# Finite Model Theory

Heinz-Dieter Ebbinghaus
Jörg Flum

June 4, 2021

## Contents



## 1 Preliminaries

### 1.1 Structures

**Vocabularies** are finite sets that consist of **relation symbols** and **constant symbols**. We denote vocabularies by $\tau$, $\sigma$,.... A **vocabulary** is relational if it does not contain constants.

#### 1.1.1 Graph

Let $\tau = \{E\}$ with a binary relation symbol $E$. A **graph** (or **undirected graph**) is a $\tau$-structure $\mathcal{G} = (G, E^G)$ satisfying

1. for all $a \in G$: not $E^G aa$

2. for all $a, b \in G$: if $E^G ab$ then $E^G ba$

By GRAPH we denote the class of **finite** graphs. If only (1) is required, we speak of a **digraph**

A subset $X$ of the universe of a graph $\mathcal{G}$ is a **clique**, if $E^G ab$ for all $a, b \in X$, $a \neq b$

Let $\mathcal{G}$ be a digraph. If $n \geq 1$ and

$$E^G a_0 a_1, E^G a_1 a_2, \dots, E^G a_{n-1} a_n$$

then $a_0, \dots, a_n$ is a **path** from $a_0$ to $a_n$ of **length** $n$. If $a_0 = a_n$, then $a_0, \dots, a_n$ is a **cycle**. A path $a_0, \dots, a_n$ is **Hamiltonian** if $G = \{a_0, \dots, a_n\}$ and $a_i \neq a_j$ for $i \neq j$. If, in addition, $E^G a_n a_0$ we speak of a **Hamiltonian circuit**

Let $\mathcal{G}$ be a graph. Write $a \sim b$ if $a = b$ or if there is a path from $a$ to $b$. The equivalence class of $a$ is called the **(connected) component** of $a$. Let CONN be the class of finite connected graphs

Denote by $d(a, b)$ the length of a shortest path from $a$ to $b$; more precisely, define the **distance function** $d : G \times G \to \mathbb{N} \cup \{\infty\}$ by

$$d(a, b) = \infty \text{ iff } a \nsim b, \quad d(a, b) = 0 \text{ iff } a = b$$

and otherwise

$$d(a, b) = \min\{n \geq 1 \mid \text{there is a path from } a \text{ to } b \text{ of length } n\}$$

We give the following definitions only for **finite** digraphs. A vertex $b$ is a successor of a vertex $a$ if $E^G ab$. The **in-degree** of a vertex is the number of its predecessors, the **out-degree** the number of its successors.

A **root** of a digraph is a vertex with in-degree 0 and a **leaf** a vertex with out-degree 0.

A **forest** is an acyclic digraph where each vertex has in-degree at most 1. A **tree** is a forest with connected underlying graph. Let TREE be the class of finite trees.

### 1.1.2 Operations on Structures

Two $\tau$-structures $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic**, written $\mathcal{C} \cong \mathcal{B}$, if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, i.e., a bijection $\pi : A \to B$ preserving relations and constants, that is

- for $n$-ary $R \in \tau$ and $a_1, \dots, a_n \in A$

$$R^A a_1 \dots a_n \quad \text{iff} \quad R^B \pi(a_i) \dots \pi(a_n)$$

- for $c \in \tau$, $\pi(c^A) = \pi(c^B)$

For relational $\tau$ we introduce the **union** (or, **disjoint union**) of structures. Assume that $\mathcal{A}$ and $\mathcal{B}$ are $\tau$-structures with $A \cap B = \emptyset$. Then $\mathcal{A} \dot{\cup} \mathcal{B}$, the **union** of $\mathcal{A}$ and $\mathcal{B}$, is the $\tau$-structure with domain $A \cup B$ and

$$R^{\mathcal{A} \dot{\cup} \mathcal{B}} := R^{\mathcal{A}} \cup R^{\mathcal{A}}$$

for any $R$ in $\tau$

Note that the union of ordered structures is not an ordered structure. The situation is different for the so-called **ordered sum**: let $\tau$ with $< \in \tau$ be relational and let $\mathcal{A}$ and $\mathcal{B}$ be ordered $\tau$-structures. Assume that $A \cap B = \emptyset$. Define $\mathcal{A} \triangleleft \mathcal{B}$, the **ordered sum** of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{A} \dot{\cup} \mathcal{B}$ but setting

$$<^{\mathcal{A} \dot{\cup} \mathcal{B}} := <^{\mathcal{A}} \cup <^{\mathcal{B}} \cup \{(a, b) \mid a \in A, b \in B\}$$

## 1.2 Syntax and Semantics of First-Order Logic

Fix a vocabulary $\tau$. Each formula of first-order logic will be a string of symbols taken from the alphabet consisting of

- variables
- connectives
- existential quantifier
- equality symbol
- )(
- the symbols in $\tau$

  Denote FO[$\tau$] the set of formulas of first-order logic of vocabulary $\tau$.

  The axioms for graphs stated above have the following formalizations in FO[$\{E\}$]

$$\forall x \neg Exx$$
$$\forall x \forall y (Exy \to Eyx)$$

When only taking into consideration finite structures, we use the notation $\Phi \models_{\text{fin}} \psi$

The **quantifier rank** $\text{qr}(\varphi)$ of a formula $\varphi$ is the maximum number of nested quantifiers occurring in it

It can be shown that every first-order formula is logically equivlent to a formula in prenex normal form, that is, to a formula of the form $Q_1 x_1, \ldots, Q_s x_s \psi$ where $Q_1, \ldots, Q_s \in \{\forall, \exists\}$, and where $\psi$ is quantifier-free. Such a formula is called $\Sigma_n$ if the string of $n$ consecutive blocks, where in each block all quantifiers all of the same type, adjacent blocks contain quantifiers of different type, and the first block is existential. $\Pi_n$ formulas are defined in the same way but now we require that the first block consists of universal quantifiers. A $\Delta_n$-formula is a formula logically equivalent to both a $\Sigma_n$-formula and a $\Pi_n$-formula

Given a formula $\varphi(x, \bar{z})$ and $n \geq 1$,

$$\exists^{\geq n} x \varphi(x, \bar{z})$$

is an abbreviation for the formula

$$\exists x_1, \ldots \exists x_n (\bigwedge_{1 \leq i \leq n} \varphi(x_i, \bar{z}) \wedge \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j)$$

We set

$$\varphi_{\geq n} := \exists^{\geq n} x \; x = x$$

Clearly

$$\mathcal{A} \models \varphi_{\geq n} \quad \text{iff} \quad \|A\| \geq n$$

## 1.3 Some Classical Results of First-Order Logic

**Theorem 1.1.** *The set of logically valid sentences of first-order logic is r.e.*

**Theorem 1.2** (Compactness Theorem)**.** $\Phi$ *is satisfiable iff every finite subset of* $\Phi$ *is satisfiable*

Neither Theorem 1.1 nor 1.2 remain valid if one only considers finite structures. A counterexample for the Compactness Theorem is given by the set $\Phi_\infty := \{\varphi_{\geq n} \mid n \geq 1\}$: Each finite subset of $\Phi_\infty$ has a finite model, but $\Phi_\infty$ has no finite model

The failure of Theorem 1.1 is documented by

**Theorem 1.3** (Trahtenbrot's Theorem). *The set of sentences of first-order logic valid in all finite structures is not r.e.*

**Lemma 1.4.** *Let $\varphi \in FO[\tau]$ and for $i \in I$, let $\Phi^i \subseteq FO[\tau]$. Assume that*

$$\models \varphi \leftrightarrow \bigvee_{i \in I} \bigwedge \Phi^i$$

*Then there is a finite $I_0 \subseteq I$ and for every $i \in I_0$, a finite $\Phi_0^i \subseteq \Phi^i$ s.t.*

$$\models \varphi \leftrightarrow \bigvee_{i \in I_0} \bigwedge \Phi_0^i$$

*Proof.* For simplicity we assume that $\varphi$ is a sentence and that every $\Phi^i$ is a set of sentences. By hypothesis, for some $i \in I$, we have $\Phi^i \models \varphi$; hence, by the Compactness Theorem, $\Phi_0^i \models \varphi$ for some finite $\Phi_0^i \subseteq \Phi^i$.

If there is not such $I_0$ with $\models \varphi \to \bigvee_{i \in I_0} \bigwedge \Phi_0^i$, then each finite subset of $\{\varphi\} \cup \{\neg \bigwedge \Phi_0^i \mid i \in I\}$ has a model. Hence by the Compactness Theorem, there is a contradiction □

**Corollary 1.5.** *Let $\Phi$ be a set of first-order sentences. Assume that any two structures that satisfy the same sentences of $\Phi$ are elementarily equivalent. Then any first-order sentence is equivalent to a boolean combination of sentences of $\Phi$*

*Proof.* For any structure $\mathcal{A}$ set

$$\Phi(\mathcal{A}) := \{\psi \mid \psi \in \Phi, \mathcal{A} \models \psi\} \cup \{\neg\psi \mid \psi \in \Phi, \mathcal{A} \models \neg\psi\}$$

Let $\varphi$ be any first-order sentence. By the preceding lemma it suffices to show that

$$\models \varphi \leftrightarrow \bigvee_{\mathcal{A} \models \varphi} \bigwedge \Phi(\mathcal{A})$$

If $\mathcal{B} \models \varphi$ then $\mathcal{B} \models \bigvee_{\mathcal{A} \models \varphi} \bigwedge \Phi(\mathcal{A})$. Suppose $\mathcal{A} \models \bigvee_{\mathcal{A} \models \varphi} \bigwedge \Phi(\mathcal{A})$. Then for some model $\mathcal{A}$ of $\varphi$, $\mathcal{B} \models \Phi(\mathcal{A})$. By the definition of $\Phi(\mathcal{A})$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of $\Phi$ □

## 1.4 Model Classes and Global Relations

Fix a vocabulary $\tau$. For a sentence $\varphi$ of FO$[\tau]$ we denote by Mod$(\varphi)$ the class of **finite** models of $\varphi$.

Mod$(\varphi)$ is closed under isomorphisms

For $\varphi(x_1, \dots, x_n) \in \mathrm{FO}[\tau]$ and a structure $\mathcal{A}$ let

$$\varphi^{\mathcal{A}}(-) := \{(a_1, \dots, a_n) \mid \mathcal{A} \vDash \varphi[a_1, \dots, a_n]\}$$

be the set of $n$-tuples **defined by $\varphi$ in $\mathcal{A}$**. For $n = 0$ this be read as

$$\varphi^{\mathcal{A}} := \begin{cases} \text{TRUE} & \text{if } \mathcal{A} \vDash \varphi \\ \text{FALSE} & \text{if } \mathcal{B} \nvDash \varphi \end{cases}$$

Use this notation we have

$$\text{if } \pi : \mathcal{A} \cong \mathcal{B} \text{ then } \pi(\varphi^{\mathcal{A}}(-) = \varphi^{\mathcal{B}}(-))$$

where for $X \subseteq A^n$ we set $\pi(X) := \{\pi(a_1), \dots, \pi(a_n) \mid (a_1, \dots, a_n) \in X\}$

**Throughout the book all classes $K$ of structures considered will tacitly be assumed to be closed under isomorphisms**, i.e.

$$\mathcal{A} \in K \text{ and } \mathcal{A} \cong \mathcal{B} \text{ implies } \mathcal{B} \in K$$

**Definition 1.6.** Let $K$ be a class of $\tau$-structures. An $n$-ary **global relation $\Gamma$ on $K$** is a mapping assigning to each $A \in K$ an $n$-ary relation $\Gamma(\mathcal{A})$ on $\mathcal{A}$ satisfying

$$\Gamma(\mathcal{A})a_1 \dots a_n \quad \text{iff} \quad \Gamma(\mathcal{B})\pi(a_1) \dots \pi(a_n)$$

for every isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$ and every $a_1, \dots, a_n \in A$. If $K$ is the class of all finite $\tau$-structures, then we just speak of an $n$-ary **global relation**

**Example 1.1.**

1. Any formula $\varphi(x_1, \dots, x_n) \in \mathrm{FO}[\tau]$ defines the global relation $\mathcal{A} \mapsto \varphi^{\mathcal{A}}(-)$

2. The "transitive closure relation" TC is the binary global relation on GRAPH with

$$\mathrm{TC}(\mathcal{G}) := \{(a, b) \mid a, b \in G, \text{ there is a path from } a \text{ to } b\}$$

3. For $m \geq 0$, $\Gamma_m$ is a unary global relation on GRAPH, where

$$\Gamma_m(\mathcal{G}) := \{a \mid \|\{b \in G \mid E^G ab\}\| = m\}$$

   is the set of elements of $\mathcal{G}$ of degree $m$

An important issue in model theory is the study of properties of classes of structures that are axiomatizable in a given logic $\mathcal{L}$ and in particular to determine what classes of structures are axiomatizable are what global relations are definable in $\mathcal{L}$.

### 1.5 Relational Databases and Query Languages

# 2 Ehrenfeucht–Fraïssé Method

### 2.1 Elementary Classes

**Proposition 2.1.** *Every finite structure can be characterized in first-order logic up to isomorphism, i.e., for every finite structure $\mathcal{A}$ there is a sentence $\varphi_{\mathcal{A}}$ of first-order logic s.t. for all structures $\mathcal{B}$ we have*

$$\mathcal{B} \models \varphi_{\mathcal{A}} \quad \textit{iff} \quad \mathcal{A} \cong \mathcal{B}$$

*Proof.* Suppose $A = \{a_1, \dots, a_n\}$. Set $\bar{a} = a_1 \dots a_n$. Let

$$\Theta_n := \{\psi \mid \psi \text{ has the form } Rx_1 \dots x_k, x = y \text{ or } c = x, \\ \text{and variables among } v_1, \dots, v_n\}$$

and

$$\varphi_{\mathcal{A}} := \exists v_1 \dots \exists v_n (\bigwedge \{\psi \mid \psi \in \Theta_n, \mathcal{A} \models \psi[\bar{a}]\} \wedge \\ \bigwedge \{\neg\psi \mid \psi \in \Theta_n, \mathcal{A} \models \neg\psi[\bar{a}]\} \wedge \forall v_{n+1}(v_{n+1} = v_n \vee \dots \vee v_{n+1} = v_n))$$

□

**Corollary 2.2.** *Let $K$ be a class of finite structures. Then there is a set $\Phi$ of first-order sentences s.t.*

$$K = \mathrm{Mod}(\Phi)$$

*that is, $K$ is the class of finite models of $\Phi$*

*Proof.* For each $n$ there is only a finite number of pairwise nonisomorphic structures of cardinality $n$. Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be a maximal subset of $K$ of pairwise nonisomorphic structures of cardinality $n$. Set

$$\psi_n := (\varphi_{=n} \to (\varphi_{\mathcal{A}_1} \vee \dots \vee \varphi_{\mathcal{A}_k}))$$

Then $K = \mathrm{Mod}(\{\psi_n \mid n \geq 1\})$ □

**Definition 2.3.** Let $K$ be a class of finite structures. $K$ is called **axiomatizable in first-order logic** or **elementary** if there is a setence $\varphi$ of first-order logic s.t. $K = \mathrm{Mod}(\varphi)$

For structures $\mathcal{A}$ and $\mathcal{B}$ and $m \in \mathbb{N}$ we write $\mathcal{A} \equiv_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are $m$**-equivalent** if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank $\leq m$

**Theorem 2.4.** *Let $K$ be a class of finite structures. Suppose that for every $m$ there are fintie structures $\mathcal{A}$ and $\mathcal{B}$ s.t.*

$$\mathcal{A} \in K, \mathcal{B} \notin K, \text{ and } \mathcal{A} \equiv_m \mathcal{B}$$

*Then $K$ is not axiomatizable in first-order logic*

*Proof.* Let $\varphi$ be any first-order sentence. Set $m := \mathrm{qr}(\varphi)$. By out assumption there are $\mathcal{A}$ and $\mathcal{B}$ s.t. $\mathcal{A} \in K, \mathcal{B} \notin K$, and $\mathcal{A} \equiv_m \mathcal{B}$. Hence $K \neq \mathrm{Mod}(\varphi)$ □

## 2.2 Ehrenfeucht's Theorem

**Definition 2.5.** Assume $\mathcal{A}$ and $\mathcal{B}$ are structures. Let $p$ be a map with $\mathrm{dom}(p) \subseteq B$. Then $p$ is said to be a **partial isomorphism** from $\mathcal{A}$ to $\mathcal{B}$ if

1. $p$ is injective
2. for every $c \in \tau$:$c^{\mathcal{A}} \in \mathrm{dom}(p)$ and $p(c^{\mathcal{A}}) = c^{\mathcal{B}}$
3. for every $n$-ary $R \in \tau$ and all $a_1, \dots, a_n \in \mathrm{dom}(p)$

$$R^{\mathcal{A}} a_1 \dots a_n \quad \text{iff} \quad R^{\mathcal{B}} p(a_1) \dots p(a_n)$$

We write $\mathrm{Part}(\mathcal{A}, \mathcal{B})$ for the set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$

In the following we identify a map $p$ with its graph $\{(a, p(a)) | a \in \mathrm{dom}(p)\}$. Then $p \subseteq q$ means that $q$ is an extension of $p$

*Remark.* 1. The empty map, $p = \emptyset$, is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ just in case the vocabulary contains no constants

2. If $p \neq \emptyset$ is a map with $\mathrm{dom}(p) \subseteq A$ and $\mathrm{ran}(p) \subseteq B$, then $p$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ iff $\mathrm{dom}(p)$ contains $c^{\mathcal{A}}$ for all constants $c \in \tau$ and $p : \mathrm{dom}(p)^{\mathcal{A}} \cong \mathrm{ran}(p)^{\mathcal{B}}$

3. For $\bar{a} = a_1 \dots a_s \in A$ and $\bar{b} = b_1 \dots b_s \in B$ the following statements are equivalent

   (a) the clauses

   $$p(a_i) = b_i \text{ for } i = 1, \dots, s$$

   and

   $$p(c^{\mathcal{A}}) = c^{\mathcal{B}} \text{ for } c \text{ in } \tau$$

   define a map, which is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ (henceforth denoted by $\bar{a} \mapsto \bar{b}$)

(b) for all quantifier-free $\varphi(v_1, \dots, v_s)$: $\mathcal{A} \models \varphi[\bar{a}]$ iff $\mathcal{B} \models \varphi[\bar{b}]$

(c) for all atomic $\varphi(v_1, \dots, v_s)$: $\mathcal{A} \models \varphi[\bar{a}]$ iff $\mathcal{B} \models \varphi[\bar{b}]$

In general, a **partial isomorphism does not preserve the validity of formulas with quantifiers**: Let $\tau = \{<\}$, $\mathcal{A} = (\{0,1,2\}, <)$, $\mathcal{B} = (\{0,1,2,3\}, <)$ whre in both cases < denotes the natural ordering. Then $p_0 := 02 \mapsto 01$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ s.t.

$$\mathcal{A} \models \exists v_3(v_1 < v_3 \wedge v_3 < v_2)[0,2]$$

but

$$\mathcal{B} \not\models \exists v_3(v_1 < v_3 \wedge v_3 < v_2)[p_0(0), p_0(2)]$$

Let $\mathcal{A}$ and $\mathcal{B}$ be $\tau$-structures, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \in \mathbb{N}$. The **Ehrenfeucht game** $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ is played by two players called the **spoiler** and the **duplicator**. Each player has to make $m$ moves in the course of a play. In his $i$-th move the spoiler first selects a structure, $\mathcal{A}$ or $\mathcal{B}$, and an element in this structure. If the spoiler chooses $e_i$ in $\mathcal{A}$ then the duplicator in his $i$-th move must choose an element $f_i$ in $\mathcal{B}$. If the spoiler chooses $f_i$ in $\mathcal{B}$ then the duplicator must choose an element $e_i$ in $\mathcal{A}$

| | $\mathcal{A}, \bar{a}$ | $\mathcal{B}, \bar{b}$ |
|---|---|---|
| first move | $e_1$ | $f_1$ |
| second move | $e_2$ | $f_2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $m$-th move | $e_m$ | $f_m$ |

The duplicator **wins** iff $\bar{a}\bar{e} \mapsto \bar{b}\bar{f} \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$.

Equivalently, the spoiler wins if after some $i \leq m$, $\bar{a}e_1 \dots e_i \mapsto \bar{b}f_1 \dots f_i$ is not a partial isomorphism. We say that a player, the spoiler or the duplicator, has a **winning strategy** in $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, or shortly, that he **wins** $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, if it is possible for him to win each play whatever choices are made by the opponent.

If $s = 0$, we denote the game by $G_m(\mathcal{A}, \mathcal{B})$

**Lemma 2.6.** *1. If $A \cong B$ then the duplicator wins $G_m(\mathcal{A}, \mathcal{B})$*

*2. If the duplicator wins $G_{m+1}(\mathcal{A}, \mathcal{B})$ and $\|A\| \leq m$ then $\mathcal{A} \cong \mathcal{B}$*

**Lemma 2.7.** *Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \geq 0$*

*1. The duplicator wins $G_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ iff $\bar{a} \mapsto \bar{b}$ is a partial isomorphism*

2. *For $m > 0$ the following are equivalent*

   (a) *The duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$*

   (b) *For all $a \in A$ there is $b \in B$ s.t. the duplicator wins the game $G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$ and for all $b \in B$ there is a $a \in A$ s.t. the duplicator wins $G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$*

   (c) *If the duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{B})$ and if $m' < m$ the duplicator wins $G_{m'}(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$*

Let $\mathcal{A}$ be given. For $\bar{a} = a_1 \dots a_s \in A$ and $m \geq 0$ we introduce a formula $\varphi^m_{\bar{a}}(v_1, \dots, v_s)$ that describes the game-theoretic properties of $\bar{a}$ in any game $G_m(\mathcal{A}, \bar{a}, \dots)$ s.t. for any $\mathcal{B}$ and $\bar{b} = b_1 \dots b_s \in B$

$$\mathcal{B} \vDash \varphi^m_{\bar{a}}[\bar{b}] \quad \text{iff} \quad \text{the duplicator wins } G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$$

**Definition 2.8.** Let $\bar{v}$ be $v_1, \dots, v_s$

$$\varphi^0_{\bar{a}} := \bigwedge \{\varphi(\bar{v}) \mid \varphi \text{ atomic or negated atmoic}, \mathcal{A} \vDash \varphi[\bar{a}]\}$$

(atomic diagram of $\mathcal{A}$) and for $m > 0$

$$\varphi^m_{\bar{a}}(\bar{v}) := \bigwedge_{a \in A} \exists v_{s+1} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1}) \wedge \forall v_{s+1} \bigvee_{a \in A} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})$$

$\varphi_{\bar{a}^0}$ describes the isomorphism type of the substructure generated by $\bar{a}$ in $\mathcal{A}$; and for $m > 0$ the formula $\varphi^m_{\bar{a}}$ tells us to which isomorphism types the tuple $\bar{a}$ can be extended in $m$ steps adding one element in each step. $\varphi^m_{\bar{a}}$ is called the $m$**-isomorphism type** (or $m$**-Hintikka formula**) of $\bar{a}$ in $\mathcal{A}$

Since $\varphi(v_1, \dots, v_s) \mid \varphi$ atomic or negated atmoic is finite, a simple induction on $m$ shows

**Lemma 2.9.** *For $s, m \geq 0$, the set $\{\varphi^m_{\mathcal{A},\bar{a}} \mid \mathcal{A}$ a structure and $\bar{a} \in A^s\}$ is finite*

**Lemma 2.10.** 1. $\mathrm{qr}(\varphi^m_{\bar{a}}) = m$

2. $\mathcal{A} \vDash \varphi^m_{\bar{a}}[\bar{a}]$

3. *For any $\mathcal{B}$ and $\bar{b}$ in $B$*

$$\mathcal{B} \vDash \varphi^0_{\bar{a}}[\bar{b}] \quad \textit{iff} \quad \bar{a} \mapsto \bar{b} \in \mathit{Part}(\mathcal{A}, \mathcal{B})$$

**Theorem 2.11** (Ehrenfeucht's Theorem)**.** *Given $\mathcal{A}$ and $\mathcal{B}$, $\bar{a} \in A^s$ and $\bar{b} \in B^s$, and $m \geq 0$, the following are equivalent*

1. *The duplicator wins* $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$

2. $\mathcal{B} \models \varphi^m_{\bar{a}}[\bar{b}]$

3. $\bar{a}$ *and* $\bar{b}$ *satisfy the same formulas of quantifier rank* $\leq m$*, that is, if* $\varphi(x_1, \dots, x_s)$ *is of quantifier rank* $\leq m$*, then*

$$\mathcal{A} \models \varphi[\bar{a}] \quad \textit{iff} \quad \mathcal{B} \models \varphi[\bar{b}] \tag{1}$$

*Proof.* $1 \leftrightarrow 2$. Induction on $m$. For $m = 0$

$$\begin{aligned} \text{the duplicator wins } G_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \quad & \text{iff} \quad \bar{a} \mapsto \bar{b} \in \text{Part}(\mathcal{A}, \mathcal{B}) \\ & \text{iff} \quad \mathcal{B} \models \varphi^0_{\bar{a}}[\bar{b}] \end{aligned}$$

For $m > 0$

$$\begin{aligned} & \text{the duplicator wins } G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \\ \text{iff} \quad & \text{for all } a \in A, \text{there is } b \in B \text{ s.t. the duplicator wins} \\ & G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b), \text{ and for all } b \in B \text{ there is } a \in A \text{ s.t.} \\ & \text{the duplicator wins } G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b) \\ \text{iff} \quad & \text{for all } a \in A, \text{ there is } b \in B \text{ with } \mathcal{B} \models \varphi^{m-1}_{\bar{a}a}[\bar{b}b] \text{ , and} \\ & \text{for all } b \in B, \text{ there is } a \in A \text{ with } \mathcal{B} \models \varphi^{m-1}_{\bar{a}a}[\bar{b}b] \\ \text{iff} \quad & \mathcal{B} \models \textstyle\bigwedge_{a \in A} \exists v_{s+1} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1}) \wedge \forall v_{s+1} \bigvee_{a \in A} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})[\bar{b}] \\ \text{iff} \quad & \mathcal{B} \models \varphi^m_{\bar{a}}[\bar{b}] \end{aligned}$$

$3 \to 1$. $\text{qr}(\varphi^m_{\bar{a}}) = m$ and $\mathcal{A} \models \varphi^m_{\bar{a}}[\bar{a}]$

$1 \to 3$. Induction on $m$. The case $m = 0$ is handled as above. Let $m > 0$ and suppose that the duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$. Clearly the set of formulas $\varphi(x_1, \dots, x_s)$ satisfying 1 contains the atomic formulas and is closed under $\neg$ and $\vee$ (Since duplicator wins the game, there are partial isomorphisms). Suppose that $\varphi(\bar{a}) = \exists y \psi$ and $\text{qr}(\varphi) \leq m$. Since $y \notin \text{free}(\varphi)$, we can assume that $y$ is distinct from the variables in $\bar{x}$. Hence $\psi = \psi(\bar{x}, y)$. Assume, for instance, $\mathcal{A} \models \varphi(\bar{a})$. Then there is $a \in A$ s.t. $\mathcal{A} \models \psi[\bar{a}, a]$. As by 1, the duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, there is $b \in B$ s.t. the duplicator wins $G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$. Since $\text{qr}(\psi) \leq m - 1$, the induction hypothesis yields $\mathcal{B} \models \psi[\bar{b}, b]$, hence $\mathcal{B} \models \varphi[\bar{b}]$ □

**Corollary 2.12.** *For structures* $\mathcal{A}, \mathcal{B}$ *and* $m \geq 0$ *the following are equivalent*

1. *The duplicator wins* $G_m(\mathcal{A}, \mathcal{B})$

2. $\mathcal{B} \models \varphi^m_{\mathcal{A}}$

3. $\mathcal{A} \equiv_m \mathcal{B}$

**Corollary 2.13.** *Let $\mathcal{A}$ be a structure with $\|A\| \leq m$. Then for all $\mathcal{B}$*

$$\mathcal{B} \models \varphi_{\mathcal{A}}^{m+1} \quad \textit{iff} \quad \mathcal{A} \cong \mathcal{B}$$

The next result shows that the formulas $\varphi_{\bar{a}}^{m}$ give a clear picture of the expressive power of first-order logic

**Theorem 2.14.** *Let $\varphi(v_1, \ldots, v_s)$ be a formula of quantifier rank $\leq m$. Then*

$$\models \varphi \leftrightarrow \bigvee\{\varphi_{\mathcal{A},\bar{a}}^{m} \mid \mathcal{A} \textit{ a structure }, \bar{a} \in \mathcal{A}, \textit{ and } \mathcal{A} \models \varphi[\bar{a}]\}$$

*Proof.* Suppose first that $\mathcal{B} \models \varphi[\bar{b}]$. Then the formula $\varphi_{\mathcal{B},\bar{b}}^{m}$ is a member of the disjunction on the right side of the equivalence, which therefore is satisfied by $\bar{b}$.

Conversely, suppose $\mathcal{B} \models \bigvee\{\varphi_{\mathcal{A},\bar{a}}^{m}[\bar{a}]\}[\bar{b}]$. Then for some $\mathcal{A}$ and $\bar{a}$ s.t. $\mathcal{A} \models \varphi[\bar{a}]$ we have $\mathcal{B} \models \varphi_{\mathcal{A},\bar{a}}^{m}[\bar{b}]$. By Theorem 2.11 $\bar{a}$ and $\bar{b}$ satisfy the same formulas of quantifier rank $\leq m$ and therefore $\mathcal{B} \models \varphi[\bar{b}]$. □

**Theorem 2.15.** *For a class $K$ of finite structures the following are equivalent*

1. *$K$ is not axiomatizable in first-order logic*
2. *For each $m$ there are finite structures $\mathcal{A}$ and $\mathcal{B}$ s.t.*

$$\mathcal{A} \in K, \mathcal{B} \notin K \textit{ and } \mathcal{A} \equiv_m \mathcal{B}$$

*Proof.* $2 \rightarrow 1$ is proved in theorem 2.4. For the converse, suppose that 2 doesn't hold, i.e., that for some $m$ and all finite $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} \in K \text{ and } \mathcal{A} \equiv_m \mathcal{B} \text{ imply } \mathcal{B} \in K$$

Then $K = \mathrm{Mod}(\bigvee\{\varphi_{\mathcal{A}}^{m} \mid \mathcal{A} \in K\})$, and thus $K$ is axiomatizable □

## 2.3 Examples and Fraïssé's Theorem

Given structures $\mathcal{A}, \mathcal{B}$ and $m \in \mathbb{N}$, let $W_n(\mathcal{A}, \mathcal{B}) :=$

$$\{\bar{a} \mapsto \bar{b} \mid s \geq 0, \bar{a} \in A^s, \bar{b} \in B^s, \text{ the duplicator wins } G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})\}$$

be the set of winning positions for the duplicator. The sequence of the $W_M(\mathcal{A}, \mathcal{B})$ has the back and forth properties as introduced in the following definition

**Definition 2.16.** Structures $\mathcal{A}$ and $\mathcal{B}$ are said to be $m$**-isomorphic**, written $\mathcal{A} \cong_m \mathcal{B}$, if there is a sequence $(I_j)_{j \leq m}$ with the following properties

1. Every $I_j$ is a nonempty set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$
2. (**Forth property**) For every $j < m$, $p \in I_{j+1}$ and $a \in A$ there is $q \in I_j$ s.t. $q \supseteq p$ and $a \in \mathrm{dom}(q)$
3. (**Back property**) For every $j < m$, $p \in I_{j+1}$, and $b \in B$ there is $q \in I_j$ s.t. $q \supseteq p$ and $b \in \mathrm{ran}(q)$

If $(I_j)_{j \leq m}$ has the properties 1,2 and 3, we write $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are $m$**-isomorphic via** $(I_j)_{j \leq m}$

*Exercise* 2.3.1. Supppose $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$. Then $(\tilde{I}_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ with $\tilde{I}_j := \{q \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \mid q \subseteq p \text{ for some } p \in I_j\}$. In particular, $\emptyset \mapsto \emptyset \in I_j$ for all $j \leq m$. Moreover $= W_j(\mathcal{A}, \mathcal{B})$