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Goal-based Investiments: A Dynamic Stochastic Programming Approach

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RESUMO

Nosso estudo tem como objetivo desenvolver uma política de investimentos que minimize a contribuição total necessária para alcançar um objetivo financeiro de longo prazo. Utilizamos uma estrutura de investimentos orientada a metas que incorpora a programação estocástica com dinâmica de retornos de ativos baseada em uma Cadeia de Markov Oculta. Essa abordagem oferece uma solução mais prática e eficaz em comparação com as técnicas convencionais de otimização de portfólio. Ao integrar a Cadeia de Markov Oculta, obtemos uma estimativa mais precisa da dinâmica dos retornos dos ativos, resultando em uma modelagem de investimentos aprimorada. Como resultado, nossa abordagem oferece uma ferramenta simples e robusta para que os investidores tomem decisões de investimento personalizadas e informadas. Ela leva em consideração o estágio atual da riqueza e as condições econômicas prevalecentes, permitindo a minimização da contribuição necessária para alcançar os objetivos financeiros desejados.

PALAVRAS CHAVE. Otimização Linear, Cadeia de Markov Escondidas, Modelo Financeiro Orientado a Objetivo de Longo Prazo.

PM - Programação Matemática, ESTMP - Estatística e Modelos Probabilísticos, BDA - Big Data e Analytics

ABSTRACT

Our study aims to develop an investment policy that minimizes the total contribution required to achieve a long-term financial goal. We employ a goal-oriented investment framework that incorporates stochastic programming with Markovian dynamics of asset returns, using a Hidden Markov Model. This approach provides a more practical and effective solution compared to conventional portfolio optimization techniques. By integrating the Hidden Markov Model, we obtain a more accurate estimation of asset return dynamics, leading to improved investment modeling. As a result, our approach offers a simple yet robust tool for investors to make personalized and informed investment decisions. It takes into account the current stage of wealth and prevailing economic conditions, allowing for the minimization of the necessary contribution to reach desired financial objectives.

KEYWORDS. Linear Optimization. Hidden Markov Model. Goal Oriented Investment.

PM - Programação Matemática, ESTMP - Estatística e Modelos Probabilísticos, BDA - Big Data e Analytics

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1. Introduction

In the realm of financial planning, one investment strategy that has gained traction and which offers significant potential is the goal-based investing framework. Such approach is focused on achieving specific financial goals, such as retirement savings, buying a home, or funding education. It considers factors like time horizon, risk tolerance, and required savings. This study proposes a personalized investment policy based on this framework.

The goal-based approach highlights risk as the probability of not reaching the desired financial outcome. In our model, achieving the target is always our aim, leading us to minimize the required contribution. Specifically, we formulate an optimization problem that utilizes a multi-stage stochastic optimization framework

$$Q_t(W_t) = \min_{c_t, \mathbf{x_t} \in X} c_t + \mathbb{E}[Q_{t+1}(W_{t+1}(\mathbf{x_t}))] \tag{1}$$

where $Q_T(W_T) = ||W_T - G||$. The model minimizes the stream contribution at time t, c_t, required to achieve a pre-determined financial goal G, while co-optimizing investment decisions $\mathbf{x_t}$ and computing the current wealth $W_{t+1}(\mathbf{x_t})$.

This study presents a pioneering approach to investment modeling that exploits machine learning techniques to enrich the accuracy of asset return dynamics. Precisely, Hidden Markov Models (HMMs) are used to capture the empirical characteristics of asset returns. By considering different economic states and blending distributions accordingly, the framework can account for uncertainty in asset returns. An example of HMM-defined economic states in the historical SP 500 is shown in Figure 1.

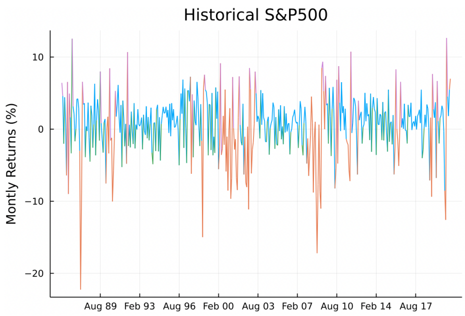

Figura 1: HMM States in the historical S&P 500

The primary aim of this study is to develop an innovative framework for goal-oriented investment that integrates cutting-edge machine learning techniques with a versatile dynamic stochastic programming model. As a result, we want to provide a dynamic and personalized investment policy that is both accessible and cost-effective.

1.1. Relevant Literature

The modern portfolio theory, based on minimizing portfolio variance or standard deviation, is commonly used to manage risk and maximize expected returns. This theory, pioneered by [Markowitz, 1952], forms the foundation of the mean-variance framework, which has been extensively studied.

However, applying the mean-variance framework directly to goal-based investing presents challenges. It overlooks the consideration of time horizons, leading to inadequate treatment of inflation risk and difficulties in defining end goals. Recent research has attempted to overcome

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this limitation. For example, one study [Yao e Chen, 2013] incorporates stochastic inflation and a broader model that accounts for varying degrees of risk across assets.

Another criticism of the mean-variance framework is its inconsistent optimal pre-commitment solution over time. To address this, researchers have explored aligning the framework with game theory principles as in [Li, 2016], [Wu, 2015], [Liu, 2015]. However, these attempts have yielded suboptimal solutions. Furthermore, the classical mean-variance framework assumes a constant level of risk and return in the financial market, which contradicts empirical evidence. Research by [Vicera, 2007] highlights that real returns are highest when stock valuations are low relative to earnings or dividends, indicating mean-reversion. Some studies, such as [Wang, 2019] and [Guan, 2015], have incorporated this observation into their models.

Despite numerous efforts to improve the Mean-Variance (MV) framework, it is primarily utilized by less constrained institutional investors, as stated by [Kim, 2019]. In contrast, the goal-based framework was specifically developed to cater to the needs of individual investors. Notable contributions to this framework include [Fowler, 2006] and [Dempster, 2016], which establish a hierarchy among multiple financial goals. To address uncertainty, multistage problems and scenario analysis are employed. [Dempster, 2016] assumes normally distributed asset returns, while [Kim, 2019] incorporates the investor's present and future cash flows in addition to prioritized financial goals.

A goal-based investing plan can be a useful tool for addressing concrete problems in investment management. One example of its application is in the management of pension schemes, which have been facing significant challenges worldwide since keeping their promises has proven too difficult. At the moment there is no unique solution to such a crisis, but a defined contribution approach appears to be offering some solutions and goal-based investing plans can offer a useful tool in this context.

2. Proposed Optimization Model

Our investment problem revolves around minimizing future contributions while addressing numerous uncertainties stemming from factors like market returns and future wealth levels across multiple stages. As it will be discussed in Section 3, computing the future cost function in multi-stage problems is a challenging task. Hence, we adopt an alternative approach.

Broadly speaking, our methodology entails approximating the objective function in the final stage through a first-order Taylor approximation. Subsequently, we utilize this approximation to construct the future cost function for the preceding stage. This iterative process continues until we reach the first stage, enabling us to optimize the overall problem in a simplified fashion.

Starting with the last stage (T), we encounter a specific optimization problem described by Equation 2. The problem involves variables such as W_T (initial wealth at stage T), c_T (contribution at stage T), c_T^+ (non-negative contribution at stage T), c_T^- (non-negative withdrawal at stage T), and G (the goal to be achieved in the last stage T).

Note that the tax rate $(1-d)$ penalizes the overachievement of the financial goal since it creates a cost for any withdrawal. Besides, in this last stage, we do not have returns to be estimated therefore we do not consider the state of the economy, we only have one function to be optimized using standard optimization techniques.

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$$
\begin{aligned}
Q_T(W_T) = \quad & \min_{c_T^+, c_T^-} \quad \lambda\, c_T \\
& \text{s.t.} \\
& \qquad G = W_T + c_T^+ - c_T^- \\
& \qquad c_T = c_T^+ - (1-d)\, c_T^-
\end{aligned}
\tag{2}
$$

At any time t, our objective is to minimize the expected contribution required for the entire investment journey. Therefore we take a backward approach in the optimization process, after estimating stage T we have the future cost function of stage T-1, so the optimization process begins for the intermediate stages. In those, we simulate returns used at each stage by a Hidden Markov Model (HMM), which generates different samples according to different HMM states.

Therefore, we estimate a future cost function, denoted as $Q_t^j(W_t)$, for intermediate states in the problem. This function is dependent on the initial wealth in the stage, represented by W_t, as well as the Hidden Markov Model (HMM) state (j) and the stage (t). The HMM, which will be explained in detail in section 2.1, offers a significant advantage by introducing a time dependency to our problem. For each state of the HMM, we simulate a set of returns (S) from a specific distribution.

Additionally, we introduce constraints related to contributions or withdrawals made at each stage, as well as an upper limit on contributions to restrict the amount demanded in a particular stage. Finally, we incorporate two balance constraints, resulting in a comprehensive problem formulation for intermediate stages.

The objective function for this problem is as follows:

$$
\begin{aligned}
Q_t^j(W_t) = \min_{c_t^+, c_t^-, x_t} \quad & c_t + \mathbb{E}[Q_{t+1}(W_{t+1}(s,k))] \\
\text{s.t} \quad & W_{t+1}(s,k) = \sum_{i \in I} x_{i,t}(1 + r_{i,t}(s,k)); \quad \forall s \in S, \forall k \in K \\
& \sum_{i \in I} x_{i,t} - (c_t^+ - c_t^-) = W_t \\
& c_t = c_t^+ - (1-d)\, c_t^- \\
& c_t \leq M
\end{aligned}
\tag{3}
$$

In this formulation, we minimize the objective function by determining optimal values for the variables c_t^+, c_t^-, and x_t. The constraints ensure the calculation of the future wealth $W_{t+1}(s,k)$, balance the contributions and withdrawals with the initial wealth W_t, and limit the value of c_t to a maximum value M. However, calculating $\mathbb{E}[Q_{t+1}(W_{t+1}(s,k))]$ is not an easy task therefore we have a particular methodology for solving it, explained in section 3.

2.1. Uncertainty Characterization

Given the nature of the uncertainty modeled in our problem, it is paramount to take into account the ”stylized facts”associated with asset returns. These are fundamental traits that have been consistently observed across various independent studies in the field, and the Hidden Markov Model (HMM) has demonstrated promising outcomes in modeling them. As stated in [Cont, 2000], the prevailing observations of financial asset returns include:

1. Daily asset returns generally do not exhibit significant autocorrelation, but longer time intervals such as weeks or months may show some degree of autocorrelation.

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2. The unconditional distribution of asset returns often displays a heavy tail, following a Pareto-like distribution. This implies that extreme events like market crashes or booms occur more frequently than predicted by a normal distribution. As a result, modeling a variable like P_t/P_{t-1} as a log-normal distribution has become widely adopted, considering P_t the price of the security at time t.

3. High-volatility events tend to cluster over time, and volatility shows positive autocorrelation over a certain period. Therefore, financial returns lack independence. HMMs effectively introduce time dependency into our model and can handle the complex volatility patterns in financial data by utilizing a mixture of distributions.

The Hidden Markov Model (HMM) is a well-established probabilistic framework that can be effectively utilized for generating observations from a specific distribution contingent upon the underlying state of an unobserved process. In our discussion we focus on the first-order Markov chain, considering q_t the state S that we are at a moment t, the stochastic process is represented in equation 4.

$$Pr(q_{t+1} = S_j | q_t = S_i, ..., q_1 = S_u) = Pr(q_{t+1} = S_j | q_t = S_i) \tag{4}$$

Given that the probability of a particular state within a Markov chain is contingent upon solely the preceding state, we can represent the state transitions from a state i to another state j, using a state transition probability $a_{i,j}$, as outlined in 5.

$$a_{i,j} = P[q_t = S_j | q_{t-1} = S_i] \tag{5}$$

As an illustrative example, suppose we consider an economy with three distinct states, denoted as S_1 representing a bull market, S_2 denoting a bear market, and S_3 signifying a neutral market. Considering the state transition matrix $\mathbf{A}$, we can gain insight into the underlying Markov process between these states, as elucidated in Figure 2.

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}$$

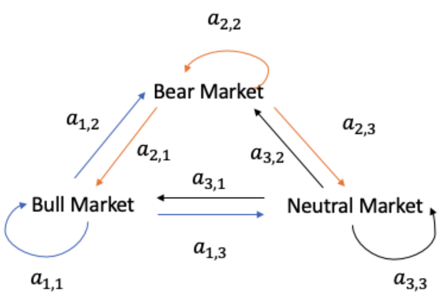

Figura 2: Stochastic Process Between States

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As we are operating within the context of a Hidden Markov Model, they are two distinct stochastic processes at play. It is worth noting that one, the underlying Markov process S_t is not directly observable, and only the other, the observations X_t is accessible to us. In the context of our concrete example, S_t would denote the state of the economy, and X_t would signify the generated asset's returns. This process is succinctly described in 6, by a conditional probability distribution.

$$b_t(j) = Pr(X_t|X_{t-1}, q_t = S_j) = Pr(X_t|q_t = S_j) \tag{6}$$

By defining these elements and adding the initial state distribution $\pi_i = Pr(S_1 = i)$ we can model a sequence of observations, specifying from which state each observation is generated. This, in turn, allows us to generate various samples from different distributions, according to the underlying state of the system.

Typically, for the HMM's mixture of distributions a common choice is to have three states corresponding to high, intermediate, and low volatility, as stated in [Holzmann, 2014]. In this work, we will use a log-normal distribution with varying scale and mean according to the state that we are in. However, determining the appropriate number of states to use in the HMM is a critical decision.

Since developing a methodology for defining the best HMM is not the focus of our study, we propose a simple test for choosing the number of states. We aim to use the most parsimonious model that passes the Markov test with the conditional coverage statistic. This statistic follows a chi-squared distribution with two degrees of freedom, and the null hypothesis of this test is that the probability of violating the HMM boundaries is the defined coverage ratio, and such occurrences are independent of each other. If we fail to reject the null hypothesis, we cannot claim that the HMM is ill-fitted.

3. Solution Methodology

A commonly used modeling approach to represent Multistage programming is the discrete event tree, which captures the decision variables at each stage, the different possible realizations of uncertainty, and the resulting scenarios. Our objective is to obtain a policy, that represents the optimal decisions to be made at each stage for all possible scenarios, providing a comprehensive framework for analyzing the decision-making process under uncertainty and developing effective strategies, as discussed in [Valladão, 2014].

When dealing with a two-stage decision problem, a common approach involves discretizing the decision variable $\mathbf{x}$ into a set of values and solving the second-stage problem for each of these values. However, even with a small number of variables, this method can result in a large number of combinations, as highlighted in [Pereira, 1989]. Furthermore, as the number of stages increases, approximating the future cost function becomes more difficult and necessitates advanced techniques. This challenge is known as the curse of dimensionality, which can be overcome by assuming the independence of the stochastic process for each stage as proposed by [Pereira, 1989].

In our problem, assuming stage-wise independence is a key aspect since it allows us to estimate a single function representing the subsequent stage, rather than relying on knowledge of various paths in a tree-like structure. As it is known our last stage function is a convex one as the intermediate problem given at equation 1. By utilizing convex functions, we can estimate $Q_t(W_t)$ through a first-order Taylor approximation. As we augment the number of points employed for the approximation, the precision of our estimation progressively enhances. In theory, if we were able to use an infinite number of points, we would achieve an exact representation of the function $Q_t(W_t)$. Due to potential computational limitations, we work with a subset of those points.

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While stage-wise independence forms a fundamental aspect of our method, we can introduce a certain degree of dependence structure without resorting to the exponential complexity associated with decision trees. To accomplish this, we incorporate a Hidden Markov Model (HMM) into our problem formulation. In this approach, we consider a distinct function for each stage and HMM state. Given an HMM state k from a set K of possible economic states, we have a corresponding sample of returns represented by the set S. In Figure 3, we illustrate the construction of one of these cost functions, which consists of selecting the maximum value among several linear functions.

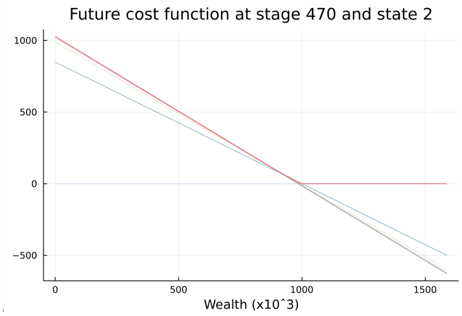

Figura 3: Estimating cost function

We incorporate a time-dependent element to approximate our future cost function. We consider an auxiliary variable $\theta_{s,k}$ for scenarios s and HMM states k that follows from a prior state j with probability $p_{j,k}$. Therefore, the approximation can be represented as $\sum_{\forall k \in K} \left(\sum_{\forall s \in S} \frac{p_s \theta_{s,k}}{1+q} \right) \times p_{j,k}$ where we incorporate the term $1+q$ to represent the adjustment of the future cost to its present value.

To ensure that each θ_k value remains above the cutoffs used to approximate their corresponding cost function for each HMM state k, we introduce a constraint for each cut employed in estimating $Q^k_{t+1}(W_{t+1}(\mathbf{x}_t))$. The first-order Taylor approximation is based on the solution at a discretized point, $\bar{W}^g_{t+1}$, and its associated dual variable, π^g_{t+1}. These values are stored in a set G obtained for each problem that defines $Q^k_{t+1}(W_{t+1}(\mathbf{x}_t))$. As a result, there exists a set $G(k)$ for each HMM state k. Finally, we also simulate possible future wealth, according to our allocation, given by $W_{t+1}(s,k)$. By calculating $\mathbb{E}[Q_{t+1}(W_{t+1}(s,k))]$ with out methodology we arrive in the following problem for intermediate states:

$$
\begin{array}{llll}
Q^j_t(W_t) = & \min\limits_{c^+_t, c^-_t, x_t} & c_t + \sum_{\forall k \in K} \left(\sum_{\forall s \in S} \frac{p_s \theta_{s,k}}{1+q} \right) \times p_{j,k} & \\
& s.t & W_{t+1}(s,k) = \sum\limits_{i \in I} x_{i,t}(1 + r_{i,t}(s,k)); \;\; \forall s \in S, & \forall k \in K \\
& & \theta_{s,k} \geq Q^k_{t+1}(\bar{W}^g_{t+1}) + \pi^g_{t+1}(W_{t+1}(s,k) - \bar{W}^g_{t+1}), & \forall s \in S, \forall g \in G(k), \forall k \in K \\
& & \theta_{s,k} \geq 0, & \forall s \in S, \forall k \in K \\
& & \sum_{i \in I} x_{i,t} - (c^+_t - c^-_t) = W_t & \\
& & c_t = c^+_t - (1-d)\, c^-_t & \\
& & c_t \leq M &
\end{array}
\tag{7}
$$

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Finally, our approach can be synthesised in the following algorithm:

Algorithm 1 Financial Planning Algorithm

Input: Goal, Contribution Upper limit, Number of stages, Fitted HMM
Output: Approximated functions for each stage
  for $t \leftarrow T$ to 1 **do**
    if $t == T$ **then**
      Run Last Stage problem with discretized points, starting with $W_T = 0$.
      Store cuts from stage T (solutions and respective duals at those points).
    else
      for $j \leftarrow 1$ to K **do**
        Simulate returns from HMM distribution in state j.
        Run intermediate problem, starting with $W_t = 0$ (use all cuts from stage $t + 1$ and the returns simulated above).
        Store cuts from stage t with HMM j.
      end for
    end if
  end for
  return Approximated functions for each stage

4. Empirical study

4.1. Case Study

Our empirical study aims to estimate a retirement savings plan for individuals by determining an investment policy consisting of monthly contributions and asset allocation. The model is designed to recommend higher contributions in the early stages of the plan to maximize compounding interest and minimize the required contribution for achieving a certain desired retirement income.

In our financial plan, contributions are limited to 20% of the salary throughout all stages, except for the final stage where the necessary amount is contributed to achieve the goal. Consequently, a penalty of $\lambda = 10$ is imposed on contributions in the last stage, denoted as T. The objective of the model is to achieve the present value required for sustaining a consistent monthly payout equivalent to 70% of the pre-retirement income until reaching the average life expectancy in Brazil. The calculation of this value involves utilizing the average monthly LIBOR rate spanning from 1986 to 2020 as the discount rate, which is set at 0.2% per month.

To determine the number of states in the HMM used in this case study, we train the model on 80% of the data and produce a probabilistic prediction with a coverage ratio of 10%, which is then tested on the remaining 20% of the S&P500 historical returns from 1986 to 2020. As a result, we have the p-value for the Markov test as follows:

Number of states	Three	Four	Five
P-value	1.70 %	7.96 %	23.4%

Tabela 1: P-value HMM

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In Figure 4, it is evident that an HMM with three states is insufficient to adequately model the data, leading to the rejection of the null hypothesis. However, when employing an HMM with four states, we successfully model the series without rejecting the null hypothesis. Thus, we opt for $K = 4$. The transition matrix of the fitted HMM can be found in Table 2, while Table 3 presents the expected real return of the SP 500.

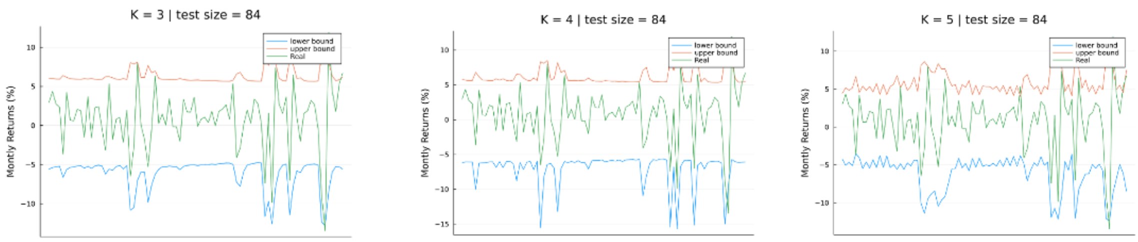

Figura 4: HMM Markov Test - K = 3 / K = 4 / K = 5

-	State 1	State 2	State 3	State 4
State 1	73.4 %	0 %	0 %	26.6 %
State 2	3.6 %	9 %	82.7 %	4.6 %
State 3	8 %	22 %	70%	0 %
State 4	25.3 %	8.3 %	61 %	5.1 %

Tabela 2: HMM transition matrix

-	State 1	State 2	State 3	State 4
Return	-2.971%	-2.732%	1.848%	6.951 %

Tabela 3: S&P Returns in Each States

The model in question provides a clear allocation policy according to the state that we are in. Those results are demonstrated by figures 5 and 6. In those we have heatmaps indicating how much is allocated in the S&P 500 for each state in every stage of the financial plan, considering that the red line represents the goal to be achieved.

In State 1, according to the HMM transition matrix , we have a high probability of staying in the same state, with negative returns for S&P500. Thanks to that, in this case, the model will always allocate 100 % in fixed income. In State 2, we have a probability of 82 % of going to the third state, which has positive returns for the S&P500. So will see the model allocate 100% in the S&P500.

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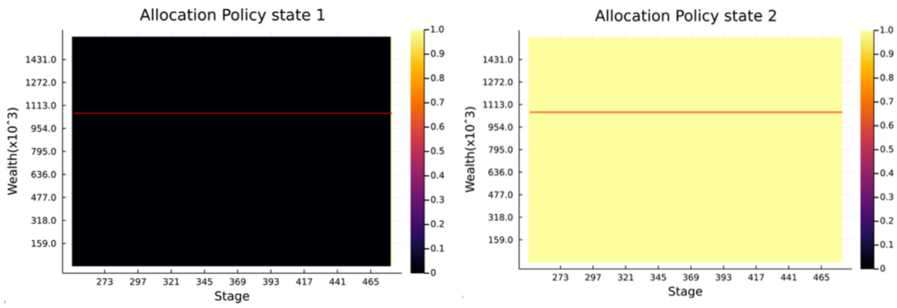

Figura 5: Allocation Policy - States 1 and 2

In State 3 we have a probability of 30 % of going to state 1 or 2, with implies a negative return for stocks, and a probability of 70 % of staying in the same state, with positive returns for the S&P 500. So we tend to allocate less in the fixed income, but it varies according to our level of wealth. The State 4 is similar to the third one although it gets a probability of 66% of going to states with positive returns for stocks, a little less than the previous case. So we can see that the model tends to allocate a little more in the fixed income.

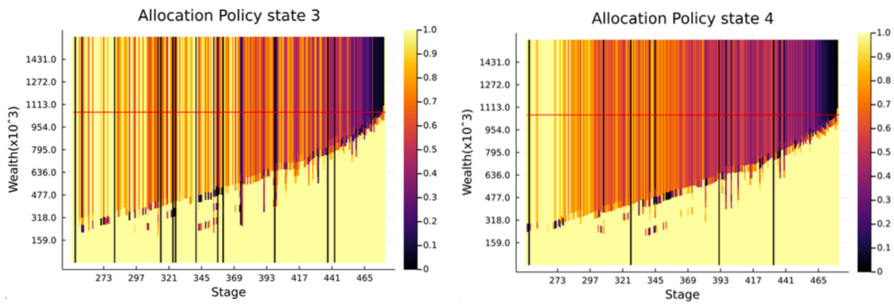

Figura 6: Allocation Policy - States 3 and 4

4.2. Sensitivity analysis

In the final stage of our model, the objective function includes a crucial parameter λ, which governs the prioritization of earlier contributions over those made in the last stage. In this section, we aimed to calibrate the value of λ through a train set and a test set (spanning two years) to evaluate the performance of our model. We compared our model with two common policies for long-term investments: the Fixed Policy, which allocates 60% to stocks and 40% to fixed income, and the Time Policy, which adjusts allocation based on the investment horizon stage. Specifically, the Time Policy allocates $100 * (t/T)\%$ in fixed income and the remainder in stocks, where t is the current stage and T is the final stage of the plan.

Our results demonstrate that the model's effectiveness is highly sensitive to the value of λ. Specifically, as depicted in Figure 7, when λ is set to 1, the models do not show a good result with most of the wealth created in the last stage thanks to a massive contribution. However, by increasing λ to 10, we observe a considerable improvement in model efficiency, achieving results that closely align with the desired target while minimizing overachievement.

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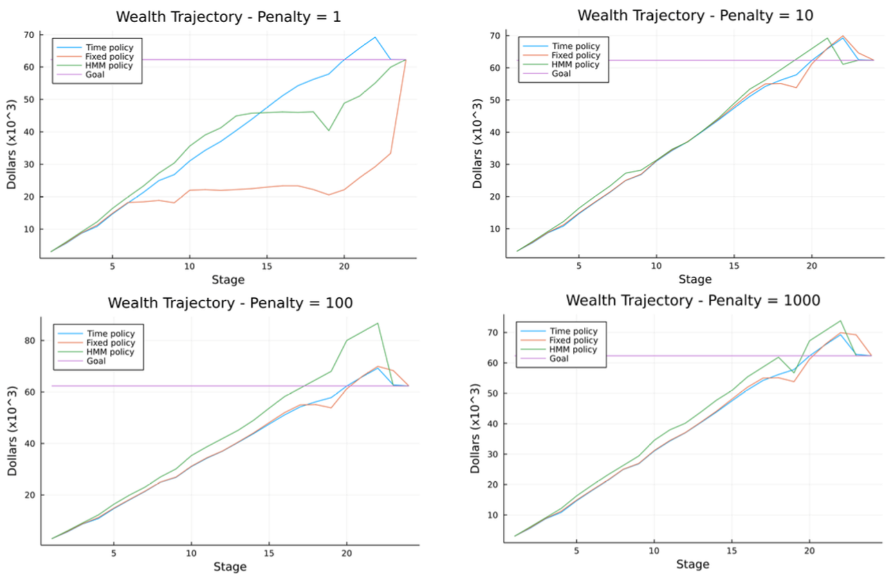

Figura 7: Parameter calibration

5. Conclusion

Our approach offers a practical and effective solution for investors, addressing the limitations of traditional portfolio optimization methods while incorporating the state of the economy for informed investment strategies. However, there are areas that require further improvement.

The calibration of the Hidden Markov Model (HMM) presented challenges, despite our study presenting a straightforward method for fitting the HMM. Exploring more efficient techniques would undoubtedly enhance its performance. Additionally, the long-term nature of our investment approach is constrained by the limited availability of data points, hampering the model's calibration. Therefore, it is crucial to employ data generation techniques to enhance our out-of-sample tests and improve the reliability of our results.

With ongoing research and advancements, our approach has the potential to become an invaluable tool for investors seeking to optimize their portfolios and achieve long-term financial goals.

Referências

Cont, R. (2000). Empirical properties of asset returns: Stylized facts and statistical issues.

Dempster, e. a., Kloppers (2016). Lifecycle goal achievement or portfolio volatility reduction?

Fowler, V. (2006). Holistic asset allocation for private individuals.

Guan, L. (2015). Mean–variance efficiency of dc pension plan under stochastic interest rate and mean-reverting returns. *Elsevier*.

Holzmann, S. (2014). Testing for the number of states in hidden markov models.

Kim, K. e. a. (2019). Personalized goal-based investing via multi-stage stochastic goal programming.

DOI: 10.59254/sbpo-2023-174924
https://proceedings.science/p/174924?lang=pt-br

Li, W. e. a. (2016). Equilibrium investment strategy for a dc plan with partial information and mean-variance criterion. *IEEE Systems Journal*.

Liu, Y. e. a. (2015). Equilibrium investment strategy for defined-contribution pension schemes with generalized mean–variance criterion and mortality risk. *Elsevier*.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*.

Pereira, P. (1989). Multi-stage stochastic optimization applied to energy planning.

Valladão, V. (2014). A multistage linear stochastic programming model for optimal corporate debt management.

Vicera (2007). Life-cycle funds.

Wang, L. e. a. (2019). Robust portfolio choice for a dc pension plan with inflation risk and mean-reverting risk premium under ambiguity. *A Journal of Mathematical Programming and Operations Research*.

Wu, Z. e. a. (2015). Nash equilibrium strategies for a defined contribution pension management. *Elsevier*.

Yao, Y. e Chen (2013). Markowitz's mean-variance defined contribution pension fund management under inflation: A continuous-time model. *Elsevier*. URL https://www.sciencedirect.com/science/article/abs/pii/S016766871300156X.

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https://proceedings.science/p/174924?lang=pt-br