Hilbert Space

A *Hilbert Space* is a complex vector space equipped with an inner product operation.

Mathematically, a Hilbert space $\mathcal{H}$ satisfies the following properties:

- **Vector Space:** $\mathcal{H}$ is a vector space over the complex numbers $\mathbb{C}$.
- **Inner Product:** There exists an inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that is linear in its first argument, conjugate symmetric, and positive-definite.
- **Completeness:** $\mathcal{H}$ is complete with respect to the norm induced by the inner product, i.e., every Cauchy sequence in $\mathcal{H}$ converges to an element in $\mathcal{H}$. This means that there are no "points missing" in an Hilbert Space.

The inner product generalizes the notion of the dot product to complex vector spaces and is fundamental in quantum theory and functional analysis. A Hilbert space generalizes the notion of Euclidean space to infinite dimensions.

In quantum mechanics, the mathematical framework is built upon Hilbert spaces:

- **States:** The possible states of a quantum system are represented by (normalized) vectors in a Hilbert space $\mathcal{H}$.
- **Observables:** Physical quantities (observables) correspond to Hermitian (self-adjoint) operators acting on $\mathcal{H}$.
- **Symmetries:** Symmetry transformations are represented by unitary operators on $\mathcal{H}$, preserving inner products and probabilities.
- **Measurements:** The act of measurement is mathematically described by orthogonal projection operators, which project state vectors onto subspaces associated with measurement outcomes.

This structure provides the foundation for the mathematical description of quantum systems.