# GENERALIZED MULTIPLE ORDER-NUMBER FUNCTION ARRAYS

By

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To my family

# ACKNOWLEDGMENTS

Dr. Rao.

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Abstract of Dissertation Presented to the Graduate School
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# CHAPTER 1
# INTRODUCTION

## 1.1 Preliminary Results

**Definition 1.1.1.** *[1, 99] [2, 41] Let $X$ be a set and $G$ a group. An* **action** *of $G$ on $X$ is a function $G \times X \to X$ given by $(g, x) \mapsto gx$, such that:*

*i. $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$;*

*ii. $1x = x$ for all $x \in X$, where $1 \in G$ is the identity.*

**Proposition 1.1.2.** *[1, 99] [2, 42] If a group $G$ acts on a set $X$ then, for every $g \in G$, the function $f_g : X \to X$ given by $f_g(x) = gx$ is a permutation of $X$. Further, the function $f : G \to S_X$ given by $f(g) = f_g$ is a homomorphism and, conversely, for any homomorphism $\phi : G \to S_X$, there is a corresponding group action given by $\phi(g)(x)$.*

**Theorem 1.1.3** (Cayley)**.** *[1, 96] [2, 120] Every group is isomorphic to a subgroup of the symmetric group $S_G$. In particular, if $|G| = n$, then $G$ is isomorphic to a subgroup of $S_n$.*

**Theorem 1.1.4.** *[1, 97] Let $H \leq G$ be a subgroup of finite index $n$. Then there exists a homomorphism $\phi : G \to S_n$ such that $\ker \phi \leq H$. In particular, when $H = \{1\}$, we get Cayley's theorem.*

**Example 1.1.5.** *[2, 122] A group acts on itself by conjugation. Let $g, h \in G$ and $x \in G$. Then $1x1^{-1} = x$ and*

$$g(hx) = g(hxh^{-1}) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = (gh)x \ .$$

*It is also immediate from the above that a group acts on its power set by conjugation. In particular, a group acts on the set of all its subgroups.*

**Definition 1.1.6.** *[1, 100] [2, 112] If $G$ acts on $X$, then the* **orbit** *of $x \in X$ is the set*

$$\mathcal{O}(x) = \{gx : g \in G\} \subseteq X \ .$$

*We say an action is* **transitive** *if there is only one orbit. The* **kernel** *of the action is the set*

$$\{g \in g : gx = x, \forall x \in X\} \ .$$

*We say an action is* **faithful** *if the kernel is the identity. The* **stabilizer** *of $x$ in $G$ is the group*

$$G_x = \{g \in g : gx = x\} \leq G \ .$$

*When a group acts on itself by conjugation, we call the orbits* **conjugacy classes***. The stabilizer of some $g \in G$ is the* **centralizer** *of $g$ in $G$, denoted $C_G(g)$. When a group acts on the set of its subgroups by conjugation, the stabilizer of a subgroup $H \leq$ is the* **normalizer** *of $H$ in $G$, denoted by $N_G(H)$.*

**Proposition 1.1.7.** *[1, 102] [2, 114] [3, 250] If $G$ acts on $X$, for $x_1, x_2 \in X$, the relation $x_1 \sim x_2$ given by $x_1 = gx_2$ is an equivalence relation. It follows immediately that the equivalence classes are the orbits of the action of $G$ on $X$ and that*

$$|X| = \sum_i |\mathcal{O}(x_i)| \ ,$$

*where $x_i$ is a single representative from each orbit.*

**Theorem 1.1.8** (Orbit-Stabilizer)**.** *[1, 102] If $G$ acts on $X$, then for each $x \in X$*

$$|\mathcal{O}(x)| = [G : G_x] \ .$$

**Corollary 1.1.9.** *[1, 103] If $G$ is finite and acts on $X$, then the size of any orbit is a divisor of $|G|$.*

**Proposition 1.1.10.** *[2, 123] The number of conjugates of a subset $S$ in a group $G$ is $|G : N_G(S)|$, the index of the normalizer of $S$. In particular, the number of conjugates of an element $s$ is $|G : C_G(s)|$, the index of the centralizer of $s$.*

**Proposition 1.1.11.** *[2, 125] Let $\sigma, \tau \in S_n$. If*

$$\sigma = (a_1 \ a_2 \ \cdots a_j)(b_1 \ b_2 \ \cdots a_k) \cdots \ ,$$

*then*

$$\tau\sigma\tau^{-1} = (\tau(a_1) \ \tau(a_2) \ \cdots \tau(a_j))(\tau(b_1) \ \tau(b_2) \ \cdots \tau(b_k)) \cdots \ .$$

**Example 1.1.12.** *[2, 127] Let $\sigma \in S_n$ be an $m$-cycle. The number of conjugates of $\sigma$ is*

$$\frac{|S_n|}{|C_{S_n}(\sigma)|} = \frac{n(n-1)(n-m+1)}{m} \ ,$$

*so that $|C_{S_n}(\sigma)| = m(n-m)!$. Since $\sigma$ commutes with its powers, and also with any permutation in $S_n$ whose cycles are disjoint from it, and there are $(n-m)!$ of those, the number computed above is the full centralizer of $\sigma$.*

**Example 1.1.13.** *[2, 132] The size of each conjugacy class in $S_n$ is*

$$\frac{n!}{\prod_r r^{n_r} n_r!} \ ,$$

*where, for each $r$-cycle, we divide by $r$ to account for the cyclical permutations of elements within a cycle. Further, if there are $n_r$ cycles of length $r$, we divide by $n_r!$ to account for the different orders in which those cycles may appear.*

**Proposition 1.1.14.** *[2, 126] Two elements in $S_n$ are conjugate if and only if they have the same cycle type. The number of conjugacy classes of $S_n$ is the number of partitions of $n$.*

**Definition 1.1.15.** *[2, 133] An isomorphism from a group $G$ to itself is called an* **automorphism** *of $G$. The group under composition of all automorphisms of $G$ is denoted by* $\mathrm{Aut}(G)$.

**Proposition 1.1.16.** *[2, 133] If $H$ is a normal subgroup of $G$, then the action of $G$ by conjugation on $H$ is, for each $g \in G$, an automorphism of $H$. The kernel of the action is $C_G(H)$. In particular, $G/c_G(H)$ is a subgroup of* $\mathrm{Aut}(H)$.

**Corollary 1.1.17.** *[2, 134] For any subgroup $H < G$ and $g \in G$, $H \cong gHg^{-1}$. Moreover, $N_G(H)/C_G(H)$ (and also $G/Z(G)$ when $G = H$) is isomorphic to a subgroup of* $\mathrm{Aut}(H)$.

**Definition 1.1.18.** *[2, 135] A subgroup $H < G$ is called* **characteristic** *if every automorphism of $G$ maps $H$ to itself.*

**Proposition 1.1.19.** *[2, 135] Characteristic subgroups are normal. Unique subgroups of a given order are characteristic. A characteristic subgroup of a normal subgroup is normal. In particular, every subgroup of a cyclic group is characteristic.*

**Proposition 1.1.20.** *[2, 135] If $G$ is cyclic of order $n$, then $\mathrm{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, and $|\mathrm{Aut}(G)| = \varphi(n)$, where $\varphi$ is Euler's totient function.*

**Corollary 1.1.21.** *[2, 136] Let $|G| = pq$, with $p \leq q$ primes. If $p \nmid q-1$, then $G$ is abelian. If, further, $p < q$, then $G$ is cyclic.*

**Proposition 1.1.22.** *[2, 136] If $|G| = p^n$, with $p$ an odd prime, then*

$$|\mathrm{Aut}(G)| = p^{n-1}(p-1)$$

*and the latter group is also cyclic. If $|G| = 2^n$ is cyclic, with $n \geq 3$, then*

$$\mathrm{Aut}(G) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2}\mathbb{Z}$$

*and the latter is not cyclic, but has a cyclic subgroup of index 2. If $V$ is the elementary abelian group of order $p^n$, then $pv = 0$ for all $v \in V$ and $V$ is an $n$-dimensional vector space over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. The automorphisms of $V$ are the nonsingular linear transformations from $V$ to itself:*

$$\mathrm{Aut}(V) \cong GL(V) \cong GL_n(\mathbb{F}_p) \ ,$$

*and if $F$ is the finite field of order $q$, then*

$$|GL_n(F)| = (q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{n-1}) \ .$$

*For all $n \neq 6$, we have $\mathrm{Aut}(S_n) \cong S_n$. Finally, $\mathrm{Aut}(D_8) \cong D_8$ and $\mathrm{Aut}(Q_8) \cong S_4$.*

**Example 1.1.23.** *[2, 137] The Klein 4-group is the elementary abelian group of order 4. It follows $\mathrm{Aut}(V_4) \cong GL_2(\mathbb{F}_2)$, and $|\mathrm{Aut}(V_4)| = 6$. Since the action of $\mathrm{Aut}(V_4)$ on $V_4$ permutes the latter's 3 nonidentity elements, by order considerations we have*

$$\mathrm{Aut}(V_4) \cong GL_2(\mathbb{F}_2) \cong S_3 \ .$$

**Proposition 1.1.24.** *[2, 314] A finite subgroup of the multiplicative group of a field is cyclic. In particular, if $F$ is a finite field, then $F^\times$ is cyclic.*

**Corollary 1.1.25.** *[2, 314] For $p$ a prime, $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.*

# CHAPTER 2
# THEORETICAL FRAMEWORK

## 2.1 Preliminary Results

**Definition 2.1.1.** *Let $x$ and $y$ be arbitrary pitches. The* ordered pitch interval *between $x$ and $y$ is given by*

$$i_{ordered}(x, y) = y - x \quad .$$

*The* ordered pitch-class interval *between $x$ and $y$ is given by*

$$i_{ordered}(\bar{x}, \bar{y}) = \bar{y} - \bar{x} \quad .$$

*The* unordered pitch interval *between $x$ and $y$ is given by*

$$i_{unordered}(x, y) = |x - y| \quad .$$

*The* unordered pitch-class interval*, or simply interval class, between $x$ and $y$ is given by*

$$i_{unordered}(\bar{x}, \bar{y}) = \min\{i_{ordered}(\bar{x}, \bar{y}), i_{ordered}(\bar{y}, \bar{x})\} \quad .$$

**Example 2.1.2.** *Put $x = 43$ and $y = -13$. Then $i_{ordered}(x, y) = -56$, $i_{ordered}(\bar{x}, \bar{y}) = \overline{11} - \bar{7} = \bar{4}$, $i_{unordered}(x, y) = 56$, and $i_{unordered}(\bar{x}, \bar{y}) = \bar{4}$.*

Whenever the context is clear, we shall drop quotient notation and subscripts. In most situations, we are interested in the interval class between $x$ and $y$, in which case we will simply write $i(7, -1) = 4$. Interval classes can also be seen graph-theoretically as the edge connecting two members of a pitch-class set displayed clockwise.

**Theorem 2.1.3** (Common-Tone)**.** *The number of common tones between a set $S$ and some transposition of itself is given by*

$$|S \cap \mathrm{T}_n(S)| = |\{x - y = n : x, y \in S\}| \quad .$$

*The number of common tones between a set $S$ and some inversion of itself is given by*

$$|S \cap \mathrm{T}_n\mathrm{I}(S)| = 2 \cdot |\{x + y = n : x, y \in S\}| + |\{a \in S : 2a = n\}| \quad .$$

*Moreover, the cardinality of the set* $\{a \in S : 2a = n\}$ *is at most 2.*

*Proof.* [4, ??] proves the second assertion as follows. We must count the occurrences of pairs of pitch classes that are interchanged by the operation at hand and double them, for if $x$ maps onto $y$ under some $\mathrm{T}_n\mathrm{I}$, then certainly $y$ maps onto $x$ under the same operation, given that every inversion operation has order two. In addition to that, we must account for the occurrences of pitch classes that may map onto themselves under the aforementioned operation. For any pair $a \neq b \in S$, it follows $a$ and $b$ are exchanged by some operation $\mathrm{T}_n\mathrm{I}$ whenever both $\mathrm{T}_n\mathrm{I}(a) = b$ and $\mathrm{T}_n\mathrm{I}(b) = a$ hold. Since $\mathrm{T}_n\mathrm{I}(a) = -a + n$ and similarly $\mathrm{T}_n\mathrm{I}(b) = -b + n$, if the pair is exchanged, we must have $-a + n = b$ and $-b + n = a$ both true. Adding the last two expressions and yields $a + b = n$, which is the first set in the right-hand side of the formula. As discussed above, the cardinality of this set must be doubled. We have for any $a$ that $\mathrm{T}_n\mathrm{I}(a) = a + n$, hence $a = \mathrm{T}_n\mathrm{I}(a) \iff a = -a + n$, that is, whenever $2a = n$. That is the second set in the formula. Finally, for any pair $(a, n)$ such that $a = \mathrm{T}_n\mathrm{I}(a)$, we also have $a + 6 = -(a + 6) + n \iff 2a = n$, so that by the above it follows $a + 6 = \mathrm{T}_n\mathrm{I}(a + 6)$. Thus the set $\{a \in S : 2a = n\}$ has cardinality at most 2, proving the last assertion. $\square$

**Example 2.1.4.** *[4, ??] Write* $S = \{0, 1, 4, 5, 8, 9\}$ *and consider some inversion operation. We have*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $2 \cdot \lvert\{x + y = n : x, y \in S\}\rvert$ | 2 | 6 | 2 | 0 | 2 | 6 | 2 | 0 | 2 | 6 | 2 | 0 |
| $\lvert\{a \in S : 2a = n\}\rvert$ | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| *Total* | 3 | 6 | 3 | 0 | 3 | 6 | 3 | 0 | 3 | 6 | 3 | 0 |

We can demonstrate the above, as well as the omitted proof of the common-tone theorem under transposition in a much simpler way with a little bit of abstract algebra. By observing the cycle decomposition the each operation at hand, if $n = 3$, then we have

$$\mathrm{T}_3\mathrm{I} = (0\ 3)(1\ 2)(4\ 11)(5\ 10)(6\ 9)(7\ 8)\ .$$

Hence, under $\mathsf{T}_3\,\mathsf{I}$, every pitch-class in $S = \{0, 1, 4, 5, 8, 9\}$ gets sent to the complement of $S$. If the operation is, for instance, $\mathsf{T}_9$, then since

$$\mathsf{T}_9 = (0\ 9\ 6\ 3)(1\ 10\ 7\ 4)(2\ 11\ 8\ 5)\ ,$$

we get straightforwardly that $S = \{0, 1, 4, 5, 8, 9\}$ shares three common tones with $\mathsf{T}_9 \circ S$, namely $0 \mapsto 9$, $4 \mapsto 1$, and $8 \mapsto 5$.

Transposition of order numbers is just rotation of pitch classes. Inversion of order numbers is equivalent to taking the retrograde and its rotations. Multiplication of order numbers modulo 12 does not in general produce a 12-tone row. Just as with the case of multiplication of pitch classes, we find that the only cases when we do get a bijective mapping are when the index of multiplication is an integer $n$ relatively prime to 12. The cases $n = 1$ and $n = 11$ gives respectively the identity operation and the eleventh rotation of the retrograde (or equivalently $T_{11} I \circ S_*$).

Question: what is the effect of $M_5$ on order numbers?

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}\{0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7\}$$

This is different than the mallalieu property. Say $S = \{0, 1, 4, \cdots\}$.

### 2.1.1 Mallalieu-Type Rows

Consider the 12-tone series $S = \{0, 1, 4, 2, 9, 5, 11, 3, 8, 10, 7, 6\}$. This series has the remarkable property that, if we include a dummy $13^{\text{th}}$ element, then taking every $n^{\text{th}}$ element of $S$ produces a transposition of it.

**Example 2.1.5.** *We have $S^* = \{0, 1, 4, \ldots, 7, 6, *\}$. Then taking every zeroth order number of $S^* \mod 13$ yields $S^*$ itself. Taking every first order number yields the series $\{1, 2, 5, \ldots, 8, 7, *\}$ which, upon removing the dummy symbol, becomes $\mathsf{T}_1 \circ S$. Repeating this procedure every $n^{th}$ order number gives the sequence of transforms $\{\mathsf{T}_i\}_{i \in S}$.*

This most peculiar property, commonly called the *mallalieu* property, was known by Babbitt since at least 1954, but first discovered by Pohlman Mallalieu (citation). It is natural to ask at this point how many different 12-tone rows are there sharing this property. Unfortunately, there is only one such 12-tone row class under $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$. We phrase below a little differently an argument given by Morris in 1975:

**Proposition 2.1.6.** *(citation) A 12-tone row has the mallalieu property if and only if it is related by* $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$ *to the row* $S = \{0, 1, 4, 2, 9, 5, 11, 3, 8, 10, 7, 6\}$.

*Proof.* One direction is just the straightforward check that every $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$ transform of $S$ possesses the mallalieu property and is left to the reader. Conversely, if a row $R$ in its untransposed prime form has the mallalieu property, then there is a transposition that takes its order numbers in zeroth rotation, that is, the set $\{0, 1, 2, \ldots, 11\}$ to its order numbers in, say, first rotation, id est, the set $\{1, 3, \ldots, 11, 0, 2, \ldots, 10\}$. We can write this transposition as a permutation $0 \mapsto 1, 1 \mapsto 3, \ldots, 11 \mapsto 10$, or in cycle notation as $T_k = (0\ 1\ 3\ 7\ 2\ 5\ 11\ 10\ 8\ 4\ 9\ 6)$. Note that $\mathsf{T}_k$ is an operation on order numbers. Since $\mathsf{T}_k$ is a transposition, there are only four candidates for $k$, namely $k \in \{1, 5, 7, 11\}$ (because these are the only indices for which a transposition in cycle notation is a 12-cycle). Moreover, we do not need to consider the cases where $k \in \{5, 7, 11\}$, as $tra_5 = \mathsf{M} \circ \mathsf{T}_1$, $\mathsf{T}_7 = \mathsf{M}\,\mathsf{I} \circ \mathsf{T}_1$, and $\mathsf{T}_{11} = \mathsf{I} \circ \mathsf{T}_1$. Hence, without loss, we can set $k = 1$. But then $S$ is the only row in untransposed prime form where $\mathsf{T}_1$ induces the permutation $\mathsf{T}_k$ from its order numbers in zeroth rotation to its order numbers in first rotation (just equate $\mathsf{T}_k$ with $\mathsf{T}_1$), completing the proof. □

Lewin (citation) provides a way of looking at mallalieu rows from the standpoint of replacing, for any 12-tone row, its order-number row $\{0, 1, \ldots, 11\}$ by the array of integers $\{1, 2, \ldots, 12, 0\}$ modulo 13. It is easy to see that such an array has the same structure as the array $S_*$ we constructed above if we substitute the asterisk by the number 12 and consider multiplication as the group operation. Obviously, this is just the isomorphism between the integers modulo 12 and the group of units modulo 13. One of the advantages of this approach is that we can dispense with the extra symbol altogether and just use the indices from 1 to $p - 1$. We shall, however, still refer to the row of order numbers as $S^*$, the context making it clear whether we are constructing it with an asterisk or not. The process of taking every $n^{\text{th}}$ element of a 12-tone row becomes then just the aforementioned multiplicative group operation on order numbers, that is, multiplying order numbers by $k \pmod{13}$ is the same as taking every $k^{\text{th}}$ element of a row.

**Example 2.1.7.** *Put $S = \{0, 1, \dots, 11\}$ and $S^* = \{1, 2, \dots, 12\}$. Then $\mathsf{M}_3 \circ S^* = \{3, 6, \dots, 10\}$, which corresponds to the row $R = \{2, 5, \dots, 9\}$. The row $R$ can be equivalently constructed by placing an asterisk as the $13^{th}$ order number of $S$ and taking every third element. The fact that $R$ and $S$ are not related by $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$ reflects the fact that neither $S$ nor $R$ have the mallalieu property.*

In addition, Lewin (citation) proposes the following:

**Proposition 2.1.8.** *For $p$ a prime, every $(p-1)$-TET system is capable of producing a mallalieu row.*

*Proof.* For every prime $p$, the group of units modulo $p$ is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$. The mallalieu property in these cases can be seen as the aforementioned isomorphism, where $\mathbb{Z}/(p-1)\mathbb{Z}$ is the group of transpositions of a row, and $(\mathbb{Z}/p\mathbb{Z})^\times$ is its multiplicative group on order numbers. The number of mallalieu rows in each $(p-1)$-TET system is then the number of isomorphisms $\mathbb{Z}/(p-1)\mathbb{Z} \to (\mathbb{Z}/p\mathbb{Z})^\times$, that is, the order of the group of automorphisms of $\mathbb{Z}/(p-1)\mathbb{Z}$. Since for every prime $p$ we have $|\operatorname{Aut}(\mathbb{Z}/(p-1)\mathbb{Z})| \geq 1$, every $(p-1)$-TET system is capable of producing a mallalieu row, as desired. $\square$

In face of 2.1.8, 2.1.6 becomes just the special case where $p = 13$, as seen in the next example:

**Example 2.1.9.** *The number of isomorphisms $\mathbb{Z}/12\mathbb{Z} \to (\mathbb{Z}/13\mathbb{Z})^\times$ is equal to $|\operatorname{Aut}\left((\mathbb{Z}/12\mathbb{Z})^\times\right)| = 4$. We can construct these isomorphisms by mapping a generator of $\mathbb{Z}/12\mathbb{Z}$, say $\bar{1}$, to the generators of $(\mathbb{Z}/13\mathbb{Z})^\times$, namely $\bar{2}, \bar{6}, \bar{7}$ and $\overline{11}$. Explicitly, we get the four maps $i \pmod{12} \mapsto 2^i \pmod{13}$, $i \pmod{12} \mapsto 6^i \pmod{13}$, $i \pmod{12} \mapsto 7^i \pmod{13}$, and $i \pmod{12} \mapsto 11^i \pmod{13}$. We leave the verification that these maps are well defined and bijective to the* *reader. Denote the first map by $\varphi$. Then*

$$\varphi(a+b) = 2^{a+b} = 2^a \cdot 2^b = \varphi(a) \cdot \varphi(b) \ ,$$

*so $\varphi$ is an isomorphism. The verification that the other three maps are isomorphisms is identical. Define $\varphi^{-1} : (\mathbb{Z}/13\mathbb{Z})^\times \to \mathbb{Z}/12\mathbb{Z}$ by $\varphi^{-1}(\log i \pmod{13}) = i \pmod{12}$. Then $\varphi^{-1}$ is easily seen to be the inverse of $\varphi$. Let $S^* = \{1, 2, \ldots, 12\}$ be a series of order numbers written multiplicatively. Then*

$$\varphi^{-1}(S^*) = \{\log 1, \log 2, \ldots, \log 12\} \pmod{13} = \{0, 1, 4, \ldots, 7, 6\} \ ,$$

*which by 2.1.6 is one of the four 12-tone rows with the mallalieu property.*

It should be of interest to many composers whether other $n$-TET systems are capable of producing mallalieu rows, and if so, how many. Unfortunately, answering this question is not as straightforward as the above discussion, since we can no longer rely on the isomorphism that constitutes the proof of 2.1.8. We shall reformulate this question at the end of the present chapter, after having covered more of what has been already done.

If, on one hand, we only get one $\mathsf{T}_n\,\mathsf{M}\,\mathsf{I}$ row class with the mallalieu property in 12 tones, we do get considerably more row classes when we relax the requirement that a row be produce a transposition of itself when taking every $n^{\text{th}}$ of its elements. This idea is explored in part by (citation – Mead), however without specifying any combinatorial aspect (in the mathematical sense) of this generalization. Moreover, we can certainly go beyond (citation – Mead) and investigate, in 12 tones, what an extension of the mallalieu property could yield under operations other than transposition.

CHAPTER 3
METHODOLOGY

## 3.1 Polya's Enumeration Formula

**Definition 3.1.1.** *[5, 85] Let $N$ be a set of $n$ beads and $R$ be a set of $r$ colors. A colored necklace is a function $f : N \to R$. Denote the set of all such functions by $R^N$, so that $|R^N| = r^n$. Define the* **weight** *associated to a coloring $f$ by*

$$w(f) = \prod_{i \in N} x_{f(i)} \quad , \tag{3–1}$$

*where $x_j$ is a variable associated with the color $j \in R$.*

**Proposition 3.1.2.** *[1, 110] Let $G$ be a group and $X = \{1, \cdots, n\}$ be a set. Let $\mathcal{C}$ be a set of $q$ colors. Then $G$ acts on the set $\mathcal{C}^n$ of $n$-tuples of colors by*

$$\tau(c_1, \cdots, c_n) = (c_{\tau 1}, \cdots, c_{\tau n}), \forall \tau \in G \quad . \tag{3–2}$$

**Proposition 3.1.3.** *[5, 85] [1, 110] If $G$ is a group acting on the set $N$, then two colorings $f$ and $f'$ are equivalent whenever $f = f' \circ g$ for some $g \in G$. This is an equivalence relation that partitions $R^N$ into equivalence classes denoted* **patterns**. *Under the conditions of 3.1.2, an orbit $(c_1, \cdots, c_n) \in \mathcal{C}^n$ is called a $(q, G)$-***coloring** *of $X$. Moreover, two equivalent colorings have the same weight, so that we may refer to the weight of a class of colorings rather than the weights of its representatives.*

**Definition 3.1.4.** *[5, 85] Let $N$ be a set of $n$ beads and $R$ be a set of $r$ colors. Let $G$ be a group acting on the set $N$ and $x_j$ the variable associated with the color $j \in R$. Let $\mathcal{M}$ be the set of of patterns. Define the pattern* **enumerator** *by*

$$w(R^N, G) = \sum_{M \in \mathcal{M}} w(M) \quad . \tag{3–3}$$

*In particular, when $x_j = 1$ for all $j \in R$, we get $w(R^N, G) = |\mathcal{M}|$.*

**Definition 3.1.5.** *[5, 86] Let $G$ be a group acting on a set $X$. Define for every $g \in G$ its* **fixed-point set** *by*

$$\mathrm{Fix}(g) = \{x \in X : gx = x\} \ . \tag{3–4}$$

**Lemma 3.1.6.** *[1, 112] Let $G < S_n$ be a group and let $\mathcal{C}$ be a set of $q$ colors. For $\tau \in G$,*

$$|\,\mathrm{Fix}(\tau)| = q^{t(\tau)} \ , \tag{3–5}$$

*where $t(\tau)$ is the number of cycles in the complete factorization of $\tau$.*

**Lemma 3.1.7** (Burnside)**.** *[1, 109] [3, 251] If $G$ acts on a finite set $X$, then the number of orbits $N$ is*

$$N = \frac{1}{|G|}\sum_{\tau \in G} |\,\mathrm{Fix}(\tau)| \ . \tag{3–6}$$

**Corollary 3.1.8.** *[1, 112] Let $G$ be a group acting on a finite set $X$. The number $N$ of $(q, G)$-colorings of $X$ is*

$$N = \frac{1}{|G|}\sum_{\tau \in G} q^{t(\tau)} \ . \tag{3–7}$$

**Definition 3.1.9.** *[5, 87] Let $g$ be a group acting on $R^N$. Define the* **cycle indicator** *of $G$ by*

$$P_G(z_1, z_2, \cdots, z_n) = \frac{1}{|G|}\sum_{g \in G} z_1^{b_1(g)} z_2^{b_2(g)} \cdots z_n^{b_n(g)} \ , \tag{3–8}$$

*where $z_i^{b_i(g)}$ corresponds to the number of cycles in the complete factorization of $g$ that have length $i$.*

**Example 3.1.10.** *The cycle indicator of $S_3$ is*

$$P_{S_3}(x_1, x_2, x_3) = \frac{1}{6}(x_1^3 + 3x_1^1x_2^1 + 2x_3^1) \ , \tag{3–9}$$

*since there are two elements in $S_3$ that comprise one cycle of length three, three elements that comprise one cycle of length one, and one cycle of length two, and one element that comprises three cycles of length one.*

**Example 3.1.11.** *Let $C_n$ be the cyclic group of order $n$. Then*

$$P_{C_n} = \frac{1}{n}\sum_{d|n} \varphi(d) z_d^{n/d} \quad . \tag{3–10}$$

**Example 3.1.12.** *For $S_n$ we have*

$$P_{S_n} = \sum_{j_1+2j_2+\cdots+nj_n=n} \frac{1}{\prod_{k=1}^{n} k^{j_k} j_k!} \prod_{k=1}^{n} a_k^{j_k} \quad . \tag{3–11}$$

*In words, there is a summand for each conjugacy class in $S_n$, and we divide each summand by the size of its corresponding conjugacy class.*

**Example 3.1.13.** *For $D_n$ we have*

$$P_{D_n} = \frac{1}{2} P_{C_n} + \begin{cases} \frac{1}{2} a_1 a_2^{(n-1)/2} & n \text{ odd} \\ \frac{1}{4}(a_1^2 a_2^{(n-2)/2} + a_2^{n/2)} & n \text{ even} \end{cases} \tag{3–12}$$

**Theorem 3.1.14** (Polya)**.** *[5, 88] [3, 256] Let $N$ be a set of cardinality $n$ and $R$ be a set of cardinality $r$. Let $G$ be a group acting on the set $N$ and $x_j$ be arbitrary variables with $j \in R$. Let $w(R^N, G)$ be the pattern enumerator for the action of $G$ on $R^N$. Then*

$$w(R^N, G) = \sum_{M \in \mathcal{M}} w(M) = P_G(\sum_{j \in R} x_j, \sum_{j \in R} x_j^2, \cdots, \sum_{j \in R} x_j^n) \quad . \tag{3–13}$$

**Corollary 3.1.15.** *[5, 89] [3, 254] Let $x_j = 1$ for all $j \in R$. Then $\sum_{j \in R} x_j^k = |R| = r$ for all $k$, hence*

$$P_G(r, \cdots, r) = |\mathcal{M}| \quad . \tag{3–14}$$

**Corollary 3.1.16.** *[5, 89] Let $|R| = r = 2$. Let $x_1 = x_{white} = x$ and $x_2 = x_{black} = 1$. Then*

$$P_G(x+1, x^2+1, \cdots, x^n+1) = \sum_{k=0}^{n} a_k x^k \quad , \tag{3–15}$$

*where $a_k$ is the number of patterns in which the color white occurs exactly $k$ times.*

**Example 3.1.17.** *[5, 89] The number of necklaces with $n$ beads and $r$ colors is*

$$\frac{1}{n}\sum_{d|n} \varphi(d) r_d^{n/d} \quad . \tag{3–16}$$

**Example 3.1.18.** *[5, 86] The definition of a weight function is useful in order to count how many necklaces contain precisely $x_j$ beads of color $j$. Set $W$ to be the color white and $B$ to be the color black. Then a 4-bead necklace with exactly 2 white and 2 black beads is expressed by $W^2B^2$. If we do not wish to account for a particular color, per 3.1.16 we may assign its weight to 1. If we chose not to account for the color black, say, then the representation above would become simply $W^2 1^2 = W^2$.*

**Example 3.1.19.** *We use Polya's theorem to describe the pattern inventory of 3-bead necklaces under the action of $S_3$. Let the colors be $A$ and $B$. Then*

$$\begin{aligned} P_{S_3}(A+B, A^2+B^2, A^3+B^3) &= \frac{1}{6}[(A+B)^3 + 3(A+B)(A^2+B^2) + 2(A^3+B^3)] \\ &= A^3 + A^2B + AB^2 + B^3 \ . \end{aligned} \tag{3–17}$$

*In words, we have one necklace with three $A$ beads, one with two $A$ and one $B$ bead, one with one $A$ and two $B$ beads, and one with three $B$ beads.*

**Example 3.1.20.** *[6] We can use Polya's theorem to count chords in 12 tones that are equivalent under transposition by setting our set of colors to be $R = \{0, 1\}$. We then disregard one of the colors, say zero, by setting $w(0) = 1$ and $w(1) = C$. The group of transpositions is just $C_{12} = \mathbb{Z}/12\mathbb{Z}$, so we get*

$$\begin{aligned} P_{C_12} &= \frac{1}{12} \sum_{d \in \{1,2,3,4,6,12\}} \varphi(d) x_d^{12/d} \\ &= \frac{1}{12}\left(x_1^{12} + x_2^6 + 2x_3^4 + 2x_4^3 + 2x_6^2 + 4x_1 2^1\right) \\ &= \frac{1}{12}\left[(1+C)^{12} + (1+C^2)^6 + 2(1+C^3)^4 + 2(1+C^4)^3 + 2(1+C^6)^2 + 4(1+C^{12})\right] \\ &= C^{12} + C^{11} + 6C^{10} + 19C^9 + 43C^8 + 66C^7 + \cdots \\ &\qquad \cdots + 80C^6 + 66C^5 + 43C^4 + 19C^3 + 6C^2 + C^1 + C^0 \ . \end{aligned} \tag{3–18}$$

*In words, we have 43 tetrachords and 19 trichords that are transpositionally equivalent.*

**Example 3.1.21.** *[3, 249] Consider the action of the dihedral group $D_8$ on the 16-element set of colorings in black and white of the corners of a square. The cycle indicator of $D_8$ is*

$$P_{D_8}(x_1, x_2, x_3, x_4) = \frac{1}{8}(x_1^4 + 2x_1^2x_2 + 3x_2^2 + x_4^1) \ . \tag{3–19}$$

*Note that, in particular, we get no $x_3$ factors by Lagrange. The identity element in $D_8$ fixes all collorings, and is represented as a permutation by four cycles of length one. By Polya's formula, we make the substitution*

$$x_1^4 = (B^1 + W^1)^4 = B^4 + 4B^3W + 6B^2W^2 + 4BW^3 + W^4 \ , \tag{3–20}$$

*where the exponent outside the parenthesis is the number of cycles, and the exponents inside the parenthesis correspond to the lengths of the cycles, for each color. The element of order four in $D_8$ has one cycle of length four, so we get*

$$x_4^1 = (B^4 + W^4)^1 = B^4 + W^4 \ , \tag{3–21}$$

*that is, the permutation $r = (1\ 2\ 3\ 4)$ fixes the squares whose corners are all black or all white. Next, there are three elements in $D_8$ which comprise two cycles of length two, so each of those yield*

$$x_2^2 = (B^2 + W^2)^2 = B^4 + 2B^2W^2 + W^4 \ . \tag{3–22}$$

*The last two elements in $D_8$ both have two cycles of length one, plus one cycle of length two, which gives*

$$x_1^2x_2 = (B + W)^2(B^2 + W^2) = B^4 + 2B^3W + 2B^2W^2 + 2BW^2 + W^4 \ . \tag{3–23}$$

*Putting it all together, we get the following pattern inventory of orbits:*

$$P_{D_8}(B+W, B^2+W^2, B^3+W^3, B^4+W^4) = B^4+4B^3W+6B^2W^2+4BW^3+W^4 \ . \tag{3–24}$$

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