Continuous First Order Logic

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Contents

1 Day 1

Continuou first order logic a.k.a. continuous logic, continuous model theory, model theory for metric structures

Logic	Mathematics
1930s: Compactness thm	1960s: diaphantine problems over local fields
ultraproducts, saturation	nonstandard analysis
1970s: shelah: classification theorem	1996: mordell-lang conjecture
stability theory	
1980s: o-minimal theory	2011: pila André-oort conjecture

background:

1960s: Applications of ultraproducts to Banach spaces. Krivine

1976: Ward Henson nonstandard hulls of Banach spaces: "Henson's logic" positive bounded formulas with an approximate semantics

Later, 2002 with Iovino. Ultraproducts in analysis

2003, Ben Yaacov, compact abstract theories, Positive model theory and compact abstract theories

2005 model theory for metric structures

many valued logic: Łukasiewicz, Chang-Keisler.

Truth values	$[0,1]$
quantifiers	$\inf, \sup$
equality	metric $d(\cdot,\cdot)$

Analogy betweeen CFO and FOL
Let (M,d) be a complete, bounded metric space
$d : M \times M \to \mathbb{R}^{\geq 0}$ is a **metric** on M

1. $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x = y$

2. $d(x,y) = d(y,x)$

3. $d(x,y) \leq d(x,z) + d(z,y)$

(M,d) is **bounded** if $\exists k > 0, \forall x,y \in M$, $d(x,y) < k$ or $\operatorname{diam}(M,d) := \sup_{x,y \in M} d(x,y) < k$
(M,d) is **complete** if every cauchy sequence convergencs
A **predicate** on M is a **uniformly continuous** function: $M^n \to [a,b] \subseteq \mathbb{R}$ for some $n \geq 1$
$P : M^n \to (0,1)$ is **uniformly continuous** if $\forall \epsilon > 0 \exists \delta > 0 \forall x,y \in M^n$ $d(x,y) < \delta \to |p(x) - p(y)| < \epsilon$
A function on M is a uniformly continuous function: $M^n \to M$ fro some $n \geq 1$.
For simplicity, (M,d) is bounded by 1, predicates have values on $[0,1]$. 0 is true and 1 is false.

Remark. In FOL, predicates: $M^n \to \{0,1\}$
In CFO, predicates: $M^n \to [0,1]$

A **metric structure** $\mathcal{M}$ based on (M,d) consists of a family $(R_i \mid i \in I)$ of predicates on M, a family $(F_j \mid j \in J)$ of functions on M and a family $(a_k \mid k \in K)$ of distinguished elements of M
We denote a metric structure as

$$\mathcal{M} = (M, R_i, F_j, a_k \mid i \in I, j \in J, k \in K)$$

Key restrictions:

1. complete bounded

2. bounded interval (R_i)

3. uniformly continuous (R_i, F_j)

Example 1.1. 1. A complete bounded metric space (M, d) with no additional structure

2. Given a first-order structure $\mathcal{M}$, define a discrete metric on it.

$$d(a,b) = \begin{cases} 1 & \text{if } a \neq b \\ 0 & \end{cases}$$

predicates taking values in $\{0,1\}$. Then $(\mathcal{M}, d)$ is a metric structure. Thus CFO is a generalization of FOL

3. Probability algebras are boolean algebras of events in probabilities space.

For each metric structure $\mathcal{M} = (M, R_i, F_j, a_k \mid i \in I, j \in J, k \in K)$, we associate a **signature**

$$L = \{p_i, f_i, c_k \mid i \in I, j \in J, k \in K\}$$

where each p_i is a **predicate symbol** corresponding to predicate R_i on $\mathcal{M}$, each f_j is a **function symbol** corresponding to predicate F_j on $\mathcal{M}$, and each c_k is a **constant symbol** corresponding to predicate a_k on $\mathcal{M}$.

We associate to each symbol its arity like in FOL.

Moreover, in CFO, we associate to each predicate symbol p a closed bounded interval I_p (usually $I_p = [0,1]$) and a **modulus of uniform continuity** Δ_p, i.e., a function $\Delta_p : (0,1] \to (0,1])$ satisfify $\forall \epsilon > 0 \forall x, y \in M^n$, if $d(x,y) < \Delta_p(\epsilon)$ (defines a δ by Δ_p) then

$$\left|p^{\mathcal{M}}(x) - p^{\mathcal{M}}(y)\right| < \epsilon$$

This Δ_p does **not** depend on $\mathcal{M}$

We associate to each function a modulus of uniformly continuity Δ_f, i.e. a function $\Delta_f : (0,1] \to (0,1]$ satisfying $\forall \epsilon > 0 \forall x, y \in M^n$, if $d(x,y) < \Delta_f(\epsilon)$ then

$$d(f^{\mathcal{M}}(x, f^{\mathcal{M}}(y))) < \epsilon$$

Finally, L provides D_L which is a bound on the diameter of (M, d). For simplicity, $D_L = 1$ and $I_p = [0,1]$.

We say $\mathcal{M}$ an L-structure

Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures. An **embedding** from $\mathcal{M}$ to $\mathcal{N}$ is a metric space **isometry** $T : (\mathcal{M}, d^{\mathcal{M}}) \to (N, d^{\mathcal{N}})$ s.t.

1. $\forall x, y \in M$, $d^{\mathcal{N}}(T(x), T(y)) = d^{\mathcal{M}}(x,y)$

2. for each n-ary predicate symbol p of L

$$\forall a_1, \ldots, a_n \in M, p^{\mathcal{N}}(T(a_1), \ldots, T(a_n)) = p^{\mathcal{M}}(a_1, \ldots, a_n)$$

3. for each n-ary function symbol f of L

$$\forall a_1, \ldots, a_n \in M, f^{\mathcal{N}}(T(a_1), \ldots, T(a_n)) = T(f^{\mathcal{M}}(a_1, \ldots, a_n))$$

4. for each constant symbol c of L

$$c^{\mathcal{N}} = T(c^{\mathcal{M}})$$

An **isomorphism** is a surjective embedding. We say that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic and write $\mathcal{M} \cong \mathcal{N}$ if there exists an isomorphism between $\mathcal{M}$ and $\mathcal{N}$

If $M \subseteq N$ and the inclusion map is an embedding of $\mathcal{M}$ into $\mathcal{N}$, then we say $\mathcal{M}$ is a **substructure** of $\mathcal{N}$ and write $\mathcal{M} \subseteq \mathcal{N}$

More on modulus of uniform continuity. See [BBHU] appendix to section 2 on pages 322-327

Fix a signature L for metric structure

Symbols of L

- nonlogical symbols: predicates, functions, constants
- logical symbols: d-binary predicate $(=)$, V_L - an infinite set of variables, $u : [0,1]^n \to [0,1]$ continuous connectives (since continus $\Rightarrow$ uniformly continuous), sup, inf (quantifiers)

The **cardinality** of L, denoted by $\mathrm{Card}(L)$ is the smallest infinite cardinal $\geq \#\{\text{nonlogical symbols}\}$

terms of L

1. variables and constant symbols
2. $f(t_1, \ldots, t_n)$

Atomic formulas of L

1. $P(t_1, \ldots, t_n)$ wher eP is an n-ary predicate symbol
2. $d(t_1, t_2)$ like "=" in FOL

Formulas of L

1. atomic formulas
2. if $u : [0,1]^n \to [0,1]$ is continuous and $\varphi_1, \dots, \varphi_n$ are L-formulas, then $u(\varphi_1, \dots, \varphi_n)$ is an L-formula
3. if φ is an L-formula and x is a variable, then $\sup_x \varphi$ and $\inf_x \varphi$ are L-formulas

This definition is not a good one

1. too general, uncountably continuous functions. we only need to concern a dense subset of it
2. too restrict, in order to develop a good nition of "definability", need formulas closed under certain kinds of limits

We write $t(x_1, \dots, x_n)$ or $\varphi(x_1, \dots, x_n)$ to indicate is free variables are among $x_1, \dots, x_n$

Example 1.2. Let D_0 denote the set of repeating decimals. Then (D_0, d_0) (d_0 is subtraction) is a **pseudometric** space. Because $d_0(0.\dot{9}, 1) = 0$ but $0.\dot{9} \neq 1$. Consider its quotient $(D, d) = (D_0, d_0)/\sim$ where $x \sim y$ if $d_0(x, y) = 0$. Then (D, d) is a metric space, but it is not complete. Actually $(D, d) = (\mathbb{Q}, d)$. Take its completion, we get $(\bar{D}, \bar{d}) = (\mathbb{R}, d)$.

pseudometric space -> metric space -> complete metric space

Fix a signature L. Let (M_0, d_0) be a pseudometric space satisfying $\operatorname{diam}(M_0, d_0) \leq D_L$. An L**-prestructure** $\mathcal{M}_0$ based on (M_0, d_0) is a structure satisfying

1. for each predicate symbol p of L $p^{\mathcal{M}_0} : M_0^n \to I_p$ has Δ_p as a modulus of uniform continuity
2. for each function symbol f of L, $f^{\mathcal{M}_0} : M_0^n \to M_0$ has Δ_f as a modulus of uniform continuity
3. for each constant symbol c of L, $c^{\mathcal{M}_0} \in M_0$

Given L-prestructure $\mathcal{M}_0$, we define its **quotient** prestructure as follows:

Let $(M, d) = (M_0, d_0)/\sim$, where $x \sim y$ iff $d_0(x, y) = 0$. Let $\pi : M_0 \to M$ be the quotient map. Then

1. for each predicate symbol p of L, define $p^{\mathcal{M}} : M^n \to I_p$ by

$$p^{\mathcal{M}}(\pi(x_1), \dots, \pi(x_n)) = p^{\mathcal{M}_0}(x_1, \dots, x_n)$$

2. for each function symbol f of L, define $f^{\mathcal{M}} : M^n \to M$ by

$$f^{\mathcal{M}}(\pi(x_1), \dots, \pi(x_n)) = \pi(f^{\mathcal{M}_0}(x_1, \dots, x_n))$$

3. for each constant symbol c of L, define $c^{\mathcal{M}} = \pi(c^{\mathcal{M}_0})$

Clearly,

1. $\operatorname{diam}(M, d) = \operatorname{diam}(M_0, d)$
2. $p^{\mathcal{M}}$ is well-defined and has Δ_p as its modulus of uniform continuity
3. $f^{\mathcal{M}}$ is well-defined and has Δ_f as its modulus of uniform continuity (these 2 proofs are in the appendix)

Thus (M, d) is an L-prestructure based on a possibly incomplete metric space.

Finally we take a **completion** of $\mathcal{M}$, denoted by L-structure $\mathcal{N}$

1. for each predicate symbol p. define $p^{\mathcal{N}} : N^n \to I_p$ as the unique extension of $p^{\mathcal{M}}$ with the same Δ_p (Check!)
2. for each f, $f^{\mathcal{N}} : N^n \to N$ is the unique extension of $f^{\mathcal{M}}$ with the same Δ_f
3. for each constant c, $c^{\mathcal{N}} = c^{\mathcal{M}}$

Let $\mathcal{M}$ be an L-prestructure and let $A \subset M$. We extend L to a signature $L(A)$ by adding a new constant symbol $c(a)$ to L for each $a \in A$. $(c(a))^{\mathcal{M}=a}$. We call $c(a)$ the **name** of a in $L(A)$. Consider $L(M)$-terms $t(x_1, \dots, x_n)$, define exactly as in FOL

$$t^{\mathcal{M}} : M^n \to M$$

The interpretation of t in $\mathcal{M}$

Key definitions of semantics in CFO

1. $(d(t_1, t_2))^{\mathcal{M}} = d^{\mathcal{M}}(t_1^{\mathcal{M}}, t_2^{\mathcal{M}})$ for all t_1, t_2
2. $(p(t_1, \dots, t_n))^{\mathcal{M}} = p^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})$ for all n-ary predicate symbol p and all $t_1, \dots, t_n$
3. for all $L(M)$-sentences $\sigma_1, \dots, \sigma_n$ and all continuous function $\mu : [0,1]^n \to [0,1]$

$$(\mu[\sigma_1, \dots, \sigma_n])^{\mathcal{M}} = \mu(\sigma_1^{\mathcal{M}}, \dots, \sigma_n^{\mathcal{M}})$$

4. for all $L(M)$-formulas $\varphi(x)$

$$(\sup_x(\varphi(x)))^{\mathcal{M}} = \sup_{a\in M}(\varphi(a))^{\mathcal{M}} \in [0,1]$$

Given $L(M)$-formula $\varphi(x_1,\dots,x_n)$, we let $\varphi^{\mathcal{M}}$ denote the function $M^n \to [0,1]$ defined by

$$\varphi^{\mathcal{M}}(a_1,\dots,a_n) = (\varphi(a_1,\dots,a_n))^{\mathcal{M}}$$

Fact: $\varphi^{\mathcal{M}}$ is a uniformly continuous function

Theorem 1.1. *Let $t(x_1,\dots,x_m)$ be an L-term and $\varphi(x_1,\dots,x_n)$ an L-formula. Then there exists functions Δ_t and $\Delta_\varphi : (0,1] \to (0,1]$ s.t. for every L-prestructure $\mathcal{M}$, Δ_t is a modulus of uniform continuity for the function $t^{\mathcal{M}} : M^n \to M$ and Δ_φ is a modulus of uniform continuity for the predicate $\varphi^{\mathcal{M}} : M^n \to [0,1]$*

Proof. Induction □

Theorem 1.2. *pseudometric space $(M_0,d_0) \to$ quotient $(M,d) \to$ completion (N,d)*

Let $t(x_1,\dots,x_n)$ be an L-term and $\varphi(x_1,\dots,x_n)$ be an L-formula. Then

1. $t^{\mathcal{M}}(\pi(a_1),\dots,\pi(a_n)) = t^{\mathcal{M}_0}(a_1,\dots,a_n)$
2. $t^{\mathcal{N}}(b_1,\dots,b_n) = t^{\mathcal{M}}(b_1,\dots,b_n)$
3. $\varphi^{\mathcal{M}}(\pi(a_1),\dots,\pi(a_n)) = \varphi^{\mathcal{M}_0}(a_1,\dots,a_n)$
4. $\varphi^{\mathcal{N}}(b_1,\dots,b_n) = \varphi^{\mathcal{M}}(b_1,\dots,b_n)$

Proof. in 3. key step is that π is surjective
in 4, key step is that $\varphi^{\mathcal{N}}$ is continuous and M is dense in N □

Two L-formulas $\varphi(x_1,\dots,x_n)$ and $\psi(x_1,\dots,x_n)$ are **logically equivalent** if

$$\varphi^{\mathcal{M}}(a_1,\dots,a_n) = \psi^{\mathcal{M}}(a_1,\dots,a_n)$$

for every L-structure $\mathcal{M}$

The **logical distance** d_L between φ and ψ is

$$d_L(\varphi,\psi) = \sup_{\mathcal{M}} \sup_{a_1,\dots,a_n\in M} \left|\varphi^{\mathcal{M}}(a_1,\dots,a_n) - \varphi^{\mathcal{M}}(a_1,\dots,a_n)\right|$$

Remark. 1. This defines a pseudometric

2. $d_L(\varphi, \psi) = 0$ iff $\varphi \sim_L \psi$

The space of L-formulas is too big. **density character** is the smallest dense subset w.r.t. logical distance.

By Weierstrass theorem, there is a countable set of functions that is dense in the set of all continuous functions w.r.t sup-distance. We may use this countable set of functions to build connectives. Then

1. the total number of constructed formulas is $\text{Card}(L)$
2. every L-formulas can be approximated arbitrarily closely in logical distance by a formula constructed using restricted connectives

2 Day 2

Definition 2.1. An L**-condition** E is of the form $\varphi = 0$, where φ is an L-formula. We call E **closed** if φ is closed, i.e., φ is an L-sentence

If E is the $L(M)$-condition $\varphi(x_1, \dots, x_n) = 0$ and $a_1, \dots, a_n \in M$, we say E is **true of** $a_1, \dots, a_n$ in $\mathcal{M}$ and we write $\mathcal{M} \vDash E[a_1, \dots, a_n]$ if $\varphi^{\mathcal{M}}(a_1, \dots, a_n) = 0$

Let E_i be the L-condition $\varphi_i(x_1, \dots, x_n) = 0$. We say E_1 and E_2 are **logically equivalent** if for every L-structure $\mathcal{M}$ and every $a_1, \dots, a_n$ we have

$$\mathcal{M} \vDash E_1[a_1, \dots, a_n] \quad \text{iff} \quad \mathcal{M} \vDash E_2[a_1, \dots, a_n]$$

$\varphi = \psi$ is an abbreviation for the condition $|\varphi - \psi| = 0$, where $|\cdot| : [0,1]^2 \to [0,1], (t_1, t_2) \mapsto |t_1 - t - 2|$ is a connective.

$\varphi \leq \psi$ iff $\varphi \dot{-} \psi = 0$

In $[0,1]$-valued logic, $\varphi \leq \psi$ is like $\varphi \to \psi$ in FOL. Since from $\psi \leq r$ we have $\varphi \leq r$ for all $r \in [0,1]$

Fix a signature L for metric structure.

Definition 2.2. A **theory** T is a set of closed L-conditions. We say $\mathcal{M}$ is a model of T and write $\mathcal{M} \vDash T$ if $\mathcal{M} \vDash E$ for every condition $E \in T$.

Let $\text{Mod}_L(T)$ be the collection of all models of T

The **theory of** $\mathcal{M}$, denoted by $\text{Th}(\mathcal{M})$, is the set of closed L-conditions that are true in $\mathcal{M}$.

If T is a theory of this form, then T is **complete**.

We say E is a **logical consequence** of T and write $T \vDash E$ if $\mathcal{M} \vDash E$ holds for every model $\mathcal{M}$ of T.

Remark. 1. models are complete metric spaces.

2. Let $\mathcal{M}_0$ be an L-prestructure s.t. $\varphi^{\mathcal{M}_0} = 0$ for every condition $\varphi = 0$ in T. Then by Theorem 1.2, the completion of the canonical quotient of $\mathcal{M}_0$ is a model of T. ($\mathcal{M}_0$ is a **premodel**)

Definition 2.3. 1. We say $\mathcal{M}$ and $\mathcal{N}$ are **elementary equivalent** and write $\mathcal{M} \equiv \mathcal{N}$ if $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$ for all L-sentences σ

2. If $\mathcal{M} \subseteq \mathcal{N}$, we say that $\mathcal{M}$ is an **elementary substructure** of $\mathcal{N}$ and write $\mathcal{M} \preceq \mathcal{N}$ if whenever $\varphi(x_1, \dots, x_n)$ is an L-formula and $a_1, \dots, a_n \in M$ we have

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \varphi^{\mathcal{N}}(a_1, \dots, a_n)$$

We also say $\mathcal{N}$ is an **elementary extension** of $\mathcal{M}$

3. $F : A \subseteq M \to N$ is an **elementary map** if whenever $\varphi(x_1, \dots, x_n)$ is an L-formula and $a_1, \dots, a_n \in \mathrm{dom}(F)$ we have

$$\varphi^{\mathcal{M}}(a_1, \dots, a_n) = \varphi^{\mathcal{N}}(F(a_1), \dots, F(a_n))$$

4. An **elementary embedding** of $\mathcal{M}$ into $\mathcal{N}$ is a function $M \to N$ that is an elementary map from $\mathcal{M}$ into $\mathcal{N}$

Remark. 1. elementary map is distance preserving, and thus is an embedding

2. $\mathcal{M} \cong \mathcal{N} \Rightarrow \mathcal{M} \equiv \mathcal{N}$

We say S of L-formulas is **dense w.r.t. logical distance** if for every L-formula $\varphi(x_1, \dots, x_n)$ and every $\epsilon > 0$ there is $\psi(x_1, \dots, x_n)$ in S s.t. for every L-structure $\mathcal{M}$ and all $a_1, \dots, a_n \in M$ we have

$$\left|\varphi^{\mathcal{M}}(a_1, \dots, a_n) - \psi^{\mathcal{M}}(a_1, \dots, a_n)\right| \le \epsilon$$

Proposition 2.4 (Tarski-Vaught Test for $\preceq$)**.** *Let S be dense w.r.t. logical distance. Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures with $\mathcal{M} \subseteq \mathcal{N}$. Then the following are equivalent*

1. $\mathcal{M} \preceq \mathcal{N}$

2. For every L-formula $e\varphi(x_1, \dots, x_n, y)$ in S and all $a \in M^n$

$$\inf\{\varphi^{\mathcal{N}}(a, b) \mid b \in N\} = \inf\{\varphi^{\mathcal{N}}(a, c) \mid c \in M\} \qquad (\star)$$

Proof. $1 \to 2$. By 1, we have

$$\begin{aligned}\inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,b) \mid b \in N\} &= \left(\inf_y \varphi(a_1,\dots,a_n,y)\right)^{\mathcal{N}} \\ &= \left((\inf_y \varphi(a_1,\dots,a_n,y))\right)^{\mathcal{M}} \\ &= \inf\{\varphi^{\mathcal{M}}(a_1,\dots,a_n,c) \mid c \in M\} \\ &= \inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,c) \mid c \in M\}\end{aligned}$$

$2 \to 1$. First we show $\star$ holds for all L-formulas $\varphi(x_1,\dots,x_n,y)$. $\forall \epsilon > 0$, take $\varphi(x_1,\dots,x_n,y) \in S$ s.t.

$$\sup_{\mathcal{M}} \sup_{a_1,\dots,a_n \in M} \left|\varphi^{\mathcal{M}}(a_1,\dots,a_n,b) - \psi^{\mathcal{M}}(a_1,\dots,a_n,b)\right| \le \epsilon$$

Let $a_1,\dots,a_n \in M$ then we have

$$\begin{aligned}\inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,b) \mid b \in M\} &\le \inf\{\psi^{\mathcal{N}}(a_1,\dots,a_n,b) \mid b \in M\} + \epsilon \\ &= \inf\{\psi^{\mathcal{N}}(a_1,\dots,a_n,c) \mid c \in N\} + \epsilon \\ &\le \inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,c) \mid c \in N\} + 2\epsilon\end{aligned}$$

Let $\epsilon \to 0$, then

$$\inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,b) \mid b \in M\} \le \inf\{\varphi^{\mathcal{N}}(a_1,\dots,a_n,c) \mid c \in N\}$$

Hence $\star$ holds for all L-formulas φ.

Then by incution on the complexities of φ and $\star$ we have $\varphi^{\mathcal{M}}(a_1,\dots,a_n) = \varphi^{\mathcal{N}}(a_1,\dots,a_n)$ for all $a_1,\dots,a_n \in M$ $\square$

Definition 2.5. Let I be a nonempty set. A **filter** on I is a collection F of subsets of I satisfies

1. $\emptyset \notin F$ and $I \in F$
2. for all $A, B \in F$, $A \cap B \in F$
3. for all $A \in F$, if $A \subseteq B \subseteq I$ then $B \in F$

A filter F is an **ultrafilter** if it is maximal under $\subseteq$ among filters on I

F is **principal** if there is a subset $A \subseteq I$ s.t. F is exactly the collection of all sets B that satisfy $A \subseteq B \subseteq I$.

non-principal is also called as **free**

Definition 2.6. S is a collection of I. We say that S has **finite intersection property** (FIP) if $\forall n \in \mathbb{N}$, $\forall$ finite subset collection $\{A_1, \dots, A_n\}$ of S, $A_1 \cap \dots \cap A_n \neq \emptyset$

Lemma 2.7. *Let I be a nonempty set and let S be a collection of subsets of I. There exists a filter F on I which contains S iff S has the FIP*

Remark. The smallest filter on I containing S is called the **filter generated by** S

Lemma 2.8. *Let F be a filter on a nonempty set I. Then F is an ultrafilter iff $\forall A \subset I$, either $A \in F$ or $A^c \in F$.*

Remark. principal ultrafilters are trivial

Theorem 2.9. *Let I be a nonempty set. Then every filter on I is contained in an ultrafilter on I.*

Proof. Zorn's lemma □

Corollary 2.10. *Let I be a nonempty set and let S be a collection of subset of I. If S has the FIP, then there is an ultrafilter on I that contains S.*

Fix a first order signature L. Let I be a nonempty set and let U be a fixed ultrafilter on I. Consider an I-indexed family of L-structures $\mathcal{A}_i$. Let $A = \prod_{i \in I} A_i$ be the Cartesian product of the sets A_i. Let $f, g \in A$. We define a relation on A

$$f \sim g \quad \text{iff} \quad \{i \mid f(i) = g(i)\} \in U$$

Lemma 2.11. *The relation $\sim$ is an equivalence relation on A*

Then $A/\sim$ is the ultraproduct of the set A_i w.r.t. the ultrafilter U on I

We let $\prod_U A_i$ denote $A/\sim$, the collection of all equivalence classes $\{[f] \mid f \in \prod_{i \in I} A_i\}$

Definition 2.12. The **ultraproduct** $\prod_U \mathcal{A}_i$ is defined to be the L-structure

1. the universe of $\prod_U \mathcal{A}_i$ is $\prod_U A_i$
2. for each constant c in L, define $f \in A$ by $f(i) = c^{\mathcal{A}_i}$
$$c^{\prod_U \mathcal{A}_i} = f/\sim$$
3. for each predicate P in L
$$P^{\prod_U \mathcal{A}_i}(f_1/\sim, \dots, f_n/\sim) \quad \text{iff} \quad \{i \in I \mid P^{\mathcal{A}_i}(f_1(i), \dots, f_n(i))\} \in U$$

4. for each function F in L

$$F^{\prod_U \mathcal{A}_i}(f_1/\sim, \dots, f_n/\sim) = f/\sim$$

where $f \in A$ is defined by $f(i) = F^{\mathcal{A}_i}(f_1(i), \dots, f_n(i))$

Need to check they are well-defined

An **ultrapower** of $\mathcal{A}$ is an ultraproduct $\prod_U \mathcal{A}_i$ with $\mathcal{A}_i = \mathcal{A}$ for all $i \in I$

Theorem 2.13 (Łoś's theorem (fundamental theorem for ultraproducts))**.** *For every L-formula $\varphi(x_1, \dots, x_n)$ and every $f/\sim = (f_1/\sim, \dots, f_n/\sim)$, we have*

$$\prod_U \mathcal{A}_i \vDash \varphi[f/\sim] \quad \textit{iff} \quad \{i \in I \mid \mathcal{A}_i \vDash \varphi[f_1(i), \dots, f_n(i)]\} \in U$$

Corollary 2.14. *if σ is an L-sentence, then $\prod_U \mathcal{A}_i \vDash \sigma$ iff $\{i \in I \mid \mathcal{A}_i \vDash \sigma\} \in U$*

Let X be a topological space and let $(x_i)_{i\in I}$ be a family of elements of X. If D is an ultrafilter on I and $x \in X$, we write $\lim_{i,D} x_i = x$ (ultra limit) and say x is the D**-limit of** $(x_i)_{i\in I}$ if $\forall$ open $U \ni x$, $\{i \in I \mid x_i \in U\} \in D$

Fact: X is a compact Hausdorff space (e.g. $X = [0,1]$) iff for every family $(x_i)_{i\in I}$ in X and every ultrafilter D on I the D-limit of $(x_i)_{i\in I}$ exists and is unique.

Lemma 2.15. *Suppose X, X' are topological spaces and $F : X \to X'$ is continuous. For every family $(x_i)_{i\in I}$ from X and every ultrafilter D on I, we have*

$$\lim_{i,D} x_i = x \Rightarrow \lim_{i,D} F(x_i) = F(x)$$

where the ultralimits are taken in X and X' respectively

Proof. Take open $U \ni F(x)$ in X'. Since F is continuous, $F^{-1}(U)$ is open in X. And $F^{-1}(U) \ni x$. If x is the D-limit of $(x_i)_{i\in I}$ there is $A \in D$ s.t. $x_i \in F^{-1}(U)$ for all $i \in A$ and thus $F(x_i) \in U$ □

Definition 2.16. Let $((M_i, d_i) \mid i \in I)$ be a family of bounded metric spaces with diameter $\leq k$. Let D be an ultrafilter on I. Define d on $\prod_{i\in I} M_i$ by $d(x,y) = \lim_{i,D} d_i(x_i, y_i)$, when $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$.

Check: d is a pseudometric on $\prod_{i\in I} M_i$

For $x, y \in \prod_{i\in I} M_i$, define $x \sim_D y$ iff $d(x,y) = 0$

Then $\sim_D$ is an equivalence relation, so we may define

$$\left(\prod_{i\in I} M_i\right)_D = \left(\prod_{i\in I} M_i\right) / \sim_D$$

Later we will see its complete

The pseudometric d on $\prod_{i\in I} M_i$ induces a metric d on $(\prod_{i\in I} M_i)_D$

The space $((\prod_{i\in I} M_i)_D, d)$ is the D-**ultraproduct** of $((M_i, d_i) \mid i \in I)$.

We denote $(x_i)_{i\in I}/\sim_D$ by $((x_i)_{i\in I})_D$

If $(M_i, d_i) = (M, d)\ \forall i \in I$. The space $(\prod_{i\in I} M_i)_D$ is called the D-**ultrapower** of M and denoted by $(M)_D$

$T : M \to (M)_D$, $x \mapsto ((x_i)_{i\in I})_D$, where $\forall i \in I$, $x_i = x$ is a **diagonal embedding**. its an isometric embedding

If (M, d) is compact, then it is easy to show $((x_i)_{i\in I})_D = T(x)$, i.e., the diagonal embedding is surjective.

Fact: every ultrapower of a closed bounded interval may be canonically identified wiht the interval itself. e.g. $([0,1])_D = [0,1]$

Proposition 2.17. *Let $((M_i, d_i) \mid i \in I)$ be a family of complete, uniformly bounded metric space. Let D be an ultrafilter on I and let (M, d) be the D-ultraproduct of $((M_i, d_i) \mid i \in I)$. The metric space (M, d) is complete*

Proof. let $(x^k)_{k\geq 1}$ be a Cauchy sequence in (M, d). WLOG, we assume that $d(x^k, x^{k+1}) < \frac{1}{2^k}$ holds for all $k \geq 1$. We want to show it has a limit.

For each $k \geq 1$ let x^k be represente by the family $(x_i^k)_{i\in I}$. For each $m \geq 1$, let $A_m = \{i \in I \mid d_i(x_i^k, x_i^{k+1}) < \frac{1}{2k} \forall k \leq m\}$. Note $A_1 \supseteq A_2 \supseteq \ldots A_n \neq \emptyset \supseteq \ldots$ and each $A_i \in D$.

We define a family $(y_i)_{i\in I}$ as follows. If $i \notin A_1$ then we take y_i arbitrarily. If $i \in A_m \setminus A_{m+1}$ for some $m \geq 1$, then we set $y_i = x_i^{m+1}$. If $\forall m \geq 1$, $i \in A_m$, then $(x_i^m)_{m\geq 1}$ is a Cauchy sequence, we take y_i to be its limit.

Then for each $m \geq 1$ each $i \in A_m$ we have $d_i(x_i^m, y_i) \leq 2^{-m+1}$. It follows that $((y_i)_{i\in I})_D$ is the limit of $(x^k)_{k\geq 1}$ in (M, d) $\square$