Lecture 7

The TT operator is defined by

$$\text{(1)} \quad TT = \lim_{x \to 0} \left[T_{+x}(x^2)\, T_{-x}(x^2) - T_{-x}(x^2)\, T_{+x}(x^2) \right] \leftarrow$$

$$\text{(2)} \quad C(t) = \langle O_{+x}(t)\, O_{-x} \rangle - \langle O_{+x} \rangle^2 = \text{constant} \quad \Rightarrow \quad \partial_t = 0$$

We want to find the equations $\{x, t\}$ so that

$$x' = x + \epsilon \quad \rightarrow \quad x + \epsilon = \tilde{x}$$
$$t' = t \quad \rightarrow \quad t = \tilde{t}$$

We introduce the following notations:

$$T_t = \exp T_{tt} \,, \quad \underline{T_t = \exp T_{tt}} \,, \quad \underline{O_t = \exp t_{tt}}$$

We make the following assumptions:

I. Local translations commute

$$\partial_t T_x = 0 \qquad T_{tx} + T_{xt} \qquad = \quad T_t T_x = 0$$

$$\partial_t T_x = 0 \quad \Rightarrow \quad \begin{array}{l} \partial_t T_{tx} + \partial_t T_{tx} = 0 \quad \Rightarrow \quad \partial T = 0 \,, \quad \partial = \partial_t \\ \underbrace{\partial_t T_{tx} + \partial_t T_{tx}}_{\partial_t^2 = \partial_t \partial_t} = 0 \quad \Rightarrow \quad \partial_t^2 = 0 \end{array}$$

$$\Rightarrow \quad \partial_t^2 = \partial_t \partial_t \quad \Rightarrow \quad \partial_t^2 = \frac{1}{2} \cdot \partial_t^2 \quad \Rightarrow$$
$$\partial_t = \partial_t \Rightarrow \partial_t^2 = \partial_t^2 \Rightarrow \partial_t^2 = 0$$

II. Local interactions

We assume that the interactions are "local" in the sense that

$$\lim_{t \to \infty} \langle O(t) O(0) \rangle = \langle O(t) \rangle \langle O(0) \rangle \qquad \forall O$$

III. Global translations

$$\underset{\substack{\text{generator of} \\ \text{translations}}}{U(t)} = e^{i \int_0^t dt\, T^{tt}} \underset{\substack{\text{is a local} \\ \text{density}}}{(x)} \quad (x) \quad \text{is a phase factor}$$

$$\Rightarrow \quad \langle O(t) \rangle = \langle \underbrace{U^\dagger(t) O(0)\, U(t) O(0)}_{O(t)} \rangle = \langle O(t) \rangle \Rightarrow \quad \text{constant}$$

$$\langle O(t) O(0) \rangle = \langle U^\dagger(t) O(0)\, U(t) O(0) \rangle = \langle O(t) O(0) \rangle \Rightarrow \langle O(t) O(0) \rangle = \langle O(t) \rangle$$

These assumptions imply that the underlying geometry is

$$\text{plane} \quad \text{or} \quad \text{cylinder}$$

$$\boxed{\begin{array}{l} \textbf{Exercise:} \text{ consider QFT that has conformal symmetry - QFT = CFT. Show that} \\ \text{II and III hold } \forall t \in \mathbb{R} \text{ both on the plane and on the cylinder. } \underline{\text{Hint:}} \text{ try} \\ \text{to exploit the conformal symmetry of the theory (map plane} \to \text{cylinder).} \end{array}}$$

① Finiteness of TT

$$TT = \lim_{x \to 0} \left[T(x) T(0) - T(0) T(x) \right] \quad \rightarrow \text{ step 1: need - will show the 1 dependence of } TT, 0$$

OPE in QFT tells us about the short distance behavior of two operators

$$O_1(x) O_2(0) \approx \underbrace{\sum_i C_i(x,0)\, O_i(0)}_{\text{Attract only up to multiplicative factor } \to C_i(x,0)} \quad \text{over all operators at the theory with } \partial_t, \partial_x$$

$$\begin{pmatrix} T_{tt}(x) T(0) & T_{tx}(x) T(0) \\ \partial_t T(x) & \partial_x T(x) \end{pmatrix} = \sum_i \begin{pmatrix} A_i(x,0) & B_i(x,0) \\ \partial_t(x,0) & C_i(x,0) \end{pmatrix} O_i(0)$$

$$T(x) \times T(x) T(0) - \partial_t O_i(0) = \sum_i \underbrace{C_i(x,0) - A_i(x,0)}_{\text{OPE}} \Big] O_i(0)$$

$$= \sum_i O_i(x) + \sum_i \underbrace{C_i(x,0)}_{\text{actually } 0} O_i(x) + \sum_i C_i^{(1)}(x,0)\, O_i(x)$$

$$\underset{0 \to 0}{\text{OPE}} \quad \underbrace{\text{actually } 0}_{\text{Attract}}$$

Assumption III tells us that $\langle O_i O_i \rangle = \underbrace{\langle O_i \rangle}_{\text{constant}} = 0$

$$\partial_t T^{tt}(x) = \partial_t T_{tt}(x) T(x) - \partial_t O_i(x) \partial_t T(x) + \partial_t O_i(x) \left[\partial_t T(x) - \partial_t O_i(x) \right]$$

$$\partial_t \langle O_i(x) \rangle = \partial_t T_{tt}(x) T(x) - \partial_t O_i(x) \partial_t T(x) + \partial_t O_i(x) \left[\partial_t T(x) - \partial_t O_i(x) \right]$$

$$\partial_t \langle O_i(x) \rangle = \langle \partial_t T_{tt}(x) \rangle \langle O_i(x) \rangle - \langle \partial_t T_{tt}(x) \rangle \langle O_i(x) \rangle$$

You can repeat the same exercise to show

$$\partial_t \langle T(x) T(0) \rangle = \langle \partial_t T_{tt}(x) \rangle \langle T(x) T(0) \rangle - \langle \partial_t T_{tt}(x) \rangle \langle T(x) T(0) \rangle$$

In these expressions we only have $\partial_t T_{tt}$ and $\partial_t T_{tt}$

$$\langle \partial_t T_{tt} \rangle \langle T(x) \rangle = 0 \,, \quad \langle \partial_t T_{tt} \rangle \langle T(x) \rangle = 0$$

$$\partial_t \langle O_i(x) \rangle = \sum_i \left[\partial_t O_i(x,0)\, \partial_t O_i - C_i(x,0)\, \partial_t O_i(x) \right] \quad \Rightarrow \quad C_i(x,0) = 0$$

$$\partial_t \langle O_i(x) \rangle = \sum_i \left[\partial_t O_i(x,0)\, \partial_t O_i - A_i(x,0)\, \partial_t O_i(x) \right]$$

$$\langle T(x) \rangle = \langle T(x) T(0) \rangle = \sum_i B_i(x) \quad \Rightarrow \quad \sum_i B_i(x) = \sum_i B_i(x)$$

$\Rightarrow$ This implies that the TT operator is well defined (finite) up to possibly divergent terms that depend on derivatives of operators

$$\Rightarrow \langle TT \rangle \text{ is well defined (finite)} \qquad \left(\langle \partial_t O \rangle = \partial_t \langle O \rangle = 0 \right)$$

$$\boxed{\langle TT \rangle = \lim_{x \to 0} \langle T(x) T(0) - \langle T(x) T(0) \rangle \rangle}$$

② Factorizability of TT

$$\langle T(x) T(0) \rangle + \langle T(x) T(0) \rangle - \langle O(x) O(0) \rangle = \langle T(x) \rangle - \langle O \rangle^2$$

First we show that $\langle T(x) T(0) \rangle$ is a constant

$$\partial_t \langle T(x) T(0) \rangle = \langle \partial_t T(x) T(0) \rangle + \langle \partial_t T(x) T(0) \rangle$$

$$= \langle \partial_t \partial_x T(x) \rangle - \partial_t \langle O(x) O(0) \rangle \quad \Big\} \text{ I}$$

$$= \langle \partial_x \partial_t T(x) \rangle - \partial_t \langle O(x) O(0) \rangle$$

$$= \underbrace{\langle \partial_x O(x) T(0) \rangle}_{\langle O(x) O(0) \rangle} - \underbrace{\partial_t \langle O(x) O(0) \rangle}_{\langle O(x) O(0) \rangle} \quad \Big\} \text{ III}$$

$$= \partial_t \langle O(x) T(0) \rangle + \partial_t \langle O(x) O(0) \rangle$$

$$= \partial_t \langle O(x) \left[\partial_t T(0) - \partial_t O(0) \right] \rangle$$

$$= 0$$

$$\partial_t \langle T \rangle = \partial_t \langle T \rangle = \partial_t \langle T \rangle + \partial_t \langle T \rangle = 0$$

$$\Rightarrow \langle T(x) \rangle = \text{constant}$$

$$\langle T(x) O(0) \rangle = \langle T(x) O(0) \rangle = \langle O(x) O(0) \rangle \Big|_{x \to 0} \quad \Big\} \text{ II}$$

$$= \langle O(x) \rangle \langle O(0) \rangle - \langle O(x) \rangle \langle O(0) \rangle$$

$$\boxed{\underbrace{\langle T(x) \rangle}_{\text{constant}} \langle T \rangle = \langle O \rangle^2}$$

- trivial for the derivative of the equation
- combination of terms we need $(\langle T \rangle - \langle O \rangle)$ is necessary to avoid divergences

Conclusion: the TT operator is well defined (finite) in any 1d Poincaré invariant QFT

In QFT, $\langle T_{tt} \rangle = 0 \Rightarrow \langle T_{tt} \rangle = 0 \quad \Rightarrow \quad \partial_t T(x) = 0 \,, \quad \partial_x T(x) = 0$

$$\partial_t = \int dx\, T(x) \qquad \partial_x = \int dx\, T(x)$$

Vacuum $\qquad$ Vacuum

$$T(x) = \sum_i e^{-iLx} L_i \qquad T(x) = \sum_i L^{-iLx}$$

$$\Rightarrow \text{in QFT,} \quad \boxed{\begin{array}{l} \left[L_n, L_m \right] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n) \delta_{n+m} \\ \left[\bar{L}_n, \bar{L}_m \right] = \\ \left[L_n, \bar{L}_m \right] = 0 \end{array}}$$

$$TT = \lim_{x \to 0} T(x) T(0) = L_n \bar{L}_n \quad \text{(on the plane)}$$

Exercise: show in QFT we can bilinear find $L_n \bar{L}_n$. Hint: the interpretation of this operator in terms of the stress tensor. Hint: use the transverse algebra and the work sequence of T and $\bar{T}$.

3) Can this be generalized to an arbitrary operator in QFT?

Generalization to other states

We want to show that factorizability of $\langle TT \rangle$ actually holds for states other than the vacuum $(|0\rangle \to |\psi\rangle)$

$$\langle \psi | TT | \psi \rangle = \lim_{x \to 0} \left[\langle \psi | T(x) T(0) | \psi \rangle - \langle \psi | \langle O(x) O(0) \rangle | \psi \rangle \right]$$

non degenerate of expectation of energy and momentum

$$= \langle \psi | T(x) \rangle \langle \psi | T(0) \rangle - \langle \psi | \langle O(x) O(0) \rangle^2 = \text{const}$$

We need to recount assumptions I - III:

I. ✓

II. This holds because $|\psi\rangle$ is an eigenstate of energy and momentum

$$U(t) |\psi\rangle = e^{i \int_0^t dt\, \underbrace{E(x,t)}_{\text{is a local density}}} |\psi\rangle \quad \text{is a phase factor}$$

$$\Rightarrow \quad \langle O(x) O(0) \rangle = \text{constant}, \quad \langle O(x) O(0) \rangle = \langle O(x) \rangle$$

Repeating the same arguments as before, we have that (using I, II)

$$\langle T_{tt}(x) \rangle = \langle O(x) \rangle \langle O(x) \rangle = \langle O(x) O(0) \rangle = \text{constant}$$

Exercise: show this and justify why it appropriate to use derivatives within the correlation function above $\langle \partial_t \rangle = \langle \partial_x \rangle$

II. We cannot use the argument of "local interactions" because $\langle O(x) O(0) \rangle |\psi\rangle$ can be understood as a "ψ" function with the ∂_t or ∂_x and infinity $\langle O(x) O(0) \rangle$

However, since $|\psi\rangle$ is an eigenstate of energy and momentum

$$\langle T_{tt}(x) \rangle = \sum_i \langle \psi | T(x) | \psi \rangle \langle \psi | \underbrace{T(0)}_{\text{constant}} | \psi \rangle - \langle O(x) \rangle \langle \psi | \langle O(x) O(0) \rangle | \psi \rangle$$

$$= \langle \psi | T(x) | \psi \rangle \langle \psi | T(0) | \psi \rangle - \langle O(x) \rangle \langle \psi | \langle O(x) O(0) \rangle | \psi \rangle$$

$$= \sum_i \langle \psi | T(x) | \psi \rangle \langle \psi | T(0) | \psi \rangle - \langle O(x) \rangle \langle \psi | \langle O(x) O(0) \rangle | \psi \rangle + \underbrace{\sum_{i \neq 0} C_i(x,0)}_{\text{independent of } |\psi\rangle \text{ in non degenerate}}$$

$$\underbrace{\langle T_i \rangle \langle T_i \rangle}_{\text{constant}} = \underbrace{\langle \psi | T(x) | \psi \rangle \langle \psi | T(0) | \psi \rangle - \langle O(x) \rangle \langle \psi | \langle O(x) O(0) \rangle | \psi \rangle}_{\text{constant}} + \underbrace{\sum_{i \neq 0} C_i(x,0)}_{\text{independent of } |\psi\rangle \text{ in non degenerate}}$$

$$\boxed{\langle O(x) O(0) \rangle = \langle \psi | T(x) | \psi \rangle \langle \psi | T(0) | \psi \rangle - \langle O(x) O(0) \rangle^2}$$

Generalization to other currents

$\underline{\partial_\mu j_\mu}$ - working flow: $\partial_t = \partial_x \partial_t$, spin: $\partial_t = \partial_x \partial_t$

Consider conserved current $\partial_\mu j_\mu = 0$

components	spin
$T_{tt} = T_{xx}$	2,2
$\partial_{tt} = \partial_{xx}$	1,1
$T_{tx} = T_{xt}$	-1,-1
$\partial_{tx} = \partial_{xt}$	-1,-1

$$\boxed{\begin{array}{l} \partial_t T_{tt}(x) = \partial_x \partial_t(x) \\ \partial_t T_{tx}(x) = \partial_x \partial_t(x) \end{array}}$$

Exercise: verify that the conservation law is compatible with the weight and spin assignments given in the table above.

For example, for a QFT current $\partial_\mu j_\mu$ and we

$$\partial_t = T_t + \partial_t \,, \quad \partial_x = T_t + \partial_t \quad \Rightarrow \quad \partial_t^2 = 0$$