MAT335 **Due: 11:59pm April 6**

Homework 4 (*March 22, 2020*)

1. For a dynamical system (T, X), a point $x \in X$ is called *eventually periodic* if there exists $m > m'$ so that $T^m x = T^{m'} x$.

   (a) Let $(T, [0, 1))$ be the doubling map and let (σ, Ω) be the full two shift. For each system, give an example of (i) a point that is eventually periodic, and (ii) a point that is eventually periodic but *not* periodic.

   (b) Let $\mathbb{C}$ be the coding function for the doubling map with the usual partition ($\mathcal{P}_0 = [0, 1/2)$ and $\mathcal{P}_1 = [1/2, 1)$).

   Prove $\mathbb{C}(x)$ is eventually periodic in (σ, Ω) if and only if x is eventually periodic in $(T, [0, 1))$.

   (c) Prove that $\mathbb{C}(x)$ is eventually periodic if and only if x is a rational number.

   (d) Prove that the binary expansion of a number in $[0, 1)$ is eventually periodic if and only if that number is rational. (*Hint:* $\mathbb{C}(x)$ *is always a binary expansion, but not all binary expansions come from codings*).

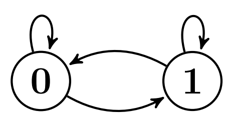
   (e) Prove that the base n expansion of a number in $[0, 1)$ is eventually periodic if an only if that number is rational.

2. Let (σ, Ω) be the full two-shift. Let $G \subseteq \Omega$ bet the set of sequences without two ones in a row. Let $X \subseteq \Omega$ be the set of sequences without *three* ones in a row.

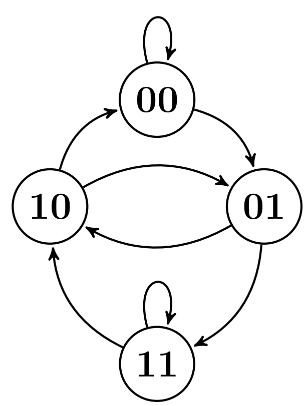
   (a) Show that (σ, X) is a subshift.

   (b) A *two-step* Markov chain is a Markov chain where the transition probabilities between states depend on the current state *and* the previous state.

   A *two-step* Markov chain on a state space S can be thought of as a one-step (regular) Markov chain on the state space $S \times S$. For example, let $\mathcal{M}$ be the Markov chain on states 0 and 1 with equal transition probabilities. Normally, $\mathcal{M}$ has a graph like this:



   However, we can also model $\mathcal{M}$ as a Markov chain $\mathcal{M}'$ with state space 00, 01, 10, and 11 and transition graph like this:



   Explain how $\mathcal{M}'$ models $\mathcal{M}$. In particular, explain why the graph for $\mathcal{M}'$ has no edge between 00 and 11.

   (c) Let $\mathcal{M}_G$ be a Markov chain whose set of realizations is G.

   i. Draw a graph for $\mathcal{M}_G$ and find its associated transition matrix.

   ii. Use the transition matrix for $\mathcal{M}_G$ to compute the entropy of (σ, G).

iii. Model $\mathcal{M}_G$ as a two-step Markov chain $\mathcal{M}'_G$, which can be viewed as a Markov chain with state space $\{00, 01, 10, 11\}$.
Draw the graph associated with $\mathcal{M}'_G$ and find its transition matrix.

iv. Using the transition matrix for $\mathcal{M}'_G$, compute the entropy of (σ, G).

(d) Can X be modeled by a one-step Markov chain? Why or why not?

(e) Find the entropy of (σ, X).

3. Let (σ, Ω) be the full two shift.

(a) Prove that (σ, Ω) is *expansive*.

(b) Prove that (σ, Ω) is *transitive*.

(c) Show that (σ, Ω) is *chaotic*.

(d) Let (T, X) be a dynamical system and let $x, y \in X$. We say x and y are *δ-correlated for n steps* if the distance between $T^i x$ and $T^i y$ is at most δ for $i = 0, \ldots, n$.
Let $x, y \in \Omega$ be points that are at distance 2^{-k} of each other. For how many steps will x and y be δ-correlated when $\delta = 1/4$?

(e) If (T, X) is a chaotic dynamical system, can two points be δ-correlated indefinitely? What implications does this have for measurement error?

4. Two dynamical systems (T, X) and (S, Y) are called *conjugate* if there exists a continuous, invertible function $\Phi : Y \to X$ so that $T = \Phi^{-1} \circ S \circ \Phi$. In this case, Φ is called a *conjugacy*.

The systems are called *semi-conjugate* if there exist continuous, onto function $\Phi : Y \to X$ so that $T \circ \Phi = \Phi \circ S$. In this case, Φ is called a *semi-conjugacy*.

(a) Let (T, X) and (S, Y) be conjugate dynamical systems.

i. Show that if there exists a point of period k in Y, there exists a point of period k in X.

ii. Show that if (S, Y) is transitive, then (T, X) is transitive.

iii. Is it true that if (S, Y) is expansive, then (T, X) is necessarily expansive?

(b) Prove that if (T, X) and (S, Y) are conjugate dynamical systems, that they are also semi-conjugate dynamical systems.

(c) Let (T, X) and (S, Y) be semi-conjugate dynamical systems.

i. Show that if there exists a point of period k in Y, there exists a point of period k in X.

ii. Show that if (S, Y) is transitive, then (T, X) is transitive.

iii. Is it true that if (S, Y) is expansive, then (T, X) is necessarily expansive?

(d) Let $(T, [0, 1))$ be the doubling map and let (σ, Ω) be the full two shift.

i. Show that $(T, [0, 1))$ and (σ, Ω) are semi-conjugate.

ii. Show that $(T, [0, 1))$ is chaotic.

(e) Let $(T, [0, 1))$ be the doubling map and let $(L, [0, 1))$ be the *logistic map* defined by $L(x) = rx(1 - x)$ for $r = 4$.

i. Define $f : [0, 1) \to [0, 1)$ by $f(x) = \sin^2(2\pi x)$. Show that f is a semi-conjugacy between $(L, [0, 1))$ and $(T, [0, 1))$.

ii. Show that $(L, [0, 1))$ contains points of every period.

iii. Show that $(L, [0, 1))$ is chaotic.

Programming Problems

For the programming problems, please use the Jupyter notebook available at

https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2020-MAT-335-webpage&subPath=homework/homework3-exercises.ipynb

Make sure to comment your code and use "Markdown" style cells to explain your answers.

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