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A fast algorithm to compute the H_∞-norm of a transfer function matrix

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Abstract: A fast algorithm is presented to compute the H_∞-norm of a transfer function matrix, based on the relation between the singular values of the transfer function matrix and the eigenvalues of a related Hamiltonian matrix. The norm is computed with guaranteed accuracy. A very simple method to compute a rather close lower bound on the H_∞-norm is given.

Keywords: H_∞-norm; L_∞-norm; H_∞-control; Hamiltonian matrix; singular values.

Notation

$[A, B, C, D]$: state-space form of a transfer function matrix

$$G(s) = C[sI - A]^{-1}B + D.$$

$G^\sim(s) = G^T(-s)$.
$G^*(s) = G^T(\bar{s})$ ($\bar{s}$ = complex conjugate of s).
σ_{max}: maximum singular value.
$\sigma_{H,max}$: maximum Hankel singular value.

1. Introduction

In the recent literature on robust analysis and control, see for instance [5,7], the H_∞-norm of a transfer function matrix plays an important role. The computation of the H_∞-norm can be necessary either in analysis of a system, or in the synthesis of a controller; see for instance [11].

Definition 1.1. Let a transfer function matrix $G(s)$ be given by $[A, B, C, D]$, and let all the eigenvalues of A have negative real part. Then the H_∞-*norm of* $G(s)$ is defined as the supremum of the maximum singular value of $G(s)$, evaluated over the right half plane:

$$\|G\|_\infty := \sup_{\mathrm{Re}(s)\geq 0} \sigma_{max}(G(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{max}(G(j\omega)). \tag{1.1}$$

The H_∞-norm is defined for systems that are analytical in the closed right half plane. Systems that have no poles on the imaginary axis have an L_∞-norm that is defined as the supremum of the maximum singular value of $G(s)$ evaluated on the imaginary axis, so the right part of (1.1) also gives the L_∞-norm in the case of unstable $G(s)$.

Until 1988 not much attention has been paid to the computation of the H_∞-norm. The 'computation' was done by a search over frequencies. The disadvantages of this approach are obvious: it cannot be used automatically within other algorithms, it takes a considerable amount of computer time, and no accuracy bound can be given. In 1988 a bisection algorithm was presented by Boyd and Balakrishnan [1,2] and Robel [10], to compute the H_∞-norm with guaranteed accuracy, using the relation between the singular values of the transfer function matrix and the eigenvalues of a related Hamiltonian matrix. This bisection algorithm is much more efficient than a search over frequencies, but for repeated use as well as for very large systems, it is still not very fast.

The authors have investigated the use of derivatives of the eigenvalues of the Hamiltonian matrix for a search algorithm. An algorithm based on this could not substantially reduce the computing time.

The algorithm presented here is considerably faster than the bisection algorithm. The algorithm is guaranteed to converge, and the local convergence is very fast.

During the preparation of this paper, the authors experienced that Boyd and Balakrishnan had results on a similar algorithm [3]. They give a

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nice proof of the local convergence being at least quadratic. Their algorithm has one essential difference in the iteration step, as well as a difference in the lower bound starting value computation.

2. Theoretical background

2.1. Hamiltonian eigenvalues and singular values

Let system $G(s)$ be given through

$$G(s) = [A,\ B,\ C,\ D] \tag{2.1}$$

and *let A not have any eigenvalues on the imaginary axis.*

For $\gamma > 0$ *not equal to a singular value of D*, we define the Hamiltonian matrix

$$H(\gamma) = \begin{bmatrix} A - BR^{-1}D^{\mathrm{T}}C & -\gamma BR^{-1}B^{\mathrm{T}} \\ \gamma C^{\mathrm{T}}S^{-1}C & -A^{\mathrm{T}} + C^{\mathrm{T}}DR^{-1}B^{\mathrm{T}} \end{bmatrix} \tag{2.2}$$

where

$$R = (D^{\mathrm{T}}D - \gamma^2 I) \quad \text{and} \quad S = (DD^{\mathrm{T}} - \gamma^2 I).$$

As stated in [2], under the assumptions made, (2.1) and (2.2) are related by the following equivalence.

Proposition 2.1. *For all $\omega_p \in \mathbb{R}$,*

$j\omega_p$ is an eigenvalue of $H(\gamma_1)$

$$\Leftrightarrow \ \gamma_1 \text{ is a singular value of } G(j\omega_p). \tag{2.3}$$

Proof. It can be shown (see Appendix A.1) that under the assumption that γ is not equal to a singular value of D, the transfer function matrix

$$\left[\gamma^2 I - G^{\sim}(s)G(s)\right]^{-1}$$

has state space realization $[\tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D}]$ with $\tilde{A} = H(\gamma)$. Also, $[\tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D}]$ can only have an unobservable or uncontrollable mode belonging to eigenvalue λ of $\tilde{A}$ if A has an eigenvalue λ or $-\lambda$ (Appendix A.2). By assumption, A has no imaginary eigenvalues, so $\tilde{A}$ can not have any unobservable or uncontrollable imaginary eigenvalues. By this we may conclude,

$j\omega_p$ is an eigenvalue of $H(\gamma)$

$$\Leftrightarrow \ j\omega_p \text{ is a pole of } \left[\gamma^2 I - G^{\sim}(s)G(s)\right]^{-1}. \tag{2.4}$$

The poles of

$$\left[\gamma^2 I - G^{\sim}(s)G(s)\right]^{-1}$$

equal the transmission zeros of $\gamma^2 I - G^{\sim}(s)G(s)$ [9].

Now λ is a transmission zero of a transfer function matrix $P(s)$ not coinciding with a pole, if and only if $\det |P(\lambda)|$ is zero. Because we assumed that $G(s)$ has no imaginary poles, $\gamma^2 I - G^{\sim}(s)G(s)$ will not have imaginary poles, so we may conclude:

$j\omega_p$ is a pole of $\left[\gamma^2 I - G^{\sim}(s)G(s)\right]^{-1}$

$$\Leftrightarrow \ \det|\gamma^2 I - G^{\sim}(j\omega)G(j\omega)| = 0. \tag{2.5}$$

Obviously,

$$\text{if } \mathrm{Re}(s) = 0 \text{ then } G^{\sim}(s) = G^*(s). \tag{2.6}$$

From the definition of singular values [4] we have

σ is a singular value of $G(s)$

$$\Leftrightarrow \ \det|\sigma^2 I - G^*(s)G(s)| = 0. \tag{2.7}$$

The proof of (2.3) follows by subsequently using (2.4), (2.5), (2.6) and (2.7):

$j\omega_p$ is an eigenvalue of $H(\gamma_1)$

$\Leftrightarrow$ $j\omega_p$ is a pole of $\left[\gamma^2 I - G^{\sim}(s)G(s)\right]^{-1}$

$\Leftrightarrow$ $\det\left|\gamma_1^2 I - G^{\sim}(j\omega_p)G(j\omega_p)\right| = 0$

$\Leftrightarrow$ $\det\left|\gamma_1^2 I - G^*(j\omega_p)G(j\omega_p)\right| = 0$

$\Leftrightarrow$ γ_1 is a singular value of $G(j\omega_p)$. □

2.2. Interpretation

Consider some strictly proper system $G(s)$, and its singular values as a function of frequency $j\omega$ (Figure 1).

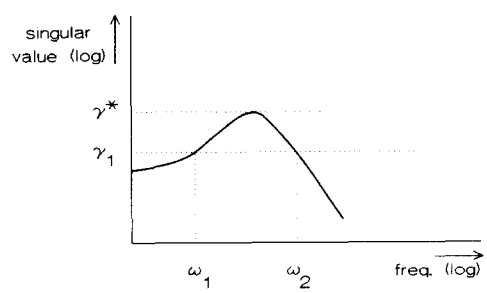

Fig. 1. Singular values of a strictly proper system.

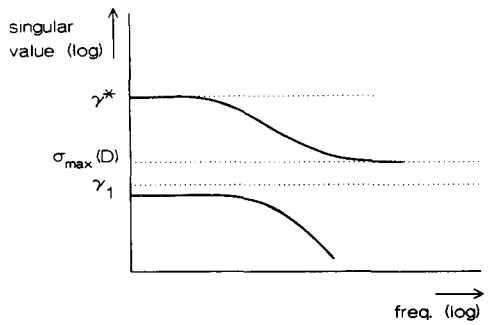

Fig. 2. Singular values of a non-strictly proper system.

If a line at value γ_1 (see Figure 1) intersects the singular values at frequencies ω_1 and ω_2, it follows from (2.3) that $H(\gamma_1)$ has imaginary eigenvalues $\pm j\omega_1$ and $\pm j\omega_2$ (γ_1 is a singular value of $G(\pm j\omega_1)$ and of $G(\pm j\omega_2)$). The H_∞-norm of this system, the supremum of the largest singular value over all frequencies $j\omega$ (Definition 1.1), is the peak value γ^*. Any line above this value will not intersect the singular value plot, so $H(\gamma)$ will not have any imaginary eigenvalues for $\gamma > \gamma^*$.

For $\omega \to \infty$ the singular values become equal to the singular values of D, so for a strictly proper system ($D = 0$) all the singular values go to zero for $\omega \to \infty$. Any line below γ^* will always intersect at least one of the singular values, so we may conclude by (2.3) that for strictly proper systems $H(\gamma)$ will have imaginary eigenvalues for any $0 < \gamma \leq \gamma^*$. For a non-strictly proper system ($D \neq 0$) at least one of the singular values will not go to zero for $\omega \to \infty$. In this case we could have the situation drawn in Figure 2.

For $\gamma = \gamma_1$ the singular value plot will not be intersected, so by (2.3) we conclude that $H(\gamma_1)$ has no imaginary eigenvalues. This shows clearly that for a non-strictly proper system there may be values $\gamma < \gamma^*$ where $H(\gamma)$ has no imaginary eigenvalues, however only for $\gamma < \sigma_{max}(D)$.

3. A fast algorithm to compute the H_∞-norm

We refer to the algorithm presented here as the 'two-step algorithm'. The algorithm starts with some lower bound $\gamma_{lb} < \| G \|_\infty$ (see (4.3)). In an iteration loop the lower bound is increased until the required accuracy is achieved. Within each iteration two steps lead to the next γ_{lb}.

3.1. Computation of next γ_{lb}

Normal singular value computation enables us to compute the singular values of a transfer function matrix at a given frequency point. Relation (2.3) enables us to do the inverse: compute the frequencies corresponding to a certain singular value γ. In the two-step algorithm both steps are taken to obtain the next γ_{lb} (see Figure 3):

Step 1. Compute the frequencies ω_1 to ω_k for lower bound $\gamma_{lb}(i)$, applying an eigenvalue computation to the Hamiltonian matrix $H(\gamma_{lb}(i))$ (2.2).

Step 2. Take frequencies m_1 to m_{k-1} with $m_i = \frac{1}{2}(\omega_i + \omega_{i+1})$, compute the singular values of $G(jm_i)$ and take as new lower bound

$$\gamma_{lb}(i+1) = \max_i \{ \sigma_{max}(G(jm_i)) \}.$$

3.2. Stop criterion

When $(\| G \|_\infty - \gamma_{lb}(i)) \leq (1+\varepsilon) \| G \|_\infty$, the required relative accuracy ε is achieved. A stop criterion is introduced by slightly altering Step 1: compute the eigenvalues of $H(\gamma)$ for γ slightly larger than γ_{lb}, $\gamma = (1+2\varepsilon)\gamma_{lb}$. If $H(\gamma)$ has imaginary eigenvalues we go on with Step 2. If $H(\gamma)$ has no imaginary eigenvalues, γ is an upper bound on the H_∞-norm, so it is guaranteed that

$$\gamma_{lb} \leq \| G \|_\infty < (1+2\varepsilon)\gamma_{lb}.$$

If we take $\| G \|_\infty = (1+\varepsilon)\gamma_{lb}$ the maximum relative error is ε.

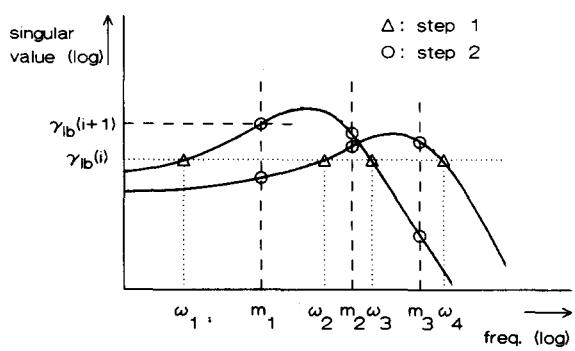

Fig. 3. Two steps to compute the next value $\gamma_{lb}(i+1)$.

3.3. The two-step algorithm

Algorithm to compute the H_∞-norm:
(a) Compute a starting value for γ_{lb}, using (4.2).
(b) *repeat* until 'break'
(b1) $\gamma = (1 + 2\varepsilon)\gamma_{lb}$;
(b2) compute the eigenvalues of $H(\gamma)$, (2.2);
(b3) *if* no imaginary eigenvalues,
$\gamma_{ub} = \gamma$, break
else
ω_1 to ω_k = imaginary eigenvalues,
$m_i = \frac{1}{2}(\omega_i + \omega_{i+1})$, $i = 1, \ldots, k-1$,
compute the singular values of $G(jm_i)$,
$\gamma_{lb} = \max_i\{\sigma_{max}(G(jm_i))\}$.
(c) $\| G \|_\infty = \frac{1}{2}(\gamma_{lb} + \gamma_{ub})$.

Remark 3.1. The algorithm is always converging because $\gamma_{lb}(i+1) > \gamma_{lb}(i)$.

Remark 3.2. The computing time for each iteration step is affected by the number of frequency points $k-1$; generally the last iteration step will take less time than the first.

Remark 3.3. Very high accuracy can be achieved in a few steps; in the first step(s) the estimation for the next γ_{lb} can sometimes be no more than a reasonable step in the right direction, but as soon as we have a γ_{lb} that intersects only twice the maximum singular value curve the estimation for the next γ_{lb} becomes very accurate. As mentioned before, quadratic convergence can be proved; see [2].

Remark 3.4. When implementing the algorithm, care must be taken of the accuracy of the eigenvalue computation (Step 2). Inaccuracy could cause the algorithm to fail.

4. New lower bound starting value

In the literature on H_∞-norm computation [1,2,3,10], two possible lower bounds are suggested:

$$\gamma_{lb} = \sigma_{max}(D), \tag{4.1}$$

$$\gamma_{lb} = \max\{\sigma_{H,max}(G(s)), \sigma_{max}(D)\} \tag{4.2}$$

where $\sigma_{H,max}$ = maximum Hankel singular value.

Bound (4.2) is derived by Enns and Glover [6,8].

The computation of (4.1) is very 'cheap' in computational effort, but too often gives a starting value not very close to $\| G \|_\infty$. Value (4.2) gives a better estimate, but the computation of the Hankel singular values is rather 'expensive', especially when the order n of the system is high, because computation of the Hankel singular values requires the solution of two Lyapunov equations with dimension n.

In the two-step algorithm, σ_{max} of $G(j\omega)$ at any frequency could be used as a starting value. Many systems reach their H_∞-norm at frequency $\omega = 0$, so a sensible choice for such a frequency would be $\omega = 0$. We propose to compute $\sigma_{max}(G(j\omega))$ at three frequencies: $\omega = 0$, $\omega = \infty$ ($\equiv \sigma_{max}(D)$) and a third frequency ω_p, depending on the poles of the transfer function matrix $G(s)$:

$$\gamma_{lb} = \max\{\sigma_{max}(G(0)), \sigma_{max}(G(j\omega_p)), \sigma_{max}(D)\} \tag{4.3}$$

where

$$\omega_p = |\lambda_i|, \quad \lambda_i \text{ a pole of } G(s), \tag{4.4}$$

with λ_i selected to

$$\text{maximize} \quad \left| \frac{\mathrm{Im}(\lambda_i)}{\mathrm{Re}(\lambda_i)} \frac{1}{|\lambda_i|} \right|$$

if $G(s)$ has poles with $\mathrm{Im}(\lambda_i) \neq 0$, or to

$$\text{minimize} \quad |\lambda_i|$$

if $G(s)$ has only real poles.

Remark 4.1. If there is a badly damped, not too fast pole, ω_p will be close to the frequency where this pole causes a peak in the singular value plot, and this might be close to the H_∞-norm.

Remark 4.2. The expression (4.4) for frequency ω_p returns a frequency within the time scale of the system. For systems with very small singular values at $\omega = 0$ and $\omega = \infty$, the maximum singular value at ω_p will be a much closer starting value than $\sigma_{max}(0)$ or $\sigma_{max}(\omega_p)$.

Remark 4.3. The poles λ_i are available (they have to be computed to check the stability of the system), so their computation takes no extra time.

Our experience is that lower bound computation (4.3) is indeed very effective. For many systems (4.3) gives a much closer bound than (4.1) and (4.2) in a very short time, at least 10 times faster then (4.2). Using starting value (4.3) often makes the computation of the H_∞-norm almost twice as fast.

5. An example

As an example we take a rather large model of a controlled wind energy conversion system, used for a robustness analysis in [12]. For the state-space matrices we refer to [12]. Some main characteristics of the model are given below.

Wind energy conversion system

- 21th order.
- input–output dimension 10×10.
- H_∞-norm: 8760.
- H_∞-norm achieved at frequency 19.3 rad/s.

5.1. Lower bound computation

In Table 1 the different lower bound values of (4.2) and (4.3) are given for this model, plus the time that their computation took (on a 12 MHz AT).

Both $\sigma_{max}(D)$ and $\sigma_{max}(0)$ give values much smaller than the H_∞-norm ($= 8760$). The maximum Hankel singular value $\sigma_{H,max}$ gives a much closer estimate, but its computation takes more than 3 minutes because of the high order of the model. The maximum singular value at ω_p, $\sigma_{max}(\omega_p)$ where ω_p is chosen according to (4.4), gives a lower bound within 3 seconds. For this example the $\sigma_{max}(\omega_p)$ value is not very close, but much better than $\sigma_{max}(D)$ and $\sigma_{max}(0)$.

Table 1
Lower bounds for the wind energy conversion model

	$\sigma_{max}(D)$	$\sigma_{H,max}$	$\sigma_{max}(0)$	$\sigma_{max}(j\omega_p)$
computing time [s]	0.05	205	1.2	3.0
value	5.2	5622	10.7	825

5.2. H_∞-norm computation

The H_∞-norm of the model is computed in three ways:

Table 2
H_∞-norm computation with relative error 10^{-4}

Algorithm	Number of iterations	Computing time [s]	Number of correct digits
Bisection	16	668	
Two-step, lower bound (4.3)	2	133	0,2,7
Two-step, lower bound (4.2)	2	354	0,2,7

(1) with the bisection algorithm as described in [2] (using (4.2) as lower bound starting value);

(2) with the two-step algorithm, computing a starting value with (4.3);

(3) with the two-step algorithm, computing a starting value with (4.2).

In Table 2 the number of iterations are given necessary to compute the H_∞-norm with maximum relative error 10^{-4}, plus the computing time for the complete algorithm. Also the number of correct digits in the lower bound after each iteration of the two-step algorithm is given.

The two-step algorithm with the new lower bound (4.3) is 5 times faster than the bisection algorithm. It is easily understood that this advantage in computing time increases significantly if we require a smaller relative error, due to the quadratic convergence properties of the two-step algorithm (usually in the last step(s) 5 correct digits are added in each iteration).

The two-step algorithm with lower bound (4.2) has a better starting value, $\sigma_{H,max}$ (see Table 1) but its computation takes more time than the complete algorithm with bound (4.3).

6. Conclusions

The two-step algorithm presented here is faster than existing algorithms. The theory underlying the algorithm guarantees fast convergence and reliability of the results.

The lower bound computation based on the poles of the system is for many systems very effective; it returns an easily computed rather close lower bound.

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Appendix A1

Lemma A.1. *Let $G(s)$ be given through $[A, B, C, D]$. Then for $\gamma > 0$ not equal to a singular value of D,*

$$\left[\gamma^2 I - G^{\sim}(s)G(s)\right]^{-1} = [\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}],$$

$$\tilde{A} = \begin{bmatrix} A - BR^{-1}D^{\mathrm{T}}C & -\gamma BR^{-1}B^{\mathrm{T}} \\ \gamma C^{\mathrm{T}}S^{-1}C & -A^{\mathrm{T}} + C^{\mathrm{T}}DR^{-1}B^{\mathrm{T}} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} BR^{-1} \\ -\gamma^{-1}C^{\mathrm{T}}DR^{-1} \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} R^{-1}D^{\mathrm{T}}C & \gamma R^{-1}B^{\mathrm{T}} \end{bmatrix}, \qquad \tilde{D} = -R^{-1},$$

where $R = (D^{\mathrm{T}}D - \gamma^2 I)$ and $S = (DD^{\mathrm{T}} - \gamma^2 I)$.

Proof. To prove this we use the following operations on transfer matrices [7]:

$$[A, B, C, D]^{\sim} = [-A^{\mathrm{T}}, -C^{\mathrm{T}}, B^{\mathrm{T}}, D^{\mathrm{T}}], \tag{i}$$

$$[A_1, B_1, C_1, D_1] \times [A_2, B_2, C_2, D_2] = \left[\begin{bmatrix} A_2 & 0 \\ B_1C_2 & A_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ B_1D_2 \end{bmatrix}, [D_1C_2 \quad C_1], [D_1D_2]\right], \tag{ii}$$

$$[A, B, C, D]^{-1} = [A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1}]. \tag{iii}$$

Using (i) and (ii) we obtain

$$G(s)^{\sim}G(s) = [-A^{\mathrm{T}}, -C^{\mathrm{T}}, B^{\mathrm{T}}, D^{\mathrm{T}}] \times [A, B, C, D]$$
$$= \left[\begin{bmatrix} A & 0 \\ -C^{\mathrm{T}}C & -A^{\mathrm{T}} \end{bmatrix}, \begin{bmatrix} B \\ -C^{\mathrm{T}}D \end{bmatrix}, [D^{\mathrm{T}}C \quad B^{\mathrm{T}}], D^{\mathrm{T}}D\right]$$

Using (iii) and $I + D(\gamma^2 I - D^{\mathrm{T}}D)^{-1}D^{\mathrm{T}} = \gamma^2(\gamma^2 I - DD^{\mathrm{T}})$ we obtain

$$\left[\gamma^2 I - G^{\sim}(s)G(s)\right]^{-1} = [\bar{A}, \bar{B}, \bar{C}, \bar{D}],$$

$$\bar{A} = \begin{bmatrix} A - BR^{-1}D^{\mathrm{T}}C & -BR^{-1}B^{\mathrm{T}} \\ \gamma^2 C^{\mathrm{T}}S^{-1}C & -A^{\mathrm{T}} + C^{\mathrm{T}}DR^{-1}B \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} BR^{-1} \\ -C^{\mathrm{T}}DR^{-1} \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} R^{-1}D^{\mathrm{T}}C & R^{-1}B^{\mathrm{T}} \end{bmatrix}, \qquad \bar{D} = -R^{-1},$$

where R and S are as stated above.

After a similarity transformation with

$$T = \begin{bmatrix} I & 0 \\ 0 & \gamma I \end{bmatrix}$$

we obtain the system $[\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$. □

Appendix A.2. Uncontrollable or unobservable modes

Lemma A.2. *$[\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ has an uncontrollable or unobservable mode belonging to eigenvalue λ of $\tilde{A}$, only if λ or $-\lambda$ is an eigenvalue of A.*

Proof. If $[\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ has an uncontrollable mode belonging to eigenvalue λ of $\tilde{A}$, there exists a nonzero vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$\begin{bmatrix} x_1^T & x_2^T \end{bmatrix} [\lambda I - \tilde{A} \quad \tilde{B}] = 0$$

$$\Leftrightarrow \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} \lambda I - A + BR^{-1}D^TC & \gamma BR^{-1}B^T & BR^{-1} \\ -\gamma C^TS^{-1}C & \lambda I + A^T - C^TDR^{-1}B^T & -\gamma^{-1}C^TDR^{-1} \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} x_1^T(\lambda I - A + BR^{-1}D^TC) - x_2^T\gamma C^TS^{-1}C = 0, & \text{(A.1)} \\ x_1^T\gamma BR^{-1}B^T + x_2^T(\lambda I + A^T - C^TDR^{-1}B^T) = 0, & \text{(A.2)} \\ x_1^TBR^{-1} - x_2^T\gamma^{-1}C^TDR^{-1} = 0. & \text{(A.3)} \end{cases}$$

Inserting (A.3) in (A.2) gives $x_2^T(\lambda I + A^T) = 0$, so either $-\lambda$ is an eigenvalue of A or $x_2 = 0$. If $x_2 = 0$, inserting (A.3) in (A.1) gives $x_1^T(\lambda I - A) = 0$, so λ is an eigenvalue of A.

In a similar way it can be shown that the same holds for unobservable modes. □

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