Topology

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1 Topological Spaces and Continuous Functions

1.1 Topological Spaces

Definition 1.1. A **topology** on a set is a collection $\mathcal{T}$ of subsets of X having the following properties

1. $\emptyset$ and X are in $\mathcal{T}$
2. The union of the elements of any subcollection of $\mathcal{T}$ is in T
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**

Example 1.1. Consider $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. $(-1/n, 1/n)$ is open but $\{0\}$ is not open in $\mathbb{R}$.

If $X$is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if $U \in \mathcal{T}$

Example 1.2. If X is any set, the collection of all subsets of X is a topology on X; it is called the **discrete topology**. The collection consisting of X and $\emptyset$ only is also a topology on X; we shall call it the **indiscrete topology**

Example 1.3. Let X be a set; let $\mathcal{T}_f$ be the collection of all subsets U of Xs.t. $X - U$ either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the **finite complement topology**. If $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$.

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Definition 1.2. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topology on a given set X. If $\mathcal{T}' \supset \mathcal{T}$ we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$ we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ or **strictly coarser**. We say $\mathcal{T}$ is **comparable** with $\mathcal{T}$ is either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$

1.2 Basis for a Topology

Definition 1.3. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis element**) s.t.

1. for each $x \in X$, there is at least one basis element B s.t. $x \in B$
2. if $x \in B_1 \cap B_2$, then there is a basis element B_3 s.t. $x \in B_3 \subset B_1 \cap B_2$

If $\mathcal{B}$ satisfies these conditions, then we define the **topology** $\mathcal{T}$ **generated by** $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis $B \in \mathcal{B}$ s.t. $x \in B \subset U$.

Now we show that $\mathcal{T}$ is indeed a topology. Take an indexed family $\{U_\alpha\}_{\alpha\in J}$ of elements of $\mathcal{T}$, we show that

$$U = \bigcup_{\alpha\in J} U_\alpha$$

belongs to $\mathcal{T}$. Given $x \in U$, there is an index α s.t. $x \in U_\alpha$. Since U_α is open, there is a basis element B s.t. $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so U is open.

If $U_1, U_2 \in \mathcal{T}$, then given $x \in U_1 \cap U_2$. we choose $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. By the second condition for a basis we have $x \in B_3 \subset B_1 \cap B_2$. Hence $x \in B_3 \subset U_1 \cap U_2$.

Lemma 1.4. *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$.

Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x for B s.t. $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$ □

Lemma 1.5. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X s.t. for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ s.t. $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. Let $x \in C_1 \cap C_2$, since C_1 and C_2 is open, $C_1 \cap C_2$ is open. Hence there exists $C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subseteq C_1 \cap C_2$

Let $\mathcal{T}$ be the collection of open sets of X; we must show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals the topology $\mathcal{T}$. If $U \in \mathcal{T}$, then there is $x \in C \subset U$. If $W \in \mathcal{T}'$, then $W = \bigcup_{x \in W} B_x$ and $B_x \in \mathcal{T}$ □

Lemma 1.6. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAW*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$*
2. *For each $x \in X$ and each basis element $x \in B \in \mathcal{B}$ there is a basis element $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$*

Proof. $2 \to 1$. Given $U \in \mathcal{T}$. Then $x \in B \subset U$ and $x \in B' \subset U$. Hence $U \in \mathcal{T}'$.

$1 \to 2$. given $x \in B \in \mathcal{B}$. Since $\mathcal{T} \subset \mathcal{T}'$ we have $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$ there is an element $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$ □

Definition 1.7. If $\mathcal{B}$ is the collection of all open intervals in the real line

$$(a, b) = \{x \mid a < x < b\}$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line. If $\mathcal{B}'$ is the collection of all half-opne intervals of the form

$$[a, b) = \{x \mid a \leq x < b\}$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** of $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$. Finally let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ is called the K**-topology** on R. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$

Lemma 1.8. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proof. Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ be the topologies of $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_K$. Given a basis element (a, b) for $\mathcal{T}$ and a point x of (a, b), the basis element $x \in [x, b) \subset (a, b)$. On the other hand, given the basis element $[x, d) \in \mathcal{T}$ there is no interval (a, b) that contains x and lies in $[x, d)$. Thus $\mathcal{T}$ is strictly finer than $\mathcal{T}$.

Given $B = (-1, 1) - K \in \mathcal{T}''$ and the point 0 of B, there is no open interval of $\mathcal{T}$ that contains 0 and lies in B

Also given B, there is no $[x, b) \in \mathcal{T}'$ s.t. $[x, b) \subset B$. □

Definition 1.9. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersection of elements of $\mathcal{S}$.

1.3 The Order Topology

Given elements <u>a and b of X</u> s.t. $a < b$,(a, b),$(a, b]$,$[a, b)$ and $[a, b]$ are **intervals**

Definition 1.10. Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X
2. All intervals of the form $[a_0, b)$ where a_0 is the smallest element of X
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element of X

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the **order topology**

1.4 The Product Topology on $X \times Y$

Definition 1.11. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y

Theorem 1.12. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection*

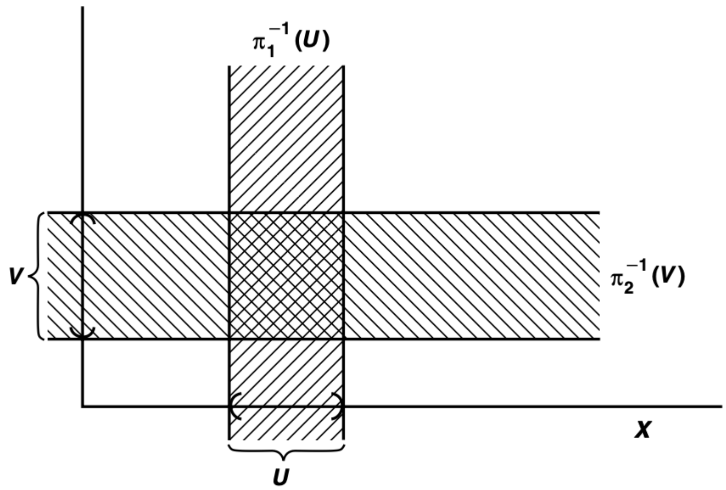
$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$

Theorem 1.13. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$



Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$; let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Then $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since

$$U \times V = \pi_1^{-1} \cap \pi_2^{-1}(V)$$

Hence $U \times V \in \mathcal{T}$ $\square$

1.5 The Subspace Topology

Definition 1.14. Let X be a topological space with topology $\mathcal{T}$. If $Y \subseteq X$, then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X

Lemma 1.15. *if $\mathcal{B}$ is a basis for the topology of X then the collection*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y

Proof. Given U open in X and given $y \in U \cap Y$, we can choose an element B of $\mathcal{B}$ s.t . $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from Lemma 1.5 that $\mathcal{B}_Y$ is a basis for the subspace topology on Y $\square$

Lemma 1.16. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X*

Theorem 1.17. *if A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U, V are open in X, Y respectively. Therefore $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

$\square$

Now let X be an ordered set in the order topology, and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. However the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X

Example 1.4. Consider the subset $Y = [0, 1]$ of the real line $\mathbb{R}$ in the *subspace* topology. Given (a, b)

$$(a, b) \cap Y = \begin{cases} (a, b) \\ [0, b) \\ (a, 1] \\ Y \text{ or } \emptyset \end{cases}$$

Sets of the second and third types are not open in the larger space $\mathbb{R}$

Note that these sets form a basis for the *order* topology on Y. Thus we see that in the case of the set $Y = [0, 1]$ its subspace topology and its order topology are the same

Given an ordered set X, a subset Y of X is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Note that intervals and rays in X are convex in X

Theorem 1.18. *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X*

Proof. Consider the ray $(a, +\infty)$ in X. If $a \in Y$ then

$$(a, +\infty) \cap Y = \{x \mid x \in Y \text{ and } x > a\}$$

this is an open ray of the ordered set Y. If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y, since Y is convex. In the former case, $(a, +\infty) \cap Y = Y$; in the latter case, it is empty

Similarly, $(-\infty, a) \cap Y$ is either an open ray of Y, or Y itself, or empty. Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y, and since each is open in the order topology,and since each is open in the order topology, the order topology contains the subspace topology

To prove the reverse, note that any open ray of Y equals the intersection of an open ray of X with Y, so it is open in the subspace topology on Y. Since the open rays of Y are a subbasis for the order topology, this topology is contained in the subspace topology □

Exercise 1.5.1. Show that if Y is a subspace of X and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X

Proof. For every open set U of topology of X, $A \cap (Y \cap U) = A \cap U$. □

Exercise 1.5.2. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X

Proof. Consider $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ which is convex in $\mathbb{Q}$ but not an interval or a ray □

1.6 Closed Sets and Limit Points

A subset A of a topological space X is said to be **closed** if the set $X - A$ is open

Theorem 1.19. *Let X be a topological space. Then the following conditions hold:*

1. *$\emptyset$ and X are closed*
2. *Arbitrary intersection of closed sets are closed*
3. *Finite unions of closed sets are closed*

Theorem 1.20. *let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y*

Proof. Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y. But $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y

Assume that A is closed in Y. Then $Y - A = U \cap Y$ for some open set U in X and $A = Y \cap (X - U)$ □

Theorem 1.21. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X*

Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A $(\bar{A})$

Theorem 1.22. *Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$*

Proof. Let B denote the closure of A in Y. The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y by Theorem 1.20. We have $B \subset (\bar{A} \cap Y)$

On the other hand, $B = C \cap Y$ for some C closed in X. Then C is a closed set of X containing A. □

A set A **intersects** a set B if the intersection $A \cap B$ is not empty

Theorem 1.23. *Let A be a subset of the topological space X*

1. *$x \in \bar{A}$ iff every open set U containing x intersects A*
2. *Suppose the topology of X is given by a basis, then $x \in \bar{A}$ iff every basis element b containing x intersects A*

Proof. 1. We consider

$x \notin \bar{A}$ iff there exists an open set U containing x that does not intersects A

If $x \notin \bar{A}$, the set $U = X - \bar{A}$ is an open set containing x that does not intersects A, as desired. Conversely, if there exsits an open set U containing x which does not intersects A, then $X - U$ is a closed set containing A. Hence $\bar{A} \subseteq X - U$ and therefore $x \notin \bar{A}$

□

U is an open set containing x equals U is a **neighborhood** of x

Example 1.5. Let X be the real line $\mathbb{R}$. If $A = (0, 1]$ then $\bar{A} = [0, 1]$ for every neighborhood of 0 intersects A, while every point outside $[0, 1]$ has a neighborhood disjoint from A.

If $B = \{1/n \mid n \in \mathbb{Z}_+\}$ then $\bar{B} = \{0\} \cup B$. If $C = \{0\} \cup (1, 2)$ then $\bar{C} = \{0\} \cup [1, 2]$. Also $\bar{\mathbb{Q}} = \mathbb{R}$.

If $A$is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** of A if every neighborhood of x intersects A in some point *other than x itself*. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$

Theorem 1.24. *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then*

$$\bar{A} = A \cup A'$$

Proof. By Theorem 1.23 $A' \subset \bar{A}$.

Suppose $x \in \bar{A} - A$. Then $x \in A'$ □

Corollary 1.25. *A subset of a topological space is closed iff it contains all its limit points*

Proof. A is closed iff $\bar{A} = A$ □

In the spaces $\mathbb{R}$ and $\mathbb{R}^2$ each one-point set $\{x_0\}$ is closed since every point different from x_0 has a neighborhood not intersecting $\{x_0\}$, so that $\{x_0\}$ is its own closure. But this fact is not true for arbitrary topological spaces. Consider the topology on the three-point set $\{a, b, c\}$ indicated in Figure 1. The one-point set $\{b\}$ is not closed, for its complement is not open

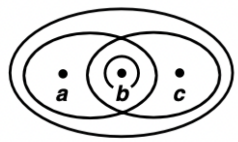


Figure 1: we

In an arbitrary topological space, one says that a sequence $x_1, x_2, \ldots$ of points of the space X **converges** to the point x of X provided that, corresponding to each neighborhood U of x there is a positive integer N s.t. $x_n \in U$ for all $n \geq N$. In $\mathbb{R}$ and $\mathbb{R}^2$ a sequence cannot converge to more than one point, but in an arbitrary space, it can. In Figure 1 the sequence defined by setting $x_n = b$ converges not only to the point b but also to the point a and c.

Definition 1.26. A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of disjoint points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint

Theorem 1.27. *Every fintie point set in a Hausdorff space X is closed.*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. $\square$

The condition that finite point sets be closed is in fact weaker than the Hausdorff condition. For example, the real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed is called the T_1 **axiom**

Theorem 1.28. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A iff every neighborhood of x contains infinitely many points*

Proof. If x is a limit point of A and suppose some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A - \{x\}$ in finitely many points; let $\{x_1, \ldots, x_m\}$ be the points of $U \cap (A - \{x\})$. The set $X - \{x_1, \ldots, x_m\}$ is an open set of X, then

$$U \cap (X - \{x_1, \ldots, x_m\})$$

is a neighborhood of x that intersects the set $A - \{x\}$ $\square$

Theorem 1.29. *If X is the Hausdorff space, then a sequence of points of X converges to at most one point of X*

Proof. Suppose that x_n is a sequence of points of X that converges to x. If $y \neq x$ let U and V be disjoint neighborhoods of x and y respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore x_n cannot converge to y. $\square$

If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$ and we say that x is the **limit** of the sequence x_n

Theorem 1.30. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

Exercise 1.6.1. Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does equality hold

Proof. It equals the closure iff both endpoints are limit points of the interval, i.e. if (a,b) is not empty and for every $x \in (a,b)$ there are $s,t \in (a,b)$ such that $a < s < x < t < b$. This is equivalent to the requirement that a has no immediate successor, and b has no immediate predecessor. Otherwise, if a has an immediate successor c then $(-\infty, c)$ is an open set containing a that does not intersect (a,b) , and, similarly, if b has an immediate predecessor c then $(c, +\infty)$ is an open set containing b that does not intersect (a,b) . $\square$

Exercise 1.6.2. Let A,B and A_α denote subsets of a space X. Prove the following

1. If $A \subset B$ then $\bar{A} \subset \bar{B}$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
3. $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails

Proof. 2. Suppose $x \notin \bar{A} \cup \bar{B}$. By Theorem 1.23 there is a neighborhoods U_A, U_B of x s.t. $U_A \cap A = U_B \cap B = \emptyset$. Let $U = U_A \cap U_B$. Then $U \cap (A \cup B) = \emptyset$.

3. Consider $A_n = (1/n, 2]$ for $n \in \mathbb{Z}_+$

$\square$

Exercise 1.6.3. Let A,B and A_α denote subsets of a space X. Determine whether the following equations hold

1. $\overline{A \cap B} = \bar{A} \cap \bar{B}$
2. $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$
3. $\overline{A - B} = \bar{A} - \bar{B}$

Proof. 1. Consider $A = (1,2)$ and $B = (0,1)$ in $\mathbb{R}$. We only have $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

3. $\overline{A-B} \supset \bar{A} - \bar{B}$. $A = (0,2), B = (0,1)$

□

Exercise 1.6.4. X is Hausdorff iff the **diagonal** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Proof. Δ is closed in $X \times X$ iff for $x \neq y$ there is a basis $x \times y \in U \times V \subset X \times X$ where U and V are neighborhoods of x and y respectively s.t. no points $(z,z) \in U \times V$ iff any pair of of different points having disjoint neighborhoods □

1.7 Continuous Functions

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y the set $f^{-1}(V)$ is an open subset of X.

Let's note that if the topology of the range space Y is given by a basis $\mathcal{B}$, then to prove continuity of f it suffices to show that the inverse image of every *basis element* is oepn.

If the topology on Y is given by a subbasis $\mathcal{S}$, to prove continuity of f it will even suffice to show that the inverse of each *subbasis* element is open.

Example 1.6. Let's consider a function

$$f : \mathbb{R} \to \mathbb{R}$$

Now we prove that our definition implies the ϵ-δ definition

Given $x_0 \in \mathbb{R}$ and given $\epsilon > 0$ the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set of the range space $\mathbb{R}$. Therefore, $f^{-1}(V)$ is an open set in the domain space $\mathbb{R}$. Because $x_0 \in f^{-1}(V)$, it contains some basis element (a,b) about x_0. We choose δ to be the smaller of the two numbers $x_0 - a$ and $b - x_0$. Then if $|x - x_0| < \delta$, the point x must be in (a,b), so that $f(x) \in V$ and $|f(x) - f(x_0)| < \epsilon$ as desired

Example 1.7. Let $\mathbb{R}$ denote the set of real numbers in its usual topology. Let

$$f : \mathbb{R} \to \mathbb{R}_l$$

by the identity function $f(x) = x$. Then f is not a continuous function. However

$$g : \mathbb{R}_l \to \mathbb{R}$$

is continuous

Theorem 1.31. *Let X and Y be topological spaces: let $f : X \to Y$. TFAE*

1. *f is continuous*
2. *for every $A \subseteq X, f(\bar{A}) \subset \overline{f(A)}$*
3. *for every closed set B of Y, the set $f^{-1}(B)$ is closed in X*

4. *for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x s.t. $f(U) \subset V$*

If the condition 4 holds for the point x of X, we say that f is ***continuous at the point*** *x*

Proof. $1 \to 2$. Assume f is continuous. Let $A \subseteq X$ and $x \in \bar{A}$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set of X containing x; it must intersect A in some point y. Then V intersects $f(A)$ in the point $f(y)$, so that $f(x) \in \overline{f(A)}$

$2 \to 3$. Let B be closed in Y and let $A = f^{-1}(B)$. We show that $\bar{A} = A$. We have $f(A) = f(f^{-1}(B)) \subset B$. Therefore if $x \in \bar{A}$

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$$

so that $x \inf^{-1}(B) = A$

$3 \to 1$. easy

$1 \to 4$. easy

$4 \to 1$. not hard 😊 □

let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a **homeomorphism**

Suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijectiive. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological embedding** or simpy an **embedding** of X in Y

Example 1.8. A bijectiive function $f : X \to Y$ can be continuous without being a homeomorphism. One such function is the identity map $g : \mathbb{R}_l \to \mathbb{R}$□ Another is the following:

Let S^1 denote the **unit circle**,

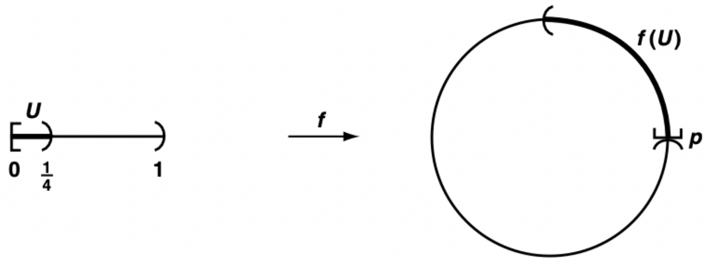
$$S^1 = \{x \times y \mid x^2 + y^2 = 1\}$$

considered as a subspace of the plane $\mathbb{R}^2$ and let

$$f : [0,1) \to S^1$$

be the map defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. f is continuous but not f^{-1}. The image under f of the open set $U = [0, \frac{1}{4})$ of the domain is not open in S^1, for the point $p = f(0)$ lies in no open set V of $\mathbb{R}^2$ s.t. $V \cap S^1 \subset f(U)$

Theorem 1.32 (Rules for constructing continuous functions)**.** *Let X, Y and Z be topological spaces*



1. *(Constant function) if $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous*
2. *(Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous*
3. *(Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous*
4. *(Restricting the domain) if $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|A : A \to Y$ is continuous*
5. *(Restricting or expanding the range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous*
6. *(Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α s.t. $f|U_\alpha$ is continuous for each α*

Proof. 1. Let V be open in Y, then $f^{-1}(V)$ equals $\emptyset$ or X

□

Theorem 1.33 (The pasting lemma)**.** *Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $A \cap B$ then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$*

The open set case of the pasting lemma is just the local formulation of continuity

Theorem 1.34 (Maps into products)**.** *Let $f : A \to X \times Y$ be given by the equation*

$$f(a) = (f_1(a), f_2(a))$$

Then f is continuous iff the functions

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

are continuous

The maps f_1 and f_2 are called the ***coordinate functions***

Proof. First note that π_1, π_2 are continuous. For $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ and these sets are open if U and V are open. Note that for each $a \in A$

$$f_1(a) = \pi_1(f(a)) \quad \text{and} \quad f_2(a) = \pi_2(f(a))$$

If f is continuous, then f_1, f_2 are continuous

Conversely, we show that for each basis element $U \times V$ for the topology $X \times Y$ its inverse image $f^{-1}(U \times V)$ is open. $a \in f^{-1}(U \times V)$ iff $f(a) \in (U \times V)$ iff $f_1(a) \in U$ and $f_2(a) \in V$. Therefore

$$f^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V)$$

□

Exercise 1.7.1. Let $F : X \times Y \to Z$. We say that F is **continuous in each variable separately** if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

Exercise 1.7.2. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0 \\ 0 & \end{cases}$$

1. Show that F is continuous in each variable separately
2. Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$
3. Show that F is not continuous

1.8 The Product Topology

Definition 1.35. Let J be an index set. Given a set X, we define J**-tuple** of elements of X to be a function $\mathbf{x} : J \to X$. If α is an element of j, we often denote the value of $\mathbf{x}$ at α by x_α; we call it the αth **coordinate** of $\mathbf{x}$. And we often denote the function $\mathbf{x}$ itself by the symbol

$$(x_\alpha)_{\alpha \in J}$$

We denote the set of all J-tuples of elements of X by X^J

Definition 1.36. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. The **cartesian product** of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of X s.t. $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$\mathbf{x} : J \to \bigcup_{\alpha \in J} A_\alpha$$

s.t. $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$

Definition 1.37. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha \in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha \in J} U_\alpha$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the **box topology**

Now we generalize the subbasis formulation of the definition. Let

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be the function assigning to each element of the product space its βth coordinate

$$\pi_\beta((x_\alpha)_{\alpha \in J}) = x_\beta$$

it is called the **projection mapping** associated with the index β

Definition 1.38. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collections

$$\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this topology $\prod_{\alpha \in J} X_\alpha$ is called a **product space**

To compare these topologies, we consider the basis $\mathcal{B}$ that $\mathcal{S}$ generates. The collection $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$. If we intersect elements belonging to the same one of the sets $\mathcal{S}_\beta$ we do not get anything new, because

$$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta)$$

We get something new only when we intersect elements from different sets $\mathcal{S}_\beta$. Thus the typical element of the basis $\mathcal{B}$ can be described as follows: let $\beta_1, \dots, \beta_n$ be a finite set of distinct indices from the index set J, and let U_{β_i} be an open set in X_{β_i} for $i = 1, \dots, n$. Then

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

is the typical element of $\mathcal{B}$

Now a point $\mathbf{x} = (x_\alpha)$ is in B iff its β_1th coordinate is in U_{β_1}, its β_2th coordinate is in U_{β_2}, and so on. As a result, we can write B as the product

$$B = \prod_{\alpha \in J} U_\alpha$$

where U_α denotes the entire space X_α if $\alpha \neq \beta_1, \dots, \beta_n$

Theorem 1.39 (Comparison of the box and product topologies)**.** *The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form U_α, where U_α is open in U_α for each α and U_α equals X_α except for finitely many values of α*

> *Whenever we consider the product X_α, we shall assume it is given the product topology unless we specifically state otherwise.*

Theorem 1.40. *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$

Theorem 1.41. *Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ is both products are given the box topology or product topology*

Theorem 1.42. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies*

Theorem 1.43. *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subseteq X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then*

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$$

Proof. Let $\mathbf{x} = (x_\alpha)$ be a point of $\prod \bar{A}_\alpha$; we show that $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains $\mathbf{x}$. Since $x_\alpha \in \bar{A}_\alpha$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$. Then $\mathbf{y} = (y_\alpha)$ belongs to both U and $\prod A_\alpha$. Since U is arbitrary, it follows that $\mathbf{x} \in \prod A_\alpha$

Conversely, suppose $\mathbf{x} = (x_\alpha)$ lies in the closure of $\prod A_\alpha$, in either topology. We show that for any given index β, we have $x_\beta \in \bar{A}_\beta$. Let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, it contains a point $\mathbf{y} = (y_\alpha)$ of $\prod A_\alpha$. Then y_β belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \in \bar{A}_\beta$ □

Theorem 1.44. *Let $f : A \to \prod_{\alpha\in J} X_\alpha$ be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha\in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous iff each function f_α is continuous

Proof. $\Rightarrow$ composition of continuous functions is continuous

$\Leftarrow$ Suppose that each coordinate function f_α is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A. A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$ where β is some index and U_β is open in X_β. now

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta)$$

because $f_\beta = \pi_\beta \circ f$. Since f_β is continuous, this set is open in A □

Example 1.9. Consider $\mathbb{R}^\omega$ and define $f : \mathbb{R} \to \mathbb{R}^\omega$

$$f(t) = (t, t, \dots)$$

f is continuous if $\mathbb{R}^\omega$ is given the box topology. Consider the basis element

$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$$

We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. $f^{-1}(B) = \{0\}$

Exercise 1.8.1. let $\mathbf{x}_1, \mathbf{x}_2, \dots$ be a sequence of the points of the products space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ iff the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \dots$ converges to $\pi_\alpha(\mathbf{x})$ for each α

Proof. Given a neighborhood $U = \prod U_\alpha$ of $\mathbf{x}$, for each α, we have N_α s.t. $\pi_\alpha(x_n) \in U_\alpha$ for all $n \geq N_\alpha$. If $U_\alpha = X_\alpha$ we take $N_\alpha = 1$. Hence in product topology we have only finitely many $N_\alpha > 1$ and we can take max. This fails in box topology as it might not have max □

Exercise 1.8.2. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero", that is, all sequences $(x_1, x_2, \dots)$ s.t. $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? justify your answer

Proof. If $\mathbb{R}^\infty$ is given the product topology, given a point $\mathbf{x} \in \mathbb{R}^\omega$ and a neighborhood $U = \bigcup_i U_i$ where U_i is a proper open subset of $\mathbb{R}$ for finitely many $i \in \omega$. Choose $y_i \in U_i$ and $y_j = 0$ if $U_j = \mathbb{R}$. Then $\mathbf{y} \in \mathbb{R}^\infty \cap U$. Hence $x \in \overline{\mathbb{R}^\infty}$

For box topology, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. □

Exercise 1.8.3. Given sequences $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \dots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots)$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology

Proof. both box and product □

1.9 The Metric Topology

Definition 1.45. A **metric** on a set X is a function

$$d : X \times X \to R$$

having the following properties

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Given a metric d on X, the number $d(x, y)$ is often called the **distance** between x and y in the metric d. Given $\epsilon > 0$ consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

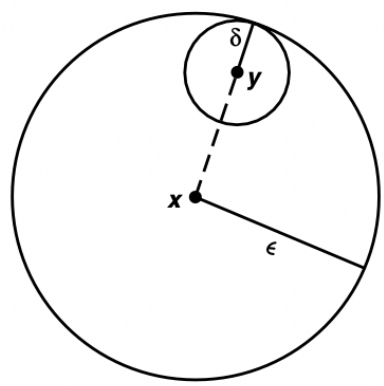
of all points y whose distance from x is less than ϵ. It is called the **ϵ-ball centered at** x

Definition 1.46. If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for a topology on X, called the **metric topology** induced by d

Check the second condition.

If $y \in B(x, \epsilon)$ then there is a basis element $B(y, \delta)$ *centered* at y that is contained in $B(x, \epsilon)$. Define δ to be $\epsilon - d(x, y)$. Then $B(y, \delta) \subset B(x, \epsilon)$, for if $z \in B(y, \delta)$ then $d(y, z) < \epsilon - d(x, y)$, from which we conclude that

$$d(x, z) \leq d(x, y) + d(y, z) < \epsilon$$



Let B_1 and B_2 be two basis element and let $y \in B_1 \cap B_2$.We have just shown that we can choose positive numbers δ_1 and δ_2 so that $B(y, \delta_1) \subset B_1$ and $B(y, \delta_2) \subset B_2$. Let $\delta = \min\{\delta_1, \delta_2\}$ we conclude $B(y, \delta) \subset B_1 \cap B_2$. Hence

> *A set U is open in the metric topology induced by d iff for each $y \in U$ there is a $\delta > 0$ s.t. $B_d(y, \delta) \subset U$*

Definition 1.47. If X is a topological space, X is said to be **metrizable** if there exists a metric d on the set X that induces the topology of X. A **metric space** is a metrizable space together with a specific metric d that gives the topology of X

Definition 1.48. Let X be a metric space with metric d. A subset A of X is said to be **bounded** if there is some number M s.t.

$$d(a_1, a_2) \leq M$$

for every pair a_1, a_2 of points of A. If A is bounded and nonempty, the **diameter** of A is defined to be the number

$$\operatorname{diam} A = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}$$

Theorem 1.49. *Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation*

$$\bar{d}(x, y) = \min\{d(x, y), 1\}$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the **standard bounded metric** corresponding to d.

Proof. Check

$$\bar{d}(x,z) \le \bar{d}(x,y) + \bar{d}(y,z)$$

If both $d(x,y)$ and $d(y,z)$ are <1. Then

$$d(x,z) \le d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$$

Note that in any metric space, the collection of ϵ-balls with $\epsilon < 1$ forms a basis for the metric topology $\square$

Definition 1.50. Given $\mathbf{x} = (x_1, \dots, x_n)$ in $\mathbb{R}^n$, we define the **norm** of $\mathbf{x}$ by

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and we define the **euclidean metric** d on $\mathbb{R}^n$ by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

We define the **square metric** ρ by

$$\rho(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, \dots, \left|x_y, y_n\right|\}$$

Check the third condition for ρ. for each $i \in \mathbb{N}_+$

$$|x_i - z_i| \le |x_i - y_i| + |y_i - z_i|$$

then

$$|x_i - z_i| \le \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z})$$

On the real line $\mathbb{R}$, these two metrics coincide with the standard metric for $\mathbb{R}$

Lemma 1.51. *Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for each $x \in X$ and each $\epsilon > 0$ there exists a $\delta > 0$ s.t.*

$$B_{d'}(x, \delta) \subset B_d(x, \epsilon)$$

Theorem 1.52. *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$*

Proof. Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be two points of $\mathbb{R}^n$. We have

$$\rho(\mathbf{x}, \mathbf{y}) \le d(\mathbf{x}, \mathbf{y}) \le \sqrt{n}\rho(\mathbf{x}, \mathbf{y})$$

The first inequality shows that

$$B_d(\mathbf{x}, \epsilon) \subset B_\rho(\mathbf{x}, \epsilon)$$

for all $\mathbf{x}$ and ϵ. Similarly

$$B_\rho(\mathbf{x}, \epsilon/\sqrt{n}) \subset B_d(\mathbf{x}, \epsilon)$$

It follows from the preceding lemma that the two metric topologies are the same

Next we show that the product topology is the same as that given by the metric ρ. First let

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

be a basis element for the product topology, and let $\mathbf{x} = (x_1, \dots, x_n) \in B$. For each i there is an ϵ_i s.t.

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$$

choose $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then $B_\rho(\mathbf{x}, \epsilon) \subset B$. □

Now we consider the infinite cartesian product $\mathbb{R}^\omega$. It is natural to try to generalize the metrics d and ρ to this space. For instance, one can attempt to define a metric d on $\mathbb{R}^\omega$ by the equation

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

But this equation does not always make sense, for the series in question need not converge. (This equation does define a metric on a certain important subset of $\mathbb{R}^\omega$, however; see the exercises.)

Similarly, one can attempt to generalize the square metric ρ to $\mathbb{R}^\omega$ by defining

$$\rho(x, y) = \sup\{|x_n - y_n|\}$$

Again, this formula does not always make sense. If however we replace the usual metric $d(x, y) = |x - y|$ on $\mathbb{R}$ by its bounded counterpart $\bar{d}(x, y) = \min\{|x - y|, 1\}$, then this definition does make sense; it gives a metric on $\mathbb{R}^\omega$ called the *uniform metric*

Definition 1.53. Given an index set J, and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let's define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. It is easy to check that $\bar{\rho}$ is indeed a metric; it is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**

Theorem 1.54. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different is J is infinite*

Proof. Suppose that we are given a point $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and a product topology basis element $\prod U_\alpha$. Let $\alpha_1, \dots, \alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$. Then for each i, choose $\epsilon_i > 0$ so that $B_{\bar{d}}(x_{\alpha_i}, \epsilon_i) \subset U_{\alpha_i}$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$, then $B_{\bar{d}}(\mathbf{x}, \epsilon) \subset \prod U_\alpha$. □

Theorem 1.55. *Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\omega}$, define*

$$D(\boldsymbol{x}, \boldsymbol{y}) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^{\omega}$

Proof. First let U be open in the metric topology and let $\mathbf{x} \in U$; Choose an ϵ-ball $B_D(\mathbf{x}, \epsilon) \subset U$. Then choose N large enough that $1/N < \epsilon$. Let V be the basis element for the product topology

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

We assert that $V \subset B_D(\mathbf{x}, \epsilon)$. Given any $\mathbf{y} \in \mathbb{R}^{\omega}$

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N} \qquad \text{for } i \geq N$$

therefore

$$D(\mathbf{x}, \mathbf{y}) \leq \max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \ldots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}$$

If $\mathbf{y} \in V$ then $D(\mathbf{x}, \mathbf{y}) < \epsilon$, so that $V \subset B_D(\mathbf{x}, \epsilon)$

Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where U_i is open in $\mathbb{R}$ in $\mathbb{R}$ for $i = \alpha_1, \ldots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices. Given $\mathbf{x} \in U$, consider an interval $(x_i - \epsilon_i, x_i + \epsilon_i) \subset U_i$ for $i = \alpha_1, \ldots, \alpha_n$; choose each $\epsilon_i \leq 1$, then define

$$\epsilon = \min\{\epsilon/i \mid i = \alpha_1, \ldots, \alpha_n\}$$

we assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U$$

let $\mathbf{y}$ be a point of $B_D(\mathbf{x}, \epsilon)$. then for all i

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon$$

Now if $i = \alpha_1, \ldots, \alpha_n$ then $\epsilon \leq \epsilon_i/i$ so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$. It follows that $|x_i - y_i| < \epsilon_i$. Therefore $\mathbf{y} \in \prod U_i$ ∎

Exercise 1.9.1. Let X be a metric space with metric d

1. $d : X \times X \to \mathbb{R}$ is continuous

2. Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X

Proof. 1. Prove that for any U open in $\mathbb{R}$ and $(x,y) \in d^{-1}(U)$ there is a basis element B of $X \times X$ s.t. $(x,y) \in B \subset d^{-1}(U)$. Suppose $d(x,y) = a$. There is a ϵ s.t. $(a-\epsilon, a+\epsilon) \subset U$. We take $B = B_d(x, \epsilon/2) \times B_d(y, \epsilon/2)$. for any $(x,y) \in B$, $d(x,y) \in (a-\epsilon, a+\epsilon)$

2. for every fixed $x \in X'$, $d_x(y) : X' \to \mathbb{R}, y \mapsto d(x,y)$ is continuous. Therefore every $B_d(x,r) = d_x^{-1}((-\infty, r))$ must be open in X'

□

Exercise 1.9.2. Consider the product, uniform and box topologies on $\mathbb{R}^\omega$

1. in which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous

$$f(t) = (t, 2t, 3t, \dots)$$
$$g(t) = (t, t, t, \dots)$$
$$h(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \dots)$$

2. in which topologies do the following sequences converge

$$\begin{array}{ll}
\mathbf{w}_1 = (1,1,1,1,\dots) & \mathbf{x}_1 = (1,1,1,1,\dots) \\
\mathbf{w}_2 = (0,2,2,2,\dots) & \mathbf{x}_2 = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots) \\
\mathbf{w}_3 = (0,0,3,3,\dots) & \mathbf{x}_3 = (0,0,\frac{1}{3},\frac{1}{3}) \\
\dots & \dots \\
\mathbf{y}_1 = (1,0,0,0,\dots) & \mathbf{z}_1 = (1,1,0,0,\dots) \\
\mathbf{y}_2 = (\frac{1}{2},\frac{1}{2},0,0,\dots) & \mathbf{z}_2 = (\frac{1}{2},\frac{1}{2},0,0,\dots) \\
\mathbf{y}_3 = (\frac{1}{3},\frac{1}{3},\frac{1}{3},0,\dots) & \mathbf{z}_3 = (\frac{1}{3},\frac{1}{3},0,0,\dots)
\end{array}$$

Proof. 1. For box topology, consider open set

$$B = (-1,1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$$

$f^{-1}(B) = g^{-1}(B) = h^{-1}(B) = \{0\}$ which is not open.

For uniform topology. First, $f^{-1}(B_{\bar{\rho}}(\mathbf{0}, 1)) \subset f^{-1}(\prod_{n \in \mathbb{Z}_+}(-1,1)) = \{0\}$. At the same time, for $k(t) = (a_1 t, a_2 t, \dots)$ equals g or h and $k(t) \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$, then for every $n \in \mathbb{Z}_+$, $|x_n - a_n t| \leq \sup_{n \in \mathbb{Z}_+} |x_n - a_n t| = \delta < \epsilon$. And for $|z| < \frac{\epsilon - \delta}{2}$

$$|x_n - a_n(t+z)| \leq |x_n - a_n t| + a_n|z| < \delta + \frac{\epsilon-\delta}{2} = \frac{\epsilon+\delta}{2} < \epsilon$$

Hence $k((t - \frac{\epsilon-\delta}{2}, t + \frac{\epsilon-\delta}{2})) \subset B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ and $k^{-1}(B_{\bar{\rho}}(\mathbf{x}, \epsilon))$ is open.

Product topology. all three

2. If a sequence converges to a point, and we change the topology to a coarser one, then the sequence still converges to the point. Therefore for each sequence we may specify the finest topology out of the three given topologies in which it converges to some point.

   For $\{\mathbf{w}_n\}$ it is the product topology, for $\{\mathbf{x}_n\}$ and $\{y_n\}$ it is the uniform topology and for $\{\mathbf{z}_n\}$ it is the product topology

   $\{\mathbf{x}_n\}$ converges to $\mathbf{0}$ in the uniform topology, as for $n > \frac{1}{\epsilon}$, $\mathbf{x}_n \in B_{\bar{\rho}}(\mathbf{0}, \epsilon)$

□

Exercise 1.9.3. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology

Proof. Let $X \in \mathbb{R}^\omega$ be the set of all sequences of real numbers that converge to 0 in $\mathbb{R}$. Note that $\mathbb{R}^\infty \subset X$. If $\mathbf{y} \notin X$, then there is $\epsilon > 0$ s.t. for every $k \in \mathbb{Z}_+$ there is $n_k \geq k$ s.t. $\left|y_{n_k}\right| \geq \epsilon$. Hence if $\mathbf{z} \in B_{\bar{\rho}}(\mathbf{y}, \frac{\epsilon}{2})$, for every $k \in \mathbb{Z}_+$, $\left|z_{n_k}\right| > \left|y_{n_k}\right| - \frac{\epsilon}{2} \geq \frac{\epsilon}{2}$ and $B_{\bar{\rho}}(\mathbf{y}, \frac{\epsilon}{2})$ doesn't contain any points of X. Therefore X is closed and contains the closure of $\mathbb{R}^\infty$.

At the same time, for every $\mathbf{x} \in X$ and $\epsilon > 0$ there is $N \in \mathbb{Z}_+$ s.t. for $n \geq N$, $|x_n| < \frac{\epsilon}{2}$ and $\mathbf{y} = (x_1, \dots, x_N, 0, 0, \dots) \in B_{\bar{\rho}(\mathbf{x}, \epsilon)} \cap \mathbb{R}^\infty$. □

1.10 The Metric Topology (continued)

subspaces of metric spaces behave the way one would wish them to; if A is a subspace of the topological space X and d is a metric for X, then the restriction of d to $A \times A$ is a metric for the topology of A

The *Hausdorff axiom* is satisfied by every metric topology

Theorem 1.56. *Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then continuity of f is equivalent to the requirement that given $x \in X$ and given $\epsilon > 0$ there exists $\delta > 0$ s.t.*

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

Proof. Suppose f is continuous. Given x and ϵ, consider the set

$$f^{-1}(B(f(x), \epsilon))$$

which is open in X and contains the point x. It contains some δ-ball $B(x, \delta)$

Conversely, suppose that the ϵ-δ condition is satisfied. Let V be open in Y; we show that $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(V)$. Since $f(x) \in V$ there is an ϵ-ball $B(f(x), \epsilon) \subset V$. By the ϵ-δ condition there is a δ-ball $B(x, \delta)$ s.t. $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Then $x \in B(x, \delta) \subset f^{-1}(V)$ so that $f^{-1}(V)$ is open . □

Lemma 1.57 (The sequence lemma)**.** *Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converge holds if X metrizable.*

Proof. Suppose $x_n \to x$ where $x_n \in A$. Then every neighborhood U of x contains a point of A.

Suppose that X is metrizable and $x \in \bar{A}$. Let d be a metric for the topology of X. For each positive integer n, take the neighborhood $B_d(x, 1/n)$ and choose x_n to be a point of its intersection with A. $\{x_n\}$ converges to x. □

Theorem 1.58. *Let $f : X \to Y$. If the function f is continuous then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is metrizable*

Proof. Assume that f is continuous. Given $x_n \to x$ we wish to show that $f(x_n) \to f(x)$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is a neighborhood of x and so there ....

Conversely, let Abe a subset of X; we show that $f(\bar{A}) = \overline{f(A)}$. If $x \in \bar{A}$ then there is a sequence x_n of points of A converging to x. Hence $f(x_n)$ converges to $f(x)$. Thus $f(x) \in \overline{f(A)}$. □

Lemma 1.59. *The addition, subtraction and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.*

Theorem 1.60. *If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous functions, then $f + g$,$f - g$ and $f \cdot g$ is continuous. If $g(x) \neq 0$ for all x, then f/g is continuous*

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by

$$h(x) = f(x) \times g(x)$$

is continuous, by Theorem 1.34. The function $f + g$ equals the composite of h and the addition operation, therefore $f + g$ is continuous. Similar arguments for others □

Definition 1.61. Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) **converges uniformly** to the function $f : X \to Y$ if given $\epsilon > 0$ there exists an integer N s.t.

$$d(f_n(x), f(x)) < \epsilon$$

for all $n > N$ and all x in X

Theorem 1.62 (Uniform limit theorem)**.** *Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous*

Proof. Let V be open in Y; let x_0 be a point of $f^{-1}(V)$. We wish to find a neighborhood U of x_0 s.t. $f(U) \subset V$.

Let $y_0 = f(x_0)$. First choose ϵ so that the $B(y_0, \epsilon) \subset V$. Then use uniform convergence, choose N so that for all $n \geq N$ and all $x \in X$

$$d(f_n(x), f(x)) < \epsilon/3$$

Finally using continuity of f_N, choose a neighborhood U of x_0 s.t. $f_N(U) \subset B(f_N(x_0), \epsilon/3)$

We claim that $f(U) \subset B(y_0, \epsilon) \subset V$. Note that if $x \in U$ then

$$\begin{aligned}
&d(f(x), f_N(x)) < \epsilon/3 \\
&d(f_N(x), f_N(x_0)) < \epsilon/3 \quad \text{by choice of } U \\
&d(f_N(x_0), f(x_0)) < \epsilon/3
\end{aligned}$$

Adding and using the triangle inequality, we see that $d(f(x), f(x_0)) < \epsilon$ □

Remark. Uniform convergence is related to the definition of the uniform metric. Consider the space $\mathbb{R}^X$ of all functions $f : X \to \mathbb{R}$ in the uniform metric $\bar{\rho}$. A sequence of functions $f_n : X \to \mathbb{R}$ converges uniformly to f iff the sequence (f_n) converges to f when they are considered as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.

Example 1.10. $\mathbb{R}^\omega$ in the box topology is not metrizable

We shall show that the sequence lemma does not hold for $\mathbb{R}^\omega$. Let A be the subset of $\mathbb{R}^\omega$ consisting of those points all of whose coordinates are positive

$$A = \{(x_1, x_2, \dots) \mid x_i > 0 \text{ for all } i \in \mathbb{Z}_i\}$$

In the box topology, $\mathbf{0} \in \bar{A}$

But we assert that there is no sequence of points of A converging to $\mathbf{0}$. For let $(\mathbf{a}_n)$ be a sequence of points of A, where

$$\mathbf{a}_n = (x_{1n}, x_{2n}, \dots)$$

Every coordinate x_{in} is positive, so we can construct a basis element B' for the box topology on $\mathbb{R}$ by setting

$$B' = -(-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \dots$$

Then $\mathbf{0} \in B'$ but it contains no member of the sequence $(\mathbf{a}_n)$;

Example 1.11. An uncountable product of $\mathbb{R}$ with itself is not metrizable

Let J be an uncountable index set; we show that $\mathbb{R}^J$ does not satisfy the sequence lemma (in the product topology)

Let A be the subset of $\mathbb{R}^J$ consisting of all points (x_α) s.t. $x_\alpha = 1$ for all but finitely many values of α.

We assert that $\mathbf{0}$ belongs to the closure of A. Let $\prod U_\alpha$ be a basis element containing $\mathbf{0}$. Then $U_\alpha \neq \mathbb{R}$ for only finitely many values of α, say for $\alpha = \alpha_1, \dots, \alpha_n$. Let (x_α) be the point of A defined by letting $x_\alpha = 0$ for $\alpha = \alpha_1, \dots, \alpha_n$ and $x_\alpha = 1$ for all other values of α; then $(x_\alpha) \in A \cap \prod U_\alpha$ as desired

But there is no sequence of points of A converging to $\mathbf{0}$. For let $\mathbf{a}_n$ be a sequence of points of A. Given n, let J_n denote the subset of J consisting of those indices α for which the αth coordinate of $\mathbf{a}_n$ is difference from 1. The union of all the sets J_n is a countable union of finite sets and therefore countable. Because J itself is uncountable, there is an index in J, say β, that does not lie in any of the sets J_n. This means that for **each** of the points $\mathbf{a}_n$, its βth coordinate equals 1

Now let U_β be the open interval $(-1, 1)$ in $\mathbb{R}$ and let U be the open set $\pi_\beta^{-1}(U_\beta)$ in $\mathbb{R}^J$. The set U is a neighborhood of $\mathbf{0}$ that contains none of the points $\mathbf{a}_n$; therefore the sequence $\mathbf{a}_n$ cannot converge to $\mathbf{0}$

2 Connectedness and Compactness

2.1 Connected Spaces

Definition 2.1. Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be connected if there does not exists a separation of X

Another way of formulating the definition of connectedness is the following

> *A space X is connected iff the only subsets of X that are both open and closed in X are the empty set and X itself.*

For if A is a nonempty proper subset of X that is both open and closed in X, then A and $X - A$ is a separation of X.

Lemma 2.2. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y*

Proof. Suppose that A and B form a separation of Y. Then A is both open closed in Y. The closure of A in Y is the set $\bar{A} \cap Y = A$. Or to say the same thing, $\bar{A} \cap B = \emptyset$

Conversely, suppose that $A$and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$ therefore we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$ $\square$

Example 2.1. Let Y denote the subspace $[-1,0)\cup(0,1]$ of the real line $\mathbb{R}$. Each of the sets $[-1,0)$ and $(0,1]$ is nonempty and open in Y; therefore they form a separation of Y.

Example 2.2. The rationals $\mathbb{Q}$ are not connected. Indeed, the only connected subspaces of $\mathbb{Q}$ are the one-point sets. If Y is a subspace of $\mathbb{Q}$ containing two points p and q, one can choose an rational number a lying between p and q and write Y as the union of the open sets

$$Y\cap(\infty,a) \quad \text{and} \quad Y\cap(a,+\infty)$$

Lemma 2.3. *If the sets C and D form a separation of X and if Y is a connected subspace of X, then Y lies entirely within either C or D*

Proof. The sets $C\cap Y$ and $D\cap Y$ are open in Y. If they are nonempty, then they form a separation. □

Theorem 2.4. *The union of a collection of connected subspaces of X that have a point in common is connected*

Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces of a space X; let $p\in\bigcap A_\alpha$. We prove that $Y=\bigcup A_\alpha$ is connected. Suppose $Y=C\cup D$ is a separation of Y. The point p is either in C or D. Suppose $p\in C$. Since A_α is connected, it must lie entirely in either C or D, it must lie entirely in either C or D, and it lie in C since $p\in C$. Hence $A_\alpha\subset C$ for all α, so that $\bigcup A_\alpha=C$ □

Theorem 2.5. *Let A be a connected subspace of X. If $A\subset B\subset\bar{A}$, then $\bar{B}$ is also connected*

Proof. Suppose $B=C\cup D$, then $A\subset C$ or $A\subset D$ by Lemma 2.3. Suppose $A\subset C$, then $\bar{A}\subset\bar{C}$; since $\bar{C}$ and D are disjoint, B cannot intersect D. A contradiction □

Theorem 2.6. *The image of a connected space under a continuous map is connected*

Proof. Let $f:X\to Y$ be a continuous map and X connected. We wish to prove the image space $Z=f(X)$ is connected. Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a continuous surjective map

$$g:X\to Z$$

Suppose that $Z=A\cup B$ is a separation of Z. Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets whose union is X, which form a separation □

Theorem 2.7. *A finite cartesian product of connected spaces is connected*

Proof. Given two connected spaces X and Y. Given $a\times b\in X\times Y$, $X\times b$, being homeomorphic with X, is connected and so is $a\times Y$. As a result

$$T_x=(X\times b)\cup(x\times Y)$$

is connected by Theorem 2.4. So is $\bigcup_{x\in X}T_x$ with (a,b) in common □

It is natural to ask whether this theorem extends to arbitrary products of connected spaces. The answer depends on which topology is used for the product, as the following examples show.

Example 2.3. Consider the cartesian product $\mathbb{R}^\omega$ in the box topology. We can write $\mathbb{R}^\omega$ as the union of the set A consisting of all bounded sequences of real numbers and the set B of all unbounded sequences.

For if $\mathbf{a} \in \mathbb{R}^\omega$

$$\mathbf{a} \in U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \dots$$

Example 2.4. Now consider $\mathbb{R}^\omega$ in the product topology. Assuming that $\mathbb{R}$ is connected, we show that $\mathbb{R}^\omega$ is connect. Let $\tilde{\mathbb{R}}^n$ denote the subspace of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x} = (x_1, x_2, \dots)$ s.t. $x_i = 0$ for $i > n$. The space $\tilde{\mathbb{R}}^n$ is clearly homeomorphic to $\mathbb{R}^n$, so that it is connected, by the preceding theorem. It follows that the space $\mathbb{R}^\infty$ is the union of the spaces $\tilde{\mathbb{R}}^n$ is connected, for these spaces have the point $\mathbf{0} = (0, 0, \dots)$ in common. We show that the closure of $\mathbb{R}^\infty$ equals all of $\mathbb{R}^\omega$, from which it follows that $\mathbb{R}^\omega$ is connected.

Exercise 2.1.1. Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ is connected

Proof. Suppose $Y \cup A$ is separate and $Y \cup A = C \cup D$. Since Y is connected, we suppose $Y \subset C$, so that $D \subset A$. We have $X = C \cup D \cup B$. No limit point of C can be in D, and no limit point of B can be in $D \subset A$, so that $B \cup C$ is closed, and D is open in X. But no limit point of $D \subset A$ can lie in C or B, so that D is closed in X. Therefore D is open and closed in X. Contradiction □

2.2 Connected Subspaces of the Real Line

Definition 2.8. A simply ordered set L having more than one element is called a **linear continuum** if the following hold:

1. L has the least upper bound property
2. if $x < y$ there exists z s.t. $x < z < y$

Theorem 2.9. *If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L*

Proof. We prove that if Y is a convex subspace of L, then Y is connected.

Suppose that Y is the union of the disjoint nonempty sets A and B, each of which is open in Y. Choose $a \in A$ and $b \in B$; suppose for convenience that $a < b$. The interval $[a, b]$ is the union of the disjoint sets

$$A_0 = A \cap [a, b] \quad \text{and} \quad B_0 = B \cap [a, b]$$

each of which is open in $[a,b]$ in the subspace topology, which is the same as the order topology. Thus A_0 and B_0 constitute a separation of $[a,b]$.

Let $c = \sup A_0$. We show that c belongs neither to A_0 nor to B_0, which contradicts the fact that $[a,b]$ is the union of A_0 and B_0.

Case 1. Suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$. In either case, it follows from the fact that B_0 is open in $[a,b]$ that there is some interval of the form $(d,c]$ contained in B_0. If $c = b$, then d is a smaller upper bound on A_0, a contradiction. If $c < b$, $(c,b] \cap A_0 = \emptyset$. then

$$(d,b] = (d,c] \cup (c,b]$$

doesn't intersect A_0. Again d is a smaller upper bound on A_0.

Case 2. Suppose that $c \in A_0$. So either $a = c$ or $a < c < b$. Because A_0 is open in $[a,b]$ there must be some interval of the form $[c,e]$ contained in A_0. We can choose $c < z < e$, contrary to the fact that c is an upper bound. □

Corollary 2.10. *The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$*

Theorem 2.11 (Intermediate value theorem)**.** *Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If $a, b \in X$ and $r \in Y$ lying between $f(a)$ and $f(b)$, then there exists a point c s.t. $f(c) = r$*

Proof. The sets

$$A = f(X) \cap (-\infty, r) \quad \text{and} \quad B = f(X) \cup (r, +\infty)$$

are disjoint and nonempty. If there were no point $c \in X$ s.t. $f(c) = r$, then $f(X)$ would be the union of A and B. Then A and B would constitute a separation of $f(X)$, contradicting the fact that the image of a connected space under a continuous map is connected □

Definition 2.12. Given points $x, y \in X$, a **path** in X from x to y is a continuous map $f : [a,b] \to X$ s.t. $f(a) = x$ and $f(b) = y$. A space X is said to be **path connected** if every pair of points of X can be joined by a path X.

A path-connected space X is connected since the image of a connected space under a continuous map is connected.

Example 2.5. Define the **unit ball** B^n in $\mathbb{R}^n$ by

$$B^n = \{\mathbf{x} \mid \|\mathbf{x}\| \leq 1\}$$

The unit ball is path connected; given $\mathbf{x}, \mathbf{x} \in B^n$, the straight-line path $f : [0,1] \to \mathbb{R}^n$ defined by

$$f(t) = (1-t)\,\mathbf{x} + t\,\mathbf{y}$$

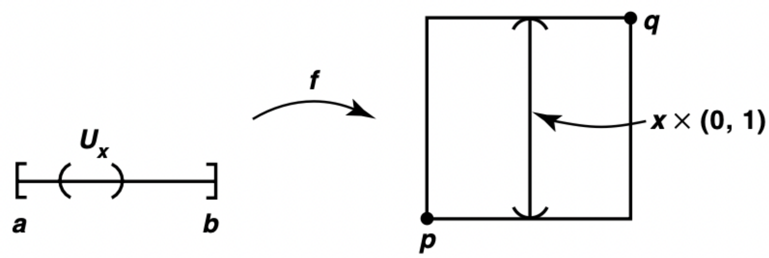
lies in B^n.

Example 2.6. The ordered square I_o^2 is connected but not path connected

Being a linear contiuum, the ordered square is connected. Let $p = 0 \times 0$ and $q = 1 \times 1$. We suppose there is a path $f : [a, b] \to I_o^2$ joining p and q and derive a contradiction. The image set $f([a, b])$must contain every point $x \times y$ of I_o^2 by the intermediate value theorem. Therefore for each $x \in I$ the set

$$U_x = f^{-1}(x \times (0, 1))$$

is a nonempty subset of $[a, b]$. By continuity, it is open in $[a, b]$



Choose for each $x \in I$ a rational number $q_x \in U_x$. Since the sets U_x are disjoint, the map $x \to q_x$ is an injective mapping of I into $\mathbb{Q}$. This contradicts the fact that the interval I is uncountable

2.3 Compact Spaces

Definition 2.13. A collection $\mathcal{A}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if $\bigcup \mathcal{A} = X$. It is called an **open covering** of X if its elements are open subsets of X

Definition 2.14. A space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to **cover** Y if the union of its elements *contains* Y

Lemma 2.15. *Let Y be a subspace of X. Then Y is compact iff every covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof. If Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X, then the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite subcollection

$$\{A_{\alpha_1} \cap Y, \dots, A_{\alpha_n} \cap Y\}$$

covers Y

Conversely. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α, $A'_\alpha = A_\alpha \cap Y$ for some A_α open in X. The collection $\{A_\alpha\}$ is a covering of Y by sets open in X. □

Theorem 2.16. *Every closed subspace of a compact space is compact*

Proof. Let Y be a closed subspace of the compact space X. Given a covering $\mathcal{A}$ of Y by sets open in X, let

$$\mathcal{B} = \mathcal{A} \cup \{X - Y\}$$

Some finite subcollection of $\mathcal{B}$ covers X. □

Theorem 2.17. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the Hausdorff space X. We shall prove $X - Y$ is open

Let x_0 be a point of $X - Y$. We show there is a neighborhood of x_0 that is disjoint from Y. For each point y of Y, choose disjoint neighborhoods U_y and V_y of the points x_0 and y. The collection $\{V_y \mid y \in Y\}$ is a covering of Y by sets open in Y; therefore, $V_{y_1}, \dots, V_{y_n}$ covers Y. The open set

$$V = Y_{y_1} \cup \cdots \cup V_{y_n}$$

contains Y, and its disjoint from the open set

$$U = U_{y_1} \cap \cdots \cap U_{y_n}$$

□

Lemma 2.18. *if Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exist disjoint open sets U and V of X containing x_0 and Y respectively*

Example 2.7. Once we prove that the interval $[a, b]$ in $\mathbb{R}$ is compact, it follows from Theorem 2.16 that any closed subspace of $[a, b]$ is compact. On the other hand, it follows from Theorem 2.17 that the intervals $(a, b]$ and (a, b) in $\mathbb{R}$ cannot be compact since they are not closed in Hausdorff space $\mathbb{R}$.

Theorem 2.19. *The image of a compact space under a continuous map is compact*

Theorem 2.20. *Let $f : X \to Y$ be a bijectiive continuous function. If X is compact and Y is Hausdorff, then f is homeomorphism*

Proof. We shall prove that images of closed sets of X under f are closed in Y. If A is closed in X, then A is compact, by Theorem 2.16. Therefore $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed in Y by Theorem 2.17 □

Theorem 2.21. *The product of finitely many compact spaces is compact*

Proof. We shall prove that the product of two compact spaces is compact;

Suppose we are given spaces X and Y, with Y compact. Suppose that x_0 is a point of X, and N is an open set of $X \times Y$ containing $x_0 \times Y$. We prove the following

There is a neighborhood W of x_0 in X s.t. N contains the entire set $W \times Y$

The set $W \times Y$ is often called a **tube** about $x_0 \times Y$

First let's cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$), lying in N. The space $x_0 \times Y$ is compact, being homeomorphic to Y. Therefore, we can cover $x_0 \times Y$ by finitely many such basis elements

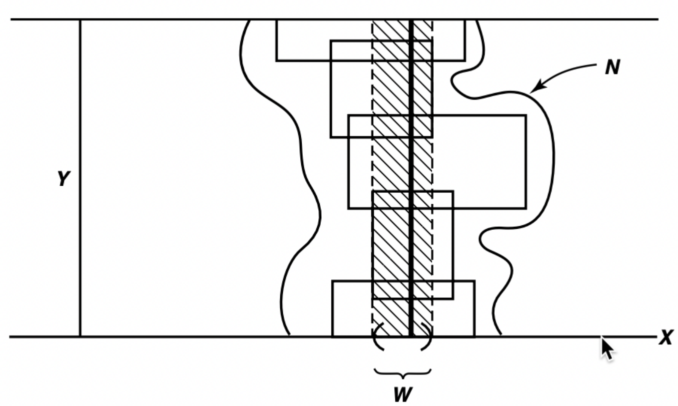
$$U_1 \times V_1, \ldots, U_n \times V_n$$

(We assume that each of the basis elements $U_i \times V_i$) actually intersects $x_0 \times Y$. Define

$$W = U_1 \cap \cdots \cap U_n$$

W is open and it contains x_0 because each set $U_i \times V_i$ intersects $x_0 \times Y$.

We assert that the sets $U_i \times V_i$, which were chosen to cover the slice $x_0 \times Y$, actually cover the tube $W \times Y$. Let $x \times y \in W \times Y$. Consider the point $x_0 \times y$ of the slice $x_0 \times Y$ having the same y-coordinate as this point. Now $x_0 \times y \in U_i \times V_i$ for some i, so that $y \in V_i$. But $x \in U_j$ for every j. Therefore we have a$x \times y \in U_i \times V_i$ as desired



Now we prove the theorem. Let X and Y be compact spaces. Let $\mathcal{A}$ be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements $A_1, \ldots, A_m \in \mathcal{A}$. $N = A_1 \cup \cdots \cup A_m$ is an open set containing $x_0 \times Y$.

The open set N contains a tube $W \times Y$ about $x_0 \times Y$ where W is open in X. Then $W \times Y$ is covered by finitely many elements $A_1, \ldots, A_m \in \mathcal{A}$.

Thus for each $x \in X$, we can choose a neighborhood W_x of x s.t. the tube $W_x \times Y$ can be covered by finitely many elements of $\mathcal{A}$. The collection of all the neighborhoods W_x is an open covering of X; therefore by compactness of X, there exists a finite subcollection

$$\{W_1, \dots, W_k\}$$

covering X. The union of the tubes

$$W_1 \times Y, \dots, W_k \times Y$$

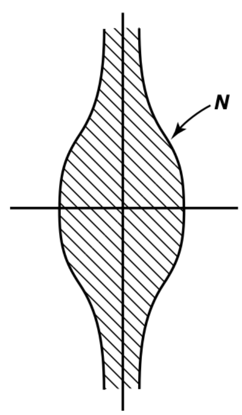
is all of $X \times Y$; since each may be covered by finitely many elements of $\mathcal{A}$. $\square$

Lemma 2.22 (The tube lemma)**.** *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$ where W is a neighborhood of x_0 in X.*

Example 2.8. The tube lemma is not true if Y is not compact. For example, let Y be the y-axis in $\mathbb{R}^2$, and

$$N = \left\{ x \times y \mid |x| < \frac{1}{y^2 + 1} \right\}$$

Then N is an open set containing the set $0 \times \mathbb{R}$, but it contains no tube about $0 \times \mathbb{R}$



Definition 2.23. A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** if for every finite subcollection

$$\{C_1, \dots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty

Theorem 2.24. *Let X be a topological space. Then X is compact iff for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty*

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} = \{X - A \mid A \in \mathcal{A}\}$$

Then the following holds

1. $\mathcal{A}$ is a collection of open sets iff $\mathcal{C}$ is a collection of closed sets
2. The collection $\mathcal{A}$ covers X iff the intersection $\bigcap_{C\in\mathcal{C}} C$ is empty
3. The finite subcollection $\{A_1, \dots, A_n\}$ of $\mathcal{A}$ covers X iff $\bigcap C_i$ is empty

Take the contrapositive of the theorem: given any collection $\mathcal{A}$ of open sets, if no finite subcollection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X, which is equivalent to: given any collection $\mathcal{C}$ of closed sets, if every finite intersection of elements of $\mathcal{C}$ is nonempty, then the intersection of all the elements of $\mathcal{C}$ is nonempty □

A special case of this theorem occurs when we have a **nested sequence** $C_1 \supset \cdots \supset C_n \supset \dots$ of closed sets in a compact space X. If each of the sets C_n is nonempty, then the collection $\mathcal{C} = \{C_n\}_{n\in\mathbb{Z}_+}$ automatically has the finite intersection property. Then the intersection

$$\bigcup_{n\in\mathbb{Z}_+} C_n$$

is nonempty

Exercise 2.3.1. 1. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the set X; suppose that $\mathcal{T}' \supset \mathcal{T}$. what does compactness of X under one of these topologies imply about compactness under the other

2. show that if X is compact Hausdorff under both $\mathcal{T}$ and $\mathcal{T}'$, then either $\mathcal{T} = \mathcal{T}'$ or they are not comparable

Proof. 1. if $(X, \mathcal{T}')$ is compact then so is $(X, \mathcal{T})$

2. Suppose one is finer than the other. Then the identity mapping from the finer one to the coarser one is a continuous and bijectiive function that maps a compact space to a Hausdorff space. Therefore it is a homeomorphism and the topologies are the same

□

Exercise 2.3.2. 1. Show that in the finite complement topology on $\mathbb{R}$, every subspace is compact

2. If $\mathbb{R}$ has the topology consisting of all sets A s.t. $\mathbb{R} - A$ is either countable or all of $\mathbb{R}$, is $[0, 1]$ a compact subspace?

Proof. 1. Given any covering $\mathcal{A}$ of the subspace, $\mathbb{R} - \bigcup \mathcal{A}$ is finite and we can choose the set one by one

2. No. Let $A_n = [0,1] - \{1/n, 1/(n+1), \dots\}$

$\square$

Exercise 2.3.3. Show that a finite union of compact subspaces of X is compact

Exercise 2.3.4. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space where not every closed bounded subspace is compact. Find a metric space in which not every closed bounded subspace is compact.

Proof. If it were not bounded, then for any ball there would be a point outside it, and while the union of all these ball does cover the whole space, there is no finite subcovering for the subspace $\square$

Exercise 2.3.5. Show that if $f : X \to Y$ is continuous, where X is compact and Y is Hausdorff, then f is a closed map (that is, f carries closed sets to closed sets)

2.4 Compact Subspaces of the Real Line

Theorem 2.25. *Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact*

Proof. Step 1. Given $a < b$, let $\mathcal{A}$ be a covering of $[a,b]$ by sets open in $[a,b]$ in the subspace topology (which is the same as the order topology). First we prove the following

> *If $x \in [a,b]$ different from b, then there is a point $y > x$ of $[a,b]$ s.t. the interval $[x,y]$ can be covered by at most two elements of $\mathcal{A}$.*

If x has an immediate successor in X, let y be this immediate successor. Then $[x,y]$ has two points x, y. If x has no immediate successor in X, choose an element $A \in \mathcal{A}$ containing x. Because $x \neq b$ and A is open, A contains an interval of the form $[x,c)$ for some $c \in [a,b]$. Choose a point $y \in (x,c)$, then $[x,y]$ can be covered by A.

Step 2. Let C be the set of all points $y > a$ of $[a,b]$ s.t. the interval $[a,y]$ can be covered by finitely many elements of $\mathcal{A}$. Applying Step 1 to the case $x = a$, we see that there exists at least one such y, so C is not empty. Let c be the least upper bound of the set C; then $a < c \leq b$

Step 3. We show that $c \in C$. Choose an element $A \in \mathcal{A}$ containing c. Since A is open, it contains an interval of the form $(d,c]$ for some $d \in [a,b]$. It $c \notin C$, there must be a point $z \in C$ lying in the interval (d,c), because otherwise $d$would be a smaller upper bound on C than c. Since $z \in C$, the interval $[a,z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$. Now $[z,c]$ lies in the single elements A of $\mathcal{A}$, hence $[a,c] = [a,z] \cup [z,c]$ can be covered by $n+1$ elements of $\mathcal{A}$. Thus $c \in C$

Step 4. Finally we show that $c = b$. Suppose $c < b$, applying Step 1 to the case $x = c$ we conclude that there exists a point $y > c$ of $[a,b]$ s.t. the interval $[c,y]$ can be covered by finitely many elements of $\mathcal{A}$. By Step 3, $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$. This means that $y \in C$, a contradiction $\square$

Corollary 2.26. *Every closed interval in $\mathbb{R}$ is compact*

Theorem 2.27. *A subspace A of $\mathbb{R}^n$ is compact iff it is closed and its bounded in the euclidean metric d or the square metric ρ*

Proof. It will suffice to consider only the metric ρ; the inequalities

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$$

imply that $A$is bounded under d iff it is bounded under ρ

Suppose that A is compact. Then by Theorem 2.17 it is closed. Consider the collection of open sets

$$\{B_\rho(\mathbf{0}, m) \mid m \in \mathbb{Z}_+\}$$

whose union is all of $\mathbb{R}^+$. Some finite subcollection covers A. It follows that $A \subset B_\rho(\mathbf{0}, M)$ for some M. Therefore, for any two points $x, y \in A$, $\rho(x, y) \leq 2M$

Conversely, suppose that A is closed and bounded under ρ; suppose $\rho(x, y) \leq N$ for every $x, y \in A$. Choose a point $x_0 \in A$ and let $\rho(x_0, \mathbf{0}) = b$. The triangle inequality implies that $\rho(x, \mathbf{0}) \leq N + b$ for every $x \in A$. If $P = N + b$, then A is a subset of the cube $[-P, P]^n$, which is compact. Being closed, A is also compact. $\square$

Theorem 2.28 (Extreme value theorem)**.** *Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X s.t. $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.*

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We show that A has a largest element M and a smallest element x.

If A has no largest element, then the collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open covering of A. Since A is compact, some finite subcollection

$$\{(-\infty, a_1), \dots, (-\infty, a_n)\}$$

covers A. If $a_i = \max\{a_1, \dots, a_n\}$, then a_i belongs to none of these sets, contrary to the fact that they cover A $\square$

Definition 2.29. Let (X, d) be a metric space; let A be a nonempty subset of X . For each $x \in X$ we define the **distance from x to A** by the equation

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}$$

Fix A, then the function $d(x, A)$ is a continuous function of x: Given $x, y \in X$, one has the inequalities

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$$

for each $a \in A$. It follows that

$$d(x, A) - d(x, y) \leq \inf d(y, a) = d(y, A)$$

so that

$$d(x, A) - d(y, A) \leq d(x, y)$$

Continuity of the function $d(x, A)$ follows

Lemma 2.30 (The Lebesgue number lemma). *Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ s.t. for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it*

The number δ is called a **Lebesgue number** for the covering $\mathcal{A}$.

Proof. Let $\mathcal{A}$ be an open covering of X. If $X \in \mathcal{A}$, then any positive number is a Lebesgue number for $\mathcal{A}$. So assume X is not an element of $\mathcal{A}$

Choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ that covers X. For each i, set $C_i = X - A_i$, and define $f : X \to \mathbb{R}$ by letting $f(x)$ be the average of the numbers $d(x, C_i)$. That is,

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$$

We show that $f(x) > 0$ for all x. Given $x \in X$, choose i so that $x \in A_i$. Then choose ϵ so the ϵ-neighborhood of x lies in A_i. Then $d(x, C_i) \geq \epsilon$, so that $f(x) \geq \epsilon/n$.

Since f is continuous, it has a minimum value δ; we show that δ is our required Lebesgue number. Let B the be a subset of X of diameter less than δ. Choose a point $x_0 \in B$; then B lies in the δ-neighborhood of x_0. Now

$$\delta \leq f(x_0) \leq d(x_0, C_m)$$

where $d(x_0, C_m) = \max\{d(x_0, C_i)\}$. Then the δ-neighborhood of x_0 is contained in the element $A_m = X - C_m$ of the covering $\mathcal{A}$. $\square$

Definition 2.31. A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be **uniformly continuous** if given $\epsilon > 0$ there is a $\delta > 0$ s.t. for every pair of points x_0, x_1 of X,

$$d_X(x_0, x_1) < \delta \implies d_Y(f(x_0), f(x_1)) < \epsilon$$

Theorem 2.32 (Uniform continuity theorem). *Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous*

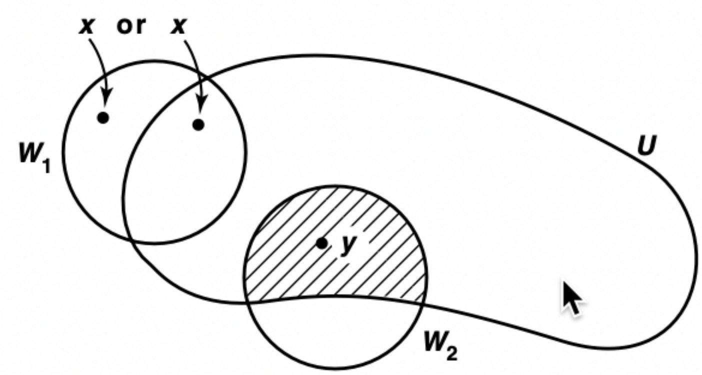
Proof. Given $\epsilon > 0$, take the open covering of Y by balls $B(y, \epsilon/2)$ of radius $\epsilon/2$. Let $\mathcal{A}$ be the open covering of X by the inverse images of these balls under f. Choose δ to be a Lebesgue number for the covering $\mathcal{A}$. Then if $x_1, x_2 \in X$ s.t. $d_X(x_1, x_2) < \delta$ the two-point set $\{x_1, x_2\}$ has diameter less than δ, so that its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then $d_Y(f(x_1), f(x_2)) < \epsilon$ as desired. $\square$

Definition 2.33. If X is a space, a point $x \in X$ is said to be an **isolated point** of X if the one-point set $\{x\}$ is open in X

Theorem 2.34. *Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable*

Proof. Step 1. We show first that given any nonempty open set U of X and any point $x \in X$, there exists a nonempty open set V contained in U s.t. $x \notin \bar{V}$

Choose a point $y \in U$ different from x; this is possible if $x \in U$ since x is not an isolated point of X and it is possible if $x \notin U$ since U is nonempty. Now choose disjoint open sets $x \in W_1$ and $y \in W_2$. Then $V = W_2 \cap U$ is the desired open set.



Step 2. We show that given $f : \mathbb{Z}_+ \to X$ the function f is not surjective. It follows that X is uncountable

Let $x_n = f(n)$. Apply Step 1 to the nonempty open set $U = X$ to choose a nonempty open set $V_1 \subset X$ s.t. $x_1 \notin \bar{V}_1$. In general, given V_{n-1} open and nonempty, choose V_n to be a nonempty open set s.t. $V_n \subset V_{n-1}$ and $\bar{V}_n$ doesn't contain x_n. Consider the nested sequence

$$\bar{V}_1 \supset \bar{V}_2 \supset \cdots$$

of nonempty closed sets of X. Because X is compact, there is a point $x \in \bigcup \bar{V}_n$, by Theorem 2.24. Now $x \neq x_n$ for any n. $\square$

Corollary 2.35. *Every closed interval in $\mathbb{R}$ is uncountable*

Exercise 2.4.1. Let X be a metric space with metric d; let $A \subset X$ be nonempty

1. Show that $d(x, A) = 0$ iff $x \in \bar{A}$
2. Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$

Proof. 2.

$\square$