1. For a dynamical system (T, X), a point $x \in X$ is called *eventually periodic* if there exists $m > m'$ so that $T^m x = T^{m'} x$.

   (a) Let $(T, [0, 1))$ be the doubling map and let (σ, Ω) be the full two shift. For each system, give an example of (i) a point that is eventually periodic, and (ii) a point that is eventually periodic but *not* periodic.

   (b) Let $\mathbb{C}$ be the coding function for the doubling map with the usual partition ($\mathcal{P}_0 = [0, 1/2)$ and $\mathcal{P}_1 = [1/2, 1)$).

   Prove $\mathbb{C}(x)$ is eventually periodic in (σ, Ω) if and only if x is eventually periodic in $(T, [0, 1))$.

   (c) Prove that $\mathbb{C}(x)$ is eventually periodic if and only if x is a rational number.

   (d) Prove that a binary expansion of a number in $[0, 1)$ is eventually periodic if and only if that number is rational. (*Hint:* $\mathbb{C}(x)$ *is always a binary expansion, but not all binary expansions come from codings*).

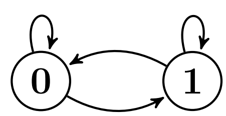
   (e) Prove that the base n expansion of a number in $[0, 1)$ is eventually periodic if an only if that number is rational.

2. Let (σ, Ω) be the full two-shift. Let $G \subseteq \Omega$ bet the set of sequences without two ones in a row. Let $X \subseteq \Omega$ be the set of sequences without *three* ones in a row.

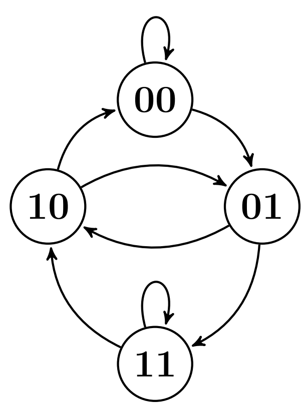
   (a) Show that (σ, X) is a subshift.

   (b) A *two-step* Markov chain is a Markov chain where the transition probabilities between states depend on the current state *and* the previous state.

   A *two-step* Markov chain on a state space S can be thought of as a one-step (regular) Markov chain on the state space $S \times S$. For example, let $\mathcal{M}$ be the Markov chain on states 0 and 1 with equal transition probabilities. Normally, $\mathcal{M}$ has a graph like this:



   However, we can also model $\mathcal{M}$ as a Markov chain $\mathcal{M}'$ with state space 00, 01, 10, and 11 and transition graph like this:



   Explain how $\mathcal{M}'$ models $\mathcal{M}$. In particular, explain why the graph for $\mathcal{M}'$ has no edge between 00 and 11.

   (c) Let $\mathcal{M}_G$ be a Markov chain whose set of realizations is G.

      i. Draw a graph for $\mathcal{M}_G$ and find its associated transition matrix.

      ii. Use the transition matrix for $\mathcal{M}_G$ to compute the entropy of (σ, G).

iii. Model $\mathcal{M}_G$ as a two-step Markov chain $\mathcal{M}'_G$, which can be viewed as a Markov chain with state space $\{00, 01, 10, 11\}$.
Draw the graph associated with $\mathcal{M}'_G$ and find its transition matrix.

iv. Using the transition matrix for $\mathcal{M}'_G$, compute the entropy of (σ, G).

(d) Can X be modeled by a one-step Markov chain? Why or why not?

(e) Find the entropy of (σ, X).

3. Let (σ, Ω) be the full two shift.

(a) Prove that (σ, Ω) is *expansive*.

(b) Prove that (σ, Ω) is *transitive*.

(c) Show that (σ, Ω) is *chaotic*.

(d) Let (T, X) be a dynamical system and let $x, y \in X$. We say x and y are *δ-correlated for n steps* if the distance between $T^i x$ and $T^i y$ is at most δ for $i = 0, \dots, n$.
Let $x, y \in \Omega$ be points that are at distance 2^{-k} of each other. For how many steps will x and y be δ-correlated when $\delta = 1/4$?

(e) If (T, X) is a chaotic dynamical system, can two points be δ-correlated indefinitely? What implications does this have for measurement error?

4. Two dynamical systems (T, X) and (S, Y) are called *conjugate* if there exists a continuous, invertible function $\Phi : Y \to X$ so that $T = \Phi^{-1} \circ S \circ \Phi$. In this case, Φ is called a *conjugacy*.

The systems are called *semi-conjugate* if there exist continuous, onto function $\Phi : Y \to X$ so that $T \circ \Phi = \Phi \circ S$. In this case, Φ is called a *semi-conjugacy*.

(a) Let (T, X) and (S, Y) be conjugate dynamical systems.

i. Show that if there exists a point of period k in Y, there exists a point of period k in X.

ii. Show that if (S, Y) is transitive, then (T, X) is transitive.

iii. Is it true that if (S, Y) is expansive, then (T, X) is necessarily expansive?

(b) Prove that if (T, X) and (S, Y) are conjugate dynamical systems, that they are also semi-conjugate dynamical systems.

(c) Let (T, X) and (S, Y) be semi-conjugate dynamical systems.

i. Show that if there exists a point of period k in Y, there exists a point of period $\leq k$ in X.

ii. Show that if (S, Y) is transitive, then (T, X) is transitive.

iii. Is it true that if (S, Y) is expansive, then (T, X) is necessarily expansive?

(d) Let $(T, [0, 1))$ be the doubling map and let (σ, Ω) be the full two shift.

i. Show that $(T, [0, 1))$ and (σ, Ω) are semi-conjugate.

ii. Show that $(T, [0, 1))$ is chaotic.

(e) Let $(T, [0, 1))$ be the doubling map and let $(L, [0, 1))$ be the *logistic map* defined by $L(x) = rx(1 - x)$ for $r = 4$.

i. Define $f : [0, 1) \to [0, 1)$ by $f(x) = \sin^2(2\pi x)$. Show that f is a semi-conjugacy between $(L, [0, 1))$ and $(T, [0, 1))$.

ii. Show that $(L, [0, 1))$ contains points of every period.

iii. Show that $(L, [0, 1))$ is chaotic.

Programming Problems

For the programming problems, please use the Jupyter notebook available at

https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2020-MAT-335-webpage&subPath=homework/homework4-exercises.ipynb

Make sure to comment your code and use "Markdown" style cells to explain your answers.

1. The *logistic map with parameter* r is the function $f : [0,1] \to [0,1]$ defined by $f(x) = rx(1-x)$ (for $r \in [0,4]$). It was invented as a simple model for population in biology. The idea is that if x is your original population, there will be some birth rate r governing population growth; however, when the population outstretches its resources, its growth will be constrained. The $1-x$ parameter models this constraint on growth.

   Despite being so simple, the logistic map can have amazingly complex behaviour!

   (a) A two-parameter logistic function `f` has been predefined. Create a function `orbit_f` which inputs a starting value, a parameter r, and an orbit length n, and returns a list with the first n points along the orbit of `f`.

   (b) Given a dynamical system (T, X), the n-orbit of x is $\mathcal{O}^n(x) = \{x, Tx, \ldots, T^{n-1}x\}$.
   Using $r = 2$, plot the 20-orbits of x for at least 10 different x's (plot time vs. value). What do you notice?

   (c) Does the logistic map with $r = 2$ have a *basin of attraction*[1]? Justify your conclusion.

   (d) Repeat part 1b using $r = 3$. What do you notice? Is there a basin of attraction consisting of a single point?

   (e) In general, a *basin of attraction* for a dynamical system is a set which all points limit to. Does the logistic map have a basin of attraction when $r = 3$? If so, describe it.

   (f) Repeat part 1b with $r = 4$. Is there a basin of attraction? Why or why not?

2. We are going to plot the basins of attraction for the logistic map as a function of r. We can approximate a basin of attraction by taking the 1000-orbit of several points, and then taking the set consisting of the last 100 points in each orbit.

   (a) Create a function `approximate_basin` which takes in a parameter r and returns a list of points that approximate the basin of attraction of the logistic map with parameter r.
   Hint: It may be useful to round your results to 3 or 4 decimal places and then use `np.unique` to get a list of manageable length.

   (b) Plot the basins of attraction of the logistic map vs. r for at 1000 different values of r between 0 and 4.

   (c) What do you notice about the basins of attraction? Did you expect this?

   (d) In the *proofs* part of this homework set, you proved that the logistic map is chaotic when $r = 4$. What does that imply about the basin of attraction?

   (e) If a population is well-modeled by the logistic map with a parameter of $r = 3.82$, what can you say about the population after 1000 days? What if it is modeled by a logistic map with parameter $r = 3.83$? Is the population more or less predictable?

[1] Look back in the notes if you've forgotten this term.