Non-Cooperative Games

A dissertation

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Abstract

This paper introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games. The games considered are n-person games represented by means of pure strategies and pay-off functions defined for the combinations of pure strategies.

The distinction between cooperative and non-cooperative games is unrelated to the mathematical description by means of pure strategies and pay-off functions of a game. Rather, it depends on the possibility or impossibility of coalitions, communication, and side-payments.

The concepts of an equilibrium point, a solution, a strong solution, a sub-solution, and values are introduced by mathematical definitions. And in later sections the interpretation of those concepts in non-cooperative games is discussed.

The main mathematical result is the proof of the existence in any game of at least one equilibrium point. Other results concern the geometrical structure of the set of equilibrium points of a game with a solution, the geometry of sub-solutions, and the existence of a symmetrical equilibrium point in a symmetrical game.

As an illustration of the possibilities for application a treatment of a simple three-man poker model is included.

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1 Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book Theory of Games and Economic Behavior (Von Neumann & Morgenstern, 1944). This books also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an equilibrium point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies".

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.

The motivation and interpretation of the mathematical concepts employed in the theory are reserved for discussion on a special section of this paper.

2 Applications

The study of n-person games for which the accepted ethics of fair play imply non-cooperative playing is, of course, an obvious direction in which to apply this theory. And poker is the most obvious target. The analysis of a more realistic poker game than our very simple model should be quite an interesting affair.

The complexity of the mathematical work needed for a complete investigation increases rather rapidly, however, with increasing complexity of the game; so that it seems that analysis of a game much more complex than the example given here would only be feasible using approximate computational methods.

A less obvious type of application is the study of cooperative games. By a cooperative game we mean a situation involving a set of players, pure strategies, and pay-offs as usual; but with the assumption that the players can and will collaborate as they do in the von Neumann and Morgenstern theory. This means the players may communicate and form coalitions which will be enforced by an umpire. It is unnecessarily restrictive, however, to assume any transferability, or even comparability of the pay-offs [which should be in utility units] to different players. Any desired transferability can be put into the game itself instead of assuming it possible in the extra-game collaboration.

The writer has developed a "dynamical" approach to the study of cooperative games based upon reduction to non-cooperative form. One proceeds by constructing a model of the pre-play negotiation so that the steps of negotiation become moves in a larger non-cooperative game [which will have an infinity of pure strategies] describing the total situation.

This larger game is then treated in terms of the theory of this paper [extended to infinite games] and if values are obtained they are taken as the values of the cooperative game. Thus the problem analyzing a cooperative game becomes the problem of obtaining a suitable, and convincing, non-cooperative model for the negotiation.

The writer has, by such a treatment, obtained values for all finite two person cooperative games, and some special n-person games.

Bibliography

Von Neumann, J., & Morgenstern, O. (1944). *Theory of Games and Economic Behavior* (1st ed.). Princeton University Press.

3 Acknowledgements