Quantum Magic Games

Luigi Soares and Roberto Rosmaninho

Department of Computer Science, UFMG - Brazil
{luigi.domenico, robertogomes}@dcc.ufmg.br

Abstract. TODO

1 Introduction

Telepathy, the ability of transmitting information from one person's mind to another's, would certainly come in handy in many situations, right? Unfortunately (or not), (to the best of our knowledge) telepathy is not a thing. At least, not according to classical physics. Certain aspects of the quantum realm, however, provide a way of communication that, for a layman, may look as "magical", or as true telepathy. This phenomenon is called quantum *pseudo-telepathy* [4].

Quantum pseudo-telepathy is observed in many contexts, usually described in the format of a game. Examples of such games are: the "impossible colouring games" [7], based on the Kochen-Specker theorem [9]; the "Deutsch-Jozsa games" [5], based on the Deutsch-Jozsa problem [8]; the "magic square" game [1], based on the Mermin-Peres proof of the Kochen-Specker theorem [11, 12, 10, 13]; and many others [7, 4]. None of these games admit a classical winning strategy (i.e. is not possible to always win), yet they can be won systematically, without any communication, provided that the players are allowed to share prior entanglement [4].

In this project, we will explore the Mermin-Peres magic square game (hereinafter referred simply as magic square). The origins of this game date back to the ninities. It was first described in the works of Mermin [11, 10] and of Peres [12, 13]. Their results provided (as per the title of Mermin's review letter) a simplied proof for the Kochen-Specker theorem [9]. Later, in 2002, Aravind demonstrated how to transform Mermin-Peres's proof of the Kochen-Specker theorem into a proof of Bell's theorem [1]. Bell's theorem [3] shows the incompatibility of hidden variables (i.e. determinism) and locality, and Kochen-Specker's theorem demonstrates the conflicting nature of determinism and noncontextuality (locality can be regarded as a special case of contextuality). These sort of results may seem challenging, but their descriptions can be greatly simplified by modeling them as quantum games. In particular, it is extremely easy to show that there cannot be a classical solution to the the magic square game, and to convince an observer that something "magical" (classically impossible) is happening in a successful implementation of a quantum winnin strategy [4].

Outline of this paper. In Section 2, we detail the intrinsics of the magic square game, classical approaches and why it is impossible to systematically win the game classically, how and why the quantum strategy works, and how to implement it. We also briefly discuss how the game translates to the proof of incompatibility of a noncontextual hidden-variable theory. Then, in Section 3, we describe a generalization of games that follow the same style of the magic square game, given by Arkhipov [2]. Finally, Section 4 concludes the presentation.

Code. The implementation of the classical and quantum solutions to the magic square game can be found at `https://github.com/luigidcsoares/quantum-magic-square`.

2 The Magic Square Game

A magic square is a 3×3 matrix whose entries are filled with ± 1's. The product of every row of a magic square must be $+1$. Similarly, the product of every column of a magic square must be -1. It is impossible to satisfy these two properties at the same time, hence the term *magic*: such a matrix cannot exist. The magic square game features two players, Alice and Bob, that must work together to fill in entries of the 3×3 table. At the beginning of the game, Alice and Bob are separated by a referee, Charlie, so that communication between them is impossible. At each round, Charlie assigns one row of the matrix at random to Alice, and asks Alice to give him the entries to fill in that particular row. After that, Charlie visits Bob and asks him to fill one column of the table, which Charlie draws at random. Bob does not know which row Alice was assigned, and Alice does not know which colum Bob was assigned. Alice and Bob win the round if their answers are valid, i.e. the product of the row filled by Alice is $+1$ and the product of the column filled by Bob is -1, and, in addition, the intersection of the row and the column agrees (i.e. both answered $+1$ or both answered -1). Before the game starts, Alice and Bob are allowed to communicate, so they can come up with an strategy. For example, they may prepare their answers for each row and column beforehand, or they may decide to follow a probabilistic strategy.

2.1 Classical Solution

Assume that Alice and Bob decided to adopt a deterministic strategy. For that, they met before the beginning of the game, and prepared their answers for each possible row and column. In this scenario, what is the best strategy that they can come up with?

Example 1 (Classical, Deterministic Solution). Alice and Bob might have agreed that, if given the first column, Bob shall fill it with three -1's. Then, whatever row Alice is given, she must fill it with either $\{-1, +1, -1\}$ or $\{-1, -1, +1\}$. Suppose that Alice decided that she would fill any row in the same way: $\{-1, -1, +1\}$. Figure 1 (a) shows how the predefined matrix would look like. Notice that Alice's rows are already defined, but Bob still has to decide how to fill the second and third columns. He can, and must, fill the second column with three -1's, so that whatever row Alice is given, the intersection agrees. Intuitively, for the same reason, he should then fill the third column with three $+1$'s. However, this is not a valid strategy, since the product of the last column will be $+1$ and not -1, as required. Thus, Bob must fill at least one of the entries of the last column with -1. By doing that, there will be one row and column whose intersection does not agree, and thus one row and column for which Alice and Bob lose, as illustrated in Figure 1 (b).

-1	-1	$+1$
-1	-1	$+1$
-1	-1	$+1$

(a) Initially, Alice and Bob agree that Bob shall fill the first column with $\{-1, -1, -1\}$. Based on this choice, Alice decides to fill all of her rows with $\{-1, -1, +1\}$.

-1	-1	± 1
-1	-1	$+1$
-1	-1	$+1$

(b) Bob has to fill the second column with -1's, to match Alice's choice. He should also match Alice's choice for each cell of the third column, but he cannot, for the product of the column must be -1.

Fig. 1: One possible classical deterministic strategy for the magic square game. Following this predefined strategy, Alice and Bob lose when assigned, respectively, the first row and third column, but win in any other scenario. That is, their probability of winning a round is $8/9$. Gray cells are entries for which Alice's and Bob's choices agree.

Example 1 illustrates one of the many deterministic strategies that Alice and Bob can adopt. It is not difficult to convince yourself that, no matter how they decide to fill in their entries, it is impossible for them to come up with predefined answers that always win. Any deterministic strategy is a pair of matrices, one for Alice and another for Bob. The only way that they could design a strategy that wins with certainty every round is to prepare two identical matrices satisfying the requirements for each row and for each column. That is, a single matrix for which the product of every row is $+1$ and the product of every column is -1. Such a matrix cannot possibly exist!

To see that, consider the constraints over each row and column:

$$\begin{aligned} m_{0,0}\, m_{0,1}\, m_{0,2} &= +1 \\ m_{1,0}\, m_{1,1}\, m_{1,2} &= +1 \\ m_{2,0}\, m_{2,1}\, m_{2,2} &= +1 \\ m_{0,0}\, m_{1,0}\, m_{2,0} &= -1 \\ m_{0,1}\, m_{1,1}\, m_{2,1} &= -1 \\ \underline{m_{0,2}\, m_{1,2}\, m_{2,2}} &\underline{= -1} \\ +1 &\neq -1 \end{aligned}$$

It turns out that the best that they can do is to win with probability $8/9$.

What if Alice and Bob decide for a probabilistic strategy? That is, Alice and Bob each carry a coin. When assigned a row/column, they flip their coins and fill the entries of the row/column based on the outcomes of the coins. Can they do better? To randomly assign values to one row (equivalently, one column) is essentially the same as randomly selecting one of the 2^9 possible predefined 3×3 grids (including those matrices that would lead to invalid answers). In other words, a probabilistic strategy for the magic square game is one in which Alice and Bob randomly select a deterministic strategy. Hence, no matter how lucky Alice and Bob are, the probability of success of any strategy that they come up with is bounded by the winning probability of the best deterministic strategy, which is $8/9$. A formal proof is given in [4] for any pseudo-telepathy game (i.e. games whose quantum strategies exhibit this notion of pseudo-telapathy).

2.2 Quantum Solution

The quantum strategy for the magic square game consists of Alice and Bob carrying entangled qubits with them. Then, when Charlie asks Alice (respectively Bob) to fill the entries of the i-th row (respectively j-th column), Alice measures her two qubits three times. The outcome of each measurement gives the value for each cell (in practice, they do not even need the third measurement, since they can determine the third value from the first two). Key to this strategy is that the outcomes of all possible measurements are ± 1. The initial, entangled state is (subscripts A and B indicate, respectively, Alice's and Bob's qubits)

$$|\psi\rangle = \frac{1}{2}\left(|00\rangle_A \otimes |00\rangle_B + |01\rangle_A \otimes |01\rangle_B + |10\rangle_A \otimes |10\rangle_B + |11\rangle_A \otimes |11\rangle_B\right).$$

Figure 2 shows the quantum circuit that can be used to prepare the initial state $|\psi\rangle$.

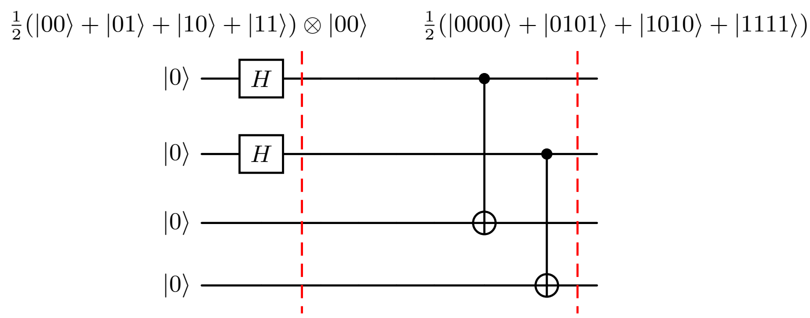


Fig. 2: Quantum circuit for preparing the initial joint state of Alice and Bob.

As seen in Section 2.1, the *magic* square is called this way because, under classical assumptions, there is no such a square: it is impossible to construct a 3×3 matrix filled with ± 1's, with the constraints that the product of every row is $+1$ and the product of every column is -1. We can, however, construct a "quantum magic" square. Figure 3 shows the 3×3 grid that Alice and Bob can use to win the magic square game systematically. Notice that the eigenvalues of the observables at each cell are ± 1. Furthermore, the product of every row is $+I$ and the product of every column is $-I$. The quantum strategy works as follows: Alice measures all the observables from the i-th row of the predefined matrix. She fills each cell with the result of each measurement. Similarly, Bob measures all the observables in the j-th column that was assigned to him, and fills each of his cells with the outcomes of each measurement. Example 2 depicts a round of the game.

$+I \otimes Z$	$+Z \otimes I$	$+Z \otimes Z$	$\prod = I \otimes I = I$
$+X \otimes I$	$+I \otimes X$	$+X \otimes X$	$\prod = I \otimes I = I$
$-X \otimes Z$	$-Z \otimes X$	$+Y \otimes Y$	$\prod = \mathrm{i}I \otimes -\mathrm{i}I = I$
$\prod = -I \otimes I = -I$	$\prod = -I \otimes I = -I$	$\prod = -\mathrm{i}I \otimes -\mathrm{i}I = -I$	

Fig. 3: Observables used by Alice and Bob to win the magic square game.

Example 2 (Quantum Solution). Suppose that Charlie assigns the first row to Alice. To decide the value of the first cell, she (based on the 3×3 grid she and Bob have prepared) measures the observable $I \otimes Z$ on her two qubits. The possible outcomes are

$$\mathrm{Out}((I \otimes Z)_A \otimes (I \otimes I)_B) = \begin{cases} +1\,(I \otimes |0\rangle\langle 0| \otimes I \otimes I) & \text{with probability } \frac{1}{2} \\ -1\,(I \otimes |1\rangle\langle 1| \otimes I \otimes I) & \text{with probability } \frac{1}{2}, \end{cases} \tag{1}$$

Suppose that Alice has observed $+1$. In this case, the initial state $|\psi\rangle$ collapses to

$$|\psi\rangle_{A1} = \frac{1}{\sqrt{2}}(|0000\rangle + |1010\rangle), \tag{2}$$

where the subscript $A1$ denotes the state after Alice's first measurement. For the second cell, Alice measures the observable $Z \otimes I$, whose possible outcomes are

$$\mathrm{Out}((Z \otimes I)_A \otimes (I \otimes I)_B) = \begin{cases} +1\,(|0\rangle\langle 0| \otimes I \otimes I \otimes I) & \text{with probability } \frac{1}{2} \\ -1\,(|1\rangle\langle 1| \otimes I \otimes I \otimes I) & \text{with probability } \frac{1}{2}. \end{cases} \tag{3}$$

Suppose, now, that Alice has observed the outcome -1. The state collapses to

$$|\psi\rangle_{A2} = |1010\rangle\,. \tag{4}$$

Finally, Alice measures the observable $Z \otimes Z$, whose possible outcomes are

$$\mathrm{Out}((Z \otimes Z)_A \otimes (I \otimes I)_B) = \begin{cases} +1\,((|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I \otimes I) & \text{with probability } 0 \\ -1\,((|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I \otimes I) & \text{with probability } 1. \end{cases} \tag{5}$$

Notice that the only possible outcome that Alice can observe is -1, and the state does not change, i.e. $|\psi\rangle_{A3} = |\psi\rangle_{A2}$. Alice, then, fills the first row with the values $\{+1, -1, -1\}$, whose product is, as required , $+1$. Upon receiving Alice's answer, Charlie goes to meet Bob. Suppose he assigns the second column to Bob. To fill in the value of the first cell (which is the intersection of the first row and second column), Bob measures the observable $Z \otimes I$. The only possible outcome, considering the state $|\psi\rangle_{A3}$, is -1 — which matches Alice's answer — and the state will not change again: $|\psi\rangle_{B1} = |\psi\rangle_{A3}$. Moving on to the second cell, Bob measures the observable $I \otimes X$, whose possible outcomes are

$$\mathrm{Out}((I \otimes I)_A \otimes (I \otimes X)_B) = \begin{cases} +1\,(I \otimes I \otimes I \otimes |+\rangle\langle +|) & \text{with probability } \frac{1}{2} \\ -1\,(I \otimes I \otimes I \otimes |-\rangle\langle -|) & \text{with probability } \frac{1}{2}. \end{cases} \tag{6}$$

Assume that Bob has observed the outcome +1. Then, the state collapses to

$$|\psi\rangle_{B2} = \frac{1}{\sqrt{2}}(|1010\rangle + |1011\rangle). \tag{7}$$

Finally, Bob measures $-Z \otimes X$ on his two qubits. The possible outcomes are

$$\mathrm{Out}((I \otimes I)_A \otimes (-Z \otimes X)_B) = \begin{cases} +1\,(I \otimes I \otimes (|1+\rangle\langle 1+| + |0-\rangle\langle 0-|)) & \text{with prob. } 1 \\ -1\,(I \otimes I \otimes (|1-\rangle\langle 1-| + |0+\rangle\langle 0+|)) & \text{with prob. } 0. \end{cases} \tag{8}$$

Bob will for sure observe the outcome +1 and the state does not change, i.e. $|\psi\rangle_{B3} = |\psi\rangle_{B2}$. Therefore, Bob answers $\{-1, +1, +1\}$. Alice's answer satisfies the property of the rows (that the product of the entries is equal to +1) and Bob's answer satisfies the property of the columns (product is equal to -1). Moreover, Alice and Bob agreed in the intersection of the row and the column. Hence, they won this round!

2.3 Formal Proof

To prove that the quantum solution indeed works, we must demonstrate two fundamental properties. First, the product of the outcomes of the three measurements on each row (equivalently, column) is equal to 1 (equiv. -1). Second, the outcome of Alice's measurement on the intersection is always equal to the outcome of Bob's. Lemma 1 formalizes the product property.

Lemma 1 (The Product Property of the Quantum Solution). *Let $M_{i,j}$ be the observable at the cell (i, j) of the 3×3 grid shown in Figure 3. Then, it follows that*

$$\text{for every } i, \prod_j \mathrm{Out}(M_{i,j}) = +1, \text{ and} \tag{9}$$

$$\text{for every } j, \prod_i \mathrm{Out}(M_{i,j}) = -1. \tag{10}$$

Proof. First, notice that $X^2 = Z^2 = Y^2 = I$, $XZ = \mathrm{i}Y$ and $ZX = -\mathrm{i}Y$ (i.e. X and Z anti-commute), and, similarly, $XY = \mathrm{i}Z$ and $YX = -\mathrm{i}Z$. Using these properties, we can see that the product of every row (equiv. column) is the identity I (equiv. $-I$), regardless of the order of the three observables. That is, the three observables of every row and every column mutually commute (or, in other words, they are compatible). Consequently, for every row and every column, it is possible to find a basis of eigenvectors that are common to the three observables, meaning that they can be diagonalized simultaneously. Tables 1 and 2 show these common eigenvectors. Let $M_{i,j}$ be the observables at each cell. Then, for every row i (equiv. column j), we have

$$\begin{aligned} M_{i,0}M_{i,1}M_{i,2} &= (P_i \Lambda_{i,0} P_i^{-1})(P_i \Lambda_{i,1} P_i^{-1})(P_i \Lambda_{i,2} P_i^{-1}) \\ &= P_i(\Lambda_{i,0}\Lambda_{i,1}\Lambda_{i,2})P_i^{-1} \\ &= P_i \Lambda_i P_i^{-1}, \end{aligned} \tag{11}$$

where P_i is the matrix whose columns are the common eigenvectors and Λ_i is the matrix whose diagonal is formed by the products of the eigenvalues of the observables, associated with the same eigenvector. What this essentially means is that any measurement $M_{i,j}$ yields one of the eigenvalues λ in the diagonal of $\Lambda_{i,j}$ and projects the state onto the λ-eigenspace. That is, it collapses the state to a linear combination of the two shared eigenvectors $|\lambda\rangle_0$ and $|\lambda\rangle_1$ in the λ-eigenspace. Then, the second measurement will collapse the state to either $|\lambda\rangle_0$ or $|\lambda\rangle_1$, eigenvectors common to the three observables. Consequently, the third measurement will not change the state any further. Therefore, the product of the three outcomes will be the product of the eigenvalues associated with either $|\lambda\rangle_0$ or $|\lambda\rangle_1$, which is one of the entries in the diagonal of Λ_i. But, notice that $P_i \Lambda_i P_i^{-1} = I$; thus, $\Lambda_i = I$ and the only possible value for the product is 1 (equivalently -1 for columns). ☐

Table 1: Eigenvectors shared by the observables at each row of the quantum magic square. The eigenvectors were taken from [6], adapting to the correct eigenvalues.

		Column 0	Column 1	Column 2
Row 0	+1	$\|00\rangle$, $\|10\rangle$	$\|00\rangle$, $\|01\rangle$	$\|00\rangle$, $\|11\rangle$
	−1	$\|01\rangle$, $\|11\rangle$	$\|10\rangle$, $\|11\rangle$	$\|01\rangle$, $\|10\rangle$
Row 1	+1	$(1/2)(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle - \|01\rangle + \|10\rangle - \|11\rangle)$	$(1/2)(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle + \|01\rangle - \|10\rangle - \|11\rangle)$	$(1/2)(\|00\rangle + \|01\rangle + \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle - \|01\rangle - \|10\rangle + \|11\rangle)$
	−1	$(1/2)(\|00\rangle - \|01\rangle - \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle + \|01\rangle - \|10\rangle - \|11\rangle)$	$(1/2)(\|00\rangle - \|01\rangle - \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle - \|01\rangle + \|10\rangle - \|11\rangle)$	$(1/2)(\|00\rangle + \|01\rangle - \|10\rangle - \|11\rangle)$, $(1/2)(\|00\rangle - \|01\rangle + \|10\rangle - \|11\rangle)$
Row 2	+1	$(1/2)(\|01\rangle - \|00\rangle + \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle + \|01\rangle - \|10\rangle + \|11\rangle)$,	$(1/2)(\|01\rangle - \|00\rangle + \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle - \|01\rangle + \|10\rangle + \|11\rangle)$,	$(1/2)(\|01\rangle - \|00\rangle + \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle + \|01\rangle + \|10\rangle - \|11\rangle)$,
	−1	$(1/2)(\|00\rangle - \|01\rangle + \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle + \|01\rangle + \|10\rangle - \|11\rangle)$,	$(1/2)(\|00\rangle + \|01\rangle - \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle + \|01\rangle + \|10\rangle - \|11\rangle)$,	$(1/2)(\|00\rangle + \|01\rangle - \|10\rangle + \|11\rangle)$, $(1/2)(\|00\rangle - \|01\rangle + \|10\rangle + \|11\rangle)$,

Table 2: Eigenvectors shared by the observables at each column of the quantum magic square. The eigenvectors were taken from [6], adapting to the correct eigenvalues.

		Row 0	Row 1	Row 2
Column 0	+1	$(1/\sqrt{2})(\|00\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|10\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle + \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle + \|11\rangle)$
	−1	$(1/\sqrt{2})(\|01\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|11\rangle)$
Column 1	+1	$(1/\sqrt{2})(\|00\rangle + \|01\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|01\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle + \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle + \|11\rangle)$
	−1	$(1/\sqrt{2})(\|10\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|10\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle - \|11\rangle)$
Column 2	+1	$(1/\sqrt{2})(\|00\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle + \|10\rangle)$	$(1/\sqrt{2})(\|01\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|11\rangle)$
	−1	$(1/\sqrt{2})(\|01\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|10\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|10\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|10\rangle)$

Lemma 1 shows that the quantum solution satisfies the requirements of the game for every row and column. Now, we move on to show that Alice and Bob always agree in the value that they assign to the intersection of the row and column.

Lemma 2 (The Intersection Property of the Quantum Solution). *For any row i assigned to Alice and column j assigned to Bob, the value ascribed to the cell (i, j) by Alice and Bob, after they have performed their measurements, is always the same.*

Proof. To prove that Alice and Bob agree at the intersection, it suffices to look to the eigenvectors shared by each set of mutually commuting operators, shown in Tables 1 and 2. Notice that, regardless of the row assigned to Alice, the state of her subsystem after her measurements is always one of the eigenvectors in that row. This state is always a linear combination of one of Bob's eigenspaces for the cell in the intersection (the same applies if we consider first Bob than Alice). Take, for instance, the second row and first column. Suppose that Alice's measurements collapsed her qubits to $(1/2)(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$. This state can be written as $(1/\sqrt{2})(|00\rangle - |10\rangle) - (1/\sqrt{2})(|01\rangle - |11\rangle)$, a linear combination of the −1-eigenspace depicted in Table 2 (column 0, row 1). What is left is to show that Bob's state will be exactly the same as Alice's, implying that Bob shall observe the same result. This follows from the fact that the initial state is already symmetric (the first qubit of Alice is equal to the first qubit of Bob; the same is true for their second qubits) and the symmetry is preserved by any measurement performed by Alice and Bob. For the first row, it is easy to see. The others are trickier, but it all boils down to doing the calculations. Consider, for instance, the second row, and suppose that the outcomes were -1, -1 and $+1$, so that Alice's state collapsed to $(1/2)(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$. The state after the first measurement is

$$\begin{aligned}&\frac{1}{2\sqrt{2}}\Big((|00\rangle - |10\rangle)\,|00\rangle + (|01\rangle - |11\rangle)\,|01\rangle + (|10\rangle - |00\rangle)\,|10\rangle + (|11\rangle - |01\rangle)\,|11\rangle\Big)\\ \equiv\, &\frac{1}{2\sqrt{2}}\Big(|00\rangle\,(|00\rangle - |10\rangle) + |01\rangle\,(|01\rangle - |11\rangle) + |10\rangle\,(|10\rangle - |00\rangle) + |11\rangle\,(|11\rangle - |01\rangle)\Big).\end{aligned} \tag{12}$$

And, the state after the second measurement is

$$\begin{aligned}&\frac{1}{4}\Big((|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|00\rangle - (|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|01\rangle -\\ &\quad (|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|10\rangle + (|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|11\rangle\Big)\\ \equiv\, &\frac{1}{4}\Big(\,|00\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle) - |01\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle) -\\ &\quad |10\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle) + |11\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle)\Big).\end{aligned} \tag{13}$$

The calculations for the other rows and columns are similar, so we omitted them. □

With Lemmas 1 and 2, we can prove Theorem 1, which states that there is a quantum strategy for the Mermin-Peres magic square game that always win.

Theorem 1 (The Mermin-Peres Magic Square Game Is Winnable). *There exists a quantum strategy for the Mermin-Peres magic square game, which consists of allowing Alice and Bob exchanging entangled qubits* before *the game begins, that wins with certainty every round.*

2.4 Implementing with Circuits

In this section, we show the circuits used to implement the quantum strategy. We implemented the solution using the `Qiskit` library. `Qiskit` only allows measurements in the computational basis. For any other basis B, we must find an unitary transformation U such that $B = U^*(Z \otimes I)U$ or $B = U^*(I \otimes Z)U$ (where U^* denotes the Hermitian adjoint of U). For example,

$$Z \otimes Z = CNOT(I \otimes Z)CNOT \tag{14}$$

and

$$X \otimes X = (H \otimes H)(Z \otimes Z)(H \otimes H). \tag{15}$$

Figure 4 shows the unitaries for each of the nine observables.

$(I \otimes I)(I \otimes Z)(I \otimes I)$	$(I \otimes I)(Z \otimes I)(I \otimes I)$	$CNOT(I \otimes Z)CNOT$
$(H \otimes I)(Z \otimes I)(H \otimes I)$	$(I \otimes H)(I \otimes Z)(I \otimes H)$	$(H \otimes H)(Z \otimes Z)(H \otimes H)$
$(H \otimes Y)(Z \otimes Z)(H \otimes Y)$	$(Y \otimes H)(Z \otimes Z)(Y \otimes H)$	$(S \otimes S)(X \otimes X)(S^* \otimes S^*)$

Fig. 4: Unitary transformations required for measuring in the different basis. The actual basis measured is the middle term of each cell, which, after expanding, is either $Z \otimes I$ or $I \otimes Z$. For example, in the last cell, $X \otimes X$ evaluates to $(H \otimes H)(Z \otimes Z)(H \otimes H)$. $Z \otimes Z$, in turn, becomes $CNOT(I \otimes Z)CNOT$, meaning that we measure the second qubit in the computational basis.

We now show the quantum circuits for each row and each column:

Row 0: When assigned the first row, Alice performs the measurements $I \otimes Z, Z \otimes I$ and $Z \otimes Z$, which the circuit in Figure 5 implements.

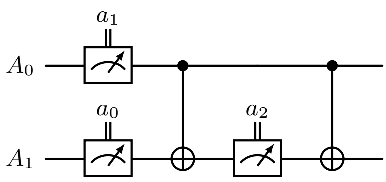


Fig. 5: Quantum circuit for measuring the first row.

Row 1: When assigned the second row, Alice performs the measurements $X \otimes I, I \otimes X$ and $X \otimes X$, which the circuit in Figure 6 implements.

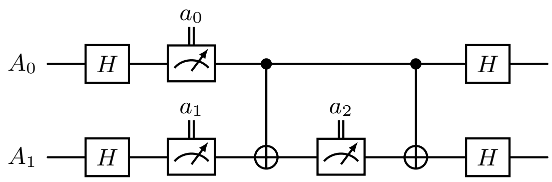


Fig. 6: Quantum circuit for measuring the second row.

Row 2: When assigned the third row, Alice performs the measurements $-X \otimes Z, -Z \otimes X$ and $Y \otimes Y$, which the circuit in Figure 7 implements.

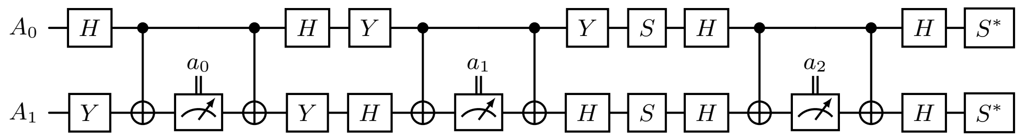


Fig. 7: Quantum circuit for measuring the third row.

Column 0: When assigned the first column, Bob performs the measurements $I \otimes Z, X \otimes I$ and $-X \otimes Z$, which the circuit in Figure 8 implements.

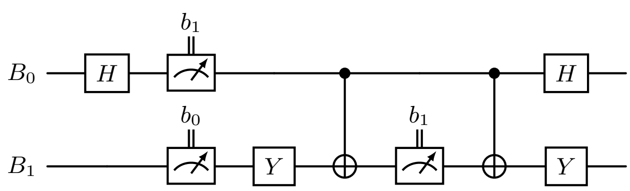


Fig. 8: Quantum circuit for measuring the first column.

Column 1: When assigned the second column, Bob performs the measurements $Z \otimes I, I \otimes X$ and $-Z \otimes X$, which the circuit in Figure 9 implements.

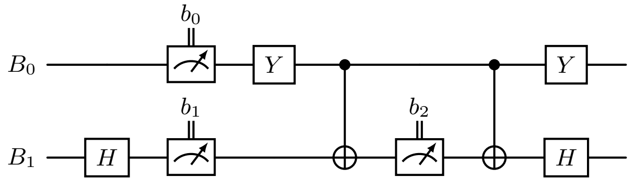


Fig. 9: Quantum circuit for measuring the second column.

Column 2: When assigned the third column, Bob performs the measurements $Z \otimes Z$, $X \otimes X$ and $Y \otimes X$, which the circuit in Figure 10 implements.

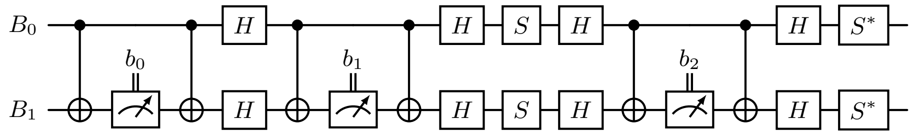


Fig. 10: Quantum circuit for measuring the third column.

2.5 A Note on Compatible Observables

As seen in the Section 2.3, the operators on each row and column are compatible. That is, for any row and any column, the observables pairwise commute. This means that they can be simultaneously diagonalized by the same eigenvectors. It turns out that the simultaneous measurement of compatible operators is equivalent to the measurement of a single nondegenerate operator [13]. That is, instead of performing three measurements, one for each cell, Alice and Bob could perform a single measurement each, whose answers would tell them to which of the four common eigenvectors their qubits would have collapsed to, had they performed the individual measurements. Based on this, they could fill each cell with the eigenvalue that corresponds to that eigenvector. To construct these single measurements for each row and column, it suffices to take the projectors onto the eigenvectors shared by the observables. Example 3 further illustrates this idea.

Example 3 (Simultaneous Measurement). Consider, for instance, the first row of the quantum solution for the magic square game, depicted in Figure 3. The observables in this row are $I \otimes Z$, $Z \otimes I$, and $Z \otimes Z$. The eigenvectors common to these operators are $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$. Alice, then, could simply construct an observable

$$M = |00\rangle\langle 00| + 2|01\rangle\langle 01| + 3|10\rangle\langle 10| + 4|11\rangle\langle 11|,$$

which corresponds to the diagonal matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

By measuring M on her qubits, she would observe (with probability $1/4$) the outcomes 1, 2, 3, or 4 , which uniquely identifies one of the four eigenvectors $|00\rangle$, $|01\rangle$, $|10\rangle$, or $|11\rangle$. Consequently, she would know the eigenvalues associated with the eigenvector "observed", and would be able to correctly fill the cells of the first row. For example, outcome 3 corresponds to $|10\rangle$. Hence, if she observes 3, she fills the first row with $\{+1, -1, -1\}$.

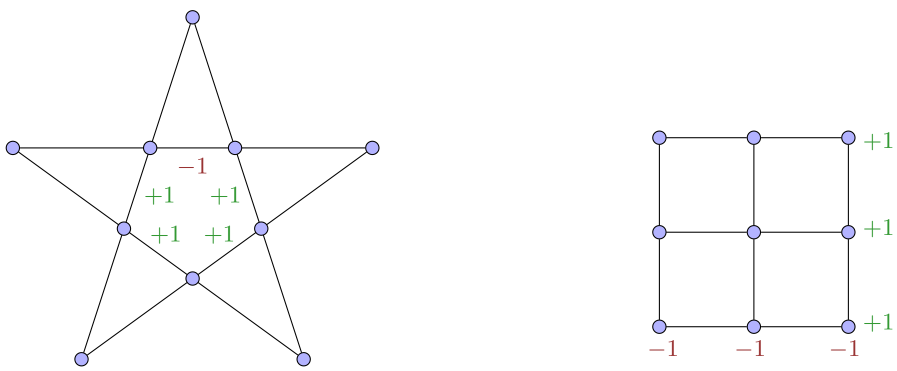
2.6 A Proof of The Kochen-Specker Theorem

The Kochen-Specker theorem [9] demonstrates the incompatilibty between an interpretation of quantum mechanics that is both deterministic — in the sense that there exists hidden variables that we do not have access to, and thus the stochasticity that we observe is merely epistemic — and, at the same time, noncontextual. Noncontextuality means that the outcomes of a measurement do not depend on the context, where a context is a set of compatible observables.

Consider the magic square game described above. In a noncontextual hidden-variable model, we should be able to ascribe pre-determined outcomes to the nine observables in each cell. That is, for each observable $M_{i,j}$ in a cell (i, j), there should be an assignment $v(M_{i,j}) = \pm 1$. However, as seen in Section 2.1, this is impossible. Hence, either the model must be contextual, meaning that the outcomes of the measurements are influenced by the context in which they are inserted — for instance, pick any observable in the 3×3 grid shown in Figure 3 and recall that it commutes with the other two observables in the row and also in the column; then, the result of measuring this observable is influenced by whether you are measuring it in the row or in the column —, or there cannot exist hidden variables governing the values assigned to each observable.

3 Characterizing Quantum Magic Games

What if we change the format of the magic square game? Maybe increase the dimensions, or even change the shape entirely? Figure 11 (a) gives an example of the magic pentagram game, in which the product of the labels (± 1) assigned to each of four vertices in a line must equal the labels of that line (the product of the four vertices in the topmost horizontal line, for example, must be -1). Notice that we are now changing the visualization format to hypergraphs. Figure 11 (b) shows the magic square as a hypergraph. As the magic square, the pentagram is also due to Mermin [10]. In this section, we explore a characterization of Mermin-style games, given by [2].



(a) The pentagram game as a hypergraph. (b) The magic square game as a hypergraph.

Fig. 11: Examples of hypergraphs representing quantum magic games [2].

We start by defining the notion of *arrangements*, which are configurations on which Mermin-style games may be played. [2] defines both unsigned and signed arrangements. Instead, we shall focus on *signed* arrangements, and thus anytime we say "arrangement" we mean the signed version.

Definition 1 (Arrangement [2]). *An arrangement $A = (V, E, \ell)$ is a finite connected hypergraph with vertex set V and hyperedge set E, where a hyperedge is a nonempty subset of V, such that every vertex $v \in V$ lies in exactly two hyperedges $e_0, e_1 \in E$ (*connected *means that the hypegraph cannot be split into two smaller disjoint hypergraphs). The labelling function $\ell \colon E \to \{+1, -1\}$ maps each hyperedge to a sign $+1$ or -1.*

We will oftenly refer to a hyperedge as a *line*. In Figure 11 (a), the lines going through sequences of four vertices are the hyperedges. Similarly, in Figure 11 (b), the horizontal and vertical lines touching the sequences of three vertices are the hyperedges. As already mentioned, arrangements define the configurations on which games may be played.

Definition 2 (Classical Realization [2]). *A classical realization of an arrangement* $A = (V, E, \ell)$ *is a labelling* $c\colon V \to \{+1, -1\}$ *that, for every hyperedge* $e \in E$*, satisfies the property*

$$\prod_{v \in e} c(v) = \ell(e).$$

Example 4 (Classical Realization). Figure 12 depicts an example of a classically realizable arrangement. That is, an arrangement for which there exists a classical realization.

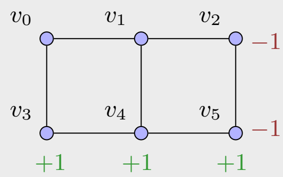


Fig. 12: Example of a classically realizable arrangement.

A classical realization c for this arrangement is

$$\begin{aligned} c(v_0) = c(v_1) = c(v_3) = c(v_4) &= +1 \\ c(v_2) = c(v_5) &= -1. \end{aligned}$$

Definition 3 (Quantum Realization [2]). *A quantum realization of an arrangement* $A = (V, E, \ell)$ *is a labelling* $q\colon V \to \mathrm{GL}(\mathcal{H})$ *mapping each vertex of the arrangement to an observable* M *on a finite-dimensional Hilbert-space* $\mathcal{H}$*, such that*

- *M is Hermitian and squares to the identity, or equivalently, has eigenvalues* ± 1*.*
- *For each hyperedge, the observable assigned to its vertices pairwise commute and.*
- *For each hyperedge* $e \in E$*, the labelling of its vertices satisfies the property*

$$\prod_{v \in e} q(v) = \ell(e) I.$$

A classical realization is simply a quantum realization in which $\mathcal{H} = \mathbb{R}$. As such, every classically realizable arrangement is quantumly realizable. An example of quantum realization was given in Section 2.2 for the magic square game. Any arrangement that is quantumly realizable with observables that all mutually commute is also classically realizable [2, Proposition 8]. The proof follows by the simultaneous diagonalization of the observables, similar to our proof for Lemma 1.

It turns out that the existence of a realization of an arrangement depends only on whether the number of -1 labels is odd or even. This is called the *parity* of the arrangement.

Definition 4 (Parity [2]). *The parity* $p(\ell)$ *of an arrangement* $A = (V, E, \ell)$ *is*

$$p(\ell) = \prod_{e \in E} \ell(E),$$

which is -1 *if there is an odd number of* -1 *labels, or* $+1$ *otherwise.*

If there exists a realization (either classical or quantum) of an arrangement $A = (V, E, \ell)$, then it is possible to construct a realization for $A' = (V, E, \ell')$ from the realization of A [2, Propositions 10 and 12]. Furthermore, an arrangement is classically realizable if, and only if, its parity is +1 [2, Proposition 11]. Despite this fundamental result, classically realizable arrangements are not as much interesting as the quantumly ones. Therefore, the goal is to characterize the gap between these two worlds. The arrangements that are quantumly realizable, but are not classically realizable, are called *magic* and are defined as follows:

Definition 5 (Magic Arrangement [2]). *An arrangement is said magic if its parity is -1 and there exists a quantum realization of it.*

The magic square and the pentagram are examples of magic arrangements. Equipped with all these tools, we now define a generalization of *parity* pseudo-telepathy games (Mermin-style games):

Definition 6 (Parity Pseudo-Telepathy Game [2]). *A parity pseudo-telepathy game on an arrangement $A = (V, E, \ell)$ is a two-players cooperative game (say, played by Alice and Bob). The game is conducted by a referee Charlie. Alice and Bob may agree on a prior strategy, but cannot communicate once the game starts. At each round,*

1. *Charlie randomly picks a vertex $u \in V$ and one of the hyperedges $e \in E$ that contains u.*
2. *Charlie sends the vertex u to Alice and the hyperedge e to Bob.*
3. *Alice labels the vertex u with $a(u) = \pm 1$ and sends $a(u)$ to Charlie.*
4. *Bob labels each vertex $v \in e$ with $b(v) = \pm 1$ and sends each $b(v)$ to Charlie.*
5. *Alice and Bob win if $a(u) = b(u)$ and $\prod_v b(v) = \ell(e)$.*

Notice that the generalization above differs a bit from the magic square game seen in Section 2, where Alice receives a hyperedge. This modification generalizes to arrangements that do not share the property exhibited by the magic square that its lines can be split into two disjoint sets (rows and columns). In this sense, one can simply understand the magic square game as a combination of two rounds. In the first "half" of the round, Charlie sends the vertex u to Alice and the line e to Bob. In the second half, Charlie sends u to Bob, but *another* line $e' \neq e$ to Alice.

There exists a strategy that wins with certainty a parity pseudo-telepathy game played on a magic arrangement that has a quantum realization in which all the operators have all real eigenvalues, provided that Alice and Bob can share entanglement in advance [2, Theorem 17]. It suffices for them to share a maximally entangled state

$$\rho_{AB} = \sum_{i=0}^{n-1} |i\rangle |i\rangle ,$$

where $\{|i\rangle\}$ is a basis for the finite-dimensional Hilbert-space $\mathcal{H}$ of dimension n. In the case of the magic square, $n = 4$ and $\{|i\rangle\} = \{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$, which, rewritten as binary strings, is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The proof is similar to our proof for Theorem 1, using the simultaneous diagonalization of operators in the same hyperedge and the symmetry of the state.

3.1 Main Result: Characterization of Magic Arrangements by Planar Graphs

We now move on to the main result of [2]. We shall derive a characterization of magic arrangements by the planarity of its corresponding intersection graph.

Definition 7 (Intersection Graph). *The intersection graph of an arrangement $A = (V, E)$ is the undirected graph $G = (V', E')$, where $V' = E$, and there is an edge between two vertices $e_0, e_1 \in V'$ for each vertex in the insection of e_0 and e_1.*

The intersection graph of an arrangement is a dual representation in which the roles of vertices and hyperedges are switched. The vertices of the intersection graph are labelled with the same values of the corresponding hyperedges in the arrangement. Similarly, a quantum realization of the intersection graph assigns to its edges the same measurements ascribed to the corresponding vertices in the arrangement. The properties observed in the arrangement are also carried to the intersection graph. In particular, the observables assigned to the edges around a vertex in the intersection graph (by around, we mean the edges that touch that vertex) mutually commute and its product equals the identity times the label of the vertex. To illustrate, the intersection graph of the magic square is the complete bipartite graph on six vertices $K_{3,3}$ and the intersection graph of the magic pentagram is the complete graph on five vertices K_5.

We are also going to need the definition of a topological minor:

Definition 8 (Topological Minor). *A graph H is a topological minor of G if there is an embedding of H in G that consists of an injective map ϕ from every vertex $v \in H$ to a vertex $\phi(v) \in G$, and a map from each edge $(u, v) \in H$ to a simple path from $\phi(u)$ to $\phi(v)$ in G, such that these paths are disjoint (except on their endpoints).*

We now restate the main result of [2] and its proof, further illustrating it with examples.

Theorem 2. *An arrangement is magic if and only if its intersection graph is not planar.*

We start by considering the direction that says that "nonplanar implies magic", or equivalently, if the intersection graph is planar, then its arrangement is not magic.

Proof. First of all, assume that there exists a quantum realization of the arrangement (if there is none, then the arrangement is not magic). The goal is to show that, if there exists a quantum realization and the intersection graph is planar, then the parity of the arrangement is $+1$ and, consequently, it is not magic. The strategy consists of repeatedly contracting edges of the intersection, and showing that contraction preserves a few properties. To contract an edge is to merge the two endpoints e_0 and e_1 into a new vertex e' (notice that we are using e to represent vertices, because these vertices correspond to hyperedges of the arrangement). Every edge that used to touch e_0, now touches e', and every edge that used to touch e_1, now touches e'. Consequently, the edge that was contracted becomes two self-loops. Moreover, we define the label of the new vertex e' as the product of the labels of e_0 and e_1. We observe that contraction preserves the following properties:

1. The resulting graph remains planar.
2. The parity (product of all the vertex labels) remains the same. This follows directly from the definition of the label of thenew vertex.
3. The product of the labels of edges around any vertex, in cyclic order (say, without loss of generality, counterclockwise), equals the identity times the label of the vertex. To see this, first notice that this property holds for the initial graph, since all observables around a vertex pairwise commute (recall that these observables correspond to the observables assigned by the quantum realization to vertices that lie in the hyperedge in the arrangement). However, the commutative property does not necessarily hold after contraction. Nevertheless, if we follow a cyclic order, we can still see that this property is preserved. Let $M_0, \dots, M_{n-1}$ be the observables around a vertex e, going counterclockwise. If $M_0 \dots M_{n-1} = \ell(e)I$, then for any starting point k, $M_k M_{k+1} \dots M_{n-1} M_0 \dots M_{k-1} = \ell(e)I$, since

$$M_k M_{k+1} \dots M_{n-1} M_0 \dots M_{k-1} = (M_{k-1} \dots M_0)(M_0 \dots M_{n-1})(M_0 \dots M_{k-1}) \quad (16)$$
$$= (M_{k-1} \dots M_0)\ell(e)I(M_0 \dots M_{k-1}) \quad (17)$$
$$= \ell(e)I, \quad (18)$$

where (16) follows from the fact that $M_i M_i = I$. Now, let X be the measurement assigned to the edge that is being contracted and let the labels around its endpoints e_0 and e_1 be $M_0 \dots M_{m-1}X$ and $XN_0 \dots N_{n-1}$ going counterclockwise. Then,

$$M_0 \dots M_{m-1}X = \ell(e_0)I$$

and

$$XN_0 \dots N_{n-1} = \ell(e_1)I.$$

Multiplying these gives

$$M_0 \dots M_{m-1}X^2N_0 \dots N_{n-1} = M_0 \dots M_{m-1}N_0 \dots N_{n-1} = \ell(e_0)\ell(e_1)I = \ell(e')I,$$

so the product of the observables around the new vertex e' still is the identity times its label. Notice that key to this strategy is the fact that the graph is planar, and thus we can draw the edges around the new vertex without mixing the edges that used to touch e_0 and e_1.

By repeatedly contracting edges, eventually we reach a graph with a single vertex. Due to how we defined the label of each new vertex (i.e. as the product of the labels of the vertices being merged), it follows that the label of the last vertex equals the product of the labels of all the original vertices, and therefore equals the parity of the arrangement. The remaining vertex has multiple self-loops, two for each measurement M ascribed to the original edges. Since $M^2 = I$, going counterclockwise we get a sequence $M_0^2 \dots M_{|E'|-1}^2 = I$. But, we saw that contraction preserves the third property, which says that the observables around a vertex multiply to the identity times the vertex label. Thus, the label of the final vertex is 1 and, consequently, the parity of the arrangement is $+1$. □

Example 5 (Planar Graph $\implies$ Not Magic). Consider the arrangement depicted in Figure 12 and assume there exists a quantum realization that assigns the observables M_i to each vertex v_i. Suppose that we do not know that the parity of this arrangement is $+1$. We are going to follow the strategy described above, to show that its parity must be $+1$. Figure 13 shows the intersection graph that corresponds to that arrangement.

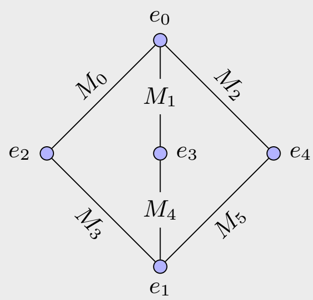


Fig. 13: Intersection graph corresponding to the arrangement seen in Figure 12.

Let us contract the edge labelled by M_0 (recall that it corresponds to the vertex v_0 in the arrangement). The resulting graph is depicted in Figure 14. Notice that $M_1M_2M_0 = \ell(e_0)I$ and $M_0M_3 = \ell(e_2)I$. After contraction, we have a new vertex e_0', with $\ell(e_0') = \ell(e_0)\ell(e_2)$ and $M_1M_2M_0^2M_3 = \ell(e_0)\ell(e_2)I = \ell(e_0')I$.

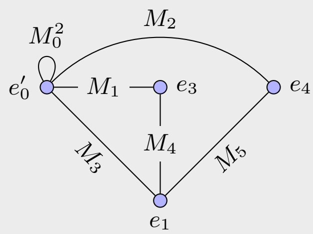


Fig. 14: Graph that results from the contraction of the edge from e_0 to e_2 in Figure 13.

Let us now contract the edge labelled by M_5. The resulting graph is shown in Figure 15.

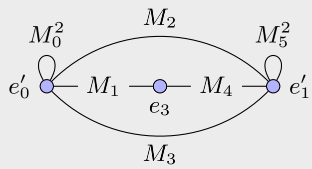


Fig. 15: Graph that results from the contraction of the edge from e_1 to e_4 in Figure 14.

If we keep contracting edges, eventually we reach a graph with a single node, whose observables going counterclockwise are $\prod_{i=0}^{5} M_i^2 = I$. And, because contracting edges preserve the cyclic product property, the product of the observables is the identity times the label of the last vertex. Hence, the parity of this arrangement is $+1$ (as we already knew).

It remains to show the other direction of the result. That is, if the intersection graph of an arrangement is nonplanar, than the arrangement is magic.

Proof. By the Pontyagin-Kuratowski theorem, everry nonplanar graph contains either the K_5 or the $K_{3,3}$ graph as a topological minor. We have already seen that the $K_{3,3}$ is the intersection graph that corresponds to the magic square and that the K_5 corresponds to the magic pentagram, and both are magic arrangements. Hence, it suffices to demonstrate that, if the intersection graph G of an arrangement has a magic intersection graph H as a topological minor, than G is magic. For that, we are going to construct a quantum realization of G from the realization of H, as follows:

1. For every vertex in H, assume that the corresponding vertex in G is labelled in the same way.
2. Assume that the label of every other vertex is $+1$.
3. For every edge of H, label each edge of the corresponding path in G with the same observable.
4. Label the remaining edges of G as the identity.

To see why items (1) and (2) are safe to assume, recall that an arrangement depends only on its parity (meaning that it is possible to construct a realization for another arrangement on the same vertices and hyperedges and of same parity), and thus it does not matter how you label the vertices of G, as long as its parity is -1. It remains to show that the strategy above gives a quantum realization of G. For that, we check each of the properties of a quantum realization:

1. Every observable M assigned to a vertex of G is either one in H or the identity. Hence, M is Hermitian and squares to the identity.
2. Every vertex of G corresponding to a vertex of H is surrounded by the same observables on its edges, plus (possibly) copies of the identity. Therefore, these operators mutually commute. Furthermore, they multiply to the identity times the label of the vertex, which is labelled in the same way as its corresponding vertex in H.
3. Every vertex of G that lies on a simple path corresponding to an edge in H touches two edges (in and out) labelled with the same operator M, which is the operator of the edge in H, plus (possibly) copies of the identity. Since $M^2 = I$, these operators all commute and multiply to $+I$. Recall that we labelled these vertices as $+1$, as they do not have a correspondence in H. Hence, the product of the operators multiply to the identity times the vertex's label, as desired.
4. Any other vertex of G is labelled as $+1$ and is only surrounded by edges labelled as $+I$, which clearly commute and multiply to $+I$, i.e. the identity times the label of the vertex.

□

4 Conclusion

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