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Attitude Error Representations for Kalman Filtering

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The quaternion has the lowest dimensionality possible for a globally nonsingular attitude representation. The quaternion must obey a unit norm constraint, though, which has led to the development of an extended Kalman filter using a quaternion for the global attitude estimate and a three-component representation for attitude errors. Various attitude error representations are considered for this multiplicative extended Kalman filter, which incorporates a nonlinear, norm preserving quaternion reset operation. Second-order bias corrections are computed in this framework.

Introduction

USE of the extended Kalman[1] filter (EKF) (see also Refs. 2 and 3) for spacecraft attitude estimation has a long history. The earliest published application[4] and a recent example[5] employ three-dimensional attitude representations, but the unavoidable fact that all three-dimensional attitude representations are singular or discontinuous for certain attitudes has led to the pursuit of alternative parameterizations.[6] The four-component unit quaternion has the lowest dimension of any globally nonsingular attitude parameterization, leading to its widespread use in Kalman filters.[7–12] Enforcing the unit norm constraint on a quaternion leaves it with the three degrees of freedom consistent with the dimensionality of the rotation group, but requires some sort of constrained quaternion estimation. Various methods for evading or enforcing the norm constraint have been proposed and tested.[9–13] The most successful method parameterizes the global attitude with a unit quaternion, while employing a three-component representation for the attitude errors.[7–9] We provide a new derivation of this filter, which has become known as the multiplicative EKF (MEKF), highlighting the interplay between the quaternion and three-component attitude representations. The main aim of this paper is to dispel the lingering suspicion that there is something amiss with the MEKF. We show that the MEKF is not really a quaternion estimator; it performs an unconstrained estimation of a three-component attitude error, with the quaternion playing the role of a reference about which the errors are defined. We then show how the proper understanding of the MEKF leads to a consistent extension to a second-order attitude filter.

The paper opens with a discussion of attitude representations, followed by a brief critical review of alternative quaternion estimation schemes. An extensive review with citations of the literature through 1981 can be found in Ref. 9. We then develop the basic equations of the MEKF, including detailed models for vector measurements and quaternion measurements. The paper continues with the construction of a second-order filter on the same foundation, followed by concluding remarks.

Attitude Parameterizations

This brief discussion will establish conventions and notation; a thorough review of attitude representations is provided in Ref. 14. We regard the 3×3 orthogonal attitude matrix, or direction cosine matrix, as the fundamental attitude representation.

Euler Axis/Angle and Rotation Vector

Euler's theorem[15] states that the most general motion of a rigid body with one point fixed is a rotation by an angle ϕ about some axis, which we specify by a unit vector $\boldsymbol{e}$. The rotation vector

$$\boldsymbol{a}_\phi \equiv \phi \boldsymbol{e} \tag{1}$$

is the first example of a general class of three-parameter attitude representations denoted by $\boldsymbol{a}$. All rotations can be mapped to points inside and on the surface of a ball of radius π in rotation vector space, with points at opposite ends of a diameter representing the same attitude. The rotation vector may jump from one end of a diameter to the other as the attitude varies smoothly, limiting its usefulness as a global attitude representation. Expanding the representation to a sphere of radius 2π can postpone these jumps but cannot avoid then entirely, because the kinematic equation for the rotation vector is singular for $\phi = 2\pi$.

Quaternions

A unit quaternion representing spacecraft attitude has a three-vector part and a scalar part, which are related to the axis and angle of rotation by

$$q = \begin{bmatrix} \boldsymbol{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}\sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix} \tag{2}$$

The quaternion components obey the unit length constraint

$$|q|^2 \equiv |\boldsymbol{q}|^2 + q_4^2 = 1 \tag{3}$$

Using q to denote a four-component quaternion rather than the magnitude of its vector part is an exception to the convention adopted in this paper of denoting the magnitude of a three-vector $\boldsymbol{v}$ by the corresponding non-boldface character v.

The four components of q can be found in a paper by Euler[16] and in unpublished notes by Gauss,[17] but Rodrigues[18] first demonstrated their general usefulness, so they are known as Euler symmetric parameters or Euler–Rodrigues parameters. Hamilton introduced the quaternion as an abstract mathematical object in 1844,[19] but there is some question as to whether he correctly understood its relation to rotations (see Ref. 20).

Unit quaternions reside on the three-dimensional sphere S^3 embedded in four-dimensional Euclidean space E^4. The attitude matrix is a homogeneous quadratic function of the components of a unit quaternion,

$$A(q) = \left(q_4^2 - |\boldsymbol{q}|^2\right) I_{3\times 3} - 2q_4[\boldsymbol{q}\times] + 2\boldsymbol{q}\boldsymbol{q}^T \tag{4}$$

where $I_{3\times 3}$ is the 3×3 identity matrix and the cross product matrix is

$$[\boldsymbol{q}\times] \equiv \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{5}$$

The quaternion representation is 2:1 because Eq. (4) shows that q and $-q$ represent the same attitude matrix. The quaternion is an ideal

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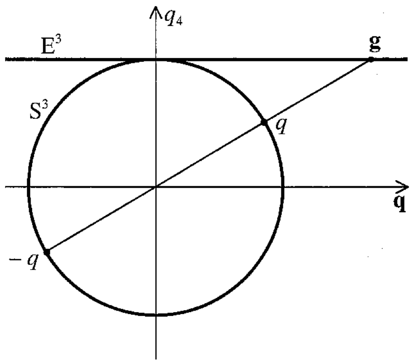

Fig. 1 Gibbs vector as a gnomonic projection.

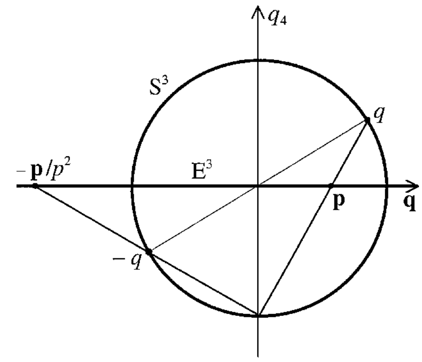

Fig. 2 Modified Rodrigues parameters as a stereographic projection.

global attitude representation because it varies continuously over S^3 as the attitude changes, avoiding the jumps required by some three-dimensional parameterizations. However, it is customary to restrict a quaternion representing an attitude *error* to the hemisphere of S^3 with $q_4 > 0$.

We follow the quaternion multiplication convention of Refs. 9 and 14,

$$q' \otimes q \equiv \begin{bmatrix} q_4' \boldsymbol{q} + q_4 \boldsymbol{q}' - \boldsymbol{q}' \times \boldsymbol{q} \\ q_4' q_4 - \boldsymbol{q}' \cdot \boldsymbol{q} \end{bmatrix} \tag{6}$$

which has the useful property that

$$A(q')A(q) = A(q' \otimes q) \tag{7}$$

This means that the rotation group and the quaternion group are almost isomorphic, the qualifier "almost" owing to the 2:1 nature of the mapping.[21] The kinematic equation for the quaternion is

$$\dot{q} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes q \tag{8}$$

where $\boldsymbol{\omega}$ is the angular velocity vector in body coordinates. With exact arithmetic, Eq. (8) preserves the normalization of q. If computational errors cause the norm constraint to be violated, it can be restored trivially by dividing q by its 2-norm.

Gibbs Vector or Rodrigues Parameters

The three components of the Gibbs vector[22]

$$\boldsymbol{g} \equiv \boldsymbol{q}/q_4 = \boldsymbol{e}\tan(\phi/2) \equiv \boldsymbol{a}_g/2 \tag{9}$$

had been introduced earlier by Rodrigues.[18] The factor of one-half in the last term of Eq. (9) ensures that the magnitude a_g is approximately equal to ϕ for small rotations. The Gibbs vector can be regarded as a gnomonic projection of the S^3 quaternion space onto three-dimensional Euclidean $\boldsymbol{g}$ space, as shown in Fig. 1. This is a 2:1 mapping of S^3, with q and $-q$ mapping to the same point. Because q and $-q$ represent the same rotation, the Gibbs vector parameterization is a 1:1 mapping of the rotations onto E^3. The Gibbs vector is infinite for 180-deg rotations (the $q_4 = 0$ equator of S^3), which is undesirable for a global representation of rotations.

Modified Rodrigues Parameters (MRPs)

The MRPs were introduced by Wiener[23]:

$$\boldsymbol{p} \equiv \boldsymbol{q}/(1 + q_4) = \boldsymbol{e}\tan(\phi/4) \equiv \boldsymbol{a}_p/4 \tag{10}$$

The factor of one-quarter in the last term ensures that a_p is approximately equal to ϕ for small rotations. Marandi and Modi[24] pointed out that these parameters can be viewed as a stereographic projection of S^3 quaternion space onto E^3, as shown in Fig. 2. One hemisphere of S^3 projects to the interior of the unit sphere in three-dimensional $\boldsymbol{p}$ space, and the other hemisphere of S^3 projects to the exterior of the unit $\boldsymbol{p}$ sphere. All rotations can be represented by MRPs inside and on the surface of the unit ball. If we extend the representation to all Euclidean $\boldsymbol{p}$ space, we have a 2:1 parameterization with $\boldsymbol{p}$ and $-\boldsymbol{p}/p^2$ representing the same rotation. This parameterization shares many characteristics with the rotation vector parameterization, including the need for discrete jumps, but avoids transcendental functions.

Alternative Quaternion Estimation Methods

Because unit quaternions reside on S^3, it would seem natural to regard the quaternion as a random variable and define its estimate as the conditional expectation

$$E\{q \mid Z\} \equiv \int_{S^3} q\rho(q \mid Z)\, \mathrm{d}^3 q \tag{11}$$

where $\rho(q \mid Z)$ is the probability density function of q on S^3, conditioned on the measurements Z. This is an unsatisfactory definition, however, because restricting the probability distribution in quaternion space to the surface of a unit sphere means that its expectation must be *inside* the sphere. The integral cannot give a unit quaternion unless the probability distribution function is concentrated at a point. This is the fundamental conceptual problem with quaternion estimation methods based either implicitly or explicitly on Eq. (11).

One proposed solution is to relax the quaternion normalization requirement of Eq. (3) and parameterize the attitude by

$$A(q) = |q|^{-2}\left\{\left(q_4^2 - |\boldsymbol{q}|^2\right)I_{3\times 3} - 2q_4[\boldsymbol{q}\times] + 2\boldsymbol{q}\boldsymbol{q}^T\right\} \tag{12}$$

which is an orthogonal matrix regardless of whether the quaternion is normalized. This approach implicitly introduces an unobservable degree of freedom, the quaternion norm, and its performance has not been encouraging.[11,12]

Other approaches simply normalize the quaternion by brute force, outside the filter.[9–12] The approach of Lefferts et al. in Sec. IX of Ref. 9 projects the 4×4 quaternion covariance onto a 3×3 matrix, arguing that this does not entail any loss of information. The resulting filter obeys the quaternion norm constraint to first order in the measurement residual, so that the second-order modification resulting from quaternion normalization is outside the purview of the first-order EKF. Vathsal built a second-order filter on this foundation, but his measurement update violates the norm constraint in second order.[25] Some quaternion estimators even relax the requirement that the attitude matrix be exactly orthogonal by employing Eq. (4) with an imperfectly normalized quaternion. However, none of these approaches is entirely satisfactory from a theoretical point of view.

The MEKF approach in Sec. XI of Ref. 9 avoids all of these problems. This is the original approach,[7,8] and it leads to the same EKF as the covariance projection approach. It rests on a firmer conceptual foundation, however, and provides the basis for a consistent second-order filter.

MEKF

The MEKF represents the true attitude as the quaternion product

$$q(t) = \delta q(\boldsymbol{a}(t)) \otimes q_{\mathrm{ref}}(t) \tag{13}$$

where $q_{\mathrm{ref}}(t)$ is some unit reference quaternion and $\delta q(\boldsymbol{a}(t))$ is a unit quaternion representing the rotation from $q_{\mathrm{ref}}(t)$ to the true attitude $q(t)$. We parameterize $\delta q(\boldsymbol{a}(t))$ by one of the three-vectors $\boldsymbol{a}(t)$ considered earlier. The two attitude representations $\boldsymbol{a}(t)$ and $q_{\mathrm{ref}}(t)$ in Eq. (13) are clearly redundant. The basic idea of the MEKF is to compute an unconstrained estimate of the three-component $\boldsymbol{a}(t)$ while using the correctly normalized four-component $q_{\mathrm{ref}}(t)$ to provide a globally nonsingular attitude representation.

Given an estimate $\hat{\boldsymbol{a}}(t)$, where a caret denotes the expectation of a random variable, Eq. (13) indicates that the corresponding estimate of the true attitude quaternion is $\delta q(\hat{\boldsymbol{a}}(t)) \otimes q_{\mathrm{ref}}(t)$. We remove the redundancy in the attitude representation by choosing the reference quaternion $q_{\mathrm{ref}}(t)$ so that $\hat{\boldsymbol{a}}(t)$ is identically zero. Because $\delta q(\mathbf{0})$ is the identity quaternion, this means that the reference quaternion is the best estimate of the true quaternion. We reiterate that the quaternion estimate in the MEKF is not defined as the expectation of a random variable as in Eq. (11). The fundamental conceptual advantage of the MEKF is that $q_{\mathrm{ref}}(t)$ is a unit quaternion by definition.

The identification of $q_{\mathrm{ref}}(t)$ as the attitude estimate means in turn that $\boldsymbol{a}(t)$ is a three-component representation of the attitude error, the difference between the truth and our estimate. This provides a consistent treatment of the attitude error statistics, with the covariance of the attitude error angles in the body frame (in radians squared) represented by the covariance of $\boldsymbol{a}(t)$. An alternative formulation, which has some advantages, reverses the order of multiplication in Eq. (13) so that $\boldsymbol{a}(t)$ represents the attitude errors in the inertial reference frame rather than in the body frame.[26]

Continuous/discrete filtering proceeds in three steps: time propagation, measurement update, and reset. The continuous-time propagation is arranged to keep $\hat{\boldsymbol{a}}(t) \equiv \mathbf{0}$, but the discrete measurement update assigns a finite postupdate value $\hat{\boldsymbol{a}}(+)$ to $\hat{\boldsymbol{a}}$. Immediately after the measurement update, the reference quaternion still retains its preupdate value $q_{\mathrm{ref}}(-)$, so that it no longer represents the optimal estimate. The reset operation corrects this situation by moving the update information from $\hat{\boldsymbol{a}}(+)$ to a postupdate reference $q_{\mathrm{ref}}(+)$ and resetting $\hat{\boldsymbol{a}}$ to zero. Because the true quaternion is not changed by this operation, Eq. (13) requires

$$\delta q(\hat{\boldsymbol{a}}(+)) \otimes q_{\mathrm{ref}}(-) = \delta q(\mathbf{0}) \otimes q_{\mathrm{ref}}(+) = q_{\mathrm{ref}}(+) \tag{14}$$

It is possible to eliminate the discrete reset and maintain $\hat{\boldsymbol{a}}(t) \equiv \mathbf{0}$ even during the measurement update by considering the update to be spread out over an infinitesimal time interval, rather than being instantaneous.[27] However, this paper treats measurement updates as instantaneous.

The reset operation is implicit in the standard EKF, which represents the true state $\boldsymbol{X}$ as the sum of the reference state $\boldsymbol{X}_{\mathrm{ref}}$, customarily called the full state estimate $\hat{\boldsymbol{X}}$, and a small error $\boldsymbol{x}$,

$$\boldsymbol{X} = \boldsymbol{X}_{\mathrm{ref}} + \boldsymbol{x} \tag{15}$$

The measurement processing produces an updated value of the error vector

$$\hat{\boldsymbol{x}}(+) = \hat{\boldsymbol{x}}(-) + \Delta\boldsymbol{x} \tag{16}$$

where $\Delta\boldsymbol{x}$ is the correction resulting from the measurement update. The reset operation moves the update information from the error state to the full state estimate by

$$\boldsymbol{X}_{\mathrm{ref}}(+) = \boldsymbol{X}_{\mathrm{ref}}(-) + \hat{\boldsymbol{x}}(+) - \hat{\boldsymbol{x}}(-) = \boldsymbol{X}_{\mathrm{ref}}(-) + \Delta\boldsymbol{x} \tag{17}$$

Because of the appearance of the final form of this equation, the update and reset are usually considered to be a single operation. The reset of the attitude must be treated explicitly in the MEKF, however, because it is not purely additive.

Attitude Error Representations

The error quaternion is parameterized by the three-component representation of Eqs. (1), (9), or (10) as

$$\delta q(\boldsymbol{a}_\phi) = \begin{bmatrix} (\boldsymbol{a}_\phi / a_\phi)\sin(a_\phi/2) \\ \cos(a_\phi/2) \end{bmatrix} \tag{18a}$$

$$\delta q(\boldsymbol{a}_g) = \left(4 + a_g^2\right)^{-\frac{1}{2}} \begin{bmatrix} \boldsymbol{a}_g \\ 2 \end{bmatrix} \tag{18b}$$

or

$$\delta q(\boldsymbol{a}_p) = \frac{1}{16 + a_p^2} \begin{bmatrix} 8\boldsymbol{a}_p \\ 16 - a_p^2 \end{bmatrix} \tag{18c}$$

respectively. A fourth parameterization $\boldsymbol{a}_q$, defined to be twice the vector part of δq, gives

$$\delta q(\boldsymbol{a}_q) = \frac{1}{2} \begin{bmatrix} \boldsymbol{a}_q \\ \left(4 - a_q^2\right)^{\frac{1}{2}} \end{bmatrix} \tag{18d}$$

This differs from the parameterization in Sec. XI of Ref. 9 only by the factor of two. These four definitions of $\boldsymbol{a}$ provide the same second-order approximations to the error quaternion,

$$\delta q(\boldsymbol{a}) \approx \begin{bmatrix} \boldsymbol{a}/2 \\ 1 - a^2/8 \end{bmatrix} \tag{19}$$

and to the attitude error matrix

$$A(\delta q(\boldsymbol{a})) \approx I_{3\times 3} - [\boldsymbol{a}\times] - \tfrac{1}{2}\left(a^2 I_{3\times 3} - \boldsymbol{a}\boldsymbol{a}^T\right) \tag{20}$$

Thus, they are equivalent for the EKF and second-order filters, but they differ in third and higher orders in $\boldsymbol{a}$.

The rotation vector has the disadvantage of requiring trigonometric function evaluations, but this has not discouraged its use in a recent application.[28] The other parameterizations avoid trigonometric functions by not employing the rotation angle ϕ explicitly. The Gibbs vector[22] has the advantage that the reset can first define the unnormalized quaternion

$$q_{\mathrm{unnorm}} \equiv \begin{bmatrix} \boldsymbol{a}_g \\ 2 \end{bmatrix} \otimes q_{\mathrm{ref}}(-) \tag{21}$$

and then update the unit quaternion by

$$q_{\mathrm{ref}}(+) = q_{\mathrm{unnorm}} / |q_{\mathrm{unnorm}}| \tag{22}$$

avoiding accumulation of numerical errors in the quaternion norm. The MRPs have the computational advantage of not requiring square roots or trigonometric functions. Equation (18d) gives a nonsensical complex result for $a_q > 2$, which may arise before the filter has converged. Even though large measurement updates violate linearity assumptions, it is desirable for a robust filter to handle them without requiring special computations.

Note that Eqs. (19) and (20) do not hold for all three-parameter attitude representations. In particular, they only hold to first order if the components of $\boldsymbol{a}$ are taken to be Euler angle[15] rotations about three orthogonal axes, as in Refs. 4 and 7. An Euler angle parameterization will lead to the same EKF, but its extension to second order will depend on the specific Euler rotation sequence used. Equations (19) and (20), however, are valid for a continuous family of parameterizations[29] interpolating between the Gibbs vector and MRPs.

Attitude and State Propagation

Because q_{ref} is a unit quaternion, it must obey a kinematic equation of the form

$$\dot{q}_{\mathrm{ref}} = \frac{1}{2} \begin{bmatrix} \boldsymbol{\omega}_{\mathrm{ref}} \\ 0 \end{bmatrix} \otimes q_{\mathrm{ref}} \tag{23}$$

where $\boldsymbol{\omega}_{\mathrm{ref}}$ has the obvious interpretation as the angular velocity of the reference attitude, and time arguments have been omitted for compactness. We now show how $\boldsymbol{\omega}_{\mathrm{ref}}$ is determined by the requirement that $\hat{\boldsymbol{a}}$ be identically zero, which is the condition that the reference attitude be the optimal attitude estimate. Computing the time derivative of Eq. (13), using Eqs. (8) and (23), gives

$$\begin{bmatrix} \boldsymbol{\omega} \\ 0 \end{bmatrix} \otimes q = 2\frac{\mathrm{d}[\delta q(\boldsymbol{a})]}{\mathrm{d}t} \otimes q_{\mathrm{ref}} + \delta q(\boldsymbol{a}) \otimes \begin{bmatrix} \boldsymbol{\omega}_{\mathrm{ref}} \\ 0 \end{bmatrix} \otimes q_{\mathrm{ref}} \tag{24}$$

Substitute Eq. (13) for q on the left side, right-multiply the entire equation by the inverse of q_{ref}, and rearrange to get

$$2\frac{\mathrm{d}[\delta q(\boldsymbol{a})]}{\mathrm{d}t} = \begin{bmatrix}\boldsymbol{\omega}\\ 0\end{bmatrix} \otimes \delta q(\boldsymbol{a}) - \delta q(\boldsymbol{a}) \otimes \begin{bmatrix}\boldsymbol{\omega}_{\text{ref}}\\ 0\end{bmatrix} \tag{25}$$

The usual EKF technique of approximating the expectation of a nonlinear function of $\boldsymbol{a}$ and $\boldsymbol{\omega}$ by the same nonlinear function of their expectations $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{\omega}}$ gives

$$2\frac{\mathrm{d}[\delta q(\hat{\boldsymbol{a}})]}{\mathrm{d}t} = \begin{bmatrix}\hat{\boldsymbol{\omega}}\\ 0\end{bmatrix} \otimes \delta q(\hat{\boldsymbol{a}}) - \delta q(\hat{\boldsymbol{a}}) \otimes \begin{bmatrix}\boldsymbol{\omega}_{\text{ref}}\\ 0\end{bmatrix} \tag{26}$$

Note that $\boldsymbol{\omega}_{\text{ref}}$ is not a random variable. The requirement that $\hat{\boldsymbol{a}}$ be zero means that $\delta q(\hat{\boldsymbol{a}})$ is the identity quaternion, which is a constant, so that Eq. (26) gives

$$\boldsymbol{\omega}_{\text{ref}}(t) = \hat{\boldsymbol{\omega}}(t) \tag{27}$$

The quaternion propagation specified by Eqs. (23) and (27) is the same as the propagation equation derived by more conventional methods.

Now we specialize to the case where a set of gyros is used to obtain angular rate information in place of models of the spacecraft dynamics.[8,9] We employ Farrenkopf's gyro dynamics error model,[30] which means that we ignore the output noise for rate-integrating gyros.[31] This is an excellent approximation for navigation-grade gyros. The angular rate vector is given in terms of the gyro output vector $\boldsymbol{\omega}_{\text{out}}(t)$ by

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_{\text{out}}(t) - \boldsymbol{b}(t) - \boldsymbol{\eta}_1(t) \tag{28}$$

where the gyro drift vector $\boldsymbol{b}(t)$ obeys

$$\dot{\boldsymbol{b}}(t) = \boldsymbol{\eta}_2(t) \tag{29}$$

and $\boldsymbol{\eta}_1(t)$ and $\boldsymbol{\eta}_2(t)$ are zero-mean white noise processes. The estimated angular velocity is

$$\hat{\boldsymbol{\omega}}(t) = \boldsymbol{\omega}_{\text{out}}(t) - \hat{\boldsymbol{b}}(t) \tag{30}$$

The Kalman filter estimates the six-component state vector

$$\boldsymbol{x}(t) \equiv \begin{bmatrix}\boldsymbol{a}(t)\\ \boldsymbol{b}(t)\end{bmatrix} \tag{31}$$

Note that the reference quaternion is not part of the state vector. The propagation of the expectation $\hat{\boldsymbol{x}}(t)$ of this state vector is trivial because $\hat{\boldsymbol{a}}(t)$ is identically zero by assumption, and Eq. (29) clearly implies that $\dot{\hat{\boldsymbol{b}}}(t) = \boldsymbol{0}$.

Covariance Propagation

The propagation of the covariance matrix

$$P \equiv E\{(\boldsymbol{x} - \hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}})^T \mid Z\} = \begin{bmatrix}P_a & P_c\\ P_c^T & P_b\end{bmatrix} \tag{32}$$

is given by

$$\dot{P} = FP + PF^T + GQG^T \tag{33}$$

where the matrices F, G, and Q are to be determined. The partitioning of the covariance matrix into 3×3 attitude, bias, and correlation submatrices will be useful later.

We consider the Gibbs vector[22] parameterization for specificity. Substituting Eq. (25) into the time derivative of

$$\boldsymbol{a}_g = 2(\delta q)_V/(\delta q)_4 \tag{34}$$

where the subscripts V and 4 denote the vector and scalar parts of the quaternion, gives

$$\dot{\boldsymbol{a}}_g = \left(I_{3\times 3} + \tfrac{1}{4}\boldsymbol{a}_g\boldsymbol{a}_g^T\right)(\boldsymbol{\omega} - \boldsymbol{\omega}_{\text{ref}}) - \tfrac{1}{2}(\boldsymbol{\omega} + \boldsymbol{\omega}_{\text{ref}}) \times \boldsymbol{a}_g \equiv \boldsymbol{f}(\boldsymbol{x}, t) \tag{35}$$

after some straightforward quaternion algebra. This is an exact kinematic equation, depending neither on the EKF approximation nor on a spacecraft attitude dynamics model. Inserting Eqs. (27), (28), and (30) and ignoring terms of higher than first order in $\boldsymbol{a}$ and the rate error

$$\boldsymbol{\omega} - \boldsymbol{\omega}_{\text{ref}} = \hat{\boldsymbol{b}} - \boldsymbol{b} - \boldsymbol{\eta}_1 \tag{36}$$

gives the linear EKF approximation

$$\boldsymbol{f}(\boldsymbol{x}, t) = \hat{\boldsymbol{b}} - \boldsymbol{b} - \boldsymbol{\eta}_1 - \boldsymbol{\omega}_{\text{ref}} \times \boldsymbol{a} \tag{37}$$

The subscript on $\boldsymbol{a}$ is omitted because the Appendix shows that Eq. (37) holds if Eq. (18c) or (18d) is used in place of Eq. (18b). In fact, it only requires a representation that satisfies Eq. (19) to at least first order. It follows that

$$F(t) \equiv \begin{bmatrix}\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{a}} & \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{b}}\\ 0_{3\times 3} & 0_{3\times 3}\end{bmatrix} = \begin{bmatrix}-[\boldsymbol{\omega}_{\text{ref}}\times] & -I_{3\times 3}\\ 0_{3\times 3} & 0_{3\times 3}\end{bmatrix} \tag{38}$$

$$G(t) \equiv \begin{bmatrix}\dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{\eta}_1} & \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{\eta}_2}\\ 0_{3\times 3} & I_{3\times 3}\end{bmatrix} = \begin{bmatrix}-I_{3\times 3} & 0_{3\times 3}\\ 0_{3\times 3} & I_{3\times 3}\end{bmatrix} \tag{39}$$

and

$$E\left\{\begin{bmatrix}\boldsymbol{\eta}_1(t)\\ \boldsymbol{\eta}_2(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}_1(t')\\ \boldsymbol{\eta}_2(t')\end{bmatrix}^T\right\} = \delta(t - t')Q(t) \tag{40}$$

where

$$\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right]_{ij} \equiv \frac{\partial f_i}{\partial x_j} \tag{41}$$

and $\delta(t - t')$ is the Dirac delta function. The matrix $Q(t)$ is block diagonal if the processes $\boldsymbol{\eta}_1(t)$ and $\boldsymbol{\eta}_2(t)$ are statistically independent, as is usually assumed. This covariance propagation is the same as that of Refs. 7–9, except for some factors of one-half.

Vector Measurement Model and Update

A vector measurement is modeled as an m-component function $\boldsymbol{h}(\boldsymbol{v}_B)$ of a vector $\boldsymbol{v}_B$ in the spacecraft body frame, corrupted by zero-mean white noise. The representation of $\boldsymbol{v}_B$ in the body frame is the mapping of its representation $\boldsymbol{v}_I$ in the inertial reference frame by the attitude matrix:

$$\boldsymbol{v}_B = A(q)\boldsymbol{v}_I \approx \left\{I_{3\times 3} - [\boldsymbol{a}\times] - \tfrac{1}{2}\left(a^2 I_{3\times 3} - \boldsymbol{a}\boldsymbol{a}^T\right)\right\}A(q_{\text{ref}})\boldsymbol{v}_I \tag{42}$$

where we have used Eqs. (7), (13), and (20). Substituting this into $\boldsymbol{h}(\boldsymbol{v}_B)$ and expanding to first order in $\boldsymbol{a}$ about the preupdate reference gives

$$\boldsymbol{h}(\boldsymbol{v}_B) = \boldsymbol{h}(\bar{\boldsymbol{v}}_B) - \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{v}}\right|_{\bar{\boldsymbol{v}}_B}[\boldsymbol{a}\times]\bar{\boldsymbol{v}}_B = \boldsymbol{h}(\bar{\boldsymbol{v}}_B) + H_a\boldsymbol{a} \tag{43}$$

where $\bar{\boldsymbol{v}}_B \equiv A(q_{\text{ref}}(-))\boldsymbol{v}_I$ is the body frame vector predicted by the preupdate quaternion and

$$H_a \equiv \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{v}}\right|_{\bar{\boldsymbol{v}}_B}[\bar{\boldsymbol{v}}_B\times] \tag{44}$$

Because the measurement does not depend explicitly on the gyro drifts, the $m \times 6$ measurement sensitivity matrix is

$$H = \begin{bmatrix}\dfrac{\partial \boldsymbol{h}}{\partial \boldsymbol{a}} & \dfrac{\partial \boldsymbol{h}}{\partial \boldsymbol{b}}\end{bmatrix} = [H_a \quad 0_{m\times 3}] \tag{45}$$

The Kalman gain matrix is given by

$$K = \begin{bmatrix}P_a(-)\\ P_c^T(-)\end{bmatrix} H_a^T\left[H_a P_a(-) H_a^T + R\right]^{-1} \tag{46}$$

where R is the covariance of the measurement white noise. The state update is given by

$$\hat{\boldsymbol{x}}(+) = \hat{\boldsymbol{x}}(-) + K[\boldsymbol{h}_{\text{obs}} - \boldsymbol{h}(\bar{\boldsymbol{v}}_B) - H_a\hat{\boldsymbol{a}}(-)] \tag{47}$$

where $\boldsymbol{h}_{\text{obs}}$ is the measured value and the predicted value is given by the preupdate expectation of Eq. (43). The covariance update is

$$P(+) = P(-) - KH_a[P_a(-) \quad P_c(-)] \tag{48}$$

Reset

The quaternion reset uses Eq. (14) with any of Eqs. (18). If a reset is performed after each measurement update, the term $H_a\hat{\boldsymbol{a}}(-)$ in Eq. (47) is identically zero. For computational efficiency, the reset is often delayed until all of the updates for a set of simultaneous measurements have been performed, in which case the $\hat{\boldsymbol{a}}(-)$ in Eq. (47) is the $\hat{\boldsymbol{a}}(+)$ from the previous update. It is imperative to perform a reset before beginning the next time propagation, however, to ensure that $\hat{\boldsymbol{a}}$ is zero at the beginning of the propagation and, thus, to avoid the necessity of propagating $\hat{\boldsymbol{a}}(t)$ between measurements. The reset does not modify the covariance because it neither increases nor decreases the total information content of the estimate; it merely moves this information from one part of the attitude representation to another.

Quaternion Measurements

A modern star tracker may track between 5 and 50 stars simultaneously, match them to stars in an internal star catalog, and compute its attitude as an inertially referenced quaternion.[32–34] The computation can also estimate the covariance of the attitude error angle vector.[35,36] It is a simple matter to transform these quantities from the star tracker reference frame to the spacecraft frame to produce a quaternion "measurement" q_{obs} and a 3×3 measurement covariance matrix R. The most convenient way to present this information to the Kalman filter is in terms of one of the three-dimensional parameterizations of the deviation between the observed and predicted attitudes[37]

$$q_{\text{obs}} = \delta q(\boldsymbol{a}_{\text{obs}}) \otimes q_{\text{ref}}(-) \tag{49}$$

The measurement model is simply

$$\boldsymbol{h}(\boldsymbol{a}) = \boldsymbol{a} \tag{50}$$

so that H_a is the 3×3 identity matrix and R is the covariance of this error angle vector. Because the predicted value of the observation is identically zero from Eq. (49), the state update simplifies to

$$\hat{\boldsymbol{x}}(+) = \hat{\boldsymbol{x}}(-) + \begin{bmatrix} P_a(-) \\ P_c^T(-) \end{bmatrix} [P_a(-) + R]^{-1}[\boldsymbol{a}_{\text{obs}} - \hat{\boldsymbol{a}}(-)] \tag{51}$$

It is important to use the same three-dimensional parameterization in the observation processing [Eq. (49)] as is used in the reset. For example, if the Gibbs vector[22] form of Eq. (34) is used for observation processing, then Eq. (18b) should be used for the reset. With this proviso, we see that, when $R \ll P_a(-)$, so that $\hat{\boldsymbol{a}}(+) = \boldsymbol{a}_{\text{obs}}$, we have $q_{\text{ref}}(+) = q_{\text{obs}}$.

Second-Order Filter

Second-order terms in the nonlinear Kalman filter can become important when nonlinearities are significant relative to the measurement and process noise terms. A first-order filter with bias correction terms obtains the essential benefit of a second-order filter without the computational penalty of additional second moment calculations.[38] This filter adds second-order corrections to the state propagation and measurement residual equations, but uses the EKF expressions for the covariance and gains. Because Eqs. (18) are exact to all orders of $\boldsymbol{a}$, the reset in the second-order filter is the same as in the MEKF.

Propagation

When second-order corrections are included, we will find that the requirement that $\hat{\boldsymbol{a}}(t)$ be identically zero no longer leads to Eq. (27) for $\boldsymbol{\omega}_{\text{ref}}$. The state estimate is propagated by

$$\dot{\hat{\boldsymbol{x}}}(t) = \begin{bmatrix} \boldsymbol{f}(\hat{\boldsymbol{x}}, t) + \hat{\boldsymbol{b}}_p(t) \\ \boldsymbol{0} \end{bmatrix} \tag{52}$$

where the propagation bias correction is given by[38]

$$\hat{\boldsymbol{b}}_p(t) \equiv \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 \boldsymbol{f}(\boldsymbol{x}, t)}{\partial x_i \partial x_j}\right|_{\hat{x}(t)} P_{ij}(t) \tag{53}$$

For the Gibbs vector[22] parameterization, Eqs. (28–32), (35), and (53) give, omitting time arguments,

$$\hat{\boldsymbol{b}}_p = \tfrac{1}{4}P_a(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_{\text{ref}}) + \boldsymbol{\omega}_c \tag{54}$$

where

$$\boldsymbol{\omega}_c \equiv \frac{1}{2}\begin{bmatrix} (P_c)_{32} - (P_c)_{23} \\ (P_c)_{13} - (P_c)_{31} \\ (P_c)_{21} - (P_c)_{12} \end{bmatrix} = \frac{1}{2}E\{(\boldsymbol{b} - \hat{\boldsymbol{b}}) \times \boldsymbol{a} \mid Z\} \tag{55}$$

Adding this second-order term to $\boldsymbol{f}(\hat{\boldsymbol{x}}, t)$ from Eq. (35) gives

$$\begin{aligned} \dot{\hat{\boldsymbol{a}}}_g &= \left[I_{3\times 3} + \tfrac{1}{4}\left(\hat{\boldsymbol{a}}_g\hat{\boldsymbol{a}}_g^T + P_a\right)\right](\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_{\text{ref}}) \\ &\quad - \tfrac{1}{2}(\hat{\boldsymbol{\omega}} + \boldsymbol{\omega}_{\text{ref}}) \times \hat{\boldsymbol{a}}_g + \boldsymbol{\omega}_c \end{aligned} \tag{56}$$

The requirement that $\hat{\boldsymbol{a}}_g$ and $\dot{\hat{\boldsymbol{a}}}_g$ be equal to zero yields

$$\boldsymbol{\omega}_{\text{ref}} = \hat{\boldsymbol{\omega}} + \left(I_{3\times 3} + \tfrac{1}{4}P_a\right)^{-1}\boldsymbol{\omega}_c \tag{57}$$

It is shown in the Appendix that the factor of $(I_{3\times 3} + \frac{1}{4}P_a)^{-1}$ depends on the specific choice of the three-dimensional parameterization of the attitude errors. Because P_a and $\boldsymbol{\omega}_c$ are both second order in the estimation errors, $(I_{3\times 3} + \frac{1}{4}P_a)$ can be replaced by the identity matrix in a second-order filter, giving

$$\boldsymbol{\omega}_{\text{ref}}(t) = \hat{\boldsymbol{\omega}}(t) + \boldsymbol{\omega}_c(t) \tag{58}$$

Time propagation between measurements is changed from the MEKF by the addition to the angular rate vector of the second-order correction $\boldsymbol{\omega}_c(t)$ arising from the skew part of the covariance between the attitude errors and gyro drift bias errors. This is equivalent to the result obtained by Vathsal.[25]

Measurement Update

In the second-order filter, the predicted measurement is approximated by

$$E\{\boldsymbol{h}(\boldsymbol{x}, t) \mid Z\} \approx \boldsymbol{h}(\hat{\boldsymbol{x}}(-), t) + \hat{\boldsymbol{b}}_m \tag{59}$$

where Z includes all of the measurements before the present one and the measurement bias term is given by[38]

$$\hat{\boldsymbol{b}}_m \equiv \frac{1}{2}\sum_{i,j} \left.\frac{\partial^2 \boldsymbol{h}(\boldsymbol{x}, t)}{\partial x_i \partial x_j}\right|_{\hat{x}(-)} P_{ij}(-) \tag{60}$$

Using Eq. (42) to expand a vector measurement $\boldsymbol{h}(\boldsymbol{v}_B)$ to second order in $\boldsymbol{a}$ gives

$$\begin{aligned} \boldsymbol{h}(\boldsymbol{v}_B) &= \boldsymbol{h}(\bar{\boldsymbol{v}}_B) - \left.\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{v}}\right|_{\bar{v}_B} \left\{[\boldsymbol{a}\times] + \frac{1}{2}\left(a^2 I_{3\times 3} - \boldsymbol{a}\boldsymbol{a}^T\right)\right\}\bar{\boldsymbol{v}}_B \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{3} \left.\frac{\partial^2 \boldsymbol{h}}{\partial v_i \partial v_j}\right|_{\bar{v}_B} ([\bar{\boldsymbol{v}}_B\times]\boldsymbol{a})_i([\bar{\boldsymbol{v}}_B\times]\boldsymbol{a})_j \end{aligned} \tag{61}$$

where the argument (–) is omitted for compactness. Inserting this into Eq. (60) and using the symmetry of P_a and the mixed second-order partial derivatives and the fact that the measurement does not depend explicitly on the gyro drift bias gives

$$\hat{\boldsymbol{b}}_m = -\frac{1}{2}\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{v}}\bigg|_{\bar{v}_B}[(\mathrm{tr}P_a)I_{3\times 3} - P_a]\bar{\boldsymbol{v}}_B + \frac{1}{2}\sum_{i,j=1}^{3}\frac{\partial^2 \boldsymbol{h}}{\partial v_i \partial v_j}\bigg|_{\bar{v}_B}\left([\bar{\boldsymbol{v}}_B\times]^T P_a[\bar{\boldsymbol{v}}_B\times]\right)_{ij} \tag{62}$$

where tr is the matrix trace. This differs from the measurement bias found by Vathsal, whose computation did not enforce the quaternion norm constraint to second order.[25]

The special case that $P_a = p_a I_{3\times 3}$ for some scalar p_a gives

$$\hat{\boldsymbol{b}}_m = p_a\left\{-\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{v}}\bigg|_{\bar{v}_B}\bar{\boldsymbol{v}}_B + \frac{1}{2}\sum_{i,j=1}^{3}\frac{\partial^2 \boldsymbol{h}}{\partial v_i \partial v_j}\bigg|_{\bar{v}_B}\left[\bar{v}_B^2\delta_{ij} - (\bar{\boldsymbol{v}}_B)_i(\bar{\boldsymbol{v}}_B)_j\right]\right\} \tag{63}$$

This case is of interest because a Kalman filter is often initialized with the covariance equal to a large multiple of the identity, and we do not want this to corrupt the update. Consider two different vector measurement models. The first models a focal plane sensor such as a star tracker or digital sun sensor,

$$\boldsymbol{h}(\boldsymbol{v}) = \begin{bmatrix} u_1/u_3 \\ u_2/u_3 \end{bmatrix} \tag{64}$$

where

$$\boldsymbol{u} = B\boldsymbol{v}_B \tag{65}$$

is the vector to the observed object in the sensor reference frame, which is rotated from the spacecraft body frame by the orthogonal transformation matrix B. The measurement sensitivity matrix is

$$H_a = \frac{1}{u_3^2}\begin{bmatrix} u_3 & 0 & -u_1 \\ 0 & u_3 & -u_2 \end{bmatrix}[\boldsymbol{u}\times]B \tag{66}$$

The first term in Eq. (63) vanishes, and the second term gives

$$\hat{\boldsymbol{b}}_m = p_a\left[1 + h^2(\bar{\boldsymbol{v}}_B)\right]\boldsymbol{h}(\bar{\boldsymbol{v}}_B) \tag{67}$$

The second measurement model is measurement of the vector itself, as by a triaxial magnetometer,

$$\boldsymbol{h}(\boldsymbol{v}) = \boldsymbol{u} \tag{68}$$

The measurement sensitivity matrix for this measurement model is

$$H_a = [\boldsymbol{u}\times]B \tag{69}$$

In this case, the second term in Eq. (63) vanishes, and the first term gives

$$\hat{\boldsymbol{b}}_m = -p_a\boldsymbol{h}(\bar{\boldsymbol{v}}_B) \tag{70}$$

These two measurement models give measurement biases of the same order of magnitude but with opposite signs. For large initialization errors of 20 deg, or 0.349 rad, the correction to the predicted measurement is 12% of the leading term $\boldsymbol{h}(\bar{\boldsymbol{v}}_B)$.

Because the input to the Kalman filter for a quaternion measurement is linear in the three-dimensional attitude parameter vector, the measurement bias $\hat{\boldsymbol{b}}_m(t)$ is identically zero in this case. The quaternion computation algorithm in the star tracker has taken account of the measurement nonlinearities, and so the Kalman filter does not see them.

Summary

The major result of this paper is to clarify the relationship between the four-component quaternion representation of attitude and the three-component representation of attitude errors in the MEKF. We view this filter as based on an apparently redundant representation of the attitude in terms of a reference quaternion and a three-vector specifying the deviation of the attitude from the reference. This apparent redundancy is removed by constraining the reference quaternion so that the expectation of the three-vector of attitude deviations is identically zero. Therefore, it is not necessary to propagate this identically zero expected value. The basic structure of the MEKF follows from constraining the reference quaternion in this fashion: The reference quaternion becomes the attitude estimate, the three-vector becomes the attitude error vector, and the covariance of the three-vector becomes the attitude covariance. Although this filter has a long history, the underlying assumptions have been unclear. Elucidating these assumptions clears the way for an extension of the MEKF to a consistent second-order filter. Several different three-dimensional parameterizations give identical results in the linear EKF and in a second-order filter, except in the reset step, where they differ in third order in the measurement update.

Appendix: Filter Dynamics in Alternative Representations

When Eq. (18c) or (18d) is employed instead of the Gibbs vector,[22] Eq. (18b) replaces Eq. (35) with

$$\dot{\boldsymbol{a}}_p = \left[\left(1 - \tfrac{1}{16}a_p^2\right)I_{3\times 3} + \tfrac{1}{8}\boldsymbol{a}_p\boldsymbol{a}_p^T\right](\boldsymbol{\omega} - \boldsymbol{\omega}_{\mathrm{ref}}) - \tfrac{1}{2}(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathrm{ref}})\times\boldsymbol{a}_p \equiv \boldsymbol{f}(\boldsymbol{x}, t) \tag{A1}$$

or

$$\dot{\boldsymbol{a}}_q = \left(1 - \tfrac{1}{4}a_q^2\right)^{\frac{1}{2}}(\boldsymbol{\omega} - \boldsymbol{\omega}_{\mathrm{ref}}) - \tfrac{1}{2}(\boldsymbol{\omega} + \boldsymbol{\omega}_{\mathrm{ref}})\times\boldsymbol{a}_q \equiv \boldsymbol{f}(\boldsymbol{x}, t) \tag{A2}$$

Both of these give Eq. (37) in the EKF approximation of ignoring terms higher than first order in $\boldsymbol{a}$ and $\boldsymbol{\omega} - \boldsymbol{\omega}_{\mathrm{ref}}$. Differentiating Eq. (A1) gives, with Eq. (53),

$$\hat{\boldsymbol{b}}_p = \left[\tfrac{1}{8}P_a - \tfrac{1}{16}(\mathrm{tr}P_a)I_{3\times 3}\right](\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_{\mathrm{ref}}) + \boldsymbol{\omega}_c \tag{A3}$$

in place of Eq. (54). The requirement that $\hat{\boldsymbol{a}}_q$ and $\dot{\hat{\boldsymbol{a}}}_q = \boldsymbol{f}(\hat{\boldsymbol{x}}, t) + \hat{\boldsymbol{b}}_p$ be zero yields

$$\boldsymbol{\omega}_{\mathrm{ref}} = \hat{\boldsymbol{\omega}} + \left[\left(1 - \tfrac{1}{16}\,\mathrm{tr}P_a\right)I_{3\times 3} + \tfrac{1}{8}P_a\right]^{-1}\boldsymbol{\omega}_c \tag{A4}$$

Differentiating Eq. (A2) gives

$$\hat{\boldsymbol{b}}_p = -\tfrac{1}{8}(\mathrm{tr}P_a)(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}_{\mathrm{ref}}) + \boldsymbol{\omega}_c \tag{A5}$$

and

$$\boldsymbol{\omega}_{\mathrm{ref}} = \hat{\boldsymbol{\omega}} + \left(1 - \tfrac{1}{8}\mathrm{tr}\,P_a\right)^{-1}\boldsymbol{\omega}_c \tag{A6}$$

The second-order approximation of either Eq. (A4) and (A6) is Eq. (58).

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