Topology

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1 Topological Spaces and Continuous Functions

1.1 Topological Spaces

Definition 1.1. A **topology** on a set is a collection $\mathcal{T}$ of subsets of X having the following properties

1. $\emptyset$ and X are in $\mathcal{T}$
2. The union of the elements of any subcollection of $\mathcal{T}$ is in T
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**

Example 1.1. Consider $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n},\frac{1}{n}) = \{0\}$. $(-1/n, 1/n)$ is open but $\{0\}$ is not open in $\mathbb{R}$.

If $X$is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if $U \in \mathcal{T}$

Example 1.2. If X is any set, the collection of all subsets of X is a topology on X; it is called the **discrete topology**. The collection consisting of X and $\emptyset$ only is also a topology on X; we shall call it the **indiscrete topology**

Example 1.3. Let X be a set; let $\mathcal{T}_f$ be the collection of all subsets U of Xs.t. $X - U$ either is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the **finite complement topology**. If $\{U_\alpha\}$ is an indexed family of nonempty elements of $\mathcal{T}_f$.

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$$

Definition 1.2. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topology on a given set X. If $\mathcal{T}' \supset \mathcal{T}$ we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$ we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ or **strictly coarser**. We say $\mathcal{T}$ is **comparable** with $\mathcal{T}$ is either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$

1.2 Basis for a Topology

Definition 1.3. If X is a set, a **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis element**) s.t.

1. for each $x \in X$, there is at least one basis element B s.t. $x \in B$
2. if $x \in B_1 \cap B_2$, then there is a basis element B_3 s.t. $x \in B_3 \subset B_1 \cap B_2$

If $\mathcal{B}$ satisfies these conditions, then we define the **topology** $\mathcal{T}$ **generated by** $\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis $B \in \mathcal{B}$ s.t. $x \in B \subset U$.

Now we show that $\mathcal{T}$ is indeed a topology. Take an indexed family $\{U_\alpha\}_{\alpha \in J}$ of elements of $\mathcal{T}$, we show that

$$U = \bigcup_{\alpha \in J} U_\alpha$$

belongs to $\mathcal{T}$. Given $x \in U$, there is an index α s.t. $x \in U_\alpha$. Since U_α is open, there is a basis element B s.t. $x \in B \subset U_\alpha$. Then $x \in B$ and $B \subset U$, so U is open.

If $U_1, U_2 \in \mathcal{T}$, then given $x \in U_1 \cap U_2$. we choose $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. By the second condition for a basis we have $x \in B_3 \subset B_1 \cap B_2$. Hence $x \in B_3 \subset U_1 \cap U_2$.

Lemma 1.4. *Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.*

Proof. Given a collection of elements of $\mathcal{B}$, they are also elements of $\mathcal{T}$. Because $\mathcal{T}$ is a topology, their union is in $\mathcal{T}$.

Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x for B s.t. $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$ $\square$