

Control Systems Engineering (EYAG-1005):

Unit 03

Luis I. Reyes-Castro

Escuela Superior Politécnica del Litoral (ESPOL)
Guayaquil - Ecuador

Semester: 2017-T1

- 1 First-order Systems
- 2 Second-order Systems
- 3 Steady-state Errors

1 First-order Systems

2 Second-order Systems

3 Steady-state Errors

Definition:

We say that a system with transfer function $G(s)$ is of first-order if:

$$G(s) = \frac{K}{s + \sigma} \quad \text{for some } K, \sigma > 0$$

First-order Systems

Step response, i.e., $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \quad \Rightarrow \quad c(t) = \mathcal{L}^{-1}[R(s) G(s)]$$

$$\Rightarrow c(t) = \mathcal{L}^{-1}\left[\frac{K}{s(s + \sigma)}\right]$$

$$\Rightarrow c(t) = \mathcal{L}^{-1}\left[\frac{K/\sigma}{s} - \frac{K/\sigma}{s + \sigma}\right]$$

$$\Rightarrow c(t) = \left[\frac{K}{\sigma} - \frac{K}{\sigma} e^{-\sigma t}\right] u(t), \quad \forall t \in \mathbb{R}$$

$$\Rightarrow c(t) = \frac{K}{\sigma} - \frac{K}{\sigma} e^{-\sigma t}, \quad \forall t \geq 0$$

- 1 First-order Systems
- 2 Second-order Systems
- 3 Steady-state Errors

Second-order Systems

Recall the following results from elementary algebra:

- For any $\alpha, \beta, \gamma \in \mathbb{R}$ the roots of the second-order polynomial

$$D(s) = \alpha s^2 + \beta s + \gamma$$

are as follows:

$$s_1 = -\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \qquad s_2 = -\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

Therefore:

- If $\beta^2 > 4\alpha\gamma$ then the roots are repeated and different.
- If $\beta^2 = 4\alpha\gamma$ then the roots are repeated and identical.
- If $\beta^2 < 4\alpha\gamma$ then the roots are complex conjugates.

Second-order Systems

- For any $a, b \in \mathbb{R}$ the roots of the second-order polynomial

$$D(s) = s^2 + a s + b$$

are as follows:

$$s_1 = -\frac{a}{2} + \frac{\sqrt{a^2 - 4b}}{2} \qquad s_2 = -\frac{a}{2} - \frac{\sqrt{a^2 - 4b}}{2}$$

Therefore:

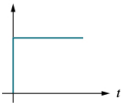


- If $a^2 > 4b$ then the roots are repeated and different.
- If $a^2 = 4b$ then the roots are repeated and identical.
- If $a^2 < 4b$ then the roots are complex conjugates.

- 1 First-order Systems
- 2 Second-order Systems
- 3 Steady-state Errors**

Steady-state Errors

Recall the reference signals:

TABLE 7.1 Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Recall the following property of the Laplace Transform:

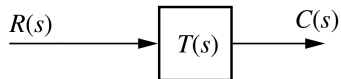
Final Value Theorem

For any function $f(t)$ defined for $t \geq 0$ with Laplace Transform $F(s)$ we have that:

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s)$$

Steady-state Errors

Steady-state errors for simple feedthrough systems:



$$E(s) = R(s) - C(s) = R(s) - T(s) R(s)$$

$$\Rightarrow E(s) = R(s) [1 - T(s)]$$

$$\Rightarrow e(\infty) = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} s R(s) [1 - T(s)]$$

Steady-state Errors

- Step input $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \quad \Longrightarrow \quad e_{step}(\infty) = \lim_{s \rightarrow 0} 1 - T(s)$$

- Ramp input $r(t) = t u(t)$:

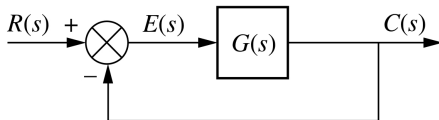
$$R(s) = \frac{1}{s^2} \quad \Longrightarrow \quad e_{ramp}(\infty) = \lim_{s \rightarrow 0} \frac{1 - T(s)}{s}$$

- Parabolic input $r(t) = (1/2) t^2 u(t)$:

$$R(s) = \frac{1}{s^3} \quad \Longrightarrow \quad e_{parabolic}(\infty) = \lim_{s \rightarrow 0} \frac{1 - T(s)}{s^2}$$

Steady-state Errors

Steady-state errors for unity-feedback systems:



$$E(s) = R(s) - C(s) = R(s) - G(s) E(s)$$

$$\Rightarrow E(s) [1 + G(s)] = R(s)$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)}$$

$$\Rightarrow e(\infty) = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} \frac{s R(s)}{1 + G(s)}$$

Steady-state Errors

- Step input $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \quad \Rightarrow \quad e_{step}(\infty) = \lim_{s \rightarrow 0} \frac{1}{1 + G(s)}$$

- Ramp input $r(t) = t u(t)$:

$$R(s) = \frac{1}{s^2} \quad \Rightarrow \quad e_{ramp}(\infty) = \lim_{s \rightarrow 0} \frac{1}{s G(s)}$$

- Parabolic input $r(t) = (1/2) t^2 u(t)$:

$$R(s) = \frac{1}{s^3} \quad \Rightarrow \quad e_{parabolic}(\infty) = \lim_{s \rightarrow 0} \frac{1}{s^2 G(s)}$$

Steady-state Errors

Furthermore, for unity-feedback systems:

- Position constant K_p :

$$K_p = \lim_{s \rightarrow 0} G(s) \quad \Rightarrow \quad e_{step}(\infty) = \frac{1}{1 + K_p}$$

- Velocity constant K_v :

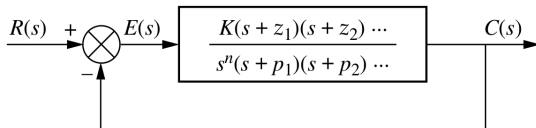
$$K_v = \lim_{s \rightarrow 0} s G(s) \quad \Rightarrow \quad e_{ramp}(\infty) = \frac{1}{K_v}$$

- Acceleration constant K_a :

$$K_a = \lim_{s \rightarrow 0} s^2 G(s) \quad \Rightarrow \quad e_{parabolic}(\infty) = \frac{1}{K_a}$$

Steady-state Errors

System Type: Number of integrators (poles at $s = 0$) on the forward path.

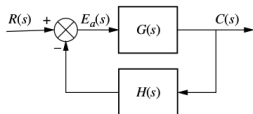


Steady-state error as a function of system type:

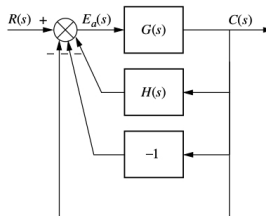
Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1 + K_p}$	$K_p = \text{Constant}$	$\frac{1}{1 + K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Steady-state Errors

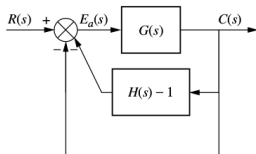
For non-unity non-unity-feedback systems, simply re-write the system in unity-feedback form:



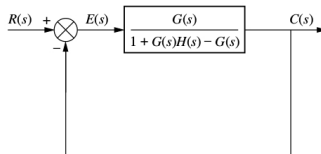
(b)



(c)



(d)



(e)