An Introduction To Algebraic Topology

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1 Introduction

1.1 Notation

$\mathbf{I} = [0, 1]$.

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

S^n is called the ***n*-sphere**. $S^n \subset \mathbb{R}^{n+1}$ (S^1 is the circle); 0-sphere S^0 consists of the two points $\{-1, 1\}$ and hence is a discrete two-point space. We may regard S^n as the **equator** of S^{n+1}

$$S^n = \mathbb{R}^{n+1} \cap S^{n+1} = \{(x_1, \dots, x_{n+2}) \in S^{n+1} : x_{n+2} = 0\}$$

The **north pole** is $(0, 0, \dots, 0, 1) \in S^n$; the **south pole** is $(0, 0, \dots, 0, -1)$. The **antipode** of $x = (x_1, \dots, x_{n+1}) \in S^n$ is the other endpoint of the diameter having one endpoint x; thus the antipode of x is $-x = (-x_1, \dots, -x_{n+1})$, for the distance from $-x$ to x is 2.

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

D^n is called the ***n*-disk** (or ***n*-ball**). Observe that $S^{n-1} \subset D^n \subset \mathbb{R}^n$; indeed S^{n-1} is the boundary of D^n in $\mathbb{R}^n$

$$\Delta^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \text{ each } x_i \geq 0 \text{ and } \sum x_i = 1\}$$

Δ^n is called the **standard *n*-simplex**. Δ^0 is a point, Δ^1 is a closed interval, Δ^2 is a triangle (with interior), Δ^3 is a (solid) tetrahedron, and so on.

There is a standard homeomorphism from S^n - {north pole} to $\mathbb{R}^n$, called **stereographic projection**. Denote the north pole by N, and define $\sigma : S^n - \{N\} \to \mathbb{R}^n$ to be the intersection of $\mathbb{R}^n$ and the line joining x and N. Points on the latter line have the form $tx + (1-t)N$, hence they have coordinates $(tx_1, \dots, tx_n, tx_{n+1} + (1-t))$. The last coordinate is zero for $t = (1 - x_{n+1})^{-1}$; hence

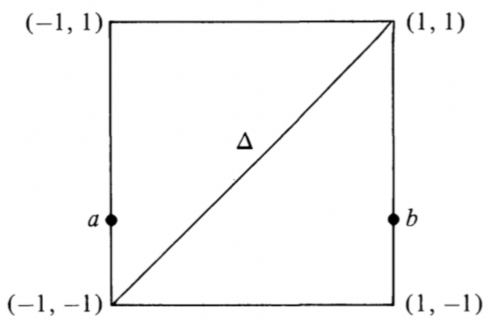
$$\sigma(x) = (tx_1, \dots, tx_n)$$

where $t = (1 - x_{n+1})^{-1}$. It is now routine to check that σ is indeed a homeomorphism. Note that $\sigma(x) = x$ iff x lies on the equator S^{n-1}

1.2 Brouwer Fixed Point Theorem

Theorem 1.1. *Every continuous $f : D^1 \to D^1$ has a fixed point*

Proof. Let $f(-1) = a$ and $f(1) = b$.If either $f(-1) = -1$ or $f(1) = 1$, we are done. Therefore we may assume that $f(-1) = a > -1$ and that $f(1) = b < 1$ as drawn. If



G is the graph of f and Δ is the graph of the identity function, then we must prove that $G \cap \Delta \neq \emptyset$. The idea is to use a connectness argument to show that every path in $D^1 \times D^1$ from a to b must cross Δ.

Since f is continuous, $G = \{(x, f(x)) : x \in D^1\}$ is connected (continuous image of connected space is connected). Define $A = \{(x, f(x)) : f(x) > x\}$ and $B = \{(x, f(x)) : f(x) < x\}$. Note that $a \in A$ and $b \in B$, so that $A \neq \emptyset$ and $B \neq \emptyset$. If $G \cap \Delta = \emptyset$, then G is the disjoint union of A and B. $\square$

Theorem 1.2 (Brouwer fixed point theorem)**.** *If $f : D^n \to D^n$ is continuous, then there exists $x \in D^n$ with $f(x) = x$*

1.3 Categories and Functors

Definition 1.3. A category $\mathcal{C}$ consists of three ingredients: a class of **objects**, obj $\mathcal{C}$; sets of **morphisms** $\mathrm{Hom}(A, B)$, one for every ordered pair $A, B \in$ obj $\mathcal{C}$; **composition** $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, denoted by $(f, g) \to g \circ f$, for every $A, B, C \in$ obj $\mathcal{C}$ satisfying the following axioms

1. the family of $\mathrm{Hom}(A, B)$'s is pairwise disjoint'
2. composition is associative when defined
3. for each $A \in$ obj $\mathcal{C}$ there exists an identity $1_A \in \mathrm{Hom}(A, A)$ satisfying $1_A \circ f = f$ for every $f \in \mathrm{Hom}(B, A)$, all $B \in$ obj $\mathcal{C}$ and $g \circ 1_A = g$ for every $g \in \mathrm{Hom}(A, C)$, all $\mathrm{C} \in$ obj $\mathcal{C}$

Definition 1.4. Let $\mathcal{C}$ and $\mathcal{A}$ be categories with $\operatorname{obj}\mathcal{C} \subset \operatorname{obj}\mathcal{A}$. If $A, B \in \operatorname{obj}\mathcal{C}$, let's denote the two possible Hom sets by $\operatorname{Hom}_{\mathcal{C}}(A, B)$ and $\operatorname{Hom}_{\mathcal{A}}(A, B)$. Then $\mathcal{C}$ is a **subcategory** of $\mathcal{A}$ if $\operatorname{Hom}_{\mathcal{C}}(A, B) \subset \operatorname{Hom}_{\mathcal{A}}(A, B)$ for all $A, B \in \operatorname{obj}\mathcal{C}$ and if composition in $\mathcal{C}$ is the same as composition in $\mathcal{A}$

Example 1.1. $\mathcal{C} = \mathbf{Top}^2$. here $\operatorname{obj}\mathcal{C}$ consists of all ordered pairs (X, A) where X is a topological space and A is a subspace of X. A morphism $f : (X, A) \to (Y, B)$ is an ordered pair (f, f') where $f : X \to Y$ is continuous and $fi = jf'$ (where i and j are inclusions)

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
B \arrow[r, hook, "j"'] & Y
\end{tikzcd}$$

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