum_{i=0}^m s_i (p_i - p_0) = \sum_{i=1}^m s_i (p_i - p_0)$$

Affine independence of $\{p_0, p_1, \ldots, p_m\}$ gives linear independence of $\{p_1 - p_0, \ldots, p_m - p_0\}$, hence $s_i = 0$ for all $i = 1, 2, \ldots, m$. Finally $\sum s_i = 0$ implies that $s_0 = 0$ as well

$2 \to 3$. Assume that $x \in A$. By Corollary 4.5,

$$x = \sum_{i=0}^m t_i p_i$$

where $\sum_{i=0}^m t_i = 1$. If also

$$x = \sum_{i=0}^m t_i' p_i$$

where $\sum_{i=0}^m t_i' = 1$, then

$$0 = \sum_{i=0}^m (t_i - t_i') p_i$$

Since $\sum (t_i - t_i') = \sum t_i - \sum t_i' = 1 - 1 = 0$, it follows that $t_i - t_i' = 0$ for all i and $t_i = t_i'$ for all i as desired

$3 \to 1$. We may assume that $m \neq 0$. Assume that each $x \in A$ has a unique expression as an affine combination of $p_0, p_1, \ldots, p_m$. We shall reach a contradiction by assuming that $\{p_1 - p_0, \dots, p_m - p_0\}$ is linearly dependent. If so, there would be real numbers r_i, not all zero, with

$$0 = \sum_{i=1}^{m} r_i(p_i - p_0)$$

Let $r_j \neq 0$ and assume its 1. Now $p_j \in A$ has two expressions as an affine combination of $p_0, p_1, \dots, p_m$

$$\begin{aligned} p_j &= 1p_j \\ p_j &= -\sum_{i \neq j} r_i p_i + \left(1 + \sum_{i \neq j} r_i\right) p_0 \end{aligned}$$

where $1 \leq i \leq m$ in the summations □

Corollary 4.8. *Affine independence is a property of the set $\{p_0, p_1, \dots, p_m\}$ that is independent of the given ordering*

Corollary 4.9. *If A is the affine set in $\mathbb{R}^n$ spanned by an affine independent set $\{p_0, p_1, \dots, p_m\}$, then A is a translate of an m-dimensional sub-vector-space V of $\mathbb{R}^n$, namely,*

$$A = V + x_0$$

for some $x_0 \in \mathbb{R}^n$

Definition 4.10. A set of points $\{a_1, \dots, a_k\}$ in $\mathbb{R}^n$ is in **general position** if every $n + 1$ of its points forms an affine independent set

Assume that $\{a_1, \dots, a_k\} \subset \mathbb{R}^n$ is in general position. If $n = 1$, we are saying that every pair $\{a_i, a_j\}$ is affine independent; that is, all the points are distinct. If $n = 2$, we are saying that no three points are collinear, and if $n = 3$, that no four points are coplanar

Let $r_0, r_1, \dots, r_m$ be real numbers. The $(m + 1) \times (m + 1)$ **Vandermonde matrix** V has as its ith column $[1, r_i, r_i^2, \dots, r_i^m]$; moreover, $\det V = \prod_{j<i}(r_i - r_j)$, hence V is nonsingular if all the r_i are distinct. If one substracts column 0 from each of the other columns of V, then the ith column (for $i > 0$) of the new matrix is

$$[0, r_i - r_0, r_i^2 - r_0^2, \dots, r_i^m - r_0^m]$$

If V^* is the southeast $m \times m$ block of this new matrix, then $\det V^* = \det V$

Theorem 4.11. *For every $k \geq 0$, euclidean space $\mathbb{R}^n$ contains k points in general position*

Proof. We may assume that $k > n+1$ (otherwise, choose the origin together with $k-1$ elements of a basis). Select k distinct reals $r_1, r_2, \dots, r_k$ and for each $i = 1, 2, \dots, k$, define

$$a_i = (r_i, r_i^2, \dots, r_i^n) \in \mathbb{R}^n$$

We claim that $\{a_1, \dots, a_k\}$ is in general position. If not, there are $n+1$ points $\{a_{i_0}, \dots, a_{i_n}\}$ not affine independent, hence $\{a_{i_1} - a_{i_0}, \dots, a_{i_n} - a_{i_0}\}$ is linearly dependent. There are thus real numbers $s_1, s_2, \dots, s_n$, not all zero, with

$$0 = \sum s_j(a_{i_j} - a_{i_0}) = (\sum s_j(r_{i_j} - r_{i_0}), \sum s_j(r_{i_j}^2 - r_{i_0}^2), \dots, \sum s_j(r_{i_j}^n - r_{i_0}^n))$$

If V^* is the $n \times n$ southeast block of the $(n+1) \times (n+1)$ Vandermonde matrix obtained from $r_{i_0}, r_{i_1}, \dots, r_{i_n}$, and if σ is the column vector $\sigma = (s_1, s_2, \dots, s_n)$, then the vector equation above is $V^*\sigma = 0$. But since all the r_i are distinct, V^* is nonsingular and $\sigma = 0$, contradicting our hypothesis that not all the s_i are zero □

Definition 4.12. Let $\{p_0, p_1, \dots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$, and let A be the affine set spanned by this subset. If $x \in A$, then Theorem 4.7 gives a unique $(m+1)$-tuple $(t_0, t_1, \dots, t_m)$ with $\sum t_i = 1$ and $x = \sum_{i=0}^m t_i p_i$. The entries of this $(m+1)$-tuple are called the **barycentric coordinates** of x (relative to the ordered set $\{p_0, p_1, \dots, p_m\}$)

Definition 4.13. Let $\{p_0, p_1, \dots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$. The convex set spanned by this set, denoted by $[p_0, p_1, \dots, p_m]$, is called the (affine) m**-simplex** with **vertices** $p_0, p_1, \dots, p_m$.

Theorem 4.14. *If $\{p_0, p_1, \dots, p_m\}$ is affine independent, then each x in the m-simplex $[p_0, p_1, \dots, p_m]$ has a unique expression of the form*

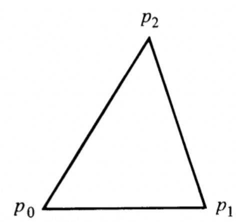
$$x = \sum t_i p_i, \quad \textit{where } \sum t_i = 1 \textit{ and each } t_i \geq 0$$

Proof. Theorem 4.4 shows that every $x \in [p_0, \dots, p_m]$ is such a convex combination □

Definition 4.15. If $\{p_0, \dots, p_m\}$ is affine independent, the **barycenter** of $[p_0, \dots, p_m]$ is $(1/m+1)(p_0 + p_1 + \dots + p_m)$

Example 4.1. The 1-simplex $[p_0, p_1] = \{tp_0 + (1-t)p_1 : t \in \mathbf{I}\}$ is the closed line segment with endpoints p_0, p_1.

Example 4.2. The 2-simplex $[p_0, p_1, p_2]$ is a triangle (with interior) with vertices p_0, p_1, p_2; the barycenter $\frac{1}{3}(p_0 + p_1 + p_2)$ is the center of gravity. Note that the three edges are $[p_0, p_1]$, $[p_1, p_2]$ and $[p_2, p_2]$



Example 4.3. The 3-simplex $[p_0, p_1, p_2, p_3]$ is the (solid) tetrahedron with vertices p_0, p_1, p_2, p_3

Example 4.4. For $i = 0, 1, \dots, n$, let e_i denote the point in $\mathbb{R}^{n+1}$ having (cartesian) coordinates all zeros except for 1 in the $(i+1)$st position. $\{e_0, e_1, \dots, e_n\}$ is affine independent. Now $[e_0, e_1, \dots, e_n]$ consists of all convex combinations $x = \sum t_i e_i$. In this case, barycentric and cartesian coordinates $(t_0, t_1, \dots, t_n)$ coincide, and $[e_0, e_1, \dots, e_n] = \Delta^n$, the standard n-simplex

Definition 4.16. Let $[p_0, p_1, \dots, p_m]$ be an m-simplex. The **face opposite** p_i is

$$[p_0, \dots, \hat{p}_i, \dots, p_m] = \left\{ \sum t_j p_j : t_j \geq 0, \sum t_j = 1, \text{ and } t_i = 0 \right\}$$

The **boundary** of $[p_0, p_1, \dots, p_m]$ is the union of its faces

Theorem 4.17. *Let S denote the n-simplex $[p_0, \dots, p_n]$*

1. *if $u, v \in S$ then $\|u - v\| \leq \sup_i \|u - p_i\|$*
2. $\operatorname{diam} S = \sup_{i,j} \left\|p_i - p_j\right\|$
3. *if b is the barycenter of S, then $\|b - p_i\| \leq (n/n+1) \operatorname{diam} S$*

Proof. 1. $v = \sum t_i p_i$, where $t_i \geq 0$ and $\sum t_i = 1$. Therefore

$$\begin{aligned} \|u - v\| &= \left\|u - \sum t_i p_i\right\| = \left\|(\sum t_i) u - \sum t_i p_i\right\| \\ &\leq \sum t_i \|u - p_i\| \leq \sum t_i \sup_i \|u - p_i\| = \sup_i \|u - p_i\| \end{aligned}$$

2. By 1, $\|u - p_i\| \leq \sup_j \left\|p_j - p_i\right\|$

3. Since $b = (1/n+1)\sum p_i$, we have

$$\begin{aligned}
\|b - p_i\| &= \left\|\sum_{j=0}^{n}(1/n+1)p_j - p_i\right\| = \left\|\sum_{j=0}^{n}(1/n+1)p_j - \left(\sum_{j=0}^{n}(1/n+1)\right)p_i\right\| \\
&= \left\|\sum_{j=0}^{n}(1/n+1)(p_j - p_i)\right\| \\
&\leq (1/n+1)\sum_{j=0}^{n}\left\|p_j - p_i\right\| \\
&\leq (n/n+1)\sup_{i,j}\left\|p_j - p_i\right\| \quad (\text{for } \left\|p_j - p_i\right\| = 0 \text{ when } j = i) \\
&= (n/n+1)\operatorname{diam} S
\end{aligned}$$

□

4.2 Affine Maps

Definition 4.18. Let $\{p_0, p_1, \ldots, p_m\} \subset \mathbb{R}^n$ be affine independent and let A denote the affine set it spans. An **affine map** $T : A \to \mathbb{R}^k$ (for some $k \geq 1$) is a function satisfying

$$T(\sum t_j p_j) = \sum t_j T(p_j)$$

whenever $\sum t_j = 1$. The restriction of T to $[p_0, p_1, \ldots, p_m]$ is also called an **affine map**

Theorem 4.19. *If $[p_0, \ldots, p_m]$ is an m-simplex, $[q_0, \ldots, q_n]$ an n-simplex, and $f : \{p_0, \ldots, p_m\} \to [q_0, \ldots, q_n]$ any function, then there exists a unique affine map $T : [p_0, \ldots, p_m] \to [q_0, \ldots, q_n]$ with $T(p_i) = f(p_i)$ for $i = 0, 1, \ldots, m$*

Exercise 4.2.1. If $T : \mathbb{R}^n \to \mathbb{R}^k$ is affine, then $T(x) = \lambda(x) + y_0$, where $\lambda : \mathbb{R}^n \to \mathbb{R}^k$ is a linear transformation and $y_0 \in \mathbb{R}^k$ is fixed

Proof. Some discussions. Link1 and Link2 □

Exercise 4.2.2. Given an explicit formula for the affine map $\theta : \mathbb{R} \to \mathbb{R}$ carrying $[s_1, s_2] \to [t_1, t_2]$ with $\theta(s_i) = t_i$, $i = 1, 2$.

5 The Fundamental Group

5.1 The Fundamental Groupoid

Definition 5.1. Let $f, g : \mathbf{I} \to X$ be paths with $f(1) = g(0)$. Define a path $f * g : \mathbf{I} \to X$ by

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ g(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

The gluing lemma shows that $f * g$ is continuous, and so $f * g$ is path in X. Our aim is to construct a group whose elements are certain homotopy classes of paths in X with binary operation $[f][g] = [f * g]$. Now if we impose the rather mild condition that X be path connected, then contractibility of $\mathbf{I}$ implies that all maps $\mathbf{I} \to X$ are homotopic (Exercise 3.2.7); thus there is only one homotopy class of maps.

Definition 5.2. Let $A \subset X$ and $f_0, f_1 : X \to Y$ be continuous maps with $f_0|A = f_1|A$. We write

$$f_0 \simeq f_1 \operatorname{rel} A$$

if there is a continuous map $F : X \times \mathbf{I} \to Y$ with $F : f_0 \simeq f_1$ and

$$F(a,t) = f_0(a) = f_1(a) \quad \text{for all } a \in A \text{ and all } t \in \mathbf{I}$$

The homotopy F above is called a **relative homotopy** (a homotopy rel A); in contrast, the original definition (which may be viewed as a homotopy rel $A = \emptyset$) is called a **free homotopy**

Definition 5.3. Let $\dot{\mathbf{I}} = \{0, 1\}$ be the boundary of $\mathbf{I}$ in $\mathbb{R}$. The equivalence class of a path $f : \mathbf{I} \to X \operatorname{rel} \dot{\mathbf{I}}$ is called the **path class** of f and is denoted by $[f]$

Theorem 5.4. *Assume that f_0, f_1, g_0, g_1 are paths in X with*

$$f_0 \simeq f_1 \text{ rel } \dot{\mathbf{I}} \quad \text{and} \quad g_0 \simeq g_1 \text{ rel } \dot{\mathbf{I}}$$

If $f_0(1) = f_1(1) = g_0(0) = g_1(0)$ then $f_0 * g_0 \simeq f_1 * g_1$ rel $\dot{\mathbf{I}}$

In path class notation, if $[f_0] = [f_1]$ and $[g_0] = [g_1]$, then $[f_0 * g_0] = [f_1 * g_1]$

Proof. If $F : f_0 \simeq f_1 \operatorname{rel} \dot{\mathbf{I}}$ and $G : g_0 \simeq g_1 \operatorname{rel} \dot{\mathbf{I}}$, then $H : \mathbf{I} \times \mathbf{I} \to X$ defined by

$$H(t,s) = \begin{cases} F(2t, s) & 0 \leq t \leq 1/2 \\ G(2t-1, s) & 1/2 \leq t \leq 1 \end{cases}$$

is a continuous map that is a relative homotopy $f_0 * g_0 \simeq f_1 * g_1 \operatorname{rel} \dot{\mathbf{I}}$ ∎

Exercise 5.1.1. Generalize Theorem 3.6 as follows. Let $A \subset X$ and $B \subset Y$ be given. Assume that $f_0, f_1 : X \to Y$ with $f_0|A = f_1|A$ and $f_i(A) \subset B$ for $i = 0, 1$; assume that $g_0, g_1 : Y \to Z$ with $g_0|B = g_1|B$. If $f_0 \simeq f_1 \operatorname{rel} A$ and $g_0 \simeq g_1 \operatorname{rel} B$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 \operatorname{rel} A$

Proof. Visibly, for all $a \in A$, $g_0 \circ f_0(a) = g_1 \circ f_1(a)$. Then follows the proof of the theorem □

Exercise 5.1.2.

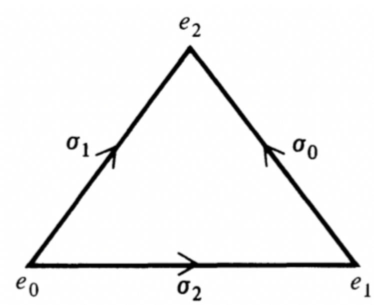
1. If $f : \mathbf{I} \to X$ is a path with $f(0) = f(1) = x_0 \in X$, then there is a continuous $f' : S^1 \to X$ given by $f'(e^{2\pi i t}) = f(t)$. If $f, g : \mathbf{I} \to X$ are paths with $f(0) = f(1) = x_0 = g(0) = g(1)$ and if $f \simeq g \operatorname{rel} \dot{\mathbf{I}}$, then $f' \simeq g' \operatorname{rel}\{1\}$ $(1 = e^0)$
2. If f and g are as above, then $f \simeq f_1 \operatorname{rel} \dot{\mathbf{I}}$ and $g \simeq g_1 \operatorname{rel} \dot{\mathbf{I}}$ implies that $f' * g' \simeq f_1' * g_1' \operatorname{rel}\{1\}$

Proof. 1. Let $k : S^1 \to \mathbf{I}$ with $k(2^{2\pi i t}) = t$. Then $f' = f \circ k$ and $g' = g \circ k$. □

Definition 5.5. If $f : \mathbf{I} \to X$ is a path from x_0 to x_1, call x_0 the **origin** of f and write $x_0 = \alpha(f)$; call x_1 the **end** of f and write $x_1 = \omega(f)$. A path f in X is **closed** at x_0 if $\alpha(f) = x_0 = \omega(f)$

Definition 5.6. If $p \in X$ then the constant function $i_p : \mathbf{I} \to X$ with $i_p(t) = p$ for all $t \in \mathbf{I}$ is called the **constant path** at p. If $f : \mathbf{I} \to X$ is a path, its **inverse** path $f^{-1}(x : \mathbf{I} \to X)$ is defined by $t \mapsto f(1-t)$

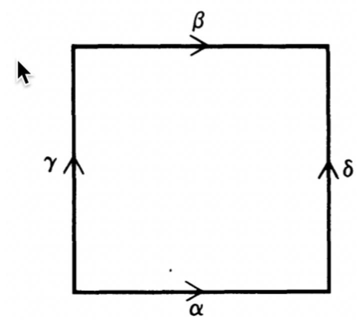
Exercise 5.1.3. Let $\sigma : \Delta^2 \to X$ be continuous, where $\Delta^2 = [e_0, e_1, e_2]$



Define $\epsilon_0 : \mathbf{I} \to \Delta^2$ as the affine map with $\epsilon_0(0) = e_1$ and $\epsilon_0(1) = e_2$; similarly, define ϵ_1 by $\epsilon_1(0) = e_0$ and $\epsilon_1(1) = e_2$ and define $\epsilon_2(0) = e_0$ and $\epsilon_2(1) = e_1$. Finally define $\sigma_i = \sigma \circ \epsilon_i$ for $i = 0, 1, 2$

1. Prove that $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ is nullhomotopic rel $\dot{\mathbf{I}}$

2. Let $F : \mathbf{I} \times \mathbf{I} \to X$ be continuous, and define paths α, β, γ, δ in X as indicated in the figure Thus $\alpha(t) = F(t, 0)$, $\beta(t) = F(t, 1)$, $\gamma(t) = F(0, t)$



and $\delta(t) = F(1, t)$. Prove that $\alpha \simeq \gamma * \beta * \delta^{-1}$ rel $\dot{\mathbf{I}}$

Exercise 5.1.4. Let $f_0 \simeq f_1$ rel $\dot{\mathbf{I}}$ and $g_0 \simeq g_1$ rel $\dot{\mathbf{I}}$ be paths in X and Y, respectively. If, for $i = 0, 1$, (f_i, g_i) is the path in $X \times Y$ defined by $t \mapsto (f_i(t), g_i(t))$, prove that $(f_0, g_0) \simeq (f_1, g_1)$ rel $\dot{\mathbf{I}}$

Exercise 5.1.5.

1. If $f \simeq g$ rel $\dot{\mathbf{I}}$ then $f^{-1} \simeq g^{-1}$ rel $\dot{\mathbf{I}}$, where f, g are paths in X

2. if f and g are paths in X with $\omega(f) = \alpha(g)$, then

$$(f * g)^{-1} = g^{-1} * f^{-1}$$

3. Given an example of a closed path f with $f * f^{-1} \neq f^{-1} * f$

4. Show that if $\alpha(f) = p$ and f is notconstant, then $i_p * f \neq f$

Proof. 3.

□

Theorem 5.7. *If X is a space, then the set of all path classes in X under the (not always defined) binary operation $[f][g] = [f * g]$ forms an algebraic system (called a **groupoid**) satisfying the following properties*

1. *each path class $[f]$ has an origin $\alpha[f] = p \in X$ and an end $\omega[f] = q \in X$ and*

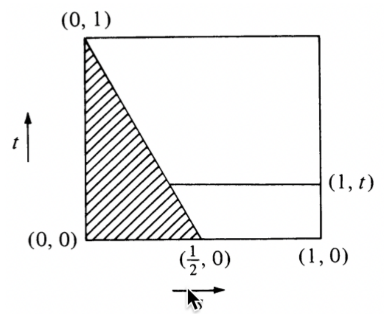
$$[i_p][f] = [f] = [f][i_q]$$

2. *associativity holds whenever possible*

3. *if* $p = \alpha[f]$ *and* $q = \omega[f]$, *then*

$$[f][f^{-1}] = [i_p] \quad \textit{and} \quad [f^{-1}][f] = [i_q]$$

Proof. 1. We show that $i_p * f \simeq f$ rel $\dot{\mathbf{I}}$ First, draw the line in $\mathbf{I} \times \mathbf{I}$ joining



(0, 1) to (1/2, 0); its equation is $2s = 1 - t$. For fixed t, define $\theta_t : [(1/t)/2, 1] \to [0, 1]$ as the affine map matching the endpoints of these intervals. By Exercise 4.2.2

□

6 Problem

7 Index