Number Systems and the Foundations of Analysis

Mathematical Adventures

Luis Francisco Gomez Lopez

2024-08-26

Table of contents

Preface

The aim of this book is to study the basic number system of mathematics using as a principal guide Mendelson (2008) and other supplementary materials.

The idea is to develop all the topics and solve the exercises proposed in Mendelson (2008) as it has done in Mathematical Adventures

1 Basic Facts and Notions of Logic and Set Theory

This chapter introduces the concepts and terminology of logic and set theory.

What sets aside mathematics from other disciplines is its reliance on proof (Bilaniuk 2009, ix). But what is a proof? In an informal way a proof is any reasoned argument accepted as such by other mathematicians (Bilaniuk 2009, ix).

In that sense mathematical logic is concerned with formalizing and analyzing the kinds of reasoning used in the rest of mathematics (Bilaniuk 2009, ix).

1.1 Propositional logic

Propositional logic attempts to make precise the relationships that certain connectives like *not*, *and*, *or*, and *if … then* (Bilaniuk 2009, x).

1.1.1 Language

First we will define the formal language of propositional logic, $\mathcal{L}_P$, by specifying the symbols and formulas

Definition 1.1 (Symbols of $\mathcal{L}_P$)**.** The symbols of $\mathcal{L}_P$ are:

- Parentheses: (and)
- Connectives: $\neg$ and $\Longrightarrow$
- Atomic formulas: $A_0, A_1, A_2, \ldots, A_n, \ldots$

Then we specify the ways in which the symbols of $\mathcal{L}_P$ can be put together.

Definition 1.2 (Formulas of $\mathcal{L}_P$)**.** The formulas of $\mathcal{L}_P$ are those finite sequences or strings of the symbols given in Definition 1.1 which satisfy the following rules:

- Every atomic formula is a formula
- If α is a formula, then $(\neg\alpha)$ is a formula

- If α and β are formulas, then $(\alpha \Longrightarrow \beta)$ is a formula
- No other sequence of symbols is a formula

In Definition 1.1 and Definition 1.2 parentheses are just punctuation where their purpose is to group other symbols together, $\neg$ and $\Longrightarrow$ represent the connectives *not* and *if … then* and the atomic formulas represent statements that cannot be broken down any further using our connectives. Finally we specify that a well formed formula can only be obtained by the first 3 rules pointed out in Definition 1.2[1].

Exercise 1.1. Show that every formula of $\mathcal{L}_P$ has the same number of left parentheses as it has of right parentheses.

Solution 1.1. By strong induction on n, the number of connectives (occurrences of $\neg$ or $\Longrightarrow$) in a formula φ of $\mathcal{L}_P$

Base step ($n = 0$): If φ is a formula of $\mathcal{L}_P$ with no connectives by Definition 1.2 is an atomic formula. Since an atomic formula has no parentheses is has the same left as right parentheses.

Induction hypothesis ($n \leq k$): Assume a formula with $n \leq k$ connectives has the same left as right parentheses.

Induction step ($n = k + 1$): Suppose φ is a formula with $n = k + 1$ connectives. From Definition 1.2 φ must be either:

- $(\neg\alpha)$ for some formula α with k connectives.
  - By the *induction hypothesis* α has the same left as right parentheses so $(\neg\alpha)$ also have the same left as right parentheses.
- $(\beta \Longrightarrow \gamma)$ for some formulas β and γ with $\leq k$ connectives each.
  - By the *induction hypothesis* β and γ have the same left as right parentheses so $(\beta \Longrightarrow \gamma)$ also has the same left as right parentheses.

Exercise 1.2. Suppose α is any formula of $\mathcal{L}_P$. Let $l(\alpha)$ be the length of α as a sequence of symbols and let $p(\alpha)$ be the number of parentheses (counting both left and right parentheses) in α. What are the minimum and maximum values of $\frac{p(\alpha)}{l(\alpha)}$?

Solution 1.2. The minimum value of $p(\alpha)$ is 0 when α is an atomic formula. Therefore the minimum value of $\frac{p(\alpha)}{l(\alpha)}$ is 0 because $p(\alpha) \geq 0$ and $l(\alpha) \geq 1$.

In the case of other values lets inspect the possible values of $p(\alpha)$ and $l(\alpha)$:

[1]See (Church 1996, 70) for more details.

- For $p(\alpha)$ the possible values are $0, 2, 4, \dots, 2m, \dots$.
- For $l(\alpha)$ we can begin with an atomic formula, A_0, and then add 3 or 4 symbols to create a well formed formula:
  - $(\neg A_0)$
  - $(A_0 \implies A_1)$

Where the possible values of $l(\alpha)$ are $1, 4, 5, 7, 8, \dots, s-1, s, s+1, \dots$ as it is shown in Figure 1.1 where the majority of duplicate branches with the same lengths are omitted but can be different in relation to $p(\alpha)$.

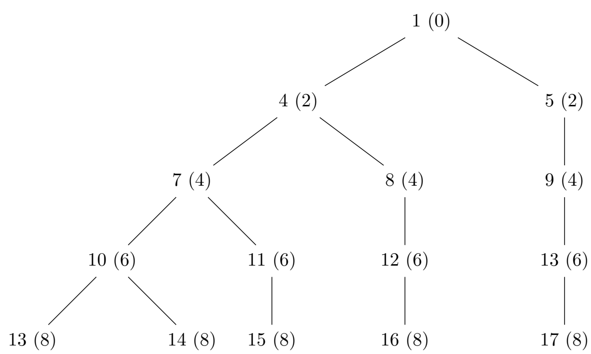

Figure 1.1: Possible values of $l(\alpha)$ and associated $p(\alpha)$ values in parentheses

Therefore we can examine the leftmost branch.

- For the lefmost branch we have for $\frac{p(\alpha)}{l(\alpha)} = \frac{0}{1}, \frac{2}{4}, \frac{4}{7}, \frac{6}{10}, \frac{8}{13}, \dots, \frac{2n}{3n+1}, \dots$. Therefore $\lim_{n\to\infty} \frac{2n}{3n+1} = \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3}$.

Therefore $0 \leq \frac{p(\alpha)}{l(\alpha)}$ and $\frac{p(\alpha)}{l(\alpha)} < \frac{2}{3}$ for the lefmost branch

Exercise 1.3. Suppose α is any formula of $\mathcal{L}_P$. Let $s(\alpha)$ be the number of atomic formulas in α (counting repetitions) and let $c(\alpha)$ be the number of occurrences of $\implies$ in α. Show that $s(\alpha) = c(\alpha) + 1$.

Solution 1.3. By strong induction on n, the number of connectives (occurrences of $\neg$ or $\implies$) in a formula φ of $\mathcal{L}_P$

Base step $(n = 0)$: If φ is a formula of $\mathcal{L}_P$ with no connectives by Definition 1.2 is an atomic formula. Since an atomic formula has no $\implies$ then $c(\varphi) = 0$ and $s(\varphi) = 1$ and so $s(\varphi) = c(\varphi)+1$.

Induction hypothesis $(n \leq k)$: Assume that for a formula with $n \leq k$ connectives $s(\varphi) = c(\varphi) + 1$.

Induction step $(n = k + 1)$: Suppose φ is a formula with $n = k + 1$ connectives. From Definition 1.2 φ must be either:

- $(\neg\alpha)$ for some formula α with k connectives.
  - By the *induction hypothesis* for α we have that $s(\alpha) = c(\alpha) + 1$. Therefore for $(\neg\alpha)$ we have also that $s((\neg\alpha)) = c((\neg\alpha)) + 1$.
- $(\beta \implies \gamma)$ for some formulas β and γ with $\leq k$ connectives each.
  - By the *induction hypothesis* for β and γ we have that $s(\beta) = c(\beta) + 1$ and $s(\gamma) = c(\gamma) + 1$. Therefore $s((\beta \implies \gamma)) = c(\beta) + c(\gamma) + 1 + 1$. But $c(\beta) + c(\gamma) + 1 = c((\beta \implies \gamma))$ so $s((\beta \implies \gamma)) = c((\beta \implies \gamma)) + 1$

Exercise 1.4. What are the possible lengths of formulas of $\mathcal{L}_P$? Prove it.

Solution 1.4. Using Solution 1.2 and Figure 1.1 the possible possible lengths of formulas of $\mathcal{L}_P$ are $1, 4, 5, 7, 8, \ldots, s-1, s, s+1, \ldots$.

Exercise 1.5. Find a way for doing without parentheses or other punctuation symbols in defining a formal language for $\mathcal{L}_P$.

Solution 1.5. See (Łukasiewicz and Borkowski 1970, 180) which is called Lukasiewicz or Polish notation

In Polish notation, logical operators are placed before their operands. This prefix positioning eliminates the need for parentheses, as the order of operations is strictly determined by the operator sequence.

So using Definition 1.2 we have that:

- Every atomic formula is a formula
- If α is a formula, then $\neg\alpha$ is a formula
- If α and β are formulas, then $\implies \alpha\beta$ is a formula
- No other sequence of symbols is a formula

For example $(((\neg\alpha) \implies \beta) \implies (\neg\gamma))$ can be written using Polish and modern notation as $\implies \implies \neg\alpha\beta\neg\gamma$

Also Polish notation is not ambiguous. For example:

- $((\alpha \implies \beta) \implies \gamma)$ is written as $\implies \implies \alpha\beta\gamma$
- $(\alpha \implies (\beta \implies \gamma))$ is written as $\implies \alpha \implies \beta\gamma$

Exercise 1.6. Show that the set of formulas of $\mathcal{L}_P$ is countable.

Solution 1.6. **Pending**

1.1.1.1 Informal conventions

We will use the symbols $\wedge$, $\vee$, and $\iff$ to represent *and*, *or* and *if and only if* respectively. Since they are not among the symbols of $\mathcal{L}_P$, we will use them as abbreviations for certain constructions involving only $\neg$ and $\implies$. Namely:

- $(\alpha \wedge \beta)$ is short for $(\neg(\alpha \implies (\neg\beta)))$,
- $(\alpha \vee \beta)$ is short for $((\neg\alpha) \implies \beta)$, and
- $(\alpha \iff \beta)$ is short for $((\alpha \implies \beta) \wedge (\beta \implies \alpha))$

Definition 1.3 (Subformulas)**.** Suppose φ is a formula of $\mathcal{L}_p$. The set of subformulas of φ, $S(\varphi)$, is defined as follows:

- If φ is an atomic formula then $S(\varphi) = \{\varphi\}$
- If φ is $(\neg\alpha)$ then $S(\varphi) = S(\alpha) \cup \{(\neg\alpha)\}$
- If φ is $(\alpha \implies \beta)$ then $S(\varphi) = S(\alpha) \cup S(\beta) \cup \{(\alpha \implies \beta)\}$

Exercise 1.7. Find all the subformulas of each of the following formulas.

- $(\neg((\neg A_{56}) \implies A_{56}))$
- $(A_9 \implies (A_8 \implies (\neg(A_{78} \implies (\neg(\neg A_0))))))$
- $(((\neg A_0) \wedge (\neg A_1)) \iff (\neg(A_0 \vee A_1)))$

Solution 1.7.

$$\begin{aligned} S((\neg((\neg A_{56}) \implies A_{56}))) &= S(((\neg A_{56}) \implies A_{56})) \cup \{(\neg((\neg A_{56}) \implies A_{56}))\} \\ &= \{A_{56}, (\neg A_{56}), ((\neg A_{56}) \implies A_{56}), \\ &\quad (\neg((\neg A_{56}) \implies A_{56}))\} \end{aligned}$$

$$\begin{aligned} S((A_9 \implies (A_8 \implies (\neg(A_{78} \implies (\neg(\neg A_0))))))) &= S(A_9) \cup S((A_8 \implies (\neg(A_{78} \implies (\neg(\neg A_0)))))) \cup \{(A_9 \implies (A_8 \implies \\ &= \{A_9, A_8, A_{78}, A_0, (\neg A_0), \\ &\quad (\neg(\neg A_0)), (A_{78} \implies (\neg(\neg A_0))), \\ &\quad (\neg(A_{78} \implies (\neg(\neg A_0)))), \\ &\quad (A_8 \implies (\neg(A_{78} \implies (\neg(\neg A_0))))), \\ &\quad (A_9 \implies (A_8 \implies (\neg(A_{78} \implies (\neg(\neg A_0))))))\} \end{aligned}$$

Let $\varphi = (((\neg A_0) \wedge (\neg A_1)) \implies (\neg(A_0 \vee A_1)))$ and $\sigma = ((\neg(A_0 \vee A_1)) \implies ((\neg A_0) \wedge (\neg A_1)))$

$$
\begin{aligned}
S((((\neg A_0) \wedge (\neg A_1)) \iff (\neg(A_0 \vee A_1)))) &= S((\varphi \implies (\neg\sigma))) \cup \{(\neg(\varphi \implies (\neg\sigma)))\} \\
&= \{A_0, A_1, (\neg A_0), (\neg A_1), (\neg(\neg A_1)), \\
&\quad (A_0 \vee A_1), ((\neg A_0) \implies (\neg(\neg A_1))), \\
&\quad (\neg(A_0 \vee A_1)), ((\neg A_0) \wedge (\neg A_1)) \\
&\quad \sigma, (\neg\sigma), \varphi, \\
&\quad (\varphi \implies (\neg\sigma)), \\
&\quad (\neg(\varphi \implies (\neg\sigma)))\}
\end{aligned}
$$

1.1.1.2 Unique Readability

To ensure that the formulas of $\mathcal{L}_p$ are unambiguous (can be read in only one way according to the rules given in Definition 1.2) one must add to Definition 1.1 the requirement that all the symbols of $\mathcal{L}_p$ are distinct and that no symbol is a subsequence of any other symbol.

Theorem 1.1 (Unique Readability)**.** *A formula of $\mathcal{L}_p$ must satisfy exactly one of the first 3 conditions in Definition 1.2.*

Proof. By strong induction on n, the number of connectives (occurrences of $\neg$ or $\implies$) in a formula φ of $\mathcal{L}_P$

Base step ($n = 0$): If φ is a formula of $\mathcal{L}_P$ with no connectives by Definition 1.2 is an atomic formula. Since an atomic formula has no connectives it can not of the form $(\neg\varphi)$ or $(\alpha \implies \beta)$.

Induction hypothesis ($n \leq k$): Assume a formula with $n \leq k$ connectives must satisfy exactly one of the first 3 conditions in Definition 1.2.

Induction step ($n = k + 1$): Suppose φ is a formula with $n = k + 1$ connectives. From Definition 1.2 φ must be either:

- $(\neg\alpha)$ for some formula α with k connectives.
  - By the *induction hypothesis* α must satisfy exactly one of the first 3 conditions in Definition 1.2.
    - Because α satisfy exactly one of the first 3 conditions in Definition 1.2 then $(\neg\alpha)$ can't be an atomic formula because an atomic formula doesn't have connectives.
    - Now assume that $(\neg\alpha)$ can be expressed as $(\beta \implies \gamma)$. Then $\neg\alpha$ will be equal to $\beta \implies \gamma$. However α satisfy exactly one of the first 3 conditions in Definition 1.2 so it will be impossible that $\neg\alpha$ and $\beta \implies \gamma$ are equal.
      - Therefore $(\neg\alpha)$ satisfy exactly the 2 condition in Definition 1.2.

- $(\beta \implies \gamma)$ for some formulas β and γ with $\leq k$ connectives each.
  - By the *induction hypothesis* β and γ must satisfy exactly one of the first 3 conditions in Definition 1.2.
    * Because β and γ satisfy exactly one of the first 3 conditions in Definition 1.2 then $(\beta \implies \gamma)$ can't be an atomic formula because an atomic formula doesn't have connectives.
    * Now assume that $(\beta \implies \gamma)$ can be expressed as $(\neg\sigma)$. Then $\neg\sigma$ will be equal to $\beta \implies \gamma$. However β and γ satisfy exactly one of the first 3 conditions in Definition 1.2 so it will be impossible that $\neg\sigma$ and $\beta \implies \gamma$ are equal.
      · Therefore $(\beta \implies \gamma)$ satisfy exactly the 3 condition in Definition 1.2.

□

1.1.2 Truth Assignments

Whether a given formula φ of $\mathcal{L}_p$ is true or false usually depends on how we interpret the atomic formulas which appear in φ (Bilaniuk 2009, 7).

Definition 1.4 (Truth assignment)**.** A truth assignment is a function v whose domain is the set of all formulas of $\mathcal{L}_p$ and whose range is the set $\{T, F\}$ of truth values, such that:

- $v(A_n)$ is defined for every atomic formula A_n.
- For any formula α:

$$v((\neg\alpha)) = \begin{cases} T & \text{if } v(\alpha) = F \\ F & \text{if } v(\alpha) = T \end{cases}$$

- For any formulas α and β:

$$v((\alpha \implies \beta)) = \begin{cases} F & \text{if } v(\alpha) = T \text{ and } v(\beta) = F \\ T & \text{otherwise} \end{cases}$$

For example, let $v(A_0) = T$ and $v(A_1) = F$ then it is possible to determine $v(((\neg A_1) \implies (A_0 \implies A_1)))$ in the following way:

- $v((\neg A_1)) = T$
- $v((A_0 \implies A_1)) = F$
- $v(((\neg A_1) \implies (A_0 \implies A_1))) = F$

Table 1.1: True table example

A_0	A_1	$(\neg A_1)$	$(A_0 \implies A_1)$	$((\neg A_1) \implies (A_0 \implies A_1))$
T	F	T	F	T

Table 1.2: Truth tables contradiction, tautology, α and β

α	β	$\bot$	$\top$	α	β
T	T	F	T	T	T
T	F	F	T	T	F
F	T	F	T	F	T
F	F	F	T	F	F

A convenient way to write out the determination of the truth value of a formula on a given truth assignment is to use a *truth table*:

Using what is described in Section 1.1.1.1 we have the following:

$$v((\alpha \wedge \beta)) = \begin{cases} T & \text{if } v(\alpha) = T \text{ and } v(\beta) = T \\ F & \text{otherwise} \end{cases}$$

$$v((\alpha \vee \beta)) = \begin{cases} F & \text{if } v(\alpha) = F \text{ and } v(\beta) = F \\ T & \text{otherwise} \end{cases}$$

$$v((\alpha \iff \beta)) = \begin{cases} T & \text{if } v(\alpha) = v(\beta) \\ F & \text{otherwise} \end{cases}$$

Also it is important to point out that in Definition 1.4 $v((\alpha \implies \beta))$ is known as *material implication* (Egré and Rott 2021). In a 2-valued propositional framework there are 16 possible truth tables for 2 formulas α and β as it is shown in Table 1.2, Table 1.3, Table 1.4 and Table 1.5

Assuming that we want:

Table 1.3: Truth tables $\neg\alpha$, $\neg\beta$, NOR and NAND

α	β	$\neg\alpha$	$\neg\beta$	$\Downarrow$	$\Uparrow$
T	T	F	F	F	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	T

Table 1.4: Truth tables AND, OR, material implication (IMPLY) and biconditional (XNOR)

α	β	$\wedge$	$\vee$	$\implies$	$\iff$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	T	T	F
F	F	F	F	T	T

Table 1.5: Truth tables converse implication, material nonimplication (NIMPLY), converse material nonimplication and XOR

α	β	$\Longleftarrow$	$\not\Longrightarrow$	$\not\Longleftarrow$	$\oplus$
T	T	T	F	F	F
T	F	T	T	F	T
F	T	F	F	T	T
F	F	T	F	F	F

- $v(\alpha) = T$, $v(\beta) = T$ and $v((\alpha \implies \beta)) = T$
- $v(\alpha) = T$, $v(\beta) = F$ and $v((\alpha \implies \beta)) = F$

From the 16 possible truth tables only 4 comply with these both conditions:

In Table 1.6 the first truth table corresponds to $\wedge$, the second corresponds to $\iff$ and the third corresponds to β where these tables are not suitable for $\implies$. Therefore the last table is the only truth table suitable for $\implies$ assuming we want the both conditions pointed above in a 2-valued propositional framework.

Proposition 1.1 (Truth values of atomic formulas)**.** *Suppose δ is any formula and u and v are truth assignments such that $u(A_n) = v(A_n)$ for all atomic formulas A_n which occur in δ. Then $u(\delta) = v(\delta)$.*

Proof. By strong induction on n, the number of connectives (occurrences of $\neg$ or $\implies$) in a formula δ of $\mathcal{L}_P$

Table 1.6: Possible truth tables for $\implies$ in a 2-valued propositional framework

α	β	$\wedge$	$\iff$	β	$\implies$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	F	F	T	T
F	F	F	T	F	T

Base step ($n = 0$): If δ is a formula of $\mathcal{L}_P$ with no connectives by Definition 1.2 is an atomic formula. So $\delta = A_i$ for some $i \in \mathbb{N}$ where $u(A_i) = v(A_i)$. Therefore $u(\delta) = v(\delta)$.

Induction hypothesis ($n \leq k$): Assume that for a formula δ with $n \leq k$ connectives we have that for u and v truth assignments such that $u(A_n) = v(A_n)$ for all atomic formulates A_n which occur in the δ we have that $u(\delta) = v(\delta)$

Induction step ($n = k + 1$): Suppose φ is a formula with $n = k + 1$ connectives. From Definition 1.2 φ must be either:

- $(\neg\delta)$ for some formula δ with k connectives.
  - Therefore $v(\delta) = u(\delta)$ and by Definition 1.4 $v(\neg\delta) = u(\neg\delta)$ which means that $v(\varphi) = u(\varphi)$.
- $(\delta \implies \beta)$ for some formulas δ and β with k connectives.
  - Therefore $v(\delta) = u(\delta)$ and $v(\beta) = u(\beta)$ and by Definition 1.4 $v(\delta \implies \delta) = u(\delta \implies \delta)$ which means that $v(\varphi) = u(\varphi)$.

□

Corollary 1.1 (Truth values of atomic formulas)**.** *Suppose u and v are truth assignments such that $u(A_n) = v(A_n)$ for every atomic formula A_n. Then $u = v$, i.e. $u(\varphi) = v(\varphi)$ for every formula φ.*

Proof. Because $u(A_n) = v(A_n)$ for every atomic formula A_n in $\mathcal{L}_p$ we can take any formula φ and apply Proposition 1.1. Thefefore $v(\varphi) = u(\varphi)$ but φ is any formula in $\mathcal{L}_p$ so it applies to every formula. □

Definition 1.5. If v is a truth assignment and φ is a formula, we will often say that v *satisfies* φ if $v(\varphi) = T$.

Similarly, if Σ is a set of formulas, we will often say that v *satisfies* Σ if $v(\sigma) = T$ for every $\sigma \in \Sigma$.

We will say that φ (respectively, Σ) is *satisfiable* if there is some truth assignment which satisfies it.

Definition 1.6. A formula φ is a *tautology* if it is satisfied by every truth assignment.

A formula ψ is a *contradiction* if there is no truth assignment which satisfies it.

For example, $(A_4 \implies A_4)$ is a tautology, $(\neg(A_4 \implies A_4))$ is a contradiction and A_4 neither of them.

Table 1.7: Example of a tautology and a contradiction

A_4	$(A_4 \implies A_4)$	$(\neg(A_4 \implies A_4))$
T	T	F
F	T	F

α	$(\neg\alpha)$	$((\neg\alpha) \vee \alpha)$	$((\neg\alpha) \wedge \alpha)$
T	F	T	F
F	T	T	F

Proposition 1.2. *If α is any formula, then $((\neg\alpha) \vee \alpha)$ is a tautology and $((\neg\alpha) \wedge \alpha)$ is a contradiction.*

Proof.

□

Proposition 1.3. *A formula β is a tautology if and only if $\neg\beta$ is a contradiction.*

Proof. For all v, where v is a true assignment, $v(\beta) = T$. By Definition 1.4 $v((\neg\beta)) = F$ so $(\neg\beta)$ is a contradiction.

For all v, where v is a true assignment, $v((\neg\beta)) = F$. By Definition 1.4 $v(\beta) = T$ so β is a tautology. □

Definition 1.7. A set of formulas Σ implies a formula φ, written as $\Sigma \vDash \varphi$, if every truth assignment v which satisfies Σ also satisfies φ.

We will often write $\Sigma \not\vDash \varphi$ if it is not the case that $\Sigma \vDash \varphi$.

In the case where Σ is empty, we will usually write $\vDash \varphi$ instead of $\emptyset \vDash \varphi$.

Similarly, if Δ and Γ are sets of formulas, then Δ implies Γ, written as $\Delta \vDash \Gamma$, if every truth assignment v which satisfies Δ also satisfies Γ.

For example, $\{A_3, (A_3 \implies (\neg A_7))\} \vDash (\neg A_7)$ because if v is a true assignment such that $v(A_3) = T$ and $v((A_3 \implies (\neg A_7))) = T$ then it must be the case that $v((\neg A_7)) = T$ by Definition 1.4.

Also, $\{A_8, (A_5 \implies A_8)\} \not\vDash A_5$, because if u is a true assignment such that $u(A_8) = T$ and $u((A_5 \implies A_8)) = T$. However, it is possible to have $u(A_5) = F$.

Remark 1.1. A formula φ is a tautology if and only if $\vDash \varphi$.

A formula φ is a contradiction if and only if $\vDash (\neg\varphi)$.

α	β	$(\alpha \Longrightarrow \beta)$	$(\neg\beta)$	$(\neg\alpha)$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Lets brake each element to understand why Remark 1.1 is true:

- If v is a truth assignment then:
  - v satisfies $\sigma \iff v(\sigma) = T$
- If Σ is a set of formulas then:
  - v satisfies $\Sigma \iff (\forall\sigma)(\sigma \in \Sigma \Longrightarrow v(\sigma) = T)$
- $\Sigma \vDash \varphi$ if and only if every truth assignment v which satisfies Σ also satisfies φ
  - $\Sigma \vDash \varphi \iff ((\forall v)((\forall\sigma)(\sigma \in \Sigma \Longrightarrow v(\sigma) = T)) \Longrightarrow v(\varphi) = T)$
- Now let's apply to $\vDash \varphi$ when φ is a tautology:
  - $\vDash \varphi \iff ((\forall v)((\forall\sigma)(\sigma \in \emptyset \Longrightarrow v(\sigma) = T)) \Longrightarrow v(\varphi) = T)$
    * $\sigma \in \emptyset$ is false so $(\sigma \in \emptyset \Longrightarrow v(\sigma) = T)$ is true by Definition 1.4.
    * $(\forall v)((\forall\sigma)(\sigma \in \emptyset \Longrightarrow v(\sigma) = T))$ will be true and $v(\sigma) = T$ is true because σ is a tautology
    * By Definition 1.4 $((\forall v)((\forall\sigma)(\sigma \in \emptyset \Longrightarrow v(\sigma) = T)) \Longrightarrow v(\varphi) = T)$ is true so we can write $\vDash \varphi$.

Proposition 1.4. *If Γ and Σ are sets of formulas such that $\Gamma \subseteq \Sigma$, then $\Sigma \vDash \Gamma$.*

Proof. If v satisfies Σ then for all $\sigma \in \Sigma$ we have that $v(\sigma) = T$. But if $\gamma \in \Gamma$ then also $\gamma \in \Sigma$. Therefore $v(\gamma) = T$ for any formula that belongs to Γ. This means that $\Sigma \vDash \Gamma$. $\square$

Exercise 1.8. How can one check whether or not $\Sigma \vDash \varphi$ for a formula φ and a finite set of formulas Σ?

Solution 1.8. We can construct a truth table and evaluate if in each row where the formulas that belongs to Σ are true φ is also true.

For example, let $\Sigma = \{(\alpha \implies \beta), (\neg\beta)\}$ and $\varphi = (\neg\alpha)$. The truth table will be:

The last row of the truth table represents a truth assignment v that satisfies Σ and also the same truth asignment v satisfies φ. Therefore we can check that $\Sigma \vDash \varphi$.

It is importa to mention that Σ must be a finite set of formulas or it will not be possible to check each row in the truth table that is build.

Proposition 1.5. *Suppose Σ is a set of formulas and ψ and ρ are formulas. Then $\Sigma \cup \{\psi\} \vDash \rho$ if and only if $\Sigma \vDash (\psi \implies \rho)$.*

Proof.

- Assume $\Sigma \vDash (\psi \implies \rho)$ and let v satisfies $\Sigma \cup \{\psi\}$.
  - By Proposition 1.4 v satisfies also Σ because $\Sigma \subseteq \Sigma \cup \{\psi\}$.
  - By assumption v satisfies $(\psi \implies \rho)$ because $\Sigma \vDash (\psi \implies \rho)$. Therefore $v((\psi \implies \rho)) = T$
  - Because $v(\psi) = T$ and $v((\psi \implies \rho)) = T$ by Definition 1.4 it must be the case that $v(\rho) = T$. Therefore $\Sigma \cup \{\psi\} \vDash \rho$.
- Assume $\Sigma \cup \{\psi\} \vDash \rho$.
  - Let v satisfies $\Sigma \cup \{\psi\}$ so $v(\sigma) = F$ for all $\sigma in \Sigma$, $v(\psi) = T$ and $v(\rho) = T$.
  - Because $v(\psi) = T$ and $v(\rho) = T$ by Definition 1.4 $v((\psi \implies \rho)) = T$.
  - Theforefore $\Sigma \vDash (\psi \implies \rho)$.

□

Proposition 1.6. *A set of formulas Σ is satisfiable if and only if there is no contradiction χ such that $\Sigma \vDash \chi$.*

Proof.

- Assume Σ is a set of formulas that is satisfiable

Pending (Possible solution Proposition 1.4) □

2 The Natural Numbers

3 The Integers

4 Rational Numbers and Ordered Fields

5 The Real Number System

References

Bilaniuk, Stefan. 2009. *A Problem Course in Mathematical Logic*. http://euclid.trentu.ca/math/sb/pcml/pcml.html.

Church, Alonzo. 1996. *Introduction to Mathematical Logic*. 10th ed. Princeton Landmarks in Mathematics and Physics. Princeton, NJ Chichester, West Sussex: Princeton University Press.

Egré, Paul, and Hans Rott. 2021. "The Logic of Conditionals." In *The Stanford Encyclopedia of Philosophy*, edited by Edward N. Zalta, Winter 2021. Metaphysics Research Lab, Stanford University. https://plato.stanford.edu/archives/win2021/entries/logic-conditionals/.

Łukasiewicz, Jan, and Ludwik Borkowski. 1970. *Selected Works*. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland Pub. Co.

Mendelson, Elliott. 2008. *Number Systems and the Foundations of Analysis*. Mineola, N.Y: Dover Publications.

A Equality

B Finite Sums and the $\sum$ notation

C Polynomials

D Finite, Infinite and Denumerable Sets and Cardinal Numbers

E Axiomatic Set Theory and the Existence of the Peano System

F Construction of the Real Numbers via Dedeking Cuts

G Complex Numbers