Non-Cooperative Games

John Forbes Nash Jr.

A DISSERTATION

Presented to the Faculty of Princeton University in Candidacy for the Degree of Doctor of Philosophy

Recommended for Acceptance by the Department of Mathematics

May, 1950

Abstract

This paper introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games. The games considered are n-person games represented by means of pure strategies and pay-off functions defined for the combinations of pure strategies.

The distinction between cooperative and non-cooperative games is unrelated to the mathematical description by means of pure strategies and pay-off functions of a game. Rather, it depends on the possibility or impossibility of coalitions, communication, and side-payments.

The concepts of an equilibrium point, a solution, a strong solution, a sub-solution, and values are introduced by mathematical definitions. And in later sections the interpretation of those concepts in non-cooperative games is discussed.

The main mathematical result is the proof of the existence in any game of at least one equilibrium point. Other results concern the geometrical structure of the set of equilibrium points of a game with a solution, the geometry of sub-solutions, and the existence of a symmetrical equilibrium point in a symmetrical game.

As an illustration of the possibilities for application a treatment of a simple three-man poker model is included.

Table of Contents

1 Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book Theory of Games and Economic Behavior [1]. This books also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an equilibrium point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies".

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.

The motivation and interpretation of the mathematical concepts employed in the theory are reserved for discussion on a special section of this paper.

2 Formal Definitions and Terminology

In this section we define the basic concepts of this paper and set up standard terminology and notation. Important definitions will be preceded[1] by a subtitle[2] indicating the concept defined[3]. The non-cooperative idea will be implicit, rather than explicit, below.

Definition 2.1 (Finite Game)**.** For us an n-person game will be a set of n players, or positions, each with an associated finite set of pure strategies; and corresponding to each player, i, a pay-off function, p_i, which maps the set of all n-tuples of pure strategies into the real numbers. When we use the term n-tuple we shall always mean a set of n items, with each item associated with a different player.

Definition 2.2 (Mixed Strategy, s_i)**.** A mixed strategy of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies.

We write $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ with $\sum_\alpha c_{i\alpha} = 1$ and $c_{i\alpha} \geq 0$ to represent such a mixed strategy, where the $\pi_{i\alpha}$'s are the pure strategies of player i. We regard the s_i's as points in a simplex whose vertices are the $\pi_{i\alpha}$'s. This simplex may be regarded as a convex subset of a real vector space, giving us a natural process of linear combination for the mixed strategies.

We shall use the suffixes i, j, k for players and α, β, γ to indicate various pure strategies of a player. The symbols s_i, t_i and r_i, etc. will indicate mixed strategies; $\pi_{i\alpha}$ will indicate the i^{th} player's α^{th} pure strategy, etc.

Definition 2.3 (Pay-off-function, p_i)**.** The pay-off function, p_i, used in the definition of a finite game above, has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player [n-linear]. This extension we shall also denote by p_i, writing $p_i(s_1, s_2, \cdots, s_n)$.

We shall write $\mathscr{s}$ or $\mathscr{t}$ to denote an n-tuple of mixed strategies and if $\mathscr{s} = (s_1, \cdots, s_n)$ then $p_i(\mathscr{s})$ shall mean $p_i(s_1, s_2, \cdots, s_n)$. Such an n-tuple, $\mathscr{s}$, will also be regarded as a point in a vector space, which space could be obtained by multiplying together the vector spaces containing the mixed strategies. And the set of all such n-tuples forms, of course, a convex polytope, the product of the simplices representing the mixed strategies.

For convenience we introduce the substitution notation $(\mathscr{s}; t_i)$ to stand for $(s_1, s_2, \cdots, s_{i-1}, t_i, s_{i+1}, \cdots, s_n)$ where $\mathscr{s} = (s_1, s_2, \cdots, s_n)$. The effect of successive substitutions $((\mathscr{s}; t_i); r_j)$ we indicate by $(\mathscr{s}; t_i; r_j)$, etc.

[1] The original spelling "preceeded" has been updated to "preceded" in this Quarto version.
[2] The original hyphenated "sub-title" has been rendered as "subtitle" to improve readability in this Quarto version.
[3] Definitions are highlighted explicitly in this Quarto version.

Definition 2.4 (Equilibrium point)**.** An n-tuple $\mathcal{s}$ is an equilibrium point if and only if for every i

$$p_i(\mathcal{s}) = \max_{\text{all } r_i\text{'s}}[p_i(\mathcal{s}; r_i)]. \tag{2.1}$$

Thus an equilibrium point is an n-tuple $\mathcal{s}$ such that each player's mixed strategy maximizes his pay-off if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others. We shall occasionally abbreviate equilibrium point by eq. pt.

We say that a mixed strategy s_i uses a pure strategy $\pi_{i\beta}$ if $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ and $c_{i\beta} > 0$. If $\mathcal{s} = (s_1, s_2, \cdots, s_n)$ and s_i uses $\pi_{i\alpha}$ we also say that $\mathcal{s}$ uses $\pi_{i\alpha}$.

From the linearity of $p_i(s_1, \cdots, s_n)$ in s_i,

$$\max_{\text{all } r_i\text{'s}}[p_i(\mathcal{s}; r_i)] = \max_\alpha[p_i(\mathcal{s}; \pi_{i\alpha})]. \tag{2.2}$$

We define $p_{i\alpha}(\mathcal{s}) = p_i(\mathcal{s}; \pi_{i\alpha})$. Then we obtain the following trivial necessary and sufficient condition for $\mathcal{s}$ to be an equilibrium point:

$$p_i(\mathcal{s}) = \max_\alpha p_{i\alpha}(\mathcal{s}). \tag{2.3}$$

If $\mathcal{s} = (s_1, s_2, \cdots, s_n)$ and $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ then $p_i(\mathcal{s}) = \sum_\alpha c_{i\alpha} p_{i\alpha}(\mathcal{s})$, consequently for 2.3 to hold we must have $c_{i\alpha} = 0$ whenever $p_{i\alpha}(\mathcal{s}) < \max_\beta p_{i\beta}(\mathcal{s})$, which is to say that $\mathcal{s}$ does not use $\pi_{i\alpha}$ unless it is an optimal pure strategy for player i. So we write

$$\text{if } \pi_{i\alpha} \text{ is used in } \mathcal{s} \text{ then } p_i(\mathcal{s}) = \max_\beta p_{i\beta}(\mathcal{s}) \tag{2.4}$$

as another necessary and sufficient condition for an equilibrium point.

Since a criterion 2.3 for an eq. pt. can be expressed as the equating of two continuous functions on the space of n-tuples $\mathcal{s}$ the eq. pts. obviously form a closed subset of this space. Actually, this subset is formed from a number of pieces of algebraic varieties, cut out by other algebraic varieties.

3 Existence of Equilibrium Points

I have previously published [Proc. N. A. S. 36 (1950) 48-49] [2] a proof of the result below based on Kakutani's generalized fixed point theorem. The proof given here uses the Brouwer theorem.

The method is to set up a sequence of continuous mappings: $\mathcal{s} \to \mathcal{s}'(\mathcal{s}, 1); \mathcal{s} \to \mathcal{s}'(\mathcal{s}, 2); \ldots$ whose fixed points have an equilibrium point as a[1] limit point. A limit mapping exists, but is discontinuous, and need not have any fixed points.

Theorem 3.1. *Every finite game has an equilibrium point.*

Proof. Using our standard notation, let $\mathcal{s}$ be an n-tuple of mixed strategies, and $p_{i\alpha}(\mathcal{s})$ the pay-off to player i if he uses his pure strategy $\pi_{i\alpha}$ and the others use their respective mixed strategies in $\mathcal{s}$. For each integer λ we define the following continuous functions of $\mathcal{s}$:

$$
\begin{aligned}
q_i(\mathcal{s}) &= \max_{\alpha} p_{i\alpha}(\mathcal{s}), \\
\phi_{i\alpha}(\mathcal{s}, \lambda) &= p_{i\alpha}(\mathcal{s}) - q_i(\mathcal{s}) + \frac{1}{\lambda}, \text{ and} \\
\phi^{+}_{i\alpha}(\mathcal{s}, \lambda) &= \max[0, \phi_{i\alpha}(\mathcal{s}, \lambda)].
\end{aligned}
$$

Now $\sum_\alpha \phi^{+}_{i\alpha}(\mathcal{s}, \lambda) \geq \max_\alpha \phi^{+}_{i\alpha}(\mathcal{s}, \lambda) = \frac{1}{\lambda} > 0$ so that $c'_{i\alpha}(\mathcal{s}, \lambda) = \frac{\phi^{+}_{i\alpha}(\mathcal{s}, \lambda)}{\sum_\beta \phi^{+}_{i\beta}(\mathcal{s}, \lambda)}$ is continuous.

Define $s'_i(\mathcal{s}, \lambda) = \sum_\alpha \pi_{i\alpha} c'_{i\alpha}(\mathcal{s}, \lambda)$ and $\mathcal{s}'(\mathcal{s}, \lambda) = (s'_1, s'_2, \ldots, s'_n)$. Since all the operations have preserved continuity, the mapping $\mathcal{s} \to \mathcal{s}'(\mathcal{s}, \lambda)$ is continuous; and since the space of n-tuples, $\mathcal{s}$, is a cell, there must be a fixed point for each λ. Hence there will be a subsequence $\mathcal{s}_\mu$, converging to $\mathcal{s}^*$, where $\mathcal{s}_\mu$ is fixed under the mapping $\mathcal{s} \to \mathcal{s}'(\mathcal{s}, \lambda_{(\mu)})$.

Now supposed $\mathcal{s}^*$ were not an equilibrium point. Then if $\mathcal{s}^* = (s^*_1, \ldots, s^*_n)$ some component s^*_i must be non-optimal against the others, which means s^*_i uses some pure strategy $\pi_{i\alpha}$ which is non-optimal [see 2.4, pg. 4]. This means that $p_{i\alpha}(\mathcal{s}^*) < q_i(\mathcal{s}^*)$ which justifies writing $p_{i\alpha}(\mathcal{s}^*) - q_i(\mathcal{s}^*) < -\epsilon$.

From continuity, if μ is large enough,

$$
\left| [p_{i\alpha}(\mathcal{s}_\mu) - q_i(\mathcal{s}_\mu)] - [p_{i\alpha}(\mathcal{s}^*) - q_i(\mathcal{s}^*)] \right| < \frac{\epsilon}{2} \text{ and } \frac{1}{\lambda_{(\mu)}} < \frac{\epsilon}{2}.
$$

[1]The article "a" has been added before "limit point" to align with modern grammatical standards in this Quarto version.

Adding, $p_{i\alpha}(\mathscr{s}_\mu) - q_i(\mathscr{s}_\mu) + \frac{1}{\lambda_{(\mu)}} < 0$ which is simply $\phi_{i\alpha}(\mathscr{s}_\mu, \lambda_{(\mu)}) < 0$, whence $\phi^+_{i\alpha}(\mathscr{s}_\mu, \lambda_{(\mu)}) = 0$, whence $c'_{i\alpha}(\mathscr{s}_\mu, \lambda_{(\mu)}) = 0$. From this last equation we know that $\pi_{i\alpha}$ is not used in $\mathscr{s}_\mu$ since $\mathscr{s}_\mu = \sum_\alpha \pi_{i\alpha} c'_{i\alpha}(\mathscr{s}_\mu, \lambda_{(\mu)})$, because $\mathscr{s}_\mu$ is a fixed point.

And since $\mathscr{s}_\mu \to \mathscr{s}^*$, $\pi_{i\alpha}$ is not used in $\mathscr{s}^*$, which contradicts our assumption.

Hence $\mathscr{s}^*$ is indeed an equilibrium point. $\square$

4 Symmetries of Games

An automorphism, or symmetry, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

If two strategies belong to a single player they must go into two strategies belonging to a single player. Thus if ϕ is the permutation of the pure strategies it induces a permutation ψ of the players.

Each n-tuple of pure strategies is therefore permuted into another n-tuple of pure strategies. We may call χ the induced permutation of these n-tupless. Let ξ denote an n-tuple of pure strategies and $p_i(\xi)$ the pay-off to player i when the n-tuple ξ is employed. We require that if

$$j = i^{\psi} \text{ then } p_j(\xi^{\chi}) = p_i(\xi)$$

which completes the definition of a symmetry.

The permutation ϕ has a unique linear extension to the mixed strategies. If $s_i = \sum_{\alpha} c_{i\alpha}\pi_{i\alpha}$ we define $(s_i)^{\phi} = \sum_{\alpha} c_{i\alpha}(\pi_{i\alpha})^{\phi}$.

The extension of ϕ to the mixed strategies clearly generates an extension of χ to the n-tuples of mixed strategies. We shall also denote this by χ.

We define a symmetric n-tuple $\mathscr{s}$ of a game by $\mathscr{s}^{\chi} = \mathscr{s}$ for all χ's it being understood that χ means a permutation derived from a symmetry ϕ.

Theorem 4.1. *Any finite game has a symmetric equilibrium point.*

Proof. First we note that $s_{i0} = \frac{\sum_{\alpha}\pi_{j\alpha}}{\sum_{\alpha} 1}$ has the property $(s_{i0})^{\phi} = s_{j0}$ where $j = i^{\psi}$, so that the n-tuple $\mathscr{s}_0 = (s_{10}, s_{20}, \dots, s_{n0})$ is fixed under any χ; hence any game has at least one symmetric n-tuple.

If $\mathscr{s} = (s_1, \dots, s_n)$ and $\mathscr{t} = (t_1, \dots, t_n)$ are symmetric then $\frac{\mathscr{s}+\mathscr{t}}{2} = \left(\frac{s_1+t_1}{2}, \dots, \frac{s_n+t_n}{2}\right)$ is so too because $\mathscr{s}^{\chi} = \mathscr{s} \iff s_j = (s_i)^{\phi}$ where $j = i^{\psi}$, hence

$$\frac{s_j + t_j}{2} = \frac{(s_i)^{\phi} + (t_i)^{\phi}}{2} = \left(\frac{s_i + t_i}{2}\right)^{\phi}, \text{ hence } \left(\frac{\mathscr{s}+\mathscr{t}}{2}\right)^{\chi} = \frac{\mathscr{s}+\mathscr{t}}{2}.$$

This shows that the set of symmetric n-tuples is a convex subset of the space of n-tuples since it is obviously closed.

Now observe that for each λ the mapping $\mathscr{s} \to \mathscr{s}'(\mathscr{s}, \lambda)$ used in the proof of existence theorem was intrinsically defined. Therefore, if $\mathscr{s}_2 = \mathscr{s}'(\mathscr{s}_1, \lambda)$ and χ is a permutation derived from[1] an automorphism of the game we will have

[1] The fragment "a permutation derived from" is included in this Quarto version as it was added by hand in the original manuscript.

$s_2^\chi = s'(s_1^\chi, \lambda)$. If s_1 is symmetric $s_1^\chi = s_1$ and therefore $s_2^\chi = s'(s_1, \lambda) = s_2$. Consequently this mapping maps the set of symmetric n-tuples into itself.

Since this set is a cell there must be a symmetric fixed point s_λ. And, as in the proof of the existence theorem we could obtain a limit point s^* which would have to be symmetric. $\square$

5 Solutions

We define here solutions, strong solutions, and sub-solutions. A non-cooperative game does not always have a solution, but when it does the solution is unique. Strong solutions are solutions with special properties. Sub-solutions always exist and have many of the properties of solutions, but lack uniqueness.

S_i will denote a set of mixed strategies of player i and $\mathcal{S}$ a set of n-tuples of mixed strategies.

Definition 5.1 (Solvability)**.** A game is solvable if its set, $\mathcal{S}$, of equilibrium points satisfies the condition

$$(\mathscr{t}; r_i) \in \mathcal{S} \text{ and } \mathscr{s} \in \mathcal{S} \implies (\mathscr{s}; r_i) \in \mathcal{S} \text{ for all } i\text{'s.} \tag{5.1}$$

This is called the interchangeability condition. The solution of a solvable game is its set, $\mathcal{S}$, of equilibrium points.

Definition 5.2 (Strong Solvability)**.** A game is strongly solvable if it has a solution, $\mathcal{S}$, such that for all i's

$$\mathscr{s} \in \mathcal{S} \text{ and } p_i(\mathscr{s}; r_i) = p_i(\mathscr{s}) \implies (\mathscr{s}; r_i) \in \mathcal{S}$$

and then $\mathcal{S}$ is called a strong solution.

Definition 5.3 (Equilibrium Strategies)**.** In a solvable game let S_i be the set of all mixed strategies s_i such that for some $\mathscr{t}$ the n-tuple $(\mathscr{t}; s_i)$ is an equilibrium point. [s_i is the i^{th} component of some equilibrium point.] We call S_i the set of equilibrium strategies of player i.

Definition 5.4 (Sub-solutions)**.** If $\mathcal{S}$ is a subset of the set of equilibrium points of a game and satisfies condition 5.1; and if $\mathcal{S}$ is maximal relative to this property then we call $\mathcal{S}$ a sub-solution.

For any sub-solution $\mathcal{S}$ we define the i^{th} factor set, S_i, as the set of all s_i's such that $\mathcal{S}$ contains $(\mathscr{t}; s_i)$ for some $\mathscr{t}$.

Note that a sub-solution, when unique, is a solution; and its factor sets are the sets of equilibrium strategies.

Theorem 5.1. *A sub-solution, $\mathcal{S}$, is the set of all n-tuples $(s_1, s_2, \ldots, s_n)$ such that each $s_i \in S_i$ where S_i is the i^{th} factor set of $\mathcal{S}$. Geometrically, $\mathcal{S}$ is the product of its factor sets.*

Proof. Consider such an n-tuple $(s_1, \dots, s_n)$. By definition $\exists \mathscr{t}_1, \mathscr{t}_2, \dots, \mathscr{t}_n$ such that for each i $(\mathscr{t}_i; s_i) \in \mathcal{S}$. Using the condition 5.1 n-1 times we obtain successively $(\mathscr{t}_1; s_1; s_2) \in \mathcal{S}, \dots, (\mathscr{t}_1; s_1; s_2; s_3; \dots; s_n) \in \mathcal{S}$ and the last is simply $(s_1, s_2, \dots, s_n) \in \mathcal{S}$, which we needed to show. □

Theorem 5.2. *The factor sets $S_1, S_2, \dots, S_n$ of a sub-solution are closed and convex as subsets of the mixed strategy spaces.*

Proof. It suffices to show two things:

(a) if s_i and $s'_i \in S_i$ then $s^*_i = \frac{(s_i + s'_i)}{2} \in S_i$;
(b) if $s^\#_i$ is a limit point of S_i then $s^\#_i \in S_i$.

Let $\mathscr{t} \in \mathcal{S}$. Then we have $p_j(\mathscr{t}; s_i) \geq p_j(\mathscr{t}; s_i; r_j)$ and $p_j(\mathscr{t}; s_i) \geq p_j(\mathscr{t}; s'_i; r_j)$ for any r_j, by using the criterion of 2.1, pg. 4[1] for an eq. pt. Adding these inequalities, using the linearity of $p_j(s_1, \dots, s_n)$ in S_i, and dividing by 2, we get $p_j(\mathscr{t}; s^*_i) \geq p_j(\mathscr{t}; s^*_i; r_j)$ since $s^*_i = \frac{(s_i + s'_i)}{2}$. From this we know that $(\mathscr{t}; s^*_i)$ is an eq. pt. for any $\mathscr{t} \in \mathcal{S}$. If the set of all such eq. pts. $(\mathscr{t}; s^*_i)$ is added to $\mathcal{S}$ the augmented set clearly satisfies condition 5.1, and since $\mathcal{S}$ was to be maximal it follows that $s^*_i \in S_i$.

To attack (b) note that the n-tuple $(\mathscr{t}; s^\#_i)$, where $\mathscr{t} \in \mathcal{S}$ will be a limit point of the set of n-tuples of the form $(\mathscr{t}; s_i)$ where $s_i \in S_i$, since $s^\#_i$ is a limit point of S_i. But this set is a set of eq. pts. and hence any point in its closure is an eq. pt., since the set of all eq. pts. is closed [see pg. 4]. Therefore $(\mathscr{t}; s^\#_i)$ is an eq. pt. and hence $s^\#_i \in S_i$ from the same argument as for s^*_i. □

Definition 5.5 (Values)**.** Let $\mathcal{S}$ be the set of equilibrium points of a game. We define

$$v_i^+ = \max_{\mathscr{s} \in \mathcal{S}} [p_i(\mathscr{s})] \ , \ v_i^- = \min_{\mathscr{s} \in \mathcal{S}} [p_i(\mathscr{s})].$$

If $v_i^+ = v_i^-$ we write $v_i = v_i^+ = v_i^-$. v_i^+ is the upper value to player i of the game; v_i^- the lower value; and v_i the value, if it exists.

Values will obviously have to exist if there is but one equilibrium point.

One can define associated values for a sub-solution by restricting $\mathcal{S}$ to the eq. pts. in the sub-solution and then using the same defining equations as above.

A two-person zero-sum game is always solvable in the sense defined above. The sets of equilibrium strategies S_1 and S_2 are simply the sets of “good” strategies. Such a game is not generally strongly solvable; strong solutions exist only when there is a “saddle point” in pure strategies.

[1]The reference to “pg. 3” in the original manuscript has been updated to “pg. 4” to match this Quarto version.

Simple examples

These are intended to illustrate the concepts defined in the paper and display special phenomena which occur in these games.

The first player has the roman letter strategies and the pay-off to the left, etc.

Example 5.1.

5	$a\alpha$	-3	Weak Solution: $\left(\frac{9}{16}a + \frac{7}{16}b, \frac{7}{17}\alpha + \frac{10}{17}\beta\right)$
-4	$a\beta$	4	
-5	$b\alpha$	5	$v_1 = -\frac{5}{17}, v_2 = +\frac{1}{2}$
3	$b\beta$	-4	

Example 5.2.

1	$a\alpha$	1	Strong Solution: (b, β)
-10	$a\beta$	10	
10	$b\alpha$	-10	$v_1 = v_2 = -1$
-1	$b\beta$	-1	

Example 5.3.

1	$a\alpha$	1	Unsolvable; equilibrium points (a, α),
-10	$a\beta$	-10	(b, β) and $\left(\frac{a}{2} + \frac{b}{2}, \frac{\alpha}{2} + \frac{\beta}{2}\right)$.
-10	$b\alpha$	-10	The strategies in the last case have
1	$b\beta$	1	maxi-min and mini-max properties.

Example 5.4.

1	$a\alpha$	1	Strong Solution: all pairs of mixed
0	$a\beta$	1	strategies
1	$b\alpha$	0	$v_1^+ = v_2^+ = 1, v_1^- = v_2^- = 0$
0	$b\beta$	0	

Example 5.5.

1	$a\alpha$	2	Unsolvable; eq. pts. $(a, \alpha), (b, \beta)$
-1	$a\beta$	-4	and $\left(\frac{1}{4}a + \frac{3}{4}b, \frac{3}{8}\alpha + \frac{5}{8}\beta\right)$. However,
-4	$b\alpha$	-1	empirical tests show a tendency
2	$b\beta$	1	toward (a, α).

Example 5.6.

1	$a\alpha$	1	Eq. pts.: (a, α) and (b, β), with
0	$a\beta$	0	(b, β) an example of instability.
0	$b\alpha$	0	
0	$b\beta$	0	

6 Geometrical Form of Solutions

In the two-person zero-sum case it has been shown that the set of "good" strategies of a player is a convex polyhedral subset of his strategy space. We shall obtain the same result for a player's set of equilibrium strategies in any solvable game.

Theorem 6.1. *The sets $S_1, S_2, \ldots, S_n$ of equilibrium strategies in a solvable game are polyhedral convex subsets of the respective mixed strategy spaces.*

Proof. An n-tuple $\mathscr{s}$ will be an equilibrium point if and only if for every i

$$p_i(\mathscr{s}) = \max_{\alpha} p_{i\alpha}(\mathscr{s}) \tag{6.1}$$

which is condition 2.3 on page 4. An equivalent condition is for every i and α

$$p_i(\mathscr{s}) - p_{i\alpha}(\mathscr{s}) \geq 0. \tag{6.2}$$

Let us now consider the form of the set S_j of the equilibrium strategies, s_j, of player j. Let $\mathscr{t}$ be any equilibrium point, then $(\mathscr{t}; s_j)$ will be an equilibrium point if and only if $s_j \in S_j$, from Theorem 5.1. We now apply condition 6.2 to $(\mathscr{t}; S_j)$, obtaining

$$s_j \in S_j \iff \text{for all } i, \alpha \quad p_i(\mathscr{t}; s_j) - p_{i\alpha}(\mathscr{t}; s_j) \geq 0. \tag{6.3}$$

Since p_i is n-linear and $\mathscr{t}$ is constant these are a set of linear inequalities of the form $F_{i\alpha}(s_j) \geq 0$. Each such inequality is either satisfied for all s_j or for those lying on and to one side of some hyperplane passing through the strategy simplex. Therefore, the complete set [which is finite] of conditions will all be satisfied simultaneously on some convex polyhedral subset of player j's strategy simplex. [Intersection of half-spaces.]

As a corollary we may conclude that S_k is the convex closure of a finite set of mixed strategies [vertices]. □

7 Dominance and Contradiction Methods

We say that s_i' dominates s_i if $p_i(\mathfrak{t}; s_i') > p_i(\mathfrak{t}; s_i)$ for every $\mathfrak{t}$.

This amounts to saying that s_i' gives player i a higher pay-off than s_i no matter what the strategies of the other players are. To see whether a strategy s_i' dominates s_i it suffices to consider only pure strategies for the other players because of the n-linearity of p_i.

It is obvious from the definitions that no equilibrium point can involve a dominated strategy s_i.

The domination of one mixed strategy by another will always entail other dominations. For suppose s_i' dominates s_i and t_i uses all of the pure strategies which have a higher coefficient in s_i than in s_i'. Then for a small enough $p > 0$

$$t_i' = t_i + p(s_i' - s_i)$$

is a mixed strategy; and t_i' dominates t_i by linearity.

One can prove a few properties of the set of undominated strategies. It is simply connected and is formed by the union of some collection of faces of the strategy simplex.

The information obtained by discovering dominances for one player may be of relevance to the others, insofar as the elimination of classes of mixed strategies as possible components of an equilibrium point is concerned. For the $\mathfrak{t}$'s whose components are all undominated are all that need be considered and this eliminating some of the strategies of one player may make possible the elimination of a new class of strategies for another player.

Another procedure which may be used in locating equilibrium points is the contradiction-type analysis. Here one assumes that an equilibrium point exists having component strategies lying within certain regions of the strategy spaces and proceeds to deduce further conditions which must be satisfied if the hypothesis is true. This sort of reasoning may be carried through several stages to eventually obtain a contradiction indicating that there is no equilibrium point satisfying the initial hypothesis.

8 Applications

The study of n-person games for which the accepted ethics of fair play imply non-cooperative playing is, of course, an obvious direction in which to apply this theory. And poker is the most obvious target. The analysis of a more realistic poker game than our very simple model should be quite an interesting affair.

The complexity of the mathematical work needed for a complete investigation increases rather rapidly, however, with increasing complexity of the game; so that it seems that analysis of a game much more complex than the example given here would only be feasible using approximate computational methods.

A less obvious type of application is the study of cooperative games. By a cooperative game we mean a situation involving a set of players, pure strategies, and pay-offs as usual; but with the assumption that the players can and will collaborate as they do in the von Neumann and Morgenstern theory. This means the players may communicate and form coalitions which will be enforced by an umpire. It is unnecessarily restrictive, however, to assume any transferability, or even comparability of the pay-offs [which should be in utility units] to different players. Any desired transferability can be put into the game itself instead of assuming it possible in the extra-game collaboration.

The writer has developed a "dynamical" approach to the study of cooperative games based upon reduction to non-cooperative form. One proceeds by constructing a model of the pre-play negotiation so that the steps of negotiation become moves in a larger non-cooperative game [which will have an infinity of pure strategies] describing the total situation.

This larger game is then treated in terms of the theory of this paper [extended to infinite games] and if values are obtained they are taken as the values of the cooperative game. Thus the problem analyzing a cooperative game becomes the problem of obtaining a suitable, and convincing, non-cooperative model for the negotiation.

The writer has, by such a treatment, obtained values for all finite two person cooperative games, and some special n-person games.

Bibliography

[1] J. Von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, 1st ed. Princeton University Press, 1944.

[2] J. F. Nash, “Equilibrium points in n-person games,” *Proceedings of the National Academy of Sciences*, vol. 36, no. 1, pp. 48–49, Jan. 1950, doi: 10.1073/pnas.36.1.48.

9 Acknowledgements

Drs. Tucker, Gale, and Zuhn gave valuable criticism and suggestions for improving the exposition of the material in this paper. David Gale suggested the investigation of symmetric games. The solution of the Poker model was a joint project undertaken by Lloyd S. Shapley and the author. Finally, the author was sustained financially by the Atomic Energy Commission in the period 1949-50 during which this work was done.