MAT335 **Due: 11:59pm April 6**

HOMEWORK 4 (*March 29, 2020*)

1. For a dynamical system $(T, X)$, a point $x \in X$ is called *eventually periodic* if there exists $m > m'$ so that $T^m x = T^{m'} x$.

   (a) Let $(T, [0, 1))$ be the doubling map and let $(\sigma, \Omega)$ be the full two shift. For each system, give an example of (i) a point that is eventually periodic, and (ii) a point that is eventually periodic but *not* periodic.

   (b) Let $\mathbb{C}$ be the coding function for the doubling map with the usual partition ($\mathcal{P}_0 = [0, 1/2)$ and $\mathcal{P}_1 = [1/2, 1)$).

   Prove $\mathbb{C}(x)$ is eventually periodic in $(\sigma, \Omega)$ if and only if $x$ is eventually periodic in $(T, [0, 1))$.

   (c) Prove that $\mathbb{C}(x)$ is eventually periodic if and only if $x$ is a rational number.

   (d) Prove that a binary expansion of a number in $[0, 1)$ is eventually periodic if and only if that number is rational. (*Hint:* $\mathbb{C}(x)$ *is always a binary expansion, but not all binary expansions come from codings*).

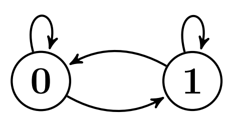
   (e) Prove that the base $n$ expansion of a number in $[0, 1)$ is eventually periodic if an only if that number is rational.

2. Let $(\sigma, \Omega)$ be the full two-shift. Let $G \subseteq \Omega$ bet the set of sequences without two ones in a row. Let $X \subseteq \Omega$ be the set of sequences without *three* ones in a row.

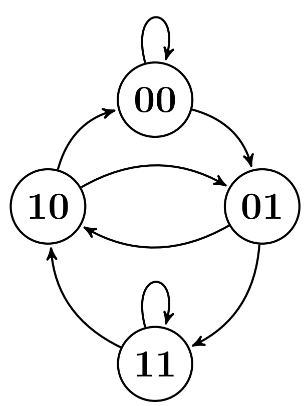
   (a) Show that $(\sigma, X)$ is a subshift.

   (b) A *two-step* Markov chain is a Markov chain where the transition probabilities between states depend on the current state *and* the previous state.

   A *two-step* Markov chain on a state space $S$ can be thought of as a one-step (regular) Markov chain on the state space $S \times S$. For example, let $\mathcal{M}$ be the Markov chain on states 0 and 1 with equal transition probabilities. Normally, $\mathcal{M}$ has a graph like this:




   However, we can also model $\mathcal{M}$ as a Markov chain $\mathcal{M}'$ with state space 00, 01, 10, and 11 and transition graph like this:




   Explain how $\mathcal{M}'$ models $\mathcal{M}$. In particular, explain why the graph for $\mathcal{M}'$ has no edge between 00 and 11.

   (c) Let $\mathcal{M}_G$ be a Markov chain whose set of realizations is $G$.

      i. Draw a graph for $\mathcal{M}_G$ and find its associated transition matrix.

      ii. Use the transition matrix for $\mathcal{M}_G$ to compute the entropy of $(\sigma, G)$.

iii. Model $\mathcal{M}_G$ as a two-step Markov chain $\mathcal{M}'_G$, which can be viewed as a Markov chain with state space $\{00, 01, 10, 11\}$.
Draw the graph associated with $\mathcal{M}'_G$ and find its transition matrix.

iv. Using the transition matrix for $\mathcal{M}'_G$, compute the entropy of $(\sigma, G)$.

(d) Can $X$ be modeled by a one-step Markov chain? Why or why not?

(e) Find the entropy of $(\sigma, X)$.

3. Let $(\sigma, \Omega)$ be the full two shift.

(a) Prove that $(\sigma, \Omega)$ is *expansive*.

(b) Prove that $(\sigma, \Omega)$ is *transitive*.

(c) Show that $(\sigma, \Omega)$ is *chaotic*.

(d) Let $(T, X)$ be a dynamical system and let $x, y \in X$. We say $x$ and $y$ are *$\delta$-correlated for $n$ steps* if the distance between $T^i x$ and $T^i y$ is at most $\delta$ for $i = 0, \dots, n$.
Let $x, y \in \Omega$ be points that are at distance $2^{-k}$ of each other. For how many steps will $x$ and $y$ be $\delta$-correlated when $\delta = 1/4$?

(e) If $(T, X)$ is a chaotic dynamical system, can two points be $\delta$-correlated indefinitely? What implications does this have for measurement error?

4. Two dynamical systems $(T, X)$ and $(S, Y)$ are called *conjugate* if there exists a continuous, invertible function $\Phi : Y \to X$ so that $T = \Phi^{-1} \circ S \circ \Phi$. In this case, $\Phi$ is called a *conjugacy*.

The systems are called *semi-conjugate* if there exist continuous, onto function $\Phi : Y \to X$ so that $T \circ \Phi = \Phi \circ S$. In this case, $\Phi$ is called a *semi-conjugacy*.

(a) Let $(T, X)$ and $(S, Y)$ be conjugate dynamical systems.

i. Show that if there exists a point of period $k$ in $Y$, there exists a point of period $k$ in $X$.

ii. Show that if $(S, Y)$ is transitive, then $(T, X)$ is transitive.

iii. Is it true that if $(S, Y)$ is expansive, then $(T, X)$ is necessarily expansive?

(b) Prove that if $(T, X)$ and $(S, Y)$ are conjugate dynamical systems, that they are also semi-conjugate dynamical systems.

(c) Let $(T, X)$ and $(S, Y)$ be semi-conjugate dynamical systems.

i. Show that if there exists a point of period $k$ in $Y$, there exists a point of period $\leq k$ in $X$.

ii. Show that if $(S, Y)$ is transitive, then $(T, X)$ is transitive.

iii. Is it true that if $(S, Y)$ is expansive, then $(T, X)$ is necessarily expansive?

(d) Let $(T, [0, 1))$ be the doubling map and let $(\sigma, \Omega)$ be the full two shift.

i. Show that $(T, [0, 1))$ and $(\sigma, \Omega)$ are semi-conjugate.

ii. Show that $(T, [0, 1))$ is chaotic.

(e) Let $(T, [0, 1))$ be the doubling map and let $(L, [0, 1))$ be the *logistic map* defined by $L(x) = rx(1 - x)$ for $r = 4$.

i. Define $f : [0, 1) \to [0, 1)$ by $f(x) = \sin^2(2\pi x)$. Show that $f$ is a semi-conjugacy between $(L, [0, 1))$ and $(T, [0, 1))$.

ii. Show that $(L, [0, 1))$ contains points of every period.

iii. Show that $(L, [0, 1))$ is chaotic.

## Programming Problems

For the programming problems, please use the Jupyter notebook available at

https://utoronto.syzygy.ca/jupyter/user-redirect/git-pull?repo=https://github.com/siefkenj/2020-MAT-335-webpage&subPath=homework/homework4-exercises.ipynb

Make sure to comment your code and use "Markdown" style cells to explain your answers.

1. The *logistic map with parameter* $r$ is the function $f : [0,1] \to [0,1]$ defined by $f(x) = rx(1-x)$ (for $r \in [0,4]$). It was invented as a simple model for population in biology. The idea is that if $x$ is your original population, there will be some birth rate $r$ governing population growth; however, when the population outstretches its resources, its growth will be constrained. The $1-x$ parameter models this constraint on growth.

   Despite being so simple, the logistic map can have amazingly complex behaviour!

   (a) A two-parameter logistic function `f` has been predefined. Create a function `orbit_f` which inputs a starting value, a parameter $r$, and an orbit length $n$, and returns a list with the first $n$ points along the orbit of `f`.

   (b) Given a dynamical system $(T, X)$, the $n$-orbit of $x$ is $\mathcal{O}^n(x) = \{x, Tx, \ldots, T^{n-1}x\}$. Using $r = 2$, plot the 20-orbits of $x$ for at least 10 different $x$'s. What do you notice?

   (c) Does the logistic map with $r = 2$ have a *basin of attraction*[1]? Justify your conclusion.

   (d) Repeat part 1b using $r = 3$. What do you notice? Is there a basin of attraction consisting of a single point?

   (e) In general, a *basin of attraction* for a dynamical system is a set which all points limit to. Does the logistic map have a basin of attraction when $r = 3$? If so, describe it.

   (f) Repeat part 1b with $r = 4$. Is there a basin of attraction? Why or why not?

2. We are going to plot the basins of attraction for the logistic map as a function of $r$. We can approximate a basin of attraction by taking the 1000-orbit of several points, and then taking the set consisting of the last 100 points in each orbit.

   (a) Create a function `approximate_basin` which takes in a parameter $r$ and returns a list of points that approximate the basin of attraction of the logistic map with parameter $r$.

   *Hint:* It may be useful to round your results to 3 or 4 decimal places and then use `np.unique` to get a list of manageable length.

   (b) Plot the basins of attraction of the logistic map vs. $r$ for at 1000 different values of $r$ between 0 and 4.

   (c) What do you notice about the basins of attraction? Did you expect this?

   (d) In the *proofs* part of this homework set, you proved that the logistic map is chaotic when $r = 4$. What does that imply about the basin of attraction?

   (e) If a population is well-modeled by the logistic map with a parameter of $r = 3.82$, what can you say about the population after 1000 days? What if it is modeled by a logistic map with parameter $r = 3.83$? Is the population more or less predictable?

[1] Look back in the notes if you've forgotten this term.