GENERALIZED MULTIPLE ORDER-NUMBER FUNCTION ARRAYS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2018

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To my family

ACKNOWLEDGMENTS

Dr. Rao.

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May 2018

Chair: Dr. James P. Sain
Major: Music Composition

Abstracts should be less than 350 words.

CHAPTER 1
INTRODUCTION

Definition 1.0.1. *(2, 99) Let X be a set and G a group. An* **action** *of G on X is a function $G \times X \to X$ given by $(g, x) \mapsto gx$, such that:*

i. $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$;

ii. $1x = x$ for all $x \in X$, where $1 \in G$ is the identity.

Proposition 1.0.2. *(2, 99) If a group G acts on a set X then, for every $g \in G$, the function $f_g : X \to X$ given by $f_g(x) = gx$ is a permutation of X. Further, the function $f : G \to S_X$ given by $f(g) = f_g$ is a homomorphism and, conversely, for any homomorphism $\phi : G \to S_X$, there is a corresponding group action given by $\phi(g)(x)$.*

Theorem 1.0.3 (Cayley)**.** *(2, 96) Every group is isomorphic to a subgroup of the symmetric group S_G. In particular, if $|G| = n$, then G is isomorphic to a subgroup of S_n.*

Theorem 1.0.4. *(2, 97) Let $H \leq G$ be a subgroup of finite index n. Then there exists a homomorphism $\phi : G \to S_n$ such that $\ker \phi \leq H$. In particular, when $H = \{1\}$, we get Cayley's theorem.*

Example 1.0.5. *(3, 122) A group acts on itself by conjugation. Let $g, h \in G$ and $x \in G$. Then $1x1^{-1} = x$ and*

$$g(hx) = g(hxh^{-1}) = ghxh^{-1}g^{-1} = ghxgh^{-1} = (gh)x \ .$$

It is also immediate from the above that a group acts on its power set by conjugation. In particular, a group acts on the set of all its subgroups.

Definition 1.0.6. *(2, 100) If G acts on X, then the* **orbit** *of $x \in X$ is the set*

$$\mathcal{O}(x) = \{gx : g \in G\} \subseteq X \ .$$

We say an action is **transitive** *if there is only one orbit. The* **stabilizer** *of x in G is the group*

$$G_x = \{g \in g : gx = x\} \leq G \ .$$

When a group acts on itself by conjugation, we call the orbits **conjugacy classes**. *The stabilizer of some $g \in G$ is the* **centralizer** *of g in G, denoted $C_G(g)$. When a group acts on the set of its subgroups by conjugation, the stabilizer of a subgroup $H \leq$ is the* **normalizer** *of H in G, denoted by $N_G(H)$.*

Proposition 1.0.7. *(2, 102) If G acts on X, for $x_1, x_2 \in X$, the relation $x_1 \sim x_2$ given by $x_1 = gx_2$ is an equivalence relation. It follows immediately that the equivalence classes are the orbits of the action of G on X and that*

$$|X| = \sum_i |\mathcal{O}(x_i)| \ ,$$

where x_i is a single representative from each orbit.

Theorem 1.0.8 (Orbit-Stabilizer)**.** *(2, 102) If G acts on X, then for each $x \in X$*

$$|\mathcal{O}(x)| = [G : G_x] \ .$$

Corollary 1.0.9. *(2, 103) If G is finite and acts on X, then the size of any orbit is a divisor of $|G|$.*

Lemma 1.0.10 (Burnside)**.** *(2, 109) If G acts on a finite set X, then the number of orbits N is*

$$N = \frac{1}{|G|} \sum_{\tau \in G} \mathrm{Fix}(\tau) \ ,$$

where $\mathrm{Fix}(\tau)$ is the cardinality of the set of $x \in X$ that are fixed by τ.

1.1 Polya's Enumeration Formula

The main application of Polya's enumeration formula in our context is as follows.

Example 1.1.1. *(1, ??) Take an un-oriented cube and color its corners in black or white. Then*

$$b^8 + b^7 w + 3b^6 w^2 + 3b^5 w^3 + 7b^4 w^4 + 3b^3 w^5 + 3b^2 w^6 + bw^7 + w^8 \ ,$$

would represent the generating function, or pattern inventory of all distinct colorings of that cube, where the coefficient of $b^i w^j$ represents the particular number of colorings with i black corners and j white corners.

CHAPTER 2
THEORETICAL FRAMEWORK

2.1 Preliminary Results

Definition 2.1.1. *Let x and y be arbitrary pitches. The* ordered pitch interval *between x and y is given by*

$$i_{ordered}(x, y) = y - x \ .$$

The ordered pitch-class interval *between x and y is given by*

$$i_{ordered}(\bar{x}, \bar{y}) = \bar{y} - \bar{x} \ .$$

The unordered pitch interval *between x and y is given by*

$$i_{unordered}(x, y) = |x - y| \ .$$

The unordered pitch-class interval, *or simply interval class, between x and y is given by*

$$i_{unordered}(\bar{x}, \bar{y}) = \min\{i_{ordered}(\bar{x}, \bar{y}), i_{ordered}(\bar{y}, \bar{x})\} \ .$$

Example 2.1.2. *Put $x = 43$ and $y = -13$. Then $i_{ordered}(x, y) = -56$, $i_{ordered}(\bar{x}, \bar{y}) = \overline{11} - \bar{7} = \bar{4}$, $i_{unordered}(x, y) = 56$, and $i_{unordered}(\bar{x}, \bar{y}) = \bar{4}$.*

Whenever the context is clear, we shall drop quotient notation and subscripts. In most situations, we are interested in the interval class between x and y, in which case we will simply write $i(7, -1) = 4$. Interval classes can also be seen graph-theoretically as the edge connecting two members of a pitch-class set displayed clockwise.

Theorem 2.1.3 (Common-Tone)**.** *The number of common tones between a set S and some transposition of itself is given by*

$$|S \cap \mathrm{T}_n(S)| = |\{x - y = n : x, y \in S\}| \ .$$

The number of common tones between a set S and some inversion of itself is given by

$$|S \cap \mathrm{T}_n\mathrm{I}(S)| = 2 \cdot |\{x + y = n : x, y \in S\}| + |\{a \in S : 2a = n\}| \ .$$

Moreover, the cardinality of the set $\{a \in S : 2a = n\}$ is at most 2.

Proof. (**?** , ??) proves the second assertion as follows. We must count the occurrences of pairs of pitch classes that are interchanged by the operation at hand and double them, for if x maps onto y under some $\mathsf{T}_n\mathsf{I}$, then certainly y maps onto x under the same operation, given that every inversion operation has order two. In addition to that, we must account for the occurrences of pitch classes that may map onto themselves under the aforementioned operation. For any pair $a \neq b \in S$, it follows a and b are exchanged by some operation $\mathsf{T}_n\mathsf{I}$ whenever both $\mathsf{T}_n\mathsf{I}(a) = b$ and $\mathsf{T}_n\mathsf{I}(b) = a$ hold. Since $\mathsf{T}_n\mathsf{I}(a) = -a + n$ and similarly $\mathsf{T}_n\mathsf{I}(b) = -b + n$, if the pair is exchanged, we must have $-a + n = b$ and $-b + n = a$ both true. Adding the last two expressions and yields $a + b = n$, which is the first set in the right-hand side of the formula. As discussed above, the cardinality of this set must be doubled. We have for any a that $\mathsf{T}_n\mathsf{I}(a) = a + n$, hence $a = \mathsf{T}_n\mathsf{I}(a) \iff a = -a + n$, that is, whenever $2a = n$. That is the second set in the formula. Finally, for any pair (a, n) such that $a = \mathsf{T}_n\mathsf{I}(a)$, we also have $a + 6 = -(a + 6) + n \iff 2a = n$, so that by the above it follows $a + 6 = \mathsf{T}_n\mathsf{I}(a + 6)$. Thus the set $\{a \in S : 2a = n\}$ has cardinality at most 2, proving the last assertion. □

Example 2.1.4. *(**?** , ??) Write $S = \{0, 1, 4, 5, 8, 9\}$ and consider some inversion operation. We have*

n	0	1	2	3	4	5	6	7	8	9	10	11
$2 \cdot \lvert\{x + y = n : x, y \in S\}\rvert$	2	6	2	0	2	6	2	0	2	6	2	0
$\lvert\{a \in S : 2a = n\}\rvert$	1	0	1	0	1	0	1	0	1	0	1	0
Total	3	6	3	0	3	6	3	0	3	6	3	0

We can demonstrate the above, as well as the omitted proof of the common-tone theorem under transposition in a much simpler way with a little bit of abstract algebra. By observing the cycle decomposition the each operation at hand, if $n = 3$, then we have

$$\mathsf{T}_3\mathsf{I} = (0\ 3)(1\ 2)(4\ 11)(5\ 10)(6\ 9)(7\ 8)\ .$$

Hence, under $\mathsf{T}_3\,\mathsf{I}$, every pitch-class in $S = \{0,1,4,5,8,9\}$ gets sent to the complement of S. If the operation is, for instance, T_9, then since

$$\mathsf{T}_9 = (0\ 9\ 6\ 3)(1\ 10\ 7\ 4)(2\ 11\ 8\ 5)\ ,$$

we get straightforwardly that $S = \{0,1,4,5,8,9\}$ shares three common tones with $\mathsf{T}_9 \circ S$, namely $0 \mapsto 9$, $4 \mapsto 1$, and $8 \mapsto 5$.

Transposition of order numbers is just rotation of pitch classes. Inversion of order numbers is equivalent to taking the retrograde and its rotations. Multiplication of order numbers modulo 12 does not in general produce a 12-tone row. Just as with the case of multiplication of pitch classes, we find that the only cases when we do get a bijective mapping are when the index of multiplication is an integer n relatively prime to 12. The cases $n = 1$ and $n = 11$ gives respectively the identity operation and the eleventh rotation of the retrograde (or equivalently $T_{11}I \circ S_*$).

Question: what is the effect of M_5 on order numbers?

$$\{0,1,2,3,4,5,6,7,8,9,10,11\}\{0,5,10,3,8,1,6,11,4,9,2,7\}$$

This is different than the mallalieu property. Say $S = \{0,1,4,\cdots\}$.

2.1.1 Mallalieu-Type Rows

Consider the 12-tone series $S = \{0,1,4,2,9,5,11,3,8,10,7,6\}$. This series has the remarkable property that, if we include a dummy 13^{th} element, then taking every n^{th} element of S produces a transposition of it.

Example 2.1.5. *We have* $S^* = \{0,1,4,\ldots,7,6,*\}$. *Then taking every zeroth order number of* $S^* \mod 13$ *yields* S^* *itself. Taking every first order number yields the series* $\{1,2,5,\ldots,8,7,*\}$ *which, upon removing the dummy symbol, becomes* $\mathsf{T}_1 \circ S$. *Repeating this procedure every* n^{th} *order number gives the sequence of transforms* $\{\mathsf{T}_i\}_{i\in S}$.

This most peculiar property, commonly called the *mallalieu* property, was known by Babbitt since at least 1954, but first discovered by Pohlman Mallalieu (citation). It is natural to ask at this point how many different 12-tone rows are there sharing this property. Unfortunately, there is only one such 12-tone row class under $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$. We phrase below a little differently an argument given by Morris in 1975:

Proposition 2.1.6. *(citation) A 12-tone row has the mallalieu property if and only if it is related by $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$ to the row $S = \{0, 1, 4, 2, 9, 5, 11, 3, 8, 10, 7, 6\}$.*

Proof. One direction is just the straightforward check that every $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$ transform of S possesses the mallalieu property and is left to the reader. Conversely, if a row R in its untransposed prime form has the mallalieu property, then there is a transposition that takes its order numbers in zeroth rotation, that is, the set $\{0, 1, 2, \dots, 11\}$ to its order numbers in, say, first rotation, id est, the set $\{1, 3, \dots, 11, 0, 2, \dots, 10\}$. We can write this transposition as a permutation $0 \mapsto 1, 1 \mapsto 3, \dots, 11 \mapsto 10$, or in cycle notation as $T_k = (0\ 1\ 3\ 7\ 2\ 5\ 11\ 10\ 8\ 4\ 9\ 6)$. Note that T_k is an operation on order numbers. Since T_k is a transposition, there are only four candidates for k, namely $k \in \{1, 5, 7, 11\}$ (because these are the only indices for which a transposition in cycle notation is a 12-cycle). Moreover, we do not need to consider the cases where $k \in \{5, 7, 11\}$, as $tra_5 = \mathsf{M} \circ \mathsf{T}_1$, $\mathsf{T}_7 = \mathsf{M}\,\mathsf{I} \circ \mathsf{T}_1$, and $\mathsf{T}_{11} = \mathsf{I} \circ \mathsf{T}_1$. Hence, without loss, we can set $k = 1$. But then S is the only row in untransposed prime form where T_1 induces the permutation T_k from its order numbers in zeroth rotation to its order numbers in first rotation (just equate T_k with T_1), completing the proof. □

Lewin (citation) provides a way of looking at mallalieu rows from the standpoint of replacing, for any 12-tone row, its order-number row $\{0, 1, \dots, 11\}$ by the array of integers $\{1, 2, \dots, 12, 0\}$ modulo 13. It is easy to see that such an array has the same structure as the array S_* we constructed above if we substitute the asterisk by the number 12 and consider multiplication as the group operation. Obviously, this is just the isomorphism between the integers modulo 12 and the group of units modulo 13. One of the advantages of this approach is that we can dispense with the extra symbol altogether and just use the indices from 1 to $p - 1$. We shall, however, still refer to the row of order numbers as S^*, the context making it clear whether we are constructing it with an asterisk or not. The process of taking every n^{th} element of a 12-tone row becomes then just the aforementioned multiplicative group operation on order numbers, that is, multiplying order numbers by $k \pmod{13}$ is the same as taking every k^{th} element of a row.

Example 2.1.7. *Put $S = \{0, 1, \dots, 11\}$ and $S^* = \{1, 2, \dots, 12\}$. Then $\mathsf{M}_3 \circ S^* = \{3, 6, \dots, 10\}$, which corresponds to the row $R = \{2, 5, \dots, 9\}$. The row R can be equivalently constructed by placing an asterisk as the 13^{th} order number of S and taking every third element. The fact that R and S are not related by $\mathsf{T}_n \mathsf{M}\,\mathsf{I}$ reflects the fact that neither S nor R have the mallalieu property.*

In addition, Lewin (citation) proposes the following:

Proposition 2.1.8. *For p a prime, every $(p-1)$-TET system is capable of producing a mallalieu row.*

Proof. For every prime p, the group of units modulo p is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$. The mallalieu property in these cases can be seen as the aforementioned isomorphism, where $\mathbb{Z}/(p-1)\mathbb{Z}$ is the group of transpositions of a row, and $(\mathbb{Z}/p\mathbb{Z})^\times$ is its multiplicative group on order numbers. The number of mallalieu rows in each $(p-1)$-TET system is then the number of isomorphisms $\mathbb{Z}/(p-1)\mathbb{Z} \to (\mathbb{Z}/p\mathbb{Z})^\times$, that is, the order of the group of automorphisms of $\mathbb{Z}/(p-1)\mathbb{Z}$. Since for every prime p we have $|\operatorname{Aut}(\mathbb{Z}/(p-1)\mathbb{Z})| \geq 1$, every $(p-1)$-TET system is capable of producing a mallalieu row, as desired. $\square$

In face of 2.1.8, 2.1.6 becomes just the special case where $p = 13$, as seen in the next example:

Example 2.1.9. *The number of isomorphisms $\mathbb{Z}/12\mathbb{Z} \to (\mathbb{Z}/13\mathbb{Z})^\times$ is equal to $|\operatorname{Aut}\left((\mathbb{Z}/12\mathbb{Z})^\times\right)| = 4$. We can construct these isomorphisms by mapping a generator of $\mathbb{Z}/12\mathbb{Z}$, say $\bar{1}$, to the generators of $(\mathbb{Z}/13\mathbb{Z})^\times$, namely $\bar{2}, \bar{6}, \bar{7}$ and $\overline{11}$. Explicitly, we get the four maps $i \pmod{12} \mapsto 2^i \pmod{13}$, $i \pmod{12} \mapsto 6^i \pmod{13}$, $i \pmod{12} \mapsto 7^i \pmod{13}$, and $i \pmod{12} \mapsto 11^i \pmod{13}$. We leave the verification that these maps are well defined and bijective to the* *reader. Denote the first map by φ. Then*

$$\varphi(a+b) = 2^{a+b} = 2^a \cdot 2^b = \varphi(a) \cdot \varphi(b) \ ,$$

so φ is an isomorphism. The verification that the other three maps are isomorphisms is identical. Define $\varphi^{-1} : (\mathbb{Z}/13\mathbb{Z})^\times \to \mathbb{Z}/12\mathbb{Z}$ by $\varphi^{-1}(\log i \pmod{13}) = i \pmod{12}$. Then φ^{-1} is easily seen to be the inverse of φ. Let $S^* = \{1, 2, \ldots, 12\}$ be a series of order numbers written multiplicatively. Then

$$\varphi^{-1}(S^*) = \{\log 1, \log 2, \ldots, \log 12\} \pmod{13} = \{0, 1, 4, \ldots, 7, 6\} \ ,$$

which by 2.1.6 is one of the four 12-tone rows with the mallalieu property.

It should be of interest to many composers whether other n-TET systems are capable of producing mallalieu rows, and if so, how many. Unfortunately, answering this question is not as straightforward as the above discussion, since we can no longer rely on the isomorphism that constitutes the proof of 2.1.8. We shall reformulate this question at the end of the present chapter, after having covered more of what has been already done.

If, on one hand, we only get one $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$ row class with the mallalieu property in 12 tones, we do get considerably more row classes when we relax the requirement that a row be produce a transposition of itself when taking every n^{th} of its elements. This idea is explored in part by (citation – Mead), however without specifying any combinatorial aspect (in the mathematical sense) of this generalization. Moreover, we can certainly go beyond (citation – Mead) and investigate, in 12 tones, what an extension of the mallalieu property could yield under operations other than transposition.

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BIOGRAPHICAL SKETCH

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