Interferometer computations and interpretations

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

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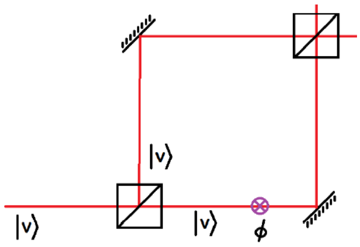
1 Interferometer (One degree of freedom)

Consider the following arrangement:



A beam (following a path $|x\rangle$) enters a beam-splitter which propagates it and also reflects it at 90 degrees, following a path $|y\rangle$. Then, we make a phase difference in the two beams by changing the phase of the beam following $|x\rangle$. After that, the beams are reflected to a second beam-splitter where a measurement is carried out to see the behavior of the probabilities of both paths.

In this case, the paths are represented by a Hilbert space H_c spanned by $(|x\rangle, |y\rangle)$, where they can be represented as column vectors $|x\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|y\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.Let's analyze how the state of the photon evolves according to the unitary operations performed on it:

The initial state is

$$|\psi_0\rangle = |x\rangle$$

Applying a 90 degrees beam-splitter is equivalent as applying a Hadamard gate:

$$BS = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then

$$|\psi_1\rangle = H|\psi_0\rangle = H|x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|y\rangle$$

Then, when we apply the phase change in the path $|y\rangle$ we represent this operator as

$$P(\phi) = |x\rangle\langle x| + e^{i\phi}|y\rangle\langle y| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

Then

$$|\psi_2\rangle = (|x\rangle\langle x| + e^{i\phi}|y\rangle\langle y|)(\frac{1}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|y\rangle)$$

$$|psi_2\rangle = \frac{1}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}e^{i\phi}|y\rangle$$

Then, when applying the second Hadamard gate representing the second beam-splitter:

$$|\psi_3\rangle = H|\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + e^{i\phi} \\ 1 - e^{i\phi} \end{pmatrix}$$

Hence, the final state of the photon is

$$|\psi_f\rangle = \frac{1}{2}(1 + e^{i\phi})|x\rangle + \frac{1}{2}(1 - e^{i\phi})|y\rangle$$

Its coefficients can be rewritten by considering the following:

$$P_x = |\frac{1}{2}(1+e^{i\phi})|^2 = \frac{1}{2}(1+e^{i\phi})\frac{1}{2}(1+e^{-i\phi}) = \frac{1}{4}(1+e^{-i\phi}+e^{i\phi}+1) = \frac{1}{4}(2+\frac{2(e^{-i\phi}+e^{i\phi})}{2}) = \frac{1}{4}(2+2\cos\frac{\phi}{2})$$

$$P_x = \frac{1}{2}(1+\cos\phi) = \cos^2\frac{\phi}{2}$$

Hence we can write the coefficient related to $|x\rangle$ as $\cos\frac{\phi}{2}$. We do the same for the coefficient related to $|y\rangle$:

$$P_y = |\frac{1}{2}(1-e^{i\phi})|^2 = \frac{1}{2}(1-e^{i\phi})\frac{1}{2}(1-e^{-i\phi}) = \frac{1}{4}(1-e^{-i\phi}-e^{i\phi}+1) = \frac{1}{4}(2-\frac{2(e^{-i\phi}+e^{i\phi})}{2}) = \frac{1}{4}(2-2\cos\frac{\phi}{2})$$

$$P_y = \frac{1}{2}(1-\cos\phi) = \sin^2\frac{\phi}{2}$$

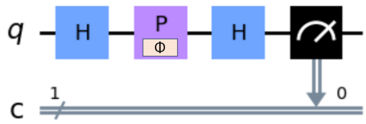
Hence we can write the coefficient related to $|y\rangle$ as $\sin\frac{\phi}{2}$. Finally, the state of the photon at the second beam-splitter is

$$\boxed{|\psi_f\rangle = \cos\frac{\phi}{2}|x\rangle + \sin\frac{\phi}{2}|y\rangle}$$

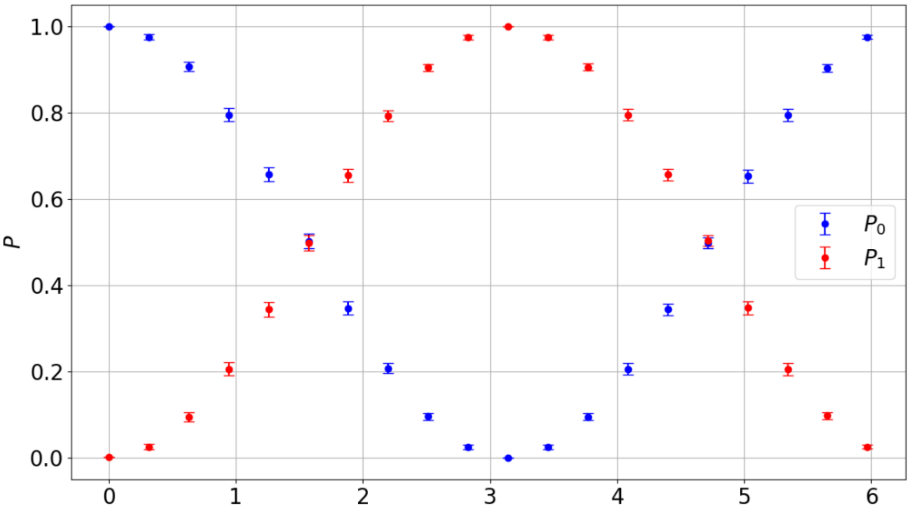
Since we have

$$|\psi_3\rangle = HP(\phi)H|\psi_0\rangle$$

the quantum circuit is implemented as follows:



The following figure shows the probabilities of measuring the photon from path $|x\rangle = |0\rangle$ (P_0) and from path $|y\rangle = |1\rangle$ (P_1) for different values of the phase ϕ (different intensities).



We can compute expressions for path distinguishability and fringe visibility by using the density matrix of $|\psi_f\rangle$:

$$\rho_f = \cos^2\frac{\phi}{2}|x\rangle\langle x| + \cos\frac{\phi}{2}\sin\frac{\phi}{2}|x\rangle\langle y| + \sin\frac{\phi}{2}\cos\frac{\phi}{2}|y\rangle\langle x| + \sin^2\frac{\phi}{2}|y\rangle\langle y| = \begin{pmatrix} \cos^2\frac{\phi}{2} & \cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\phi}{2} & \sin^2\frac{\phi}{2} \end{pmatrix}$$

Then, from 'Two interferometric complementarities' paper, path distinguishability for preparation $|\psi_f\rangle$ is

$$D(|\psi_f\rangle) = (1-4|\rho_{12}|^2)^{1/2} = (1-4|\sin\frac{\phi}{2}\cos\frac{\phi}{2}|^2)^{1/2} = (1-4\sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2})^{1/2} = (1-(2\sin\frac{\phi}{2}\cos\frac{\phi}{2})^2)^{1/2}$$

$$D(|\psi_f\rangle) = (1-\sin^2\phi)^{1/2} = \cos\phi$$

and fringe visibility is

$$v = 2|\rho_{12}| = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2} = \sin\phi$$

Interpretations:

- $\phi = 0$.- We can see from the figure that for this phase value the probability of measuring the photon as coming from $|x\rangle = |0\rangle$ is 1. Hence, path distinguishability should be equal to 1 and fringe visibility should be 0, showing completely particle behavior:

  $$D = \cos 0 = 1$$

  and

  $$v = \sin 0 = 0$$

  satisfying

  $$D^2 + v^2 = 1$$

  since $|\psi_1\rangle$ is a pure simple state.

- $\phi = \frac{\pi}{4}$.- We can see from the figure that for this phase value the probability of measuring the photon as coming from $|x\rangle = |0\rangle$ is about 0.8, whereas from $|y\rangle = |1\rangle$ is about 0.2. Hence, we have partial knowledge about both path distinguishability and fringe visibility!

  $$D = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

  $$v = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

  Since probability for the photon of coming from $|x\rangle$ is higher it is not certain, and thus there should be partial information about the pattern visibility. Again, wave-particle duality is valid since

  $$D^2 + v^2 = \frac{1}{2} + \frac{1}{2} = 1$$

  is satisfied.

- $\phi = \frac{\pi}{2}$.- We can see from the figure that for this phase value the probability of measuring the photon as coming from $|x\rangle = |0\rangle$ is 0.5, same from $|y\rangle = |1\rangle$ which is 0.5. Hence, we have no knowledge about both which path the photon is traversing, but fringe visibility manifests with certainty!

  $$D = \cos\frac{\pi}{2} = 0$$

  $$v = \sin\frac{\pi}{2} = 0$$

and again

$$D^2 + v^2 = 1$$

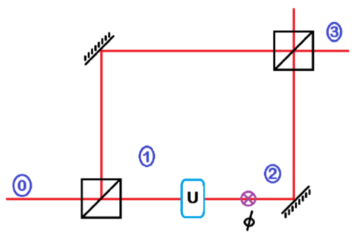
If we compare this case with that one for $\phi = \frac{\pi}{4}$ we can see that wave behavior manifests as $\phi \longrightarrow \frac{\pi}{2}$. In some sense, we began with completely particle behavior, then we changed the phase and found this duality, but wave behavior raised when we kept increasing ϕ.

The same effects occur when we keep changing the phase values, for $\phi = \frac{3\pi}{4}$ we have again an intermediate effect representing the wave-particle duality but now measuring the photon from $|y\rangle = |1\rangle$ has more chances; and for $\phi = 2\pi$ again we recover the particle behavior of measuring the photon from $|y\rangle = |1\rangle$ with certainty.

2 Interferometer (Two degrees of freedom - case 1)

In this case we consider the latter arrangement, but now we add the photon polarization as a new degree of freedom which can be affected. The path is described by a vector in a Hilbert space H_c as the latter example, and polarization by a vector in H_p spanned by $(|h\rangle, |v\rangle)$. Then, the whole system is described by a vector in $H_c \otimes H_p$.

Consider the following arrangement:



the photon initial state is

$$|\psi_0\rangle = |x\rangle|h\rangle$$

After the first beam splitter, which is modeled by the operator $H \otimes I$ (since it only splits the path states), the state is

$$|\psi_1\rangle = (H \otimes I)|\psi_0\rangle = (H \otimes I)|x\rangle \otimes |h\rangle = H|x\rangle \otimes I|h\rangle$$

then

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |y\rangle) \otimes |h\rangle = \frac{1}{\sqrt{2}}|x\rangle|h\rangle + \frac{1}{\sqrt{2}}|y\rangle|h\rangle$$

Then, polarization of the beam following path $|x\rangle$ is changed (we generate a superposition). To model this operation, we use $\bar{U} = |x\rangle\langle x| \otimes U + |y\rangle\langle y| \otimes I$, where U is the operation that will change polarization. We consider it as a rotation in H_p defined by

$$U = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

We can also write it as

$$U = \cos\frac{\theta}{2}|h\rangle\langle h| + (-\sin\frac{\theta}{2})|h\rangle\langle v| + \sin\frac{\theta}{2}|v\rangle\langle h| + \cos\frac{\theta}{2}|v\rangle\langle v|$$

This form for $\bar{U}$ makes sense, since if the term of the path state in along $|x\rangle$ then projecting $|x\rangle\langle x|$ will apply U to the polarization state, if the term is along $|y\rangle$ this operation will be zero; on the other hand, for the projection $|y\rangle\langle y|$, if the term is along $|x\rangle$ its operation will be zero, and along $|y\rangle$ it will apply I to the polarization state.

Then

$$|\psi_2\rangle = (|x\rangle\langle x| \otimes U + |y\rangle\langle y| \otimes I)(\frac{1}{\sqrt{2}}|x\rangle|h\rangle + \frac{1}{\sqrt{2}}|y\rangle|h\rangle) = \frac{1}{\sqrt{2}}|x\rangle \otimes U|h\rangle + \frac{1}{\sqrt{2}}|y\rangle \otimes I|h\rangle$$

The only terms that are not zero when applying U are as follows

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|x\rangle \otimes (\cos\frac{\theta}{2}|h\rangle + \sin\frac{\theta}{2}|v\rangle) + \frac{1}{\sqrt{2}}|y\rangle \otimes |h\rangle$$

After that, we generate a phase change in path $|x\rangle$ by using the operator $P(\phi) \otimes I$, where

$$P(\phi) = |x\rangle\langle x|e^{i\phi} + |y\rangle\langle y| = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}$$

Then, the state becomes

$$|\psi_3\rangle = [(|x\rangle\langle x|e^{i\phi} + |y\rangle\langle y|) \otimes I][\frac{1}{\sqrt{2}}|x\rangle \otimes (\cos\frac{\theta}{2}|h\rangle + \sin\frac{\theta}{2}|v\rangle) + \frac{1}{\sqrt{2}}|y\rangle \otimes |h\rangle]$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}e^{i\phi}|x\rangle \otimes (\cos\frac{\theta}{2}|h\rangle + \sin\frac{\theta}{2}|v\rangle) + \frac{1}{\sqrt{2}}|y\rangle \otimes |h\rangle$$

Finally, when both beams achieve the second beam-splitter:

$$|\psi_4\rangle = (H \otimes I)|\psi_3\rangle = (H \otimes I)(\frac{1}{\sqrt{2}}e^{i\phi}|x\rangle \otimes (\cos\frac{\theta}{2}|h\rangle + \sin\frac{\theta}{2}|v\rangle) + \frac{1}{\sqrt{2}}|y\rangle \otimes |h\rangle)$$

$$= \frac{1}{\sqrt{2}}e^{i\phi}H|x\rangle \otimes (\cos\frac{\theta}{2}|h\rangle + \sin\frac{\theta}{2}|v\rangle) + \frac{1}{\sqrt{2}}H|y\rangle \otimes |h\rangle$$

$$= \frac{1}{2}e^{i\phi}(|x\rangle + |y\rangle)(\cos\frac{\theta}{2}|h\rangle + \sin\frac{\theta}{2}|v\rangle) + \frac{1}{2}(|x\rangle - |y\rangle)|h\rangle$$

Rearranging terms:

$$\boxed{|\psi_f\rangle = \frac{1}{2}(1 + e^{i\phi}\cos\frac{\theta}{2})|x\rangle|h\rangle + \frac{1}{2}e^{i\phi}\sin\frac{\theta}{2}|x\rangle|v\rangle + \frac{1}{2}(e^{i\phi}\cos\frac{\theta}{2} - 1)|y\rangle|h\rangle + +\frac{1}{2}e^{i\phi}\sin\frac{\theta}{2}|y\rangle|v\rangle}$$

Hence, the corresponding probabilities are

$$P_{xh} = \frac{1}{2}(1 + e^{i\phi}\cos\frac{\theta}{2})\frac{1}{2}(1 + e^{-i\phi}\cos\frac{\theta}{2}) = \frac{1}{4}[1 + \cos\frac{\theta}{2}2\frac{(e^{i\phi} + e^{-i\phi})}{2} + \cos^2\frac{\theta}{2}]$$

$$P_{xh} = \frac{1}{4}[1 + 2\cos\frac{\theta}{2}\cos\phi + \cos^2\frac{\theta}{2}]$$

$$P_{xv} = \frac{1}{2}e^{i\phi}\sin\frac{\theta}{2}\frac{1}{2}e^{-i\phi}\sin\frac{\theta}{2} = \frac{1}{4}\sin^2\frac{\theta}{2}$$

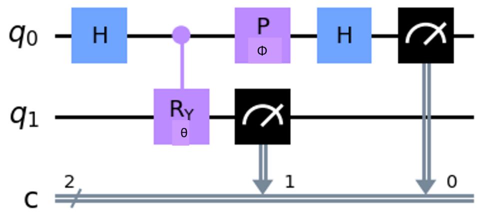
$$P_{yh} = \frac{1}{2}(e^{i\phi}\cos\frac{\theta}{2} - 1)\frac{1}{2}(e^{-i\phi}\cos\frac{\theta}{2} - 1) = \frac{1}{4}[1 - \cos\frac{\theta}{2}2\frac{(e^{i\phi} + e^{-i\phi})}{2} + \cos^2\frac{\theta}{2}]$$

$$P_{yh} = \frac{1}{4}[1 - 2\cos\frac{\theta}{2}\cos\phi + \cos^2\frac{\theta}{2}]$$

$$P_{yv} = \frac{1}{2}e^{i\phi}\sin\frac{\theta}{2}\frac{1}{2}e^{-i\phi}\sin\frac{\theta}{2} = \frac{1}{4}\sin^2\frac{\theta}{2}$$

P_{xv} and P_{yv} are phase independent!

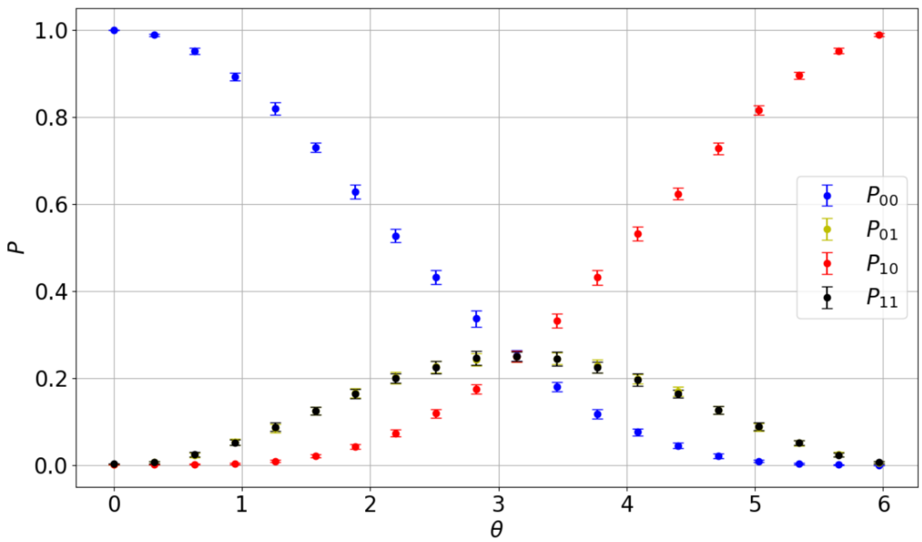
The quantum circuit we have to implement for this arrangement is shown in the following figure.



The following figures shows the probabilities of measuring the photon from paths $|x\rangle = |0\rangle$ and $|y\rangle = |1\rangle$ with polarization $|h\rangle$ and $|v\rangle$ for different values of rotations in the polarization space θ (different polarization) and for different phase difference values. Here we consider $P_{00} = P_{xh}$, $P_{01} = P_{xv}$, $P_{10} = P_{yh}$ and $P_{11} = P_{yv}$.

Interpretations:

- $\phi = 0$



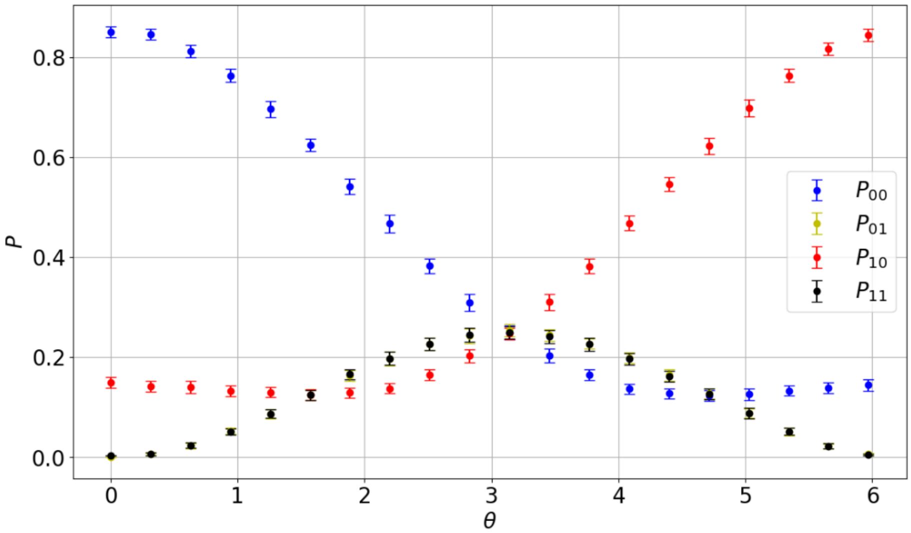
From the figure we can see that:

- $\theta = 0$.- State $|x\rangle|h\rangle$ is certain to be measured. Then we have a completely particle behavior.
- $\theta = \frac{\pi}{4}$.- States with polarization $|v\rangle$ start being able to be measured, and so does the state $|y\rangle|h\rangle$ but with probability close to 0. Hence, if photon is measured to have polarization $|h\rangle$, then it will come from path $|x\rangle$.

– $\theta = \frac{9\pi}{10}$.- It is not obvious now to tell that if polarization is measured to be $|h\rangle$ then it comes from $|x\rangle$. There is an intermediate effect of duality between path and polarization wave-particle behavior: we can partially know which path is more likely to be followed nor which polarization it will have.

– $\theta = \pi$.- All possible states have same probability to occur. Then, the photon has entirely a wave behavior: we cannot tell neither the path nor the polarization.

- $\phi = \frac{\pi}{4}$



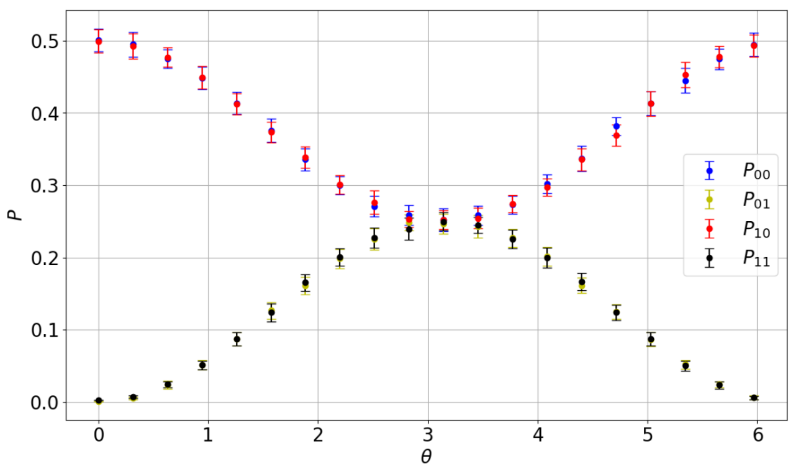
From the figure we can see that:

– $\theta = 0$.- In this case there is no a completely particle behavior as in the latter case. In this case we see the intermediate effects of the duality for the path degree of freedom only.

– $\theta = \frac{9\pi}{10}$.- We can see that probabilities for getting states $|x\rangle|v\rangle$ and $|y\rangle|v\rangle$ are greater than the probability of getting $|y\rangle|h\rangle$ which was the opposite before $\theta = \frac{\pi}{2}$. At this point we can still partially tell which states are more likely to measure. Hence we sill having this wave-particle duality.

– $\theta = \pi$.- All possible states have same probability to occur. Then, the photon has entirely a wave behavior: we cannot tell neither the path nor the polarization.

- $\phi = \frac{\pi}{2}$

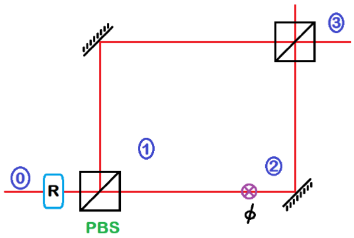


From the figure we can see that:

- $\theta = 0$.- There is a wave behavior for the path degree of freedom (completely uncertain to tell if it comes from $|x\rangle$ or from $|y\rangle$) but a particle behavior for the polarization (it is certain to measure $|h\rangle$).
- $\theta = \frac{9\pi}{10}$.- Intermediate effects for both degrees of freedom are shown. We can say though that it is more probable to measure the photon with polarization $|h\rangle$ (partial information again).
- $\theta = \pi$.- All possible states have same probability to occur. Then, the photon has entirely a wave behavior: we cannot tell neither the path nor the polarization.

3 Interferometer (Two degrees of freedom - case 2)

In this case, we consider a similar arrangement but operations on the photon are in different order, as shown in the following figure



Here the Hilbert space is the same as in case 1, spanned by the same vector bases. First, we consider the initial state to be $|\psi_0\rangle = |x\rangle|h\rangle$. Then we apply an operation which makes a superposition of the polarization states, given by $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Then

$$|\psi_1\rangle = (I \otimes R(\theta))|\psi_0\rangle = |x\rangle \otimes R(\theta)|h\rangle$$

$$R(\theta)|h\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\theta|h\rangle + \sin\theta|v\rangle$$

Hence,

$$|\psi_1\rangle = |x\rangle \otimes (\cos\theta|h\rangle + \sin\theta|v\rangle)$$

Now, the beam follows the path $|x\rangle$ towards the PBS, which is a beam-splitter which splits the beam into paths according to its polarization:

$$PBS = I \otimes |h\rangle\langle h| + \sigma_x \otimes |v\rangle\langle v|$$

This is, if the state has polarization $|h\rangle$, the it will follow path $|x\rangle$; if it has polarization $|v\rangle$, then it will follow path $|y\rangle = \sigma_x|x\rangle$:

$$|\psi_2\rangle = PBS|\psi_1\rangle = (I \otimes |h\rangle\langle h| + \sigma_x \otimes |v\rangle\langle v|)(|x\rangle \otimes (\cos\theta|h\rangle + \sin\theta|v\rangle))$$

$$= |x\rangle \otimes \cos\theta|h\rangle + \sigma_x|x\rangle \otimes \sin\theta|v\rangle$$

But $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |x\rangle\langle y| + |y\rangle\langle x|$. Then

$$\sigma_x|x\rangle = |x\rangle\langle y|x\rangle + |y\rangle|\langle x|x\rangle = |y\rangle$$

Hence

$$|\psi_2\rangle = \cos\theta|x\rangle|h\rangle + \sin\theta|y\rangle|v\rangle$$

As we can see, the PBS has entangled paths and polarization!!!. Actually, for $\theta = \frac{\pi}{4}$ this state represents one of the Bell basis. (This is interesting, realizing how a Bell state is generated by using an interferometer!).

After that, we apply a phase change to the path $|x\rangle$:

$$|\psi_3\rangle = (P(\phi)\otimes I)|\psi_2\rangle = (P(\phi)\otimes I)(\cos\theta|x\rangle|h\rangle + \sin\theta|y\rangle|v\rangle) = \cos\theta P(\phi)|x\rangle\otimes|h\rangle + \sin\theta P(\phi)|y\rangle\otimes|v\rangle$$

$$|\psi_3\rangle = \cos\theta e^{i\phi}|x\rangle|h\rangle + \sin\theta|y\rangle|v\rangle$$

where $P(\phi) = e^{i\phi}|x\rangle\langle x| + |y\rangle\langle y|$.

Finally, we apply a normal beam-splitter that considers only the paths:

$$|\psi_4\rangle = BS|\psi_3\rangle = (H\otimes I)(\cos\theta e^{i\phi}|x\rangle|h\rangle + \sin\theta|y\rangle|v\rangle) = \cos\theta e^{i\phi}H|x\rangle\otimes|h\rangle + \sin\theta H|y\rangle\otimes|v\rangle$$

$$= \cos\theta e^{i\phi}\frac{1}{\sqrt{2}}(|x\rangle + |y\rangle)|h\rangle + \sin\theta\frac{1}{\sqrt{2}}(|x\rangle - |y\rangle)|v\rangle$$

Then

$$\boxed{|\psi_4\rangle = \frac{1}{\sqrt{2}}e^{i\phi}\cos\theta|x\rangle|h\rangle + \frac{1}{\sqrt{2}}\sin\theta|x\rangle|v\rangle + \frac{1}{\sqrt{2}}e^{i\phi}\cos\theta|y\rangle|h\rangle - \frac{1}{\sqrt{2}}\sin\theta|y\rangle|v\rangle}$$

The corresponding probabilities are

$$P_{xh} = \langle\psi_4|\pi_x\otimes\pi_h|\psi_4\rangle = |\frac{1}{\sqrt{2}}e^{i\phi}\cos\theta|^2 = \frac{1}{2}\cos^2\theta$$

$$P_{xv} = \langle\psi_4|\pi_x\otimes\pi_h|\psi_4\rangle = |\frac{1}{\sqrt{2}}\sin\theta|^2 = \frac{1}{2}\sin^2\theta$$

One can realize that

$$P_x = P_{xh} + P_{xv} = \frac{1}{2}$$

$$P_{yh} = \langle\psi_4|\pi_x\otimes\pi_h|\psi_4\rangle = |\frac{1}{\sqrt{2}}e^{i\phi}\cos\theta|^2 = \frac{1}{2}\cos^2\theta$$

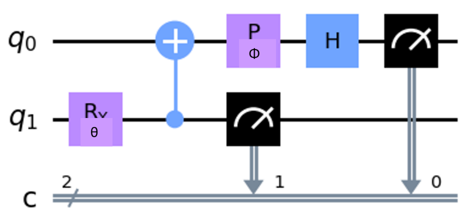
$$P_{yv} = \langle\psi_4|\pi_x\otimes\pi_h|\psi_4\rangle = |-\frac{1}{\sqrt{2}}\sin\theta|^2 = \frac{1}{2}\sin^2\theta$$

One can realize that

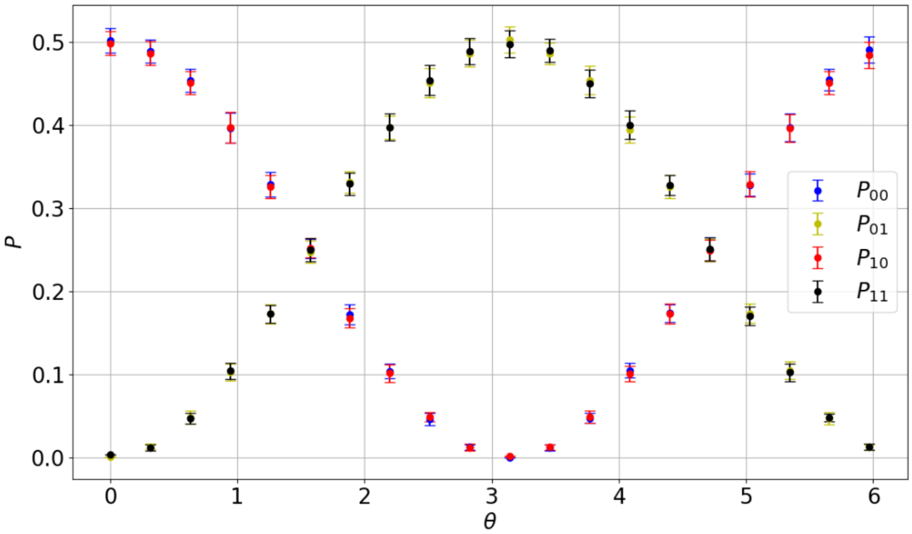
$$P_y = P_{yh} + P_{yv} = \frac{1}{2}$$

We note that in this case probabilities are all phase independent. Also, P_{xh} and P_{xv} have same probability since they come from the same quantum correlation. Same for P_{xv} and P_{yv}.

The quantum circuit we have to implement for this arrangement is shown in the following figure.



The following figure shows the probabilities of measuring the photon from paths $|x\rangle = |0\rangle$ and $|y\rangle = |1\rangle$ with polarizations $|h\rangle$ and $|v\rangle$ for different values of rotations in the polarization space θ (different polarizations). Here we consider $P_{00} = P_{xh}$, $P_{01} = P_{xv}$, $P_{10} = P_{yh}$ and $P_{11} = P_{yv}$.



Interpretations:

- $\theta = 0$.- From the figure we can see that only polarization in $|h\rangle$ is probable to be measured, and this photon is coming from paths $|x\rangle$ and $|y\rangle$ each with probability 0.5. This shows a wave behavior for path states, since path distinguishability is zero, i.e. we cannot tell with certainty which path has taken our measured photon with polarization $|h\rangle$. Since it is certain we will measure $|h\rangle$ there is a notion of particle behavior for these polarization states in this case.

- $\theta = \frac{\pi}{4}$.- We can see from the figure that for this value we have a probability is 0.4 for measuring a photon with polarization $|h\rangle$ coming from either $|x\rangle$ or $|y\rangle$, whereas the probability of measuring it with polarization $|v\rangle$ is 0.1 for either paths $|x\rangle$ and $|y\rangle$. Then, there is a duality between these states with polarization $|h\rangle$ to be measured in either of the two paths, and the states with polarization $|v\rangle$ to be measured as well in either of the two paths. Indeed, the duality is highlighted in polarizations.

- $\theta = \frac{\pi}{2}$.- We can see from the figure that for this value the probability is the same (0.5) for measuring the photon coming from paths $|x\rangle$ and $|y\rangle$ for either polarizations $|h\rangle$ or $|v\rangle$. Hence, there is a complete wave behavior for the photon since we cannot tell with certainty which path and polarization it has.

Duality is characterized by polarization since probabilities corresponding to same polarization but different paths are the same.