Stat 641 Project: Exponential and Gamma Distributions

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Introduction:

The Gamma Function:

The Gamma Function is defined as:

$$\Gamma(\alpha) = \int\limits_0^\infty t^{\alpha-1} e^{-t} dt,$$

where $\alpha > 0$.

Useful Identities:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0$$

$$\Gamma(n) = (n-1)!, \quad n \,\epsilon\, \mathbb{N}$$

$$\Gamma(1) = 1$$

$$\Gamma(\frac{1}{2}) = \sqrt(\pi)$$

Derivation of the Gamma Distribution:

Consider

$$f(t) = \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)}, \quad 0 \ < \ t \ < \infty.$$

Integrating f(t) from 0 *to* ∞, we get

$$\begin{aligned}\int\limits_0^\infty f(t)dt = \int\limits_0^\infty \frac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)} dt &= \frac{\int_0^\infty t^{\alpha-1}e^{-t}dt}{\Gamma(\alpha)} \\ &= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \\ &= 1. \qquad (1)\end{aligned}$$

Therefore, f(t) is a pdf.

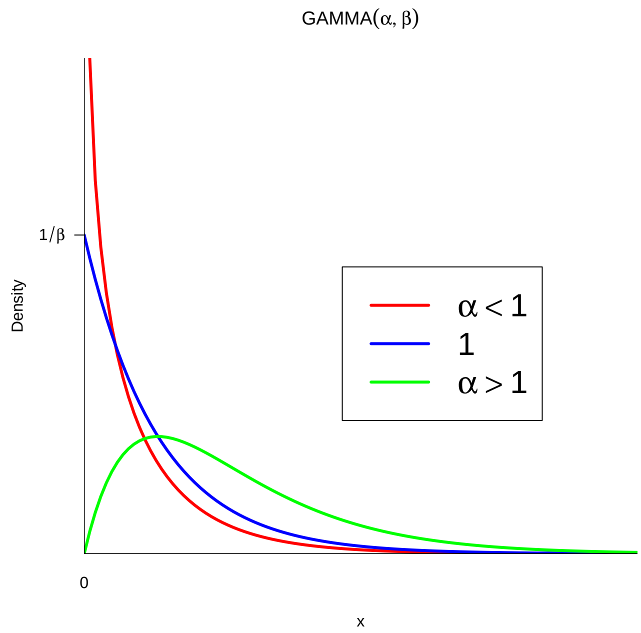
Derivation:

pdf:

Let $X \sim \beta T$ in (1), where $\beta > 0$. Then the *gamma*(α, β) family is defined below as:

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, \quad \alpha > 0, \quad \beta > 0.$$

The parameters α and β are referred to as the shape and scale parameters respectively as the α most influences the peakedness of the distribution, while β most influenxes the spread of the distribution.



Integration to 1:

Let $X \sim \Gamma(\alpha, \beta)$, $x, \alpha, \beta > 0$. Then,

$$\int_0^\infty f(x)dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^\infty x^{\alpha-1} e^{-x/\beta} dt$$

$$= \frac{\beta\beta^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \int_0^\infty \frac{x^{\alpha-1}}{\beta^{\alpha-1}} e^{-x/\beta} d\frac{x}{\beta}$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)\beta^{\alpha}} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\frac{x}{\beta}} d\frac{x}{\beta}$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-t} dt$$

$$\frac{1}{\Gamma(\alpha)} \Gamma(\alpha) = 1.$$

Therefore, f(x) integrates to 1.

CDF

The Gamma Cumulative Distribution is function is given by:

$$\frac{\gamma(k, x/\theta)}{\Gamma(k)},$$

where $k > 0$, and

$$\gamma(k, x/\theta) = \int_0^{x/\theta} t^{k-1} e^{-t} dt$$

Mean and Variance:

$$E[X] = \alpha\beta$$

Proof:

$$E[X] = \int_0^\infty x \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{\alpha+1} e^{-x/\beta} dt$$

$$= \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha}$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \beta$$

$$= \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \beta$$

$$= \alpha\beta$$

$$Var[X] = \alpha\beta^2$$

Proof:

$$E[X^2] = \int_0^\infty x^2 \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{(\alpha+2)-1} e^{-x/\beta} dx$$

$$\frac{\Gamma(\alpha+2)\beta^{\alpha+2}}{\Gamma(\alpha)\beta^\alpha}$$

$$= (\alpha+1)\alpha\beta^2$$

Therefore,

$$Var[X] = E[X^2] - (E[X])^2$$

$$= (\alpha+1)\alpha\beta^2 - (\alpha\beta)^2$$

$$= \alpha\beta^2$$

Moment Generating Function:

$$M_x(t) = E[e^{tx}] = \left(\frac{1}{1-\beta^t}\right)^\alpha$$

$$M_x(t) = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-x/\beta} x^{\alpha-1} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-(x/\beta)+tx} x^{\alpha-1} dx$$

$$= \frac{1}{\beta^\alpha} \int_0^\infty \frac{\left(\frac{1}{(1/\beta)-t}\right)^\alpha}{\left(\Gamma(\alpha)\frac{1}{(1/\beta)-t}\right)^\alpha} e^{-x/\frac{1}{(1/\beta)-t}} x^{\alpha-1} dx$$

$$= \left(\frac{1}{(1/\beta)-t}\right)^\alpha / \beta^\alpha$$

$$= \left(\frac{1}{[(1/\beta)-t]\beta}\right)^\alpha$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha$$

Relation to Other Distributions:

Poisson-Gamma. The Gamma distribution is a conjugate prior for the mean of the Poisson distribution.
Exponential-Inverse Gamma. The Inverse Gamma distribution is a conjugate prior for the mean of the exponential distribution.

Exponential. $Gamma(1, \beta) = EXP(\beta)$.
Chi-Sqaured. $Gamma(p/2, 2) = \chi^2(p)$.
Maxwell. Let X be distributed as a Gamma. If $\alpha = 3/2$, then $Y = \sqrt{X/\beta}$ is distributed as a Maxwell.
Inverse Gamma. Let $X \sim Gamma(\alpha, \beta)$. Then $Y = 1/X \sim InverseGamma(\alpha, \beta)$.

More on Poisson-Gamma. The Negative Binomial can be derived as a gamma mixture of Poissons.

Specifically, if

$$X \sim NB(r, \beta)$$

and

$$X|\lambda \sim POI(\lambda),$$

then

$$\lambda \sim Gamma(r, \beta).$$

Applications of the Gamma Distribution:

The Gamma distribution is positive and right-skewed, so it is can be used to model a variety of events, including the following:

- The amount of rainfall accumulated in a reservoir
- The size of loan defaults of aggregate insurance claims
- The flow of items through manufacturing and distribution processes
- The load on web servers
- The many and varied forms of telecom exchange

Example:

Suppose that for a graduate Statistical Computing course, the average time required for every student to come up with a feasible solution for each exam problem is two hours. Compute the probability that a student, say Mickey, will take 8 or more hours to come up with the solutions to all four problems of the exam.

One problem every 2 hours means we would expect to "solve" $\beta = 1/2$ exam problems every hour on average. Using $\beta = 1/2$ and $\alpha = 16$, we can compute this as follows:

$$P(X \geq 8) = \int_{8}^{\infty} \frac{x^{16-1} e^{-x/.5}}{\Gamma(16)(1/2)^{16}} dx = .466745$$