Set Theory

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Contents

1 Background Material

1.1 The Axioms of Set Theory

We work in predicate logic with $\mathcal{L} = \{\in\}$.

Axiom 1. Extensionality.

$$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$$

Axiom 2. Foundation.

$$\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))$$

Axiom 3. Comprehension Scheme. For each formula, φ, without y free

$$\exists y \forall x(x \in y \leftrightarrow x \in v \wedge \varphi(x))$$

Axiom 4. Pairing.

$$\exists z(x \in z \wedge y \in z)$$

Axiom 5. Union

$$\exists A \forall Y \forall x(x \in Y \wedge Y \in \mathcal{F} \rightarrow x \in A)$$

Axiom 6. Replacement Scheme. For each formula φ, without B free,

$$\forall x \in A \exists ! y \varphi(x, y) \rightarrow \exists B \forall x \in A \exists y \in B \varphi(x, y)$$

On the basis of Axioms 1,3,4,5, define $\subseteq, \emptyset, S, \cap$ and $\mathrm{SING}(x)$ (x is a singleton) by

$$\begin{array}{rcl}
x \subseteq y & \Leftrightarrow & \forall z(z \in x \rightarrow z \in y) \\
x = \emptyset & \Leftrightarrow & \forall z(z \notin x) \\
y = S(x) & \Leftrightarrow & \forall z(z \in y \Leftrightarrow z \in x \vee z = x) \\
y = v \cap w & \Leftrightarrow & \forall x(x \in y \Leftrightarrow x \in v \wedge x \in w) \\
\mathrm{SING}(x) & \Leftrightarrow & \exists y \in x \forall z \in x(z = y)
\end{array}$$

Axiom 7. Infinity.

$$\exists x(\emptyset \in x \wedge \forall y \in x(S(y) \in x))$$

Axiom 8. Power Set.

$$\exists y \forall z(z \subseteq x \rightarrow z \in y)$$

Axiom 9. Chioce, or AC.

$$\emptyset \notin F \wedge \forall x \in F \forall y \in F(x \neq y \rightarrow x \cap y = \emptyset) \rightarrow \exists C \forall x \in F(\mathrm{SING}(C \cap x))$$

- ZFC = Axioms 1-9. ZF=Axioms 1-8
- ZC and Z are ZFC and ZF, respectively, with the Replacement Scheme deleted
- X^{-1} denotes X without Foundation Axiom
- $X - P$ denote X without the Power Set Axiom
- $X - inf$ denotes X without Axiom of Infinity

Definition 1.1. BST ("Basic Set Theory") denotes the axioms of Extensionality, Foundation, Comprehension, Pairing and Union, plus the disjunction: the Power Set Axiom holds or the Replacement Axioms holds.

Definition 1.2. AC^{+} is the statement that every set can be well-ordered

1.2 Extensionality, Comprehension, Pairing, Union

Definition 1.3. $\text{int}(v, w, y) \leftrightarrow \forall x(x \in y \leftrightarrow x \in v \wedge x \in w)$

Introducing a defined relation such as $tint(v, w, y)$ requires no justification, although defining $v \cap w$ to be *the* y such that $\text{int}(v, w, y)$ *does* require a justification, namely

Lemma 1.4. $\forall v, w \exists! y int(v, w, y)$

Proof. To prove $\exists y \text{int}(v, w, y)$, use Comprehension, with φ the formula $x \in w$:

$$\forall v, w[\exists y \forall x(x \in y \leftrightarrow x \in v \wedge x \in w)]$$

To prove that y is unique, observe, from the definition of $\text{int}(v, w, y)$, that

$$\text{int}(v, w, y_1) \wedge \text{int}(v, w, y_2) \rightarrow \forall x(x \in y_1 \leftrightarrow x \in y_2)$$

so that $y_1 = y_2$ by Extensionality $\square$

This justifies:

Definition 1.5. $v \cap w$ is the unique y s.t. $\text{int}(v, w, y)$

Before giving a name to an object satisfying some property, we must prove that property really is held by a unique object

Definition 1.6. For any formula $\varphi(x)$

- $\{x : \varphi(x)\}$ is, informally, called a **class**
- if there is a set A s.t. $\forall x[x \in A \leftrightarrow \varphi(x)]$, then A is unique by Extensionality, and we denote this set by $\{x : \varphi(x)\}$, and we say that $\{x : \varphi(x)\}$ exists
- if there is no such set, then we say that $\{x : \varphi(x)\}$ doesn't exist, *or* forms a proper class
- $\{x \in v : \varphi(x)\}$ abbreviates $\{x : x \in v \wedge \varphi(x)\}$

Comprehension asserts that sets of the form $\{x : x \in v \wedge \varphi(x)\}$ always exists

Definition 1.7. Given v, w

$$v \cap w := \{x \in v : x \in w\}$$
$$v \setminus w := \{x \in v : x \notin w\}$$

Definition 1.8. $\text{emp}(y)$ iff $\forall x(x \notin y)$

Definition 1.9. $\emptyset$ denotes the (unique) y s.t. $\text{emp}(y)$

Justification. Fix any v and form $y = \{x \in v : x \neq x\}$, which is empty, so $\exists y \text{emp}(y)$. Then y is unique by Extensionality. □

Definition 1.10. The **universe**, $V := \{x : x = x\}$

Lemma 1.11. *V doesn't exists, and neither does $R := \{x : x \notin x\}$*

Proof. For R, we need no axioms at all; if we had an R s.t. $\forall x[x \in R \leftrightarrow x \notin x]$, then $R \in R \leftrightarrow R \notin R$, a contradiction. For V, we use Comprehension; if we had a V with $\forall x[x \in V]$, then we could form R as $\{x \in V : x \notin x\}$ □

Definition 1.12.

$$\begin{aligned} &\{x, y\} = \{w : w = x \vee w = y\} \\ &\{x\} = \{x, x\} \\ &\langle x, y\rangle = (x, y) = \{\{x\}, \{x, y\}\} \end{aligned}$$

Lemma 1.13. $\langle x, y\rangle = \langle x', y'\rangle \to x = x' \wedge y = y'$

Definition 1.14. $0 = \emptyset, 1 = \{0\} = \{\emptyset\}, 2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$

The axioms so far don't let us construct any sets with more than two elements. To do that we use the Union Axiom; then we can get $3 = 2 \cup \{2\} = \{0, 1, 2\}$. The Union Axiom can be used to justify infinite unions as well.

$$\forall \mathcal{F} \exists A \forall Y \forall x[x \in Y \wedge Y \in \mathcal{F} \to x \in A]$$

Definition 1.15. $\bigcup \mathcal{F} = \bigcup_{Y \in \mathcal{F}} Y = \{x : \exists Y \in \mathcal{F}(x \in Y)\}$