Category Theory

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1 Categories

1.1 Examples of categories

Definition 1.1. A functor

$$F: \mathbf{C} \to \mathbf{C}$$

between categories \boldsymbol{C} and \boldsymbol{D} is a mapping of objects to objects and arrows to arrows, in such a way that

1.
$$F(f:A \rightarrow B) = F(f): F(A) \rightarrow F(B)$$

2.
$$F(1_A) = 1_{F(A)}$$

3.
$$F(g \circ f) = F(g) \circ F(f)$$

2 Abstract structures

2.1 Products

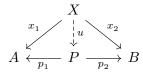
Definition 2.1. In any category C, a **product diagram** for the objects A and B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP: Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique $u: X \rightarrow P$ making the diagram



2.2 Categories with products

Let **C** be a category that has a product diagram for every pair of objects. Suppose we have objects and arrows

$$\begin{array}{cccccc} A \xleftarrow{p_1} & A \times A' & \stackrel{p_2}{\longrightarrow} & A' \\ f \downarrow & & \downarrow f' \\ B \xleftarrow{q_1} & B \times B' & \stackrel{q_2}{\longrightarrow} & B' \end{array}$$

with indicated products. Then we write

$$f \times f' : A \times A' \to B \times B$$

for
$$f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle$$

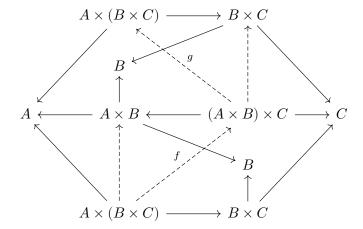
In this way, if we choose a product for each pair of objects, we get a functor

$$\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

To prove

$$(A\times B)\times C\cong A\times (B\times C)$$

Consider



Given no objects, there is an object 1 with no maps, and give nany other object X and no maps, there is a unique arrow

$$!:X\to 1$$

Definition 2.2. A category **C** is said to **have all finite products**, if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category **C** has all (small) products if every set of objects in **C** has a product

2.3 Hom-sets

In this section, we assume that all categories are locally small Given any objects A and B in category C, we write

$$\operatorname{Hom}(A,B) = \{ f \in \mathbf{C} \mid f : A \to B \}$$

and call such a set of arrows a Hom-set

Note that any arrow $g: B \to B'$ in **C** induces a function

$$\operatorname{Hom}(A,g):\operatorname{Hom}(A,B)\to\operatorname{Hom}(A,B')$$

$$(f:A\to B)\mapsto (g\circ f:A\to B\to B')$$

Let's show that this determines a functor

$$\operatorname{Hom}(A,-):\mathbf{C}\to\operatorname{\mathbf{Sets}}$$

called the (covariant) **representable functor** of *A*. We need to show that

$$\operatorname{Hom}(A,1_X)=1_{\operatorname{Hom}(A,X)}$$

and that

$$\operatorname{Hom}(A, g \circ f) = \operatorname{Hom}(A, g) \circ \operatorname{Hom}(A, f)$$

For any object P, a pair of arrows $p_1:P\to A$ and $p_2:P\to B$ determine an element (p_1,p_2) of the set

$$\operatorname{Hom}(P,A) \times \operatorname{Hom}(P,B)$$

Now given any arrow

$$x:X\to P$$

composing with p_1 and p_2 gives a pair of arrows $x_1=p_1\circ x:X\to A$ and $x_2=p_2\circ x:X\to B$

In this way, we have a function

$$\theta_X = (\operatorname{Hom}(X,p_1),\operatorname{Hom}(X,p_2)):\operatorname{Hom}(X,P) \to \operatorname{Hom}(X,A) \times \operatorname{Hom}(X,B)$$

defined by

$$\theta_X(x) = (x_1, x_2)$$

Proposition 2.3. A diagram of the form

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

is a product for A and B iff for every object X, the canonical function θ_X is an isomorphism

$$\theta_X: \operatorname{Hom}(X,P) \cong \operatorname{Hom}(X,A) \times \operatorname{Hom}(X,B)$$