Control Systems Engineering (EYAG-1005): **Unit 05:** Frequency Response Techniques

Luis I. Reyes-Castro

Escuela Superior Politécnica del Litoral (ESPOL)
Guayaquil - Ecuador

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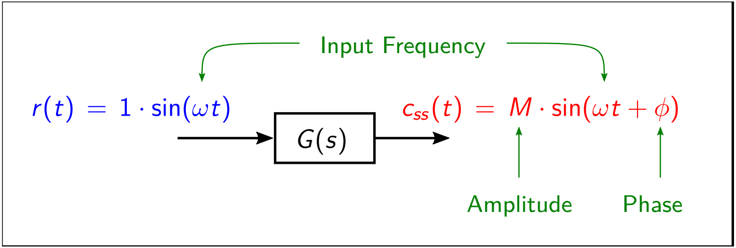
1 Bode Plots

- Introduction
- Elementary Systems
- Gain and Phase Margins
- Phase Margin and Damping Ratio
- Steady-state Errors

Bode Plots

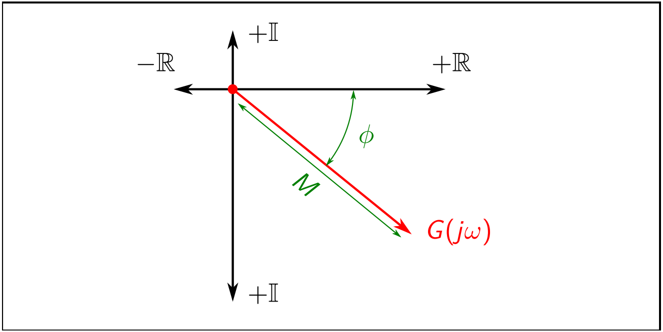
Consider a stable system with transfer function $G(s)$.

- Suppose the input $r(t)$ is a sinusoid with frequency ω and unit amplitude.
- Then the steady-state output $c_{ss}(t)$ must be a a sinusoid with the same frequency ω but with a particular amplitude M and phase ϕ which depend on the transfer function $G(s)$ and on the input frequency ω.



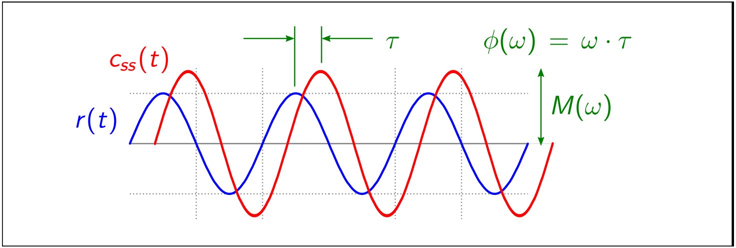
Bode Plots

- Given $G(s)$ and a frequency ω we can evaluate the amplitude and phase of the steady-state output by computing the phasor $G(j\omega)$. In particular:
  - The phasor's magnitude yields the amplitude $M(\omega)$.
  - The phasor's angle with $+\mathbb{R}$ yields the phase $\phi(\omega)$.



Bode Plots

- We can also estimate $M(\omega)$ and $\phi(\omega)$ experimentally:



Bode Plots

- Furthermore, notice that:
  - Amplitude $M(\omega)$ is always positive.
    - If $M(\omega) \in (0, 1)$ we get attenuation.
    - If $M(\omega) = 1$ we get amplitude matching.
    - If $M(\omega) > 1$ we get amplification.
  - Phase $\phi(\omega)$ may be negative, zero or postive.
    - If $\phi(\omega) < 0$ then the output lags the input.
    - If $\phi(\omega) = 0$ then the output matches the input.
    - If $\phi(\omega) > 0$ then the output leads the input.

Bode Plots

Bode Plots show amplitude $M(\omega)$ and phase $\phi(\omega)$ as functions of frequency ω. More precisely, they consist of the following two plots.

- Magnitude Plot
  - Amplitude $M(\omega)$ versus frequency ω.
  - The x-axis is frequency ω in decades, *i.e.*, $x = \log_{10}(\omega)$.
  - The y-axis is amplitude $M(\omega)$ in decibels, *i.e.*, $y = 20 \cdot \log_{10}(M(\omega))$.
- Phase Plot
  - Phase $\phi(\omega)$ versus frequency ω.
  - The x-axis is frequency ω in decades, *i.e.*, $x = \log_{10}(\omega)$.
  - The y-axis is phase in degrees, *i.e.*, $y = \phi(\omega)$.

Notice that when sketching Bode Plots by hand, we usually don't draw exactly the functions $M(\omega)$ and $\phi(\omega)$ but instead sketch *asymptotic approximations*.

Bode Plots: Elementary Systems

Bode plot for a simple amplifier: $G(s) = K$

- Magnitude plot is constant at $y = 20 \cdot \log_{10}(K)$ decibels.
- Phase plot is constant at $y = 0$ degree.

Bode Plots: Elementary Systems

Bode plot for an integrator: $G(s) = \dfrac{1}{s}$

- Phasor as a function of ω :

$$G(j\omega) \; = \; \frac{1}{j\omega} \; = \; -\frac{j}{\omega} \qquad \Longrightarrow \qquad M(\omega) \; = \; \frac{1}{\omega} \quad \& \quad \phi(\omega) \; = \; -90^\circ$$

- Magnitude plot is $y = -20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of -20 decibels per decade which hits zero decibels at $\omega = 1$ rad/s.
- Phase plot is constant at $y = -90$ degree.

Bode Plots: Elementary Systems

Bode plot for a differentiator: $G(s) = s$

- Phasor as a function of ω :

$$G(j\omega) \; = \; j\omega \qquad \Longrightarrow \qquad M(\omega) \; = \; \omega \quad \& \quad \phi(\omega) \; = \; +90^\circ$$

- Magnitude plot is $y = +20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of $+20$ decibels per decade which hits zero decibels at $\omega = 1$ rad/s.

- Phase plot is constant at $y = +90$ degree.

Bode Plots: Elementary Systems

Bode plot for a single-pole system: $G(s) = \dfrac{a}{s+a}$

- Phasor as a function of ω :

$$G(j\omega) \; = \; \frac{a}{j\omega + a} \cdot \frac{(-j\omega + a)}{(-j\omega + a)} \; = \; \frac{a^2 - j\omega a}{\omega^2 + a^2}$$

- Behavior at the break-away frequency, *i.e.*, $\omega = a$:

$$G(j\omega) \; = \; \frac{a^2 - ja^2}{2a^2} \; = \; \frac{1}{2} - j\left(\frac{1}{2}\right)$$

$$\implies \quad M(\omega) \; = \; \frac{1}{\sqrt{2}} \quad \& \quad \phi(\omega) = -45^\circ$$

  - In addition, phase $\phi(\omega)$ decreases at -45° per decade on the range of frequencies from $\omega = 0.1a$ to $\omega = 10a$.

Bode Plots: Elementary Systems

- Behavior at low frequencies, *i.e.*, $\omega << a$:

$$G(j\omega) \;=\; \frac{a^2 - j\omega a}{\omega^2 + a^2} \cdot \frac{(1/a^2)}{(1/a^2)} \;=\; \frac{1 - j(\omega/a)}{(\omega/a)^2 + 1} \;\longrightarrow\; 1$$

  - Magnitude plot is constant at $y = 0$ decibels.
  - Phase plot is constant at $y = 0$ degree.

- Behavior at high frequencies, *i.e.*, $\omega >> a$:

$$G(j\omega) \;=\; \frac{a^2 - j\omega a}{\omega^2 + a^2} \cdot \frac{(1/\omega^2)}{(1/\omega^2)} \;=\; \frac{(a/\omega)^2 - j(1/\omega)}{1 + (a/\omega)^2} \;\longrightarrow\; -\frac{j}{\omega}$$

  - Magnitude plot is $y = -20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of -20 decibels per decade.
  - Phase plot is constant at $y = -90$ degree.

Bode Plots: Elementary Systems

Bode plot for a single-zero system: $G(s) = \dfrac{s+a}{a}$

- Phasor as a function of ω :

$$G(j\omega) \;=\; \frac{j\omega + a}{a} \;=\; 1 + j\left(\frac{\omega}{a}\right)$$

- Behavior at the break-away frequency, *i.e.*, $\omega = a$:

$$G(j\omega) \;=\; 1 + j \quad \Longrightarrow \quad M(\omega) \;=\; \sqrt{2} \quad \& \quad \phi(\omega) = -45^\circ$$

  - In addition, phase $\phi(\omega)$ increases at $+45^\circ$ per decade on the range of frequencies from $\omega = 0.1a$ to $\omega = 10a$.

Bode Plots: Elementary Systems

- Behavior at low frequencies, *i.e.*, $\omega << a$:

$$G(j\omega) \;=\; 1 + j\left(\frac{\omega}{a}\right) \;\longrightarrow\; 1$$

  - Magnitude plot is constant at $y = 0$ decibels.
  - Phase plot is constant at $y = 0$ degree.

- Behavior at high frequencies, *i.e.*, $\omega >> a$:

$$G(j\omega) \;=\; 1 + j\left(\frac{\omega}{a}\right) \;\longrightarrow\; j\omega$$

  - Magnitude plot is $y = +20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of $+20$ decibels per decade.
  - Phase plot is constant at $y = +90$ degree.

Bode Plots: Elementary Systems

Bode plot for a second-order underdamped system: $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}$

- Phasor as a function of ω :

$$G(j\omega) \;=\; \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + 2\zeta\omega_n\omega\, j} \;=\; \frac{\omega_n^2\left[\,(\omega_n^2 - \omega^2) - 2\zeta\omega_n\omega\, j\,\right]}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}$$

- Behavior at the natural frequency, *i.e.*, $\omega = \omega_n$:

$$G(j\omega) \;=\; -\left(\frac{1}{2\zeta}\right) j$$

$$\implies \quad M(\omega) \;=\; \frac{1}{2\zeta} \quad \& \quad \phi(\omega) = -90^\circ$$

  - In addition, phase $\phi(\omega)$ decreases at -90° per decade on the range of frequencies from $\omega = 0.1\,\omega_n$ to $\omega = 10\,\omega_n$.

Bodeplots: Elementary Systems

- Behavior at low frequencies, *i.e.*, $\omega << \omega_n$:

$$G(j\omega) \;=\; \frac{\omega_n^2\left[\,(\omega_n^2-\omega^2)-2\zeta\omega_n\omega\, j\,\right]}{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}\cdot\frac{(1/\omega_n^4)}{(1/\omega_n^4)} \;\longrightarrow\; 1$$

  - Magnitude plot is constant at $y = 0$ decibels.
  - Phase plot is constant at $y = 0$ degree.

- Behavior at high frequencies, *i.e.*, $\omega >> \omega_n$:

$$G(j\omega) \;=\; \frac{\omega_n^2\left[\,(\omega_n^2-\omega^2)-2\zeta\omega_n\omega\, j\,\right]}{(\omega_n^2-\omega^2)^2+4\zeta^2\omega_n^2\omega^2}\cdot\frac{(1/\omega^4)}{(1/\omega^4)} \;\longrightarrow\; -\frac{1}{\omega^2}$$

  - Magnitude plot is $y = -40 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of -40 decibels per decade.
  - Phase plot is constant at $y = -180$ degree.

- Behavior at the peak frequency:
  - Peak frequency ω_p is the frequency for which the amplitude is maximum. It is very close to the natural frequency ω_n but it is generally not the same.
  - Peak frequency:

$$\omega_p \triangleq \underset{\omega > 0}{\arg\max} \; M(\omega) = \omega_n \sqrt{1 - 2\zeta^2}$$

  - Peak amplitude:

$$M_p \triangleq M(\omega_p) = \frac{1}{2\zeta \sqrt{1 - \zeta^2}}$$

Bode Plots: Elementary Systems

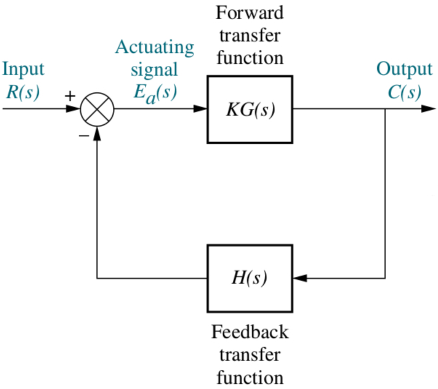
- Bandwidth:
  - In general bandwidth ω_{BW} is defined as the frequency for which the amplitude is 3 decibels below the low-frequency asymptote.
  - For a second-order underdamped system of the form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n\, s + \omega_n^2}$$

  it can be calculated as:

$$\omega_{BW} \; = \; \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Recall:


Forward
transfer
function
Actuating
signal
$E_a(s)$
Input
$R(s)$
+
−
$KG(s)$
Output
$C(s)$
$H(s)$
Feedback
transfer
function

Bode Plots: Gain and Phase Margins

Recall:

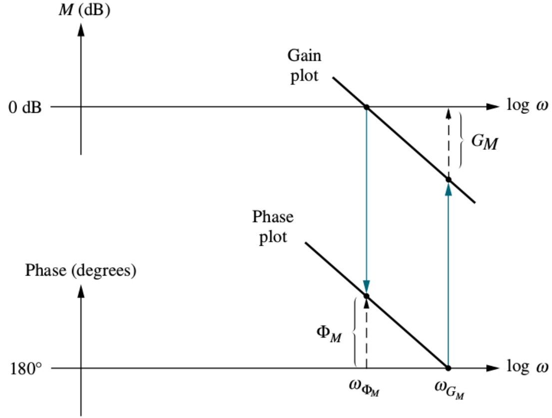
- Open loop transfer function: $K\,G(s)\,H(s)$
- Closed-loop transfer function:

$$T(s) \;=\; \frac{K\,G(s)}{1+K\,G(s)\,H(s)}$$

Margin definitions:

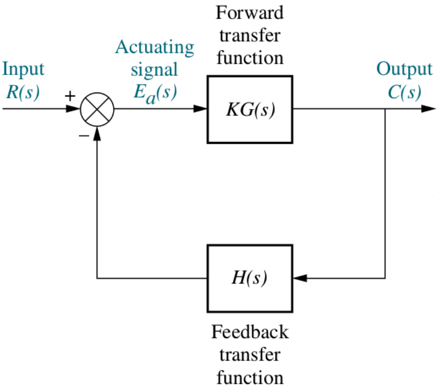
- Gain margin G_M is the factor by which we can increase K before the closed-loop transfer function $T(s)$ becomes unstable.
- Phase margin ϕ_M is closedly related to the delay T we can tolerate in the sensor $H(s)$ before $T(s)$ becomes unstable.
    - Unfortunately the gain margin does not correspond to an intuitive concept in the time domain.

Margin evaluations:



Bode Plots: Phase Margin and Damping Ratio

Recall:



Bode Plots: Phase Margin and Damping Ratio

Relationship between Phase Margin and Damping Ratio:

- The following two quantities are related:
  - The phase margin ϕ_M of the open-loop transfer function $K\,G(s)\,H(s)$.
  - The damping ratio ζ of the pair of dominant poles of the closed-loop transfer function $T(s)$.
    - This quantity is tightly related to the percent overshoot ($\%OS$).
- More precisely, the relationship is:

$$\phi_M \;=\; \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}\right)$$

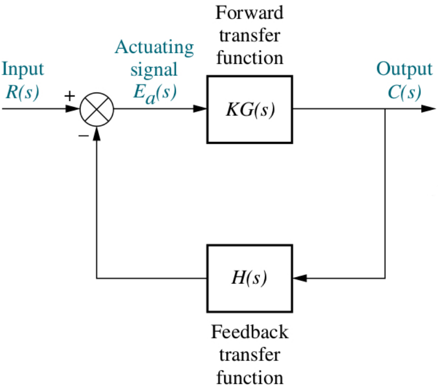
Bode Plots: Phase Margin and Damping Ratio

Using this Relationship for Design:

1. Define the desired percent overshoot ($\%OS$) of the closed-loop transfer function $T(s)$.
2. Calculate the phase margin ϕ_M that the open-loop transfer function $K\,G(s)\,H(s)$ needs to have.
3. Add lead or lag compensators to increase or decrease the phase margin of the open-loop transfer function $K\,G(s)\,H(s)$.
    - While designing, strive for a compensator that achieves the desired phase margin while minimally impacting the amplitude diagram.

Bode Plots: Steady-state Errors

Recall:



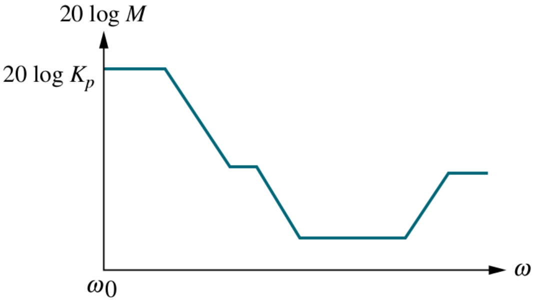
Bode Plots: Steady-state Errors

Relationship between Low-frequency Asymptote and Steady-state Error:

- The following two quantities are related:
  - The low-frequency asymptote value $M(\omega \to 0)$ of the open-loop transfer function $K\,G(s)\,H(s)$.
  - The step-input error constant K_p of the open-loop transfer function.
  - The steady-state error of the closed-loop transfer function $T(s)$ when the input is a unit step, denoted $e_{step}(\infty)$.
- More precisely, the relationship is, in decibels:

$$M(\omega \to 0) \; = \; K_p \; = \; \frac{1}{e_{step}(\infty)} - 1$$

Bode Plots: Steady-state Errors



Bode Plots: Steady-state Errors

Using this Relationship for Design:

1. Define the desired steady-state error of the closed-loop transfer function $T(s)$ when the input is a unit step.
2. Calculate the step-input error constant K_p that the open-loop transfer function $K\,G(s)\,H(s)$ needs to have.
3. Add lag or lead compensators to increase or decrease the low-frequency asymptote of the open-loop transfer function $K\,G(s)\,H(s)$.
   - While designing, strive for a compensator that achieves the desired low-frequency asymptote while minimally impacting the phase diagram.