Category Theory

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1 Categories

1.1 Examples of categories

Definition 1.1. A **functor**

$$F : \mathbf{C} \to \mathbf{C}$$

between categories **C** and **D** is a mapping of objects to objects and arrows to arrows, in such a way that

1. $F(f : A \to B) = F(f) : F(A) \to F(B)$
2. $F(1_A) = 1_{F(A)}$
3. $F(g \circ f) = F(g) \circ F(f)$

1.2 Free categories

The "Kleene closure" of A is defined to be the set

$$A^* = \{\text{words over } A\}$$

Also

$$i : A \to A^*$$

defined by $i(a) = a$ and called the "intersection of generators"

A monoid M is **freely generated** by a subset A of M if the following conditions hold:

1. Every element $m \in M$ can be written as a product of elements of A
2. No "nontrivial" relations hold in M, that is, if $a_1 \dots a_j = a'_1 \dots a'_k$,then this is required by the axioms for monoids

Every monoid N has an underlying set $|N|$, and every monoid homomorphism $f : N \to M$ has an underlying function $|f| : |N| \to |M|$. The free monoid $M(A)$ on a set A is by definition "the" monoid with the following UMP

Universal Mapping Property of $M(A)$

There is a function $i : A \to |M(A)|$, and given any monoid N and any function $f : A \to |N|$, there is a **unique** monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $\left|\overline{f}\right| \circ i = f$

in **Mon**

$$M(A) \xrightarrow{\overline{f}} N$$

in **Sets**

$$\begin{tikzcd} |M(A)| \arrow[r, "|\overline{f}|"] & |N| \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 1.2. *A^* has the UMP of the free monoid on A*

Proof. Given $f : A \to |N|$, define $\overline{f} : A^* \to N$ by

$$\overline{f}(-) = u_N, \qquad \text{the unit of } N$$
$$\overline{f}(a_1 \dots a_i) = f(a_1) \cdot_N \cdots \cdot_N f(a_i)$$

$\square$

2 Abstract structures

2.1 Initial and terminal objects

Example 2.1. A **Boolean algebra** is a poset B equipped with distinguished elements 0,1, binary operations $a \vee b$ of join and $a \wedge b$ of meet, and a unary operation $\neg b$ of complementation. These are required to satisfy the conditions

$$\begin{aligned} 0 &\leq a \\ a &\leq 1 \\ a \leq c \quad \text{and} \quad b \leq c \quad &\text{iff} \quad a \vee b \leq c \\ c \leq a \quad \text{and} \quad c \leq b \quad &\text{iff} \quad c \leq a \wedge b \\ a \leq \neg b \quad &\text{iff} \quad a \wedge b = 0 \\ \neg\neg a &= a \end{aligned}$$

2 $= \{0, 1\}$ is an initial elements of **BA**. **BA** has as arrows the Boolean homomorphisms that $h(0) = 0, h(a \vee b) = h(a) \vee h(b)$, etc.

2.2 Products

Definition 2.1. In any category **C**, a **product diagram** for the objects A and B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP:
Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique $u : X \to P$ making the diagram

$$\begin{tikzcd} & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

2.3 Categories with products

Let **C** be a category that has a product diagram for every pair of objects. Suppose we have objects and arrows

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}
$$

with indicated products. Then we write

$$f \times f' : A \times A' \to B \times B$$

for $f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle$

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times f'"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}
$$

In this way, if we choose a product for each pair of objects, we get a functor

$$\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times f'"] \arrow[dd, dashed, bend right] & A' \arrow[d, "f'"] \\
B \arrow[d, "g"'] & B \times B' \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, dashed, "g \times g'"] & B' \arrow[d, "g'"] \\
C & C \times C' \arrow[l, "o_1"'] \arrow[r, "o_2"] & C'
\end{tikzcd}
$$

$(g \circ f) \times (g' \circ f') = (f \times f') \circ (g \times g')$

To prove

$$(A \times B) \times C \cong A \times (B \times C)$$

Consider

$$
\begin{tikzcd}
 & A \times (B \times C) \arrow[r] \arrow[ddl] & B \times C \arrow[dl] \arrow[ddr] & \\
 & B & & \\
A & A \times B \arrow[l] \arrow[u] \arrow[dr] & (A \times B) \times C \arrow[l] \arrow[r] \arrow[uu, dashed] \arrow[uul, dashed, "g"'] & C \\
 & & B & \\
 & A \times (B \times C) \arrow[r] \arrow[uul] \arrow[uu, dashed] \arrow[uur, dashed, "f"] & B \times C \arrow[u] \arrow[uur] &
\end{tikzcd}
$$

Given no objects, there is an object 1 with no maps, and give nany other object X and no maps, there is a unique arrow

$$! : X \to 1$$

Definition 2.2. A category **C** is said to **have all finite products**, if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category **C has all (small) products** if every set of objects in **C** has a product

2.4 Hom-sets

In this section, we assume that all categories are locally small

Given any objects A and B in category **C**,we write

$$\mathrm{Hom}(A, B) = \{f \in \mathbf{C} \mid f : A \to B\}$$

and call such a set of arrows a **Hom-set**

Note that any arrow $g : B \to B'$ in **C** induces a function

$$\mathrm{Hom}(A, g) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$$
$$(f : A \to B) \mapsto (g \circ f : A \to B \to B')$$

Let's show that this determines a functor

$$\mathrm{Hom}(A, -) : \mathbf{C} \to \mathbf{Sets}$$

called the (covariant) **representable functor** of A. We need to show that

$$\mathrm{Hom}(A, 1_X) = 1_{\mathrm{Hom}(A,X)}$$

and that

$$\mathrm{Hom}(A, g \circ f) = \mathrm{Hom}(A, g) \circ \mathrm{Hom}(A, f)$$

For any object P, a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$ determine an element (p_1, p_2) of the set

$$\mathrm{Hom}(P, A) \times \mathrm{Hom}(P, B)$$

Now given any arrow

$$x : X \to P$$

composing with p_1 and p_2 gives a pair of arrows $x_1 = p_1 \circ x : X \to A$ and $x_2 = p_2 \circ x : X \to B$

In this way, we have a function

$$\theta_X = (\mathrm{Hom}(X, p_1), \mathrm{Hom}(X, p_2)) : \mathrm{Hom}(X, P) \to \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

defined by

$$\theta_X(x) = (x_1, x_2)$$

Proposition 2.3. *A diagram of the form*

$$A \xleftarrow[p_1]{} P \xrightarrow[p_2]{} B$$

is a product for A and B iff for every object X, the canonical function θ_X is an isomorphism

$$\theta_X : \mathrm{Hom}(X, P) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

Proof. Note that we are talking about isomorphism on the set □

Definition 2.4. Let **C**, **D** be categories with binary products. A functor F : **C** $\to$ **D** is said to **preserve binary products** if it takes every product diagram

$$A \xleftarrow[p_1]{} A \times B \xrightarrow[p_2]{} B$$

to a product diagram

$$FA \xleftarrow[Fp_1]{} F(A \times B) \xrightarrow[Fp_2]{} FB$$

F preserves products just if

$$F(A \times B) \cong FA \times FB$$

Corollary 2.5. *For any object X in a category **C** with products, the (covariant) representable functor*

$$\mathrm{Hom}_{\mathbf{C}}(X, -) : \mathbf{C} \to \mathbf{Sets}$$

preserves products

3 Duality

3.1 Coproducts

Definition 3.1. A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A, B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ with $u \circ q_i = z_i$

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "z_1"] \arrow[r, "q_1"] & Q \arrow[u, dotted, "u"] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"] \end{tikzcd}$$

written as $A + B$

In **Sets**, every finite set A is a coproduct

$$A \cong 1 + 1 + \cdots + 1 \quad (n\text{-times})$$

Example 3.1. If $M(A)$ and $M(B)$ are free monoids on sets A and B, then in **Mon** we can construct their coproduct as

$$M(A) + M(B) \cong M(A + B)$$

$$\begin{tikzcd} & N & \\ M(A) \arrow[ur] \arrow[r] & M(A+B) \arrow[u, dotted] & M(B) \arrow[ul] \arrow[l] \\ A \arrow[u, "\eta_A"] \arrow[r] & A+B \arrow[u, "\eta_{A+B}"] & B \arrow[u, "\eta_B"'] \arrow[l] \end{tikzcd}$$

Here we are working in two different categories. Half below is in **Sets**, the other is **Mon**

Product of two powerset Boolean algebras $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is also a powerset

$$\mathcal{P}(A) \times \mathcal{P}(B) \cong \mathcal{P}(A + B)$$

Example 3.2. Two monoids $M(|A| + |B|)$ is strings over the disjoint union $|A| + |B|$ of the underlying sets. That is, the elements of A and B and the equivalence relation $v \sim w$ is the least one containing all instances of the following equations

$$\begin{aligned}
(\ldots x u_A y \ldots) &= (\ldots xy \ldots) \\
(\ldots x u_B y \ldots) &= (\ldots xy \ldots) \\
(\ldots aa' \ldots) &= (\ldots a \cdot_A a' \ldots) \\
(\ldots bb' \ldots) &= (\ldots b \cdot_B b' \ldots)
\end{aligned}$$

The unit is the equivalence class $[-]$ of the empty word. Multiplication is

$$[x \ldots y] \cdot [x' \ldots y'] = [x \ldots yx' \ldots y']$$

The coproduct injections $i_A : A \to A + B$ and $i_B : B \to A + B$ are

$$i_A(a) = [a], \quad i_B(b) = [b]$$

Given any homomorphisms $f : A \to M$ and $g : B \to M$ into a monoid, the unique homomorphism

$$[f, g] : A + B \to M$$

is defined by

$$|A| + |B| \xrightarrow{[|f|+|g|]} |M|$$

$$\begin{tikzcd}
M(|A| + |B|) \arrow[r, "{[f,g]'}"] \arrow[d, two heads] & M \\
M(|A| + |B|)/\sim \arrow[ur, dotted, "{[f,g]}"'] &
\end{tikzcd}$$