* **Problem 14.** For X a simple random variable, show that $E(X)$ as defined by Definition 1 equals $E(X)$ as defined by Definition 8.

A glance at Definition 1, which is needed to make sense of Definition 5 and Definition 8, shows that the definition of expectation involves the underlying probability measure P. Thus, we have been somewhat imprecise in speaking of 'the' expectation operator; changing the underlying probability space changes the expectation operator. This lack of precision is justified in those circumstances when the underlying probability space is clear from the context. And, as we will soon see, the expectation of a random variable is determined by its distribution, so when the distribution of a random variable X is known, knowledge of the underlying probability space is not necessary for the calculation of $E(X)$.

Nevertheless, it is sometimes necessary to distinguish expectation operators defined for different probability spaces. (See Theorem 15 for an example.) In such cases, the expectation operator on the space of $\overline{\mathbb{R}}$-valued random variables with underlying probability space $(\Omega, \mathcal{F}, P)$ is denoted by E_P and is called the *expectation operator with respect to* P. The quantity $E_P(X)$ is called the *expectation of* X *with respect to* P.

4.2. Linearity and positivity

In this section, we prove linearity and other basic properties of the expectation operator. Preliminary versions of some of these properties have appeared in Lemma 3, Lemma 4, Lemma 6, and Lemma 7.

The following convention, already used in Lemma 6, will simplify our discussion. We will adhere to it throughout the book.

CONVENTION. Unless explicitly stated otherwise, the products $0 \cdot \infty$ and $\infty \cdot 0$ will be interpreted to equal 0.

Theorem 9. *Let X and Y be $\overline{\mathbb{R}}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, and let a, b, and c be real constants.*

(i) If $aX(\omega) + bY(\omega)$ is defined for all ω, then $E(aX + bY) = aE(X) + bE(Y)$, provided the expression on the right is meaningful.
(ii) If $X = c$ a.s., then $E(X) = c$.
(iii) If $X = Y$ a.s., then either the expectations of X and Y both exist and are equal, or neither exists.
(iv) If $X \leq Y$ a.s. and either $E(X)$ exists and is different from $-\infty$ or $E(Y)$ exists and is different from ∞, then the other of $E(Y)$ and $E(X)$ exists and $E(X) \leq E(Y)$.
(v) If $E(X) = E(Y)$ is finite and $X \leq Y$ a.s., then $X = Y$ a.s.
(vi) If $E(X)$ exists, then $|E(X)| \leq E(|X|)$.
(vii) If $E(X)$ does not exist, then $E(|X|) = \infty$.
(viii) If $X(\omega) + Y(\omega)$ is defined for all ω, then $E(|X + Y|) \leq E(|X|) + E(|Y|)$.

PARTIAL PROOF. We first prove a special case of (i) —namely tha

$$E(X+Y) = E(X) + E(Y) \tag{4.3}$$

whenever the expression on the right is meaningful. We use the followin

$$(X+Y)^+ + Y^- + X^- = (X+Y)^- + Y^+ + X^+ .$$

On each side we have a sum of three $\overline{\mathbb{R}}^+$-valued random variables. By

$$E((X+Y)^+) + E(Y^-) + E(X^-) = E((X+Y)^-) + E(Y^+) + E($$

If all terms are finite, they can be rearranged to give the desired conc

$$E((X+Y)^+) - E((X+Y)^-) = (E(X^+) - E(X^-)) + (E(Y^+) - E$$

Consideration of the cases involving some infinite terms is left for the

We also leave it to the reader to use the definition of $E(X)$ in co with Lemma 6 to prove that $E(aX) = aE(X)$ for all real a, provided t exists. Assertion (i) follows from this fact and (4.3).

We leave it to the reader to prove (iv). Then (iii) follows from two ap of (iv), one using (iv) as it stands and the other using (iv) with interchanged. Since the expected value of a constant random variab constant, (ii) as a special case of (iii). The contrapositive of (vii) is a co of (iv) and the inequality $X \leq |X|$. Thus, it remains for us to prove and (viii).

Assertion (viii) follows from the inequality $|X+Y| \leq |X| + |Y|$, (iv) consequence $E(|X| + |Y|) = E(|X|) + E(|Y|)$ of (i).

Suppose that $E(X)$ exists. By (i), $E(-X) = -E(X)$ also exists. applications of (iv)—one to X and $|X|$ and the other to $-X$ and $|X$ (vi).

To prove (v) we suppose that $E(X) = E(Y)$ is finite and $X \leq Y$ expected value of the nonnegative random variable $(Y - X)$ exists, so th be applied to the equality $Y = X + (Y - X)$ to give $E(Y) = E(X) + $ from which it follows that $E(Y - X) = 0$. Thus, for any $\varepsilon > 0$, t function that equals 0 when $Y - X < \varepsilon$ and equals ε when $Y - X \geq$ expectation. Therefore,

$$P(\{\omega\colon Y(\omega) - X(\omega) \geq \varepsilon\}) = 0 .$$

Let $\varepsilon \searrow 0$ through a sequence and use the Continuity of Measure Th obtain $P(\{\omega\colon Y(\omega) - X(\omega) > 0\}) = 0$, as desired. $\square$

Problem 15. Complete the proof of the preceding theorem.