Quantum Pseudo-Telepathy

The Magic Square Game

Luigi Soares and Roberto Rosmaninho

Department of Computer Science, UFMG - Brazil
{luigi.domenico, robertogomes}@dcc.ufmg.br

1 Introduction

Telepathy, the ability of transmitting information from one person's mind to another's, would certainly come in handy in many situations, right? Unfortunately (or not), (to the best of our knowledge) telepathy is not a thing. At least, not according to classical physics. Certain aspects of the quantum realm, however, provide a way of communication that for a layman looks as magical as "true" telepathy. This phenomenon is called quantum *pseudo-telepathy* [1].

Quantum pseudo-telepathy is observed in many contexts, usually described in the format of a game: the "impossible colouring games" [1, 3]; the parity games, in which $n \geq 3$ players are given bit-strings and, without communicating to each other, they output one of their bits, winning if their outputs combined obey certain parity conditions [1, 4]; the Deutsch-Jozsa games, where Alice and Bob are given bit strings x and y, and must output bit strings a and b such that $a = b$ if and only if $x = y$ [1, 2]; and, the Magic Square game [1, 4]. None of these games admit a classical winning strategy (i.e. is not possible to always win), yet they can be won systematically, without any communication, provided that the players share prior entanglement [1].

In this project, we shall explore the Mermin-Peres Magic Square game (hereinafter referred simply as Magic Square). The origins of this game date back to the ninities. It was first described — albeit not in the format of a game — in the works of Mermin [**mermin:1990**] and of Peres [**peres:1990**]. Their results provided (as per the title of Mermin's review letter) a simplied proof for the Kochen-Specker theorem [**Kocher1975**]. Later, in 2002, Aravind demonstrated how to transform Mermin-Peres's proof of then Kochen-Specker theorem into a proof of Bell's theorem [**aravind:xxx**]. Bell's theorem shows the incompatibility of hidden variables (i.e. determinism) and locality, whereas Kochen-Specker's theorem demonstrates the conflicting nature of determinism and non-contextuality. These sort of results may seem challenging, but their descriptions can be greatly simplified by modeling them as quantum games. In particular, it is extremely easy to show that there cannot be a classical solution to the the Magic Square game, and to convince an observer that something "magical" (classically impossible) is happening in a successful implementation of a quantum winnin strategy [1].

TODO: FIX: they talk about contextuality only (both theorems)

Outline of this paper.

2 Contextuality

3 The Magic Square Game

A magic square is a 3×3 matrix whose entries are filled with ± 1's. The product of every row of a magic square must be $+1$. Similarly, the product of every column of a magic square must be -1. Is impossible to satisfy these two properties at the same time, hence the term *magic*: such a matrix cannot exist. The Magic Square game features two players, Alice and Bob, that must work together to fill in entries of the 3×3 table. At the beginning of the game, Alice and Bob are separated by a referee, Charlie, so that communication between them is impossible. At each round, Charlie assigns one row of the matrix at random to Alice, and asks Alice to give him the entries to fill in that particular row. After that, Charlie visits Bob and asks him to fill one column of the table, which Charlie draws at random. Bob does not know which row Alice was assigned, and Alice does not know which colum Bob was assigned. Alice and Bob win the round if their answers are valid, i.e. the product of the row filled by Alice is $+1$ and the product of the column filled by Bob is -1, and, in addition, the intersection of the row and the column agrees (i.e. both answered $+1$ or both answered -1). Before the game starts, Alice and Bob are allowed to communicate, so they can come up with an strategy. For example, they may prepare their answers for each row and column beforehand, or they may decide to follow a probabilistic strategy.

3.1 Classical Solution

Assume that Alice and Bob decided to adopt a deterministic strategy. For that, they met before the beginning of the game, and prepared their answers for each possible row and column. In this scenario, what is the best strategy that they can come up with?

Example 1 (Classical, Deterministic Solution). Alice and Bob might have agreed that, if given the first column, Bob shall fill it with three -1's. Then, whatever row Alice is given, she must fill it with either $\{-1, +1, -1\}$ or $\{-1, -1, +1\}$. Suppose that Alice decided that she would fill any row in the same way: $\{-1, -1, +1\}$. Figure 1 (a) shows how the predefined matrix would look like. Notice that Alice's rows are already defined, but Bob still has to decide how to fill the second and third columns. He can, and must, fill the second column with three -1's, so that whatever row Alice is given, the intersection agrees. Intuitively, for the same reason, he should then fill the third column with three $+1$'s. However, this is not a valid strategy, since the product of the last column will be $+1$ and not -1, as required. Thus, Bob must fill at least one of the entries of the last column with -1. By doing that, there will be one row and column whose intersection does not agree, and thus one row and column for which Alice and Bob lose, as illustrated in Figure 1 (b).

-1	-1	$+1$
-1	-1	$+1$
-1	-1	$+1$

(a) Initially, Alice and Bob agree that Bob shall fill the first column with $\{-1, -1, -1\}$. Based on this choice, Alice decides to fill all of her rows with $\{-1, -1, +1\}$.

-1	-1	± 1
-1	-1	$+1$
-1	-1	$+1$

(b) Bob has to fill the second column with -1's, to match Alice's choice. He should also match Alice's choice for each cell of the third column, but he cannot, for the product of the column must be -1.

Fig. 1: One possible classical deterministic strategy for the Magic Square game. Following this predefined strategy, Alice and Bob lose when assigned, respectively, the first row and third column, but win in any other scenario. That is, their probability of winning a round is $8/9$. Gray cells are entries for which Alice's and Bob's choices agree.

Example 1 illustrates one of the many deterministic strategies that Alice and Bob can adopt. It is not difficult to convince yourself that, no matter how they decide to fill in their entries, it is impossible for them to come up with predefined answers that always win. Any deterministic strategy is a pair of matrices, one for Alice and another for Bob. The only way that they could design a strategy that wins with certainty every round is to prepare two identical matrices satisfying the requirements for each row and for each column. That is, a single matrix for which the product of every row is $+1$ and the product of every column is -1. Such a matrix cannot possibly exist! It turns out that the best that they can do is to win with probability $8/9$.

What if Alice and Bob decide for a probabilistic strategy? That is, Alice and Bob each carry a coin. When assigned a row/column, they flip their coins and fill the entries of the row/column based on the outcomes of the coins. Can they do better? To randomly assign values to one row (equivalently, one column) is essentially the same as randomly selecting one of the 2^9 possible predefined 3×3 grids (including those matrices that would lead to invalid answers). In other words, a probabilistic strategy for the Magic Square game is one in which Alice and Bob randomly select a deterministic strategy. Hence, no matter how lucky Alice and Bob are, the probability of success of any strategy that they come up with is bounded by the winning probability of the best deterministic strategy, which is $8/9$. Theorem **??** formalizes this argument.

3.2 Quantum Solution

The quantum strategy for the Magic Square game consists of Alice and Bob carrying entangled qubits with them. For each round, they need a pair of qubits each. Then, when Charlie asks Alice (respectively Bob) to fill the entries of the i-th row (respectively j-th column), Alice measures her two qubits three times. The outcome of each measurement gives the value for each cell (in practice, they do not even need the third measurement, since they can determine the third value from the first two). Key to this strategy is that the outcomes of all possible measurements are ± 1. The initial, entangled state is (subscripts A and B indicate, respectively, Alice's and Bob's qubits)

$$|\psi\rangle = \frac{1}{2}\left(|00\rangle_A \otimes |00\rangle_B + |01\rangle_A \otimes |01\rangle_B + |10\rangle_A \otimes |10\rangle_B + |11\rangle_A \otimes |11\rangle_B\right). \tag{1}$$

Figure 2 shows the quantum circuit that can be used to prepare the initial state $|\psi\rangle$.

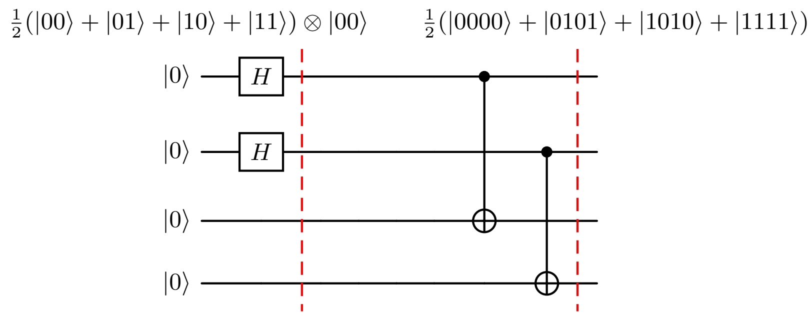

Fig. 2

As seen in Section 3.1, the *magic* square is called this way because, under classical assumptions, there is no such a square: it is impossible to construct a 3×3 matrix filled with ± 1's, with the constraints that the product of every row is $+1$ and the product of every column is -1. We can, however, construct a "quantum magic" square. Figure 3 shows the 3×3 grid that Alice and Bob can use to win the Magic Square game systematically. Notice that the eigenvalues of the observables at each cell are ± 1. Furthermore, the product of every row is $+I$ and the product of every column is $-I$.

$+I \otimes Z$	$+Z \otimes I$	$+Z \otimes Z$	$\prod = I \otimes I = I$
$+X \otimes I$	$+I \otimes X$	$+X \otimes X$	$\prod = I \otimes I = I$
$-X \otimes Z$	$-Z \otimes X$	$+Y \otimes Y$	$\prod = -\mathrm{i}I \otimes \mathrm{i}I = I$
$\prod = -I \otimes I = -I$	$\prod = -I \otimes I = -I$	$\prod = \mathrm{i}I \otimes \mathrm{i}I = -I$	

Fig. 3

Example 2 (Quantum Solution). Suppose that Charlie assigns the first row to Alice. To decide the value of the first cell, she (based on the 3×3 grid she and Bob have prepared) measures the observable $I \otimes Z$ on her two qubits. The possible outcomes are

$$\mathrm{Out}((I \otimes Z)_A \otimes (I \otimes I)_B) = \begin{cases} +1, & \text{proj. onto } \{|i0jk\rangle \mid i,j,k \in \{0,1\}\} \text{ with prob. } \frac{1}{2} \\ -1, & \text{proj. onto } \{|i1jk\rangle \mid i,j,k \in \{0,1\}\} \text{ with prob. } \frac{1}{2}, \end{cases} \tag{2}$$

Suppose that Alice has observed $+1$. Notice that only two of the components of the $+1$-eigenspace are seen in the initial state $|\psi\rangle$: $|0000\rangle$ and $|1010\rangle$. Therefore, $|\psi\rangle$ collapses to

$$|\psi\rangle_{A1} = \frac{1}{\sqrt{2}}(|0000\rangle + |1010\rangle), \tag{3}$$

where the subscript $A1$ denotes the state after Alice's first measurement. For the second cell, Alice measures the observable $Z \otimes I$, whose possible outcomes are

$$\mathrm{Out}((Z \otimes I)_A \otimes (I \otimes I)_B) = \begin{cases} +1, & \text{proj. onto } \{|0ijk\rangle \mid i,j,k \in \{0,1\}\} \text{ with prob. } \frac{1}{2} \\ -1, & \text{proj. onto } \{|1ijk\rangle \mid i,j,k \in \{0,1\}\} \text{ with prob. } \frac{1}{2}. \end{cases} \tag{4}$$

Suppose, now, that Alice has observed the outcome -1. Notice that only one component of the -1-eigenspace is seen in the state $|\psi\rangle_{A1}$. Hence, the state collapses to

$$|\psi\rangle_{A2} = |1010\rangle\,. \tag{5}$$

Finally, Alice measures the observable $Z \otimes Z$, whose only possible outcomes is

$$\mathrm{Out}((Z \otimes Z)_A \otimes (I \otimes I)_B) = \begin{cases} -1, & \text{proj. onto } \{|i\bar{i}jk\rangle \mid i,j,k \in \{0,1\}\} \text{ with prob. } 1, \end{cases} \tag{6}$$

where $\bar{i}$ is the `not` operator. Notice that, because $|\psi\rangle_{A2}$ falls exactly into the -1-eigenspace, the only possible outcome is -1, and the state does not change, i.e. $|\psi\rangle_{A3} = |\psi\rangle_{A2}$. Alice, then, fills the first row with the values $\{+1, -1, -1\}$, whose product is, as required, $+1$. Upon receiving Alice's answer, Charlie goes to meet Bob. Suppose he assigns the second column to Bob. To fill in the value of the first cell (which is the intersection of the first row and second column), Bob measures the observable $Z \otimes I$. Notice that $|\psi\rangle_{A3}$ is entirely in the -1-eigenspace; thus, Bob will for sure observe the outcome -1 — which matches Alice's answer — and the state will not change: $|\psi\rangle_{B1} = |\psi\rangle_{A3}$. Moving on to the second cell, Bob measures the observable $I \otimes X$, whose possible outcomes are (for readability, we are showing only the components of the eigenspaces that are not orthogonal to $|\psi\rangle_{B1}$)

$$\mathrm{Out}((I \otimes I)_A \otimes (I \otimes X)_B) = \begin{cases} +1, & \text{proj. onto } \frac{1}{\sqrt{2}}(|1010\rangle + |1011\rangle) \text{ with prob. } \frac{1}{2} \\ -1, & \text{proj. onto } \frac{1}{\sqrt{2}}(|1010\rangle - |1011\rangle) \text{ with prob. } \frac{1}{2}. \end{cases} \tag{7}$$

Assume that Bob has observed the outcome $+1$. Then, the state collapses to

$$|\psi\rangle_{B2} = \frac{1}{\sqrt{2}}(|1010\rangle + |1011\rangle). \tag{8}$$

Finally, Bob measures $-Z \otimes X$ on his two qubits. The only possible outcome is

$$\mathrm{Out}((I \otimes I)_A \otimes (-Z \otimes X)_B) = \begin{cases} +1, & \text{proj. onto } \frac{1}{\sqrt{2}}(|1010\rangle + |1011\rangle) \text{ with prob. } 1, \end{cases} \tag{9}$$

and the state does not change, i.e. $|\psi\rangle_{B3} = |\psi\rangle_{B2}$. Alice's answer satisfies the property of the rows (that the product of the entries is equal to $+1$) and Bob's answer satisfies the property of the columns (product is equal to -1). Moreover, Alice and Bob agreed in the intersection of the row and the column. Hence, they won this round!

4 Quantum Unitaries (? research)

References

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