Quantum Magic Games

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1 Introduction

Telepathy, the ability of transmitting information from one person's mind to another's, would certainly come in handy in many situations, right? Unfortunately (or not), (to the best of our knowledge) telepathy is not a thing. At least, not according to classical physics. Certain aspects of the quantum realm, however, provide a way of communication that for a layman looks as magical as "true" telepathy. This phenomenon is called quantum *pseudo-telepathy* [1].

Quantum pseudo-telepathy is observed in many contexts, usually described in the format of a game: the "impossible colouring games" [1, 3]; the parity games, in which $n \geq 3$ players are given bit-strings and, without communicating to each other, they output one of their bits, winning if their outputs combined obey certain parity conditions [1, 4]; the Deutsch-Jozsa games, where Alice and Bob are given bit strings x and y, and must output bit strings a and b such that $a = b$ if and only if $x = y$ [1, 2]; and, the Magic Square game [1, 4]. None of these games admit a classical winning strategy (i.e. is not possible to always win), yet they can be won systematically, without any communication, provided that the players share prior entanglement [1].

In this project, we shall explore the Mermin-Peres Magic Square game (hereinafter referred simply as Magic Square). The origins of this game date back to the ninities. It was first described — albeit not in the format of a game — in the works of Mermin [**mermin:1990**] and of Peres [**peres:1990**]. Their results provided (as per the title of Mermin's review letter) a simplied proof for the Kochen-Specker theorem [**Kocher1975**]. Later, in 2002, Aravind demonstrated how to transform Mermin-Peres's proof of then Kochen-Specker theorem into a proof of Bell's theorem [**aravind:xxx**]. Bell's theorem shows the incompatibility of hidden variables (i.e. determinism) and locality, whereas Kochen-Specker's theorem demonstrates the conflicting nature of determinism and non-contextuality. These sort of results may seem challenging, but their descriptions can be greatly simplified by modeling them as quantum games. In particular, it is extremely easy to show that there cannot be a classical solution to the the Magic Square game, and to convince an observer that something "magical" (classically impossible) is happening in a successful implementation of a quantum winnin strategy [1].

TODO: FIX: they talk about contextuality only (both theorems)

Outline of this paper.

2 The Magic Square Game

A magic square is a 3×3 matrix whose entries are filled with ± 1's. The product of every row of a magic square must be $+1$. Similarly, the product of every column of a magic square must be -1. Is impossible to satisfy these two properties at the same time, hence the term *magic*: such a matrix cannot exist. The Magic Square game features two players, Alice and Bob, that must work together to fill in entries of the 3×3 table. At the beginning of the game, Alice and Bob are separated by a referee, Charlie, so that communication between them is impossible. At each round, Charlie assigns one row of the matrix at random to Alice, and asks Alice to give him the entries to fill in that particular row. After that, Charlie visits Bob and asks him to fill one column of the table, which Charlie draws at random. Bob does not know which row Alice was assigned, and Alice does not know which colum Bob was assigned. Alice and Bob win the round if their answers are valid, i.e. the product of the row filled by Alice is $+1$ and the product of the column filled by Bob is -1, and, in addition, the intersection of the row and the column agrees (i.e. both answered $+1$ or both answered -1). Before the game starts, Alice and Bob are allowed to communicate, so they can come up with an strategy. For example, they may prepare their answers for each row and column beforehand, or they may decide to follow a probabilistic strategy.

2.1 Classical Solution

Assume that Alice and Bob decided to adopt a deterministic strategy. For that, they met before the beginning of the game, and prepared their answers for each possible row and column. In this scenario, what is the best strategy that they can come up with?

Example 1 (Classical, Deterministic Solution). Alice and Bob might have agreed that, if given the first column, Bob shall fill it with three -1's. Then, whatever row Alice is given, she must fill it with either $\{-1, +1, -1\}$ or $\{-1, -1, +1\}$. Suppose that Alice decided that she would fill any row in the same way: $\{-1, -1, +1\}$. Figure 1 (a) shows how the predefined matrix would look like. Notice that Alice's rows are already defined, but Bob still has to decide how to fill the second and third columns. He can, and must, fill the second column with three -1's, so that whatever row Alice is given, the intersection agrees. Intuitively, for the same reason, he should then fill the third column with three $+1$'s. However, this is not a valid strategy, since the product of the last column will be $+1$ and not -1, as required. Thus, Bob must fill at least one of the entries of the last column with -1. By doing that, there will be one row and column whose intersection does not agree, and thus one row and column for which Alice and Bob lose, as illustrated in Figure 1 (b).

-1	-1	$+1$
-1	-1	$+1$
-1	-1	$+1$

(a) Initially, Alice and Bob agree that Bob shall fill the first column with $\{-1, -1, -1\}$. Based on this choice, Alice decides to fill all of her rows with $\{-1, -1, +1\}$.

-1	-1	± 1
-1	-1	$+1$
-1	-1	$+1$

(b) Bob has to fill the second column with -1's, to match Alice's choice. He should also match Alice's choice for each cell of the third column, but he cannot, for the product of the column must be -1.

Fig. 1: One possible classical deterministic strategy for the Magic Square game. Following this predefined strategy, Alice and Bob lose when assigned, respectively, the first row and third column, but win in any other scenario. That is, their probability of winning a round is $8/9$. Gray cells are entries for which Alice's and Bob's choices agree.

Example 1 illustrates one of the many deterministic strategies that Alice and Bob can adopt. It is not difficult to convince yourself that, no matter how they decide to fill in their entries, it is impossible for them to come up with predefined answers that always win. Any deterministic strategy is a pair of matrices, one for Alice and another for Bob. The only way that they could design a strategy that wins with certainty every round is to prepare two identical matrices satisfying the requirements for each row and for each column. That is, a single matrix for which the product of every row is $+1$ and the product of every column is -1. Such a matrix cannot possibly exist! To see that, consider the constraints over each row and column:

$$\begin{aligned}
m_{0,0}\, m_{0,1}\, m_{0,2} &= +1 \\
m_{1,0}\, m_{1,1}\, m_{1,2} &= +1 \\
m_{2,0}\, m_{2,1}\, m_{2,2} &= +1 \\
m_{0,0}\, m_{1,0}\, m_{2,0} &= -1 \\
m_{0,1}\, m_{1,1}\, m_{2,1} &= -1 \\
\underline{m_{0,2}\, m_{1,2}\, m_{2,2}} &\underline{= -1} \\
+1 &\neq -1
\end{aligned}$$

It turns out that the best that they can do is to win with probability $8/9$

What if Alice and Bob decide for a probabilistic strategy? That is, Alice and Bob each carry a coin. When assigned a row/column, they flip their coins and fill the entries of the row/column based on the outcomes of the coins. Can they do better? To randomly assign values to one row (equivalently, one column) is essentially the same as randomly selecting one of the 2^9 possible predefined 3×3 grids (including those matrices that would lead to invalid answers). In other words, a probabilistic strategy for the Magic Square game is one in which Alice and Bob randomly select a deterministic strategy. Hence, no matter how lucky Alice and Bob are, the probability of success of any strategy that they come up with is bounded by the winning probability of the best deterministic strategy, which is $8/9$. A formal proof is given in **??** for any pseudo-telepathy game (i.e. games whose quantum strategies exhibit this notion of pseudo-telapathy).

2.2 Quantum Solution

The quantum strategy for the Magic Square game consists of Alice and Bob carrying entangled qubits with them. For each round, they need a pair of qubits each. Then, when Charlie asks Alice (respectively Bob) to fill the entries of the i-th row (respectively j-th column), Alice measures her two qubits three times. The outcome of each measurement gives the value for each cell (in practice, they do not even need the third measurement, since they can determine the third value from the first two). Key to this strategy is that the outcomes of all possible measurements are ± 1. The initial, entangled state is (subscripts A and B indicate, respectively, Alice's and Bob's qubits)

$$|\psi\rangle = \frac{1}{2}\left(|00\rangle_A \otimes |00\rangle_B + |01\rangle_A \otimes |01\rangle_B + |10\rangle_A \otimes |10\rangle_B + |11\rangle_A \otimes |11\rangle_B\right). \tag{1}$$

Figure 2 shows the quantum circuit that can be used to prepare the initial state $|\psi\rangle$.

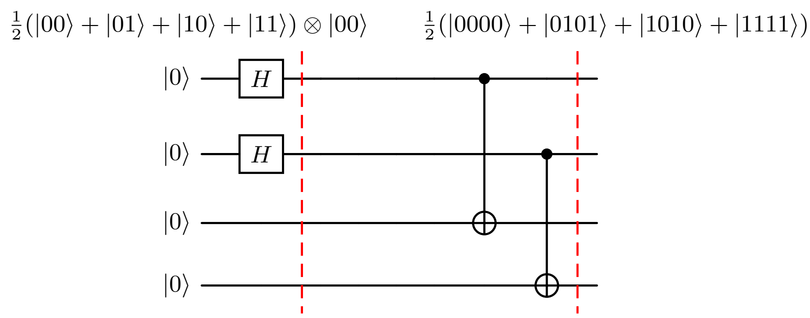


Fig. 2: Quantum circuit for preparing the initial joint state of Alice and Bob.

As seen in Section 2.1, the *magic* square is called this way because, under classical assumptions, there is no such a square: it is impossible to construct a 3×3 matrix filled with ± 1's, with the constraints that the product of every row is $+1$ and the product of every column is -1. We can, however, construct a "quantum magic" square. Figure 3 shows the 3×3 grid that Alice and Bob can use to win the Magic Square game systematically. Notice that the eigenvalues of the observables at each cell are ± 1. Furthermore, the product of every row is $+I$ and the product of every column is $-I$. The quantum strategy works as follows: Alice measures all the observables from the i-th row of the predefined matrix. She fills each cell with the result of each measurement. Similarly, Bob measures all the observables in the j-th column that was assigned to him, and fills each cell of his cells with the outcomes of each measurement. Example 2 depicts a round of the game.

$+I \otimes Z$	$+Z \otimes I$	$+Z \otimes Z$	$\prod = I \otimes I = I$
$+X \otimes I$	$+I \otimes X$	$+X \otimes X$	$\prod = I \otimes I = I$
$-X \otimes Z$	$-Z \otimes X$	$+Y \otimes Y$	$\prod = \mathrm{i}I \otimes -\mathrm{i}I = I$
$\prod = -I \otimes I = -I$	$\prod = -I \otimes I = -I$	$\prod = -\mathrm{i}I \otimes -\mathrm{i}I = -I$	

Fig. 3: Observables used by Alice and Bob to win the Magic Square game.

Example 2 (Quantum Solution). Suppose that Charlie assigns the first row to Alice. To decide the value of the first cell, she (based on the 3×3 grid she and Bob have prepared) measures the observable $I \otimes Z$ on her two qubits. The possible outcomes are

$$\mathrm{Out}((I \otimes Z)_A \otimes (I \otimes I)_B) = \begin{cases} +1\,(I \otimes |0\rangle\langle 0| \otimes I \otimes I) & \text{with probability } \frac{1}{2} \\ -1\,(I \otimes |1\rangle\langle 1| \otimes I \otimes I) & \text{with probability } \frac{1}{2}, \end{cases} \tag{2}$$

Suppose that Alice has observed $+1$. In this case, the initial state $|\psi\rangle$ collapses to

$$|\psi\rangle_{A1} = \frac{1}{\sqrt{2}}(|0000\rangle + |1010\rangle), \tag{3}$$

where the subscript $A1$ denotes the state after Alice's first measurement. For the second cell, Alice measures the observable $Z \otimes I$, whose possible outcomes are

$$\mathrm{Out}((Z \otimes I)_A \otimes (I \otimes I)_B) = \begin{cases} +1\,(|0\rangle\langle 0| \otimes I \otimes I \otimes I) & \text{with probability } \frac{1}{2} \\ -1\,(|1\rangle\langle 1| \otimes I \otimes I \otimes I) & \text{with probability } \frac{1}{2}. \end{cases} \tag{4}$$

Suppose, now, that Alice has observed the outcome -1. The state collapses to

$$|\psi\rangle_{A2} = |1010\rangle\,. \tag{5}$$

Finally, Alice measures the observable $Z \otimes Z$, whose possible outcomes are

$$\mathrm{Out}((Z \otimes Z)_A \otimes (I \otimes I)_B) = \begin{cases} +1\,((|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I \otimes I) & \text{with probability } 0 \\ -1\,((|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I \otimes I) & \text{with probability } 1. \end{cases} \tag{6}$$

Notice that the only possible outcome that Alice can observe is -1, and the state does not change, i.e. $|\psi\rangle_{A3} = |\psi\rangle_{A2}$. Alice, then, fills the first row with the values $\{+1, -1, -1\}$, whose product is, as required, $+1$. Upon receiving Alice's answer, Charlie goes to meet Bob. Suppose he assigns the second column to Bob. To fill in the value of the first cell (which is the intersection of the first row and second column), Bob measures the observable $Z \otimes I$. The only possible outcome, considering the state $|\psi\rangle_{A3}$, is -1 — which matches Alice's answer — and the state will not change again: $|\psi\rangle_{B1} = |\psi\rangle_{A3}$. Moving on to the second cell, Bob measures the observable $I \otimes X$, whose possible outcomes are

$$\mathrm{Out}((I \otimes I)_A \otimes (I \otimes X)_B) = \begin{cases} +1\,(I \otimes I \otimes I \otimes |+\rangle\langle +|) & \text{with probability } \frac{1}{2} \\ -1\,(I \otimes I \otimes I \otimes |-\rangle\langle -|) & \text{with probability } \frac{1}{2}. \end{cases} \tag{7}$$

Assume that Bob has observed the outcome $+1$. Then, the state collapses to

$$|\psi\rangle_{B2} = \frac{1}{\sqrt{2}}(|1010\rangle + |1011\rangle). \tag{8}$$

Finally, Bob measures $-Z \otimes X$ on his two qubits. The possible outcomes are

$$\mathrm{Out}((I \otimes I)_A \otimes (-Z \otimes X)_B) = \begin{cases} +1\,(I \otimes I \otimes (|1+\rangle\langle 1+| + |0-\rangle\langle 0-|)) & \text{with prob. } 1 \\ -1\,(I \otimes I \otimes (|1-\rangle\langle 1-| + |0+\rangle\langle 0+|)) & \text{with prob. } 0. \end{cases} \tag{9}$$

Bob will for sure observe the outcome $+1$ and the state does not change, i.e. $|\psi\rangle_{B3} = |\psi\rangle_{B2}$. Therefore, Bob answers $\{-1, +1, +1\}$. Alice's answer satisfies the property of the rows (that the product of the entries is equal to $+1$) and Bob's answer satisfies the property of the columns (product is equal to -1). Moreover, Alice and Bob agreed in the intersection of the

2.3 Formal Proof

To prove that the quantum solution indeed works, we must demonstrate two fundamental properties. First, the product of the outcomes of the three measurements on each row (equivalently, column) is equal to 1 (equiv. -1). Second, the outcome of Alice's measurement on the intersection is always equal to the outcome of Bob's measurement. Lemma 1 formalizes the product property.

Lemma 1 (The Product Property of the Quantum Solution). *Let $M_{i,j}$ be the observable at the cell (i,j) of the 3×3 grid shown in Figure 3. Then, it follows that*

$$\text{for every } i, \prod_j \mathrm{Out}(M_{i,j}) = +1, \text{ and} \tag{10}$$

$$\text{for every } j, \prod_i \mathrm{Out}(M_{i,j}) = -1. \tag{11}$$

Proof. First, notice that $X^2 = Z^2 = Y^2 = I$, $XZ = \mathrm{i}Y$ and $ZX = -\mathrm{i}Y$ (i.e. X and Z anti-commute), and, similarly, $XY = \mathrm{i}Z$ and $YX = -\mathrm{i}Z$. Using these properties, we can see that the product of every row (equiv. column) is the identity I (equiv. $-I$), regardless of the order of the three observables. That is, the three observables of every row and every column mutually commute (or, in other words, they are compatible). Consequently, for every row and every column, it is possible to find a basis of eigenvectors that are common to the three observables, meaning that they can be diagonalized simultaneously. Tables 1 and 2 show these common eigenvectors. Let $M_{i,j}$ be the observables at each cell. Then, for every row i (equiv. column j), we have

$$\begin{aligned} M_{i,0}M_{i,1}M_{i,2} &= (P_i\Lambda_{i,0}P_i^{-1})(P_i\Lambda_{i,1}P_i^{-1})(P_i\Lambda_{i,2}P_i^{-1}) \\ &= P_i(\Lambda_{i,0}\Lambda_{i,1}\Lambda_{i,2})P_i^{-1} \\ &= P_i\Lambda_iP_i^{-1}, \end{aligned} \tag{12}$$

where P_i is the matrix whose columns are the common eigenvectors and Λ_i is the matrix whose diagonal is formed by the products of the eigenvalues of the observables, associated with the same eigenvector. What this essentially means is that any measurement $M_{i,j}$ yields one of the eigenvalues λ in the diagonal of $\Lambda_{i,j}$ and projects the state onto the λ-eigenspace. That is, it collapses the state to a linear combination of the two shared eigenvectors $|\lambda\rangle_0$ and $|\lambda\rangle_1$ in the λ-eigenspace. Then, the second measurement will collapse the state to either $|\lambda\rangle_0$ or $|\lambda\rangle_1$, eigenvectors common to the three observables. Consequently, the third measurement will not change the state any further. Therefore, the product of the three outcomes will be the product of the eigenvalues associated with either $|\lambda\rangle_0$ or $|\lambda\rangle_1$, which is one of the entries in the diagonal of Λ_i. But, notice that $P_i\Lambda_iP_i^{-1} = I$; thus, $\Lambda_i = I$ and the only possible value for the product is 1 (equivalently -1 for columns). ☐

Table 1: Eigenvectors shared by the observables at each row of the quantum magic square. The eigenvectors were taken from **??**, adapting to the correct eigenvalues.

		Column 1	Column 2	Column 3
Row 1	$+1$	$\|00\rangle$, $\|10\rangle$	$\|00\rangle$, $\|01\rangle$	$\|00\rangle$, $\|11\rangle$
	-1	$\|01\rangle$, $\|11\rangle$	$\|10\rangle$, $\|11\rangle$	$\|01\rangle$, $\|10\rangle$
Row 2	$+1$	$(1/2)(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)$	$(1/2)(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)$	$(1/2)(\|00\rangle+\|01\rangle+\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)$
	-1	$(1/2)(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)$	$(1/2)(\|00\rangle-\|01\rangle-\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)$	$(1/2)(\|00\rangle+\|01\rangle-\|10\rangle-\|11\rangle)$, $(1/2)(\|00\rangle-\|01\rangle+\|10\rangle-\|11\rangle)$
Row 3	$+1$	$(1/2)(\|01\rangle-\|00\rangle+\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle+\|01\rangle-\|10\rangle+\|11\rangle)$,	$(1/2)(\|01\rangle-\|00\rangle+\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle-\|01\rangle+\|10\rangle+\|11\rangle)$,	$(1/2)(\|01\rangle-\|00\rangle+\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle+\|01\rangle+\|10\rangle-\|11\rangle)$,
	-1	$(1/2)(\|00\rangle-\|01\rangle+\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle+\|01\rangle+\|10\rangle-\|11\rangle)$,	$(1/2)(\|00\rangle+\|01\rangle-\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle+\|01\rangle+\|10\rangle-\|11\rangle)$,	$(1/2)(\|00\rangle+\|01\rangle-\|10\rangle+\|11\rangle)$, $(1/2)(\|00\rangle-\|01\rangle+\|10\rangle+\|11\rangle)$,

Table 2: Eigenvectors shared by the observables at each column of the quantum magic square. The eigenvectors were taken from **??**, adapting to the correct eigenvalues.

		Row 1	Row 2	Row 3
Column 1	$+1$	$(1/\sqrt{2})(\|00\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|10\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle + \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle + \|11\rangle)$
	-1	$(1/\sqrt{2})(\|01\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|11\rangle)$
Column 2	$+1$	$(1/\sqrt{2})(\|00\rangle + \|01\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|01\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle + \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle + \|11\rangle)$
	-1	$(1/\sqrt{2})(\|10\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|10\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|01\rangle)$, $(1/\sqrt{2})(\|10\rangle - \|11\rangle)$
Column 3	$+1$	$(1/\sqrt{2})(\|00\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|11\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle + \|10\rangle)$	$(1/\sqrt{2})(\|01\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|00\rangle - \|11\rangle)$
	-1	$(1/\sqrt{2})(\|01\rangle + \|10\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|10\rangle)$	$(1/\sqrt{2})(\|00\rangle - \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|10\rangle)$	$(1/\sqrt{2})(\|00\rangle + \|11\rangle)$, $(1/\sqrt{2})(\|01\rangle - \|10\rangle)$

Lemma 1 shows that the quantum solution satisfies the requirements of the game for every row and column. Now, we move on to show that Alice and Bob always agree in the value that they assign to the intersection of the row and column.

Lemma 2 (The Intersection Property of the Quantum Solution). *For any row i assigned to Alice and column j assigned to Bob, the value ascribed to the cell (i, j) by Alice and Bob, after they have performed their measurements, is always the same.*

Proof. To prove that Alice and Bob agree at the intersection, it suffices to look to the eigenvectors shared by each set of mutually commuting operators, shown in Tables 1 and 2. Notice that, regardless of the row assigned to Alice, the state of her subsystem after her measurements is always one of the eigenvectors in that row. This state is always a linear combination of one of Bob's eigenspaces for the cell in the intersection (the same applies if we consider first Bob than Alice). Take, for instance, the second row and first column. Suppose that Alice's measurements collapsed her qubits to $(1/2)(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$. This state can be written as $(1/\sqrt{2})(|00\rangle - |10\rangle) - (1/\sqrt{2})(|01\rangle - |11\rangle)$, a linear combination of the -1-eigenspace depicted in Table 2 (column 1, row 2). What is left is to show that Bob's state will be exactly the same as Alice's, implying that Bob shall observe the same result. This follows from the fact that the initial state is already symmetric (the first qubit of Alice is equal to the first qubit of Bob; the same is true for their second qubits) and the symmetry is preserved by any measurement performed by Alice and Bob. For the first row, it is easy to see. The others are trickier, but it all boils down to doing the calculations. Consider, for instance, the second row, and suppose that the outcomes were -1, -1 and $+1$, so that Alice's state collapsed to $(1/2)(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$. The state after the first measurement is

$$\begin{aligned}
&\frac{1}{2\sqrt{2}}\Big((|00\rangle - |10\rangle)\,|00\rangle + (|01\rangle - |11\rangle)\,|01\rangle + (|10\rangle - |00\rangle)\,|10\rangle + (|11\rangle - |01\rangle)\,|11\rangle \Big) \\
&\equiv \frac{1}{2\sqrt{2}}\Big(|00\rangle\,(|00\rangle - |10\rangle) + |01\rangle\,(|01\rangle - |11\rangle) + |10\rangle\,(|10\rangle - |00\rangle) + |11\rangle\,(|11\rangle - |01\rangle) \Big).
\end{aligned} \tag{13}$$

And, the state after the second measurement is

$$\begin{aligned}
&\frac{1}{4}\Big((|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|00\rangle - (|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|01\rangle - \\
&\quad (|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|10\rangle + (|00\rangle - |01\rangle - |10\rangle + |11\rangle)\,|11\rangle \Big) \\
&\equiv \frac{1}{4}\Big(|00\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle) - |01\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle) - \\
&\quad |10\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle) + |11\rangle\,(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \Big).
\end{aligned} \tag{14}$$

The calculations for the other rows and columns are similar, so we omitted them. □

With Lemmas 1 and 2, we can prove Theorem 1, which states that there is a quantum strategy for the Mermin-Peres magic square game that always win.

Theorem 1 (The Mermin-Peres Magic Square Game Is Winnable). *There exists a quantum strategy for the Mermin-Peres magic square game, which consists of allowing Alice and Bob to exchange entangled qubits* before *the game begins, that wins with certainty every round.*

2.4 Implementing with Circuits

In this section, we show the circuits used to implement the quantum strategy. We implemented the solution using the `Qiskit` library. `Qiskit` only allows measurements in the computational basis. For any other basis B, we must find an unitary transformation U such that $B = U^*(Z \otimes I)U$ (where U^* denotes the Hermitian adjoint of U). For example, $Z \otimes Z = CNOT(I \otimes Z)CNOT$ and $X \otimes X = (H \otimes H)(Z \otimes Z)(H \otimes H)$. Figure 4 shows the unitaries for each of the nine observables.

$(I \otimes I)(I \otimes Z)(I \otimes I)$	$(I \otimes I)(Z \otimes I)(I \otimes I)$	$CNOT(I \otimes Z)CNOT$
$(H \otimes I)(Z \otimes I)(H \otimes I)$	$(I \otimes H)(I \otimes Z)(I \otimes H)$	$(H \otimes H)(Z \otimes Z)(H \otimes H)$
$(H \otimes Y)(Z \otimes Z)(H \otimes Y)$	$(Y \otimes H)(Z \otimes Z)(Y \otimes H)$	$(S \otimes S)(X \otimes X)(S^* \otimes S^*)$

Fig. 4: Unitary transformations required for measuring in the different basis. The actual basis measured is the middle term of each cell, which, after expanding, is either $Z \otimes I$ or $I \otimes Z$. For example, in the last cell, $X \otimes X$ evaluates to $(H \otimes H)(Z \otimes Z)(H \otimes H)$. $Z \otimes Z$, in turn, becomes $CNOT(I \otimes Z)CNOT$, meaning that we measure the second qubit in the computational basis.

We now show the quantum circuits for each row and each column:

Row 0: When assigned the first row, Alice performs the measurements $I \otimes Z, Z \otimes I$ and $Z \otimes Z$, which the circuit in Figure 5 implements.

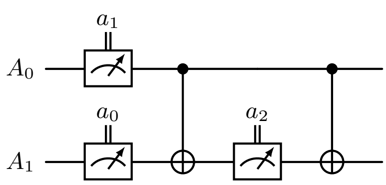


Fig. 5: Quantum circuit for measuring the first row.

Row 1: When assigned the second row, Alice performs the measurements $X \otimes I, I \otimes X$ and $X \otimes X$, which the circuit in Figure 6 implements.

Row 2: When assigned the third row, Alice performs the measurements $-X \otimes Z, -Z \otimes X$ and $Y \otimes Y$, which the circuit in Figure 7 implements.

Column 0: When assigned the first column, Bob performs the measurements $I \otimes Z, X \otimes I$ and $-X \otimes Z$, which the circuit in Figure 8 implements.

Column 1: When assigned the second column, Bob performs the measurements $Z \otimes I, I \otimes X$ and $-Z \otimes X$, which the circuit in Figure 9 implements.

Column 2: When assigned the third column, Bob performs the measurements $Z \otimes Z, X \otimes X$ and $Y \otimes X$, which the circuit in Figure 10 implements.

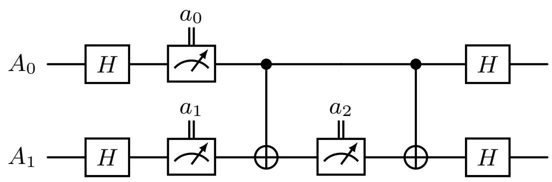


Fig. 6: Quantum circuit for measuring the second row.

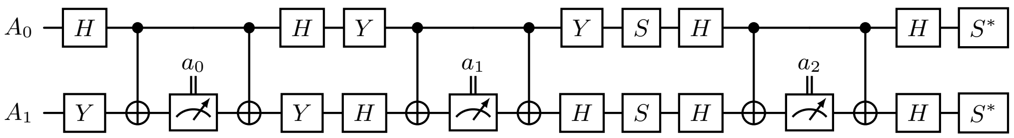


Fig. 7: Quantum circuit for measuring the third row.

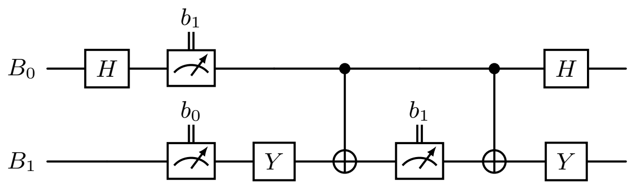


Fig. 8: Quantum circuit for measuring the first column.

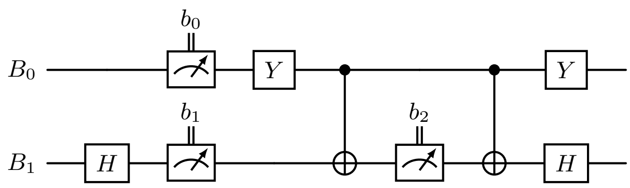


Fig. 9: Quantum circuit for measuring the second column.

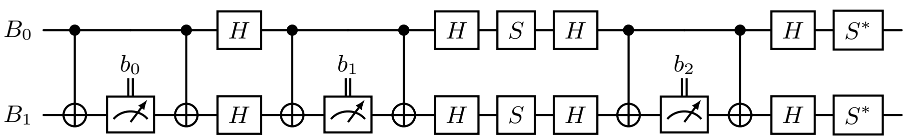
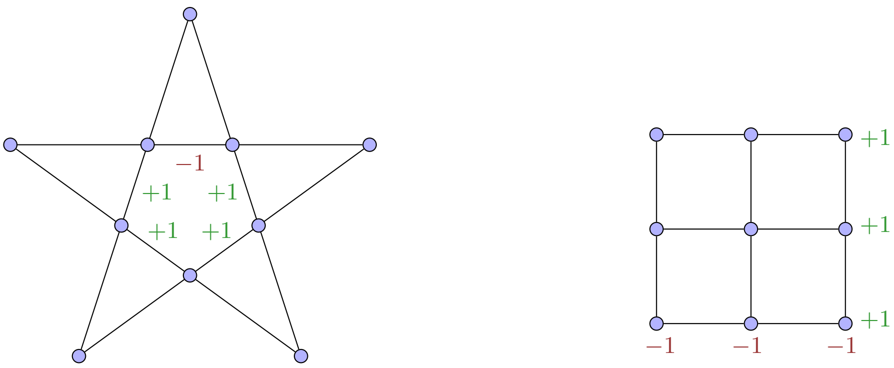


Fig. 10: Quantum circuit for measuring the third column.

3 Contextuality

4 Characterizing Quantum Magic Games

What if we change the format of the Mermin-Peres magic square game? Maybe increase the dimensions, or even change the shape entirely? Figure 11 (a) gives an example of the magic pentagram game, in which the product of the labels (± 1) assigned to each of four vertices in a line must equal the labels of that line (the product of the four vertices in the topmost horizontal line, for example, must be -1). Notice that we are now changing the visualization format to hypergraphs. Figure 11 (b) shows the magic square as a hypergraph. As the magic square, the pentagram is also due to Mermin **??**. In this section, we explore a characterization of Mermin-style games, given by [**arkhipov2012**].



(a) The pentagram game as a hypergraph. (b) The magic square game as a hypergraph.

Fig. 11: Examples of hypergraphs representing quantum magic games [**arkhipov2012**].

We start by defining the notion of *arrangements*, which are configurations on which Mermin-style games may be played. [**arkhipov2012**] defines both unsigned and signed arrangements. Instead, we shall focus on *signed* arrangements, and thus anytime we say "arrangement" we mean the signed version.

Definition 1 (Arrangement [arkhipov2012]). *An arrangement $A = (V, E, \ell)$ is a finite connected hypergraph with vertex set V and hyperedge set E, where a hyperedge is a nonempty subset of V, such that every vertex $v \in V$ lies in exactly two hyperedges $e_0, e_1 \in E$ (*connected *means that the hypegraph cannot be split into two smaller disjoint hypergraphs). The labelling function $\ell\colon E \to \{+1, -1\}$ maps each hyperedge to a sign $+1$ or -1.*

We will oftenly refer to a hyperedge as a *line*. In Figure 11 (a), the lines going through sequences of four vertices are the hyperedges. Similarly, in Figure 11 (b), the horizontal and vertical lines touching the sequences of three vertices are the hyperedges. As already mentioned, arrangements define the configurations on which games may be played.

Definition 2 (Classical Realization [arkhipov2012]). *A classical realization of an arrangement $A = (V, E, \ell)$ is a labelling $c\colon V \to \{+1, -1\}$ that, for every hyperedge $e \in E$, satisfies the property*

$$\prod_{v \in e} c(v) = \ell(e).$$

Example 3 (Classical Realization). Figure 12 depicts an example of a classically realizable arrangement. That is, an arrangement for which there exists a classical realization.

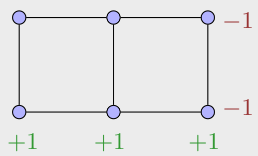


Fig. 12: Example of a classically realizable arrangement.

A classical realization c for this arrangement is

$$\begin{aligned} c(v_{00}) = c(v_{01}) = c(v_{10}) = c(v_{11}) &= +1 \\ c(v_{02}) = c(v_{12}) &= -1. \end{aligned}$$

Definition 3 (Quantum Realization [arkhipov2012]). *A quantum realization of an arrangement $A = (V, E, \ell)$ is a labelling $q\colon V \to \mathrm{GL}(\mathcal{H})$ is a mapping from each vertex to an observable M on a fixed finite-dimensional Hilbert-space $\mathcal{H}$, such that*

- *M is Hermitian and squares to the identity, or equivalently, has eigenvalues ± 1;*
- *For each hyperedge, the observable assigned to its vertices pairwise commute and*
- *For each hyperedge $e \in E$, the labelling of its vertices satisfies the property*

$$\prod_{v \in e} q(v) = \ell(e) I.$$

A classical realization is simply a quantum realization in which $\mathcal{H} = \mathbb{R}$. As such, every classically realizable arrangement is quantumly realizable. An example of quantum realization was given in Section 2.2 for the magic square game. Any arrangement that is quantumly realizable with observables that all mutually commute, then it is also classically realizable [**arkhipov2012**]. The proof follows by the simultaneous diagonalization of the observables, similar to our proof for Lemma 1.

It turns out that the existence of a classical or quantum realization of an arrangement depends only on whether the number of -1 labels is odd or even. This is called the *parity* of the arrangement.

Definition 4 (Parity [arkhipov2012]). *The parity $p(\ell)$ of an arrangement $A = (V, E, \ell)$ is*

$$p(\ell) = \prod_{e \in E} \ell(E),$$

which is -1 if there is an odd number of -1 labels, or $+1$ otherwise.

If there exists a realization (either classical or quantum) of an arrangement $A = (V, E, \ell)$, then it is possible to construct a realization for $A' = (V, E, \ell')$ from the realization of A [**arkhipov2012**]. Furthermore, an arrangement is classically realizable if, and only if, its parity is $+1$ [**arkhipov2012**]. Despite this fundamental result, Classically realizable arrangements are not as much interesting as the quantumly ones. Therefore, the goal is to characterize the gap between these two worlds. The arrangements that are quantumly realizable, but are not classically realizable, are called *magic* and are defined as follows:

Definition 5 (Magic Arrangement [arkhipov]). *An arrangement is said magic its parity is -1 and it is quantumly realizable.*

The magic square and the pentagram are examples of magic arrangements. Equipped with all these tools, we now define a generalization of *parity* pseudo-telepathy games (Mermin-style games):

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