Topology

Munkres

June 29, 2021

Contents

1 Topological Spaces and Continuous Functions

1.1 Topological Spaces

Definition 1.1. A **topology** on a set is a collection $\mathcal{T}$ of subsets of X having the following properties

1. $\emptyset$ and X are in $\mathcal{T}$
2. The union of the elements of any subcollection of $\mathcal{T}$ is in T
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**

Example 1.1. Consider $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. $(-1/n, 1/n)$ is open but $\{0\}$ is not open in $\mathbb{R}$.

If $X$is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if $U \in \mathcal{T}$

Example 1.2. If X is any set, the collection of all subsets of X is a topology on X; it is called the **discrete topology**. The collection consisting of X and $\emptyset$ only is also a topology on X; we shall call it the **indiscrete topology**

Example 1.3. Let X be a set; let $\mathcal{T}_f$ be the collection of all subsets U of Xs.t. $X - U$ either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the **finite complement topology**. If $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$.

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Definition 1.2. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topology on a given set X. If $\mathcal{T}' \supset \mathcal{T}$ we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$ we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ or **strictly coarser**. We say $\mathcal{T}$ is **comparable** with $\mathcal{T}$ is either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$

1.2 Basis for a Topology

Definition 1.3. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis element**) s.t.

1. for each $x \in X$, there is at least one basis element B s.t. $x \in B$
2. if $x \in B_1 \cap B_2$, then there is a basis element B_3 s.t. $x \in B_3 \subset B_1 \cap B_2$

If $\mathcal{B}$ satisfies these conditions, then we define the **topology** $\mathcal{T}$ **generated by** $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis $B \in \mathcal{B}$ s.t. $x \in B \subset U$.

Now we show that $\mathcal{T}$ is indeed a topology. Take an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of $\mathcal{T}$, we show that

$$U = \bigcup_{\alpha \in J} U_\alpha$$

belongs to $\mathcal{T}$. Given $x \in U$, there is an index α s.t. $x \in U_\alpha$. Since U_α is open, there is a basis element B s.t. $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so U is open.

If $U_1, U_2 \in \mathcal{T}$, then given $x \in U_1 \cap U_2$. we choose $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. By the second condition for a basis we have $x \in B_3 \subset B_1 \cap B_2$. Hence $x \in B_3 \subset U_1 \cap U_2$.

Lemma 1.4. *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$.

Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x for B s.t. $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$ $\square$

Lemma 1.5. *Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X s.t. for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ s.t. $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis for the topology of X.*

Proof. Let $x \in C_1 \cap C_2$, since C_1 and C_2 is open, $C_1 \cap C_2$ is open. Hence there exists $C_3 \in \mathcal{C}$ s.t. $x \in C_3 \subseteq C_1 \cap C_2$

Let $\mathcal{T}$ be the collection of open sets of X; we must show that the topology $\mathcal{T}'$ generated by $\mathcal{C}$ equals the topology $\mathcal{T}$. If $U \in \mathcal{T}$, then there is $x \in C \subset U$. If $W \in \mathcal{T}'$, then $W = \bigcup_{x \in W} B_x$ and $B_x \in \mathcal{T}$ □

Lemma 1.6. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. TFAW*

1. *$\mathcal{T}'$ is finer than $\mathcal{T}$*
2. *For each $x \in X$ and each basis element $x \in B \in \mathcal{B}$ there is a basis element $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$*

Proof. $2 \to 1$. Given $U \in \mathcal{T}$. Then $x \in B \subset U$ and $x \in B' \subset U$. Hence $U \in \mathcal{T}'$.

$1 \to 2$. given $x \in B \in \mathcal{B}$. Since $\mathcal{T} \subset \mathcal{T}'$ we have $B \in \mathcal{T}'$. Since $\mathcal{T}'$ is generated by $\mathcal{B}'$ there is an element $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$ □

Definition 1.7. If $\mathcal{B}$ is the collection of all open intervals in the real line

$$(a, b) = \{x \mid a < x < b\}$$

the topology generated by $\mathcal{B}$ is called the **standard topology** on the real line. If $\mathcal{B}'$ is the collection of all half-opne intervals of the form

$$[a, b) = \{x \mid a \leq x < b\}$$

where $a < b$, the topology generated by $\mathcal{B}'$ is called the **lower limit topology** of $\mathbb{R}$. When $\mathbb{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_l$. Finally let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b) along with all sets of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ is called the ***K*-topology** on R. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$

Lemma 1.8. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.*

Proof. Let $\mathcal{T}, \mathcal{T}', \mathcal{T}''$ be the topologies of $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_K$. Given a basis element (a, b) for $\mathcal{T}$ and a point x of (a, b), the basis element $x \in [x, b) \subset (a, b)$. On the other hand, given the basis element $[x, d) \in \mathcal{T}$ there is no interval (a, b) that contains x and lies in $[x, d)$. Thus $\mathcal{T}$ is strictly finer than $\mathcal{T}$.

Given $B = (-1, 1) - K \in \mathcal{T}''$ and the point 0 of B, there is no open interval of $\mathcal{T}$ that contains 0 and lies in B

Also given B, there is no $[x, b) \in \mathcal{T}'$ s.t. $[x, b) \subset B$. □

Definition 1.9. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersection of elements of $\mathcal{S}$.

1.3 The Order Topology

Given elements $\underline{a \text{ and } b \text{ of } X}$ s.t. $a < b$,(a,b),$(a,b]$,$[a,b)$ and $[a,b]$ are **intervals**

Definition 1.10. Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

1. All open intervals (a,b) in X
2. All intervals of the form $[a_0, b)$ where a_0 is the smallest element of X
3. All intervals of the form $(a, b_0]$ where b_0 is the largest element of X

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the **order topology**

1.4 The Product Topology on $X \times Y$

Definition 1.11. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y

Theorem 1.12. *If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection*

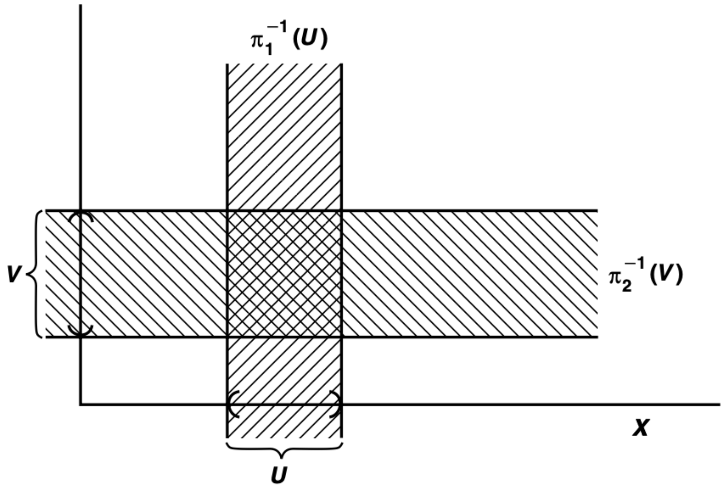
$$\mathcal{D} = \{B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$

Theorem 1.13. *The collection*

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$



Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$; let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Then $\mathcal{T}' \subset \mathcal{T}$. On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since

$$U \times V = \pi_1^{-1} \cap \pi_2^{-1}(V)$$

Hence $U \times V \in \mathcal{T}$ □

1.5 The Subspace Topology

Definition 1.14. Let X be a topological space with topology $\mathcal{T}$. If $Y \subseteq X$, then

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology**. With this topology, Y is called a **subspace** of X

Lemma 1.15. *if $\mathcal{B}$ is a basis for the topology of X then the collection*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y

Proof. Given U open in X and given $y \in U \cap Y$, we can choose an element B of $\mathcal{B}$ s.t . $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. It follows from Lemma 1.5 that $\mathcal{B}_Y$ is a basis for the subspace topology on Y □

Lemma 1.16. *Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X*

Theorem 1.17. *if A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$*

Proof. The set $U \times V$ is the general basis element for $X \times Y$, where U, V are open in X, Y respectively. Therefore $(U \times V) \cap (A \times B)$ is the general basis element for the subspace topology on $A \times B$. Now

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

$\square$

Now let X be an ordered set in the order topology, and let Y be a subset of X. The order relation on X, when restricted to Y, makes Y into an ordered set. However the resulting order topology on Y need not be the same as the topology that Y inherits as a subspace of X

Example 1.4. Consider the subset $Y = [0, 1]$ of the real line $\mathbb{R}$ in the *subspace* topology. Given (a, b)

$$(a, b) \cap Y = \begin{cases} (a, b) \\ [0, b) \\ (a, 1] \\ Y \text{ or } \emptyset \end{cases}$$

Sets of the second and third types are not open in the larger space $\mathbb{R}$

Note that these sets form a basis for the *order* topology on Y. Thus we see that in the case of the set $Y = [0, 1]$ its subspace topology and its order topology are the same

Given an ordered set X, a subset Y of X is **convex** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Note that intervals and rays in X are convex in X

Theorem 1.18. *Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X*

Proof. Consider the ray $(a, +\infty)$ in X. If $a \in Y$ then

$$(a, +\infty) \cap Y = \{x \mid x \in Y \text{ and } x > a\}$$

this is an open ray of the ordered set Y. If $a \notin Y$, then a is either a lower bound on Y or an upper bound on Y, since Y is convex. In the former case, $(a, +\infty) \cap Y = Y$; in the latter case, it is empty

Similarly, $(-\infty, a) \cap Y$ is either an open ray of Y, or Y itself, or empty. Since the sets $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the subspace topology on Y, and since each is open in the order topology,and since each is open in the order topology, the order topology contains the subspace topology

To prove the reverse, note that any open ray of Y equals the intersection of an open ray of X with Y, so it is open in the subspace topology on Y. Since the open rays of Y are a subbasis for the order topology, this topology is contained in the subspace topology □

Exercise 1.5.1. Show that if Y is a subspace of X and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X

Proof. For every open set U of topology of X, $A \cap (Y \cap U) = A \cap U$. □

Exercise 1.5.2. Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X

Proof. Consider $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ which is convex in $\mathbb{Q}$ but not an interval or a ray □

1.6 Closed Sets and Limit Points

A subset A of a topological space X is said to be **closed** if the set $X - A$ is open

Theorem 1.19. *Let X be a topological space. Then the following conditions hold:*

1. *$\emptyset$ and X are closed*
2. *Arbitrary intersection of closed sets are closed*
3. *Finite unions of closed sets are closed*

Theorem 1.20. *let Y be a subspace of X. Then a set A is closed in Y iff it equals the intersection of a closed set of X with Y*

Proof. Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that $(X - C) \cap Y$ is open in Y. But $(X - C) \cap Y = Y - A$. Hence $Y - A$ is open in Y

Assume that A is closed in Y. Then $Y - A = U \cap Y$ for some open set U in X and $A = Y \cap (X - U)$ □

Theorem 1.21. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X*

Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A ($\bar{A}$)

Theorem 1.22. *Let Y be a subspace of X; let A be a subset of Y; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$*

Proof. Let B denote the closure of A in Y. The set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y by Theorem 1.20. We have $B \subset (\bar{A} \cap Y)$

On the other hand, $B = C \cap Y$ for some C closed in X. Then C is a closed set of X containing A. ☐

A set A **intersects** a set B if the intersection $A \cap B$ is not empty

Theorem 1.23. *Let A be a subset of the topological space X*

1. *$x \in \bar{A}$ iff every open set U containing x intersects A*

2. *Suppose the topology of X is given by a basis, then $x \in \bar{A}$ iff every basis element b containing x intersects A*

Proof. 1. We consider

$x \notin \bar{A}$ iff there exists an open set U containing x that does not intersects A

If $x \notin \bar{A}$, the set $U = X - \bar{A}$ is an open set containing x that does not intersects A, as desired. Conversely, if there exsits an open set U containing x which does not intersects A, then $X - U$ is a closed set containing A. Hence $\bar{A} \subseteq X - U$ and therefore $x \notin \bar{A}$

☐

U is an open set containing x equals U is a **neighborhood** of x

Example 1.5. Let X be the real line $\mathbb{R}$. If $A = (0, 1]$ then $\bar{A} = [0, 1]$ for every neighborhood of 0 intersects A, while every point outside $[0, 1]$ has a neighborhood disjoint from A.

If $B = \{1/n \mid n \in \mathbb{Z}_+\}$ then $\bar{B} = \{0\} \cup B$. If $C = \{0\} \cup (1, 2)$ then $\bar{C} = \{0\} \cup [1, 2]$. Also $\bar{\mathbb{Q}} = \mathbb{R}$.

If $A$is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** of A if every neighborhood of x intersects A in some point *other than x itself*. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$

Theorem 1.24. *Let A be a subset of the topological space X; let A' be the set of all limit points of A. Then*

$$\bar{A} = A \cup A'$$

Proof. By Theorem 1.23 $A' \subset \bar{A}$.

Suppose $x \in \bar{A} - A$. Then $x \in A'$ ☐

Corollary 1.25. *A subset of a topological space is closed iff it contains all its limit points*

Proof. A is closed iff $\bar{A} = A$ ☐

In the spaces $\mathbb{R}$ and $\mathbb{R}^2$ each one-point set $\{x_0\}$ is closed since every point different from x_0 has a neighborhood not intersecting $\{x_0\}$, so that $\{x_0\}$ is its own closure. But this fact is not true for arbitrary topological spaces. Consider the topology on the three-point set $\{a, b, c\}$ indicated in Figure 1. The one-point set $\{b\}$ is not closed, for its complement is not open

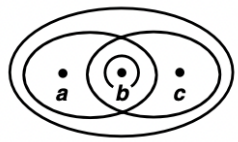


Figure 1: we

In an arbitrary topological space, one says that a sequence $x_1, x_2, \dots$ of points of the space X **converges** to the point x of X provided that, corresponding to each neighborhood U of x there is a positive integer N s.t. $x_n \in U$ for all $n \geq N$. In $\mathbb{R}$ and $\mathbb{R}^2$ a sequence cannot converge to more than one point, but in an arbitrary space, it can. In Figure 1 the sequence defined by setting $x_n = b$ converges not only to the point b but also to the point a and c.

Definition 1.26. A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of disjoint points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint

Theorem 1.27. *Every fintie point set in a Hausdorff space X is closed.*

Proof. It suffices to show that every one-point set $\{x_0\}$ is closed. □

The condition that finite point sets be closed is in fact weaker than the Hausdorff condition. For example, the real line $\mathbb{R}$ in the finite complement topology is not a Hausdorff space, but it is a space in which finite point sets are closed. The condition that finite point sets be closed is called the T_1 **axiom**

Theorem 1.28. *Let X be a space satisfying the T_1 axiom; let A be a subset of X. Then the point x is a limit point of A iff every neighborhood of x contains infinitely many points*

Proof. If x is a limit point of A and suppose some neighborhood U of x intersects A in only finitely many points. Then U also intersects $A - \{x\}$ in finitely many points; let $\{x_1, \dots, x_m\}$ be the points of $U \cap (A - \{x\})$. The set $X - \{x_1, \dots, x_m\}$ is an open set of X, then

$$U \cap (X - \{x_1, \dots, x_m\})$$

is a neighborhood of x that intersects the set $A - \{x\}$ □

Theorem 1.29. *If X is the Hausdorff space, then a sequence of points of X converges to at most one point of X*

Proof. Suppose that x_n is a sequence of points of X that converges to x. If $y \neq x$ let U and V be disjoint neighborhoods of x and y respectively. Since U contains x_n for all but finitely many values of n, the set V cannot. Therefore x_n cannot converge to y. □

If the sequence x_n of points of the Hausdorff space X converges to the point x of X, we often write $x_n \to x$ and we say that x is the **limit** of the sequence x_n

Theorem 1.30. *Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.*

Exercise 1.6.1. Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does equality hold

Proof. It equals the closure iff both endpoints are limit points of the interval, i.e. if (a,b) is not empty and for every $x \in (a,b)$ there are $s,t \in (a,b)$ such that $a < s < x < t < b$. This is equivalent to the requirement that a has no immediate successor, and b has no immediate predecessor. Otherwise, if a has an immediate successor c then $(-\infty, c)$ is an open set containing a that does not intersect (a,b) , and, similarly, if b has an immediate predecessor c then $(c, +\infty)$ is an open set containing b that does not intersect (a,b) . □

Exercise 1.6.2. Let A,B and A_α denote subsets of a space X. Prove the following

1. If $A \subset B$ then $\bar{A} \subset \bar{B}$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
3. $\overline{\bigcup A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails

Proof.

2. Suppose $x \notin \bar{A} \cup \bar{B}$. By Theorem 1.23 there is a neiborhoods U_A, U_B of x s.t. $U_A \cap A = U_B \cap B = \emptyset$. Let $U = U_A \cap U_B$. Then $U \cap (A \cup B) = \emptyset$.
3. Consider $A_n = (1/n, 2]$ for $n \in \mathbb{Z}_+$

□

Exercise 1.6.3. Let A,B and A_α denote subsets of a space X. Determine whether the following equations hold

1. $\overline{A \cap B} = \bar{A} \cap \bar{B}$
2. $\overline{\bigcap A_\alpha} = \bigcap \bar{A}_\alpha$
3. $\overline{A - B} = \bar{A} - \bar{B}$

Proof.

1. Consider $A = (1,2)$ and $B = (0,1)$ in $\mathbb{R}$. We only have $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$

3. $\overline{A-B} \supset \bar{A} - \bar{B}$. $A = (0,2), B = (0,1)$

□

Exercise 1.6.4. X is Hausdorff iff the **diagonal** $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Proof. Δ is closed in $X \times X$ iff for $x \neq y$ there is a basis $x \times y \in U \times V \subset X \times X$ where U and V are neighborhoods of x and y respectively s.t. no points $(z,z) \in U \times V$ iff any pair of of different points having disjoint neighborhoods □

1.7 Continuous Functions

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for each open subset V of Y the set $f^{-1}(V)$ is an open subset of X.

Let's note that if the topology of the range space Y is given by a basis $\mathcal{B}$, then to prove continuity of f it suffices to show that the inverse image of every *basis element* is oepn.

If the topology on Y is given by a subbasis $\mathcal{S}$, to prove continuity of f it will even suffice to show that the inverse of each *subbasis* element is open.

Example 1.6. Let's consider a function

$$f : \mathbb{R} \to \mathbb{R}$$

Now we prove that our definition implies the ϵ-δ definition

Given $x_0 \in \mathbb{R}$ and given $\epsilon > 0$ the interval $V = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set of the range space $\mathbb{R}$. Therefore, $f^{-1}(V)$ is an open set in the domain space $\mathbb{R}$. Because $x_0 \in f^{-1}(V)$, it contains some basis element (a, b) about x_0. We choose δ to be the smaller of the two numbers $x_0 - a$ and $b - x_0$. Then if $|x - x_0| < \delta$, the point x must be in (a, b), so that $f(x) \in V$ and $|f(x) - f(x_0)| < \epsilon$ as desired

Example 1.7. Let $\mathbb{R}$ denote the set of real numbers in its usual topology. Let

$$f : \mathbb{R} \to \mathbb{R}_l$$

by the identity function $f(x) = x$. Then f is not a continuous function. However

$$g : \mathbb{R}_l \to \mathbb{R}$$

is continuous

Theorem 1.31. *Let X and Y be topological spaces: let $f : X \to Y$. TFAE*

1. *f is continuous*
2. *for every $A \subseteq X$, $f(\bar{A}) \subset \overline{f(A)}$*
3. *for every closed set B of Y, the set $f^{-1}(B)$ is closed in X*

4. *for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x s.t. $f(U) \subset V$*

*If the condition 4 holds for the point x of X, we say that f is **continuous at the point** x*

Proof. $1 \to 2$. Assume f is continuous. Let $A \subseteq X$ and $x \in \bar{A}$. Let V be a neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open set of X containing x; it must intersect A in some point y. Then V intersects $f(A)$ in the point $f(y)$, so that $f(x) \in \overline{f(A)}$

$2 \to 3$. Let B be closed in Y and let $A = f^{-1}(B)$. We show that $\bar{A} = A$. We have $f(A) = f(f^{-1}(B)) \subset B$. Therefore if $x \in \bar{A}$

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$$

so that $x \inf^{-1}(B) = A$

$3 \to 1$. easy

$1 \to 4$. easy

$4 \to 1$. not hard 😊 □

let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a **homeomorphism**

Suppose that $f : X \to Y$ is an injective continuous map, where X and Y are topological spaces. Let Z be the image set $f(X)$, considered as a subspace of Y; then the function $f' : X \to Z$ obtained by restricting the range of f is bijectiive. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological embedding** or simpy an **embedding** of X in Y

Example 1.8. A bijectiive function $f : X \to Y$ can be continuous without being a homeomorphism. One such function is the identity map $g : \mathbb{R}_l \to \mathbb{R}$□ Another is the following:

Let S^1 denote the **unit circle**,

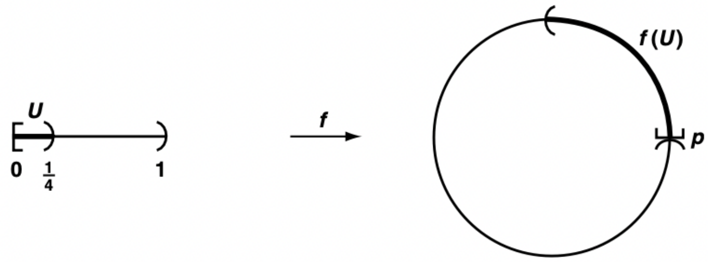
$$S^1 = \{x \times y \mid x^2 + y^2 = 1\}$$

considered as a subspace of the plane $\mathbb{R}^2$ and let

$$f : [0, 1) \to S^1$$

be the map defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. f is continuous but not f^{-1}. The image under f of the open set $U = [0, \frac{1}{4})$ of the domain is not open in S^1, for the point $p = f(0)$ lies in no open set V of $\mathbb{R}^2$ s.t. $V \cap S^1 \subset f(U)$

Theorem 1.32 (Rules for constructing continuous functions)**.** *Let X, Y and Z be topological spaces*



1. $($*Constant function*$)$ *if* $f : X \to Y$ *maps all of* X *into the single point* y_0 *of* Y*, then* f *is continuous*
2. $($*Inclusion*$)$ *If* A *is a subspace of* X*, the inclusion function* $j : A \to X$ *is continuous*
3. $($*Composites*$)$ *If* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then the map* $g \circ f : X \to Z$ *is continuous*
4. $($*Restricting the domain*$)$ *if* $f : X \to Y$ *is continuous, and if* A *is a subspace of* X*, then the restricted function* $f|A : A \to Y$ *is continuous*
5. $($*Restricting or expanding the range*$)$ *Let* $f : X \to Y$ *be continuous. If* Z *is a subspace of* Y *containing the image set* $f(X)$*, then the function* $g : X \to Z$ *obtained by restricting the range of* f *is continuous. If* Z *is a space having* Y *as a subspace, then the function* $h : X \to Z$ *obtained by expanding the range of* f *is continuous*
6. $($*Local formulation of continuity*$)$ *The map* $f : X \to Y$ *is continuous if* X *can be written as the union of open sets* U_α *s.t.* $f|U_\alpha$ *is continuous for each* α

Proof. 1. Let V be open in Y, then $f^{-1}(V)$ equals $\emptyset$ or X

□

Theorem 1.33 (The pasting lemma)**.** *Let* $X = A \cup B$*, where* A *and* B *are closed in* X*. Let* $f : A \to Y$ *and* $g : B \to Y$ *be continuous. If* $f(x) = g(x)$ *for every* $A \cap B$ *then* f *and* g *combine to give a continuous function* $h : X \to Y$*, defined by setting* $h(x) = f(x)$ *if* $x \in A$ *and* $h(x) = g(x)$ *if* $x \in B$

The open set case of the pasting lemma is just the local formulation of continuity

Theorem 1.34 (Maps into products)**.** *Let* $f : A \to X \times Y$ *be given by the equation*

$$f(a) = (f_1(a), f_2(a))$$

Then f *is continuous iff the functions*

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

are continuous

The maps f_1 *and* f_2 *are called the* ***coordinate functions***

Proof. First note that π_1, π_2 are continuous. For $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ and these sets are open if U and V are open. Note that for each $a \in A$

$$f_1(a) = \pi_1(f(a)) \quad \text{and} \quad f_2(a) = \pi_2(f(a))$$

If f is continuous, then f_1, f_2 are continuous

Conversely, we show that for each basis element $U \times V$ for the topology $X \times Y$ its inverse image $f^{-1}(U \times V)$ is open. $a \in f^{-1}(U \times V)$ iff $f(a) \in (U \times V)$ iff $f_1(a) \in U$ and $f_2(a) \in V$. Therefore

$$f^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V)$$

□

Exercise 1.7.1. Let $F : X \times Y \to Z$. We say that F is **continuous in each variable separately** if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

Exercise 1.7.2. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0 \\ 0 \end{cases}$$

1. Show that F is continuous in each variable separately
2. Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$
3. Show that F is not continuous

1.8 The Product Topology

Definition 1.35. Let J be an index set. Given a set X, we define J**-tuple** of elements of X to be a function $\mathbf{x} : J \to X$. If α is an element of j, we often denote the value of $\mathbf{x}$ at α by x_α; we call it the αth **coordinate** of $\mathbf{x}$. And we often denote the function $\mathbf{x}$ itself by the symbol

$$(x_\alpha)_{\alpha \in J}$$

We denote the set of all J-tuples of elements of X by X^J

Definition 1.36. Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. The **cartesian product** of this indexed family, denoted by

$$\prod_{\alpha \in J} A_\alpha$$

is defined to be the set of all J-tuples $(x_\alpha)_{\alpha\in J}$ of elements of X s.t. $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$\mathbf{x} : J \to \bigcup_{\alpha\in J} A_\alpha$$

s.t. $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$

Definition 1.37. Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space

$$\prod_{\alpha\in J} X_\alpha$$

the collection of all sets of the form

$$\prod_{\alpha\in J} U_\alpha$$

where U_α is open in X_α, for each $\alpha \in J$. The topology generated by this basis is called the **box topology**

Now we generalize the subbasis formulation of the definition. Let

$$\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$$

be the function assigning to each element of the product space its βth coordinate

$$\pi_\beta((x_\alpha)_{\alpha\in J}) = x_\beta$$

it is called the **projection mapping** associated with the index β

Definition 1.38. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ open in } X_\beta\}$$

and let $\mathcal{S}$ denote the union of these collections

$$\mathcal{S} = \bigcup_{\beta\in J} \mathcal{S}_\beta$$

The topology generated by the subbasis $\mathcal{S}$ is called the **product topology**. In this topology $\prod_{\alpha\in J} X_\alpha$ is called a **product space**

To compare these topologies, we consider the basis $\mathcal{B}$ that $\mathcal{S}$ generates. The collection $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$. If we intersect elements belonging to the same one of the sets $\mathcal{S}_\beta$ we do not get anything new, because

$$\pi_\beta^{-1}(U_\beta) \cap \pi_\beta^{-1}(V_\beta) = \pi_\beta^{-1}(U_\beta \cap V_\beta)$$

We get something new only when we intersect elements from different sets $\mathcal{S}_\beta$. Thus the typical element of the basis $\mathcal{B}$ can be described as follows: let $\beta_1, \dots, \beta_n$ be a finite set of distinct indices from the index set J, and let U_{β_i} be an open set in X_{β_i} for $i = 1, \dots, n$. Then

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

is the typical element of $\mathcal{B}$

Now a point $\mathbf{x} = (x_\alpha)$ is in B iff its β_1th coordinate is in U_{β_1}, its β_2th coordinate is in U_{β_2}, and so on. As a result, we can write B as the product

$$B = \prod_{\alpha \in J} U_\alpha$$

where U_α denotes the entire space X_α if $\alpha \neq \beta_1, \dots, \beta_n$

Theorem 1.39 (Comparison of the box and product topologies)**.** *The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form U_α, where U_α is open in U_α for each α and U_α equals X_α except for finitely many values of α*

> *Whenever we consider the product X_α, we shall assume it is given the product topology unless we specifically state otherwise.*

Theorem 1.40. *Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology on $\prod_{\alpha \in J} X_\alpha$

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$

Theorem 1.41. *Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ is both products are given the box topology or product topology*

Theorem 1.42. *If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both the box and product topologies*

Theorem 1.43. *Let $\{X_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subseteq X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then*

$$\prod \bar{A}_\alpha = \overline{\prod A_\alpha}$$

Proof. Let $\mathbf{x} = (x_\alpha)$ be a point of $\prod \bar{A}_\alpha$; we show that $\mathbf{x} \in \overline{\prod A_\alpha}$. Let $U = \prod U_\alpha$ be a basis element for either the box or product topology that contains $\mathbf{x}$. Since $x_\alpha \in \bar{A}_\alpha$, we can choose a point $y_\alpha \in U_\alpha \cap A_\alpha$. Then $\mathbf{y} = (y_\alpha)$ belongs to both U and $\prod A_\alpha$. Since U is arbitrary, it follows that $\mathbf{x} \in \overline{\prod A_\alpha}$

Conversely, suppose $\mathbf{x} = (x_\alpha)$ lies in the closure of $\prod A_\alpha$, in either topology. We show that for any given index β, we have $x_\beta \in \bar{A}_\beta$. Let V_β be an arbitrary open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ in either topology, it contains a point $\mathbf{y} = (y_\alpha)$ of $\prod A_\alpha$. Then y_β belongs to $V_\beta \cap A_\beta$. It follows that $x_\beta \in \bar{A}_\beta$ □

Theorem 1.44. *Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous iff each function f_α is continuous

Proof. $\Rightarrow$ composition of continuous functions is continuous

$\Leftarrow$ Suppose that each coordinate function f_α is continuous. To prove that f is continuous, it suffices to prove that the inverse image under f of each subbasis element is open in A. A typical subbasis element for the product topology on $\prod X_\alpha$ is a set of the form $\pi_\beta^{-1}(U_\beta)$ where β is some index and U_β is open in X_β. now

$$f^{-1}(\pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta)$$

because $f_\beta = \pi_\beta \circ f$. Since f_β is continuous, this set is open in A □

Example 1.9. Consider $\mathbb{R}^\omega$ and define $f : \mathbb{R} \to \mathbb{R}^\omega$

$$f(t) = (t, t, \dots)$$

f is continuous if $\mathbb{R}^\omega$ is given the box topology. Consider the basis element

$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$$

We assert that $f^{-1}(B)$ is not open in $\mathbb{R}$. $f^{-1}(B) = \{0\}$

Exercise 1.8.1. let $\mathbf{x}_1, \mathbf{x}_2, \dots$ be a sequence of the points of the products space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ iff the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \dots$ converges to $\pi_\alpha(\mathbf{x})$ for each α

Proof. Given a neighborhood $U = \prod U_\alpha$ of $\mathbf{x}$, for each α, we have N_α s.t. $\pi_\alpha(x_n) \in U_\alpha$ for all $n \geq N_\alpha$. If $U_\alpha = X_\alpha$ we take $N_\alpha = 1$. Hence in product topology we have only finitely many $N_\alpha > 1$ and we can take max. This fails in box topology as it might not have max □

Exercise 1.8.2. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero", that is, all sequences $(x_1, x_2, \dots)$ s.t. $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? justify your answer

Proof. $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$ for both

If $\mathbb{R}^\infty$ is given the product topology, given a point $\mathbf{x} \in \mathbb{R}^\omega$ and a neighborhood $U = \bigcup_i U_i$ where U_i is a proper open subset of $\mathbb{R}$ for finitely many $i \in \omega$. Choose $y_i \in U_i$ and $y_j = 0$ if $U_j = \mathbb{R}$. Then $\mathbf{y} \in \mathbb{R}^\infty \cap U$. Hence $x \in \overline{\mathbb{R}^\infty}$ $\square$

Exercise 1.8.3. Given sequences $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \dots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots)$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology

Proof. both box and product $\square$

1.9 The Metric Topology

Definition 1.45. A **metric** on a set X is a function

$$d : X \times X \to R$$

having the following properties

1. $d(x, y) \geq 0$ for all $x, y \in X$; equality holds iff $x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Given a metric d on X, the number $d(x, y)$ is often called the **distance** between x and y in the metric d. Given $\epsilon > 0$ consider the set

$$B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$$

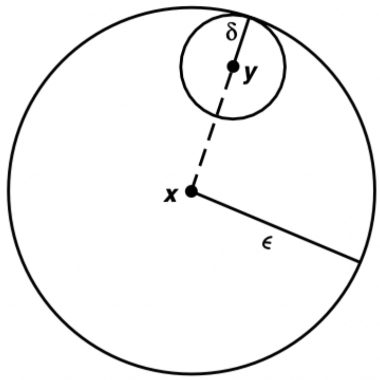
of all points y whose distance from x is less than ϵ. It is called the **ϵ-ball centered at x**

Definition 1.46. If d is a metric on the set X, then the collection of all ϵ-balls $B_d(x, \epsilon)$ for $x \in X$ and $\epsilon > 0$ is a basis for a topology on X, called the **metric topology** induced by d

Check the second condition.

If $y \in B(x,\epsilon)$ then there is a basis element $B(y,\delta)$ *centered* at y that is contained in $B(x,\epsilon)$. Define δ to be $\epsilon - d(x,y)$. Then $B(y,\delta) \subset B(x,\epsilon)$, for if $z \in B(y,\delta)$ then $d(y,z) < \epsilon - d(x,y)$, from which we conclude that

$$d(x,z) \le d(x,y) + d(y,z) < \epsilon$$



Let B_1 and B_2 be two basis element and let $y \in B_1 \cap B_2$.We have just shown that we can choose positive numbers δ_1 and δ_2 so that $B(y,\delta_1) \subset B_1$ and $B(y,\delta_2) \subset B_2$. Let $\delta = \min\{\delta_1,\delta_2\}$ we conclude $B(y,\delta) \subset B_1 \cap B_2$. Hence

> *A set U is open in the metric topology induced by d iff for each $y \in U$ there is a $\delta > 0$ s.t. $B_d(y,\delta) \subset U$*

Definition 1.47. If X is a topological space, X is said to be **metrizable** if there exists a metric d on the set X that induces the topology of X. A **metric space** is a metrizable space together with a specific metric d that gives the topology of X

Definition 1.48. Let X be a metric space with metric d. A subset A of X is said to be **bounded** if there is some number M s.t.

$$d(a_1,a_2) \le M$$

for every pair a_1, a_2 of points of A. If A is bounded and nonempty, the **diameter** of A is defined to be the number

$$\operatorname{diam} A = \sup\{d(a_1,a_2) \mid a_1,a_2 \in A\}$$

Theorem 1.49. *Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation*

$$\bar{d}(x,y) = \min\{d(x,y),1\}$$

Then $\bar{d}$ is a metric that induces the same topology as d.

The metric $\bar{d}$ is called the **standard bounded metric** corresponding to d.

Proof. Check

$$\bar{d}(x,z) \leq \bar{d}(x,y) + \bar{d}(y,z)$$

If both $d(x,y)$ and $d(y,z)$ are <1. Then

$$d(x,z) \leq d(x,y) + d(y,z) = \bar{d}(x,y) + \bar{d}(y,z)$$

Note that in any metric space, the collection of ϵ-balls with $\epsilon < 1$ forms a basis for the metric topology □

Definition 1.50. Given $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, we define the **norm** of $\mathbf{x}$ by

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

and we define the **euclidean metric** d on $\mathbb{R}^n$ by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

We define the **square metric** ρ by

$$\rho(\mathbf{x},\mathbf{y}) = \max\{|x_1 - y_1|, \ldots, \left|x_y, y_n\right|\}$$

Check the third condition for ρ. for each $i \in \mathbb{N}_+$

$$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$$

then

$$|x_i - z_i| \leq \rho(\mathbf{x},\mathbf{y}) + \rho(\mathbf{y},\mathbf{z})$$

On the real line $\mathbb{R}$, these two metrics coincide with the standard metric for $\mathbb{R}$

Lemma 1.51. *Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ iff for each $x \in X$ and each $\epsilon > 0$ there exists a $\delta > 0$ s.t.*

$$B_{d'}(x,\delta) \subset B_d(x,\epsilon)$$

Theorem 1.52. *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$*

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be two points of $\mathbb{R}^n$. We have

$$\rho(\mathbf{x},\mathbf{y}) \leq d(\mathbf{x},\mathbf{y}) \leq \sqrt{n}\rho(\mathbf{x},\mathbf{y})$$

The first inequality shows that

$$B_d(\mathbf{x},\epsilon) \subset B_\rho(\mathbf{x},\epsilon)$$

for all $\mathbf{x}$ and ϵ. Similarly

$$B_\rho(\mathbf{x}, \epsilon/\sqrt{n}) \subset B_d(\mathbf{x}, \epsilon)$$

It follows from the preceding lemma that the two metric topologies are the same

Next we show that the product topology is the same as that given by the metric ρ. First let

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n)$$

be a basis element for the product topology, and let $\mathbf{x} = (x_1, \ldots, x_n) \in B$. For each i there is an ϵ_i s.t.

$$(x_i - \epsilon_i, x_i + \epsilon_i) \subset (a_i, b_i)$$

choose $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$. Then $B_\rho(\mathbf{x}, \epsilon) \subset B$. □

Now we consider the infinite cartesian product $\mathbb{R}^\omega$. It is natural to try to generalize the metrics d and ρ to this space. For instance, one can attempt to define a metric d on $\mathbb{R}^\omega$ by the equation

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

But this equation does not always make sense, for the series in question need not converge. (This equation does define a metric on a certain important subset of $\mathbb{R}^\omega$, however; see the exercises.)

Similarly, one can attempt to generalize the square metric ρ to $\mathbb{R}^\omega$ by defining

$$\rho(x, y) = \sup\{|x_n - y_n|\}$$

Again, this formula does not always make sense. If however we replace the usual metric $d(x, y) = |x - y|$ on $\mathbb{R}$ by its bounded counterpart $\bar{d}(x, y) = \min\{|x - y|, 1\}$, then this definition does make sense; it gives a metric on $\mathbb{R}^\omega$ called the *uniform metric*

Definition 1.53. Given an index set J, and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let's define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by

$$\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. It is easy to check that $\bar{\rho}$ is indeed a metric; it is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology**

Theorem 1.54. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different is J is infinite*

Proof. Suppose that we are given a point $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and a product topology basis element $\prod U_\alpha$. Let $\alpha_1, \ldots, \alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$. Then for each i, choose $\epsilon_i > 0$ so that $B_{\bar{d}}(x_{\alpha_i}, \epsilon_i) \subset U_{\alpha_i}$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, then $B_{\bar{d}}(\mathbf{x}, \epsilon) \subset \prod U_\alpha$. □

Theorem 1.55. *Let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^\omega$, define*

$$D(\boldsymbol{x}, \boldsymbol{y}) = \sup\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$

Proof. First let U be open in the metric topology and let $\mathbf{x} \in U$; Choose an ϵ-ball $B_D(\mathbf{x}, \epsilon) \subset U$. Then choose N large enough that $1/N < \epsilon$. Let V be the basis element for the product topology

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

We assert that $V \subset B_D(\mathbf{x}, \epsilon)$. Given any $\mathbf{y} \in \mathbb{R}^\omega$

$$\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N} \qquad \text{for } i \geq N$$

therefore

$$D(\mathbf{x}, \mathbf{y}) \leq \max\left\{\frac{\bar{d}(x_1, y_1)}{1}, \ldots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\right\}$$

If $\mathbf{y} \in V$ then $D(\mathbf{x}, \mathbf{y}) < \epsilon$, so that $V \subset B_D(\mathbf{x}, \epsilon)$

Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

for the product topology, where U_i is open in $\mathbb{R}$ in $\mathbb{R}$ for $i = \alpha_1, \ldots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices. Given $\mathbf{x} \in U$, consider an interval $(x_i - \epsilon_i, x_i + \epsilon_i) \subset U_i$ for $i = \alpha_1, \ldots, \alpha_n$; choose each $\epsilon_i \leq 1$, then define

$$\epsilon = \min\{\epsilon/i \mid i = \alpha_1, \ldots, \alpha_n\}$$

we assert that

$$\mathbf{x} \in B_D(\mathbf{x}, \epsilon) \subset U$$

let $\mathbf{y}$ be a point of $B_D(\mathbf{x}, \epsilon)$. then for all i

$$\frac{\bar{d}(x_i, y_i)}{i} \leq D(\mathbf{x}, \mathbf{y}) < \epsilon$$

Now if $i = \alpha_1, \ldots, \alpha_n$ then $\epsilon \leq \epsilon_i / i$ so that $\bar{d}(x_i, y_i) < \epsilon_i \leq 1$. It follows that $|x_i - y_i| < \epsilon_i$. Therefore $\mathbf{y} \in \prod U_i$ □

Exercise 1.9.1. Let X be a metric space with metric d

1. $d : X \times X \to \mathbb{R}$ is continuous

2. Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X

Proof. 1. Prove that for any U open in $\mathbb{R}$ and $(x, y) \in d^{-1}(U)$ there is a basis element B of $X \times X$ s.t. $(x, y) \in B \subset d^{-1}(U)$. Suppose $d(x, y) = a$. There is a ϵ s.t. $(a - \epsilon, a + \epsilon) \subset U$. We take $B = B_d(x, \epsilon/2) \times B_d(y, \epsilon/2)$. for any $(x, y) \in B$, $d(x, y) \in (a - \epsilon, a + \epsilon)$

2. for every fixed $x \in X'$, $d_x(y) : X' \to \mathbb{R}, y \mapsto d(x, y)$ is continuous. Therefore every $B_d(x, r) = d_x^{-1}((-\infty, r))$ must be open in X'

□

Exercise 1.9.2. Consider the product, uniform and box topologies on $\mathbb{R}^\omega$

1. in which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous

$$f(t) = (t, 2t, 3t, \dots)$$
$$g(t) = (t, t, t, \dots)$$
$$h(t) = (t, \frac{1}{2}t, \frac{1}{3}t, \dots)$$

2. in which topologies do the following sequences converge

$$\begin{array}{ll}
\mathbf{w}_1 = (1, 1, 1, 1, \dots) & \mathbf{x}_1 = (1, 1, 1, 1, \dots) \\
\mathbf{w}_2 = (0, 2, 2, 2, \dots) & \mathbf{x}_2 = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots) \\
\mathbf{w}_3 = (0, 0, 3, 3, \dots) & \mathbf{x}_3 = (0, 0, \frac{1}{3}, \frac{1}{3}) \\
\dots & \dots \\
\mathbf{y}_1 = (1, 0, 0, 0, \dots) & \mathbf{z}_1 = (1, 1, 0, 0, \dots) \\
\mathbf{y}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots) & \mathbf{z}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots) \\
\mathbf{y}_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \dots) & \mathbf{z}_3 = (\frac{1}{3}, \frac{1}{3}, 0, 0, \dots)
\end{array}$$

Proof. 1. For box topology, consider open set

$$B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$$

$f^{-1}(B) = g^{-1}(B) = h^{-1}(B) = \{0\}$ which is not open.

For uniform topology. First, $f^{-1}(B_{\bar{\rho}}(\mathbf{0}, 1)) \subset f^{-1}(\prod_{n \in \mathbb{Z}_+}(-1, 1)) = \{0\}$. At the same time, for $k(t) = (a_1 t, a_2 t, \dots)$ equals g or h and $k(t) \in B_{\bar{\rho}}(\mathbf{x}, \epsilon)$, then for every $n \in \mathbb{Z}_+$, $|x_n - a_n t| \le \sup_{n \in \mathbb{Z}_+} |x_n - a_n t| = \delta < \epsilon$. And for $|z| < \frac{\epsilon - \delta}{2}$

$$|x_n - a_n(t+z)| \le |x_n - a_n t| + a_n|z| < \delta + \frac{\epsilon - \delta}{2} = \frac{\epsilon + \delta}{2} < \epsilon$$

Hence $k((t - \frac{\epsilon-\delta}{2}, t + \frac{\epsilon-\delta}{2})) \subset B_{\bar{\rho}}(\mathbf{x}, \epsilon)$ and $k^{-1}(B_{\bar{\rho}}(\mathbf{x}, \epsilon))$ is open.

Product topology. all three

2. If a sequence converges to a point, and we change the topology to a coarser one, then the sequence still converges to the point. Therefore for each sequence we may specify the finest topology out of the three given topologies in which it converges to some point.

   For $\{\mathbf{w}_n\}$ it is the product topology, for $\{\mathbf{x}_n\}$ and $\{y_n\}$ it is the uniform topology and for $\{\mathbf{z}_n\}$ it is the product topology

   $\{\mathbf{x}_n\}$ converges to $\mathbf{0}$ in the uniform topology, as for $n > \frac{1}{\epsilon}$, $\mathbf{x}_n \in B_{\bar{\rho}}(\mathbf{0}, \epsilon)$

□