UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola Superior d'Enginyeries Industrial,
Aeroespacial i Audiovisual de Terrassa

Departamento de Matemáticas
1 de diciembre de 2024

Erasmus+ Traineeship Report

Luigi Gisolfi (Unipd/UPC)

1. Introduction

Astrodynamics aims at studying the motion of a body m under the gravitational influence of other major bodies, as well as under the non gravitational forces (if any) acting on it. A solution of the type $\vec{x(t)}$, where $\vec{x}$ is a 6−dimensional vector made of positions and velocities of m, is sought as a function of time. The solution $\vec{x(t)}$ can be found via integration of the Equations of Motion (**EoM**), which are of the type:

$$\frac{\mathrm{d}^2\vec{q}}{\mathrm{d}t^2} = \vec{F_g} + \vec{F_{ng}} \tag{1}$$

where q is the spatial component of $\vec{x}$, and $\vec{F_g}$ and $\vec{F_{ng}}$ indicate the gravitational and non gravitational contributions of the forces acting on the body, respectively. The former are related to the shape and mass of the bodies that make up the gravitational field in which m is moving, while the latter could be represented, for example, by the solar radiation pressure or some drag force acting on m along its trajectory. Seen as a dynamical system, this problem can be formulated as

$$\dot{x} = \mathcal{F}(x, \xi, t) \tag{2}$$

where $\mathcal{F}$ is a smooth vector field $\mathcal{F} = \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^n$, with $n = 6$. In order to make the study of the EoM (1) more affordable, different models (often referred to as "problems") have been devised over the past centuries and decades. Each model is tailored to a different scenario and gives precious information on the motion of m under the gravitational of N of bodies. For the sake of this work, we will consider purely gravitational models, for which equation (1) reads

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \vec{F_g}. \tag{3}$$

Our goal will be to quantify which of the chosen astrodynamics models better describes the motion of m in some relevant scenarios. In Section 2, an overview of the adopted astrodynamics models is given, and the explicit form of the EoM is written down for each model. Section 4 describes the SPICE toolkit ADD REFERENCE NASA SPICE toolkit which - as it will be seen - is a key-tool that will be used to compare the reality to the results yielded by the models. Section 5 sets the reference frames in which our results are given, and how to go from one to the other. The quantities used to compare different astrodynamics models are introduced in Section 6, and the obtained results are finally discussed in Section 7.

2. Adopted Models

2.1. The Restricted N-body Problem

Determining the state vector $x(t)$ of a body in the gravitational field of N bodies constitutes one of the most interesting problems of physics. The N-body problem considers n point masses m and m_j, $j = 1, 2, ..., N-1$ moving in an inertial reference frame in three dimensional space moving under the influence of their mutual gravitational attraction. Each mass m_j is associated with a position vector q_j. Then, assuming spherical bodies, Newton's second law ADD REFERENCE PINCIPIA MATHEMATICA NEWTON implies that m is subject to a total force F_{tot} producing the acceleration $a(t) = \ddot{x(t)}$

$$\vec{a_m(t)} = \ddot{\vec{x(t)}} = -G\sum_{j=1}^{N} \frac{m_j \vec{q_{jm}}}{q_{jm}^3} \tag{4}$$

where G is the gravitational constant ($G = 6{,}6743 \pm 0{,}00015 \times 10^{-11} m^3 kg^{-1} s^{-2}$), and $q_{\vec{j}} m$ is the relative position vector between the m_j and m:

$$q_{\vec{jm}} = q_{\vec{m}} - q_{\vec{j}} \tag{5}$$

where all vectors $\vec{q}$ are defined with respect to an inertially fixed origin O assumed to coincide with the position of the barycenter of the N bodies. (4) is equivalent to six first-order differential equations: three for the position and three for the velocity. As it is well known, the $N-body$ problem is not solvable analytically. Nonetheless, if we have access to the positions $q_j\vec{(t)}$ of each of the N-1 bodies and $q_m(t)$ at some time t, then $x_m(t) = [q_m(t), \dot{q_m}(t)]$ can be found by numerical integration of (4). The positions $q_j\vec{(t)}$ can be retrieved from what is known as a "Development Ephemeris"(DE).

2.1.1. Development Ephemeris

Development Ephemeris (DE) have been produced over decades at the Jet Propulsion Laboratory (JPL) in Pasadena, California. These models provide numerical representations of positions, velocities, and accelerations of major Solar System bodies, tabulated at regular intervals. They include barycentric coordinates for the Sun, eight major planets, Pluto, and coordinates for the Moon. Each ephemeris is created by numerically integrating the equations of motion from a set of initial conditions. Due to the precision required by modern observational data, numerical integration replaced the less accurate analytical method of general perturbations. This approach solves the n-body problem, effectively simulating the Solar System's dynamics on a computer. Initial conditions include constants like planetary masses from external sources and parameters like initial positions and velocities, adjusted for best fit to observations using a least-squares technique. ADD REFERENCE Newhall, X. X.; Standish, E. M.; Williams, J. G. (1983). "DE 102 - A numerically integrated ephemeris of the moon and planets spanning forty-four centuries". Astronomy and Astrophysics. As of DE441 (the most recent one), the model includes perturbations from 343 asteroids, covering about 90 % of the main asteroid belt's mass. ADD REFERENCE Folkner, William (February 15, 2014). "The Planetary and Lunar Ephemerides, DE430 and DE431"(PDF). The physics modeled incorporates mutual Newtonian gravitational accelerations with relativistic corrections, tidal distortions of the Earth, accelerations from the Earth's and Moon's figure, and lunar librations. More information on the JPL DE versions can be found at: `https://ssd.jpl.nasa.gov/planets/eph_export.html`.

2.2. The Circular Restricted Three-Body Problem

2.2.1. Inertial and Rotating Reference Frames

Numerical integration of the N-body problem is cumbersome and complex. Moreover, for the motion of a spacecraft or asteroid in the Earth-Moon-Sun system, a three-body or four-body problem formulation might be enough to describe the evolution of $\vec{x_m(t)}$, as all other Solar System bodies are much further than the Sun, Earth or Moon and thus their gravitational contributions to the EoM (4) are generally small. For this reason, simplifications of the N-body problem have been formulated during the years, the most famous of which goes by the name of Circular Restricted Three Body Problem (CR3BP). The CR3BP constitutes the basis of many models found in the literature, and describes the case where two bodies - the primary, with mass m_1, and the secondary, with $mass m_2$, $m_1 > m_2$ - move about each other in circular orbits. For the way the problem is set, it is convenient to define a reference frame **X-Y-Z** centered on the barycenter of the system, with **X-Y** being the orbital plane of the primaries. As the two primaries orbit about each other, the barycenter is fixed at the origin, and the primaries start moving. In this inertial coordinates, their positions change. However, a rotating frame **x-y-z** can be defined, with **z** = **Z**, the **x**-axis connecting the two primaries at each time t and with **y**-axis such that $\hat{x} \times \hat{z}$ completes the right-handed orthogonal frame. ADD FIGURE as in libro.pdf, pag 27.
The bodies m_1 and m_2 orbit around each other over one period T and the corresponding mean motion is given by $n = \frac{2\pi}{T}$. This means that, assuming the $x-y-z$ rotating frame to coincide with the $X-Y-Z$ inertial frame at $t = 0$, the two will also coincide after any integer multiple of the period. The two coordinate systems share the same origin - the barycenter of the primaries - and the rotating coordinate axis are found by applying a counterclockwise rotation given by the matrix $A(t)$:

$$A(t) = \begin{pmatrix} \cos(nt) & \sin(nt) & 0 \\ -\sin(nt) & \cos(nt) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Hence, the coordinates of point P as expressed in the inertial frame will satisfy:

$$\begin{pmatrix} X_p \\ Y_p \\ Z_p \end{pmatrix} = A(t)^{-1} \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} \tag{7}$$

where (x_p, y_p, z_p) are the coordinates of P as expressed in the rotational frame. This can be checked by expressing the inertial coordinates unit vectors in terms of the rotating ones. In pretty much the same way, it can also be shown that the velocities change as:

$$\begin{pmatrix} \dot{X}_p \\ \dot{Y}_p \\ \dot{Z}_p \end{pmatrix} = A(t)^{-1} \begin{pmatrix} \dot{x}_p - y_p \\ \dot{y}_p + x_p \\ \dot{z}_p \end{pmatrix} \tag{8}$$

For the explicti computation, please refer to ADD REFERENCE libro.pdf, pp 27-28.

2.2.2. Adimensional Coordinates

The equations of motion in the CR3BP are often given in adimensional coordinates. It is thus needed to normalize length, time, and mass units. First, the unit of length $[LU]$ is chosen to be the constant separation between m_1 and m_2 (this is possible because of the circular orbit assumption in the CR3BP), $[LU] = d$; then the time unit is chosen to be $[TU] = 1/n$, so that the period of the system is $T = 2\pi$. Masses are given in terms of the combined mass of the system, meaning $[MU] = M = m_1 + m_2$. In this way, the primary will be placed at $x_1 = \mu$ and the secondary at $x_2 = \mu - 1$, where $\mu = \frac{m_2}{M}$ is the so-called "mass parameter". Moreover, with this convention, $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$ are the normalized masses of the primaries. Please note that, having normalized lengths, times and masses, the universal constant G value becomes $G = 1$ ($[G] = m^3 kg^{-1} s^{-2}$ in the physical system) . The reference system so defined is shown in figure ADD FIGURE showing the adimensional system.

2.2.3. Derivation of the EoM (CR3BP)

In this subsection, we will give a derivation of the $CR3BP$ EoM, and we will do so by using the Hamiltonian formalism. As it is well-known ADD REFERENCE From Lagrangian to Hamiltonian, the Hamiltonian $\mathcal{H}(q_i, p_i)$ of a system can be obtained by applying the Legendre transformation to the Lagrangian $\mathcal{L}(q_i, p_i)$:

$$\mathcal{H}(q_i, p_i) = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}(q_i, p_i) \tag{9}$$

where

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{10}$$

are the linear momenta of the system, and the Hamiltonian equations are given by

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \tag{11}$$

$$\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i} \tag{12}$$

$$\tag{13}$$

The Lagrangian of the system (kinetic minus potential energy) in the inertial frame takes the form

$$L(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}) = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - U(X, Y, Z, t) \tag{14}$$

and it is time dependent because - in normalized units - the gravitational potential takes the form:

$$U = -\frac{\mu_1}{r_1(t)} - \frac{\mu_2}{r_2(t)} - \frac{\mu_1\mu_2}{2} \tag{15}$$

where $r_1(t)$ and $r_2(t)$ are the distances of the spacecraft from m_1 and m_2 respectively and their time dependence reflect the movement of the two primaries in the inertial reference frame.
Moving to the rotational frame has the advantage of removing this time dependence, as in the rotating frame the two primaries are fixed and lie on the x-axis. Using equations (8) the Lagrangian then becomes:

$$\mathcal{L}(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}((\dot{x} - y)^2 + (\dot{y} + x)^2 + \dot{Z}^2) - \mathcal{U}(x, y, z) \tag{16}$$

where $\mathcal{U}$ has the same form as U in equation (15), but with r_1 and r_2 expressed in the rotating coordinates (x, y, z):

$$r_1 = \sqrt{(x-\mu)^2 + y^2 + z^2}, \tag{17}$$

$$r_2 = \sqrt{(x-\mu+1)^2 + y^2 + z^2} \tag{18}$$

In this specific case, the linear momenta are given by:

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial \dot{y}} \\ \frac{\partial L}{\partial \dot{z}} \end{pmatrix} = \begin{pmatrix} \dot{x} - y \\ \dot{y} + x \\ \dot{z} \end{pmatrix} \tag{19}$$

and the corresponding Hamiltonian function becomes:

$$H(x, y, z, p_x, p_y, p_z) = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \mathcal{L} \tag{20}$$

Hence, the Hamiltonian equations (11) read:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{p_x} \\ \dot{p_y} \\ \dot{p_z} \end{pmatrix} = \begin{pmatrix} p_x + y \\ p_y - x \\ p_z \\ p_y - x - \tilde{\mathcal{U}}_x \\ -p_x - x - \tilde{\mathcal{U}}_y \\ -\tilde{\mathcal{U}}_z \end{pmatrix} \tag{21}$$

where

$$\tilde{\mathcal{U}} = -\frac{1}{2}(x^2 + y^2) + \mathcal{U}(x, y, z) \tag{22}$$

is the *effective potential* and $U_i, i = x, y, z$ denote its partial derivatives. This system is equivalent to the result given by the Euler-Lagrange equations:

$$\frac{\mathrm{d}\partial L}{\mathrm{d}\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{23}$$

applied to the Lagrangian $\mathcal{L}$ of eq. (16):

$$\frac{\mathrm{d}(\dot{x} - y)}{\mathrm{d}t} = \dot{y} + x - U_x \tag{24}$$

$$\frac{\mathrm{d}(\dot{y} + x)}{\mathrm{d}t} = -\dot{x} + y - U_y \tag{25}$$

$$\frac{\mathrm{d}\dot{z}}{\mathrm{d}t} = -U_z \tag{26}$$

which, after simplification, gives the most-famous form of the CR3BP equations of motion:

$$\ddot{x} = 2\dot{y} - \tilde{\mathcal{U}}_x \tag{27}$$

$$\ddot{y} = -2\dot{x} - \tilde{\mathcal{U}}_y \tag{28}$$

$$\ddot{z} = -\tilde{\mathcal{U}}_z. \tag{29}$$

2.2.4. Jacobi Integral

2.3. The Hill Three-Body Problem

When $\mu \ll 1$, the orbit of the test particle is basically Keplerian with respect to the primary whenever the test particle is far from the secondary, and it is perturbed only when the close to the secondary. The Hill's model describes the motion of the spacecraft (test particle) around the secondary, effectively constituting an approximation to the CR3BP close to the smaller of the two primaries. Here, we show how it is derived. Starting from equations (27):

$$\ddot{x} - 2\dot{y} - x = -\frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu[x-(\mu-1)]}{r_2^3} \tag{30}$$

$$\ddot{y} + 2\dot{x} - y = -\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \tag{31}$$

$$\ddot{z} = -\tilde{\mathcal{U}}_z \tag{32}$$

we first shift the origin of the coordinate system to the location of the secondary, using $x' = x - \mu + 1$, and $y' = y$ (the secondary is located at $x = \mu - 1$). Substituting these in (30):

$$\ddot{x}' - 2\dot{y}' = x' - \mu + 1 - \frac{(1-\mu)(x'-1)}{r_1^3} - \frac{\mu x'}{r_2^3} \tag{33}$$

$$\ddot{y} + 2\dot{x} - y = -\frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3} \tag{34}$$

$$\ddot{z} = -\tilde{\mathcal{U}}_z \tag{35}$$

Of course, now we have also:

$$r_1 = \sqrt{(x'-1)^2 + y'^2 + z'^2}, \tag{36}$$

$$r_2 = \sqrt{(x')^2 + y'^2 + z'^2} \tag{37}$$

At this point, we take the limit $\mu \ll 1$, so that $1 - \mu \approx 1 + \mu \approx 1$, and $x' + \mu \approx x' - \mu \approx x'$, but we keep terms of the order $x' \sim \mu^{1/3} > \mu$ (since $\mu \ll |x'| \ll 1$). We get:

$$\ddot{x}' - 2\dot{y} = x' + 1 - \frac{x'-1}{r_1^3} - \frac{\mu x'}{r_2^3} \tag{38}$$

$$\ddot{y}' + 2\dot{x}' - y' = -\frac{y'}{r_1^3} - \frac{\mu y'}{r_2^3} \tag{39}$$

$$\ddot{z}' = \tilde{\mathcal{U}}_z \tag{40}$$

KEEP WRITING...

2.4. The Elliptical Three-Body Problem

2.5. The Hill Elliptical Three-Body Problem

2.6. The Bicircular Problem

3. Common Framework

4. SPICE Ephemerides

5. Changing References Frames

It is customary to present the results of a $C3TBP$-like analysis in the synodic adimensional reference frame described in 2.2.2. However, SPICE data is related to the physical world: spatial coordinates are always given in kilometers (km), and times are given in seconds (s). Moreover, as explained in 4, the user has to choose both a reference frame and an observer from which to compute the coordinates and velocities of the wanted object. Not all reference frames (e.g. $J2000$, $ECLIPJ2000$) [1] and observers are available, and their selection has to be made accordingly to the available kernels. For the sake of our work, we will always retrieve planetary SPICE data from the $J2000$ frame, whose origin will be set on the Solar System Barycenter (SSB). Results in this reference frame will have to be converted in results in the synodic $CR3BP$-like reference system to allow for comparisons and discussions. In 5.1, we briefly describe the $ECLIPJ2000$ reference system, while in 5.2 show how to go from one frame to another without loss of generality.

5.1. The $ECLIPJ2000$ Reference System

The $ECLIPJ2000$ reference system has its origin in the center of the Earth, and:

- its X-axis pointing from the center of the Earth to the mean vernal equinox[2] at Julian year 2000.0[3];
- Z-axis pointing toward the Northern Hemisphere in the normal direction to the ecliptic plane at Julian year 2000.0;
- Y-axis perpendicular to the x- and z- axes, forming a right-handed coordinate system.

It is therefore an Earth-centered inertial reference frame, and it does not rotate with the Earth. It being fixed with respect to distant celestial objects, it is suitable for describing the motion of celestial bodies and spacecraft. ADD IMAGE $ECLIPJ2000$

5.2. Reference Frame Conversion

Going from coordinates as expressed in the $ECLIPJ2000$ inertial reference frame to the synodic, $CR3BP$-like coordinates involves taking into account both a rotation (the synodic system is rotating together with the two primaries) and a translation of the origin to the barycenter of the synodic system (the origin of the $ECLIPJ2000$ corresponds to the Earth center). Then, when working with real data (such as the data retrieved from SPICE), not only one needs to apply a rotation and a translation to the inertial coordinates: both a distance conversion (from km to adimensional units) and time conversion (from physical times in seconds to the adimensional system, where time is expressed in terms of revolutions) are needed too. As we shall see, this conversion will be most important when transforming velocities and accelerations from one system to the other.

5.2.1. Time Conversion

As we have already mentioned in 2.2.2, the unit of time for the synodic adimensional system is given by $[TU] = 1/n$, where $n = \sqrt{\frac{Gm_1+Gm_2}{a}} = \frac{2\pi}{T}$ is the mean motion of the system made up of the two primaries with semi-major axis a and revolving around each other with period T[4]. Times in the $ECLIPJ2000$ reference frame - which we will denote with t^* - and times as expressed in the adimensional reference - denoted t - frame will thus be related by:

$$t = nt^* = \frac{t^*}{[TU]} \tag{41}$$

Given this rule, the derivatives f' in the inertial system are related to the adimensional ones ($\dot{f}$) by:

[1] visit https://naif.jpl.nasa.gov/pub/naif/toolkit$_docs/C/req/frames.htmlformore.$
[2] The vernal equinox line, which is drawn from the center of the Earth to the center of the Sun on the first day of spring.
[3] this corresponds exactly to 12:00 TT (close to but not exactly Greenwich mean noon) on January 1, 2000 in the Gregorian calendar.
[4] For example, $n \sim \frac{2\pi}{27}$ $days^{-1}$ for the Earth-Moon system and $n \sim \frac{2\pi}{365}$ $days^{-1}$ for the Sun-Earth system.

$$f' = \frac{df}{dt^*} = \frac{df}{dt} \cdot \frac{dt}{dt^*} = n\dot{f} \tag{42}$$

$$f'' = \frac{df^2}{dt^2} = n^2\ddot{f} \tag{43}$$

$$f''' = n^3\dddot{f}. \tag{44}$$

5.2.2. Position Conversion

The general change of coordinates between two reference systems with coordinates R and a is:

$$\vec{R} = kC\vec{a} + \vec{b} \tag{45}$$

where k is a change of scale factor, C is an orthogonal matrix (a rotation) and $\vec{b}$ is the vector that translates the origin of the system ($\vec{b} = \vec{0}$ if the origins of the two systems coincide). In our specific case, $\vec{R}$ refers to the positions in the inertial $ECLIPJ2000$ frame, while $\vec{a}$ denotes position vectors in the synodical adimensional reference system. If we indicate with $\vec{R_1}$ and $\vec{R_2}$ the positions of the two primaries in the inertial reference system, and $\vec{R_{21}} = \vec{R_2} - \vec{R_1}$, then $\vec{b} = \vec{R_1} + \mu\vec{R_{21}}$ is the location of the primaries' barycenter in the inertial coordinates.
As for the matrix C, it is a 3×3 matrix composed of three column vectors $\vec{C_1}, \vec{C_2}, \vec{C_3}$:

$$C = \begin{pmatrix} C_1 & C_2 & C_3 \end{pmatrix} \tag{46}$$

with

$$\vec{C_1} = \frac{\vec{R}_{21}}{\|\vec{R}_{21}\|}, \tag{47}$$

$$\vec{C_3} = \frac{\vec{R}_{21} \times \vec{R}'_{21}}{\|\vec{R}_{21} \times \vec{R}'_{21}\|}, \tag{48}$$

$$\vec{C_2} = \vec{C_3} \times \vec{C_1} \tag{49}$$

where $\vec{R_{21}}' = \vec{R_1}' - \vec{R_2}'$. The scaling factor k is the distance between the primary and the secondary:

$$k = \left\|\vec{R_{21}}\right\| \tag{50}$$

5.2.3. Velocity Conversion

Taking equation (45) and deriving it with respect to the inertial time t^*, we get:

$$\vec{R}' = k'C\vec{a} + kC'\vec{a} + kC\frac{d\vec{a}}{dt^*} + \vec{b}' = k'C\vec{a} + kC'\vec{a} + nkC\frac{d\vec{a}}{dt} + \vec{b}' \tag{51}$$

where we have used relation (41).

In a more compact form:

$$\vec{R}' = k'C\vec{a} + kC'\vec{a} + nkC\dot{a} + \vec{b}'. \tag{52}$$

The derivative C' appears in equation (52). This is a 3×3 matrix:

$$C' = \begin{pmatrix} \vec{C_1}' & \vec{C_2}' & \vec{C_3}' \end{pmatrix} \tag{53}$$

that is found by derivation of the components C_1, C_2, C_3 of C. Let's start with the first component, namely C_1:

$$\vec{C_1} = \frac{\vec{R}_{21}}{\|\vec{R}_{21}\|} \tag{54}$$

Its derivative is:

$$\vec{C_1}' = \frac{\vec{R_{21}'}}{k} - \frac{\vec{R_{21}}(\vec{R_{21}} \cdot \vec{R_{21}'})}{\left\|\vec{R_{21}}\right\|^3} \tag{55}$$

which can be rewritten as:

$$\vec{C_1}' = \frac{\vec{R_{21}'} - (k' \cdot \vec{C_1})}{k}, \tag{56}$$

since

$$k' = \frac{dk}{dt} = \frac{d\left\|\vec{R_{21}}\right\|}{dt} = \frac{\vec{R_{21}} \cdot \vec{R_{21}'}}{k}. \tag{57}$$

so we have got an expression for C_1'.

Moving to C_3',

$$\vec{C_3} = \frac{\vec{R}_{21} \times \vec{R}_{21}'}{\left\|\vec{R}_{21} \times \vec{R}_{21}'\right\|} \tag{58}$$

we derive the equation to get[5]:

$$\vec{C_3}' = \frac{\vec{R_{21}} \times \vec{R_{21}''}}{\left\|\vec{R_{21}} \times \vec{R_{21}'}\right\|} - \frac{\vec{R_{21}} \times \vec{R_{21}}'}{\left\|\vec{R_{21}} \times \vec{R_{21}'}\right\|^3}(\vec{R_{21}} \times \vec{R_{21}}') \cdot (\vec{R_{21}} \times \vec{R_{21}}''). \tag{59}$$

This can be simplified upon defining the angular momentum vector $\vec{h}$ and its first derivative (note that here we also provide the second derivative of h as it will turn out to be useful later, but this does not enter the expression for C_3'):

$$\vec{h} = \vec{R_{21}} \times \vec{R_{21}}' \tag{60}$$

$$\vec{h}' = \frac{(\vec{R_{21}} \times \vec{R_{21}}'') \cdot (\vec{R_{21}} \times \vec{R_{21}}')}{h} \tag{61}$$

$$\vec{h}'' = \frac{(\vec{R_{21}} \times \vec{R_{21}}'')^2 + (\vec{R_{21}} \times \vec{R_{21}})' \cdot (\vec{R_{21}}' \times \vec{R_{21}}'' + \vec{R_{21}} \times \vec{R_{21}}''') - \vec{h}' \cdot \vec{h}'}{h} \tag{62}$$

These are the norm of the angular momentum $\vec{h}$ and its first derivative $\vec{h}'$. With these substitutions, the last term of equation (59) becomes:

$$-\frac{\vec{R_{21}} \times \vec{R_{21}}'}{\left\|\vec{R_{21}} \times \vec{R_{21}'}\right\|^3}(\vec{R_{21}} \times \vec{R_{21}'}) \cdot (\vec{R_{21}} \times \vec{R_{21}}'') = \frac{\vec{h}' \cdot \vec{C_3}}{h} \tag{63}$$

thus we finally get:

$$\vec{C_3}' = \frac{(\vec{R_{21}} \times \vec{R_{21}}'') - (\vec{h}' \cdot \vec{C_3})}{h} \tag{64}$$

As for $\vec{C_2}'$, its derivative can be computed as:

$$\vec{C_2}' = \vec{C_3}' \times \vec{C_1} + \vec{C_3} \times \vec{C_1}' \tag{65}$$

in virtue of equation (47). Wrapping up the results obtained:

[5] In deriving this expression, we have used: $\frac{d\|f(t)\|}{dt} = \frac{f(t)}{\|f(t)\|}\frac{df(t)}{dt}$

$$\vec{C_1}' = \frac{\vec{R_{21}'} - (k' \cdot \vec{C_1})}{k} \tag{66}$$

$$\vec{C_3}' = \frac{(\vec{R_{21}} \times \vec{R_{21}}'') - (\vec{h}' \cdot \vec{C_3})}{h} \tag{67}$$

$$\vec{C_2}' = \vec{C_3}' \times \vec{C_1} + \vec{C_3} \times \vec{C_1}'. \tag{68}$$

5.2.4. Acceleration Conversion

To convert accelerations from the synodic adimensional reference system to the inertial physical one, one needs to take the second derivative of the conversion rule (45) (or, equivalently, the derivative of (52)). This yields:

$$\vec{R}'' = (k''C + 2k'C' + kC'')\vec{a} + (2k'C + 2kC')n\vec{a}' + n^2kC\vec{a}'' + \vec{b}'' \tag{69}$$

where again we have used the time transformation rule (41).
The matrix C'' components are computed in the following way. Starting from:

$$\vec{C_1}' = \frac{\vec{R_{21}'} - (k' \cdot \vec{C_1})}{k}, \tag{70}$$

we derive it once to get:

$$C_1'' = \frac{\vec{R_{21}}''}{\left\|\vec{R_{21}}\right\|} - \frac{\vec{R_{21}}'(\vec{R_{21}} \cdot \vec{R_{21}}')}{\|\vec{_{21}}\|^3} - \frac{\vec{R_{21}}'(\vec{R_{21}} \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^3} - \frac{\vec{R_{21}}(\vec{R_{21}}' \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^3} - \frac{\vec{R_{21}}(\vec{R_{21}} \cdot \vec{R_{21}}'')}{\left\|\vec{R_{21}}\right\|^3} + 3\frac{\vec{R_{21}}(\vec{R_{21}} \cdot \vec{R_{21}}') \cdot (\vec{R_{21}} \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^5}. \tag{71}$$

The second and third term on the right hand side can be summed up together:

$$C_1'' = \frac{\vec{R_{21}}''}{\left\|\vec{R_{21}}\right\|} - 2\frac{\vec{R_{21}}'(\vec{R_{21}} \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^3} \underbrace{- \frac{\vec{R_{21}}(\vec{R_{21}}' \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^3} - \frac{\vec{R_{21}}(\vec{R_{21}} \cdot \vec{R_{21}}'')}{\left\|\vec{R_{21}}\right\|^3} + 3\frac{\vec{R_{21}}(\vec{R_{21}} \cdot \vec{R_{21}}') \cdot (\vec{R_{21}} \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^5}}. \tag{72}$$

We also note that the underbraced term in equation (72) is equal to:

$$\underbrace{- \frac{\vec{R_{21}}(\vec{R_{21}}' \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^3} - \frac{\vec{R_{21}}(\vec{R_{21}} \cdot \vec{R_{21}}'')}{\left\|\vec{R_{21}}\right\|^3} + 3\frac{\vec{R_{21}}(\vec{R_{21}} \cdot \vec{R_{21}}') \cdot (\vec{R_{21}} \cdot \vec{R_{21}}')}{\left\|\vec{R_{21}}\right\|^5}} = \frac{\vec{R_{21}}}{k^2}\left[\frac{2k' \cdot k'}{k} - k''\right] \tag{73}$$

where k'' is the derivative of k'.

$$k'' = \frac{\vec{R_{21}}' \cdot \vec{R_{21}}'}{k} + \frac{\vec{R_{21}} \cdot \vec{R_{21}}''}{k} - \frac{(\vec{R_{21}} \cdot \vec{R_{21}}') \cdot (\vec{R_{21}} \cdot \vec{R_{21}}')}{k^3} \tag{74}$$

or, more simply:

$$k'' = \frac{\vec{R_{21}}' \cdot \vec{R_{21}}' + \vec{R_{21}} \cdot \vec{R_{21}}'' - k'^2}{k} \tag{75}$$

Putting the information together and substituting k' where needed:

$$C_1'' = \frac{\vec{R_{21}}''}{k} - 2\vec{R_{21}}\frac{k'}{k^2} + \vec{R_{21}}\frac{(2k' \cdot k' - k \cdot k'')}{k^3}. \tag{76}$$

We now seek an expression for C_3''. This will be given by deriving the expression we had previously found for C_3':

$$\vec{C_3}' = \frac{(\vec{\vec{R_{21}}} \times \vec{\vec{R_{21}''}}) - (\vec{h}' \cdot \vec{C_3})}{h}. \tag{77}$$

We compute it as:

$$C_3'' = \frac{(\vec{R_{21}}' \times \vec{R_{21}}'' + \vec{R_{21}} \times \vec{R_{21}}''') - h'' \cdot C_3 - h' \cdot C_3'}{h} - \frac{(\vec{R_{21}} \times \vec{R_{21}}'') - (h' \cdot C_3)}{h^3}(h \cdot h') \tag{78}$$

which, upon careful substitution of C_3, C_3' and h'' (see eq. (47), (64), (60)), can be rewritten as:

$$C_3'' = \frac{(\vec{R_{21}}' \times \vec{R_{21}}'' + \vec{R_{21}} \times \vec{R_{21}}''')}{h} - 2\frac{h'(\vec{R_{21}} \times \vec{R_{21}}'')}{h^2} + (\vec{R_{21}} \times \vec{R_{21}}')\frac{2(h' \cdot h') - (h \cdot h'')}{h^3} \tag{79}$$

Finally, deriving C_2' in equation (65), we get:

$$C_2'' = C_3'' \times C_1 + 2C_3' \times C_1 + C_3 \times C_1''. \tag{80}$$

Therefore, the following column vectors make up the matrix C'':

$$C_1'' = \frac{\vec{R_{21}}''}{k} - 2\vec{R_{21}}\frac{k'}{k^2} + \vec{R_{21}}\frac{(2k' \cdot k' - k \cdot k'')}{k^3} \tag{81}$$

$$C_3'' = \frac{(\vec{R_{21}}' \times \vec{R_{21}}'' + \vec{R_{21}} \times \vec{R_{21}}''') - h'' \cdot C_3 - h' \cdot C_3'}{h} - \frac{(\vec{R_{21}} \times \vec{R_{21}}'') - (h' \cdot C_3)}{h^3}(h \cdot h') \tag{82}$$

$$C_2'' = C_3'' \times C_1 + 2C_3' \times C_1 + C_3 \times C_1''. \tag{83}$$

6. Comparing Different Models

6.1. The Residual Acceleration

7. Results

7.1. Scenario 1: Orbits About the Primaries

7.2. Scenario 2: Orbits Far from the Primaries

7.3. Scenario 3: Periodic Orbits