An Introduction To Algebraic Topology

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1 Introduction

1.1 Notation

$\mathbf{I} = [0, 1]$.

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

S^n is called the ***n*-sphere**. $S^n \subset \mathbb{R}^{n+1}$ (S^1 is the circle); 0-sphere S^0 consists of the two points $\{-1, 1\}$ and hence is a discrete two-point space. We may regard S^n as the **equator** of S^{n+1}

$$S^n = \mathbb{R}^{n+1} \cap S^{n+1} = \{(x_1, \dots, x_{n+2}) \in S^{n+1} : x_{n+2} = 0\}$$

The **north pole** is $(0, 0, \dots, 0, 1) \in S^n$; the **south pole** is $(0, 0, \dots, 0, -1)$. The **antipode** of $x = (x_1, \dots, x_{n+1}) \in S^n$ is the other endpoint of the diameter having one endpoint x; thus the antipode of x is $-x = (-x_1, \ldots, -x_{n+1})$, for the distance from $-x$ to x is 2.

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

D^n is called the n**-disk** (or n**-ball**). Observe that $S^{n-1} \subset D^n \subset \mathbb{R}^n$; indeed S^{n-1} is the boundary of D^n in $\mathbb{R}^n$

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \text{ each } x_i \geq 0 \text{ and } \sum x_i = 1\}$$

Δ^n is called the **standard** n**-simplex**. Δ^0 is a point, Δ^1 is a closed interval, Δ^2 is a triangle (with interior), Δ^3 is a (solid) tetrahedron, and so on.

There is a standard homeomorphism from S^n - {north pole} to $\mathbb{R}^n$, called **stereographic projection**. Denote the north pole by N, and define $\sigma : S^n - \{N\} \to \mathbb{R}^n$ to be the intersection of $\mathbb{R}^n$ and the line joining x and N. Points on the latter line have the form $tx + (1-t)N$, hence they have coordinates $(tx_1, \ldots, tx_n, tx_{n+1} + (1-t))$. The last coordinate is zero for $t = (1 - x_{n+1})^{-1}$; hence

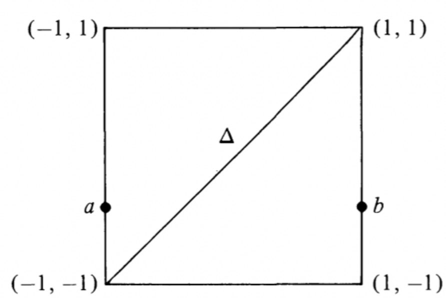
$$\sigma(x) = (tx_1, \ldots, tx_n)$$

where $t = (1 - x_{n+1})^{-1}$. It is now routine to check that σ is indeed a homeomorphism. Note that $\sigma(x) = x$ iff x lies on the equator S^{n-1}

1.2 Brouwer Fixed Point Theorem

Theorem 1.1. *Every continuous $f : D^1 \to D^1$ has a fixed point*

Proof. Let $f(-1) = a$ and $f(1) = b$.If either $f(-1) = -1$ or $f(1) = 1$, we are done. Therefore we may assume that $f(-1) = a > -1$ and that $f(1) = b < 1$ as drawn. If G is the graph of f and Δ is the graph of the identity



function, then we must prove that $G \cap \Delta \neq \emptyset$. The idea is to use a connectness argument to show that every path in $D^1 \times D^1$ from a to b must cross Δ.

Since f is continuous, $G = \{(x, f(x)) : x \in D^1\}$ is connected (continuous image of connected space is connected). Define $A = \{(x, f(x)) : f(x) > x\}$ and $B = \{(x, f(x)) : f(x) < x\}$. Note that $a \in A$ and $b \in B$, so that $A \neq \emptyset$ and $B \neq \emptyset$. If $G \cap \Delta = \emptyset$, then G is the disjoint union of A and B. $\square$

Definition 1.2. A subspace X of a topological space Y is a **retract** of Y if there is a continuous map $r : Y \to X$ with $r(x) = x$ for all $x \in X$; such a map r is called a **retraction**

Theorem 1.3 (Brouwer fixed point theorem). *If $f : D^n \to D^n$ is continuous, then there exists $x \in D^n$ with $f(x) = x$*

2 Categories and Functors

Definition 2.1. A category $\mathcal{C}$ consists of three ingredients: a class of **objects**, obj $\mathcal{C}$; sets of **morphisms** $\mathrm{Hom}(A, B)$, one for every ordered pair $A, B \in$ obj $\mathcal{C}$; **composition** $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$, denoted by $(f, g) \to g \circ f$, for every $A, B, C \in$ obj $\mathcal{C}$ satisfying the following axioms

1. the family of $\mathrm{Hom}(A, B)$'s is pairwise disjoint
2. composition is associative when defined
3. for each $A \in$ obj $\mathcal{C}$ there exists an identity $1_A \in \mathrm{Hom}(A, A)$ satisfying $1_A \circ f = f$ for every $f \in \mathrm{Hom}(B, A)$, all $B \in$ obj $\mathcal{C}$ and $g \circ 1_A = g$ for every $g \in \mathrm{Hom}(A, C)$, all $C \in$ obj $\mathcal{C}$

Definition 2.2. Let $\mathcal{C}$ and $\mathcal{A}$ be categories with obj $\mathcal{C} \subset$ obj $\mathcal{A}$. If $A, B \in$ obj $\mathcal{C}$, let's denote the two possible Hom sets by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{A}}(A, B)$. Then $\mathcal{C}$ is a **subcategory** of $\mathcal{A}$ if $\mathrm{Hom}_{\mathcal{C}}(A, B) \subset \mathrm{Hom}_{\mathcal{A}}(A, B)$ for all $A, B \in$ obj $\mathcal{C}$ and if composition in $\mathcal{C}$ is the same as composition in $\mathcal{A}$

Example 2.1. $\mathcal{C} = \mathbf{Top}^2$. here obj $\mathcal{C}$ consists of all ordered pairs (X, A) where X is a topological space and A is a subspace of X. A morphism $f : (X, A) \to (Y, B)$ is an ordered pair (f, f') where $f : X \to Y$ is continuous and $fi = jf'$ (where i and j are inclusions)

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ B \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

and composition is coordinatewise. $\mathbf{Top}^2$ is called the category of **pairs** (of topological spaces)

Example 2.2. $\mathcal{C} = \mathbf{Top}_*$. Here obj $\mathcal{C}$ consists of all ordered pairs (X, x_0) where X is a topological space and x_0 is a point of X. $\mathbf{Top}_*$ is a subcategory of $\mathbf{Top}^2$ and it is called the category of **pointed spaces**; x_0 is called the **basepoint** of (X, x_0) and morphisms are called **pointed maps** (or **basepoint preserving maps**). The category $\mathbf{Sets}_*$ of pointed sets is defined similarly

Exercise 2.0.1. Let $f \in \mathrm{Hom}(A, B)$ be a morphism in a category $\mathcal{C}$. If f has a left inverse g ($g \in \mathrm{Hom}(B, A)\backslash$ and $g \circ f = 1_A$) and a right inverse h ($h \in \mathrm{Hom}(B, A)$ and $f \circ h = 1_B$), then $g = h$

Exercise 2.0.2. A set X is called **quasi-ordered** (or **pre-ordered**) if X has a transitive and reflexive relation $\leq$ (such a set is partially ordered if $\leq$ is antisymmetric). Prove that the following construction gives a category $\mathcal{C}$. Define obj $\mathcal{C} = X$, if $x, y \in X$ and $x \not\leq y$, define $\mathrm{Hom}(x, y) = \emptyset$; if $x \leq y$, define $\mathrm{Hom}(x, y)$ to be a set with exactly one element, denoted by i_y^x; if $x \leq y \leq z$ define composition by $i_z^y \circ i_y^x = i_z^x$

Exercise 2.0.3. Let G be a **monoid**, that is, a semigroup with 1. Show that the following gives a category $\mathcal{C}$. Let obj $\mathcal{C}$ have exactly one element, denoted by *; define $\mathrm{Hom}(*, *) = G$ and define composition $G \times G \to G$ as the given multiplication in G

Definition 2.3. A **diagram** in a category $\mathcal{C}$ is a directed graph whose vertices are labeled by objects of $\mathcal{C}$ and whose directed edges are labeled by morphisms in $\mathcal{C}$. A **commutative diagram** in $\mathcal{C}$ is a diagram in which, for each pair of vertices, every two paths (composites) between them are equal as morphisms.

Exercise 2.0.4. Given a category $\mathcal{C}$, shows that the following construction gives a category $\mathcal{M}$. First, an object of $\mathcal{M}$ is a morphism of $\mathcal{C}$. Next, if $f, g \in$ obj $\mathcal{M}$, say $f : A \to B$ and $g : C \to D$, then a morphism in $\mathcal{M}$ is an ordered pair (h, k) of morphisms in $\mathcal{C}$ s.t. the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

commutes. Define composition coordinatewise

$$(h', k') \circ (h, k) = (h' \circ h, k' \circ k)$$

Definition 2.4. A **congruence** on a category $\mathcal{C}$ is an equivalence relation $\sim$ on the class $\bigcup_{(A,B)} \mathrm{Hom}(A,B)$ of all morphisms in $\mathcal{C}$ s.t.

1. $f \in \mathrm{Hom}(A,B)$ and $f \sim f'$ implies $f' \in \mathrm{Hom}(A,B)$
2. $f \sim f'$, $g \sim g'$ and the composite $g \circ f$ exists imply that
$$g \circ f \sim g' \circ f'$$

Theorem 2.5. *Let $\mathcal{C}$ be a category with congruence $\sim$ and let $[f]$ denote the equivalence class of a morphism f. Define $\mathcal{C}'$ as follows*

$$\begin{aligned} \mathrm{obj}\, \mathcal{C}' &= \mathrm{obj}\, \mathcal{C} \\ \mathrm{Hom}_{\mathcal{C}'}(A,B) &= \{[f] : f \in \mathrm{Hom}_{\mathcal{C}}(A,B)\} \\ [g] \circ [f] &= [g \circ f] \end{aligned}$$

Then $\mathcal{C}'$ is a category

Proof. Property 1 in the definition of congruence shows that $\sim$ partitions each set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ and this implies that $\mathrm{Hom}_{\mathcal{C}'}(A,B)$ is a set; moreover, the family of these sets is pairwise disjoint. Property 2 in the definition of congruence shows that composition in $\mathcal{C}'$ is well. $\mathcal{C}'$ is associative and $[1_A]$ is the identity is not hard □

The category $\mathcal{C}'$ is called a **quotient category** of $\mathcal{C}$; one usually denotes $\mathrm{Hom}_{\mathcal{C}'}(A,B)$ by $[A,B]$

Exercise 2.0.5. Let G be a group and let $\mathcal{C}$ be the one-object category it defines: $\mathrm{obj}\, \mathcal{C} = \{*\}$, $\mathrm{Hom}(*,*) = G$ and composition is the group operation. If H is a normal subgroup of G, define $x \sim y$ to mean $xy^{-1} \in H$. Show that $\sim$ is a congruence on $\mathcal{C}$ and that $[*,*] = G/H$ in the corresponding quotient category

Definition 2.6. If $\mathcal{A}$ and $\mathcal{C}$ are categories, a **functor** $T : \mathcal{A} \to \mathcal{C}$ is a function, that is,

1. $A \in \mathrm{obj}\, \mathcal{A}$ implies $TA \in \mathrm{obj}\, \mathcal{C}$
2. if $f : A \to A'$ is a morphism in $\mathcal{A}$, then $Tf : TA \to TA'$ is a morphism in $\mathcal{C}$
3. if f, g are morphisms in $\mathcal{A}$ for which $g \circ f$ is defined, then
$$T(g \circ f) = (Tg) \circ (Tf)$$

4. $T(1_A) = 1_{TA}$ for every $A \in \text{obj}\,\mathcal{A}$

Example 2.3. If $\mathcal{C}$ is a category, the **identity functor** $J : \mathcal{C} \to \mathcal{C}$ is defined by $JA = A$ for every object A and $Jf = f$ for every morphism

Example 2.4. If M is a fixed topological space, Then T_m : **Top** $\to$ **Top** is a functor, where $T_M(X) = X \times M$ and if $f : X \to Y$ is continuous, then $T_M(f) : X \times M \to Y \times M$ is defined by $(x, m) \mapsto (f(x), m)$

Example 2.5. Fix an object A in category $\mathcal{C}$. Then $\text{Hom}(A, -) : \mathcal{C} \to$ **Sets** is a functor assigning to each object B the set $\text{Hom}(A, B)$ and to each morphism $f : B \to B'$ the **induced map** $\text{Hom}(A, f) : \text{Hom}(A, B) \to \text{Hom}(A, B')$ defined by $g \mapsto f \circ g$. One usually denotes the induced map $\text{Hom}(A, f)$ by f_*

Functors as just defined are also called **covariant functors** to distinguish them from **contravariant functors** that reverse the direction of arrows. Thus the functor of the example is sometimes called a **covariant Hom functor**.

Definition 2.7. if $\mathcal{A}$ and $\mathcal{C}$ are categories, a **contravariant functor** $S : \mathcal{A} \to \mathcal{C}$ is a function that

1. $A \in \text{obj}\,\mathcal{A}$ implies $SA \in \text{obj}\,\mathcal{C}$
2. if $f : A \to A'$ is a morphism in $\mathcal{C}$, then $Sf : SA' \to SA$ is a morphism in $\mathcal{C}$
3. if f, g are morphisms in $\mathcal{A}$ for which $g \circ f$ is defined, then
$$S(g \circ f) = S(f) \circ S(g)$$
4. $S(1_A) = 1_{SA}$ for every $A \in \text{obj}\,\mathcal{A}$

Example 2.6. Fix an object B in a category $\mathcal{C}$. Then $\text{Hom}(-, B) : \mathcal{C} \to$ **Sets** is a contravariant functor assigning to each object A the set $\text{Hom}(A, B)$ and to each morphism $g : A \to A'$ the **induced map** $\text{Hom}(g, B) : \text{Hom}(A', B) \to \text{Hom}(A, B)$ defined by $h \mapsto h \circ g$. One usually denotes the induced map $\text{Hom}(g, B)$ by g^*; $\text{Hom}(-, B)$ is called a **contravariant Hom functor**

Definition 2.8. An **equivalence** in a category $\mathcal{C}$ is a morphism $f : A \to B$ for which there exists a morphism $g : B \to A$ with $f \circ g = 1_B$ and $g \circ f = 1_A$

Theorem 2.9. *If $\mathcal{A}$ and $\mathcal{C}$ are categories and $T : \mathcal{A} \to \mathcal{C}$ is a functor of either variance, then f an equivalence in $\mathcal{A}$ implies that Tf is an equivalence in $\mathcal{C}$*

Exercise 2.0.6. Let $\mathcal{C}$ and $\mathcal{A}$ be categories, let $\sim$ be a congruence on $\mathcal{C}$. If $T : \mathcal{C} \to \mathcal{A}$ is a functor with $T(f) = T(g)$ whenever $f \sim g$, then T defines a functor $T' : \mathcal{C}' \to \mathcal{A}$ (where $\mathcal{C}'$ is the quotient category) by $T'(X) = T(X)$ for every object X and $T'([f]) = T(f)$ for every morphism f.

Exercise 2.0.7.

1. if X is a topological space, show that $C(X)$, the set of all continuous real-valued functions on X, is a commutative ring with 1 under pointwise operations

   $$f + g : x \mapsto f(x) + g(x) \quad \text{and} \quad f \cdot g \mapsto f(x)g(x)$$

   for all $x \in X$

2. show that $X \mapsto C(X)$ gives a (contravariant) functor **Top** $\to$ **Rings**

Proof.

2. From exercise 2.0.4

$\square$

3 Some Basic Topological Notions

3.1 Homotopy

Definition 3.1. If X and Y are spaces and if f_0, f_1 are continuous maps from X to Y, then f_0 is **homotopic** to f_1, denoted by $f_0 \simeq f_1$ if there is a continuous map $F : X \times \mathbf{I} \to Y$ with

$$F(x,0) = f_0(x) \quad \text{and} \quad F(x,1) = f_1(x) \quad \text{for all } x \in X$$

Such a map F is called a **homotopy**, written as $F : f_0 \simeq f_1$

If $f_t : X \to Y$ is defined by $f_t(x) = F(x,t)$, then a homotopy F gives a one-parameter family of continuous maps deforming f_0 into f_1

Lemma 3.2 (Gluing lemma)**.** *Assume that a space X is a finite union of closed subsets $X = \bigcup_{i=1}^{n} X_i$. If, for some space Y, there are continuous maps $f_i : X_i \to Y$ that agree on overlaps ($f_i|X_i \cap X_j = f_j|X_i \cap X_j$ for all i, j), then there exists a unique continuous $f : X \to Y$ with $f|X_i = f_i$ for all i*

Proof. If C is closed in Y, then

$$\begin{aligned} f^{-1}(C) &= X \cap f^{-1}(C) = (\bigcup X_i) \cap f^{-1}(C) \\ &= \bigcup (X_i \cap f^{-1}(C)) \\ &= \bigcup (X_i \cap f_i^{-1}(C)) = \bigcup f_i^{-1}(C) \end{aligned}$$

Since each f_i is continuous, $f_i^{-1}(C)$ is closed in X_i. Since X_i is closed in X, $f_i^{-1}(C)$ is closed in X, therefore $f^{-1}(C)$ is closed in X and f is continuous $\square$

Lemma 3.3 (Gluing Lemma). *Assume that a space X has a (possibly infinite) open cover $X = \bigcup X_i$. If for some space Y, there are continuous maps $f_i : X_i \to Y$ that agree on overlaps, then there exists a unique continuous $f : X \to Y$ with $f|X_i = f_i$ for all i*

Theorem 3.4. *Homotopy is an equivalence relation on the set of all continuous maps $X \to Y$*

Proof. Reflexivity. If $f : X \to Y$, define $F : X \times \mathbf{I} \to Y$ by $F(x,t) = f(x)$ for all $x \in X$ and $t \in \mathbf{I}$; clearly $F : f \simeq f$

Symmetry: Assume that $f \simeq g$, so there is a continuous $F : X \times \mathbf{I} \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. Define $G : X \times \mathbf{I} \to Y$ by $G(x,t) = F(x,1-t)$, and note that $G : g \simeq f$.

Transivity: assume that $F : f \simeq g$ and $G : g \simeq h$. Define $H : X \times \mathbf{I} \to Y$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq 1/2 \\ G(x,2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

Because these functions agree on the overlap $\{(x,1/2) : x \in X\}$, the gluing lemma shows that H is continuous. Therefore $H : f \simeq h$ $\square$

Definition 3.5. If $f : X \to Y$ is continuous, its **homotopy class** is the equivalence class

$$[f] = \{\text{continuous } g : X \to Y : g \simeq f\}$$

The family of all such homotopy classes is denoted by $[X,Y]$

Theorem 3.6. *Let $f_i : X \to Y$ and $g_i : Y \to Z$, for $i = 0,1$, be continuous. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$; that is, $[g_0 \circ f_0] = [g_1 \circ f_1]$*

Proof. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be homotopies. First, we show that

$$g_0 \circ f_0 \simeq g_1 \circ f_0$$

Define $H : X \times \mathbf{I} \to Z$ by $H(x,t) = G(f_0(x),t)$. Clearly, H is continuous; moreover, $H(x,0) = G(f_0(x),0) = g_0(f_0(x))$ and $H(x,1) = G(f_0(x),1) = g_1(f_0(x))$. Now observe that

$$K : g_1 \circ f_0 \sim g_1 \circ f_1$$

where $K : X \times \mathbf{I} \to Z$ is the composite $g_1 \circ F$. Now use the transitivity of the homotopy relation, we have $g_0 \circ f_0 \simeq g_1 \circ f_1$ $\square$

Corollary 3.7. *Homotopy is a congruence on the category* ***Top****.*

It follows from Theorem 2.5 that there is a quotient category whose objects are topological spaces X, whose Hom sets are $\mathrm{Hom}(X, Y) = [X, Y]$ and whose composition is $[g] \circ [f] = [g \circ f]$

Definition 3.8. The quotient category just described is called the **homotopy category**, and it is denoted by **hTop**

All the functors $T : \mathbf{Top} \to \mathcal{A}$ that we shall construct, where $\mathcal{A}$ is some "algebraic" category (e.g. **Ab**, **Groups**, **Rings**) will have the property that $f \simeq g$ implies $T(f) = T(g)$. This fact, aside from a natural wish to identify homotopic maps, makes homotopy valuable, *beacause it guarantees that the algebraic problem in $\mathcal{A}$ arising from a topological problem via T is simpler than the original problem*

Definition 3.9. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there is a continuous map $g : Y \to X$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Two spaces X and Y have the **same homotopy type** if there is a homotopy equivalence $f : X \to Y$

f is a homotopy equivalence iff $[f] \in [X, Y]$ is an equivalence in **hTop**. ()

Definition 3.10. Let X and Y be spaces, and let $y_0 \in Y$. The **constant map** at y_0 is the function $c : X \to Y$ with $c(x) = y_0$ for all $x \in X$. A continuous map $f : X \to Y$ is **nullhomotopic** if there is a constant map $c : X \to Y$ with $f \simeq c$

Theorem 3.11. *Let $\mathbb{C}$ denote the complex numbers, let $\Sigma_\rho \subset \mathbb{C} \approx \mathbb{R}^2$ denote the circle with center at the origin 0 and radius ρ, and let $f^n_\rho : \Sigma_\rho \to \mathbb{C} - \{0\}$ denote the restriction to Σ^ρ of $z \mapsto z^n$. If none of the maps f^n_ρ is nullhomotopic ($n \geq 1$ and $\rho > 0$) then the fundamental theorem of algebra is true (i.e., every nonconstant complex polynomial has a complex root)*

Proof. Consider the polynomial with complex coefficients

$$g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

Choose $\rho > \max\{1, \sum_{i=1}^{n-1} |a_i|\}$ and define $F : \Sigma_\rho \times \mathbf{I} \to \mathbb{C}$

$$F(z, t) = z^n + \sum_{i=0}^{n-1} (1 - t) a_i z^i$$

It's obvious that $F : g|\Sigma_\rho \simeq f_\rho^n$ if we can show that the image of F is contained in $\mathbb{C} - \{0\}$. that is, $F(z,t) \neq 0$. If, on the contrary, $F(z,t) = 0$ for some $t \in \mathbf{I}$ and some z with $|z| = \rho$, then $z^n = -\sum_{i=0}^{n-1}(1-t)a_i z^i$. The triangle inequality gives

$$\rho^n \leq \sum_{i=0}^{n-1}(1-t)|a_i|\rho^i \leq \sum_{i=0}^{n-1}|a_i|\rho^i \leq \left(\sum_{i=0}^{n-1}|a_i|\right)\rho^{n-1}$$

for $\rho > 1$ implies that $\rho^i \leq \rho^{n-1}$. Canceling ρ^{n-1} gives $\rho \leq \sum_{i=0}^{n-1}|a_i|$, a contradiction.

Assume now that g has no complex roots. Define $G : \Sigma_\rho \times \mathbf{I} \to \mathbb{C} - \{0\}$ by $G(z,t) = g((1-t)z)$. (Since g has no roots, the values of G do lie in $\mathbb{C} - \{0\}$) Visibly, $G : g|\Sigma_\rho \simeq k$, where k is the constant function at a_0. Therefore $g|\Sigma_\rho$ is nullhomotopic and by transitivity, f_ρ^n is nullhomotopic, contradicting the hypothesis. $\square$

A common problem involves extending a map $f : X \to Z$ to a larger space Y; the picture is

$$\begin{tikzcd}
Y \arrow[dr, dashed, "g"] & \\
X \arrow[u, hook] \arrow[r, "f"'] & Z
\end{tikzcd}$$

Homotopy itself raises such a problem: if $f_0, f_1 : X \to Z$ then $f_0 \simeq f_1$ if we can extend $f_0 \cup f_1 : X \times \{0\} \cup X \times \{1\} \to Z$ to all of $X \times \mathbf{I}$

Theorem 3.12. *Let $f : S^n \to Y$ be a continuous map into some space Y. TFAE*

1. *f is nullhomotopic*
2. *f can be extended to a continuous map $D^{n+1} \to Y$*
3. *if $x_0 \in S^n$ and $k : S^n \to Y$ is the constant map at $f(x_0)$, then there is a homotopy $F : f \simeq k$ with $F(x_0, t) = f(x_0)$ for all $t \in \mathbf{I}$*

Proof. $1 \to 2$. Assume that $F : f \simeq c$, where $c(x) = y_0$ for all $x \in S^n$. Define $g : D^{n+1} \to Y$ by

$$g(x) = \begin{cases} y_0 & 0 \leq \|x\| \leq 1/2 \\ F(x/\|x\|, 2 - 2\|x\|) & 1/2 \leq \|x\| \leq 1 \end{cases}$$

if $x \neq 0$, then $x/\|x\| \in S^n$; if $1/2 \leq \|x\| \leq 1$ then $2 - 2\|x\| \in \mathbf{I}$; if $\|x\| = 1/2$ then $2 - 2\|x\| = 1$ and $F(x/\|x\|, 1) = c(x/\|x\|) = y_0$. The gluing lemma shows that g is continuous. Finally g does extend f: if $x \in S^n$, then $\|x\| = 1$ and $g(x) = F(x, 0) = f(x)$.

$2 \to 3$. Assume that $g : D^{n+1} \to Y$ extends f. Define $F : S^n \times \mathbf{I} \to Y$ by $F(x, t) = g((1-t)x + tx_0)$; note that $(1-t)x + tx_0 \in D^{n+1}$. Visibly F is continuous. Now $F(x, 0) = g(x) = f(x)$ while $F(x, 1) = g(x_0) = f(x_0)$ for all $x \in S^n$; hence $F : f \simeq k$ where $k : S^n \to Y$ is the constant map at $f(x_0)$. Finally, $F(x_0, t) = g(x_0) = f(x_0)$ for all $t \in \mathbf{I}$

$3 \to 1$ obvious ☐

3.2 Convexity, Contractibility, and Cones

Definition 3.13. A subset X of $\mathbb{R}^m$ is **convex** if for each pair of points $x, y \in X$ the line segment joining x and y is contained in X. In other words, if $x, y \in X$, then $tx + (1-t)y \in X$ for all $t \in \mathbf{I}$

Definition 3.14. A space X is **contractible** if 1_X is nullhomotopic

Theorem 3.15. *Every convex set X is contractible*

Proof. Choose $x_0 \in X$, and define $c : X \to X$ by $c(x) = x_0$ for all $x \in X$. Define $F : X \times \mathbf{I} \to X$ by $F(x, t) = tx_0 + (1-t)x$. Hence $F : 1_X \simeq c$. ☐

A hemisphere is contractible but not convex, so that the converse of Theorem 3.15 is not true

Exercise 3.2.1. Let $R : S^1 \to S^1$ be rotation by α radians. Prove that $R \simeq 1_S$. Conclude that every continuous map $f : S^1 \to S^1$ is homotopic to a continuous map $g : S^1 \to S^1$ with $g(1) = 1$ (where $1 = e^{2\pi i 0} \in S^1$)

Proof. Let $F : S^1 \times \mathbf{I} \to S^1$ be

$$F((\cos\theta, \sin\theta), t) = (\cos(\theta + \alpha(1-t)), \sin(\theta + \alpha(1-t)))$$

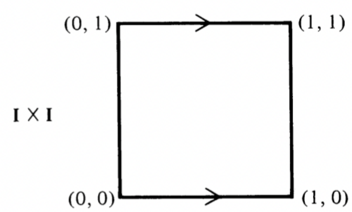
☐

Exercise 3.2.2. Let $X = \{0\} \cup \{1, 1/2, 1/3, \dots, 1/n, \dots\}$ and let Y be a countable discrete space. Show that X and Y do not have the same homotopy type.

Definition 3.16. Let X be a topological space and let $X' = \{X_j : j \in J\}$ be a partition of X. The **natural map** $\nu : X \to X'$ is defined by $\nu(x) = X_j$ where $x \in X_j$. The **quotient topology** on X' is the family of all subsets U' of X' for which $\nu^{-1}(U')$ is open in X

$\nu : X \to X'$ is continuous when X' has the quotient topology. There are two special cases that we wish to mention. If A is a subset of X, then we write X/A for X', where the partition of X consists of A together with all the one-point subsets of $X - A$. The second special case arises from an equivalence relation $\sim$ on X. In this case, the partition consists of the equivalence classes, the natural map is given by $\nu : x \mapsto [x]$, and the quotient space is denoted by $X/\sim$.

Example 3.1. Let $X = \mathbf{I} \times \mathbf{I}$ and define $(x, 0) \sim (x, 1)$ for every $x \in \mathbf{I}$. Then



$X/\sim$ is homeomorphic to the cylinder $S^1 \times \mathbf{I}$. Furthermore, suppose we define a second equivalence relation on $\mathbf{I} \times \mathbf{I}$ by $(x, 0) \sim (x, 1)$ for all $x \in \mathbf{I}$ and $(0, y) \sim (1, y)$ for all $y \in \mathbf{I}$. Then $\mathbf{I} \times \mathbf{I}/\sim$ is the **torus** $S^1 \times S^1$

Example 3.2. If $h : X \to Y$ is a function, then **ker *h*** is the equivalence relation on X defined by $x \sim x'$ if $h(x) = h(x')$. The corresponding quotient space is denoted by $X/\ker h$. Note that, given $h : X \to Y$ there always exists an injection $\varphi : X/\ker h \to Y$ making the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "\nu"'] & & Y \\ & X/\ker h \arrow[ur, "\varphi"'] & \end{tikzcd}$$

namely, $\varphi([x]) = h(x)$

Definition 3.17. A continuous surjection $f : X \to Y$ is an **identification** if a subset U of Y is open iff $f^{-1}(U)$ is open in X

Example 3.3. If $\sim$ is an equivalence relation on $X$and $X/\sim$ is given the quotient topology, then the natural map $\nu : X \to X/\sim$ is an identification

Example 3.4. If $f : X \to Y$ is a continuous surjective map having a **section** (i.e., there is a continuous $s : Y \to X$ with $fs = 1_Y$), then f is an identification

Theorem 3.18. *Let $f : X \to Y$ be a continuous surjection. Then f is an identification iff for all spaces Z and all functions $g : Y \to Z$, one has g continuous iff gf is continuous*

$$\begin{tikzcd} X \arrow[rr, "gf"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Assume f is an identification. If g is continuous, then gf is continuous. Conversely, if f is continuous and let V be open in Z. Then $f^{-1}g^{-1}(V)$ is open in X; since f is an identification, $g^{-1}(V)$ is open in Y

Assume the condition. Let $Z/\ker f$, let $\nu : X \to X/\ker f$ be the natural map and let $\varphi : X/\ker f \to Y$ be the injection of Example 3.2. Note that φ is surjective because f is. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\nu"] \arrow[dr, "f"'] & & X/\ker f \\ & Y \arrow[ur, "\varphi^{-1}"'] & \end{tikzcd}$$

Then $\varphi^{-1}f = \nu$ is continuous implies that φ^{-1} is continuous, by hypothesis. Also φ is continuous because ν is an identification. We conclude that φ is a homeomorphism □

Definition 3.19. Let $f : X \to Y$ be a function and let $y \in Y$. Then $f^{-1}(y)$ is called the **fiber** over y

Corollary 3.20. *Let $f : X \to Y$ be an identification and, for some space Z, let $h : X \to Z$ be a continuous function that is constant on each fiber of f. Then $hf^{-1} : Y \to Z$ is continuous*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "hf^{-1}"'] & \end{tikzcd}$$

Moreover, hf^{-1} is an open (closed) map iff $h(U)$ is open (closed) in Z whenever U is an open (closed) set in X of the form $U = f^{-1}f(U)$

Proof. h is constant on each fiber of f implies that hf^{-1} is well-defined. hf^{-1} is continuous because $(hf^{-1})(f) = h$ is continuous, and Theorem 3.18 applies. Finally if V is open in Y, then $f^{-1}(V)$ is an open set of the stated form $f^{-1}(V) = f^{-1}f(f^{-1}(V))$ □

Corollary 3.21. *Let X and Z be spaces and let $h : X \to Z$ be an identification. Then the map $\varphi : X/\ker h \to Z$ defined by $[x] \mapsto h(x)$ is a homeomorphism*

Proof. φ is a bijection. φ is continuous by Corollary 3.20. The $\nu : X \to X/\ker h$ be the natural map. Let U open in $X/\ker h$. Then $h^{-1}\varphi(U) = \nu^{-1}(U)$ is an open set in X, because ν is continuous and hence $\varphi(U)$ is open, because h is an identification $\square$

4 Problem

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