The Exponential and Gamma Distributions

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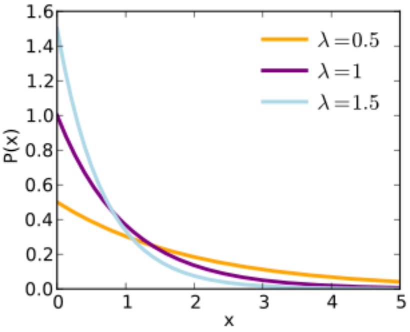
The Exponential Distribution

$$f(x|\beta) = \frac{1}{\beta}e^{(-x/\beta)}, 0 \leq x < \infty, \beta > 0$$

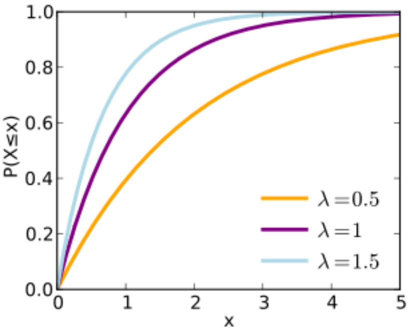
The parameter β is a scale parameter.



Note: $\lambda = \frac{1}{\beta}$

CDF

$$F(x) = 1 - e^{(-x/\beta)}, 0 \leq x < \infty$$



Note: $\lambda = \frac{1}{\beta}$

Exponential Mean, Variance, and MGF

$$E(X) = \beta$$

$$Var(X) = \beta^2$$

$$M_x(t) = \frac{1}{1 - \beta t}, t < \frac{1}{\beta}$$

Relationship to Other Distributions

The Exponential Distribution is related to the Uniform, Gamma and Weibull Distributions.

If $X \sim \textit{Uniform}$ and $Y = -\textit{log}(X)$, then $Y \sim \textit{Exp}$.
If $X \sim \textit{Gamma}(\alpha = 1, \beta)$, then $X \sim \textit{Exp}(\beta)$.
If $X \sim \textit{Exp}$ and $Y = X^{(1/\gamma)}$, then $Y \sim \textit{Weibull}$.

The Exponential Distribution is the continuous form of the Geometric Distribution.

Example of Exponential Distribution

If the average waiting time at a doctor's office is $\beta = 20$ minutes, what is the probability that you will have to wait less than 10 minutes?

$$P(X \leq 10) = \int_0^{10} f(x)\mathrm{d}x = 0.3934693$$

More about the Exponential Distribution

The exponential distribution can also be used to model time between Poisson events and lifetimes.

An interesting characteristic is that the Exponential distribution is memoryless.

$$P(X > s | X > t) = P(X > s - t)$$

For example: Given you have waited 20 minutes, the probability you'll wait 30 minutes is the same as the probability you'll wait 10 minutes.

Gamma Function:

The Gamma Function is defined as:

$$\Gamma(\alpha) = \int_{0}^{\infty} t^{\alpha-1} e^{-t} dt,$$

where $\alpha > 0$.

Useful Identities:

Properties of the Gamma Function:

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \quad \alpha > 0$$

$$\Gamma(n) = (n-1)!, \quad n \,\epsilon\, \mathbb{N}$$

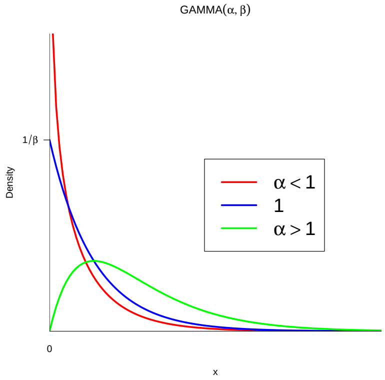
$$\Gamma(1) = 1$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Gamma Distribution:

If $X \sim Gamma(\alpha, \beta)$, then the pdf of X is:

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}$$



Gamma Distribution:

And the CDF is:

$$\frac{\gamma(k, x/\theta)}{\Gamma(k)},$$

where $k > 0$, and

$$\gamma(k, x/\theta) = \int_0^{x/\theta} t^{k-1} e^{-t} dt$$

Gamma Mean, Variance, and Moment Generating Function:

Mean:

$$E[X] = \alpha\beta$$

Variance:

$$Var[X] = \alpha\beta^2$$

MGF:

$$M_x(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}$$

Relation to Other Distributions:

Poisson-Gamma. The Gamma distribution is a conjugate prior for the mean of the Poisson distribution.
Exponential-Inverse Gamma. The Inverse Gamma distribution is a conjugate prior for the mean of the exponential distribution.

Exponential. $Gamma(1, \beta) = EXP(\beta)$.
Chi-Sqaured. $Gamma(p/2, 2) = \chi^2(p)$.
Maxwell. Let X be distributed as a Gamma. If $\alpha = 3/2$, then $Y = \sqrt{X/\beta}$ is distributed as a Maxwell.
Inverse Gamma. Let $X \sim Gamma(\alpha, \beta)$. Then $Y = 1/X \sim InverseGamma(\alpha, \beta)$.

Relation to Other Distributions:

More on Poisson-Gamma. The Negative Binomial can be derived as a gamma mixture of Poissons.

Specifically, if

$$X \sim NB(r, \beta)$$

and

$$X|\lambda \sim POI(\lambda),$$

then

$$\lambda \sim Gamma(\alpha, \beta).$$

Applications of the Gamma Distribution:

The Gamma distribution is positive and right-skewed, so it is can be used to model a variety of events, including the following:

- The amount of rainfall accumulated in a reservoir
- The size of loan defaults of aggregate insurance claims
- The flow of items through manufacturing and distribution processes
- The load on web servers
- The many and varied forms of telecom exchange

Very Hypothetical Situation:

Suppose that for a graduate Statistical Computing course, the average time required for every student to come up with a feasible solution for each exam problem is two hours. Compute the probability that a student, say Mickey, will take 6 or more hours to come up with the solutions to all four problems of the exam.

One problem every 2 hours means we would expect to "solve" $\beta = 1/2$ exam problems every hour on average. Using $\beta = 1/2$ and $\alpha = 16$, we can compute this as follows:

$$P(X \geq 8) = \int_{8}^{\infty} \frac{x^{4-1} e^{-x/.5}}{\Gamma(4) 2^4} dx = .466745$$

Hypothetical Inference???