GENERALIZED MULTIPLE ORDER-NUMBER FUNCTION ARRAYS

By

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To my family

ACKNOWLEDGMENTS

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CHAPTER 1
INTRODUCTION

Music has become an almost arbitrary matter, and composers will no longer be bound by laws and rules, but avoid the names of School and Law as they would Death itself...

– Johann Joseph Fux

1.1 Derivation and Polyphony

Derivation is the process of extracting ordered segments from rows in order to generate new compositional materials. It is a technique that dates back to the Second Viennese School. In its most incipient form, a composer may simply extract these segments from a row, and combine them to form another, as we see in Berg's *Lulu*. The basic row used by Berg is $\mathrm{S} = \{10, 2, 3, 0, 5, 7, 4, 6, 9, 8, 1, 11\}$. In the Prologue, however, one is greeted with the row $\{10, 3, 4, 9, 2, 7, 8, 1, 0, 5, 6, 11\}$, as depicted in Fig. 1-1. It is clear that the segments that constitute the Prologue's row are ordered segments in the basic row form, and the fact that one row cannot obtained from another via row operations is irrelevant.

Figure 1-1. Derived rows in Alban Berg's *Lulu* [1, 182].

Less naive approaches to derivation ofted involve a derived row that will have more structure. In them, one will often see a combination matrix where derived and original rows are matched with some tewlve-tone or order operation of themselves, or both. This is illustrated in Ex. 1.1.1.

Example 1.1.1. *We may use the basic row in* Lulu *as motivation for a basic derivation procedure. The first step is to create a* 2×24 *array where the first row is* S *followed by* $\mathrm{R} \circ \mathrm{S}$*,* *and the second row is initially undefined.*

$$\left[\begin{array}{cccccccccccc|cccccccccccc} 10 & 2 & 3 & 0 & 5 & 7 & 4 & 6 & 9 & 8 & 1 & 11 & 11 & 1 & 8 & 9 & 6 & 4 & 7 & 5 & 0 & 3 & 2 & 10 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \end{array}\right]. \tag{1–1}$$

Next, one chooses an arbitrary segment, and separates it from the the top row by placing in in the bottom row:

$$\left[\begin{array}{cccccccccccc|cccccccccccc} . & 2 & 3 & . & . & . & 4 & 6 & 9 & 8 & 1 & 11 & . & . & . & . & . & . & 7 & 5 & 0 & . & . & 10 \\ 10 & . & . & 0 & 5 & 7 & . & . & . & . & . & . & 11 & 1 & 8 & 9 & 6 & 4 & . & . & . & 3 & 2 & . \end{array}\right]. \tag{1–2}$$

Let $\mathrm{T} = \{10, 3, 4, 6, 9, 8, 1, 11, 7, 5, 0, 2\}$*. Then* T *is a row derived from* S*. In particular, the ordered segment* $\{10, 0, 5, 7\}$ *in* S *is preserved by* $\mathrm{R} \circ \mathrm{T}$*.*

It is of interest to note at this point that, in this kind of construction, the choice of a particular segment is already an important compositional decision. This choice bears relevance in that it extablishes motivic material, that is, the segment itself. It also potentially introduces complementary harmonic regions, one given by the segment, the other given by its set complement. Moreover, and perhaps more importantly, it presents an opportunity for exploring syntax. There are a multitude of ways in which a composer may obtain syntax from a simple derivation procedure such as the one given in the example above. One way would be to find an operation that makes the chosen segment invariant. In particular, it is easily checked that $\mathrm{S}_1 = \{10, 0, 5, 7\} = \mathrm{R}\,\mathrm{T}_5\,\mathrm{I} \circ \mathrm{S}_1$. One can then extend Ex. 1.1.1 into the combination array $[\mathrm{S} \mid \mathrm{R} \circ \mathrm{S} \mid \mathrm{R}\,\mathrm{T}_5\,\mathrm{I} \circ \mathrm{S} \mid \mathrm{T}_5\,\mathrm{I} \circ \mathrm{S}_1]$. In the extended array, the segment S_1 would be preserved, but the row derived from $\mathrm{R}\,\mathrm{T}_5\,\mathrm{I} \circ \mathrm{S}$ would not be a transform of T. If the set complement of S_1 in T were parsed to produce more than one harmonic region, then the complement of S_1 under this new derived row would produce different harmonic regions. This can be very pertinent compositionally, as one would be capable of producing contrasting harmonic regions while maintaining motivic coherence under the S_1 segment.

Yet another way of generating syntax from derivation would be to follow S with T itself. One would then derive a new row from T, say Q, and eventually follow T with Q. Repeating this procedure *ad libitum* could generate many contrasting harmonic regions. In particular, this tipe of derivation is seen in Donald Martino's *Notturno* of 1974, a composition that won the Pulitzer Prize in the following year [1, 181]. If, by compostional choice, the chain of derived rows picked always the same order numbers, then a potential for rhythmic and agogic coherence could also be explored.

In one of the seminal academic works in the field of 12-tone theory, [1] utilizes a mostly set-theoretic framework to understand and categorize rows and procedures involved in producing derivation, polyphony, and self-derived combination matrices. The main objective is similar to ours, in it has a bias toward unveiling self derivation, which unfortunately still remains a somewhat obscure topic. The set-theoretic approach revolves around the idea of looking at collections from the standpoint of their order constraints: a totally constrained set with no precedence contradictions is a 12-tone row; a completely unconstrained set of 12 tones represents the free aggregate; a maximally constrained one is what the author calls the simmultaneous aggregate, that is, a 12-tone cluster. Sets that live in between can often be projected in the middle and background of a composition, fact that amounts to a Schenkerian-flavored view of the whole process.

Mathematically, the ideas in [1] translate into considering the set U of all ordered pairs of pitch classes. There are 12 choices for the first position, and 12 choices for the second position. As both choices are independent, this set has cardinality $12^2 = 144$. An element of U is called an order constraint, and a subset C of U is called a pitch-class relation. The latter can be viewed as a 12×12 matrix where the entry $\mathrm{c_{ij}}$ is equal to one whenever $\{\mathrm{i}, \mathrm{j}\} \in \mathrm{C}$, and zero otherwise. One can the apply biwise operations to these matrices in a very computationally efficient manner. For any pitch classes x and y, we define a relation $\mathrm{x} \sim \mathrm{y}$ on the power set of U by the set inclusion of the element $\{\mathrm{x}, \mathrm{y}\}$. A subset C will then be reflexive if, whenever an element of C (which is a set) contains the pitch class x, then $\{\mathrm{x}, \mathrm{x}\} \in \mathrm{C}$. In words, reflexivity means that if a reflexive collection C of notes contains an element x, then x precedes (and follows) itself in C. The free aggregate is a minimal reflexive subset of U that contains all 12 tones. The relation $\sim$ will be symmetric if $\{\mathrm{x}, \mathrm{y}\} \in \mathrm{C}$ implies $\{\mathrm{y}, \mathrm{x}\} \in \mathrm{C}$, and antisymmetric whenever $\{\mathrm{x}, \mathrm{y}\} \in \mathrm{C}$ implies $\{\mathrm{y}, \mathrm{x}\} \notin \mathrm{C}$, for $\mathrm{x} \neq \mathrm{y} \in \mathbb{Z}/12\mathbb{Z}$. Similarly, transitivity is defined as $\{\mathrm{x}, \mathrm{y}\} \in \mathrm{C}$ and $\{\mathrm{y}, \mathrm{z}\} \in \mathrm{C}$, then $\{\mathrm{x}, \mathrm{z}\} \in \mathrm{C}$; and trichotomy is defined as either $\{\mathrm{x}, \mathrm{y}\} \in \mathrm{C}$ or $\{\mathrm{y}, \mathrm{x}\} \in \mathrm{C}$ for any $\mathrm{x} \neq \mathrm{y} \in \mathbb{Z}/12\mathbb{Z}$. The relation $\sim$ is, of course, an order relation on the set of 12 tones by definition. A partial order is one that is reflexive, transitive, and antisymmetric, while a total order (a row), is a partial order that satisfies trichotomy.

Often, pitch-class relations will contain many redundancies due to transitivity. In order to express these relations as oriented graphs, one must first remove, or prune such redundancies. This process can be reversed and a pitch-class relation can be extended to the point of its transitive closure. It is also common for a pitch-class relation to be absent of any order constraint involving both $\{\mathrm{x}, \mathrm{y}\}$, in which case we say x and y are incomparable. Such x and y are bound to be struck together, or else be *linearized* by the injection of some constraint that will make them comparable, as long as the is still a partial order, that is, as long as it does not introduce a symmetry, for instance. The set of all total orderings that can be linearized out of some partial order is called its total order class. In a completely analogous manner, one can *verticalize* a pitch-class relation by removing constraints, and again minding that the result is still transitive and symmetric. We say a partial order covers another whenever the former is a verticalization of the latter. A simple procedure to guarantee that a verticalization will remain a partial order is to take its union with the free aggregate, then subject this union to an extension operation, thus providing reflexivity in the first step, as well as transitivity in the second. We can say the following about covering and about unions and intersections of pitch-class relations:

Theorem 1.1.2. *[1, 193]*

i. Covering is transitive;

ii. A pitch-class relation is covered by its extension;

iii. If a pitch-class relation covers another, then the extension of the former covers the extension of the latter.

Theorem 1.1.3. *[1, 194] Let* A *and* B *be partial orders and denote by* $\mathrm{Toc}(\mathrm{A})$ *and* $\mathrm{Toc}(\mathrm{B})$ *their respective total order classes. Then*

$$\mathrm{Toc}(\mathrm{A}) \cap \mathrm{Toc}(\mathrm{B}) = \mathrm{Toc}(\mathrm{Ext}(\mathrm{A} \cup \mathrm{B})) \ , \tag{1–3}$$

where Ext *is the extension operator.*

Theorem 1.1.4. *[1, 194] The intersection of two partial orders is again a partial order.*

1.2 The Mallalieu Property

Consider the 12-tone series $\mathrm{S} = \{0, 1, 4, 2, 9, 5, 11, 3, 8, 10, 7, 6\}$. This series has the remarkable property that, if we include a dummy 13^{th} element, then taking every n^{th} element of S produces a transposition of it.

Example 1.2.1. *We have* $\mathrm{S}^* = \{0, 1, 4, \dots, 7, 6, *\}$. *Then taking every zeroth order number of* $\mathrm{S}^* \mod 13$ *yields* S^* *itself. Taking every first order number yields the series* $\{1, 2, 5, \dots, 8, 7, *\}$ *which, upon removing the dummy symbol, becomes* $\mathrm{T}_1 \circ \mathrm{S}$. *Repeating this procedure every* n^{th} *order number gives the sequence of transforms* $\{\mathrm{T}_\mathrm{i}\}_{\mathrm{i} \in \mathrm{S}}$.

This most peculiar property, commonly called the *mallalieu* property, was first discovered by Pohlman Mallalieu [2, 285]. It is natural to ask at this point how many different 12-tone rows are there sharing this property. Unfortunately, there is only one such 12-tone row class under $\mathrm{T_n\, M\, I}$. We phrase below a little differently an argument given in [3, 17].

[2, 278] provides a way of looking at mallalieu rows from the standpoint of replacing, for any 12-tone row, its order-number row $\{0, 1, \dots, 11\}$ by the array of integers $\{1, 2, \dots, 12, 0\}$ modulo 13. It is easy to see that such an array has the same structure as the array S_* we constructed above if we substitute the asterisk by the number 12 and consider multiplication as the group operation. Obviously, this is just the isomorphism between the integers modulo 12 and the group of units modulo 13. One of the advantages of this approach is that we can dispense with the extra symbol altogether and just use the indices from 1 to $\mathrm{p} - 1$. We shall, however, still refer to the row of order numbers as S^*, the context making it clear whether we are constructing it with an asterisk or not. The process of taking every n^{th} element of a 12-tone row becomes then just the aforementioned multiplicative group operation on order numbers, that is, multiplying order numbers by $\mathrm{k} \pmod{13}$ is the same as taking every k^{th} element of a row.

Example 1.2.2. *Put* $\mathrm{S} = \{0, 1, \dots, 11\}$ *and* $\mathrm{S}^* = \{1, 2, \dots, 12\}$. *Then*

$$\mathrm{M}_3 \circ \mathrm{S}^* = \{3, 6, \dots, 10\} \ , \tag{1–4}$$

which corresponds to the row $\mathrm{R} = \{2, 5, \dots, 9\}$*. The row* R *can be equivalently constructed by placing an asterisk as the* 13^{th} *order number of* S *and taking every third element. The fact that* R *and* S *are not related by* $\mathrm{T_n\,M\,I}$ *reflects the fact that neither* S *nor* R *have the mallalieu property.*

It should be of interest to many composers whether other n-TET systems are capable of producing mallalieu rows, and if so, how many. Unfortunately, answering this question is not as straightforward as the above discussion, since we can no longer rely on the isomorphism that constitutes the proof of 2.1.27. We shall reformulate this question at the end of the present chapter, after having covered more of what has been already done.

If, on one hand, we only get one $\mathrm{T_n\,M\,I}$ row class with the mallalieu property in 12 tones, we do get considerably more row classes when we relax the requirement that a row produce a transposition of itself when taking every n^{th} of its elements. This idea is explored in part by [4], however without specifying any combinatorial aspect (in the mathematical sense) of this generalization. Moreover, we can certainly go beyond [4] and investigate, in 12 tones, what an extension of the mallalieu property could yield under operations other than transposition.

CHAPTER 2
THEORETICAL FRAMEWORK

2.1 Group Actions

Definition 2.1.1. *[5, 99] [6, 41] Let X be a set and G a group. An **action** of G on X is a function $\mathrm{G} \times \mathrm{X} \to \mathrm{X}$ given by $(\mathrm{g}, \mathrm{x}) \mapsto \mathrm{gx}$, such that:*

i. $(\mathrm{gh})\mathrm{x} = \mathrm{g}(\mathrm{hx})$ *for all* $\mathrm{g}, \mathrm{h} \in \mathrm{G}$ *and* $\mathrm{x} \in \mathrm{X}$*;*

ii. $1\mathrm{x} = \mathrm{x}$ *for all* $\mathrm{x} \in \mathrm{X}$*, where* $1 \in \mathrm{G}$ *is the identity.*

Proposition 2.1.2. *[5, 99] [6, 42] If a group G acts on a set X then, for every $\mathrm{g} \in \mathrm{G}$, the function $\mathrm{f_g} : \mathrm{X} \to \mathrm{X}$ given by $\mathrm{f_g}(\mathrm{x}) = \mathrm{gx}$ is a permutation of X. Further, the function $\mathrm{f} : \mathrm{G} \to \mathrm{S_X}$ given by $\mathrm{f}(\mathrm{g}) = \mathrm{f_g}$ is a homomorphism and, conversely, for any homomorphism $\phi : \mathrm{G} \to \mathrm{S_X}$, there is a corresponding group action given by $\phi(\mathrm{g})(\mathrm{x})$.*

Theorem 2.1.3 (Cayley)**.** *[5, 96] [6, 120] Every group is isomorphic to a subgroup of the symmetric group $\mathrm{S_G}$. In particular, if $|\mathrm{G}| = \mathrm{n}$, then G is isomorphic to a subgroup of $\mathrm{S_n}$.*

Theorem 2.1.4. *[5, 97] Let $\mathrm{H} \leq \mathrm{G}$ be a subgroup of finite index n. Then there exists a homomorphism $\phi : \mathrm{G} \to \mathrm{S_n}$ such that $\ker \phi \leq \mathrm{H}$. In particular, when $\mathrm{H} = \{1\}$, we get Cayley's theorem.*

Example 2.1.5. *[6, 122] A group acts on itself by conjugation. Let $\mathrm{g}, \mathrm{h} \in \mathrm{G}$ and $\mathrm{x} \in \mathrm{G}$. Then $1\mathrm{x}1^{-1} = \mathrm{x}$ and*

$$\mathrm{g}(\mathrm{hx}) = \mathrm{g}(\mathrm{hxh}^{-1}) = \mathrm{ghxh}^{-1}\mathrm{g}^{-1} = (\mathrm{gh})\mathrm{x}(\mathrm{gh})^{-1} = (\mathrm{gh})\mathrm{x} \ . \tag{2–1}$$

It is also immediate from the above that a group acts on its power set by conjugation. In particular, a group acts on the set of all its subgroups.

Definition 2.1.6. *[5, 100] [6, 112] If G acts on X, then the **orbit** of $\mathrm{x} \in \mathrm{X}$ is the set*

$$\mathcal{O}(\mathrm{x}) = \{\mathrm{gx} : \mathrm{g} \in \mathrm{G}\} \subseteq \mathrm{X} \ . \tag{2–2}$$

*We say an action is **transitive** if there is only one orbit. The **kernel** of the action is the set*

$$\{\mathrm{g} \in \mathrm{g} : \mathrm{gx} = \mathrm{x}, \forall \mathrm{x} \in \mathrm{X}\} \ . \tag{2–3}$$

*We say an action is **faithful** if the kernel is the identity. The **stabilizer** of* x *in* G *is the group*

$$\mathrm{G_x} = \{\mathrm{g} \in \mathrm{g} : \mathrm{gx} = \mathrm{x}\} \leq \mathrm{G} \ . \tag{2–4}$$

*When a group acts on itself by conjugation, we call the orbits **conjugacy classes**. The stabilizer of some* $\mathrm{g} \in \mathrm{G}$ *is the **centralizer** of* g *in* G*, denoted* $\mathrm{C_G(g)}$*. When a group acts on the set of its subgroups by conjugation, the stabilizer of a subgroup* $\mathrm{H} \leq$ *is the **normalizer** of* H *in* G*, denoted by* $\mathrm{N_G(H)}$*.*

Proposition 2.1.7. *[5, 102] [6, 114] [7, 250] If* G *acts on* X*, for* $\mathrm{x_1}, \mathrm{x_2} \in \mathrm{X}$*, the relation* $\mathrm{x_1} \sim \mathrm{x_2}$ *given by* $\mathrm{x_1} = \mathrm{gx_2}$ *is an equivalence relation. It follows immediately that the equivalence classes are the orbits of the action of* G *on* X *and that*

$$|\mathrm{X}| = \sum_{\mathrm{i}} |\mathcal{O}(\mathrm{x_i})| \ , \tag{2–5}$$

where $\mathrm{x_i}$ *is a single representative from each orbit.*

Theorem 2.1.8 (Orbit-Stabilizer)**.** *[5, 102] If* G *acts on* X*, then for each* $\mathrm{x} \in \mathrm{X}$

$$|\mathcal{O}(\mathrm{x})| = [\mathrm{G} : \mathrm{G_x}] \ . \tag{2–6}$$

Corollary 2.1.9. *[5, 103] If* G *is finite and acts on* X*, then the size of any orbit is a divisor of* $|\mathrm{G}|$*.*

Proposition 2.1.10. *[6, 123] The number of conjugates of a subset* S *in a group* G *is* $|\mathrm{G} : \mathrm{N_G(S)}|$*, the index of the normalizer of* S*. In particular, the number of conjugates of an element* s *is* $|\mathrm{G} : \mathrm{C_G(s)}|$*, the index of the centralizer of* s*.*

Proposition 2.1.11. *[6, 125] Let* $\sigma, \tau \in \mathrm{S_n}$*. If*

$$\sigma = (\mathrm{a_1}\ \mathrm{a_2}\ \cdots \mathrm{a_j})(\mathrm{b_1}\ \mathrm{b_2}\ \cdots \mathrm{a_k}) \cdots \ , \tag{2–7}$$

then

$$\tau\sigma\tau^{-1} = (\tau(\mathrm{a_1})\ \tau(\mathrm{a_2})\ \cdots \tau(\mathrm{a_j}))(\tau(\mathrm{b_1})\ \tau(\mathrm{b_2})\ \cdots \tau(\mathrm{b_k})) \cdots \ . \tag{2–8}$$

Example 2.1.12. *[6, 127] Let $\sigma \in \mathrm{S_n}$ be an m-cycle. The number of conjugates of σ is*

$$\frac{|\mathrm{S_n}|}{|\mathrm{C_{S_n}}(\sigma)|} = \frac{\mathrm{n(n-1)(n-m+1)}}{\mathrm{m}} \ , \tag{2–9}$$

so that $|\mathrm{C_{S_n}}(\sigma)| = \mathrm{m(n-m)!}$. Since σ commutes with its powers, and also with any permutation in $\mathrm{S_n}$ whose cycles are disjoint from it, and there are $\mathrm{(n-m)!}$ of those, the number computed above is the full centralizer of σ.

Example 2.1.13. *[6, 132] The size of each conjugacy class in $\mathrm{S_n}$ is*

$$\frac{\mathrm{n!}}{\prod_{\mathrm{r}} \mathrm{r^{n_r} n_r!}} \ , \tag{2–10}$$

where, for each r-cycle, we divide by r to account for the cyclical permutations of elements within a cycle. Further, if there are $\mathrm{n_r}$ cycles of length r, we divide by $\mathrm{n_r!}$ to account for the different orders in which those cycles may appear.

Proposition 2.1.14. *[6, 126] Two elements in $\mathrm{S_n}$ are conjugate if and only if they have the same cycle type. The number of conjugacy classes of $\mathrm{S_n}$ is the number of partitions of n.*

Definition 2.1.15. *[6, 133] An isomorphism from a group G to itself is called an **automorphism** of G. The group under composition of all automorphisms of G is denoted by $\mathrm{Aut(G)}$.*

Proposition 2.1.16. *[6, 133] If H is a normal subgroup of G, then the action of G by conjugation on H is, for each $\mathrm{g} \in \mathrm{G}$, an automorphism of H. The kernel of the action is $\mathrm{C_G(H)}$. In particular, $\mathrm{G/c_G(H)}$ is a subgroup of $\mathrm{Aut(H)}$.*

Corollary 2.1.17. *[6, 134] For any subgroup $\mathrm{H} < \mathrm{G}$ and $\mathrm{g} \in \mathrm{G}$, $\mathrm{H} \cong \mathrm{gHg^{-1}}$. Moreover, $\mathrm{N_G(H)/C_G(H)}$ (and also $\mathrm{G/Z(G)}$ when $\mathrm{G = H}$) is isomorphic to a subgroup of $\mathrm{Aut(H)}$.*

Definition 2.1.18. *[6, 135] A subgroup $\mathrm{H} < \mathrm{G}$ is called **characteristic** if every automorphism of G maps H to itself.*

Proposition 2.1.19. *[6, 135] Characteristic subgroups are normal. Unique subgroups of a given order are characteristic. A characteristic subgroup of a normal subgroup is normal. In particular, every subgroup of a cyclic group is characteristic.*

Proposition 2.1.20. *[6, 135] If* G *is cyclic of order* n*, then* $\mathrm{Aut(G)} \cong (\mathbb{Z}/\mathrm{n}\mathbb{Z})^\times$*, and* $|\mathrm{Aut(G)}| = \varphi(\mathrm{n})$*, where* φ *is Euler's totient function.*

Corollary 2.1.21. *[6, 136] Let* $|\mathrm{G}| = \mathrm{pq}$*, with* $\mathrm{p} \leq \mathrm{q}$ *primes. If* $\mathrm{p} \nmid \mathrm{q} - 1$*, then* G *is abelian. If, further,* $\mathrm{p} < \mathrm{q}$*, then* G *is cyclic.*

Proposition 2.1.22. *[6, 136] If* $|\mathrm{G}| = \mathrm{p^n}$*, with* p *an odd prime, then*

$$|\mathrm{Aut(G)}| = \mathrm{p^{n-1}(p-1)} \tag{2–11}$$

and the latter group is also cyclic. If $|\mathrm{G}| = 2^\mathrm{n}$ *is cyclic, with* $\mathrm{n} \geq 3$*, then*

$$\mathrm{Aut(G)} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{\mathrm{n}-2}\mathbb{Z} \tag{2–12}$$

and the latter is not cyclic, but has a cyclic subgroup of index 2. If V *is the elementary abelian group of order* $\mathrm{p^n}$*, then* $\mathrm{pv} = 0$ *for all* $\mathrm{v} \in \mathrm{V}$ *and* V *is an* n*-dimensional vector space over the field* $\mathbb{F}_\mathrm{p} = \mathbb{Z}/\mathrm{p}\mathbb{Z}$*. The automorphisms of* V *are the nonsingular linear transformations from* V *to itself:*

$$\mathrm{Aut(V)} \cong \mathrm{GL(V)} \cong \mathrm{GL_n}(\mathbb{F}_\mathrm{p}) \ , \tag{2–13}$$

and if F *is the finite field of order* q*, then*

$$|\mathrm{GL_n(F)}| = \mathrm{(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})} \ . \tag{2–14}$$

For all $\mathrm{n} \neq 6$*, we have* $\mathrm{Aut(S_n)} \cong \mathrm{S_n}$*. Finally,* $\mathrm{Aut(D_8)} \cong \mathrm{D_8}$ *and* $\mathrm{Aut(Q_8)} \cong \mathrm{S_4}$*.*

Example 2.1.23. *[6, 137] The Klein 4-group is the elementary abelian group of order 4. It follows* $\mathrm{Aut(V_4)} \cong \mathrm{GL_2}(\mathbb{F}_2)$*, and* $|\mathrm{Aut(V_4)}| = 6$*. Since the action of* $\mathrm{Aut(V_4)}$ *on* $\mathrm{V_4}$ *permutes the latter's 3 nonidentity elements, by order considerations we have*

$$\mathrm{Aut(V_4)} \cong \mathrm{GL_2}(\mathbb{F}_2) \cong \mathrm{S_3} \ . \tag{2–15}$$

Proposition 2.1.24. *[6, 314] A finite subgroup of the multiplicative group of a field is cyclic. In particular, if* F *is a finite field, then* $\mathrm{F}^\times$ *is cyclic.*

Corollary 2.1.25. *[6, 314] For* p *a prime,* $(\mathbb{Z}/\mathrm{p}\mathbb{Z})^\times$ *is cyclic.*

Proposition 2.1.26. *[3, 17] A 12-tone row has the mallalieu property if and only if it is related by* $\mathrm{T_n\,M\,I}$ *to the row* $\mathrm{S} = \{0, 1, 4, 2, 9, 5, 11, 3, 8, 10, 7, 6\}$.

Proof. One direction is just the straightforward check that every $\mathrm{T_n\,M\,I}$ transform of S possesses the mallalieu property and is left to the reader. Conversely, if a row R in its untransposed prime form has the mallalieu property, then there is a transposition that takes its order numbers in zeroth rotation, that is, the set $\{0, 1, 2, \dots, 11\}$ to its order numbers in, say, first rotation, id est, the set $\{1, 3, \dots, 11, 0, 2, \dots, 10\}$. We can write this transposition as a permutation $0 \mapsto 1, 1 \mapsto 3, \dots, 11 \mapsto 10$, or in cycle notation as $\mathrm{T_k} = (0\ 1\ 3\ 7\ 2\ 5\ 11\ 10\ 8\ 4\ 9\ 6)$. Note that $\mathrm{T_k}$ is an operation on order numbers. Since $\mathrm{T_k}$ is a transposition, there are only four candidates for k, namely $\mathrm{k} \in \{1, 5, 7, 11\}$ (because these are the only indices for which a transposition in cycle notation is a 12-cycle). Moreover, we do not need to consider the cases where $\mathrm{k} \in \{5, 7, 11\}$, as $\mathrm{T_5} = \mathrm{M} \circ \mathrm{T_1}$, $\mathrm{T_7} = \mathrm{M\,I} \circ \mathrm{T_1}$, and $\mathrm{T_{11}} = \mathrm{I} \circ \mathrm{T_1}$. Hence, without loss, we can set $\mathrm{k} = 1$. But then S is the only row in untransposed prime form where $\mathrm{T_1}$ induces the permutation $\mathrm{T_k}$ from its order numbers in zeroth rotation to its order numbers in first rotation (just equate $\mathrm{T_k}$ with $\mathrm{T_1}$), completing the proof. □

Proposition 2.1.27. *[2, 285] For* p *a prime, every* $(\mathrm{p}-1)$*-TET system is capable of producing a mallalieu row.*

Proof. For every prime p, the group of units modulo p is isomorphic to $\mathbb{Z}/(\mathrm{p}-1)\mathbb{Z}$. The mallalieu property in these cases can be seen as the aforementioned isomorphism, where $\mathbb{Z}/(\mathrm{p}-1)\mathbb{Z}$ is the group of transpositions of a row, and $(\mathbb{Z}/\mathrm{p}\mathbb{Z})^\times$ is its multiplicative group on order numbers. The number of mallalieu rows in each $(\mathrm{p}-1)$-TET system is then the number of isomorphisms $\mathbb{Z}/(\mathrm{p}-1)\mathbb{Z} \to (\mathbb{Z}/\mathrm{p}\mathbb{Z})^\times$, that is, the order of the group of automorphisms of $\mathbb{Z}/(\mathrm{p}-1)\mathbb{Z}$. Since for every prime p we have $|\,\mathrm{Aut}(\mathbb{Z}/(\mathrm{p}-1)\mathbb{Z})| \geq 1$, every $(\mathrm{p}-1)$-TET system is capable of producing a mallalieu row, as desired. □

Example 2.1.28. *[8, 8] [9, 9] In face of 2.1.27, 2.1.26 becomes just the special case where* $p = 13$*. The number of isomorphisms* $\mathbb{Z}/12\mathbb{Z} \to (\mathbb{Z}/13\mathbb{Z})^{\times}$ *is equal to*

$$|\operatorname{Aut}\left((\mathbb{Z}/12\mathbb{Z})^{\times}\right)| = 4 \ . \tag{2–16}$$

We can construct these isomorphisms by mapping a generator of $\mathbb{Z}/12\mathbb{Z}$*, say* $\bar{1}$*, to the generators of* $(\mathbb{Z}/13\mathbb{Z})^{\times}$*, namely* $\bar{2}, \bar{6}, \bar{7}$ *and* $\overline{11}$*. Explicitly, we get the four maps* $i \pmod{12} \mapsto 2^i \pmod{13}$, $i \pmod{12} \mapsto 6^i \pmod{13}$, $i \pmod{12} \mapsto 7^i \pmod{13}$, *and* $i \pmod{12} \mapsto 11^i \pmod{13}$*. We leave the verification that these maps are well defined and bijective to the reader. Denote the first map by* φ*. Then*

$$\varphi(a + b) = 2^{a+b} = 2^a \cdot 2^b = \varphi(a) \cdot \varphi(b) \ , \tag{2–17}$$

so φ *is an isomorphism. The verification that the other three maps are isomorphisms is identical. Define* $\varphi^{-1} : (\mathbb{Z}/13\mathbb{Z})^{\times} \to \mathbb{Z}/12\mathbb{Z}$ *by* $\varphi^{-1}(\log i \pmod{13}) = i \pmod{12}$*. Then* φ^{-1} *is easily seen to be the inverse of* φ*. Let* $S^* = \{1, 2, \dots, 12\}$ *be a series of order numbers written multiplicatively. Then*

$$\varphi^{-1}(S^*) = \{\log 1, \log 2, \dots, \log 12\} \pmod{13} = \{0, 1, 4, \dots, 7, 6\} \ , \tag{2–18}$$

which by 2.1.26 is one of the four 12-tone rows with the mallalieu property.

2.2 Polya's Enumeration Formula

Definition 2.2.1. *[10, 85] Let* N *be a set of* n *beads and* R *be a set of* r *colors. A colored necklace is a function* $f : N \to R$*. Denote the set of all such functions by* R^N*, so that* $|R^N| = r^n$*. Define the* ***weight*** *associated to a coloring* f *by*

$$w(f) = \prod_{i \in N} x_{f(i)} \ , \tag{2–19}$$

where x_j *is a variable associated with the color* $j \in R$*.*

Proposition 2.2.2. *[5, 110] Let G be a group and $\mathrm{X} = \{1, \cdots, \mathrm{n}\}$ be a set. Let $\mathcal{C}$ be a set of q colors. Then G acts on the set $\mathcal{C}^{\mathrm{n}}$ of n-tuples of colors by*

$$\tau(\mathrm{c}_1, \cdots, \mathrm{c}_\mathrm{n}) = (\mathrm{c}_{\tau 1}, \cdots, \mathrm{c}_{\tau \mathrm{n}}), \forall \tau \in \mathrm{G} \ . \tag{2–20}$$

Proposition 2.2.3. *[10, 85] [5, 110] If G is a group acting on the set N, then two colorings f and f' are equivalent whenever $\mathrm{f} = \mathrm{f}' \circ \mathrm{g}$ for some $\mathrm{g} \in \mathrm{G}$. This is an equivalence relation that partitions R^N into equivalence classes denoted **patterns**. Under the conditions of 2.2.2, an orbit $(\mathrm{c}_1, \cdots, \mathrm{c}_\mathrm{n}) \in \mathcal{C}^\mathrm{n}$ is called a (q, G)-**coloring** of X. Moreover, two equivalent colorings have the same weight, so that we may refer to the weight of a class of colorings rather than the weights of its representatives.*

Definition 2.2.4. *[10, 85] Let N be a set of n beads and R be a set of r colors. Let G be a group acting on the set N and $\mathrm{x_j}$ the variable associated with the color $\mathrm{j} \in \mathrm{R}$. Let $\mathcal{M}$ be the set of of patterns. Define the pattern **enumerator** by*

$$\mathrm{w}(\mathrm{R}^\mathrm{N}, \mathrm{G}) = \sum_{\mathrm{M} \in \mathcal{M}} \mathrm{w}(\mathrm{M}) \ . \tag{2–21}$$

In particular, when $\mathrm{x_j} = 1$ for all $\mathrm{j} \in \mathrm{R}$, we get $\mathrm{w}(\mathrm{R}^\mathrm{N}, \mathrm{G}) = |\mathcal{M}|$.

Definition 2.2.5. *[10, 86] Let G be a group acting on a set X. Define for every $\mathrm{g} \in \mathrm{G}$ its **fixed-point set** by*

$$\mathrm{Fix}(\mathrm{g}) = \{\mathrm{x} \in \mathrm{X} : \mathrm{gx} = \mathrm{x}\} \ . \tag{2–22}$$

Lemma 2.2.6. *[5, 112] Let $\mathrm{G} < \mathrm{S_n}$ be a group and let $\mathcal{C}$ be a set of q colors. For $\tau \in \mathrm{G}$,*

$$|\mathrm{Fix}(\tau)| = \mathrm{q}^{\mathrm{t}(\tau)} \ , \tag{2–23}$$

where $\mathrm{t}(\tau)$ is the number of cycles in the complete factorization of τ.

Lemma 2.2.7 (Burnside)**.** *[5, 109] [7, 251] If G acts on a finite set X, then the number of orbits N is*

$$\mathrm{N} = \frac{1}{|\mathrm{G}|} \sum_{\tau \in \mathrm{G}} |\mathrm{Fix}(\tau)| \ . \tag{2–24}$$

Corollary 2.2.8. *[5, 112] Let G be a group acting on a finite set X. The number N of (q, G)-colorings of X is*

$$\mathrm{N} = \frac{1}{|\mathrm{G}|} \sum_{\tau \in \mathrm{G}} \mathrm{q}^{\mathrm{t}(\tau)} \quad . \tag{2–25}$$

Corollary 2.2.9. *[11, 54] [12, 127] The number of n-tone rows under $\mathrm{R\,T_n\,I}$ is*

$$\begin{cases} \frac{1}{4}\left[(\mathrm{n}-1)! + (\mathrm{n}-1)(\mathrm{n}-3)\cdots(2)\right] & \mathrm{n} \textit{ odd} \\ \frac{1}{4}\left[(\mathrm{n}-1)! + (\mathrm{n}-2)(\mathrm{n}-4)\cdots(2)(1+\frac{\mathrm{n}}{2})\right] & \mathrm{n} \textit{ even} \end{cases} \tag{2–26}$$

In particular, there are 9985920 twelve-tone rows.

Definition 2.2.10. *[10, 87] Let g be a group acting on R^{N}. Define the **cycle indicator** of G by*

$$\mathrm{P_G}(\mathrm{z}_1, \mathrm{z}_2, \cdots, \mathrm{z}_\mathrm{n}) = \frac{1}{|\mathrm{G}|} \sum_{\mathrm{g} \in \mathrm{G}} \mathrm{z}_1^{\mathrm{b}_1(\mathrm{g})} \mathrm{z}_2^{\mathrm{b}_2(\mathrm{g})} \cdots \mathrm{z}_\mathrm{n}^{\mathrm{b}_\mathrm{n}(\mathrm{g})} \quad , \tag{2–27}$$

where $\mathrm{z}_\mathrm{i}^{\mathrm{b}_\mathrm{i}(\mathrm{g})}$ corresponds to the number of cycles in the complete factorization of g that have length i.

Example 2.2.11. *The cycle indicator of S_3 is*

$$\mathrm{P}_{\mathrm{S}_3}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = \frac{1}{6}(\mathrm{x}_1^3 + 3\mathrm{x}_1^1\mathrm{x}_2^1 + 2\mathrm{x}_3^1) \quad , \tag{2–28}$$

since there are two elements in S_3 that comprise one cycle of length three, three elements that comprise one cycle of length one, and one cycle of length two, and one element that comprises three cycles of length one.

Example 2.2.12. *Let $\mathrm{C_n}$ be the cyclic group of order n. Then*

$$\mathrm{P}_{\mathrm{C_n}} = \frac{1}{\mathrm{n}} \sum_{\mathrm{d}|\mathrm{n}} \varphi(\mathrm{d}) \mathrm{z}_\mathrm{d}^{\mathrm{n}/\mathrm{d}} \quad . \tag{2–29}$$

Example 2.2.13. *For $\mathrm{S_n}$ we have*

$$\mathrm{P}_{\mathrm{S_n}} = \sum_{\mathrm{j}_1 + 2\mathrm{j}_2 + \cdots + \mathrm{n}\mathrm{j}_\mathrm{n} = \mathrm{n}} \frac{1}{\prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{k}^{\mathrm{j}_\mathrm{k}} \mathrm{j}_\mathrm{k}!} \prod_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{a}_\mathrm{k}^{\mathrm{j}_\mathrm{k}} \quad . \tag{2–30}$$

In words, there is a summand for each conjugacy class in S_n, and we divide each summand by the size of its corresponding conjugacy class.

Example 2.2.14. *For D_n we have*

$$P_{D_n} = \frac{1}{2}P_{C_n} + \begin{cases} \frac{1}{2}a_1a_2^{(n-1)/2} & n \text{ odd} \\ \frac{1}{4}(a_1^2a_2^{(n-2)/2} + a_2^{n/2)} & n \text{ even} \end{cases} \tag{2–31}$$

In particular, we obtain

$$P_{D_{24}}(1+x,\cdots,1+x^{12}) = \frac{1}{24}(x_1^{12} + 6x_1^2x_2^5 + 7x_2^6 + 2x_3^4 + 2x_4^3 + 2x_6^2 + 4x_{12}) \ . \tag{2–32}$$

Example 2.2.15. *[12, 120] For $\mathrm{Aff}_1(C_{12})$ we have*

$$\begin{aligned} P_{\mathrm{Aff}_1(C_{12})}(1+x,\cdots,1+x^{12}) = \frac{1}{48}(&x_1^{12} + 2x_1^6x_2^3 + 3x_1^4x_2^4 + 6x_1^2x_2^5 \\ &+ 12x_2^6 + 4x_2^3x_6 + 2x_3^4 + 8x_4^3 + 6x_6^2 + 4x_{12}) \ . \end{aligned} \tag{2–33}$$

Theorem 2.2.16 (Polya)**.** *[10, 88] [7, 256] Let N be a set of cardinality n and R be a set of cardinality r. Let G be a group acting on the set N and x_j be arbitrary variables with $j \in R$. Let $w(R^N, G)$ be the pattern enumerator for the action of G on R^N. Then*

$$w(R^N, G) = \sum_{M\in\mathcal{M}} w(M) = P_G(\sum_{j\in R} x_j, \sum_{j\in R} x_j^2, \cdots, \sum_{j\in R} x_j^n) \ . \tag{2–34}$$

Corollary 2.2.17. *[10, 89] [7, 254] Let $x_j = 1$ for all $j \in R$. Then $\sum_{j\in R} x_j^k = |R| = r$ for all k, hence*

$$P_G(r,\cdots,r) = |\mathcal{M}| \ . \tag{2–35}$$

Corollary 2.2.18. *[10, 89] Let $|R| = r = 2$. Let $x_1 = x_{white} = x$ and $x_2 = x_{black} = 1$. Then*

$$P_G(x+1, x^2+1, \cdots, x^n+1) = \sum_{k=0}^{n} a_kx^k \ , \tag{2–36}$$

where a_k is the number of patterns in which the color white occurs exactly k times.

Example 2.2.19. *[10, 89] The number of necklaces with* n *beads and* r *colors is*

$$\frac{1}{\mathrm{n}}\sum_{\mathrm{d}|\mathrm{n}}\varphi(\mathrm{d})\mathrm{r}_{\mathrm{d}}^{\mathrm{n}/\mathrm{d}}\quad . \tag{2–37}$$

Example 2.2.20. *[10, 86] The definition of a weight function is useful in order to count how many necklaces contain precisely* $\mathrm{x_j}$ *beads of color* j*. Set* W *to be the color white and* B *to be the color black. Then a 4-bead necklace with exactly 2 white and 2 black beads is expressed by* $\mathrm{W^2B^2}$*. If we do not wish to account for a particular color, per 2.2.18 we may assign its weight to 1. If we chose not to account for the color black, say, then the representation above would become simply* $\mathrm{W^2 1^2 = W^2}$*.*

Example 2.2.21. *We use Polya's theorem to describe the pattern inventory of 3-bead necklaces under the action of* $\mathrm{S_3}$*. Let the colors be* A *and* B*. Then*

$$\begin{aligned}\mathrm{P_{S_3}(A+B, A^2+B^2, A^3+B^3)} &= \frac{1}{6}[\mathrm{(A+B)^3 + 3(A+B)(A^2+B^2) + 2(A^3+B^3)}] \\ &= \mathrm{A^3 + A^2B + AB^2 + B^3}\quad .\end{aligned} \tag{2–38}$$

In words, we have one necklace with three A *beads, one with two* A *and one* B *bead, one with one* A *and two* B *beads, and one with three* B *beads.*

Example 2.2.22. *We can use Polya's theorem to count chords in 12 tones that are equivalent under transposition by setting our set of colors to be* $\mathrm{R = \{r_0, r_1\}}$*. We then disregard one of the colors, say* $\mathrm{r_0}$*, by setting* $\mathrm{w(r_0) = 1}$ *and* $\mathrm{w(r_1) = C}$*. The group of transpositions is just*

$C_{12} = \mathbb{Z}/12\mathbb{Z}$, *so we get*

$$\begin{aligned}
P_{C_{12}}(1+x,\cdots,1+x^{12}) &= \frac{1}{12}\sum_{d\in\{1,2,3,4,6,12\}} \varphi(d)x_d^{12/d} \\
&= \frac{1}{12}\left(x_1^{12}+x_2^6+2x_3^4+2x_4^3+2x_6^2+4x_{12}\right) \\
&= \frac{1}{12}\left[(1+C)^{12}+(1+C^2)^6+2(1+C^3)^4\right. \\
&\qquad\left.+\,2(1+C^4)^3+2(1+C^6)^2+4(1+C^{12})\right] \\
&= C^{12}+C^{11}+\cdots \\
&\qquad\cdots+80C^6+66C^5+43C^4+19C^3+6C^2+C^1+C^0 \ .
\end{aligned} \tag{2–39}$$

In particular, there are 43 tetrachords and 19 trichords that are transpositionally equivalent.

Example 2.2.23. *[11, 53] More generally, the number of* k*-chords under the action of* C_n *is*

$$\frac{1}{n}\sum_{j|(n,k)} \varphi(j)\binom{n/j}{k/j} \ . \tag{2–40}$$

Under the action of the dihedral group D_{2n}*, we obtain*

$$\begin{cases}
\frac{1}{2n}\left[\sum_{j|(n,k)}\varphi(j)\binom{n/j}{k/j}+n\binom{(n-1)/2}{k/2}\right] & n \textit{ odd} \\
\frac{1}{2n}\left[\sum_{j|(n,k)}\varphi(j)\binom{n/j}{k/j}+n\binom{n/2}{k/2}\right] & n \textit{ even and } k \textit{ even} \\
\frac{1}{2n}\left[\sum_{j|(n,k)}\varphi(j)\binom{n/j}{k/j}+n\binom{(n/2)-1}{k/2}\right] & n \textit{ even and } k \textit{ odd}
\end{cases} \tag{2–41}$$

In particular, the pattern inventory of k*-chords in 12 tones that are equivalent under* $T_n I$ *is*

$$P_{D_{24}}(1+x,\cdots,1+x^{12}) = C^{12}+\cdots+50C^6+38C^5+29C^4+12C^3+6C^2+C^1+C^0 \ . \tag{2–42}$$

Example 2.2.24. *[7, 249] Consider the action of the dihedral group* D_8 *on the 16-element set of colorings in black and white of the corners of a square. The cycle indicator of* D_8 *is*

$$P_{D_8}(x_1,x_2,x_3,x_4) = \frac{1}{8}(x_1^4+2x_1^2x_2+3x_2^2+x_4^1) \ . \tag{2–43}$$

Note that, in particular, we get no x_3 *factors by Lagrange. The identity element in* D_8 *fixes all collorings, and is represented as a permutation by four cycles of length one. By Polya's formula, we make the substitution*

$$\mathrm{x}_1^4 = (\mathrm{B}^1 + \mathrm{W}^1)^4 = \mathrm{B}^4 + 4\mathrm{B}^3\mathrm{W} + 6\mathrm{B}^2\mathrm{W}^2 + 4\mathrm{B}\mathrm{W}^3 + \mathrm{W}^4 \ , \tag{2–44}$$

where the exponent outside the parenthesis is the number of cycles, and the exponents inside the parenthesis correspond to the lengths of the cycles, for each color. The element of order four in D_8 *has one cycle of length four, so we get*

$$\mathrm{x}_4^1 = (\mathrm{B}^4 + \mathrm{W}^4)^1 = \mathrm{B}^4 + \mathrm{W}^4 \ , \tag{2–45}$$

that is, the permutation $\mathrm{r} = (1\ 2\ 3\ 4)$ *fixes the squares whose corners are all black or all white. Next, there are three elements in* D_8 *which comprise two cycles of length two, so each of those yield*

$$\mathrm{x}_2^2 = (\mathrm{B}^2 + \mathrm{W}^2)^2 = \mathrm{B}^4 + 2\mathrm{B}^2\mathrm{W}^2 + \mathrm{W}^4 \ . \tag{2–46}$$

The last two elements in D_8 *both have two cycles of length one, plus one cycle of length two, which gives*

$$\mathrm{x}_1^2\mathrm{x}_2 = (\mathrm{B} + \mathrm{W})^2(\mathrm{B}^2 + \mathrm{W}^2) = \mathrm{B}^4 + 2\mathrm{B}^3\mathrm{W} + 2\mathrm{B}^2\mathrm{W}^2 + 2\mathrm{B}\mathrm{W}^2 + \mathrm{W}^4 \ . \tag{2–47}$$

Putting it all together, we get the following pattern inventory of orbits:

$$\mathrm{P}_{\mathrm{D}_8}(\mathrm{B} + \mathrm{W}, \mathrm{B}^2 + \mathrm{W}^2, \mathrm{B}^3 + \mathrm{W}^3, \mathrm{B}^4 + \mathrm{W}^4) = \mathrm{B}^4 + 4\mathrm{B}^3\mathrm{W} + 6\mathrm{B}^2\mathrm{W}^2 + 4\mathrm{B}\mathrm{W}^3 + \mathrm{W}^4 \ . \tag{2–48}$$

APPENDIX A

A.1 ELEMENTARY TWELVE-TONE THEORY

Definition A.1.1. *Let* x *and* y *be arbitrary pitches. The* ***ordered pitch interval*** *between* x *and* y *is given by*

$$\mathrm{i}(\mathrm{x},\mathrm{y}) = \mathrm{y} - \mathrm{x} \ . \tag{A–1}$$

The ***ordered pitch-class interval*** *between* x *and* y *is given by*

$$\mathrm{i}(\bar{\mathrm{x}},\bar{\mathrm{y}}) = \bar{\mathrm{y}} - \bar{\mathrm{x}} \ . \tag{A–2}$$

The ***unordered pitch interval*** *between* x *and* y *is given by*

$$\bar{\mathrm{i}}(\mathrm{x},\mathrm{y}) = |\mathrm{x} - \mathrm{y}| \ . \tag{A–3}$$

The ***unordered pitch-class interval****, or simply interval class, between* x *and* y *is given by*

$$\bar{\mathrm{i}}(\bar{\mathrm{x}},\bar{\mathrm{y}}) = \min\{\mathrm{i}(\bar{\mathrm{x}},\bar{\mathrm{y}}),\mathrm{i}(\bar{\mathrm{y}},\bar{\mathrm{x}})\} \ . \tag{A–4}$$

Example A.1.2. *Put* $\mathrm{x} = 43$ *and* $\mathrm{y} = -13$*. Then*

$$\mathrm{i}(\mathrm{x},\mathrm{y}) = -56 \tag{A–5}$$

$$\mathrm{i}(\bar{\mathrm{x}},\bar{\mathrm{y}}) = \overline{11} - \bar{7} = \bar{4} \tag{A–6}$$

$$\bar{\mathrm{i}}(\mathrm{x},\mathrm{y}) = 56 \tag{A–7}$$

$$\bar{\mathrm{i}}(\bar{\mathrm{x}},\bar{\mathrm{y}}) = \bar{4} \tag{A–8}$$

Whenever the context is clear, we shall drop quotient notation and subscripts. In most situations, we are interested in the interval class between x *and* y*, in which case we will simply write* $\mathrm{i}(7,-1) = 4$*. Interval classes can also be seen graph-theoretically as the edge connecting two members of a pitch-class set, displayed clockwise.*

Theorem A.1.3. *[13, 10] The number of common tones between a set S and some transposition of itself is given by*

$$|\mathrm{S} \cap \mathrm{T_n}(\mathrm{S})| = |\{\mathrm{x} - \mathrm{y} = \mathrm{n} : \mathrm{x}, \mathrm{y} \in \mathrm{S}\}| \ . \tag{A–9}$$

The number of common tones between a set S and some inversion of itself is given by

$$|\mathrm{S} \cap \mathrm{T_n\,I}(\mathrm{S})| = 2 \cdot |\{\mathrm{x} + \mathrm{y} = \mathrm{n} : \mathrm{x}, \mathrm{y} \in \mathrm{S}\}| + |\{\mathrm{a} \in \mathrm{S} : 2\mathrm{a} = \mathrm{n}\}| \ . \tag{A–10}$$

Moreover, the cardinality of the set $\{\mathrm{a} \in \mathrm{S} : 2\mathrm{a} = \mathrm{n}\}$ is at most 2.

Proof. We must count the occurrences of pairs of pitch classes that are interchanged by the operation at hand and double them, for if x maps onto y under some $\mathrm{T_n\,I}$, then certainly y maps onto x under the same operation, given that every inversion operation has order two. In addition to that, we must account for the occurrences of pitch classes that may map onto themselves under the aforementioned operation. For any pair $\mathrm{a} \neq \mathrm{b} \in \mathrm{S}$, it follows a and b are exchanged by some operation $\mathrm{T_n\,I}$ whenever both $\mathrm{T_n\,I(a)} = \mathrm{b}$ and $\mathrm{T_n\,I(b)} = \mathrm{a}$ hold. Since $\mathrm{T_n\,I(a)} = -\mathrm{a} + \mathrm{n}$ and similarly $\mathrm{T_n\,I(b)} = -\mathrm{b} + \mathrm{n}$, if the pair is exchanged, we must have $-\mathrm{a} + \mathrm{n} = \mathrm{b}$ and $-\mathrm{b} + \mathrm{n} = \mathrm{a}$ both true. Adding the last two expressions and yields $\mathrm{a} + \mathrm{b} = \mathrm{n}$, which is the first set in the right-hand side of the formula. As discussed above, the cardinality of this set must be doubled. We have for any a that $\mathrm{T_n\,I(a)} = \mathrm{a} + \mathrm{n}$, hence $\mathrm{a} = \mathrm{T_n\,I(a)} \iff \mathrm{a} = -\mathrm{a} + \mathrm{n}$, that is, whenever $2\mathrm{a} = \mathrm{n}$. That is the second set in the formula. Finally, for any pair (a, n) such that $\mathrm{a} = \mathrm{T_n\,I(a)}$, we also have $\mathrm{a} + 6 = -(\mathrm{a} + 6) + \mathrm{n} \iff 2\mathrm{a} = \mathrm{n}$, so that by the above it follows $\mathrm{a} + 6 = \mathrm{T_n\,I(a + 6)}$. Thus the set $\{\mathrm{a} \in \mathrm{S} : 2\mathrm{a} = \mathrm{n}\}$ has cardinality at most 2, proving the last assertion. □

Example A.1.4. *[13, 11] Write* $\mathrm{S} = \{0,1,4,5,8,9\}$ *and consider some inversion operation. We have*

$$\begin{array}{c|cccccccccccc} \mathrm{n} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2\cdot|\{\mathrm{x}+\mathrm{y}=\mathrm{n}:\mathrm{x},\mathrm{y}\in\mathrm{S}\}| & 2 & 6 & 2 & 0 & 2 & 6 & 2 & 0 & 2 & 6 & 2 & 0 \\ |\{\mathrm{a}\in\mathrm{S}:2\mathrm{a}=\mathrm{n}\}| & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \text{Total} & 3 & 6 & 3 & 0 & 3 & 6 & 3 & 0 & 3 & 6 & 3 & 0 \end{array} \tag{A–11}$$

Example A.1.5. *We can demonstrate A.1.4, as well as the omitted proof of A.1.3 under transposition in a much simpler way with a little bit of abstract algebra. By observing the cycle decomposition the each operation at hand, if* $\mathrm{n} = 3$*, then we have*

$$\mathrm{T}_3\,\mathrm{I} = (0\ 3)(1\ 2)(4\ 11)(5\ 10)(6\ 9)(7\ 8)\ . \tag{A–12}$$

Hence, under $\mathrm{T}_3\,\mathrm{I}$*, every pitch-class in* $\mathrm{S} = \{0,1,4,5,8,9\}$ *maps to the complement of* S*. If the operation is, for instance,* T_9*, then since*

$$\mathrm{T}_9 = (0\ 9\ 6\ 3)(1\ 10\ 7\ 4)(2\ 11\ 8\ 5)\ , \tag{A–13}$$

we get straightforwardly that $\mathrm{S} = \{0,1,4,5,8,9\}$ *shares three common tones with* $\mathrm{T}_9 \circ \mathrm{S}$*, namely* $0 \mapsto 9$*,* $4 \mapsto 1$*, and* $8 \mapsto 5$*.*

A.2 ELEMENTARY GROUP THEORY

Proposition A.2.1. *[12, 127] We have the following isomorphisms:*

$$\langle \mathrm{T} \rangle \cong \mathrm{C}_{12} \tag{A–14}$$

$$\langle \mathrm{T}, \mathrm{I} \rangle \cong \mathrm{D}_{24} \tag{A–15}$$

$$\langle \mathrm{I}, \mathrm{R} \rangle \cong \mathrm{V}_4 \tag{A–16}$$

$$\langle \mathrm{T}, \mathrm{I}, \mathrm{R} \rangle \cong \mathrm{D}_{24} \oplus \mathbb{F}_2 \tag{A–17}$$

$$\langle \mathrm{T}, \mathrm{I}, \mathrm{M} \rangle \cong \mathrm{Aff}_1(\mathrm{C}_{12}) \tag{A–18}$$

$$\langle \mathrm{T}, \mathrm{I}, \mathrm{M}, \mathrm{R} \rangle \cong \mathrm{Aff}_1(\mathrm{C}_{12}) \oplus \mathbb{F}_2 \tag{A–19}$$

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BIOGRAPHICAL SKETCH

Luis F. Vieira Damiani earned in 2002 a Bachelor of Music degree in violin performance from Rio Grande do Sul's Federal University, under the orientation of Prof. Marcello Guerchfeld, a former student of Galamian. In 2003, Luis entered Porto Alegre Symphony, with which he performed as soloist in the 2005 season. In 2004, he was appointed Assistant Concertmaster of the Catholic University of Rio Grande do Sul's Philharmonic Orchestra, and from 2009 until 2011, Luis was a member of Minas Gerais Philharmonic, one of the top Brazilian groups of its kind. In Italy, he studied with Boris Belkin and Giuliano Carmignola at the Accademia Chigiana. Other teachers include Jennifer John at the Aspen Music Festival in 1996, and Kurt Sassmannshaus at University of Cincinnati's College-Conservatory of Music in 1999. As a performer, Luis researched, engraved and premiered works by 20th Century Southern Brazilian composers Bruno Kiefer and Armando Albuquerque, as well as recorded in 2006 of the *Duo for Violin and Cello* (2000) by composer Pablo Castellar, a recording that won the Açorianos Prize from the Municipal Office of Culture of Porto Alegre in 2008.

As a composer, Luis wrote in 2008 the original music for the motion picture *O Guri*. For television, his commissioned works include the soundtrack for the short film *O Sabiá*, aired in 2010. In the same year, he received the prestigious Classical Composition Award from the National Foundation of Arts in Brazil for his *Solo Violin Suite*, making him take a turn from a well-established orchestral career into pursuing graduate studies in the USA. Luis' awards since include Best Feature Soundtrack at the 6th Cinefantasy International Fantastic Film Festival in 2011 and the University of South Florida's 2012 Percussion Composition Prize, among numerous academic awards. In 2013, he received a Master of Music degree in music composition from the University of South Florida, under the orientation of Dr. Baljinder Sekhon. In the same year, Luis was awarded a fellowship to pursue Ph.D. studies in music composition at the University of Florida, where he currently works as a graduate teaching assistant, under the orientation of Dr. James Paul Sain.