Category Theory

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Contents

1 Categories

1.1 Examples of categories

Definition 1.1. A **functor**

$$F : \mathbf{C} \to \mathbf{C}$$

between categories **C** and **D** is a mapping of objects to objects and arrows to arrows, in such a way that

1. $F(f : A \to B) = F(f) : F(A) \to F(B)$
2. $F(1_A) = 1_{F(A)}$
3. $F(g \circ f) = F(g) \circ F(f)$

2 Abstract structures

2.1 Products

Definition 2.1. In any category **C**, a **product diagram** for the objects A and B consists of an object P and arrows

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following UMP:
Given any diagram of the form

$$A \xleftarrow{x_1} X \xrightarrow{x_2} B$$

there exists a unique $u : X \to P$ making the diagram

$$\begin{tikzcd}
& X \arrow[dl, "x_1"'] \arrow[d, dashed, "u"] \arrow[dr, "x_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}$$

2.2 Categories with products

Let **C** be a category that has a product diagram for every pair of objects. Suppose we have objects and arrows

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}$$

with indicated products. Then we write

$$f \times f' : A \times A' \to B \times B$$

for $f \times f' = \langle f \circ p_1, f' \circ p_2 \rangle$

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times A' \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, dashed, "f \times f'"] & A' \arrow[d, "f'"] \\
B & B \times B' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B'
\end{tikzcd}$$

In this way, if we choose a product for each pair of objects, we get a functor

$$\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

To prove

$$(A \times B) \times C \cong A \times (B \times C)$$

Consider

$$\begin{tikzcd}
 & A \times (B \times C) \arrow[r] \arrow[ddl] & B \times C \arrow[dl] \arrow[ddr] & \\
 & B & & \\
A & A \times B \arrow[l] \arrow[u] \arrow[dr] & (A \times B) \times C \arrow[l] \arrow[r] \arrow[uul, dashed, "g"] \arrow[uu, dashed] & C \\
 & & B & \\
 & A \times (B \times C) \arrow[r] \arrow[uul] \arrow[uu, dashed] \arrow[uur, dashed, "f"] & B \times C \arrow[u] \arrow[uur] &
\end{tikzcd}$$

Given no objects, there is an object 1 with no maps, and give nany other object X and no maps, there is a unique arrow

$$! : X \to 1$$

Definition 2.2. A category **C** is said to **have all finite products**, if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category **C has all (small) products** if every set of objects in **C** has a product

2.3 Hom-sets

In this section, we assume that all categories are locally small

Given any objects A and B in category **C**,we write

$$\mathrm{Hom}(A, B) = \{f \in \mathbf{C} \mid f : A \to B\}$$

and call such a set of arrows a **Hom-set**

Note that any arrow $g : B \to B'$ in **C** induces a function

$$\mathrm{Hom}(A, g) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$$
$$(f : A \to B) \mapsto (g \circ f : A \to B \to B')$$

Let's show that this determines a functor

$$\mathrm{Hom}(A, -) : \mathbf{C} \to \mathbf{Sets}$$

called the (covariant) **representable functor** of A. We need to show that

$$\mathrm{Hom}(A, 1_X) = 1_{\mathrm{Hom}(A,X)}$$

and that

$$\mathrm{Hom}(A, g \circ f) = \mathrm{Hom}(A, g) \circ \mathrm{Hom}(A, f)$$

For any object P, a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$ determine an element (p_1, p_2) of the set

$$\mathrm{Hom}(P, A) \times \mathrm{Hom}(P, B)$$

Now given any arrow

$$x : X \to P$$

composing with p_1 and p_2 gives a pair of arrows $x_1 = p_1 \circ x : X \to A$ and $x_2 = p_2 \circ x : X \to B$

In this way, we have a function

$$\theta_X = (\mathrm{Hom}(X, p_1), \mathrm{Hom}(X, p_2)) : \mathrm{Hom}(X, P) \to \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$

defined by

$$\theta_X(x) = (x_1, x_2)$$

Proposition 2.3. *A diagram of the form*

$$A \xleftarrow[p_1]{} P \xrightarrow[p_2]{} B$$

is a product for A and B iff for every object X, the canonical function θ_X is an isomorphism

$$\theta_X : \mathrm{Hom}(X, P) \cong \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B)$$