Finite Model Theory

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1 Preliminaries

1.1 Structures

Vocabularies are finite sets that consist of **relation symbols** and **constant symbols**. We denote vocabularies by τ, σ,.... A **vocabulary** is relational if it does not contain constants.

1.1.1 Graph

Let $\tau = \{E\}$ with a binary relation symbol E. A **graph** (or **undirected graph**) is a τ-structure $\mathcal{G} = (G, E^G)$ satisfying

1. for all $a \in G$: not $E^G aa$
2. for all $a, b \in G$: if $E^G ab$ then $E^G ba$

By GRAPH we denote the class of **finite** graphs. If only (1) is required, we speak of a **digraph**

A subset X of the universe of a graph $\mathcal{G}$ is a **clique**, if $E^G ab$ for all $a, b \in X$, $a \neq b$

Let $\mathcal{G}$ be a digraph. If $n \geq 1$ and

$$E^G a_0 a_1, E^G a_1 a_2, \dots, E^G a_{n-1} a_n$$

then $a_0, \dots, a_n$ is a **path** from a_0 to a_n of **length** n. If $a_0 = a_n$, then $a_0, \dots, a_n$ is a **cycle**. A path $a_0, \dots, a_n$ is **Hamiltonian** if $G = \{a_0, \dots, a_n\}$ and $a_i \neq a_j$ for $i \neq j$. If, in addition, $E^G a_n a_0$ we speak of a **Hamiltonian circuit**

Let $\mathcal{G}$ be a graph. Write $a \sim b$ if $a = b$ or if there is a path from a to b. The equivalence class of a is called the **(connected) component** of a. Let CONN be the class of finite connected graphs

Denote by $d(a, b)$ the length of a shortest path from a to b; more precisely, define the **distance function** $d : G \times G \to \mathbb{N} \cup \{\infty\}$ by

$$d(a, b) = \infty \text{ iff } a \nsim b, \quad d(a, b) = 0 \text{ iff } a = b$$

and otherwise

$$d(a, b) = \min\{n \geq 1 \mid \text{there is a path from } a \text{ to } b \text{ of length } n\}$$

We give the following definitions only for **finite** digraphs. A vertex b is a successor of a vertex a if $E^G ab$. The **in-degree** of a vertex is the number of its predecessors, the **out-degree** the number of its successors.

A **root** of a digraph is a vertex with in-degree 0 and a **leaf** a vertex with out-degree 0.

A **forest** is an acyclic digraph where each vertex has in-degree at most 1. A **tree** is a forest with connected underlying graph. Let TREE be the class of finite trees.

1.1.2 Orderings

Let $\tau = \{<\}$ with a binary relation symbol. A τ-structure $\mathcal{A} = (A, <^A)$ is called an **ordering** if for all $a, b, c \in A$:

1. not $a <^A a$
2. $a <^A b$ or $b <^A a$ or $a = b$
3. if $a <^A b$ and $b <^A c$ then $a <^A c$

1.1.3 Operations on Structures

Two τ-structures $\mathcal{A}$ and $\mathcal{B}$ are **isomorphic**, written $\mathcal{C} \cong \mathcal{B}$, if there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, i.e., a bijection $\pi : A \to B$ preserving relations and constants, that is

- for n-ary $R \in \tau$ and $a_1, \ldots, a_n \in A$

$$R^A a_1 \ldots a_n \quad \text{iff} \quad R^B \pi(a_i) \ldots \pi(a_n)$$

- for $c \in \tau$, $\pi(c^A) = \pi(c^B)$

For relational τ we introduce the **union** (or, **disjoint union**) of structures. Assume that $\mathcal{A}$ and $\mathcal{B}$ are τ-structures with $A \cap B = \emptyset$. Then $\mathcal{A} \dot{\cup} \mathcal{B}$, the **union** of $\mathcal{A}$ and $\mathcal{B}$, is the τ-structure with domain $A \cup B$ and

$$R^{\mathcal{A} \dot{\cup} \mathcal{B}} := R^{\mathcal{A}} \cup R^{\mathcal{A}}$$

for any R in τ

Note that the union of ordered structures is not an ordered structure. The situation is different for the so-called **ordered sum**: let τ with $< \in \tau$ be relational and let $\mathcal{A}$ and $\mathcal{B}$ be ordered τ-structures. Assume that $A \cap B = \emptyset$. Define $\mathcal{A} \lhd \mathcal{B}$, the **ordered sum** of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{A} \dot{\cup} \mathcal{B}$ but setting

$$<^{\mathcal{A} \dot{\cup} \mathcal{B}} := <^{\mathcal{A}} \cup <^{\mathcal{B}} \cup \{(a, b) \mid a \in A, b \in B\}$$

1.2 Syntax and Semantics of First-Order Logic

Fix a vocabulary τ. Each formula of first-order logic will be a string of symbols taken from the alphabet consisting of

- variables

- connectives
- existential quantifier
- equality symbol
-)(
- the symbols in τ

  Denote $\mathrm{FO}[\tau]$ the set of formulas of first-order logic of vocabulary τ.

  The axioms for graphs stated above have the following formalizations in $\mathrm{FO}[\{E\}]$

  $$\forall x \neg Exx$$
  $$\forall x \forall y (Exy \to Eyx)$$

  When only taking into consideration finite structures, we use the notation $\Phi \vDash_{\mathrm{fin}} \psi$

  The **quantifier rank** $\mathrm{qr}(\varphi)$ of a formula φ is the maximum number of nested quantifiers occurring in it

  It can be shown that every first-order formula is logically equivlent to a formula in prenex normal form, that is, to a formula of the form $Q_1 x_1, \dots, Q_s x_s \psi$ where $Q_1, \dots, Q_s \in \{\forall, \exists\}$, and where ψ is quantifier-free. Such a formula is called Σ_n if the string of n consecutive blocks, where in each block all quantifiers all of the same type, adjacent blocks contain quantifiers of different type, and the first block is existential. Π_n formulas are defined in the same way but now we require that the first block consists of universal quantifiers. A Δ_n-formula is a formula logically equivalent to both a Σ_n-formula and a Π_n-formula

  Given a formula $\varphi(x, \bar{z})$ and $n \geq 1$,

  $$\exists^{\geq n} x \varphi(x, \bar{z})$$

  is an abbreviation for the formula

  $$\exists x_1, \dots \exists x_n (\bigwedge_{1 \leq i \leq n} \varphi(x_i, \bar{z}) \wedge \bigwedge_{1 \leq i < j \leq n} \neg x_i = x_j)$$

  We set

  $$\varphi_{\geq n} := \exists^{\geq n} x \; x = x$$

  Clearly

  $$\mathcal{A} \vDash \varphi_{\geq n} \quad \text{iff} \quad \|A\| \geq n$$

1.3 Some Classical Results of First-Order Logic

Theorem 1.1. *The set of logically valid sentences of first-order logic is r.e.*

Theorem 1.2 (Compactness Theorem)**.** Φ *is satisfiable iff every finite subset of* Φ *is satisfiable*

Neither Theorem 1.1 nor 1.2 remain valid if one only considers finite structures. A counterexample for the Compactness Theorem is given by the set $\Phi_\infty := \{\varphi_{\geq n} \mid n \geq 1\}$: Each finite subset of Φ_∞ has a finite model, but Φ_∞ has no finite model

The failure of Theorem 1.1 is documented by

Theorem 1.3 (Trahtenbrot's Theorem)**.** *The set of sentences of first-order logic valid in all finite structures is not r.e.*

Lemma 1.4. *Let* $\varphi \in FO[\tau]$ *and for* $i \in I$*, let* $\Phi^i \subseteq FO[\tau]$*. Assume that*

$$\models \varphi \leftrightarrow \bigvee_{i \in I} \bigwedge \Phi^i$$

Then there is a finite $I_0 \subseteq I$ *and for every* $i \in I_0$*, a finite* $\Phi^i_0 \subseteq \Phi^i$ *s.t.*

$$\models \varphi \leftrightarrow \bigvee_{i \in I_0} \bigwedge \Phi^i_0$$

Proof. For simplicity we assume that φ is a sentence and that every Φ^i is a set of sentences. By hypothesis, for some $i \in I$, we have $\Phi^i \models \varphi$; hence, by the Compactness Theorem, $\Phi^i_0 \models \varphi$ for some finite $\Phi^i_0 \subseteq \Phi^i$.

If there is not such I_0 with $\models \varphi \rightarrow \bigvee_{i \in I_0} \bigwedge \Phi^i_0$, then each finite subset of $\{\varphi\} \cup \{\neg \bigwedge \Phi^i_0 \mid i \in I\}$ has a model. Hence by the Compactness Theorem, there is a contradiction □

Corollary 1.5. *Let* Φ *be a set of first-order sentences. Assume that any two structures that satisfy the same sentences of* Φ *are elementarily equivalent. Then any first-order sentence is equivalent to a boolean combination of sentences of* Φ

Proof. For any structure $\mathcal{A}$ set

$$\Phi(\mathcal{A}) := \{\psi \mid \psi \in \Phi, \mathcal{A} \models \psi\} \cup \{\neg\psi \mid \psi \in \Phi, \mathcal{A} \models \neg\psi\}$$

Let φ be any first-order sentence. By the preceding lemma it suffices to show that

$$\models \varphi \leftrightarrow \bigvee_{\mathcal{A} \models \phi} \bigwedge \Phi(\mathcal{A})$$

If $\mathcal{B} \models \varphi$ then $\mathcal{B} \models \bigvee_{\mathcal{A}\models\varphi} \bigwedge \Phi(\mathcal{A})$. Suppose $\mathcal{A} \models \bigvee_{\mathcal{A}\models\varphi} \bigwedge \Phi(\mathcal{A})$. Then for some model $\mathcal{A}$ of φ, $\mathcal{B} \models \Phi(\mathcal{A})$. By the definition of $\Phi(\mathcal{A})$, $\mathcal{A}$ and $\mathcal{B}$ satisfy the same sentences of Φ $\square$

1.4 Model Classes and Global Relations

Fix a vocabulary τ. For a sentence φ of FO$[\tau]$ we denote by Mod(φ) the class of **finite** models of φ.

Mod(φ) is closed under isomorphisms

For $\varphi(x_1, \dots, x_n) \in$ FO$[\tau]$ and a structure $\mathcal{A}$ let

$$\varphi^{\mathcal{A}}(-) := \{(a_1, \dots, a_n) \mid \mathcal{A} \models \varphi[a_1, \dots, a_n]\}$$

be the set of n-tuples **defined by φ in $\mathcal{A}$**. For $n = 0$ this be read as

$$\varphi^{\mathcal{A}} := \begin{cases} \text{TRUE} & \text{if } \mathcal{A} \models \varphi \\ \text{FALSE} & \text{if } \mathcal{B} \not\models \varphi \end{cases}$$

Use this notation we have

$$\text{if } \pi : \mathcal{A} \cong \mathcal{B} \text{ then } \pi(\varphi^{\mathcal{A}}(-) = \varphi^{\mathcal{B}}(-))$$

where for $X \subseteq A^n$ we set $\pi(X) := \{\pi(a_1), \dots, \pi(a_n) \mid (a_1, \dots, a_n) \in X\}$

Throughout the book all classes K of structures considered will tacitly be assumed to be closed under isomorphisms, i.e.

$$\mathcal{A} \in K \text{ and } \mathcal{A} \cong \mathcal{B} \text{ implies } \mathcal{B} \in K$$

Definition 1.6. Let K be a class of τ-structures. An n-ary **global relation Γ on K** is a mapping assigning to each $A \in K$ an n-ary relation $\Gamma(\mathcal{A})$ on $\mathcal{A}$ satisfying

$$\Gamma(\mathcal{A})a_1 \dots a_n \quad \text{iff} \quad \Gamma(\mathcal{B})\pi(a_1) \dots \pi(a_n)$$

for every isomorphism $\pi : \mathcal{A} \cong \mathcal{B}$ and every $a_1, \dots, a_n \in A$. If K is the class of all finite τ-structures, then we just speak of an n-ary **global relation**

Example 1.1.

1. Any formula $\varphi(x_1, \dots, x_n) \in$ FO$[\tau]$ defines the global relation $\mathcal{A} \mapsto \varphi^{\mathcal{A}}(-)$

2. The "transitive closure relation" TC is the binary global relation on GRAPH with

$$\text{TC}(\mathcal{G}) := \{(a, b) \mid a, b \in G, \text{ there is a path from } a \text{ to } b\}$$

3. For $m \geq 0$, Γ_m is a unary global relation on GRAPH, where

$$\Gamma_m(\mathcal{G}) := \{a \mid \|\{b \in G \mid E^G ab\}\| = m\}$$

is the set of elements of $\mathcal{G}$ of degree m

An important issue in model theory is the study of properties of classes of structures that are axiomatizable in a given logic $\mathcal{L}$ and in particular to determine what classes of structures are axiomatizable are what global relations are definable in $\mathcal{L}$.

1.5 Relational Databases and Query Languages

2 Ehrenfeucht–Fraïssé Method

2.1 Elementary Classes

Proposition 2.1. *Every finite structure can be characterized in first-order logic up to isomorphism, i.e., for every finite structure $\mathcal{A}$ there is a sentence $\varphi_{\mathcal{A}}$ of first-order logic s.t. for all structures $\mathcal{B}$ we have*

$$\mathcal{B} \models \varphi_{\mathcal{A}} \quad \textit{iff} \quad \mathcal{A} \cong \mathcal{B}$$

Proof. Suppose $A = \{a_1, \dots, a_n\}$. Set $\bar{a} = a_1 \dots a_n$. Let

$$\Theta_n := \{\psi \mid \psi \text{ has the form } Rx_1 \dots x_k, x = y \text{ or } c = x, \\ \text{and variables among } v_1, \dots, v_n\}$$

and

$$\varphi_{\mathcal{A}} := \exists v_1 \dots \exists v_n (\bigwedge\{\psi \mid \psi \in \Theta_n, \mathcal{A} \models \psi[\bar{a}]\} \wedge \\ \bigwedge\{\neg\psi \mid \psi \in \Theta_n, \mathcal{A} \models \neg\psi[\bar{a}]\} \wedge \forall v_{n+1}(v_{n+1} = v_n \vee \cdots \vee v_{n+1} = v_n))$$

□

Corollary 2.2. *Let K be a class of finite structures. Then there is a set Φ of first-order sentences s.t.*

$$K = \mathrm{Mod}(\Phi)$$

that is, K is the class of finite models of Φ

Proof. For each n there is only a finite number of pairwise nonisomorphic structures of cardinality n. Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be a maximal subset of K of pairwise nonisomorphic structures of cardinality n. Set

$$\psi_n := (\varphi_{=n} \rightarrow (\varphi_{\mathcal{A}_1} \vee \cdots \vee \varphi_{\mathcal{A}_k}))$$

Then $K = \mathrm{Mod}(\{\psi_n \mid n \geq 1\})$ $\square$

Definition 2.3. Let K be a class of finite structures. K is called **axiomatizable in first-order logic** or **elementary** if there is a setence φ of first-order logic s.t. $K = \mathrm{Mod}(\varphi)$

For structures $\mathcal{A}$ and $\mathcal{B}$ and $m \in \mathbb{N}$ we write $\mathcal{A} \equiv_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are m**-equivalent** if $\mathcal{A}$ and $\mathcal{B}$ satisfy the same first-order sentences of quantifier rank $\leq m$

Theorem 2.4. *Let K be a class of finite structures. Suppose that for every m there are fintie structures $\mathcal{A}$ and $\mathcal{B}$ s.t.*

$$\mathcal{A} \in K, \mathcal{B} \notin K, \text{ and } \mathcal{A} \equiv_m \mathcal{B}$$

Then K is not axiomatizable in first-order logic

Proof. Let φ be any first-order sentence. Set $m := \mathrm{qr}(\varphi)$. By out assumption there are $\mathcal{A}$ and $\mathcal{B}$ s.t. $\mathcal{A} \in K, \mathcal{B} \notin K$, and $\mathcal{A} \equiv_m \mathcal{B}$. Hence $K \neq \mathrm{Mod}(\varphi)$ $\square$

2.2 Ehrenfeucht's Theorem

Definition 2.5. Assume $\mathcal{A}$ and $\mathcal{B}$ are structures. Let p be a map with $\mathrm{dom}(p) \subseteq B$. Then p is said to be a **partial isomorphism** from $\mathcal{A}$ to $\mathcal{B}$ if

1. p is injective
2. for every $c \in \tau$:$c^{\mathcal{A}} \in \mathrm{dom}(p)$ and $p(c^{\mathcal{A}}) = c^{\mathcal{B}}$
3. for every n-ary $R \in \tau$ and all $a_1, \dots, a_n \in \mathrm{dom}(p)$

$$R^{\mathcal{A}} a_1 \dots a_n \quad \text{iff} \quad R^{\mathcal{B}} p(a_1) \dots p(a_n)$$

We write $\mathrm{Part}(\mathcal{A}, \mathcal{B})$ for the set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$

In the following we identify a map p with its graph $\{(a, p(a)) | a \in \mathrm{dom}(p)\}$. Then $p \subseteq q$ means that q is an extension of p

Remark. 1. The empty map, $p = \emptyset$, is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ just in case the vocabulary contains no constants

2. If $p \neq \emptyset$ is a map with $\mathrm{dom}(p) \subseteq A$ and $\mathrm{ran}(p) \subseteq B$, then p is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ iff $\mathrm{dom}(p)$ contains $c^{\mathcal{A}}$ for all constants $c \in \tau$ and $p : \mathrm{dom}(p)^{\mathcal{A}} \cong \mathrm{ran}(p)^{\mathcal{B}}$

3. For $\bar{a} = a_1 \dots a_s \in A$ and $\bar{b} = b_1 \dots b_s \in B$ the following statements are equivalent

   (a) the clauses
   $$p(a_i) = b_i \text{ for } i = 1, \dots, s$$
   and
   $$p(c^{\mathcal{A}}) = c^{\mathcal{B}} \text{ for } c \text{ in } \tau$$
   define a map, which is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ (henceforth denoted by $\bar{a} \mapsto \bar{b}$)

   (b) for all quantifier-free $\varphi(v_1, \dots, v_s)$: $\mathcal{A} \vDash \varphi[\bar{a}]$ iff $\mathcal{B} \vDash \varphi[\bar{b}]$

   (c) for all atomic $\varphi(v_1, \dots, v_s)$: $\mathcal{A} \vDash \varphi[\bar{a}]$ iff $\mathcal{B} \vDash \varphi[\bar{b}]$

In general, a **partial isomorphism does not preserve the validity of formulas with quantifiers**: Let $\tau = \{<\}$, $\mathcal{A} = (\{0,1,2\}, <)$, $\mathcal{B} = (\{0,1,2,3\}, <)$ whre in both cases < denotes the natural ordering. Then $p_0 := 02 \mapsto 01$ is a partial isomorphism from $\mathcal{A}$ to $\mathcal{B}$ s.t.

$$\mathcal{A} \vDash \exists v_3(v_1 < v_3 \wedge v_3 < v_2)[0,2]$$

but

$$\mathcal{B} \nvDash \exists v_3(v_1 < v_3 \wedge v_3 < v_2)[p_0(0), p_0(2)]$$

Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \in \mathbb{N}$. The **Ehrenfeucht game** $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ is played by two players called the **spoiler** and the **duplicator**. Each player has to make m moves in the course of a play. In his i-th move the spoiler first selects a structure, $\mathcal{A}$ or $\mathcal{B}$, and an element in this structure. If the spoiler chooses e_i in $\mathcal{A}$ then the duplicator in his i-th move must choose an element f_i in $\mathcal{B}$. If the spoiler chooses f_i in $\mathcal{B}$ then the duplicator must choose an element e_i in $\mathcal{A}$

	$\mathcal{A}, \bar{a}$	$\mathcal{B}, \bar{b}$
first move	e_1	f_1
second move	e_2	f_2
$\vdots$	$\vdots$	$\vdots$
m-th move	e_m	f_m

The duplicator **wins** iff $\bar{a}\bar{e} \mapsto \bar{b}\bar{f} \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$.

Equivalently, the spoiler wins if after some $i \le m$, $\bar{a}e_1 \dots e_i \mapsto \bar{b}f_1 \dots f_i$ is not a partial isomorphism. We say that a player, the spoiler or the duplicator, has a **winning strategy** in $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, or shortly, that he **wins** $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, if it is possible for him to win each play whatever choices are made by the opponent.

If $s = 0$, we denote the game by $G_m(\mathcal{A}, \mathcal{B})$

Lemma 2.6. *1. If $A \cong B$ then the duplicator wins $G_m(\mathcal{A}, \mathcal{B})$*

2. If the duplicator wins $G_{m+1}(\mathcal{A}, \mathcal{B})$ and $\|A\| \le m$ then $\mathcal{A} \cong \mathcal{B}$

Lemma 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be structures, $\bar{a} \in A^s$, $\bar{b} \in B^s$, and $m \ge 0$*

1. *The duplicator wins $G_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ iff $\bar{a} \mapsto \bar{b}$ is a partial isomorphism*
2. *For $m > 0$ the following are equivalent*
   - *(a) The duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$*
   - *(b) For all $a \in A$ there is $b \in B$ s.t. the duplicator wins the game $G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$ and for all $b \in B$ there is a $a \in A$ s.t. the duplicator wins $G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$*
   - *(c) If the duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{B})$ and if $m' < m$ the duplicator wins $G_{m'}(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$*

Let $\mathcal{A}$ be given. For $\bar{a} = a_1 \dots a_s \in A$ and $m \ge 0$ we introduce a formula $\varphi_{\bar{a}}^m(v_1, \dots, v_s)$ that describes the game-theoretic properties of $\bar{a}$ in any game $G_m(\mathcal{A}, \bar{a}, \dots)$ s.t. for any $\mathcal{B}$ and $\bar{b} = b_1 \dots b_s \in B$

$$\mathcal{B} \models \varphi_{\bar{a}}^m[\bar{b}] \quad \text{iff} \quad \text{the duplicator wins } G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$$

Definition 2.8. Let $\bar{v}$ be $v_1, \dots, v_s$

$$\varphi_{\bar{a}}^0 := \bigwedge \{\varphi(\bar{v}) \mid \varphi \text{ atomic or negated atmoic}, \mathcal{A} \models \varphi[\bar{a}]\}$$

(atomic diagram of $\mathcal{A}$) and for $m > 0$

$$\varphi_{\bar{a}}^m(\bar{v}) := \bigwedge_{a \in A} \exists v_{s+1} \varphi_{\bar{a}a}^{m-1}(\bar{v}, v_{s+1}) \wedge \forall v_{s+1} \bigvee_{a \in A} \varphi_{\bar{a}a}^{m-1}(\bar{v}, v_{s+1})$$

$\varphi_{\bar{a}^0}$ describes the isomorphism type of the substructure generated by $\bar{a}$ in $\mathcal{A}$; and for $m > 0$ the formula $\varphi_{\bar{a}}^m$ tells us to which isomorphism types the tuple $\bar{a}$ can be extended in m steps adding one element in each step. $\varphi_{\bar{a}}^m$ is called the **m-isomorphism type** (or **m-Hintikka formula**) of $\bar{a}$ in $\mathcal{A}$

Since $\varphi(v_1, \dots, v_s) \mid \varphi$ atomic or negated atmoic is finite, a simple induction on m shows

Lemma 2.9. *For $s, m \geq 0$, the set $\{\varphi^m_{\mathcal{A},\bar{a}} \mid \mathcal{A}$ a structure and $\bar{a} \in A^s\}$ is finite*

Lemma 2.10.
1. $\mathrm{qr}(\varphi^m_{\bar{a}}) = m$
2. $\mathcal{A} \vDash \varphi^m_{\bar{a}}[\bar{a}]$
3. *For any $\mathcal{B}$ and $\bar{b}$ in B*

$$\mathcal{B} \vDash \varphi^0_{\bar{a}}[\bar{b}] \quad \textit{iff} \quad \bar{a} \mapsto \bar{b} \in Part(\mathcal{A}, \mathcal{B})$$

Theorem 2.11 (Ehrenfeucht's Theorem)**.** *Given $\mathcal{A}$ and $\mathcal{B}$, $\bar{a} \in A^s$ and $\bar{b} \in B^s$, and $m \geq 0$, the following are equivalent*

1. *The duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$*
2. $\mathcal{B} \vDash \varphi^m_{\bar{a}}[\bar{b}]$
3. *$\bar{a}$ and $\bar{b}$ satisfy the same formulas of quantifier rank $\leq m$, that is, if $\varphi(x_1, \dots, x_s)$ is of quantifier rank $\leq m$, then*

$$\mathcal{A} \vDash \varphi[\bar{a}] \quad \textit{iff} \quad \mathcal{B} \vDash \varphi[\bar{b}] \tag{1}$$

Proof. $1 \leftrightarrow 2$. Induction on m. For $m = 0$

$$\begin{aligned} \text{the duplicator wins } G_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \quad & \text{iff} \quad \bar{a} \mapsto \bar{b} \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \\ & \text{iff} \quad \mathcal{B} \vDash \varphi^0_{\bar{a}}[\bar{b}] \end{aligned}$$

For $m > 0$

$$\begin{aligned}
& \text{the duplicator wins } G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b}) \\
\text{iff} \quad & \text{for all } a \in A, \text{there is } b \in B \text{ s.t. the duplicator wins} \\
& G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b), \text{ and for all } b \in B \text{ there is } a \in A \text{ s.t.} \\
& \text{the duplicator wins } G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b) \\
\text{iff} \quad & \text{for all } a \in A, \text{ there is } b \in B \text{ with } \mathcal{B} \vDash \varphi^{m-1}_{\bar{a}a}[\bar{b}b] \text{ , and} \\
& \text{for all } b \in B, \text{ there is } a \in A \text{ with } \mathcal{B} \vDash \varphi^{m-1}_{\bar{a}a}[\bar{b}b] \\
\text{iff} \quad & \mathcal{B} \vDash \textstyle\bigwedge_{a \in A} \exists v_{s+1} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1}) \wedge \forall v_{s+1} \bigvee_{a \in A} \varphi^{m-1}_{\bar{a}a}(\bar{v}, v_{s+1})[\bar{b}] \\
\text{iff} \quad & \mathcal{B} \vDash \varphi^m_{\bar{a}}[b]
\end{aligned}$$

$3 \to 1$. $\mathrm{qr}(\varphi^m_{\bar{a}}) = m$ and $\mathcal{A} \models \varphi^m_{\bar{a}}[\bar{a}]$

$1 \to 3$. Induction on m. The case $m = 0$ is handled as above. Let $m > 0$ and suppose that the duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$. Clearly the set of formulas $\varphi(x_1, \ldots, x_s)$ satisfying 1 contains the atomic formulas and is closed under $\neg$ and $\vee$ (Since duplicator wins the game, there are partial isomorphisms). Suppose that $\varphi(\bar{a}) = \exists y \psi$ and $\mathrm{qr}(\varphi) \leq m$. Since $y \notin \mathrm{free}(\varphi)$, we can assume that y is distinct from the variables in $\bar{x}$. Hence $\psi = \psi(\bar{x}, y)$. Assume, for instance, $\mathcal{A} \models \varphi(\bar{a})$. Then there is $a \in A$ s.t. $\mathcal{A} \models \psi[\bar{a}, a]$. As by 1, the duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$, there is $b \in B$ s.t. the duplicator wins $G_{m-1}(\mathcal{A}, \bar{a}a, \mathcal{B}, \bar{b}b)$. Since $\mathrm{qr}(\psi) \leq m - 1$, the induction hypothesis yields $\mathcal{B} \models \psi[\bar{b}, b]$, hence $\mathcal{B} \models \varphi[\bar{b}]$ □

Corollary 2.12. *For structures $\mathcal{A}$, $\mathcal{B}$ and $m \geq 0$ the following are equivalent*

1. *The duplicator wins $G_m(\mathcal{A}, \mathcal{B})$*
2. $\mathcal{B} \models \varphi^m_{\mathcal{A}}$
3. $\mathcal{A} \equiv_m \mathcal{B}$

Corollary 2.13. *Let $\mathcal{A}$ be a structure with $\|A\| \leq m$. Then for all $\mathcal{B}$*

$$\mathcal{B} \models \varphi^{m+1}_{\mathcal{A}} \quad \textit{iff} \quad \mathcal{A} \cong \mathcal{B}$$

The next result shows that the formulas $\varphi^m_{\bar{a}}$ give a clear picture of the expressive power of first-order logic

Theorem 2.14. *Let $\varphi(v_1, \ldots, v_s)$ be a formula of quantifier rank $\leq m$. Then*

$$\models \varphi \leftrightarrow \bigvee \{\varphi^m_{\mathcal{A},\bar{a}} \mid \mathcal{A} \textit{ a structure }, \bar{a} \in \mathcal{A}, \textit{ and } \mathcal{A} \models \varphi[\bar{a}]\}$$

Proof. Suppose first that $\mathcal{B} \models \varphi[\bar{b}]$. Then the formula $\varphi^m_{\mathcal{B},\bar{b}}$ is a member of the disjunction on the right side of the equivalence, which therefore is satisfied by $\bar{b}$.

Conversely, suppose $\mathcal{B} \models \bigvee \{\varphi^m_{\mathcal{A},\bar{a}}[\bar{a}]\}[\bar{b}]$. Then for some $\mathcal{A}$ and $\bar{a}$ s.t. $\mathcal{A} \models \varphi[\bar{a}]$ we have $\mathcal{B} \models \varphi^m_{\mathcal{A},\bar{a}}[\bar{b}]$. By Theorem 2.11 $\bar{a}$ and $\bar{b}$ satisfy the same formulas of quantifier rank $\leq m$ and therefore $\mathcal{B} \models \varphi[\bar{b}]$. □

Theorem 2.15. *For a class K of finite structures the following are equivalent*

1. *K is not axiomatizable in first-order logic*
2. *For each m there are finite structures $\mathcal{A}$ and $\mathcal{B}$ s.t.*

$$\mathcal{A} \in K, \mathcal{B} \notin K \textit{ and } \mathcal{A} \equiv_m \mathcal{B}$$

Proof. $2 \to 1$ is proved in theorem 2.4. For the converse, suppose that 2 doesn't hold, i.e., that for some m and all finite $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} \in K \text{ and } \mathcal{A} \equiv_m \mathcal{B} \text{ imply } \mathcal{B} \in K$$

Then $K = \mathrm{Mod}(\bigvee\{\varphi^m_{\mathcal{A}} \mid \mathcal{A} \in K\})$, and thus K is axiomatizable $\square$

2.3 Examples and Fraïssé's Theorem

Given structures $\mathcal{A}, \mathcal{B}$ and $m \in \mathbb{N}$, let $W_m(\mathcal{A}, \mathcal{B}) :=$

$$\{\bar{a} \mapsto \bar{b} \mid s \geq 0, \bar{a} \in A^s, \bar{b} \in B^s, \text{ the duplicator wins } G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})\}$$

be the set of winning positions for the duplicator. The sequence of the $W_M(\mathcal{A}, \mathcal{B})$ has the back and forth properties as introduced in the following definition

Definition 2.16. Structures $\mathcal{A}$ and $\mathcal{B}$ are said to be ***m*-isomorphic**, written $\mathcal{A} \cong_m \mathcal{B}$, if there is a sequence $(I_j)_{j \leq m}$ with the following properties

1. Every I_j is a nonempty set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$
2. (**Forth property**) For every $j < m$, $p \in I_{j+1}$ and $a \in A$ there is $q \in I_j$ s.t. $q \supseteq p$ and $a \in \mathrm{dom}(q)$
3. (**Back property**) For every $j < m$, $p \in I_{j+1}$, and $b \in B$ there is $q \in I_j$ s.t. $q \supseteq p$ and $b \in \mathrm{ran}(q)$

If $(I_j)_{j \leq m}$ has the properties 1,2 and 3, we write $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are ***m*-isomorphic via** $(I_j)_{j \leq m}$

Exercise 2.3.1. Supppose $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$. Then $(\tilde{I}_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ with $\tilde{I}_j := \{q \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \mid q \subseteq p \text{ for some } p \in I_j\}$. In particular, $\emptyset \mapsto \emptyset \in I_j$ for all $j \leq m$. Moreover
$\widetilde{W_j(\mathcal{A}, \mathcal{B})} = W_j(\mathcal{A}, \mathcal{B})$

Proof. Forth property: Suppose $j < m$, $p \in \tilde{I}_{j+1}$ and $a \in A$. Then $p \subseteq p' \in I_{j+1}$. Then we have $q' \in I_j$ with $a \in \mathrm{dom}(q')$ and $p' \subseteq q'$. We construct $q = p \cup \{(a, q'(a))\}$. $q \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ since $q' \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$. $\square$

Theorem 2.17. *For structures $\mathcal{A}$ and $\mathcal{B}$, $\bar{a} \in A^s$, $\bar{b} \in B^s$ and $m \geq 0$ the following are equivalent*

1. *The duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$*

2. $\bar{a} \mapsto \bar{b} \in W_m(\mathcal{A}, \mathcal{B})$ *and* $(W_j(\mathcal{A}, \mathcal{B}))_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$

3. *There is* $(I_j)_{j \leq m}$ *with* $\bar{a} \mapsto \bar{b} \in I_m$ *s.t.* $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$

4. $\mathcal{B} \models \varphi^m_{\bar{a}}[\bar{b}]$

5. $\bar{a}$ *satisfies in* $\mathcal{A}$ *the same formulas of quantifier rank* $\leq m$ *as* $\bar{b}$ *in* $\mathcal{B}$

Proof. $1 \to 2$. For each $\bar{a} \mapsto \bar{b} \in W_m(\mathcal{A}, \mathcal{B})$, by Lemma 2.7 the duplicator wins $G_0(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ as the duplicator wins $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$. For $j < m$, $\bar{a} \mapsto \bar{b} \in W_{j+1}(\mathcal{A}, \mathcal{B})$ and $a \in A$. We have $\bar{a}a \mapsto \bar{b}b \in W_j(\mathcal{A}, \mathcal{B})$ for some $b \in B$.

$2 \to 3$. Obvious

$3 \to 1$. Suppose that $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ and $\bar{a} \mapsto \bar{b} \in I_m$. We describe a winning strategy in $G_m(\mathcal{A}, \bar{a}, \mathcal{B}, \bar{b})$ for the duplicator: in his i-th move he should choose the element e_i s.t. for $p_i : \bar{a}e_1 \dots e_i \mapsto \bar{b}f_1 \dots f_i$ it is true that $p_i \subseteq q$ for some $q \in I_{m-i}$; this is always possible because of the back and forth properties of $(I_j)_{j \leq m}$ □

Corollary 2.18. *For structures* $\mathcal{A}, \mathcal{B}$ *and* $m \geq 0$ *the following are equivalent*

1. *the duplicator wins* $G_m(\mathcal{A}, \mathcal{B})$

2. $(W_j(\mathcal{A}, \mathcal{B}))_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$

3. $\mathcal{A} \cong_m \mathcal{B}$

4. $\mathcal{B} \models \varphi^m_{\mathcal{A}}$

5. $\mathcal{A} \equiv_m \mathcal{B}$

the equivalence of 3 and 5 is known as Fraïssé's Theorem.

Example 2.1. Let τ be the empty vocabulary and $\mathcal{A}$ and $\mathcal{B}$ be τ-structures. Suppose $\|A\| \geq m$ and $\|B\| \geq m$. Then $\mathcal{A} \cong_m \mathcal{B}$. In fact, $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ with $I_j := \{p \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \mid \|\mathrm{dom}(p)\| \leq m - j\}$

As a consequence the class $\mathrm{EVEN}[\tau]$ of finite τ-structures of even cardinality is not axiomatizable in first-order logic. In fact, for each $m > 0$, let $\mathcal{A}_m$ be a structure of cardinality m. Then $\mathcal{A}_m \in \mathrm{EVEN}[\tau]$ iff $\mathcal{A}_{m+1} \notin \mathrm{EVEN}[\tau]$, but $\mathcal{A}_m \cong_m \mathcal{A}_{m+1}$. Now apply Theorem 2.15.

Prove that for arbitrary τ that $\mathrm{EVEN}[\tau]$ is not axiomatizable.

Example 2.2. Let $\tau = \{<, \min, \max\}$ be a vocabulary for finite orderings. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite orderings, $\|A\| > 2^m$ and $\|B\| > 2^m$ and

$m \geq 1$. Then $\mathcal{A} \cong_m \mathcal{B}$. Hence the class of finite orderings of even cardinality is not axiomatizable in first-order logic. If we consider orderings as $\{<, S, \min, \max\}$-structures, the last statements remains true

Given any ordering $\mathcal{C}$, we define its distance function d by

$$d(a, a') := \|\{b \in C \mid (a < b \leq a') \text{ or } (a' < b \leq a)\}\|$$

And for $j \geq 0$, we introduce the "truncated" j-distance function d_j on $C \times C$ by

$$d_j(a, a') := \begin{cases} d(a, a') & \text{if } d(a, a') < 2^j \\ \infty \end{cases}$$

Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are finite orderings with $\|A\|, \|B\| > 2^m$. For $j \leq m$ set

$$I_j := \{p \in \mathrm{Part}(\mathcal{A}, \mathcal{B}) \mid d_j(a, a') = d_j(p(a), p(a')) \text{ for } a, a' \in \mathrm{dom}(p)\}$$

Then $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$: by assumption on the cardinalities of $\mathcal{A}$ and $\mathcal{B}$ we have $\{(\min^A, \min^B), (\max^A, \max^B)\} \in I_j$ for every $j \leq m$. To give a proof of the forth property of $(I_j)_{j \leq m}$, suppose $j < m, p \in I_{j+1}$ and $a \in A$. We distinguish two cases, depending on whether or the following condition

$$\text{there is an } a' \in \mathrm{dom}(p) \text{ s.t. } d_j(a, a') < 2^j$$

is satisfied. If the condition holds then there is exactly one $b \in B$ for which $p \cup \{(a, b)\})$ is a partial isomorphism preserving d_j-distances. Now assume that the condition doesn't hold. let $\mathrm{dom}(p) = \{a_1, \ldots, a_r\}$ with $a_1 < \cdots < a_r$. We restrict outselves to the case $a_i < a < a_{i+1}$ for some i. Then $d_j(a_1, a) = \infty$ and $d_j(a, a_{i+1}) = \infty$; hence $d_{j+1}(a_i, a_{i+1}) = \infty$ and therefore $d_{j+1}(p(a_i), p(a_{i+1})) = \infty$. Thus there is a b s.t. $p(a_i) < b < p(a_{i+1}), d_j(p(a_i), b) = \infty$ and $d_j(b, p(a_{i+1})) = \infty$. We can verify that $q := p \cup \{(a, b)\}$ is a partial isomorphism

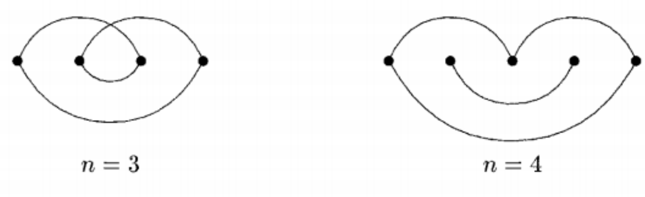
Example 2.3. Let $\tau = \{<, \min, \max\}$ be as in the preceding example and $\sigma = \tau \cup \{E\}$ with a binary relation symbol E. For $n \geq 3$ let $\mathcal{A}_n$ be the ordered τ-structure with $A_n = \{0, \ldots, n\}$, $\min^{A_n} = 0$, $\max^{A_n} = n$, where $<^{A_n}$ is the natural ordering on A_n and

$$E^{A_n} = \{(i, j) \mid |i - j| = 2\} \cup \{(0, n), (n, 0), (1, n-1), (n-1, 1)\}$$

(A_n, E^{A_n}) is a graph that is connected iff n is odd. Now let $m \geq 2$ and $l, k \geq 2^m$

3 Problem List

2.1



$n = 3$
$n = 4$