Stone Spaces

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1 Preliminaries

1.1 Lattices

Let A be a poset in which every finite subset has a join. Then the binary operation $\vee$ and the element 0 defiend above satisfy the equations

1. $a \vee a = a$
2. $a \vee b = b \vee a$
3. $a \vee (b \vee c) = (a \vee b) \vee c$
4. $a \vee 0 = a$

for all a, b, c. We can say that $(A, \vee, 0)$ is a commutative monoid where every element is idempotent. Conversely we have

Theorem 1.1. *Let $(A, \vee, 0)$ be a commutative monoid in which every element is idempotent. Then there exists a unique partial order on A s.t. $a \vee b$ is the join of a and b, and 0 is the least element*

Proof. If such a partial order exists, we must have $a \leq b$ iff $a \vee b = b$. Take this as a definition of $\leq$. Suppose $a \leq b$ and $b \leq c$. Then

$$\begin{aligned} a \vee c &= a \vee (b \vee c) \\ &= (a \vee b) \vee c \\ &= b \vee c = c \end{aligned}$$

so $a \leq c$.

Now let a, b be any two elements of A. Then $a \vee (a \vee b) = (a \vee a) \vee b = a \vee b$, so $a \leq a \vee b$ and similarly $b \leq a \vee b$. But if $a \leq c$ and $b \leq c$, then $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$, so $a \vee b \leq c$; i.e. $a \vee b$ is the join of a and b. □

A set with the structure described in the theorem is called a **semilattice**. A semilattice homomorphism $f : A \to B$ (i.e. a map preserving the distinguished element 0 and the operation $\vee$) is necessarily an order-preserving map.

A **lattice** is a poset A in which every finite subset has both a join and a meet.

Proposition 1.2. *Suppose $(A, \vee, 0)$ and $(A, \wedge, 1)$ are semilattices. Then $(A, \vee, \wedge, 0, 1)$ is a lattice iff the* ***absorptive laws***

$$a \wedge (a \vee b) = a, \quad a \vee (a \wedge b) = a$$

are satisfied for all $a, b \in A$

Proof. If $a \vee b = b$, then $a \wedge b = a \wedge (a \vee b) = a$. So the two partial orders on A agree. □

distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Lemma 1.3. *If the distributive law holds in a lattice, then so does its dual, i.e. the identity*

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Proof.

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) \\ &= \end{aligned}$$

□