

Non-Cooperative Games

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Abstract

This paper introduces the concept of a non-cooperative game and develops methods for the mathematical analysis of such games. The games considered are n-person games represented by means of pure strategies and pay-off functions defined for the combinations of pure strategies.

The distinction between cooperative and non-cooperative games is unrelated to the mathematical description by means of pure strategies and pay-off functions of a game. Rather, it depends on the possibility or impossibility of coalitions, communication, and side-payments.

The concepts of an equilibrium point, a solution, a strong solution, a sub-solution, and values are introduced by mathematical definitions. And in later sections the interpretation of those concepts in non-cooperative games is discussed.

The main mathematical result is the proof of the existence in any game of at least one equilibrium point. Other results concern the geometrical structure of the set of equilibrium points of a game with a solution, the geometry of sub-solutions, and the existence of a symmetrical equilibrium point in a symmetrical game.

As an illustration of the possibilities for application a treatment of a simple three-man poker model is included.

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1 Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book Theory of Games and Economic Behavior [1]. This book also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an equilibrium point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing “good strategies”.

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.

The motivation and interpretation of the mathematical concepts employed in the theory are reserved for discussion on a special section of this paper.

2 Formal Definitions and Terminology

In this section we define the basic concepts of this paper and set up standard terminology and notation. Important definitions will be preceded¹ by a subtitle² indicating the concept defined³. The non-cooperative idea will be implicit, rather than explicit, below.

Definition 2.1 (Finite Game). For us an n-person game will be a set of n players, or positions, each with an associated finite set of pure strategies; and corresponding to each player, i , a pay-off function, p_i , which maps the set of all n-tuples of pure strategies into the real numbers. When we use the term n-tuple we shall always mean a set of n items, with each item associated with a different player.

Definition 2.2 (Mixed Strategy, s_i). A mixed strategy of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies.

We write $s_i = \sum_{\alpha} c_{i\alpha} \pi_{i\alpha}$ with $\sum_{\alpha} c_{i\alpha} = 1$ and $c_{i\alpha} \geq 0$ to represent such a mixed strategy, where the $\pi_{i\alpha}$'s are the pure strategies of player i . We regard the s_i 's as points in a simplex whose vertices are the $\pi_{i\alpha}$'s. This simplex may be regarded as a convex subset of a real vector space, giving us a natural process of linear combination for the mixed strategies.

We shall use the suffixes i, j, k for players and α, β, γ to indicate various pure strategies of a player. The symbols s_i, t_i and r_i , etc. will indicate mixed strategies; $\pi_{i\alpha}$ will indicate the i^{th} player's α^{th} pure strategy, etc.

Definition 2.3 (Pay-off-function, p_i). The pay-off function, p_i , used in the definition of a finite game above, has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player [n-linear]. This extension we shall also denote by p_i , writing $p_i(s_1, s_2, \dots, s_n)$.

We shall write σ or τ to denote an n-tuple of mixed strategies and if $\sigma = (s_1, \dots, s_n)$ then $p_i(\sigma)$ shall mean $p_i(s_1, s_2, \dots, s_n)$. Such an n-tuple, σ , will also be regarded as a point in a vector space, which space could be obtained by multiplying together the vector spaces containing the mixed strategies. And the set of all such n-tuples forms, of course, a convex polytope, the product of the simplices representing the mixed strategies.

For convenience we introduce the substitution notation $(\sigma; t_i)$ to stand for $(s_1, s_2, \dots, s_{i-1}, t_i, s_{i+1}, \dots, s_n)$ where $\sigma = (s_1, s_2, \dots, s_n)$. The effect of successive substitutions $((\sigma; t_i); r_j)$ we indicate by $(\sigma; t_i; r_j)$, etc.

¹The original spelling “preceeded” has been updated to “preceded” in this [Quarto](#) version.

²The original hyphenated “sub-title” has been rendered as “subtitle” to improve readability in this [Quarto](#) version.

³Definitions are highlighted explicitly in this [Quarto](#) version.

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Definition 2.4 (Equilibrium point). An n-tuple \mathbf{s} is an equilibrium point if and only if for every i

$$p_i(\mathbf{s}) = \max_{\text{all } r_i \text{'s}} [p_i(\mathbf{s}; r_i)]. \quad (2.1)$$

Thus an equilibrium point is an n-tuple \mathbf{s} such that each player's mixed strategy maximizes his pay-off if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others. We shall occasionally abbreviate equilibrium point by eq. pt.

We say that a mixed strategy s_i uses a pure strategy $\pi_{i\beta}$ if $s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha}$ and $c_{i\beta} > 0$. If $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and s_i uses $\pi_{i\alpha}$ we also say that \mathbf{s} uses $\pi_{i\alpha}$.

From the linearity of $p_i(s_1, \dots, s_n)$ in s_i ,

$$\max_{\text{all } r_i \text{'s}} [p_i(\mathbf{s}; r_i)] = \max_\alpha [p_i(\mathbf{s}; \pi_{i\alpha})]. \quad (2.2)$$

We define $p_{i\alpha}(\mathbf{s}) = p_i(\mathbf{s}; \pi_{i\alpha})$. Then we obtain the following trivial necessary and sufficient condition for \mathbf{s} to be an equilibrium point:

$$p_i(\mathbf{s}) = \max_\alpha p_{i\alpha}(\mathbf{s}). \quad (2.3)$$

If $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha}$ then $p_i(\mathbf{s}) = \sum_\alpha c_{i\alpha} p_{i\alpha}(\mathbf{s})$, consequently for 2.3 to hold we must have $c_{i\alpha} = 0$ whenever $p_{i\alpha}(\mathbf{s}) < \max_\beta p_{i\beta}(\mathbf{s})$, which is to say that \mathbf{s} does not use $\pi_{i\alpha}$ unless it is an optimal pure strategy for player i . So we write

$$\text{if } \pi_{i\alpha} \text{ is used in } \mathbf{s} \text{ then } p_i(\mathbf{s}) = \max_\beta p_{i\beta}(\mathbf{s}) \quad (2.4)$$

as another necessary and sufficient condition for an equilibrium point.

Since a criterion 2.3 for an eq. pt. can be expressed as the equating of two continuous functions on the space of n-tuples \mathbf{s} the eq. pts. obviously form a closed subset of this space. Actually, this subset is formed from a number of pieces of algebraic varieties, cut out by other algebraic varieties.

3 Existence of Equilibrium Points

I have previously published [Proc. N. A. S. 36 (1950) 48-49] [2] a proof of the result below based on Kakutani's generalized fixed point theorem. The proof given here uses the Brouwer theorem.

The method is to set up a sequence of continuous mappings: $s \rightarrow s'(s, 1); s \rightarrow s'(s, 2); \dots$ whose fixed points have an equilibrium point as a¹ limit point. A limit mapping exists, but is discontinuous, and need not have any fixed points.

Theorem 3.1. *Every finite game has an equilibrium point.*

Proof. Using our standard notation, let s be an n-tuple of mixed strategies, and $p_{i\alpha}(s)$ the pay-off to player i if he uses his pure strategy $\pi_{i\alpha}$ and the others use their respective mixed strategies in s . For each integer λ we define the following continuous functions of s :

$$\begin{aligned} q_i(s) &= \max_{\alpha} p_{i\alpha}(s), \\ \phi_{i\alpha}(s, \lambda) &= p_{i\alpha}(s) - q_i(s) + \frac{1}{\lambda}, \text{ and} \\ \phi_{i\alpha}^+(s, \lambda) &= \max[0, \phi_{i\alpha}(s, \lambda)]. \end{aligned}$$

Now $\sum_{\alpha} \phi_{i\alpha}^+(s, \lambda) \geq \max_{\alpha} \phi_{i\alpha}^+(s, \lambda) = \frac{1}{\lambda} > 0$ so that $c'_{i\alpha}(s, \lambda) = \frac{\phi_{i\alpha}^+(s, \lambda)}{\sum_{\beta} \phi_{i\beta}^+(s, \lambda)}$ is continuous.

Define $s'_i(s, \lambda) = \sum_{\alpha} \pi_{i\alpha} c'_{i\alpha}(s, \lambda)$ and $s'(s, \lambda) = (s'_1, s'_2, \dots, s'_n)$. Since all the operations have preserved continuity, the mapping $s \rightarrow s'(s, \lambda)$ is continuous; and since the space of n-tuples, s , is a cell, there must be a fixed point for each λ . Hence there will be a subsequence s_{μ} , converging to s^* , where s_{μ} is fixed under the mapping $s \rightarrow s'(s, \lambda_{(\mu)})$.

Now suppose s^* were not an equilibrium point. Then if $s^* = (s_1^*, \dots, s_n^*)$ some component s_i^* must be non-optimal against the others, which means s_i^* uses some pure strategy $\pi_{i\alpha}$ which is non-optimal [see 2.4, pg. 4]. This means that $p_{i\alpha}(s^*) < q_i(s^*)$ which justifies writing $p_{i\alpha}(s^*) - q_i(s^*) < -\epsilon$.

From continuity, if μ is large enough,

$$|[p_{i\alpha}(s_{\mu}) - q_i(s_{\mu})] - [p_{i\alpha}(s^*) - q_i(s^*)]| < \frac{\epsilon}{2} \text{ and } \frac{1}{\lambda_{(\mu)}} < \frac{\epsilon}{2}.$$

¹The article "a" has been added before "limit point" to align with modern grammatical standards in this Quarto version.

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Adding, $p_{i\alpha}(s_\mu) - q_i(s_\mu) + \frac{1}{\lambda_{(\mu)}} < 0$ which is simply $\phi_{i\alpha}(s_\mu, \lambda_{(\mu)}) < 0$, whence $\phi_{i\alpha}^+(s_\mu, \lambda_{(\mu)}) = 0$, whence $c'_{i\alpha}(s_\mu, \lambda_{(\mu)}) = 0$. From this last equation we know that $\pi_{i\alpha}$ is not used in s_μ since $s_\mu = \sum_\alpha \pi_{i\alpha} c'_{i\alpha}(s_\mu, \lambda_{(\mu)})$, because s_μ is a fixed point.

And since $s_\mu \rightarrow s^*$, $\pi_{i\alpha}$ is not used in s^* , which contradicts our assumption.

Hence s^* is indeed an equilibrium point. \square

4 Symmetries of Games

An automorphism, or symmetry, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

If two strategies belong to a single player they must go into two strategies belonging to a single player. Thus if ϕ is the permutation of the pure strategies it induces a permutation ψ of the players.

Each n-tuple of pure strategies is therefore permuted into another n-tuple of pure strategies. We may call χ the induced permutation of these n-tuples. Let ξ denote an n-tuple of pure strategies and $p_i(\xi)$ the pay-off to player i when the n-tuple ξ is employed. We require that if

$$j = i^\psi \text{ then } p_j(\xi^\chi) = p_i(\xi)$$

which completes the definition of a symmetry.

The permutation ϕ has a unique linear extension to the mixed strategies. If $s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha}$ we define $(s_i)^\phi = \sum_\alpha c_{i\alpha} (\pi_{i\alpha})^\phi$.

The extension of ϕ to the mixed strategies clearly generates an extension of χ to the n-tuples of mixed strategies. We shall also denote this by χ .

We define a symmetric n-tuple s of a game by $s^\chi = s$ for all χ 's it being understood that χ means a permutation derived from a symmetry ϕ .

Theorem 4.1. *Any finite game has a symmetric equilibrium point.*

Proof. First we note that $s_{i0} = \frac{\sum_\alpha \pi_{j\alpha}}{\sum_\alpha 1}$ has the property $(s_{i0})^\phi = s_{j0}$ where $j = i^\psi$, so that the n-tuple $s_0 = (s_{10}, s_{20}, \dots, s_{n0})$ is fixed under any χ ; hence any game has at least one symmetric n-tuple.

If $s = (s_1, \dots, s_n)$ and $t = (t_1, \dots, t_n)$ are symmetric then $\frac{s+t}{2} = \left(\frac{s_1+t_1}{2}, \dots, \frac{s_n+t_n}{2} \right)$ is so too because $s^\chi = s \iff s_j = (s_i)^\phi$ where $j = i^\psi$, hence

$$\frac{s_j + t_j}{2} = \frac{(s_i)^\phi + (t_i)^\phi}{2} = \left(\frac{s_i + t_i}{2} \right)^\phi, \text{ hence } \left(\frac{s+t}{2} \right)^\chi = \frac{s+t}{2}.$$

This shows that the set of symmetric n-tuples is a convex subset of the space of n-tuples since it is obviously closed.

Now observe that for each λ the mapping $s \rightarrow s'(s, \lambda)$ used in the proof of existence theorem was intrinsically defined. Therefore, if $s_2 = s'(s_1, \lambda)$ and χ is a permutation derived from¹ an automorphism of the game we will have

¹The fragment “a permutation derived from” is included in this [Quarto](#) version as it was added by hand in the original manuscript.

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$s_2^\chi = s'(s_1^\chi, \lambda)$. If s_1 is symmetric $s_1^\chi = s_1$ and therefore $s_2^\chi = s'(s_1, \lambda) = s_2$. Consequently this mapping maps the set of symmetric n-tuples to itself.

Since this set is a cell there must be a symmetric fixed point of s_λ . And, as in the proof of the existence theorem we could obtain a limit point s^* which would have to be symmetric. \square

5 Applications

The study of n-person games for which the accepted ethics of fair play imply non-cooperative playing is, of course, an obvious direction in which to apply this theory. And poker is the most obvious target. The analysis of a more realistic poker game than our very simple model should be quite an interesting affair.

The complexity of the mathematical work needed for a complete investigation increases rather rapidly, however, with increasing complexity of the game; so that it seems that analysis of a game much more complex than the example given here would only be feasible using approximate computational methods.

A less obvious type of application is the study of cooperative games. By a cooperative game we mean a situation involving a set of players, pure strategies, and pay-offs as usual; but with the assumption that the players can and will collaborate as they do in the von Neumann and Morgenstern theory. This means the players may communicate and form coalitions which will be enforced by an umpire. It is unnecessarily restrictive, however, to assume any transferability, or even comparability of the pay-offs [which should be in utility units] to different players. Any desired transferability can be put into the game itself instead of assuming it possible in the extra-game collaboration.

The writer has developed a “dynamical” approach to the study of cooperative games based upon reduction to non-cooperative form. One proceeds by constructing a model of the pre-play negotiation so that the steps of negotiation become moves in a larger non-cooperative game [which will have an infinity of pure strategies] describing the total situation.

This larger game is then treated in terms of the theory of this paper [extended to infinite games] and if values are obtained they are taken as the values of the cooperative game. Thus the problem analyzing a cooperative game becomes the problem of obtaining a suitable, and convincing, non-cooperative model for the negotiation.

The writer has, by such a treatment, obtained values for all finite two person cooperative games, and some special n-person games.

Bibliography

- [1] J. Von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior*, 1st ed. Princeton University Press, 1944.
- [2] J. F. Nash, “Equilibrium points in n-person games,” *Proceedings of the National Academy of Sciences*, vol. 36, no. 1, pp. 48–49, Jan. 1950, doi: [10.1073/pnas.36.1.48](https://doi.org/10.1073/pnas.36.1.48).

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