Chapter 2
Basic Navigational Mathematics, Reference Frames and the Earth's Geometry

Navigation algorithms involve various coordinate frames and the transformation of coordinates between them. For example, inertial sensors measure motion with respect to an inertial frame which is resolved in the host platform's body frame. This information is further transformed to a navigation frame. A GPS receiver initially estimates the position and velocity of the satellite in an inertial orbital frame. Since the user wants the navigational information with respect to the Earth, the satellite's position and velocity are transformed to an appropriate Earth-fixed frame. Since measured quantities are required to be transformed between various reference frames during the solution of navigation equations, it is important to know about the reference frames and the transformation of coordinates between them. But first we will review some of the basic mathematical techniques.

2.1 Basic Navigation Mathematical Techniques

This section will review some of the basic mathematical techniques encountered in navigational computations and derivations. However, the reader is referred to (Chatfield 1997; Rogers 2007 and Farrell 2008) for advanced mathematics and derivations. This section will also introduce the various notations used later in the book.

2.1.1 Vector Notation

In this text, a vector is depicted in bold lowercase letters with a superscript that indicates the coordinate frame in which the components of the vector are given. The vector components do not appear in bold, but they retain the superscript. For example, the three-dimensional vector r for a point in an arbitrary frame *k* is depicted as

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$$\mathbf{r}^k = \begin{bmatrix} x^k \\ y^k \\ z^k \end{bmatrix} \tag{2.1}$$

In this notation, the superscript k represents the k-frame, and the elements (x^k, y^k, z^k) denote the coordinate components in the k-frame. For simplicity, the superscript is omitted from the elements of the vector where the frame is obvious from the context.

2.1.2 Vector Coordinate Transformation

Vector transformation from one reference frame to another is frequently needed in inertial navigation computations. This is achieved by a transformation matrix. A matrix is represented by a capital letter which is not written in bold. A vector of any coordinate frame can be represented into any other frame by making a suitable transformation. The transformation of a general k-frame vector $\mathbf{r}^k$ into frame m is given as

$$\mathbf{r}^m = R_k^m \mathbf{r}^k \tag{2.2}$$

where R_k^m represents the matrix that transforms vector $\mathbf{r}$ from the k-frame to the m-frame. For a valid transformation, the superscript of the vector that is to be transformed must match the subscript of the transformation matrix (in effect they cancel each other during the transformation).

The inverse of a transformation matrix R_k^m describes a transformation from the m-frame to the k-frame

$$\mathbf{r}^k = \left(R_k^m\right)^{-1} \mathbf{r}^m = R_m^k \mathbf{r}^m \tag{2.3}$$

If the two coordinate frames are mutually orthogonal, their transformation matrix will also be orthogonal and its inverse is equivalent to its transpose. As all the computational frames are orthogonal frames of references, the inverse and the transpose of their transformation matrices are equal. Hence for a transformation matrix R_k^m we see that

$$R_k^m = \left(R_m^k\right)^T = \left(R_m^k\right)^{-1} \tag{2.4}$$

A square matrix (like any transformation matrix) is orthogonal if all of its vectors are mutually orthogonal. This means that if

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2.5}$$

where

$$\mathbf{r}_1 = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}, \mathbf{r}_2 = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{33} \end{bmatrix}, \mathbf{r}_3 = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \tag{2.6}$$

then for matrix R to be orthogonal the following should be true

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = 0,\ \mathbf{r}_1 \cdot \mathbf{r}_3 = 0,\ \mathbf{r}_2 \cdot \mathbf{r}_3 = 0 \tag{2.7}$$

2.1.3 Angular Velocity Vectors

The angular velocity of the rotation of one computational frame about another is represented by a three component vector $\boldsymbol{\omega}$. The angular velocity of the k-frame relative to the m-frame, as resolved in the p-frame, is represented by ω_{mk}^{p} as

$$\boldsymbol{\omega}_{mk}^{p} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{2.8}$$

where the subscripts of $\boldsymbol{\omega}$ denote the direction of rotation (the k-frame with respect to the m-frame) and the superscripts denote the coordinate frame in which the components of the angular velocities $\left(\omega_x, \omega_y, \omega_z\right)$ are given.

The rotation between two coordinate frames can be performed in two steps and expressed as the sum of the rotations between two different coordinate frames, as shown in Eq. (2.9). The rotation of the k-frame with respect to the p-frame can be performed in two steps: firstly a rotation of the m-frame with respect to the p-frame and then a rotation of the k-frame with respect to the m-frame

$$\boldsymbol{\omega}_{pk}^{k} = \boldsymbol{\omega}_{pm}^{k} + \boldsymbol{\omega}_{mk}^{k} \tag{2.9}$$

For the above summation to be valid, the inner indices must be the same (to cancel each other) and the vectors to be added or subtracted must be in the same reference frame (i.e. their superscripts must be the same).

2.1.4 Skew-Symmetric Matrix

The angular rotation between two reference frames can also be expressed by a skew-symmetric matrix instead of a vector. In fact this is sometimes desired in order to change the cross product of two vectors into the simpler case of matrix multiplication. A vector and the corresponding skew-symmetric matrix forms of an angular velocity vector $\boldsymbol{\omega}_{mk}^{p}$ are denoted as

$$\underbrace{\boldsymbol{\omega}_{mk}^{p} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\text{Angular velocity vector}} \Rightarrow \underbrace{\Omega_{mk}^{p} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{\text{Skew-symmetric form of angular the velocity vector}} \tag{2.10}$$

Similarly, a velocity vector $\mathbf{v}^p$ can be represented in skew-symmetric form V^p as

$$\underbrace{\mathbf{v}^{p} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}}_{\text{Velocity vector}} \Rightarrow \underbrace{\mathrm{V}^{p} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}}_{\text{Skew-symmetric form of the velocity vector}} \tag{2.11}$$

Note that the skew-symmetric matrix is denoted by a non-italicized capital letter of the corresponding vector.

2.1.5 Basic Operations with Skew-Symmetric Matrices

Since a vector can be expressed as a corresponding skew-symmetric matrix, the rules of matrix operations can be applied to most vector operations. If **a**, **b** and **c** are three-dimensional vectors with corresponding skew-symmetric matrices A, B and C, then following relationships hold

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \mathbf{b}^T \mathbf{a} \tag{2.12}$$

$$\mathbf{a} \times \mathbf{b} = \mathrm{A}\mathbf{b} = B^T \mathbf{a} = -\mathrm{B}\mathbf{a} \tag{2.13}$$

$$[\![\mathrm{A}\mathbf{b}]\!] = \mathrm{AB} - \mathrm{BA} \tag{2.14}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a}^T \mathrm{B}\mathbf{c} \tag{2.15}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathrm{AB}\mathbf{c} \tag{2.16}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathrm{AB}\mathbf{c} - \mathrm{BA}\mathbf{c} \tag{2.17}$$

where $[\![\mathrm{A}\mathbf{b}]\!]$ in Eq. (2.14) depicts the skew-symmetric matrix of vector A**b**.

2.1.6 Angular Velocity Coordinate Transformations

Just like any other vector, the coordinates of an angular velocity vector can be transformed from one frame to another. Hence the transformation of an angular velocity vector $\boldsymbol{\omega}_{mk}$ from the k-frame to the p-frame can be expressed as

$$\boldsymbol{\omega}^{p}_{mk} = R^{p}_{k}\boldsymbol{\omega}^{k}_{mk} \tag{2.18}$$

The equivalent transformation between two skew-symmetric matrices has the special form

$$\Omega^{p}_{mk} = R^{p}_{k}\Omega^{k}_{mk}R^{k}_{p} \tag{2.19}$$

2.1.7 Least Squares Method

The method of least squares is used to solve a set of equations where there are more equations than the unknowns. The solution minimizes the sum of the squares of the residual vector. Suppose we want to estimate a vector $\mathbf{x}$ of n parameters $(x_1, x_2, \ldots, x_n)$ from vector $\mathbf{z}$ of m noisy measurements $(z_1, z_2, \ldots, z_m)$ such that $m > n$. The measurement vector is linearly related to parameter $\mathbf{x}$ with additive error vector ε such that

$$\mathbf{z} = H\mathbf{x} + \varepsilon \tag{2.20}$$

where H is a known matrix of dimension $m \times n$, called the design matrix, and it is of rank n (linearly independent row or columns).

In the method of least square, the sum of the squares of the components of the residual vector $(\mathbf{z} - H\mathbf{x})$ is minimized in estimating vector $\mathbf{x}$, and is denoted by $\hat{\mathbf{x}}$. Hence

$$\text{minimize } \|\mathbf{z} - H\hat{\mathbf{x}}\|^2 = (\mathbf{z} - H\hat{\mathbf{x}})^T_{1\times m}(\mathbf{z} - H\hat{\mathbf{x}})_{m\times 1} \tag{2.21}$$

This minimization is achieved by differentiating the above equation with respect to $\hat{\mathbf{x}}$ and setting it to zero. Expanding the above equation gives

$$\|\mathbf{z} - H\hat{\mathbf{x}}\|^2 = \mathbf{z}^T\mathbf{z} - \mathbf{z}^T H\hat{\mathbf{x}} - \hat{\mathbf{x}}^T H^T\mathbf{z} + \hat{\mathbf{x}}^T H^T H\hat{\mathbf{x}} \tag{2.22}$$

Using the following relationships

$$\frac{\partial(\mathbf{x}^T\mathbf{a})}{\partial\mathbf{x}} = \frac{\partial(\mathbf{a}^T\mathbf{x})}{\partial\mathbf{x}} = \mathbf{a}^T \tag{2.23}$$

and

$$\frac{\partial(\mathbf{x}^T A\mathbf{x})}{\partial\mathbf{x}} = (A\mathbf{x})^T + \mathbf{x}^T A \tag{2.24}$$

the derivative of the scalar quantity represented by Eq. (2.22) is obtained as follows

$$\begin{aligned}
\frac{\partial\left(\|\mathbf{z}-H\hat{\mathbf{x}}\|^2\right)}{\partial\hat{\mathbf{x}}} &= 0-\mathbf{z}^TH-(H^T\mathbf{z})^T+(H^TH\hat{\mathbf{x}})^T+\hat{\mathbf{x}}^TH^TH \\
\frac{\partial\left(\|\mathbf{z}-H\hat{\mathbf{x}}\|^2\right)}{\partial\hat{\mathbf{x}}} &= -\mathbf{z}^TH-\mathbf{z}^TH+\hat{\mathbf{x}}^TH^TH+\hat{\mathbf{x}}^TH^TH \\
\frac{\partial\left(\|\mathbf{z}-H\hat{\mathbf{x}}\|^2\right)}{\partial\hat{\mathbf{x}}} &= -2(\mathbf{z}^TH+\hat{\mathbf{x}}^TH^TH)
\end{aligned} \tag{2.25}$$

To obtain the maximum, the derivative is set to zero and solved for $\hat{\mathbf{x}}$

$$-2(\mathbf{z}^TH-\hat{\mathbf{x}}^TH^TH)=0 \tag{2.26}$$

$$\begin{aligned}
\hat{\mathbf{x}}^TH^TH &= \mathbf{z}^TH \\
(\hat{\mathbf{x}}^TH^TH)^T &= (\mathbf{z}^TH)^T \\
H^TH\hat{\mathbf{x}} &= H^T\mathbf{z}
\end{aligned} \tag{2.27}$$

and finally

$$\hat{\mathbf{x}}=(H^TH)^{-1}H^T\mathbf{z} \tag{2.28}$$

We can confirm that the above value of $\hat{\mathbf{x}}$ produces the minimum value of the cost function (2.22) by differentiating Eq. (2.25) once more that results in $2H^TH$ which is positive definite.

2.1.8 Linearization of Non-Linear Equations

The non-linear differential equations of navigation must be linearized in order to be usable by linear estimation methods such as Kalman filtering. The non-linear system is transformed to a linear system whose states are the deviations from the nominal value of the non-linear system. This provides the estimates of the errors in the states which are added to the estimated state.

Suppose we have a non-linear differential equation

$$\dot{\mathbf{x}}=f(\mathbf{x},t) \tag{2.29}$$

and that we know the nominal solution to this equation is $\tilde{\mathbf{x}}$ and we let $\delta\mathbf{x}$ be the error in the nominal solution, then the new estimated value can be written as

$$\mathbf{x}=\tilde{\mathbf{x}}+\delta\mathbf{x} \tag{2.30}$$

The time derivative of the above equation provides

$$\dot{\mathbf{x}}=\dot{\tilde{\mathbf{x}}}+\delta\dot{\mathbf{x}} \tag{2.31}$$

Substituting the above value of $\dot{\mathbf{x}}$ in the original Eq. (2.29) gives

$$\dot{\tilde{\mathbf{x}}} + \delta\dot{\mathbf{x}} = f(\tilde{\mathbf{x}} + \delta\mathbf{x}, t) \tag{2.32}$$

Applying the Taylor series expansion to the right-hand side about the nominal value $\tilde{\mathbf{x}}$ yields

$$f(\tilde{\mathbf{x}} + \delta\mathbf{x}, t) = f(\tilde{\mathbf{x}}, t) + \left.\frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\tilde{\mathbf{x}}} \delta\mathbf{x} + HOT \tag{2.33}$$

where the *HOT* refers to the higher order terms that have not been considered. Substituting Eq. (2.32) for the left-hand side gives

$$\dot{\tilde{\mathbf{x}}} + \delta\dot{\mathbf{x}} \approx f(\tilde{\mathbf{x}}, t) + \left.\frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\tilde{\mathbf{x}}} \delta\mathbf{x} \tag{2.34}$$

and since $\tilde{\mathbf{x}}$ also satisfies Eq. (2.29)

$$\dot{\tilde{\mathbf{x}}} = f(\tilde{\mathbf{x}}, t) \tag{2.35}$$

substituting this in Eq. (2.34) gives

$$\dot{\tilde{\mathbf{x}}} + \delta\dot{\mathbf{x}} \approx \dot{\tilde{\mathbf{x}}} + \left.\frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\tilde{\mathbf{x}}} \delta\mathbf{x} \tag{2.36}$$

The linear differential equations whose states are the errors in the original states is give as

$$\delta\dot{\mathbf{x}} \approx \left.\frac{\partial f(\mathbf{x}, t)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\tilde{\mathbf{x}}} \delta\mathbf{x} \tag{2.37}$$

After solving this, we get the estimated errors that are added to the estimated state in order to get the new estimate of the state.

2.2 Coordinate Frames

Coordinate frames are used to express the position of a point in relation to some reference. Some useful coordinate frames relevant to navigation and their mutual transformations are discussed next.

2.2.1 Earth-Centered Inertial Frame

An inertial frame is defined to be either stationary in space or moving at constant velocity (i.e. no acceleration). All inertial sensors produce measurements relative to an inertial frame resolved along the instrument's sensitive axis. Furthermore, we require an inertial frame for the calculation of a satellite's position and velocity in its orbit around the Earth. The frame of choice for near-Earth environments is the Earth-centered inertial (ECI) frame. This is shown in Fig. 2.1 and defined[1] by (Grewal et al. 2007; Farrell 1998) as

a. The origin is at the center of mass of the Earth.
b. The z-axis is along axis of the Earth's rotation through the conventional terrestrial pole (CTP).
c. The x-axis is in the equatorial plane pointing towards the vernal equinox.[2]
d. The y-axis completes a right-handed system.

In Fig. 2.1, the axes of the ECI frame are depicted with superscript i as x^i, y^i, z^i, and in this book the ECI frame will be referred to as the i-frame.

2.2.2 Earth-Centered Earth-Fixed Frame

This frame is similar to the i-frame because it shares the same origin and z-axis as the i-frame, but it rotates along with the Earth (hence the name *Earth-fixed*). It is depicted in Fig. 2.1 along with the i-frame and can be defined as

a. The origin is at the center of mass of the Earth.
b. The z-axis is through the CTP.
c. The x-axis passes through the intersection of the equatorial plane and the reference meridian (i.e. the Greenwich meridian).
d. The y-axis completes the right-hand coordinate system in the equatorial plane.

In Fig. 2.1 the axes of the Earth-Centered Earth-Fixed Frame (ECEF) are shown as X^e, Y^e, Z^e and $(t - t_0)$ represents the elapsed time since reference epoch t_0. The term ω_{ie}^e represents the Earth's rotation rate with respect to the inertial frame resolved in the ECEF frame. In this book the ECEF frame will be referred to as the e-frame.

2.2.3 Local-Level Frame

A local-level frame (LLF) serves to represent a vehicle's attitude and velocity when on or near the surface of the Earth. This frame is also known as the local geodetic or navigation frame. A commonly used LLF is defined as follows

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[1] Strictly speaking this definition does not satisfy the requirement defined earlier for an inertial frame because, in accordance with Kepler's second law of planetary motion, the Earth does not orbit around the sun at a fixed speed; however, for short periods of time it is satisfactory.

[2] The vernal equinox is the direction of intersection of the equatorial plane of the Earth with the ecliptic (the plane of Earth's orbit around the sun).

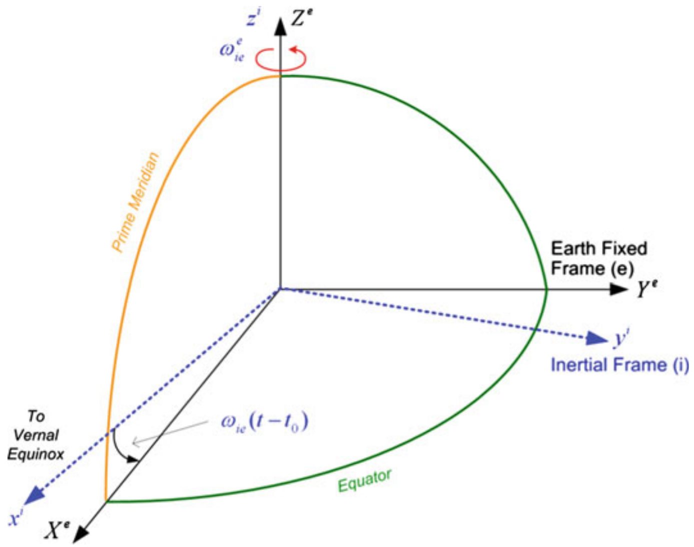


Fig. 2.1 An illustration of the ECI and ECEF coordinate frames

a. The origin coincides with the center of the sensor frame (origin of inertial sensor triad).
b. The y-axis points to true north.
c. The x-axis points to east.
d. The z-axis completes the right-handed coordinate systems by pointing up, perpendicular to reference ellipsoid.

This frame is referred to as ENU since its axes are aligned with the east, north and up directions. This frame is shown in Fig. 2.2. There is another commonly used LLF that differs from the ENU only in that the z axis completes a left-handed coordinate system and therefore points downwards, perpendicular to the reference ellipsoid. This is therefore known as the NED (north, east and down) frame. This book will use the ENU convention, and the LLF frame will be referred to as the l-frame.

2.2.4 Wander Frame

In the l-frame the y-axis always points towards true north, so higher rotation rates about the z-axis are required in order to maintain the orientation of the l-frame in the polar regions (higher latitudes) than near the equator (lower latitudes). As is apparent in Fig. 2.3b, the l-frame must rotate at higher rates to maintain its orientation when moving towards the pole, reaching its maximum when it crosses the north pole. This rate can even become infinite (a singularity condition) if the l-frame passes directly over the pole. The wander frame avoids higher rotation rates and singularity problems. Instead of always pointing northward, this rotates

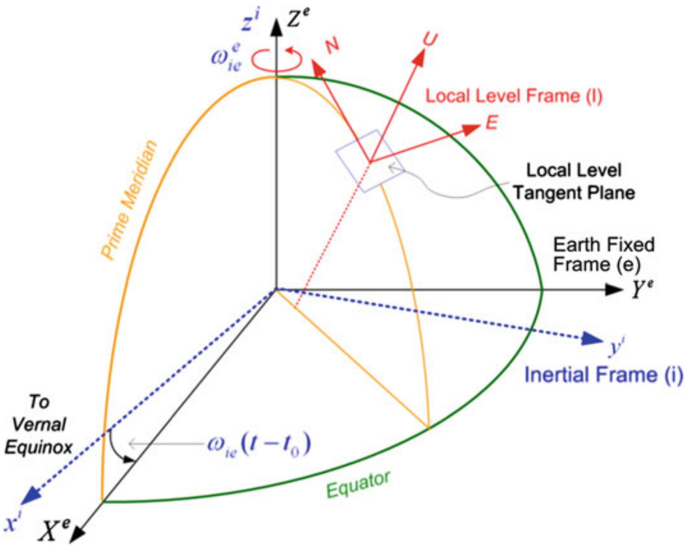


Fig. 2.2 The local-level ENU reference frame in relation to the ECI and ECEF frames

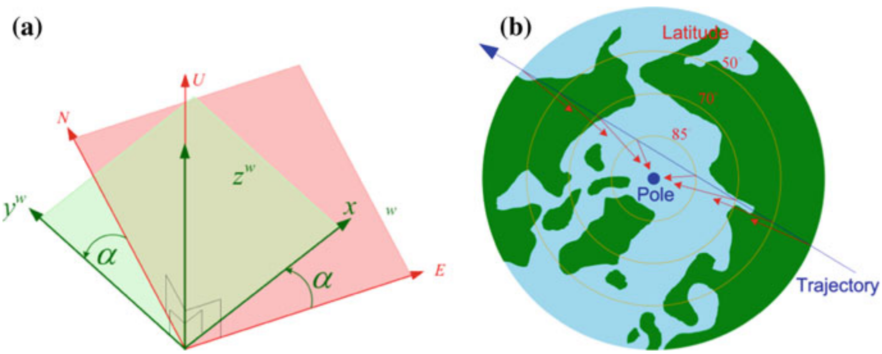


Fig. 2.3 **a** The wander frame shown with respect to the local-level frame. **b** Rotation of the y-axis of the local-level frame (shown with *red/dark arrows*) for a near polar crossing trajectory at various latitudes

about the z-axis with respect to the l-frame. The angle between the y-axis of the wander frame and north is known as the wander angle α. The rotation rate of this angle is given as

$$\dot{\alpha} = -\dot{\lambda} \sin \varphi \tag{2.38}$$

The wander frame (in relation to the l-frame) is shown in Fig. 2.3a, and is defined as

a. The origin coincides with the center of the sensor frame (origin of inertial sensor triad).

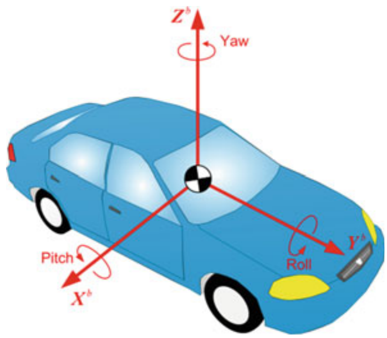


Fig. 2.4 The body frame of a moving platform

b. The z-axis is orthogonal to the reference ellipsoid pointing upward.
c. The y-axis rotates by an angle α anticlockwise from north.
d. The x-axis is orthogonal to the y and z axes and forms a right-handed coordinate frame.

In this book the wander frame is referred to as the w-frame.

2.2.5 Computational Frame

For the ensuing discussion, the computational frame is defined to be any reference frame used in the implementation of the equations of motion. It can be any of the abovementioned coordinate frames, and is referred to as the k-frame.

2.2.6 Body Frame

In most applications, the sensitive axes of the accelerometer sensors are made to coincide with the axes of the moving platform in which the sensors are mounted. These axes are usually known as the body frame.

The body frame used in this book is shown in Fig. 2.4, and is defined as

a. The origin usually coincides with the center of gravity of the vehicle (this simplifies derivation of kinematic equations).
b. The y-axis points towards the forward direction. It is also called the roll axis as the roll angle is defined around this axis using the right-hand rule.
c. The x-axis points towards the transverse direction. It is also called the pitch axis, as the pitch angle corresponds to rotations around this axis using the right-hand rule.
d. The z-axis points towards the vertical direction completing a right-handed coordinate system. It is also called the yaw axis as the yaw angle corresponds to rotations around this axis using the right-hand rule.

In this book the body frame is referred to as b-frame.

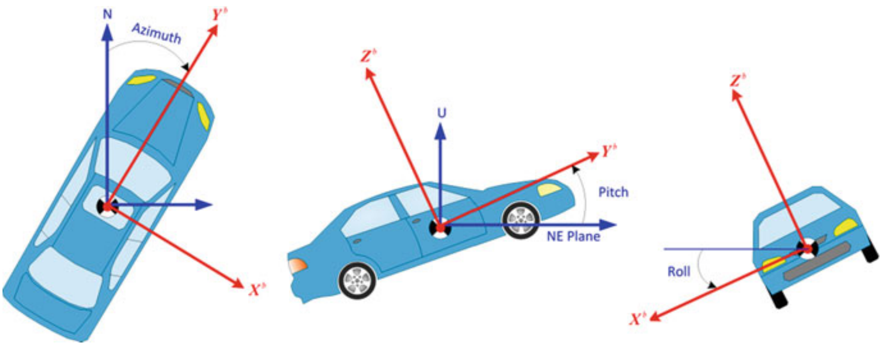


Fig. 2.5 A depiction of a vehicle's azimuth, pitch and roll angles. The body axes are shown in *red*

2.2.6.1 Vehicle Attitude Description

Apart from a vehicle's position, we are also interested in its orientation in order to describe its heading and tilt angles. This involves specifying its rotation about the vertical (z), transversal (x) and forward (y) axes of the b-frame with respect to the l-frame. In general, the rotation angles about the axes of the b-frame are called the Euler angles. For the purpose of this book, the following convention is applied to vehicle attitude angles (Fig. 2.5)

a. **Azimuth** (**or yaw**): Azimuth is the deviation of the vehicle's forward (y) axis from north, measured clockwise in the E-N plane. The yaw angle is similar, but is measured counter clockwise from north. In this book, the azimuth angle is denoted by '*A*' and the yaw angle by '*y*'. Owing to this definition, the vertical axis of the b-frame is also known as the yaw axis (Fig. 2.4).
b. **Pitch**: This is the angle that the forward (y) axis of the b-frame makes with the E-N plane (i.e. local horizontal) owing to a rotation around its transversal (x) axis. This axis is also called the pitch axis, the pitch angle is denoted by '*p*' and follows the right-hand rule (Fig. 2.5).
c. **Roll**: This is the rotation of the b-frame about its forward (y) axis, so the forward axis is also called the roll axis and the roll angle is denoted by '*r*' and follows the right-hand rule.

2.2.7 Orbital Coordinate System

This is a system of coordinates with Keplerian elements to locate a satellite in inertial space. It is defined as follows

a. The origin is located at the focus of an elliptical orbit that coincides with the center of the mass of the Earth.

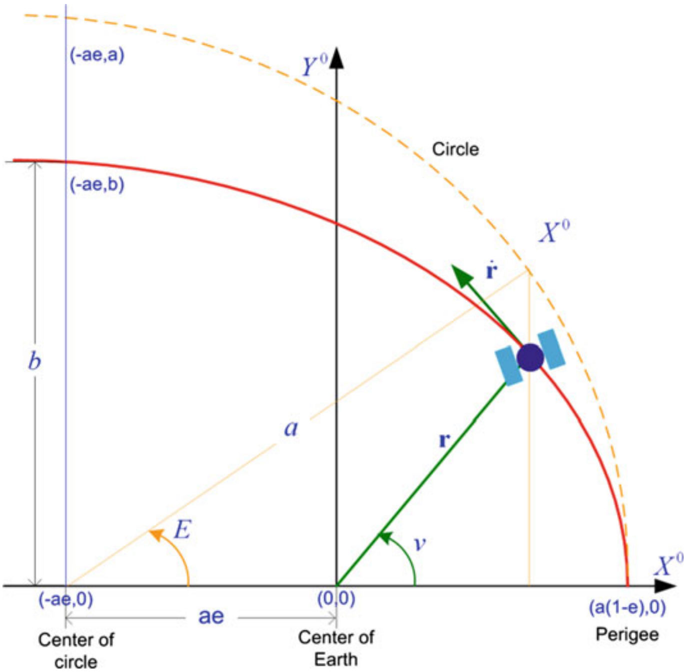


Fig. 2.6 The orbital coordinate system for a satellite

b. The y-axis points towards the descending node, parallel to the minor axis of the orbital ellipse.
c. The x-axis points to the perigee (the point in the orbit nearest the Earth's center) and along the major axis of the elliptical orbit of the satellite.
d. The z-axis is orthogonal to the orbital plane.

The orbital coordinate system is illustrated in Fig. 2.6. It is mentioned here to complete the discussion of the frames used in navigation (it will be discussed in greater detail in Chap. 3).

2.3 Coordinate Transformations

The techniques for transforming a vector from one coordinate frame into another can use direction cosines, rotation (Euler) angles or quaternions. They all involve a rotation matrix which is called either the transformation matrix or the direction cosine matrices (DCM), and is represented as R_k^l where the subscript represents the frame from which the vector originates and the superscript is the target frame. For example, a vector $\mathbf{r}^k$ in a coordinate frame k can be represented by another vector $\mathbf{r}^l$ in a coordinate frame l by applying a rotation matrix R_k^l as follows

$$\mathbf{r}^l = R_k^l \mathbf{r}^k \tag{2.39}$$

If Euler angles are used, these readily yield the elementary matrices required to construct the DCM.

2.3.1 Euler Angles and Elementary Rotational Matrices

A transformation between two coordinate frames can be accomplished by carrying out a rotation about each of the three axes. For example, a transformation from the reference frame a to the new coordinate frame b involves first making a rotation of angle γ about the z-axis, then a rotation of an angle β about the new x-axis, and finally a rotation of an angle α about the new y-axis. In these rotations, α, β and γ are the Euler angles.

To transform a vector $\mathbf{r}^a = [x^a, y^a, z^a]$ from frame a to frame d where the two frames are orientated differently in space, we align frame a with frame d using the three rotations specified above, each applying a suitable direction cosine matrix. The individual matrices can be obtained by considering each rotation, one by one.

First we consider the x-y plane of frame a in which the projection of vector r (represented by r_1) makes an angle θ_1 with the x-axis. We therefore rotate frame a around its z-axis through an angle γ to obtain the intermediate frame b, as illustrated in Fig. 2.7.

According to this figure, the new coordinates are represented by x^b, y^b, z^b and can be expressed as

$$x^b = r_1 \cos(\theta_1 - \gamma) \tag{2.40}$$

$$y^b = r_1 \sin(\theta_1 - \gamma) \tag{2.41}$$

Since the rotation was performed around the z-axis, this remains unchanged

$$z^b = z^a \tag{2.42}$$

Using the following trigonometric identities

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \end{aligned} \tag{2.43}$$

Equations (2.40) and (2.41) can be written as

$$x^b = r_1 \cos\theta_1 \cos\gamma + r_1 \sin\theta_1 \sin\gamma \tag{2.44}$$

$$y^b = r_1 \sin\theta_1 \cos\gamma - r_1 \cos\theta_1 \sin\gamma \tag{2.45}$$

The original coordinates of vector $\mathbf{r}_1$ in the x-y plane can be expressed in terms of angle θ_1 as

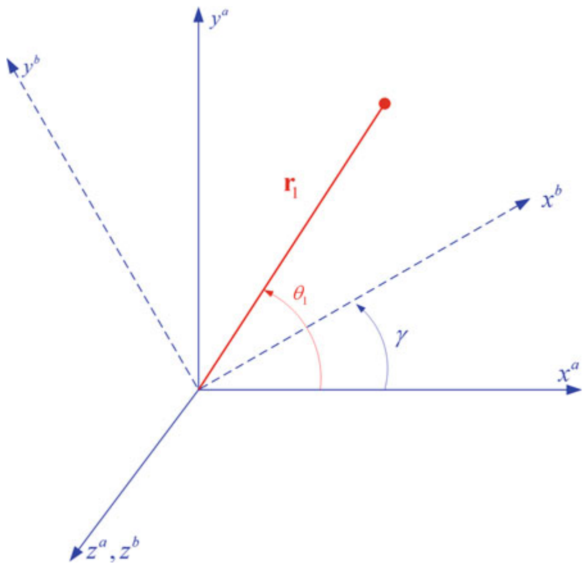


Fig. 2.7 The first rotation of frame ‘a’ about its z^a-axis

$$x^a = r_1 \cos\theta_1 \tag{2.46}$$

$$y^a = r_1 \sin\theta_1 \tag{2.47}$$

Substituting the above values in Eqs. (2.44) and (2.45) produces

$$x^b = x^a \cos\gamma + y^a \sin\gamma \tag{2.48}$$

$$y^b = -x^a \sin\gamma + y^a \cos\gamma \tag{2.49}$$

and we have shown that

$$z^b = z^a \tag{2.50}$$

In matrix form, the three equations above can be written as

$$\begin{bmatrix} x^b \\ y^b \\ z^b \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^a \\ y^a \\ z^a \end{bmatrix} \tag{2.51}$$

$$\begin{bmatrix} x^b \\ y^b \\ z^b \end{bmatrix} = R_a^b \begin{bmatrix} x^a \\ y^a \\ z^a \end{bmatrix} \tag{2.52}$$

where R_a^b is the elementary DCM which transforms the coordinates x^a, y^a, z^a to x^b, y^b, z^b in a frame rotated by an angle γ around the z-axis of frame a.

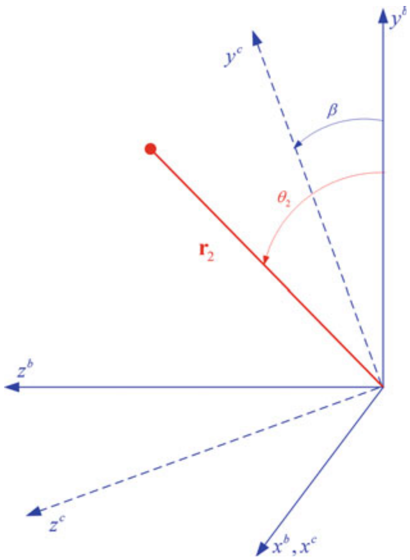


Fig. 2.8 The second rotation of rotated frame 'b' about x^b-axis

For the second rotation, we consider the y-z plane of the new coordinate frame b, and rotate it by an angle β around its x-axis to an intermediate frame c as shown in Fig. 2.8.

In a similar fashion it can be shown that the new coordinates x^c, y^c, z^c can be expressed in terms of x^b, y^b, z^b as follows

$$\begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x^b \\ y^b \\ z^b \end{bmatrix} \tag{2.53}$$

$$\begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = R_b^c \begin{bmatrix} x^b \\ y^b \\ z^b \end{bmatrix} \tag{2.54}$$

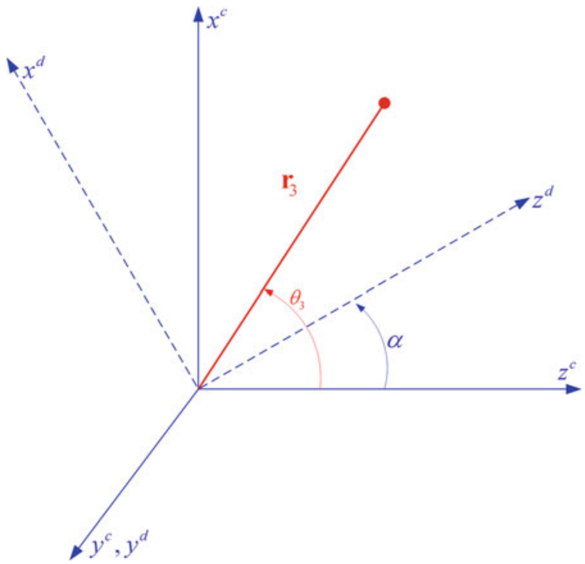
where R_b^c is the elementary DCM which transforms the coordinates x^b, y^b, z^b to x^c, y^c, z^c in a frame rotated by an angle β around the x-axis of frame b.

For the third rotation, we consider the x-z plane of new coordinate frame c, and rotate it by an angle α about its y-axis to align it with coordinate frame d as shown in Fig. 2.9.

The final coordinates x^d, y^d, z^d can be expressed in terms of x^c, y^c, z^c as follows

$$\begin{bmatrix} x^d \\ y^d \\ z^d \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} \tag{2.55}$$

Fig. 2.9 The third rotation of rotated frame ‘c’ about y^c-axis



$$\begin{bmatrix} x^d \\ y^d \\ z^d \end{bmatrix} = R_c^d \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} \tag{2.56}$$

where R_c^d is the elementary DCM which transforms the coordinates x^c, y^c, z^c to x^d, y^d, z^d in the final desired frame d rotated by an angle α around the y-axis of frame c.

We can combine all three rotations by multiplying the cosine matrices into a single transformation matrix as

$$R_a^d = R_c^d R_b^c R_a^b \tag{2.57}$$

The final DCM for these particular set of rotations can be given as

$$R_a^d = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.58}$$

$$R_a^d = \begin{bmatrix} \cos\alpha\cos\gamma - \sin\beta\sin\alpha\sin\gamma & \cos\alpha\sin\gamma + \cos\gamma\sin\beta\sin\alpha & -\cos\beta\sin\alpha \\ -\cos\beta\sin\gamma & \cos\beta\cos\gamma & \sin\beta \\ \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta & \cos\beta\cos\alpha \end{bmatrix} \tag{2.59}$$

The inverse transformation from frame d to a is therefore

$$\begin{aligned} R_d^a = \left(R_a^d\right)^{-1} = \left(R_a^d\right)^T &= \left(R_c^d R_b^c R_a^b\right)^T \\ &= \left(R_a^b\right)^T \left(R_b^c\right)^T \left(R_c^d\right)^T \end{aligned} \tag{2.60}$$

It should be noted that the final transformation matrix is contingent upon the order of the applied rotations, as is evident from the fact that $R_b^c R_a^b \neq R_a^b R_b^c$. The order of rotations is dictated by the specific application. We will see in Sect. 2.3.6 that a different order of rotation is required and the elementary matrices are multiplied in a different order to yield a different final transformation matrix.

For small values of α, β and γ we can use the following approximations

$$\cos\theta \approx 1\,,\ \sin\theta \approx \theta \tag{2.61}$$

Using these approximations and ignoring the product of the small angles, we can reduce the DCM to

$$\begin{aligned} R_a^d &\approx \begin{bmatrix} 1 & \gamma & -\alpha \\ -\gamma & 1 & \beta \\ \alpha & -\beta & 1 \end{bmatrix} \\ R_a^d &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\gamma & \alpha \\ \gamma & 0 & -\beta \\ -\alpha & \beta & 0 \end{bmatrix} \\ R_a^d &= I - \Psi \end{aligned} \tag{2.62}$$

where Ψ is the skew-symmetric matrix for the small Euler angles. For the small-angle approximation, the order of rotation is no longer important since in all cases the final result will always be the matrix of the Eq. (2.62). Similarly, it can be verified that

$$\begin{aligned} R_d^a &\approx \begin{bmatrix} 1 & \gamma & -\alpha \\ -\gamma & 1 & \beta \\ \alpha & -\beta & 1 \end{bmatrix}^T \\ R_d^a &= I - \Psi^T \end{aligned} \tag{2.63}$$

2.3.2 Transformation Between ECI and ECEF

The angular velocity vector between the i-frame and the e-frame as a result of the rotation of the Earth is

$$\boldsymbol{\omega}_{ie}^e = (0, 0, \omega_e)^T \tag{2.64}$$

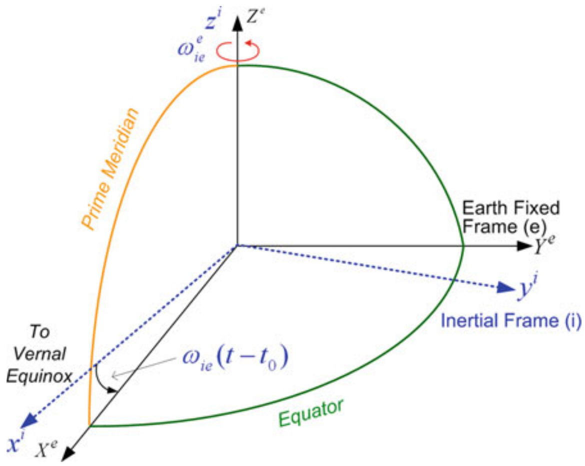


Fig. 2.10 Transformation between the e-frame and the i-frame

where ω_e denotes the magnitude of the Earth's rotation rate. The transformation from the i-frame to the e-frame is a simple rotation of the i-frame about the z-axis by an angle $\omega_e t$ where t is the time since the reference epoch (Fig. 2.10). The rotation matrix corresponds to the elementary matrix R_a^b, and when denoted R_i^e which can be expressed as

$$R_i^e = \begin{bmatrix} \cos\omega_e t & \sin\omega_e t & 0 \\ -\sin\omega_e t & \cos\omega_e t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.65}$$

Transformation from the e-frame to the i-frame can be achieved through R_e^i, the inverse of R_i^e. Since rotation matrices are orthogonal

$$R_e^i = \left(R_i^e\right)^{-1} = \left(R_i^e\right)^T \tag{2.66}$$

2.3.3 Transformation Between LLF and ECEF

From Fig. 2.11 it can be observed that to align the l-frame with the e-frame, the l-frame must be rotated by $\varphi - 90$ degrees around its x-axis (east direction) and then by $-90 - \lambda$ degrees about its z-axis (up direction).

For the definition of elementary direction cosine matrices, the transformation from the l-frame to the e-frame is

$$R_l^e = R_a^b(-\lambda - 90)R_b^c(\varphi - 90) \tag{2.67}$$

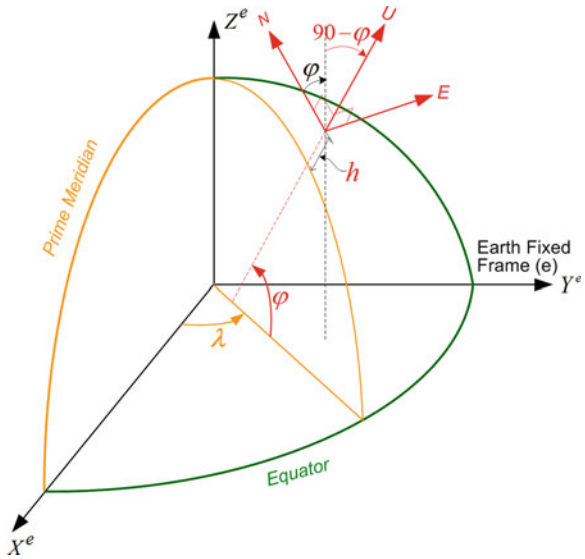


Fig. 2.11 The LLF in relation to the ECEF frame

$$R_l^e = \begin{bmatrix} \cos(-\lambda - 90) & \sin(-\lambda - 90) & 0 \\ -\sin(-\lambda - 90) & \cos(-\lambda - 90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi - 90) & \sin(\varphi - 90) \\ 0 & -\sin(\varphi - 90) & \cos(\varphi - 90) \end{bmatrix} \tag{2.68}$$

$$R_l^e = \begin{bmatrix} -\sin\lambda & -\cos\lambda & 0 \\ \cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\varphi & -\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix} \tag{2.69}$$

$$R_l^e = \begin{bmatrix} -\sin\lambda & -\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ \cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix} \tag{2.70}$$

The reverse transformation is

$$R_e^l = \left(R_l^e\right)^{-1} = \left(R_l^e\right)^T \tag{2.71}$$

2.3.4 Transformation Between LLF and Wander Frame

The wander frame has a rotation about the z-axis of the l-frame by a wander angle α, as depicted in Fig. 2.12. Thus the transformation matrix from the w-frame frame to the l-frame corresponds to the elementary matrix R_a^b with an angle $-\alpha$, and is expressed as

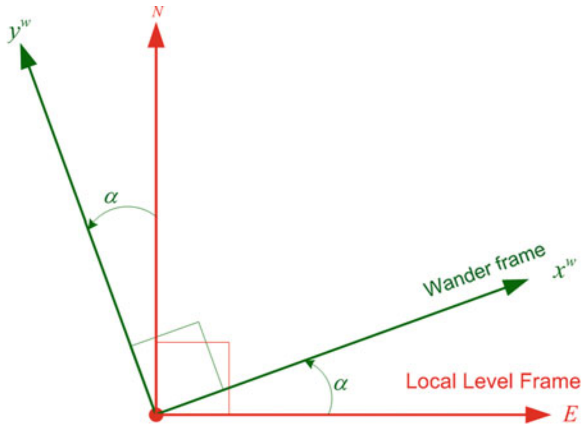


Fig. 2.12 The relationship between the l-frame and the w-frame (the third axes of these the frames are not shown because they coincide and point out of the page towards the reader)

$$R_w^l = \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ -\sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.72}$$

$$R_w^l = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.73}$$

and

$$R_l^w = \left(R_w^l\right)^{-1} = \left(R_w^l\right)^T \tag{2.74}$$

2.3.5 Transformation Between ECEF and Wander Frame

This transformation is obtained by first going from the w-frame to the l-frame and then from the l-frame to the e-frame

$$R_w^e = R_l^e R_w^l \tag{2.75}$$

$$R_w^e = \begin{bmatrix} -\sin\lambda & -\sin\varphi\cos\lambda & \cos\varphi\cos\lambda \\ \cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi\sin\lambda \\ 0 & \cos\varphi & \sin\varphi \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.76}$$

$$R_w^e = \begin{bmatrix} -\sin\lambda\cos\alpha - \cos\lambda\sin\varphi\sin\alpha & \sin\lambda\sin\alpha - \cos\lambda\sin\varphi\cos\alpha & \cos\lambda\cos\varphi \\ \cos\lambda\cos\alpha - \sin\lambda\sin\varphi\sin\alpha & -\cos\lambda\sin\alpha - \sin\lambda\sin\varphi\cos\alpha & \sin\lambda\cos\varphi \\ \cos\varphi\sin\alpha & \cos\varphi\cos\alpha & \sin\varphi \end{bmatrix} \tag{2.77}$$

The inverse is

$$R_e^w = \left(R_w^e\right)^{-1} = \left(R_w^e\right)^T \tag{2.78}$$

2.3.6 Transformation Between Body Frame and LLF

One of the important direction cosine matrices is R_b^l, which transforms a vector from the b-frame to the l-frame, a requirement during the mechanization process. This is expressed in terms of yaw, pitch and roll Euler angles. According to the definitions of these specific angles and the elementary direction cosine matrices, R_b^l can be expressed as

$$\begin{aligned} R_b^l &= \left(R_l^b\right)^{-1} = \left(R_l^b\right)^T = \left(R_c^d R_b^c R_a^b\right)^T \\ &= \left(R_a^b\right)^T \left(R_b^c\right)^T \left(R_c^d\right)^T \end{aligned} \tag{2.79}$$

Substituting the elementary matrices into this equation gives

$$R_b^l = \begin{bmatrix} \cos y & \sin y & 0 \\ -\sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos p & \sin p \\ 0 & -\sin p & \cos p \end{bmatrix}^T \begin{bmatrix} \cos r & 0 & -\sin r \\ 0 & 1 & 0 \\ \sin r & 0 & \cos r \end{bmatrix}^T \tag{2.80}$$

$$R_b^l = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos p & -\sin p \\ 0 & \sin p & \cos p \end{bmatrix} \begin{bmatrix} \cos r & 0 & \sin r \\ 0 & 1 & 0 \\ -\sin r & 0 & \cos r \end{bmatrix} \tag{2.81}$$

$$R_b^l = \begin{bmatrix} \cos y\cos r - \sin y\sin p\sin r & -\sin y\cos p & \cos y\sin r + \sin y\sin p\cos r \\ \sin y\cos r + \cos y\sin p\sin r & \cos y\cos p & \sin y\sin r - \cos y\sin p\cos r \\ -\cos p\sin r & \sin p & \cos p\cos r \end{bmatrix} \tag{2.82}$$

where 'p', 'r' and 'y' are the pitch, roll and yaw angles. With a known R_b^l, these angles can be calculated as

$$p = \tan^{-1}\left\{\frac{R_b^l(3,2)}{\sqrt{\left[R_b^l(1,2)\right]^2+\left[R_b^l(2,2)\right]^2}}\right\} \tag{2.83}$$

$$y = -\tan^{-1}\left[\frac{R_b^l(1,2)}{R_b^l(2,2)}\right] \tag{2.84}$$

$$r = -\tan^{-1}\left[\frac{R_b^l(3,1)}{R_b^l(3,3)}\right] \tag{2.85}$$

A transformation from the l-frame to the b-frame can be achieved by the inverse rotation matrix, R_b^l, as follows

$$R_l^b = \left(R_b^l\right)^{-1} = \left(R_b^l\right)^T \tag{2.86}$$

2.3.7 Transformation From Body Frame to ECEF and ECI Frame

Two other important transformations are from the b-frame to the e-frame and the i-frames. Their rotation matrices can be computed from those already defined as follows.

For the body frame to the e-frame

$$R_b^e = R_l^e R_b^l \tag{2.87}$$

For the body frame to the i-frame

$$R_b^i = R_e^i R_b^e \tag{2.88}$$

Their inverses are

$$R_e^b = \left(R_b^e\right)^{-1} = \left(R_b^e\right)^T \tag{2.89}$$

$$R_i^b = \left(R_b^i\right)^{-1} = \left(R_b^i\right)^T \tag{2.90}$$

2.3.8 Time Derivative of the Transformation Matrix

If a coordinate reference frame k rotates with angular velocity $\boldsymbol{\omega}$ relative to another frame m, the transformation matrix between the two is composed of a set of time variable functions. The time rate of change of the transformation matrix $\dot{R}_k^m$ can be described using a set of differential equations. The frame in which the time differentiation occurs is usually identified by the superscript of the variable.

At time t, the two frames m and k are related by a DCM $R_k^m(t)$. After a short time δt, frame k rotates to a new orientation and the new DCM at time $t+\delta t$ is $R_k^m(t+\delta t)$. The time derivative of the DCM R_k^m is therefore

$$\begin{aligned}\dot{R}_k^m &= \lim_{\delta t\to 0}\frac{\delta R_k^m}{\delta t}\\ \dot{R}_k^m &= \lim_{\delta t\to 0}\frac{R_k^m(t+\delta t)-R_k^m(t)}{\delta t}\end{aligned} \tag{2.91}$$

The transformation at time $t+\delta t$ is the outcome of the transformation up to time t followed by the small change of the m frame that occurs during the brief interval δt. Hence $R_k^m(t+\delta t)$ can be written as the product of two matrices

$$R_k^m(t+\delta t) = \delta R^m R_k^m(t) \tag{2.92}$$

From Eq. (2.62), the small angle transformation can be given as

$$\delta R^m = I - \Psi^m \tag{2.93}$$

Substituting Eq. (2.93) into (2.92) gives

$$R_k^m(t+\delta t) = (I-\Psi^m)R_k^m(t) \tag{2.94}$$

and substituting this back into Eq. (2.91) produces

$$\begin{aligned}\dot{R}_k^m &= \lim_{\delta t\to 0}\frac{(I-\Psi^m)R_k^m(t)-R_k^m(t)}{\delta t}\\ \dot{R}_k^m &= \lim_{\delta t\to 0}\frac{(I-\Psi^m-I)R_k^m(t)}{\delta t}\\ \dot{R}_k^m &= \lim_{\delta t\to 0}\frac{-\Psi^m R_k^m(t)}{\delta t}\\ \dot{R}_k^m &= -\left(\lim_{\delta t\to 0}\frac{\Psi^m}{\delta t}\right)R_k^m(t)\end{aligned} \tag{2.95}$$

When $\delta t\to 0$, $\Psi^m/\delta t$ is the skew-symmetric form of the angular velocity vector of the m frame with respect to the k frame during the time increment δt. Due to the limiting process, the angular velocity can also be referenced to the k frame

$$\lim_{\delta t\to 0}\frac{\Psi^m}{\delta t} = \Omega_{km}^m \tag{2.96}$$

Substituting Eq. (2.96) into (2.95) gives

$$\dot{R}_k^m = -\Omega_{km}^m R_k^m \tag{2.97}$$

From the relation $\Omega_{km}^m = -\Omega_{mk}^m$, this becomes

$$\dot{R}_k^m = \Omega_{mk}^m R_k^m \tag{2.98}$$

From Eq. (2.19)

$$\Omega_{mk}^{m} = R_{k}^{m}\Omega_{mk}^{k}R_{m}^{k} \tag{2.99}$$

Substituting this into (2.98) gives

$$\dot{R}_{k}^{m} = R_{k}^{m}\Omega_{mk}^{k}R_{m}^{k}R_{k}^{m} \tag{2.100}$$

Finally we get the important equation for the rate of change of the DCM as

$$\dot{R}_{k}^{m} = R_{k}^{m}\Omega_{mk}^{k} \tag{2.101}$$

This implies that the time derivative of the rotation matrix is related to the angular velocity vector $\boldsymbol{\omega}$ of the relative rotation between the two coordinate frames. If we have the initial transformation matrix between the body and inertial frames R_{b}^{i}, then we can update the change of the rotation matrix using gyroscope output Ω_{ib}^{b}.

2.3.9 *Time Derivative of the Position Vector in the Inertial Frame*

For a position vector $\mathbf{r}^{b}$, the transformation of its coordinates from the b-frame to the inertial frame is

$$\mathbf{r}^{i} = R_{b}^{i}\mathbf{r}^{b} \tag{2.102}$$

Differentiating both sides with respect to time leads to

$$\dot{\mathbf{r}}^{i} = \dot{R}_{b}^{i}\mathbf{r}^{b} + R_{b}^{i}\dot{\mathbf{r}}^{b} \tag{2.103}$$

Substituting the value of $\dot{R}_{b}^{i}$ from Eq. (2.101) into (2.103) gives

$$\dot{\mathbf{r}}^{i} = \left(R_{b}^{i}\Omega_{ib}^{b}\right)\mathbf{r}^{b} + R_{b}^{i}\dot{\mathbf{r}}^{b} \tag{2.104}$$

A rearrangement of the terms gives

$$\dot{\mathbf{r}}^{i} = R_{b}^{i}\left(\dot{\mathbf{r}}^{b} + \Omega_{ib}^{b}\mathbf{r}^{b}\right) \tag{2.105}$$

which describes the transformation of the velocity vector from the b-frame to the inertial frame. This is often called the Coriolis equation.

2.3.10 *Time Derivative of the Velocity Vector in the Inertial Frame*

The time derivative of the velocity vector is obtained by differentiating Eq. (2.105) as follows

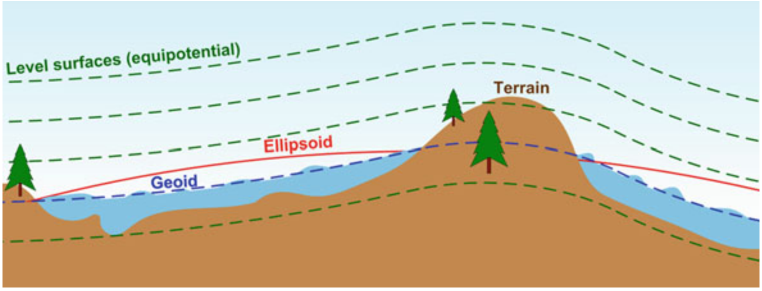


Fig. 2.13 A depiction of various surfaces of the Earth

$$\begin{aligned}\ddot{\mathbf{r}}^i &= \dot{R}_b^i \dot{\mathbf{r}}^b + R_b^i \ddot{\mathbf{r}}^b + \dot{R}_b^i \Omega_{ib}^b \mathbf{r}^b + \left(\dot{\Omega}_{ib}^b \mathbf{r}^b + \Omega_{ib}^b \dot{\mathbf{r}}^b\right) R_b^i \\ \ddot{\mathbf{r}}^i &= \dot{R}_b^i \dot{\mathbf{r}}^b + R_b^i \ddot{\mathbf{r}}^b + \dot{R}_b^i \Omega_{ib}^b \mathbf{r}^b + R_b^i \dot{\Omega}_{ib}^b \mathbf{r}^b + R_b^i \Omega_{ib}^b \dot{\mathbf{r}}^b\end{aligned} \tag{2.106}$$

Substituting the value of $\dot{R}_b^i$ from Eq. (2.101) yields

$$\begin{aligned}\ddot{\mathbf{r}}^i &= R_b^i \Omega_{ib}^b \dot{\mathbf{r}}^b + R_b^i \ddot{\mathbf{r}}^b + R_b^i \Omega_{ib}^b \Omega_{ib}^b \mathbf{r}^b + R_b^i \dot{\Omega}_{ib}^b \mathbf{r}^b + R_b^i \Omega_{ib}^b \dot{\mathbf{r}}^b \\ \ddot{\mathbf{r}}^i &= R_b^i \left(\Omega_{ib}^b \dot{\mathbf{r}}^b + \ddot{\mathbf{r}}^b + \Omega_{ib}^b \Omega_{ib}^b \mathbf{r}^b + \dot{\Omega}_{ib}^b \mathbf{r}^b + \Omega_{ib}^b \dot{\mathbf{r}}^b\right) \\ \ddot{\mathbf{r}}^i &= R_b^i \left(2\Omega_{ib}^b \dot{\mathbf{r}}^b + \ddot{\mathbf{r}}^b + \Omega_{ib}^b \Omega_{ib}^b \mathbf{r}^b + \dot{\Omega}_{ib}^b \mathbf{r}^b\right)\end{aligned} \tag{2.107}$$

and rearrangement gives

$$\ddot{\mathbf{r}}^i = R_b^i \left(\ddot{\mathbf{r}}^b + 2\Omega_{ib}^b \dot{\mathbf{r}}^b + \dot{\Omega}_{ib}^b \mathbf{r}^b + \Omega_{ib}^b \Omega_{ib}^b \mathbf{r}^b\right) \tag{2.108}$$

where

$\ddot{\mathbf{r}}^b$	is the acceleration of the moving object in the b-frame
Ω_{ib}^b	is the angular velocity of the moving object measured by a gyroscope
$2\Omega_{ib}^b \dot{\mathbf{r}}^b$	is the Coriolis acceleration
$\dot{\Omega}_{ib}^b \mathbf{r}^b$	is the tangential acceleration
$\Omega_{ib}^b \Omega_{ib}^b \mathbf{r}^b$	is the centripetal acceleration

2.4 The Geometry of the Earth

Although the Earth is neither a sphere nor a perfect ellipsoid, it is approximated by an ellipsoid for computational convenience. The ellipsoid and various surfaces that are useful for understanding the geometry of the Earth's shape are depicted in Fig. 2.13.

2.4.1 Important Definitions

At this point, it will be useful to describe some of the important definitions which will assist in understanding the ensuing analysis. For further details, the reader is referred to (Titterton and Weston 2005; Vanícek and Krakiwsky 1986).

- **Physical Surface—Terrain**": This is defined as the interface between the solid and fluid masses of the Earth and its atmosphere. It is the actual surface that we walk or float on.
- **Geometric Figure—Geoid**": This is the equipotential surface (surface of constant gravity) best fitting the average sea level in the least squares sense (ignoring tides and other dynamical effects in the oceans). It can be thought of as the idealized mean sea level extended over the land portion of the globe. The geoid is a smooth surface but its shape is irregular and it does not provide the simple analytic expression needed for navigational computations.
- **Reference Ellipsoid—Ellipsoid**": This mathematically defined surface approximates the geoid by an ellipsoid that is made by rotating an ellipse about its minor axis, which is coincident with the mean rotational axis of the Earth. The center of the ellipsoid is coincident with the Earth's center of mass.

The ellipsoid is the most analytically convenient surface to work with for navigational purposes. Its shape is defined by two geometric parameters called the semimajor axis and the semiminor axis. These are typically represented by a and b respectively, as in Fig. 2.14. The geoid height N is the distance along the ellipsoidal normal from the surface of the ellipsoid to the geoid. The orthometric height H is the distance from the geoid to the point of interest. The geodetic height h (also known as altitude) is the sum of the geoid and orthometric heights ($h = H + N$). Various parameter sets have been defined to model the ellipsoid. This book uses the world geodetic system (WGS)-84 whose defining parameters (Torge 1980; Vanícek and Krakiwsky 1986) are

Semimajor axis (equatorial radius) $a = 6,378,137.0\,\text{m}$
Reciprocal flattening $\frac{1}{f} = 298.257223563$
Earth's rotation rate $\omega_e = 7.292115 \times 10^{-5}\,\text{rad/s}$
Gravitational constant $GM = 3.986004418 \times 10^{14}\,\text{m}^3/\text{s}^2$

Other derived parameters of interest are

Flatness $f = \frac{a-b}{a} = 0.00335281$
Semiminor axis $b = a(1-f) = 6356752.3142\,\text{m}$
Eccentricity $e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{f(2-f)} = 0.08181919$

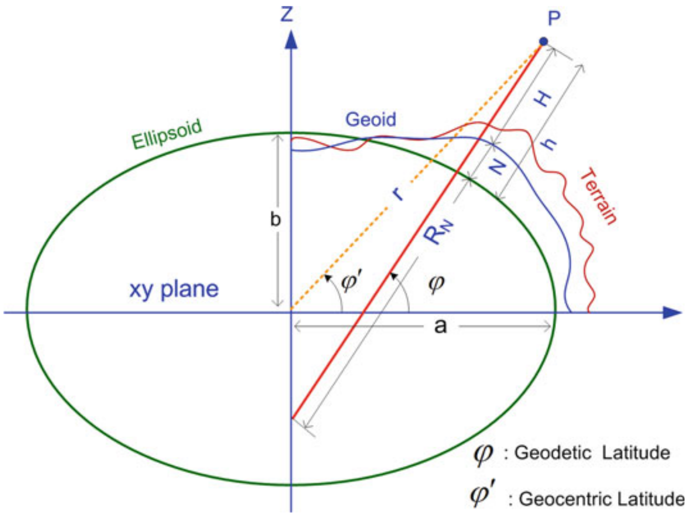


Fig. 2.14 The relationship between various Earth surfaces (highly exaggerated) and a depiction of the ellipsoidal parameters

2.4.2 Normal and Meridian Radii

In navigation two radii of curvature are of particular interest, the normal radius and the meridian radius. These govern the rates at which the latitude and longitude change as a navigating platform moves on or near the surface of the Earth.

The normal radius R_N is defined for the east-west direction, and is also known as the great normal or the radius of curvature of the prime vertical

$$R_N = \frac{a}{\left(1 - e^2 \sin^2 \varphi\right)^{\frac{1}{2}}} \tag{2.109}$$

The meridian radius of curvature is defined for the north-south direction and is the radius of the ellipse

$$R_M = \frac{a(1 - e^2)}{\left(1 - e^2 \sin^2 \varphi\right)^{\frac{3}{2}}} \tag{2.110}$$

A derivation of these radii can be found in Appendix A, and for further insight the reader is referred to (Grewal et al. 2007; Rogers 2007).

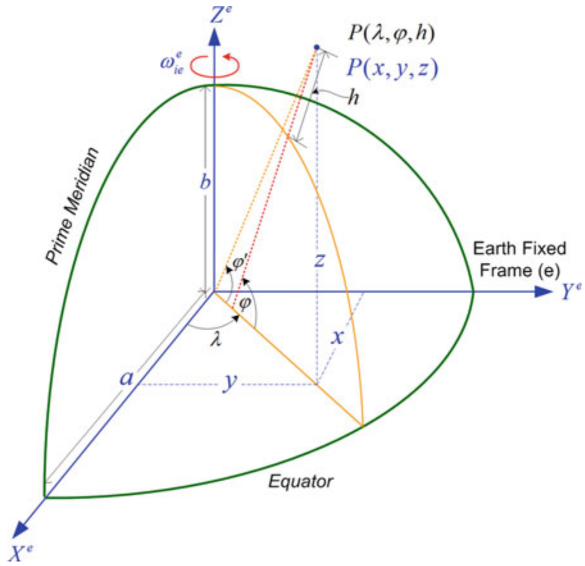


Fig. 2.15 Two types of ECEF coordinates and their interrelationship

2.5 Types of Coordinates in the ECEF Frame

It is important to distinguish between the two common coordinate systems of the e-frame, known as the 'rectangular' and 'geodetic' systems.

2.5.1 Rectangular Coordinates in the ECEF Frame

Rectangular coordinates are like traditional Cartesian coordinates, and represent the position of a point with its x, y and z vector components aligned parallel to the corresponding e-frame axes (Fig. 2.15).

2.5.2 Geodetic Coordinates in the ECEF Frame

Geodetic (also referred to as ellipsoidal or curvilinear) coordinates are defined in a way that is more intuitive for positioning applications on or near the Earth. These coordinates are defined (Farrell 2008) as

a. Latitude (φ) is the angle in the meridian plane from the equatorial plane to the ellipsoidal normal at the point of interest.
b. Longitude (λ) is the angle in the equatorial plane from the prime meridian to the projection of the point of interest onto the equatorial plane.

c. Altitude h is the distance along the ellipsoidal normal between the surface of the ellipsoid and the point of interest.

The two types of e-frame coordinates and their interrelationship are illustrated in Fig. 2.15.

2.5.3 *Conversion From Geodetic to Rectangular Coordinates in the ECEF Frame*

In navigation, it is often necessary to convert from geodetic e-frame coordinates to rectangular e-frame coordinates. The following relationship (see Appendix B for a derivation) accomplishes this

$$\begin{bmatrix} x^e \\ y^e \\ z^e \end{bmatrix} = \begin{bmatrix} (R_N + h)\cos\varphi\cos\lambda \\ (R_N + h)\cos\varphi\sin\lambda \\ \{R_N(1 - e^2) + h\}\sin\varphi \end{bmatrix} \tag{2.111}$$

where

(x^e, y^e, z^e)	are the e-frame rectangular coordinates
R_N	is the normal radius
h	is the ellipsoidal height
λ	is the longitude
φ	is the latitude
e	is the eccentricity.

2.5.4 *Conversion From Rectangular to Geodetic Coordinates in the ECEF Frame*

Converting rectangular to geodetic coordinates is not straightforward, because the analytical solution results in a fourth-order equation. There are approximate closed form solutions but an iterative scheme is usually employed.

2.5.4.1 Closed-Form Algorithm

This section will describe a closed form algorithm to calculate e-frame geodetic coordinates directly from e-frame rectangular coordinates though series expansion (Hofmann-Wellenhof et al. 2008). An alternate method is detailed in Appendix C.

Longitude is

$$\lambda = 2 \arctan \frac{y^e}{x^e + \sqrt{(x^e)^2 + (y^e)^2}} \tag{2.112}$$

latitude is

$$\varphi = \arctan \frac{z^e + (e')^2 b \sin^3 \theta}{p - e^2 a \cos^3 \theta} \tag{2.113}$$

where

$$\theta = \arctan \frac{z^e a}{pb}$$

$$e' = \sqrt{\frac{a^2 - b^2}{b^2}}$$

$$p = \sqrt{(x^e)^2 + (y^e)^2}$$

and height is

$$h = \frac{p}{\cos \varphi} - N \tag{2.114}$$

where

$$N = \frac{a^2}{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}}$$

2.5.4.2 Iterative Algorithm

The iterative algorithm is derived in Appendix D and implemented by taking the following steps

a. Initialize the altitude as

$$h_0 = 0 \tag{2.115}$$

b. Choose an arbitrary value of latitude either from a previous measurement (if one is available) or by using the approximation

$$\varphi_0 = \tan^{-1}\left[\frac{z^e}{P^e(1 - e^2)}\right] \tag{2.116}$$

c. The geodetic longitude is calculated as

$$\lambda = \tan^{-1}\left(\frac{y^e}{x^e}\right) \tag{2.117}$$

d. Starting from $i = 1$, iterate as follows

$$R_{N_i} = \frac{a}{\left(1 - e^2 \sin^2 \varphi_{i-1}\right)^{1/2}} \tag{2.118}$$

$$h_i = \frac{\sqrt{(x^e)^2 + (y^e)^2}}{\cos \varphi_{i-1}} - R_{N_i} \tag{2.119}$$

$$\varphi_i = \tan^{-1}\left\{\frac{z^e}{\sqrt{(x^e)^2 + (y^e)^2}} \cdot \frac{(R_{N_i} + h_i)}{R_{N_i}(1 - e^2) + h_i}\right\} \tag{2.120}$$

e. Compare φ_i, φ_{i-1} and h_i, h_{i-1}; if convergence has been achieved then stop, otherwise repeat step d using the new values.

2.6 Earth Gravity

The gravity field vector is different from the gravitational field vector. Due to the Earth's rotation, the gravity field is used more frequently and is defined as

$$\mathbf{g} = \bar{\mathbf{g}} - \Omega_{ie}\Omega_{ie}\mathbf{r} \tag{2.121}$$

where $\bar{\mathbf{g}}$ is the gravitational vector, Ω_{ie} is the skew-symmetric representation of the Earth's rotation vector $\boldsymbol{\omega}_{ie}$ with respect to the i-frame, and $\mathbf{r}$ is the geocentric position vector. The second term in the above equation denotes the centripetal acceleration due to the rotation of the Earth around its axis. Usually, the gravity vector is given in the l-frame. Because the normal gravity vector on the ellipsoid coincides with the ellipsoidal normal, the east and the north components of the normal gravity vector are zero and only third component is non-zero

$$\mathbf{g}^l = [0 \quad 0 \quad -g]^T \tag{2.122}$$

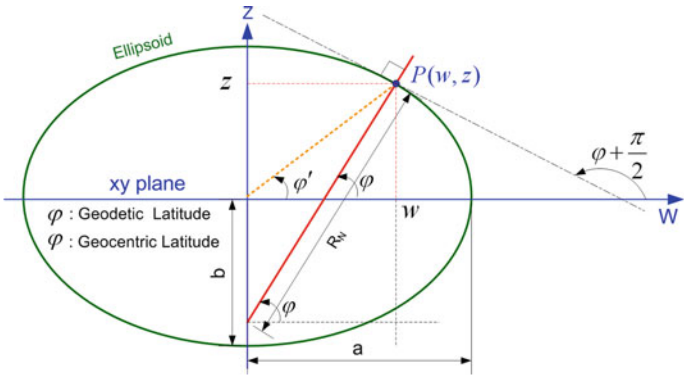


Fig. 2.16 A meridian cross section of the reference ellipsoid containing the projection of the point of interest P

The magnitude of the normal gravity vector over the surface of the ellipsoid can be computed as a function of latitude and height by a closed form expression known as the Somigliana formula (Schwarz and Wei Jan 1999), which is

$$\gamma = a_1\left(1 + a_2 \sin^2 \varphi + a_3 \sin^4 \varphi\right) + \left(a_4 + a_5 \sin^2 \varphi\right)h + a_6 h^2 \tag{2.123}$$

where h is the height above the Earth's surface and the coefficients a_1 through a_6 for the 1980 geographic reference system (GRS) are defined as

$$\begin{array}{ll} a_1 = 9.7803267714 \text{ m/s}^2; & a_4 = -0.0000030876910891/\text{s}^2; \\ a_2 = 0.0052790414; & a_5 = 0.0000000043977311/\text{s}^2; \\ a_3 = 0.0000232718; & a_6 = 0.0000000000007211/\text{ms}^2 \end{array}$$

Appendix A

Derivation of Meridian Radius and Normal Radius

For Earth ellipsoids, every meridian is an ellipse with equatorial radius a (called the semimajor axis) and polar radius b (called the semiminor axis). Figure 2.16 shows a meridian cross section of one such ellipse (Rogers 2007).

This ellipse can be described by the equation

$$\frac{w^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{2.124}$$

and the slope of the tangent to point P can be derived by differentiation

$$\frac{2wdw}{a^2} + \frac{2zdz}{b^2} = 0 \tag{2.125}$$

$$\frac{dz}{dw} = -\frac{b^2 w}{a^2 z} \tag{2.126}$$

An inspection of the Fig. 2.16 shows that the derivative of the curve at point P, which is equal to the slope of the tangent to the curve at that point, is

$$\frac{dz}{dw} = \tan\left(\frac{\pi}{2} + \varphi\right) \tag{2.127}$$

$$\frac{dz}{dw} = \frac{\sin\left(\frac{\pi}{2} + \varphi\right)}{\cos\left(\frac{\pi}{2} + \varphi\right)} = \frac{\cos\varphi}{-\sin\varphi} = -\frac{1}{\tan\varphi} \tag{2.128}$$

Therefore

$$\frac{1}{\tan\varphi} = \frac{b^2 w}{a^2 z} \tag{2.129}$$

From the definition of eccentricity, we have

$$\begin{aligned} e^2 &= 1 - \frac{b^2}{a^2} \\ \frac{b^2}{a^2} &= 1 - e^2 \end{aligned} \tag{2.130}$$

and Eq. (2.129) becomes

$$z = w\left(1 - e^2\right)\tan\varphi \tag{2.131}$$

The ellipse described by Eq. (2.124) gives

$$w^2 = a^2\left(1 - \frac{z^2}{b^2}\right) \tag{2.132}$$

Substituting the value of z from Eq. (2.131) yields

$$\begin{aligned} w^2 &= \left(a^2 - \frac{a^2 w^2 (1-e^2)^2 \tan^2\varphi}{b^2}\right) \\ w^2 &+ \frac{a^2 w^2 (1-e^2)^2 \tan^2\varphi}{b^2} = a^2 \\ w^2 &\left(\frac{b^2 + a^2(1-e^2)^2 \tan^2\varphi}{b^2}\right) = a^2 \end{aligned}$$

$$
\begin{aligned}
w^2 &= \frac{a^2b^2}{b^2 + a^2(1-e^2)^2\tan^2\varphi} \\
w^2 &= \frac{a^2[a^2(1-e^2)]}{a^2(1-e^2) + a^2(1-e^2)^2\tan^2\varphi} \\
w^2 &= \frac{a^2}{1 + (1-e^2)\tan^2\varphi} \\
w^2 &= \frac{a^2}{1 + (1-e^2)\frac{\sin^2\varphi}{\cos^2\varphi}} \\
w^2 &= \frac{a^2\cos^2\varphi}{\cos^2\varphi + (1-e^2)\sin^2\varphi} \\
w^2 &= \frac{a^2\cos^2\varphi}{1 - e^2\sin^2\varphi}
\end{aligned} \tag{2.133}
$$

$$
w = \frac{a\cos\varphi}{\left(1 - e^2\sin^2\varphi\right)^{\frac{1}{2}}} \tag{2.134}
$$

Substituting this expression for w in Eq. (2.131) produces

$$
z = \frac{a(1-e^2)\sin\varphi}{\left(1 - e^2\sin^2\varphi\right)^{\frac{1}{2}}} \tag{2.135}
$$

which will be used later to derive the meridian radius.

It can easily be proved from Fig. 2.16 that

$$
w = R_N\cos\varphi \tag{2.136}
$$

From this and Eq. (2.134) we have the expression for the normal radius, R_N, also known as the radius of curvature in the prime vertical

$$
R_N = \frac{a}{\left(1 - e^2\sin^2\varphi\right)^{\frac{1}{2}}} \tag{2.137}
$$

The radius of curvature of an arc of constant longitude is

$$
R_M = \frac{\left[1 + \left(\frac{dz}{dw}\right)^2\right]^{\frac{3}{2}}}{\pm\frac{d^2z}{dw^2}} \tag{2.138}
$$

The second derivative of Eq. (2.126) provides

$$
\begin{aligned}
\frac{dz}{dw} &= -\frac{b^2 w}{a^2 z} = -\frac{b^2 w}{a^2}\left(b^2 - \frac{b^2 w^2}{a^2}\right)^{-1/2} \\
\frac{d^2 z}{dw^2} &= -\frac{b^2}{a^2}\left(b^2 - \frac{b^2 w^2}{a^2}\right)^{-1/2} - \frac{b^2 w}{a^2}\left(-\frac{1}{2}\right)\left(b^2 - \frac{b^2 w^2}{a^2}\right)^{-3/2}\left(-\frac{b^2}{a^2}\right)2w \\
\frac{d^2 z}{dw^2} &= -\frac{b^2}{a^2 z} - \frac{b^4 w^2}{a^4 z^3} \\
\frac{d^2 z}{dw^2} &= \frac{-b^2 a^2 z^2 - b^4 w^2}{a^4 z^3} = \frac{-b^2 a^2\left(b^2 - \frac{b^2 w^2}{a^2}\right) - b^4 w^2}{a^4 z^3} \\
\frac{d^2 z}{dw^2} &= \frac{-b^4 a^2 + b^4 w^2 - b^4 w^2}{a^4 z^3}
\end{aligned}
$$

which simplifies to

$$\frac{d^2 z}{dw^2} = -\frac{b^4}{a^2 z^3} \tag{2.139}$$

Substituting Eqs. (2.139) and (2.126) into (2.138) yields

$$R_M = \frac{\left[1 + \frac{b^4 w^2}{a^4 z^2}\right]^{\frac{3}{2}}}{\frac{b^4}{a^2 z^3}} \tag{2.140}$$

and since $\frac{b^2}{a^2} = 1 - e^2$

$$R_M = \frac{\left[1 + \frac{\left(1-e^2\right)^2 w^2}{z^2}\right]^{\frac{3}{2}}}{\frac{b^2\left(1-e^2\right)}{z^3}}$$

$$R_M = \frac{\left[\frac{z^2 + \left(1-e^2\right)^2 w^2}{z^2}\right]^{\frac{3}{2}}}{\frac{b^2\left(1-e^2\right)}{z^3}}$$

$$R_M = \frac{\left[z^2 + \left(1 - e^2\right)^2 w^2\right]^{\frac{3}{2}}}{b^2\left(1 - e^2\right)} \tag{2.141}$$

Substituting the value of w from Eq. (2.134) and z from (2.135) gives

$$R_M = \frac{\left[\frac{a^2\left(1-e^2\right)^2\sin^2\varphi}{1-e^2\sin^2\varphi} + (1 - e^2)^2 \frac{a^2\cos^2\varphi}{1-e^2\sin^2\varphi}\right]^{\frac{3}{2}}}{b^2(1 - e^2)}$$

$$R_M = \frac{\left[\frac{a^2\left(1-e^2\right)^2\sin^2\varphi + \left(1-e^2\right)^2 a^2\cos^2\varphi}{1-e^2\sin^2\varphi}\right]^{\frac{3}{2}}}{b^2(1 - e^2)}$$

$$R_M = \frac{\left[a^2(1 - e^2)^2\left(\sin^2\varphi + \cos^2\varphi\right)\right]^{\frac{3}{2}}}{b^2(1 - e^2)\left(1 - e^2\sin^2\varphi\right)^{\frac{3}{2}}}$$

$$R_M = \frac{a^3(1 - e^2)^3}{b^2(1 - e^2)\left(1 - e^2\sin^2\varphi\right)^{\frac{3}{2}}}$$

$$R_M = \frac{a(1 - e^2)^3}{(1 - e^2)(1 - e^2)\left(1 - e^2\sin^2\varphi\right)^{\frac{3}{2}}}$$

leading to the meridian radius of curvature

$$R_M = \frac{a(1-e^2)}{\left(1-e^2\sin^2\varphi\right)^{\frac{3}{2}}} \tag{2.142}$$

Appendix B

Derivation of the Conversion Equations From Geodetic to Rectangular Coordinates in the ECEF Frame

Figure 2.17 shows the relationship between geodetic and rectangular coordinates in the ECEF frame.

It is evident that

$$\mathbf{r}_P = \mathbf{r}_Q + h\mathbf{n} \tag{2.143}$$

where

$$\mathbf{r}_Q = \begin{bmatrix} R_N\cos\varphi\cos\lambda \\ R_N\cos\varphi\sin\lambda \\ R_N\sin\varphi - R_N e^2\sin\varphi \end{bmatrix} = R_N\begin{bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi - R_N e^2\sin\varphi \end{bmatrix} \tag{2.144}$$

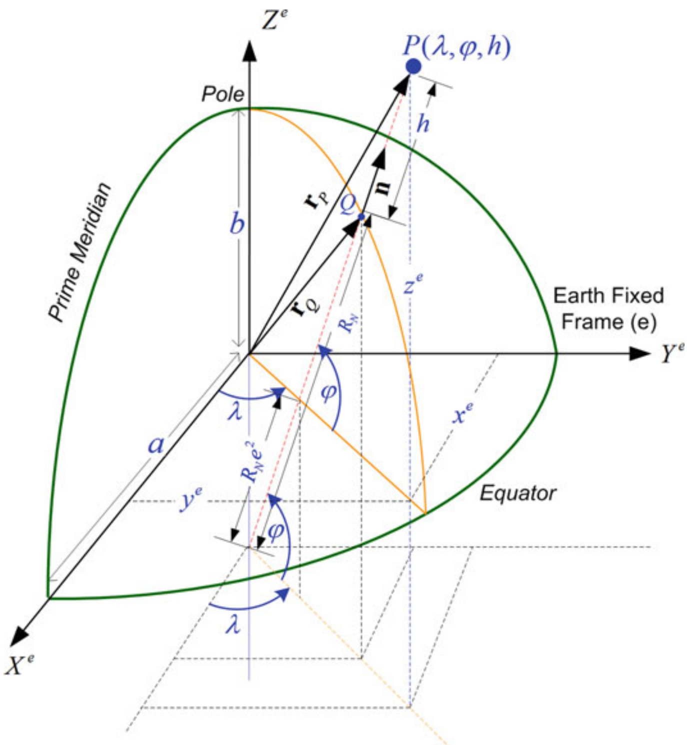


Fig. 2.17 The relationship between geodetic and rectangular coordinates in the ECEF frame

Also, the unit vector along the ellipsoidal normal is

$$\mathbf{n} = \begin{bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{bmatrix} \tag{2.145}$$

Substituting Eqs. (2.144) and (2.145) into (2.143) gives

$$\mathbf{r}_P = R_N \begin{bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi - R_N e^2 \sin\varphi \end{bmatrix} + h \begin{bmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{bmatrix} \tag{2.146}$$

$$\mathbf{r}_P = \begin{bmatrix} (R_N + h)\cos\varphi\cos\lambda \\ (R_N + h)\cos\varphi\sin\lambda \\ \left\{R_N(1 - e^2) + h\right\}\sin\varphi \end{bmatrix} \tag{2.147}$$

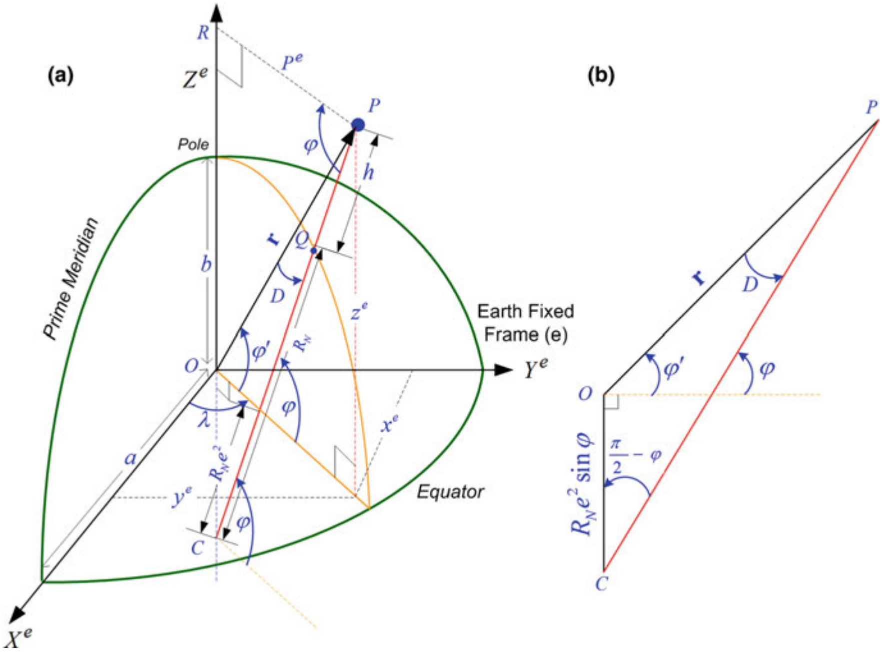


Fig. 2.18 Reference diagram for conversion of rectangular coordinates to geodetic coordinates in the ECEF frame through a closed-form method

$$\begin{bmatrix} x^e \\ y^e \\ z^e \end{bmatrix} = \begin{bmatrix} (R_N + h)\cos\varphi\cos\lambda \\ (R_N + h)\cos\varphi\sin\lambda \\ \{R_N(1 - e^2) + h\}\sin\varphi \end{bmatrix} \quad (2.148)$$

Appendix C

Derivation of Closed Form Equations From Rectangular to Geodetic Coordinates in the ECEF Frame

Here we derive a closed form algorithm which uses a series expansion (Schwarz and Wei Jan 1999) to compute e-frame geodetic coordinates directly from e-frame rectangular coordinates.

From Fig. 2.18a it can is evident that, for given rectangular coordinates, the calculation of geocentric coordinates (λ, φ', r) is simply

$$\mathbf{r} = \sqrt{(x^e)^2 + (y^e)^2 + (z^e)^2} \quad (2.149)$$

$$\lambda = \tan^{-1}\left(\frac{y^e}{x^e}\right) \tag{2.150}$$

$$\varphi' = \sin^{-1}\left(\frac{z^e}{r}\right) \tag{2.151}$$

From the triangle POC (elaborated in Fig. 2.18b) it is apparent that the difference between the geocentric latitude φ' and the geodetic latitude φ is

$$\begin{aligned} &D + \varphi' + \frac{\pi}{2} + \left(\frac{\pi}{2} - \varphi\right) = \pi \\ &D + \varphi' + \pi - \varphi = \pi \\ &D = \varphi - \varphi' \end{aligned} \tag{2.152}$$

where D is the angle between the ellipsoidal normal at Q and the normal to the sphere at P.

Applying the law of sines to the triangle in Fig. 2.18b provides

$$\begin{aligned} &\frac{\sin D}{R_N e^2 \sin\varphi} = \frac{\sin\left(\frac{\pi}{2} - \varphi\right)}{r} \\ &\frac{\sin D}{R_N e^2 \sin\varphi} = \frac{\cos\varphi}{r} \\ &\sin D = \frac{R_N e^2 \sin\varphi\cos\varphi}{r} \\ &D = \sin^{-1}\left(\frac{R_N e^2 \sin\varphi\cos\varphi}{r}\right) \end{aligned} \tag{2.153}$$

$$D = \sin^{-1}\left(\frac{R_N e^2 \frac{1}{2}\sin 2\varphi}{r}\right) \tag{2.154}$$

Substituting the definition of the normal radius R_N given in Eq. (2.137)

$$R_N = \frac{a}{\left(1 - e^2 \sin^2\varphi\right)^{1/2}} \tag{2.155}$$

into Eq. (2.154) gives

$$D = \sin^{-1}\left(\frac{\frac{a}{\left(1-e^2\sin^2\varphi\right)^{1/2}} e^2 \frac{1}{2}\sin 2\varphi}{r}\right) \tag{2.156}$$

$$D = \sin^{-1}\left(\frac{k\sin 2\varphi}{\left(1 - e^2\sin^2\varphi\right)^{1/2}}\right) \tag{2.157}$$

where

$$k = \frac{e^2 a}{2} \tag{2.158}$$

To achieve a first approximation, if $\varphi = \varphi'$ then D can be computed by using the geocentric latitude

$$D_c = \sin^{-1}\left(\frac{k \sin 2\varphi'}{\left(1 - e^2 \sin^2 \varphi'\right)^{1/2}}\right) \tag{2.159}$$

Expanding Eq. (2.157) for this approximation, D_c, gives

$$\begin{aligned} D = D(\varphi = \varphi') + D'(\varphi)|_{\varphi=\varphi'}(\varphi - \varphi') + D''(\varphi)|_{\varphi=\varphi'}\frac{(\varphi - \varphi')^2}{2!} \\ + D'''(\varphi)|_{\varphi=\varphi'}\frac{(\varphi - \varphi')^3}{3!} + \ldots \end{aligned} \tag{2.160}$$

$$D = D_c + D'(\varphi')D + \frac{1}{2!}D''(\varphi')D^2 + \frac{1}{3!}D'''(\varphi')D^3 + \ldots \tag{2.161}$$

For a very small value of D (less than 0.005), it can be assumed that $R_N \simeq r$, and hence

$$D = \frac{e^2}{2}\sin 2\varphi \tag{2.162}$$

So for this situation, let $k = \frac{e^2}{2}$ so that

$$D = k \sin 2\varphi \tag{2.163}$$

The series used above can therefore be truncated after the fourth-order term and be considered as a polynomial equation. Solving this polynomial provides

$$D = \frac{D_c}{1 - 2k\cos 2\varphi' + 2k^2 \sin^2 \varphi'} \tag{2.164}$$

The geodetic latitude can be given as

$$\varphi = \varphi' + D = \sin^{-1}\frac{z^e}{r} + D \tag{2.165}$$

The geodetic longitude is the same as shown earlier

$$\lambda = \tan^{-1}\left(\frac{y^e}{x^e}\right) \tag{2.166}$$

The ellipsoidal height is calculated from the triangle PRC of Fig. 2.18a

$$P^e = (R_N + h)\cos\varphi \tag{2.167}$$

$$h = \frac{P^e}{\cos\varphi} - R_N \tag{2.168}$$

$$h = \frac{\sqrt{(x^e)^2 + (y^e)^2}}{\cos\varphi} - R_N \tag{2.169}$$

Equations (2.165), (2.166) and (2.169) are therefore closed-form expressions to convert from rectangular coordinates to geodetic coordinates in the e-frame.

Appendix D

Derivation of the Iterative Equations From Rectangular to Geodetic Coordinates in the ECEF Frame

From Eq. (2.148), which relates geodetic and rectangular coordinates in the e-frame, we see that

$$x^e = (R_N + h)\cos\varphi\cos\lambda \tag{2.170}$$

$$y^e = (R_N + h)\cos\varphi\sin\lambda \tag{2.171}$$

$$z^e = \left\{R_N\left(1 - e^2\right) + h\right\}\sin\varphi \tag{2.172}$$

Given the rectangular coordinates and these equations, the geodetic longitude is

$$\lambda = \tan^{-1}\left(\frac{y^e}{x^e}\right) \tag{2.173}$$

Equation (2.169) specifies the relationship for the altitude as

$$h = \frac{\sqrt{(x^e)^2 + (y^e)^2}}{\cos\varphi} - R_N \tag{2.174}$$

Also, from Eqs. (2.170) and (2.171) it can be shown that

$$(x^e)^2 + (y^e)^2 = (R_N + h)^2 \cos\varphi^2\left[\cos^2\lambda + \sin^2\lambda\right] \tag{2.175}$$

$$\sqrt{(x^e)^2 + (y^e)^2} = (R_N + h)\cos\varphi \tag{2.176}$$

And finally, dividing Eq. (2.172) by Eq. (2.176) yields

$$\frac{z^e}{\sqrt{(x^e)^2 + (y^e)^2}} = \frac{\left[R_N(1 - e^2) + h\right]}{(R_N + h)}\tan\varphi$$

$$\varphi = \tan^{-1}\left\{\frac{z^e(R_N + h)}{[R_N(1 - e^2) + h]\sqrt{(x^e)^2 + (y^e)^2}}\right\} \tag{2.177}$$

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