1. For a dynamical system (T, X), a point $x \in X$ is called *eventually periodic* if there exists $m > m'$ so that $T^m x = T^{m'} x$.

   (a) Let $(T, [0, 1))$ be the doubling map and let (σ, Ω) be the full two shift. For each system, give an example of (i) a point that is eventually periodic, and (ii) a point that is eventually periodic but *not* periodic.

   (b) Let $\mathbb{C}$ be the coding function for the doubling map with the usual partition ($\mathcal{P}_0 = [0, 1/2)$ and $\mathcal{P}_1 = [1/2, 1)$).

   Prove $\mathbb{C}(x)$ is eventually periodic in (σ, Ω) if and only if x is eventually periodic in $(T, [0, 1))$.

   (c) Prove that $\mathbb{C}(x)$ is eventually periodic if and only if x is a rational number.

   (d) Prove that the binary expansion of a number in $[0, 1)$ is eventually periodic if and only if that number is rational. (*Hint:* $\mathbb{C}(x)$ *is always a binary expansion, but not all binary expansions come from codings*).

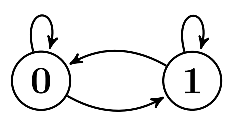
   (e) Prove that the base n expansion of a number in $[0, 1)$ is eventually periodic if an only if that number is rational.

2. Let (σ, Ω) be the full two-shift. Let $G \subseteq \Omega$ bet the set of sequences without two ones in a row. Let $X \subseteq \Omega$ be the set of sequences without *three* ones in a row.

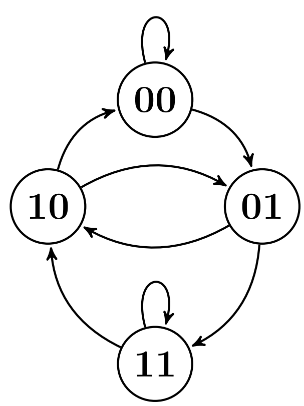
   (a) Show that (σ, X) is a subshift.

   (b) A *two-step* Markov chain is a Markov chain where the transition probabilities between states depend on the current state *and* the previous state.

   A *two-step* Markov chain on a state space S can be thought of as a one-step (regular) Markov chain on the state space $S \times S$. For example, let $\mathcal{M}$ be the Markov chain on states 0 and 1 with equal transition probabilities. Normally, $\mathcal{M}$ has a graph like this:



   However, we can also model $\mathcal{M}$ as a Markov chain $\mathcal{M}'$ with state space 00, 01, 10, and 11 and transition graph like this:



   Explain how $\mathcal{M}'$ models $\mathcal{M}$. In particular, explain why the graph for $\mathcal{M}'$ has no edge between 00 and 11.

   (c) Let $\mathcal{M}_G$ be a Markov chain whose set of realizations is G.

   i. Draw a graph for $\mathcal{M}_G$ and find its associated transition matrix.

   ii. Use the transition matrix for $\mathcal{M}_G$ to compute the entropy of (σ, G).

   iii. Model $\mathcal{M}_G$ as a two-step Markov chain $\mathcal{M}'_G$, which can be viewed as a Markov chain with state space $\{00, 01, 10, 11\}$.
   Draw the graph associated with $\mathcal{M}'_G$ and find its transition matrix.
   iv. Using the transition matrix for $\mathcal{M}'_G$, compute the entropy of (σ, G).

(d) Can X be modeled by a one-step Markov chain? Why or why not?

(e) Find the entropy of (σ, X).

3. Let (σ, Ω) be the full two shift.

   (a) Prove that (σ, Ω) is *expansive*.

   (b) Prove that (σ, Ω) is *transitive*.

   (c) Show that (σ, Ω) is *chaotic*.

   (d) Let (T, X) be a dynamical system and let $x, y \in X$. We say x and y are *δ-correlated for n steps* if the distance between $T^i x$ and $T^i y$ is at most δ for $i = 0, \dots, n$.
   Let $x, y \in \Omega$ be points that are at distance 2^{-k} of each other. For how many steps will x and y be δ-correlated when $\delta = 1/4$?

   (e) If (T, X) is a chaotic dynamical system, can two points be δ-correlated indefinitely? What implications does this have for measurement error?

4. Two dynamical systems (T, X) and (S, Y) are called *conjugate* if there exists a continuous, invertible function $\Phi : X \to Y$ so that $T = \Phi^{-1} \circ S \circ \Phi$. In this case, Φ is called a *conjugacy*.

   (a) Let (T, X) and (S, Y) be conjugate dynamical systems.
      i. Show that if there exists a point of period k in X, there exists a point of period k in Y.
      ii. Show that if (T, X) is transitive, then (S, Y) is transitive.
      iii. Is it true that if (T, X) is expansive, then (S, Y) is expansive?

   (b) Let $(T, [0, 1))$ be the doubling map and let $(L, [0, 1))$ be the *logistic map* defined by $L(x) = rx(1 - x)$ for $r = 4$.
      i. Define $f : [0, 1) \to [0, 1)$ by $f(x) = \sin^2(2\pi x)$. Show that f is a conjugacy between $(T, [0, 1))$ and $(L, [0, 1))$.
      ii. Show that $(L, [0, 1))$ is chaotic.
      iii. (bonus) Show that $(L, [0, 1))$ contains points of every period.

Programming Problems

For the programming problems, please use the Jupyter notebook available at

https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2020-MAT-335-webpage&subPath=homework/homework3-exercises.ipynb

Make sure to comment your code and use "Markdown" style cells to explain your answers.

1.