Naive Set Theory

Paul R. Halmos

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Preface by the Editor

As the book title says, this is the famous set theory book *Naive Set Theory* by Paul Richard Halmos, first published in 1960 by D. Van Nostrand Company, INC., part of a series called *The University Series in Undergraduate Mathematics*.

What the title doesn't say is that this version is an independent re-edition. The original work is currently public domain in Hathi Trust Digital Library — the reader probably found (or could find) the original digitized book on Google by just searching for its title. This version was written in LaTeX and first released on July 14, 2023, available for free to download on my Github repository. After the initial release, some people have already made contributions in fixing typos and further improving the re-edition. I extend here my gratitude to these people for helping me and keeping the spirit of the re-edition alive.

Even though the book was freely available online, there are three reasons for this project. First, the book in its digitized state is perfectly readable, but it doesn't allow searching words with `Ctrl-F` (windows), `Command-F` (mac), `Ctrl-F` (linux) and doesn't have an interactable summary with it. The second is to update the book by correcting the errors in the original version, following the published errata - as noticed, and updated in this edition, by Michał Zdunek. The third reason is purely personal, I have a passion and gratitude for this book and, while I want to learn OCR, I decided to re-edit it as a homage.

Some notes on this re-edition are necessary. The book page format is B5 paper with font size 12pt. The margins of the book should be perfectly suitable for printing. The mainly differences with the original editions are the cover and the chapters title page designs. The mathematical symbol which denotes *set inclusion* in the original is (ϵ), but I opted to use ($\in$) since it's used regularly nowadays for this case. Besides this, I didn't change anything from the text. Therefore, any mistakes — which I hope are non-existent or, at least, few — are solely mine, and if someone finds any please contact me via e-mail.

As mentioned, the original book is public domain and, so, freely available in the internet. Therefore, the resulting re-edition of the book at the end of this project has no lucrative ends by any means. This re-edition cannot be used for any commercial purposes.

I thought about writting a short story about Paul R. Halmos, since it's common for books to do this specially after the author has deceased. However, I couldn't do a better job than someone just searching on Google and/or Wikipedia. So, for now, I will just say that this book has a special place in my heart. It was one of the first works that introduced and helped me push througth writting proofs. And at the end, I fell in love not only with it, but with mathematics overall. I hope that anybody that found this version can have the same outcome as I did. Now read it, absorb it and forget it.

Matheus Girola Macedo Barbosa - 01/07/2024

Preface by the Author

Every mathematician agrees that every mathematician must know some set theory; the disagreement begins in trying to decide how much is some. This book contains my answer to that question. The purpose of the book is to tell the beginning student of advanced mathematics the basic set-theoretic facts of life, and to do so with the minimum of philosophical discourse and logical formalism. The point of view throughout is that prospective mathematician anxious to study groups, or integrals, or manifolds. From this point of view the concepts and methods of this book are merely some of the standard mathematical tools; the expert specialist will find nothing new here.

Scholarly bibliographical credits and references are out of place in a purely expository book such as this one. The student who gets interested in set theory for its own sake should know, bowever, that there is much more to the subject than there is in this book. One of the most beautiful sources of set-theoretic wisdom is still Hausdorff's *Set theory*. A recent and highly readable addition to the literature, with an extensive and up-to-date bibliography, is *Axiomatic set theory* by Suppes.

In set theory "naive" and "axiomatic" are contrasting words. The present treatment might best be described as axiomatic set theory from the naive point of view. It is axiomatic in that some axioms for set theory are stated and used as the basis of all subsequent proofs. It is naive in that the language and notation are those of ordinary informal (but formalizable) mathematics. A more important way in which the naive point view predominates is that set theory is regarded as a body of facts, of which the axioms are a brief and convenient summary; in the orthodox axiomatic view the logical relations among various axioms are the central objects of study. Analogously, a study of geometry might be regarded purely naive if it proceeded on the paper-folding kind of intuition alone; the other extreme, the purely axiomatic one, is the one in which axioms for the various non-Euclidean geometries are studied with the same amount of attention as Euclid's. The analogue of the point of view of this book is the study of just one sane set of axioms with the intention of describing Euclidean geometry only.

Instead of *Naive set theory* a more honest title for the book would have been *An outline of the elements of naive set theory.* "Elements" would warn the reader that not everything is here; "outline" would warn him that even what is here needs filling in. The style is usually informal to the point of conversational. There are very few displayed theorems; most of the facts are just stated and followed by a sketch of a proof, very much as they might be in a general descriptive lecture. There are only a few exercises, officially so labelled, but, in fact, most of the book is nothing but a long chain of exercises with hints. The reader should continually ask himself whether he knows how to jump from one hint to the next, and, accordingly, he should not be discouraged if he finds that his reading rate is considerably slower than normal.

This is not to say that the contents of this book are unusually difficult or profound. What is true is that the concepts are very general and very abstract, and that, therefore, they may take some getting used to. It is a mathematical truism, however, that the more generally a theorem applies, the less deep it is. The student's task in learning set theory is to steep himself in unfamiliar but essentially shallow generalities till they become so familiar that they can be used with almost no conscious effort. In other words, general set theory is pretty trivial stuff really, but, if you want to be a mathematician, you need some, and here it is; read it, absorb it, and forget it.

P. R. H.

1 The Axiom of Extension

A pack of wolves, a bunch of grapes, or a flock of pigeons are all examples of sets of things. The mathematical concept of a set can be used as the foundation for all known mathematics. The purpose of this little book is to develop the basic properties of sets. Incidentally, to avoid terminological monotony, we shall sometimes say *collection* instead of *set*. The word "class" is also used in this context, but there is a slight danger in doing so. The reason is that in some approaches to set theory "class" has a special technical meaning. We shall have occasion to refer to this again a little later.

One thing that the development will not include is a definition of sets. The situation is analogous to the familiar axiomatic approach to elementary geometry. That approach does not offer a definition of points and lines; instead it describes what it is that one can do with those objects. The semi-axiomatic point of view adopted here assumes that the reader has the ordinary, human, intuitive (and frequently erroneous) understanding of what sets are; the purpose of the exposition is to delineate some of the many things that one can correctly do with them.

Sets, as they are usually conceived, have *elements* or *members*. An element of a set may be a wolf, a grape, or a pigeon. It is important to know that a set itself may also be an element of some other set. Mathematics is full of examples of sets of sets. A line, for instance; is a set of points; the set of all lines in the plane is a natural example of a set of sets (of points). What may be surprising is not so much that sets may occur as elements, but that for mathematical purposes no other elements need ever be considered. In this book, in particular, we shall study set, and sets of sets, and similar towers of sometimes frightening height and complexity — and nothing else. By way of examples we might occasionally speak of sets of cabbages, and kings, and the like, but such usage is always to be construed as an illuminating parable only, and not as a part of the theory that is being developed.

The principal concept of set theory, the one that in completely axiomatic studies is the principal primitive (undefined) concept, is that of *belonging*. If x belongs to A (x is an element of A, x is *contained* in A), we shall write

$$x \in A.$$

This version of the Greek letter epsilon is so often used to denote belonging that its use to denote anything else is almost prohibited. Most authors relegate $\in$ to its set-theoretic use forever and use ε when they need the fifth letter of the Greek alphabet.

Perhaps a brief digression on alphabetic etiquette in set theory might be helpful. There is no compelling reason for using small and capital letters as in the preceding paragraph; we might have written, and often will write, things like $x \in y$ and $A \in B$. Whenever possible, however, we shall informally indicate the status of a set in a particular hierarchy under consideration by means of the convention that letters at the beginning of the alphabet denote elements, and letters at the end denote sets containing them; similarly letters of a relatively simple kind denote elements, and letters of the larger and gaudier fonts denote sets containing them. Examples: $x \in A$, $A \in X$, $X \in \mathcal{C}$.

A possible relation between sets, more elementary than belonging, is *equality*. The equality of two sets A and B is universally denoted by the familiar symbol

$$A = B;$$

the fact that A and B are not equal is expressed by writing

$$A \neq B.$$

The most basic property of belonging is its relation to equality, which can be formulated as follows.

Axiom 1.1 (Axiom of extension)**.** *Two sets are equal if and only if they have the same elements.*

With greater pretentiousness and less clarity: a set is determined by its extension.

It is valuable to understand that the axiom of extension is not just a logically necessary property of equality but a non-trivial statement about belonging. One way to come to understand the point is to consider a partially analogous situation in which the analogue of the axiom of extension does not hold. Suppose, for instance, that we consider human beings instead of sets, and that, if x and

A are human beings, we write $x \in A$ whenever x is an ancestor of A. (The ancestors of a human being are his parents, his parents' parents, their parents, etc., etc.) The analogue of the axiom of extension would say here that if two human beings are equal, then they have the same ancestors (this is the "only if" part, and it is true), and also that if two human being the same ancestors, then they are equal (this is the "if" part, and it is false).

If A and B are sets and if every element of A is an element of B, we say that A is a *subset* of B, or B *includes* A, and we write

$$A \subset B$$

or

$$A \supset B.$$

The wording of the definition implies that each set must be considered to be included in itself ($A \subset A$); this fact is described by saying that set inclusion is *reflexive*. (Note that; in the same sense of the word, equality also is reflexive.) If A and B are sets such that $A \subset B$ and $A \neq B$, the word *proper* is used (proper subset, proper inclusion). If A, B, and C are sets such that $A \subset B$ and $B \subset C$, then $A \subset C$; this fact is described by saying that set inclusion is *transitive*. (This property is also shared by equality.)

If A and B are sets such that $A \subset B$ and $B \subset A$, then A and B have the same elements and therefore, by the axiom of extension, $A = B$. This fact is described by saying that set inclusion is *antisymmetric*. (In this respect set inclusion behaves differently from equality. Equality is *symmetric*, in the sense that if $A = B$, then necessarily $B = A$.) The axiom of extension can, in fact, be reformulated in these terms: if A and B are sets, then a necessary and sufficient condition that $A = B$ is that both $A \subset B$ and $B \subset A$. Correspondingly , almost all proofs of equalities between two sets A and B are split into two parts; first show that $A \subset B$, and then show that $B \subset A$.

Observe that belonging ($\in$) and inclusion ($\subset$) are conceptually very different indeed. One important difference has already manifested itself above: inclusion is always reflexive, whereas it is not at all clear that belonging is ever reflexive. That is: $A \subset A$ is always true; is $A \in A$ ever true? It is certainly not true of any reasonable set that anyone has ever seen. Observe, along the same lines, that inclusion is transitive, whereas belonging is not. Everyday examples, involving, for instance, super-organizations whose members are organizations, will readily occur to the interested reader.

2 The Axiom of Specification

All the basic principles of set theory, except only the axiom of extension, are designed to make new sets out of old ones. The first and most important of these basic principles of set manufacture says, roughly speaking, that anything intelligent one can assert about the elements of a set specifies a subset, namely, the subset of those elements about which the assertion is true.

Before formulating this principle in exact terms, we look at a heuristic example. Let A be the set of all men. The sentence "x is married" is true for some of the elements x of A and false for others. The principle we are illustrating is the one that justifies the passage from the given set A to the subset (namely, the set of all married men) specified by the given sentence. To indicate the generation of the subset, it is usually denoted by

$$\{x \in A : x \textit{ is married}\}.$$

Similarly

$$\{x \in A : x \textit{ is not married}\}$$

is the get of all bachelors;

$$\{x \in A : \textit{ the father of } x \textit{ is Adam}\}$$

is the set that contains Seth, Cain and Abel and nothing else; and

$$\{x \in A : x \textit{ is the father of Abel}\}$$

is the set that contains Adam and nothing else. Warning: a box that contains a hat and nothing else is not the same thing as a hat, and, in the same way, the last set in this list of examples is not to be confused with Adam. The analogy between sets and boxes has many weak points, but sometimes it gives a helpful picture of the facts.

All that is lacking for the precise general formulation that underlies the examples above is a definition of *sentence* . Here is a quick and informal one. There are two basic types of sentences, namely, assertions of belonging,

$$x \in A,$$

and assertions of equality,

$$A = B;$$

all other sentences are obtained from such *atomic* sentences by repeated applications of the usual logical operators, subject only to the minimal courtesies of grammar and unambiguity. To make the definition more explicit (and longer) it is necessary to append to it a list of the "usual logical operators" and the rules of syntax. An adequate (and, in fact, redundant) list of the former contains seven items:

and,
or (in the sense of "either — or — or both"),
not,
if—then—(or implies),
if and only if,
for some (or there exists),
for all.

As for the rules of sentence construction, they can be described as follows. (i) Put "not" before a sentence and enclose the result between parentheses. (The reason for parentheses, here and below, is to guarantee unambiguity. Note, incidentally, that they make all other punctuation marks unnecessary. The complete parenthetical equipment that the definition of sentences calls for is rarely needed. We shall always omit as many parentheses as it seems safe to omit without leading to confusion. In normal mathematical practice, to be followed in this book, several different sizes and shapes of parentheses are used, but that is for visual convenience only.) (ii) Put "and" or "or" or "if and only if" between two sentences and enclose the result between parentheses. (iii) Replace the dashes in "if—then—" by sentences and enclose the result in parentheses. (iv) Replace the dash in "for some—" or in "for all—" by a letter, follow the result by a sentence, and enclose the whole in parentheses. (If the letter used does not occur in the sentence, no harm is done. According to the usual and natural convention "for some y ($x \in A$)" just means "$x \in A$". It is equally harmless if the letter used has already been used with "for some—." Recall that "for some x ($x \in A$)" means the same as "for some y ($y \in A$)"; it follows that a judicious change of notation will always avert alphabetic collisions.)

We are now ready to formulate the major principle of set theory, often referred to by its German name *Aussonderungsaxiom.*

Axiom 2.1 (Axiom of specification)**.** *To every set A and to every condition $S(x)$ corresponds a set B whose elements are exactly those elements x of A for which $S(x)$ holds.*

A "condition" here is just a sentence. The symbolism is intended to indicate the letter x is *free* in the sentence $S(x)$; that means that x occurs in $S(x)$ at least once without being introduced by one of the phrases "for some x" or "for all x". It is an immediate consequence of the axiom of extension that the axiom of specification determines the set B uniquely. To indicate the way B is obtained from A and from $S(x)$ it is customary to write

$$B = \{x \in A : S(x)\}.$$

To obtain an amusing and instructive application of the axiom of specification, consider, in the role of $S(x)$, the sentence

$$\text{not } (x \in x).$$

It will be convenient, here and throughout, to write "$x \notin A$" instead of "not ($x \in A$)"; in this notation, the role of $S(x)$ is now played by

$$x \notin x.$$

It follows that, whatever the set A may be, if $B = x \in A : x \notin x$, then, for all y,

$$y \in B \textit{ if and only if } (y \in A \textit{ and } y \notin y). \tag{2.1}$$

Can it be that $B \in A$? We proceed to prove that the answer is no. Indeed, if $B \in A$, then either $B \in B$ also (unlikely, but not obviously impossible), or else $B \notin B$. If $B \in B$, then, by Equation 2.1, the assumption $B \in A$ yields $B \notin B$—a contradiction. If $B \notin B$, then, by Equation 2.1 again, the assumption $B \in A$ yields $B \in B$—a contradiction again. This completes the proof that is impossible, so that we must have $B \notin A$. The most interesting part of this conclusion is that there exists something (namely B) that does not belong to A. The set A in this argument was quite arbitrary. We have proved, in other words, that

nothing contains everything,

or, more spectacularly,

there is no universe.

"Universe" here is used in the sense of "universe of discourse," meaning, in any particular discussion, a set that contains all the objects that enter into that discussion.

In older (pre-axiomatic) approaches to set theory, the existence of universe was taken for granted, and the argument in the preceding paragraph was known as the *Russell's paradox*. The moral is that it is impossible, especially in mathematics, to get something for nothing. To specify a set, it is not enough to pronounce some magic words (which may form a sentence such as "$x \notin x$"); it is necessary also to have at hand a set to whose elements the magic words apply.

3 Unordered Pairs

For all that has been said so far, we might have been operating in a vacuum. To give the discussion some substance, let us now officially assume that

there exists a set.

Since later on we shall formulate a deeper and more useful existential assumption, this assumption plays a temporary role only. One consequence of this innocuous seeming assumption is that there exists a set without any elements at all. Indeed, if A is a set, apply the axiom of specification to A with the sentence "$x \neq x$" (or, for that matter, with any other universally false sentence). The result is the set $\{x \in A : x \neq x\}$, and that set, clearly, has no elements. The axiom of extension implies that there can be only one set with no elements. The usual symbol for that set is

$$\emptyset;$$

the set is called the *empty set*.

The empty set is a subset of every set, or, in other words, $\emptyset \subset A$ for every A. To establish this, we might argue as follows. It is to be proved that every element in $\emptyset$ belongs to A; since there are no elements in $\emptyset$, the condition is automatically fulfilled. The reasoning is correct but perhaps unsatisfying. Since it is a typical example of a frequent phenomenon, a condition holding in the "vacuous" sense, a word of advice to the inexperienced reader might be in order. To prove that something is true about the empty set, prove that it cannot be false. How, for instance, could it be false that $\emptyset \subset A$? It could be false only if $\emptyset$ had an element that did not belong to A. Since $\emptyset$ has no elements at all, this is absurd. Conclusion: $\emptyset \subset A$ is not false, and therefore $\emptyset \subset A$ for every A.

The set theory developed so far is still a pretty poor thing; for all we know there is only one set and that one is empty. Are there enough sets to ensure that every set is an element of some set? Is it true that for any two sets there is a third one that they both belong to? What about three sets, or four, or any number? We need a new principle of set construction to resolve such questions. The following principle is a good beginning.

Axiom 3.1 (Axiom of pairing)**.** *For any two sets there exists a set that they both belong to.*

Note that this is just the affirmative answer to the second question above.

To reassure worriers, let us hasten to observe that words such as "two," "three," and "four," used above, do not refer to the mathematical concepts bearing those names, which will be defined later; at present such words are merely the ordinary linguistic abbreviations for "something and then something else" repeated an appropriate number of times. Thus, for instance, the axiom of pairing, in unabbreviated form, says that if a and b are sets, then there exists a set A such that $a \in A$ and $b \in A$.

One consequence (in fact an equivalent formulation) of the axiom of pairing is that for any two sets there exists a set that contains both of them and nothing else. Indeed, if a and b are sets, and if A is a set such that $a \in A$ and $b \in A$, then we can apply the axiom of specification to A with the sentence "$x = a$ *or* $x = b$." The result is the set

$$\{x \in A : x = a \text{ } or \text{ } x = b\},$$

and that set, clearly, contains just a and b. The axiom of extension implies that there can be only one set with this property. The usual symbol for that set is

$$\{a, b\};$$

the set is called the *pair* (or, by way of emphatic comparison with a subsequent concept, the *unordered pair*) formed by a and b.

If, temporarily, we refer to the sentence "$x = a$ *or* $x = b$" as $S(x)$, we may express the axiom of pairing by saying that there exists a set B such that

$$x \in B \text{ } if \text{ } and \text{ } only \text{ } if \text{ } S(x). \tag{3.1}$$

The axiom of specification, applied to a set A, asserts the existence of a set B such that

$$x \in B \text{ if and only if } (x \in A \text{ and } S(x)). \tag{3.2}$$

The relation between Equation 3.1 and Equation 3.2 typifies something that occurs quite frequently. All the remaining principles of set construction are pseudo-special cases of the axiom of specification in the sense in which Equation 3.1 is a pseudo-special case of Equation 3.2. They all assert the existence of a set specified by a certain condition; if it were known in advance that there exists a set containing all the specified elements, then the existence of a set containing just them would indeed follow as a special case of the axiom of specification.

If a is a set, we may form the unordered pairs $\{a, a\}$. That unordered pair is denoted by

$$\{a\}$$

and is called the *singleton* of a; it is uniquely characterized by the statement that it has a as its only element. Thus, for instance, $\emptyset$ and $\{\emptyset\}$ are very different sets; the former has no elements, whereas the latter has the unique element $\emptyset$. To say that $a \in A$ is equivalent to saying that $\{a\} \subset A$.

The axiom of pairing ensures that every set is an element of some set and that any two sets are simultaneously elements of some one and the same set. (The corresponding questions for three and four and more sets will be answered later.) Another pertinent comment is that from the assumptions we have made so far we can infer the existence of very many sets indeed. For examples consider the sets $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}$, etc.; consider the pairs, such as $\{\emptyset, \{\emptyset\}\}$, formed by any two of them; consider the pairs formed by any two such pairs, or else the mixed pairs formed by any singleton and any pair; proceed so on ad infinitum.

Exercise 3.1. Are all the sets obtained in this way distinct from one another?

Before continuing our study of set theory, we pause for a moment to discuss a notational matter. It seems natural to denote the set B described in Equation 3.1 by $\{x : S(x)\}$; in the special case that was there considered

$$\{x : x = a \text{ or } x = b\} = \{a, b\}.$$

We shall use this symbolism whenever it is convenient and permissible to do so. If, that is, $S(x)$ is a condition on x such that the x's that $S(x)$ specifies constitute a set, then we may denote that set by

$$\{x : S(x)\}.$$

In case A is a set and $S(x)$ is $(x \in A)$, then it is permissible to form $\{x : S(x)\}$; in fact

$$\{x : x \in A\} = A.$$

If A is a set and $S(x)$ is an arbitrary sentence, it is permissible to form $\{x : x \in A \textit{ and } S(x)\}$; this set is the same as $\{x \in A : S(x)\}$. As further examples, we note that

$$\{x : x \neq x\} = \emptyset$$

and

$$\{x : x = a\} = \{a\}.$$

In case $S(x)$ is $(x \notin x)$, or in case $S(x)$ is $(x = x)$, the specified x's do not constitute a set.

Despite the maxim about never getting something for nothing, it seems a little harsh to be told that certain sets are not really sets and even their names must never be mentioned. Some approaches to set theory try to soften the blow by making systematic use of such illegal sets but just not calling them sets; the customary word is "class". A precise explanation of what classes really are and how they are used is irrelevant in the present approach. Roughly speaking, a class may be identified with a condition (sentence), or, rather, with the "extension" of a condition.

4 Unions and Intersections

If A and B are sets, it is sometimes natural to wish to unite their elements into one comprehensive set. One way of describing such a comprehensive set is to require it to contain all the elements that belong to at least one of the two members of the pair $\{A, B\}$. This formulation suggests a sweeping generalization of itself; surely a similar construction should apply to arbitrary collections of sets and not just to pairs of them. What is wanted, in other words, is the following principle of set construction.

Axiom 4.1 (Axiom of unions)**.** *For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.*

Here it is again: for every collection $\mathcal{C}$ there exists a set U such that if $x \in X$ for some X in $\mathcal{C}$, then $x \in U$. (Note that "at least one" is the same as "some.")

The comprehensive set U described above may be too comprehensive; it may contain elements that belong to none of the sets X in the collection $\mathcal{C}$. This is easy to remedy; just apply the axiom of specification to form the set

$$\{x \in U : x \in X \textit{ for some } X \textit{ in } \mathcal{C}\}.$$

The condition here is a translation into idiomatic usage of the mathematically more acceptable "*for some* X $(x \in X$ *and* $X \in \mathcal{C})$.") It follows that, for every x, a necessary and sufficient condition that x belong to this set is that x belong to X for some X in $\mathcal{C}$. If we change notation and call the new set U again, then

$$U = \{x : x \in X \textit{ for some } X \textit{ in } \mathcal{C}\}.$$

This set U is called the *union* of the collection $\mathcal{C}$ of sets; note that the axiom of extension guarantees its uniqueness. The simplest symbol for U that is in use at all is not very popular in mathematical circles; it is

$$\bigcup \mathcal{C}.$$

Most mathematicians prefer something like

$$\bigcup\{X : X \in \mathcal{C}\}$$

or

$$\bigcup_{X \in \mathcal{C}} X.$$

Further alternatives are available in certain important special cases; they will be described in due course.

For the time being we restrict our study of the theory of unions to the simplest facts only. The simplest fact of all is that

$$\bigcup\{X : X \in \emptyset\} = \emptyset,$$

and the next simplest fact is that

$$\bigcup\{X : X \in \{A\}\} = A.$$

In the brutally simple notation mentioned above these facts are expressed by

$$\bigcup \emptyset = \emptyset$$

and

$$\bigcup\{A\} = A.$$

The proofs are immediate from the definitions.

There is a little more substance in the union of pairs of sets (which is what started this whole discussion anyway). In that case special notation is used:

$$\bigcup\{X : X \in \{A, B\}\} = A \cup B.$$

The general definition of unions implies in the special case that $x \in A \cup B$ if and only if x belongs to either A or B or both; it follows that

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Here are some easily proved facts about the unions of pairs:

$$A \cup \emptyset = A,$$

$$A \cup B = B \cup A \ (\textit{commutatitivity}),$$

$$A \cup (B \cup C) = (A \cup B) \cup C \ (\textit{associativity}),$$

$$A \cup A = A \ (\textit{idempotence}),$$

$$A \subset B \textit{ if and only if } A \cup B = B.$$

Every student of mathematics should prove these things for himself at least once in his life. The proofs are based on the corresponding elementary properties of the logical operator *or*.

An equally simple but quite suggestive fact is that

$$\{a\} \cup \{b\} = \{a, b\}.$$

What this suggests is the way to generalize pairs. Specifically, we write

$$\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}.$$

The equation defines its left side. The right side should by rights have at least one pair of parentheses in it, but, in view of the associative law, their omission can lead to no misunderstanding. Since it is easy to prove that

$$\{a, b, c\} = \{x : x = a \text{ or } x = b \text{ or } x = c\},$$

we know now that for every three sets there exists a set that contains them and nothing else; it is natural to call that uniquely determined set the (*unordered*) *triple* formed by them. The extension of the notation and terminology thus introduced to more terms (*quadruples*, etc.) is obvious.

The formation of unions has many points of similarity with another set-theoretic operation. If A and B are sets, the *intersection* of A and B is the set

$$A \cap B$$

defined by

$$A \cap B = \{x \in A : x \in B\}.$$

The definition is symmetric in A and B even if it looks otherwise; we have

$$A \cap B = \{x \in B : x \in A\},$$

and, in fact, since $x \in A \cap B$ if and only if x belongs to both A and B, it follows that

$$A \cap B = \{x : x \in A \textit{ and } x \in B\}.$$

The basic facts about intersections, as well as their proofs, are similar to the basic facts about unions:

$$\begin{gathered} A \cap \emptyset = \emptyset, \\ A \cap B = B \cap A, \\ A \cap (B \cap C) = (A \cap B) \cap C, \\ A \cap A = A, \\ A \subset B \textit{ if and only if } A \cap B = A. \end{gathered}$$

Pairs of sets with an empty intersection occur frequently enough to justify the use of a special word: if $A \cap B = \emptyset$, the sets A and B are called *disjoint*. The same word is sometimes applied to a collection of sets to indicate that any two distinct sets of the collection are disjoint; alternatively we may speak in such a situation of a *pairwise disjoint* collection.

Two useful facts about unions and intersections involve both the operations at the same time:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

These identities are called the *distributive laws.* By way of a sample of a set-theoretic proof, we prove the second one. If x belongs to the left side, then x belongs either to A or to both B and C; if x is in A, then x is in both $A \cup B$ and $A \cup C$, and if x is in both B and C, then, again, x is in both $A \cup B$ and $A \cup C$; it follows that, in any case, x belongs to the right side. This proves that the right side includes the left. To prove the reverse inclusion, just observe that if x belongs to both $A \cup B$ and $A \cup C$, then x belongs either to A or to both B and C.

The formation of the intersection of two sets A and B, or, we might as well say, the formation of the intersection of a pair $\{A, B\}$ of sets, is a special case of a much more general operation. (This is another respect in which the theory of intersections imitates that of unions.) The existence of the general operation of intersection depends on the fact that for each non-empty collection of sets there exists a set that contains exactly those elements that belong to every set of the given collection. In other words: for each collection $\mathcal{C}$, other than $\emptyset$, there exists a set V such that $x \in V$ if and only if $x \in X$ for every X in $\mathcal{C}$. To prove this assertion, let A be any particular set in $\mathcal{C}$ (this step is justified by the fact that $\mathcal{C} \neq \emptyset$) and write

$$V = \{x \in A : x \in X \textit{ for every } X \textit{ in } \mathcal{C}\}.$$

(The condition means "*for all X (if $X \in \mathcal{C}$, then $x \in X$).*") The dependence of V on the arbitrary choice of A is illusory; in fact

$$V = \{x \in X : x \in X \textit{ for every } X \textit{ in } \mathcal{C}\}.$$

The set V is called the *intersection* of the collection $\mathcal{C}$ of sets; the axiom of extension guarantees its uniqueness. The customary notation is similar the one for unions: instead of the unobjectionable but unpopular

$$\bigcap \mathcal{C},$$

the set V is usually denoted by

$$\bigcap \{X : X \in \mathcal{C}\}$$

or

$$\bigcap_{X \in \mathcal{C}} X.$$

Exercise 4.1. A necessary and sufficient condition that $(A \cap B) \cup C = A \cap (B \cup C)$ is that $C \subset A$. Observe that the condition has nothing to do with the set B.

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