(100 points, **due on Wed March 22nd at 4pm**). Consider again the google trends data in **googletrendsUCSC.csv** (available online). Let $\mathcal{M}_0$ be the model given by

$$\begin{aligned} y_t &= \theta_t + \nu_t, \quad \nu_t \sim N(0, v), \\ \theta_t &= \theta_{t-1} + w_t, \quad w_t \sim N(0, vW_t), \end{aligned}$$

with v unknown and W_t specified by the optimal discount factor $\hat{\delta}_0 \in (0, 1]$ that maximizes the observed predictive density.

1. (65 points) Consider a DLM with the following two components: (a) a second order polynomial to capture the trend in the data and (b) a full seasonal Fourier model with fundamental period $p = 12$, unknown observational variance v and a system covariance matrix specified through a single optimal discount factor $\hat{\delta}_1 \in (0, 1]$ that maximizes the observed predictive density for this model form. Refer to this model as $\mathcal{M}_1$. Note that this model should have a state vector $\boldsymbol{\theta}_t$ of dimension 13 (2 parameters for the second order polynomial and 11 parameters for the seasonal component).

   (a) (35 points) Write down the DLM form, find the optimal discount factor $\hat{\delta}_1$, and provide summaries of the following distributions: the marginal filtering distributions for the polynomial trend and each harmonic in the seasonal component; the one-step ahead forecast distribution $(\mathbf{y}_t|\mathcal{D}_{t-1})$, for $t = 1 : T$; the marginal smoothing distributions for the polynomial trend and each harmonic in the seasonal component; the forecast distributions $(y_{T+k}|\mathcal{D}_T)$ for $T = 156$ and $k = 1 : 12$.

   (b) (15 points) Discuss your results, particularly in comparison with your analysis using model $\mathcal{M}_0$.

   (c) (15 points) Assess the importance of the harmonic components. Are there any harmonic components that could be discarded? Justify your answer. If there are components that can be discarded, fit the reduced model, referred to as $\mathcal{M}_2$, with an optimal discount factor $\hat{\delta}_2$ that maximizes the observed predictive density for this model form. Summarize the same distributions for model $\mathcal{M}_2$ that you summarized for model $\mathcal{M}_1$ (except, of course, those for the harmonic components that were discarded).

2. (35 points) Now consider the following model, referred to as model $\mathcal{M}_3$:

$$\begin{aligned} y_t &= \alpha + \beta t + x_t + \epsilon_t, \quad \epsilon_t \sim N(0, v), \\ x_t &= \sum_{j=1}^{q} \phi_j x_{t-j} + \nu_t, \quad \nu_t \sim N(0, 30), \end{aligned}$$

   with $\alpha \sim N(0, vw)$, and $\beta \sim N(0, vw)$, with v, w, unknown and q known. Assuming conjugate priors for v, w, $(x_0, x_{-1}, \ldots, x_{1-q})'$ and $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_q)'$ (please specify the form of these distributions and the values of the hyperparameters):

(a) (15 points) Write down the steps of a MCMC algorithm to obtain samples from the distribution of $(x_{1:T}, \alpha, \beta, \phi, v, w|\mathcal{D}_T)$. Note: Your algorithm must use FFBS.

(b) (15 points) Implement this algorithm for a q of your choice (justify your choice) and provide summaries of the posterior distributions for following parameters:

i. $(x_{1:T}|\mathcal{D}_T)$;

ii. $(\alpha, \beta|\mathcal{D}_T)$;

iii. $(v|\mathcal{D}_T)$ and $(w|\mathcal{D}_T)$;

iv. the moduli and the periods of the quasi-periodic roots (if any) of the characteristic polynomial ordered by period.

v. $(y_{T+k}|\mathcal{D}_T)$, for $k = 1:12$.

(c) (5 points) Discuss the results from this model particularly in comparison with the results you obtained for models $\mathcal{M}_1$ and $\mathcal{M}_2$.