Category Theory

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1 Categories

1.1 Examples of categories

Definition 1.1. A **functor**

$$F : \mathbf{C} \to \mathbf{C}$$

between categories **C** and **D** is a mapping of objects to objects and arrows to arrows, in such a way that

1. $F(f : A \to B) = F(f) : F(A) \to F(B)$
2. $F(1_A) = 1_{F(A)}$
3. $F(g \circ f) = F(g) \circ F(f)$

1.2 Free categories

The "Kleene closure" of A is defined to be the set

$$A^* = \{\text{words over } A\}$$

Also

$$i : A \to A^*$$

defined by $i(a) = a$ and called the "intersection of generators"

A monoid M is **freely generated** by a subset A of M if the following conditions hold:

1. Every element $m \in M$ can be written as a product of elements of A
2. No "nontrivial" relations hold in M, that is, if $a_1 \dots a_j = a_1' \dots a_k'$,then this is required by the axioms for monoids

Every monoid N has an underlying set $|N|$, and every monoid homomorphism $f : N \to M$ has an underlying function $|f| : |N| \to |M|$. The free monoid $M(A)$ on a set A is by definition "the" monoid with the following UMP

Universal Mapping Property of $M(A)$

There is a function $i : A \to |M(A)|$, and given any monoid N and any function $f : A \to |N|$, there is a **unique** monoid homomorphism $\overline{f} : M(A) \to N$ s.t. $\left|\overline{f}\right| \circ i = f$

in **Mon**

$$\begin{tikzcd} M(A) \arrow[r, dotted, "\overline{f}"] & N \end{tikzcd}$$

in **Sets**

$$\begin{tikzcd} {|M(A)|} \arrow[r, "|\overline{f}|"] & {|N|} \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 1.2. *A^* has the UMP of the free monoid on A*

Proof. Given $f : A \to |N|$, define $\overline{f} : A^* \to N$ by

$$\overline{f}(-) = u_N, \quad \text{the unit of } N$$
$$\overline{f}(a_1 \dots a_i) = f(a_1) \cdot_N \cdots \cdot_N f(a_i)$$

□

2 Abstract structures

2.1 Initial and terminal objects

Example 2.1. A **Boolean algebra** is a poset B equipped with distinguished elements 0,1, binary operations $a \vee b$ of join and $a \wedge b$ of meet, and a unary operation $\neg b$ of complementation. These are required to satisfy the conditions

$$
\begin{array}{rcl}
 & & 0 \leq a \\
 & & a \leq 1 \\
a \leq c \quad \text{and} \quad b \leq c & \text{iff} & a \vee b \leq c \\
c \leq a \quad \text{and} \quad c \leq b & \text{iff} & c \leq a \wedge b \\
a \leq \neg b & \text{iff} & a \wedge b = 0 \\
 & & \neg\neg a = a
\end{array}
$$

2 $= \{0, 1\}$ is an initial elements of **BA**. **BA** has as arrows the Boolean homomorphisms that $h(0) = 0, h(a \vee b) = h(a) \vee h(b)$, etc.

2.2 Products

Definition 2.1. In any category **C**, a **product diagram** for the objects A and B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP:
Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique $u : X \to P$ making the diagram

$$
\begin{tikzcd}
 & X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

2.3 Categories with products

Let **C** be a category that has a product diagram for every pair of objects. Suppose we have objects and arrows

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}
$$

with indicated products. Then we write

$$f \times f' : A \times A' \to B \times B$$

for $f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle$

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times f'"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}
$$

In this way, if we choose a product for each pair of objects, we get a functor

$$\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

$$
\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times f'"] \arrow[dd, dashed, bend right] & A' \arrow[d, "f'"] \\
B \arrow[d, "g"'] & B \times B' \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, dashed, "g \times g'"] & B' \arrow[d, "g'"] \\
C & C \times C' \arrow[l, "o_1"'] \arrow[r, "o_2"] & C'
\end{tikzcd}
$$

$(g \circ f) \times (g' \circ f') = (f \times f') \circ (g \times g')$

To prove

$$(A \times B) \times C \cong A \times (B \times C)$$

Consider

$$
\begin{tikzcd}
 & A \times (B \times C) \arrow[rr] \arrow[ddl] & & B \times C \arrow[dll] \arrow[ddr] & \\
 & B & & & \\
A & A \times B \arrow[l] \arrow[u] \arrow[dr, shift left=0] \arrow[drr] & & (A \times B) \times C \arrow[ll] \arrow[r] \arrow[uull, dashed, "g"'] \arrow[uu, dashed] & C \\
 & & & B & \\
 & A \times (B \times C) \arrow[rr] \arrow[uul] \arrow[uu, dashed] \arrow[uurr, dashed, "f"] & & B \times C \arrow[u] \arrow[uur] &
\end{tikzcd}
$$

Given no objects, there is an object 1 with no maps, and give nany other object X and no maps, there is a unique arrow

$$! : X \to 1$$

Definition 2.2. A category **C** is said to **have all finite products**, if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category **C has all (small) products** if every set of objects in **C** has a product

2.4 Hom-sets

In this section, we assume that all categories are locally small

Given any objects A and B in category **C**,we write

$$\mathrm{Hom}(A, B) = \{f \in \mathbf{C} \mid f : A \to B\}$$

and call such a set of arrows a **Hom-set**

Note that any arrow $g : B \to B'$ in **C** induces a function

$$\begin{gathered}\mathrm{Hom}(A, g) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B') \\ (f : A \to B) \mapsto (g \circ f : A \to B \to B')\end{gathered}$$

Let's show that this determines a functor

$$\mathrm{Hom}(A, -) : \mathbf{C} \to \mathbf{Sets}$$

called the (covariant) **representable functor** of A. We need to show that

$$\mathrm{Hom}(A, 1_X) = 1_{\mathrm{Hom}(A,X)}$$

and that

$$\mathrm{Hom}(A, g \circ f) = \mathrm{Hom}(A, g) \circ \mathrm{Hom}(A, f)$$

For any object P, a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$ determine an element (p_1, p_2) of the set

$$\mathrm{Hom}(P, A) \times \mathrm{Hom}(P, B)$$

Now given any arrow

$$x : X \to P$$

composing with p_1 and p_2 gives a pair of arrows $x_1 = p_1 \circ x : X \to A$ and $x_2 = p_2 \circ x : X \to B$

In this way, we have a function

$$\theta_X = (\mathrm{Hom}(X, p_1), \mathrm{Hom}(X, p_2)) : \mathrm{Hom}(X, P) \to \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

defined by

$$\theta_X(x) = (x_1, x_2)$$

Proposition 2.3. *A diagram of the form*

$$A \xleftarrow[p_1]{} P \xrightarrow[p_2]{} B$$

is a product for A and B iff for every object X, the canonical function θ_X is an isomorphism

$$\theta_X : \mathrm{Hom}(X, P) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

Proof. Note that we are talking about isomorphism on the set □

Definition 2.4. Let **C**, **D** be categories with binary products. A functor F : **C** $\to$ **D** is said to **preserve binary products** if it takes every product diagram

$$A \xleftarrow[p_1]{} A \times B \xrightarrow[p_2]{} B$$

to a product diagram

$$FA \xleftarrow[Fp_1]{} F(A \times B) \xrightarrow[Fp_2]{} FB$$

F preserves products just if

$$F(A \times B) \cong FA \times FB$$

Corollary 2.5. *For any object X in a category **C** with products, the (covariant) representable functor*

$$\mathrm{Hom}_{\mathbf{C}}(X, -) : \mathbf{C} \to \mathbf{Sets}$$

preserves products

3 Duality

3.1 Coproducts

Definition 3.1. A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a coproduct of A, B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ with $u \circ q_i = z_i$

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dotted, "u"] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"] \end{tikzcd}$$

written as $A + B$

In **Sets**, every finite set A is a coproduct

$$A \cong 1 + 1 + \cdots + 1 \quad (n\text{-times})$$

Example 3.1. If $M(A)$ and $M(B)$ are free monoids on sets A and B, then in **Mon** we can construct their coproduct as

$$M(A) + M(B) \cong M(A + B)$$

$$\begin{tikzcd} & N & \\ M(A) \arrow[ur] \arrow[r] & M(A+B) \arrow[u, dotted] & M(B) \arrow[ul] \arrow[l] \\ A \arrow[u, "\eta_A"] \arrow[r] & A+B \arrow[u, "\eta_{A+B}"] & B \arrow[u, "\eta_B"'] \arrow[l] \end{tikzcd}$$

Here we are working in two different categories. Half below is in **Sets**, the other is **Mon**

Product of two powerset Boolean algebras $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is also a powerset

$$\mathcal{P}(A) \times \mathcal{P}(B) \cong \mathcal{P}(A + B)$$

Example 3.2. Two monoids $M(|A| + |B|)$ is strings over the disjoint union $|A| + |B|$ of the underlying sets. That is, the elements of A and B and the equivalence relation $v \sim w$ is the least one containing all instances of the following equations

$$\begin{aligned}(\dots xu_A y \dots) &= (\dots xy \dots)\\ (\dots xu_B y \dots) &= (\dots xy \dots)\\ (\dots aa' \dots) &= (\dots a \cdot_A a' \dots)\\ (\dots bb' \dots) &= (\dots b \cdot_B b' \dots)\end{aligned}$$

The unit is the equivalence class $[-]$ of the empty word. Multiplication is

$$[x \dots y] \cdot [x' \dots y'] = [x \dots yx' \dots y']$$

The coproduct injections $i_A : A \to A + B$ and $i_B : B \to A + B$ are

$$i_A(a) = [a], \quad i_B(b) = [b]$$

Given any homomorphisms $f : A \to M$ and $g : B \to M$ into a monoid, the unique homomorphism

$$[f, g] : A + B \to M$$

is defined by first extending the function $[|f|, |g|] : |A| + |B| \to |M|$ to one $[f, g]'$ on the free monoid $M(|A| + |B|)$

$$|A| + |B| \xrightarrow{[|f|+|g|]} |M|$$

$$\begin{tikzcd} M(|A| + |B|) \arrow[r, "{[f,g]'}"] \arrow[d, two heads] & M \\ M(|A| + |B|)/\sim \arrow[ur, dotted, "{[f,g]}"'] & \end{tikzcd}$$

If $v \sim w$ in $M(|A| + |B|)/\sim$ then $[f, g]'(v) = [f, g]'(w)$. Thus $[f, g]'$ extends to the quotient to yield the desired map $[f, g] : M(|A| + |B|)/\sim \to M$

This construction also works to give coproducts in **Groups**, where it is called the **free product** of A and B and written as $A \oplus B$.

Proposition 3.2. *In the category **Ab** of abelian groups, there is a canonical isomorphism between the binary coproduct and product*

$$A + B \cong A \times B$$

Proof. Take $1_A : A \to A$ and $O_B : A \to B$. we get

$$\theta = [\langle 1_A, 0_B \rangle, \langle 0_A, 1_B \rangle] : A + B \to A \times B$$

Then given any $(a, b) \in A + B$, we have

$$\begin{aligned}
\theta(a, b) &= [\langle 1_A, 0_B \rangle, \langle 0_A, 1_B \rangle](a, b) \\
&= \langle 1_A, 0_B \rangle(a) + \langle 0_A, 1_B \rangle(b) \\
&= (1_A(a), 0_B(A)) + (0_A(b), 1_B(b)) \\
&= (a, 0_B) + (0_A, b) \\
&= (a + 0_A, 0_B + b) \\
&= (a, b)
\end{aligned}$$

□

Proposition 3.3. *Coproducts are unique up to isomorphism*

3.2 Equalizers

Definition 3.4. In any category **C**, given parallel arrows

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

an **equalizer** of f and g consists of an object E and an arrow $e : E \to A$, universal s.t.

$$f \circ e = g \circ e$$

That is, given any $z : Z \to A$ with $f \circ z = g \circ z$, there is a **unique** $u : Z \to E$ with $e \circ u = z$, all as in the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dotted, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Example 3.3. Suppose we have the functions $f, g : \mathbb{R}^2 \rightrightarrows \mathbb{R}$, where

$$\begin{aligned}
f(x, y) &= x^2 + y^2 \\
g(x, y) &= 1
\end{aligned}$$

and we take the equalizer, say in **Top**. This is the subspace

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \hookrightarrow \mathbb{R}^2$$

For, given any "generalized element" $z : Z \to \mathbb{R}^2$ we get a pair of such "elements" $z_1, z_2 : Z \to \mathbb{R}$ just by composing with the two projections, $z = \langle z_1, z_2 \rangle$ and for these we then have

$$\begin{aligned} f(z) = g(z) \quad &\text{iff} \quad z_1^2 + z_2^2 = 1 \\ &\text{iff} \quad \langle z_1, z_2 \rangle = z \in S \end{aligned}$$

where the last line really means that there is a factorization $z = \overline{z} \circ i$ of z through the inclusion $i : S \hookrightarrow \mathbb{R}^2$, as indicated in the following diagram

$$\begin{tikzcd} S \arrow[r, hook, "i"] & \mathbb{R}^2 \arrow[r, shift left, "x^2+y^2"] \arrow[r, shift right, "1"'] & \mathbb{R} \\ Z \arrow[u, dotted, "\overline{z}"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Since the inclusion i is monic, such a factorization, if it exists, is necessarily unique, and thus $S \hookrightarrow \mathbb{R}^2$ is indeed the equalizer of f and g

Example 3.4. In **Sets**, given any functions $f, g : A \rightrightarrows B$, their equalizer is the inclusion into A of the equationally defined subset

$$\{x \in A \mid f(x) = g(x)\} \hookrightarrow A$$

Let

$$2 = \{\top, \bot\}$$

Then consider the **characteristic function**

$$\chi_U : A \to 2$$

defined for $x \in A$ by

$$\chi_U(x) = \begin{cases} \top & x \in U \\ \bot & x \notin U \end{cases}$$

So the following is an equalizer

$$U \longrightarrow A \underset{\chi_U}{\overset{\top!}{\rightrightarrows}} 2$$

where $\top! = \top \circ ! : U \xrightarrow{!} 1 \xrightarrow{\top} 2$

Moreover, for every function,

$$\varphi : A \to 2$$

we can form the variety

$$V_\varphi = \{x \in A \mid \varphi(x) = \top\}$$

as an equalizer.

It is easy to see that these operations χ_U and V_φ are mutually inverse

$$\begin{aligned} V_{\chi_U} &= \{x \in A \mid \chi_U(x) = \top\} \\ &= \{x \in A \mid x \in U\} \\ &= U \end{aligned}$$

for any $U \subseteq A$, and given any $\varphi : A \to 2$

$$\begin{aligned} \chi_{V_\varphi}(x) &= \begin{cases} \top & x \in V_\varphi \\ \bot & x \notin V_\varphi \end{cases} \\ &= \begin{cases} \top & \varphi(x) = \top \\ \bot & \varphi(x) = \bot \end{cases} \\ &= \varphi(x) \end{aligned}$$

Thus we have the familiar isomorphism

$$\mathrm{Hom}(A, 2) \cong P(A)$$

mediated by taking equalizers

Proposition 3.5. *In any category, if $e : E \to A$ is an equalizer of some pair of arrows, then e is monic*

Proof. Consider

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, shift left, "x"] \arrow[u, shift right, "y"'] \arrow[ur, "z"'] & & \end{tikzcd}$$

Suppose $ex = ey$, we want to show $x = y$. Put $z = ex = ey$. Then $fz = fex = gex = gz$, so there is a unique $u : Z \to E$ s.t. $eu = z$. So $x = u = y$ $\square$

Example 3.5. In abelian groups, one has an alternate description of the equalizer, using the fact that

$$f(x) = g(x) \quad \text{iff} \quad (f - g)(x) = 0$$

3.3 Coequalizers

Definition 3.6. For any parallel arrows $f, g : A \to B$ in a category $\mathbf{C}$, a **coequalizer** consists of Q and $q : B \to Q$, universal with the property $qf = qg$ as in

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dotted, "u"] \\ & & Z \end{tikzcd}$$

That is, given any Z and $z : B \to Z$ if $zf = zg$, then there exists a unique $u : Q \to Z$ s.t. $uq = z$

Proposition 3.7. *If $q : B \to Q$ is a coequalizer of some pair of arrows, then q is epic*

We can therefore think of a coequalizer $q : B \twoheadrightarrow Q$ as a "collapse" of B by "identifying" all pairs $f(a) = g(a)$

Example 3.6. Let $R \subseteq X \times X$ be an equivalence relation on a set X, and consider the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "f_2"'] & X \end{tikzcd}$$

where the r's are the two projections of the inclusion $R \subseteq X \times X$

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[d, hook] \arrow[dr, "r_2"] & \\ X & X \times X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

The quotient projection

$$\pi : X \to X/R$$

defined by $x \mapsto [x]$ is then a coequalizer of r_1 and r_2. For given an $f : X \to Y$ as in

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "\pi"] \arrow[dr, "f"'] & X/R \arrow[d, dotted, "\overline{f}"] \\ & & Y \end{tikzcd}$$

there exists a function $\overline{f}$ s.t.

$$\overline{f}\pi(x) = f(x)$$

whenever f respects R in the sense that $(x, x') \in R$ implies $f(x) = f(x')$. But this condition just says that $f \circ r_1 = f \circ r_2$ since $f \circ r_1(x, x') = f(x')$ and $f \circ r_2(x, x') = f(x')$ for all $(x, x') \in R$. Moreover, if it exists, such a function $\overline{f}$ is then necessarily unique, since π is an epimorphism

The coequalizer in **Sets** of an arbitrary parallel pair of function $f, g : A \twoheadrightarrow B$ can be constructed by quotienting B by the equivalence relation generated by the equations $f(x) = g(x)$for all $x \in A$

Consider

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \longrightarrow Q = B/(f = g)$$

where the equivalence relation R on b is generated by the pairs $(f(x), g(x))$ for all $x \in A$. That is, R is the intersection of all equivalence relations on B containing all such pairs

Example 3.7. Taken in posets

$$1 \overset{0_P}{\underset{0_Q}{\rightrightarrows}} P + Q \longrightarrow P + Q/(0_P = 0_Q)$$

$(0_P = 0_Q)$ is the equivalent closure of $(0_P(1), 0_Q(1))$.

Example 3.8 (Presentations of algebras)**.** Suppose we are given

Generators: x, y, z
Relations: $xy = z, y^2 = 1$

To build an algebra on these generators and satisfying these relations, start with the free algebra

$$F(3) = F(x, y, z)$$

and then "force" the relation $xy = z$ to hold by taking a coequalizer of the maps

$$F(1) \overset{xy}{\underset{z}{\rightrightarrows}} F(3) \xrightarrow{q} Q$$

We use the fact that maps $F(1) \to A$ correspond to elements $a \in A$ by $v \mapsto a$, where v is the single generator of $F(1)$. Now similarly for the equation $y^2 = 1$, taking the coequalizer

$$F(1) \overset{q(y^2)}{\underset{q(1)}{\rightrightarrows}} Q \longrightarrow Q'$$

These two steps can actually be done simultaneously. Let

$$F(2) = F(1) + F(1)$$

$$F(2) \overset{f}{\underset{g}{\rightrightarrows}} F(3)$$

where $f = [xy, y^2]$ and $g = [z, 1]$