Control Systems Engineering (EYAG-1005): **Unit 03**

Luis I. Reyes-Castro

Escuela Superior Politécnica del Litoral (ESPOL)
Guayaquil - Ecuador

Semester: 2017-T1

First-order Systems

Definition:

We say that a system with transfer function $G(s)$ is of first-order if:

$$G(s) \; = \; \frac{K}{s+\sigma} \qquad \text{for some } K, \sigma > 0$$

First-order Systems

Step response, *i.e.*, $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \quad \implies \quad c(t) = \mathcal{L}^{-1}[\, R(s)\, G(s) \,]$$

$$\implies \quad c(t) = \mathcal{L}^{-1}\left[\frac{K}{s(s+\sigma)} \right]$$

$$\implies \quad c(t) = \mathcal{L}^{-1}\left[\frac{K/\sigma}{s} - \frac{K/\sigma}{s+\sigma} \right]$$

$$\implies \quad c(t) = \left[\frac{K}{\sigma} - \frac{K}{\sigma}\, e^{-\sigma t} \right] u(t)\,, \quad \forall t \in \mathbb{R}$$

$$\implies \quad c(t) = \frac{K}{\sigma} - \frac{K}{\sigma}\, e^{-\sigma t}, \quad \forall t \geq 0$$

Second-order Systems

Recall the following results from elementary algebra:

- For any $\alpha, \beta, \gamma \in \mathbb{R}$ the roots of the second-order polinomial

$$D(s) = \alpha\, s^2 + \beta\, s + \gamma$$

are as follows:

$$s_1 = -\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 - 4\,\alpha\,\gamma}}{2\,\alpha} \qquad\qquad s_2 = -\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 - 4\,\alpha\,\gamma}}{2\,\alpha}$$

Therefore:

  - If $\beta^2 > 4\,\alpha\,\gamma$ then the roots are repeated and different.
  - If $\beta^2 = 4\,\alpha\,\gamma$ then the roots are repeated and identical.
  - If $\beta^2 < 4\,\alpha\,\gamma$ then the roots are complex conjugates.

Second-order Systems

- For any $a, b \in \mathbb{R}$ the roots of the second-order polinomial

$$D(s) \; = \; s^2 + a\,s + b$$

are as follows:

$$s_1 \; = \; -\frac{a}{2} + \frac{\sqrt{a^2 - 4b}}{2} \qquad\qquad s_2 \; = \; -\frac{a}{2} - \frac{\sqrt{a^2 - 4b}}{2}$$

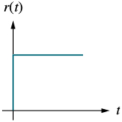
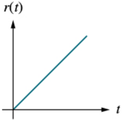
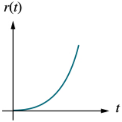
Therefore:

  - If $a^2 > 4b$ then the roots are repeated and different.
  - If $a^2 = 4b$ then the roots are repeated and identical.
  - If $a^2 < 4b$ then the roots are complex conjugates.

Steady-state Errors

Recall the reference signals:

TABLE 7.1 Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

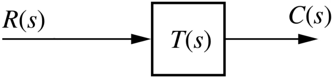
Recall the following property of the Laplace Transform:

Final Value Theorem

For any function $f(t)$ defined for $t \geq 0$ with Laplace Transform $F(s)$ we have that:

$$\lim_{t \to \infty} f(t) \quad = \quad \lim_{s \to 0} s\,F(s)$$

Steady-state errors for simple feedthrough systems:



$$E(s) \;=\; R(s) - C(s) \;=\; R(s) - T(s)\,R(s)$$

$$\implies E(s) \;=\; R(s)\,[\,1 - T(s)\,]$$

$$\implies e(\infty) \;=\; \lim_{t\to\infty} e(t) \;=\; \lim_{s\to 0} s\,E(t) \;=\; \lim_{s\to 0}\; s\,R(s)\,[\,1 - T(s)\,]$$

Steady-state Errors

- Step input $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \qquad \Longrightarrow \qquad e_{step}(\infty) = \lim_{s \to 0} 1 - T(s)$$

- Ramp input $r(t) = t\,u(t)$:

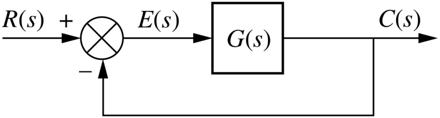
$$R(s) = \frac{1}{s^2} \qquad \Longrightarrow \qquad e_{ramp}(\infty) = \lim_{s \to 0} \frac{1 - T(s)}{s}$$

- Parabolic input $r(t) = (1/2)\,t^2\,u(t)$:

$$R(s) = \frac{1}{s^3} \qquad \Longrightarrow \qquad e_{parabolic}(\infty) = \lim_{s \to 0} \frac{1 - T(s)}{s^2}$$

Steady-state Errors

Steady-state errors for unity-feedback systems:



$$E(s) \;=\; R(s) - C(s) \;=\; R(s) - G(s)\,E(s)$$

$$\Longrightarrow \; E(s)\,[\,1 + G(s)\,] \;=\; R(s)$$

$$\Longrightarrow \; E(s) \;=\; \frac{R(s)}{1 + G(s)}$$

$$\Longrightarrow \; e(\infty) \;=\; \lim_{t\to\infty} e(t) \;=\; \lim_{s\to 0} s\,E(t) \;=\; \lim_{s\to 0} \frac{s\,R(s)}{1 + G(s)}$$

Steady-state Errors

- Step input $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \qquad \Longrightarrow \qquad e_{step}(\infty) = \lim_{s \to 0} \frac{1}{1 + G(s)}$$

- Ramp input $r(t) = t\, u(t)$:

$$R(s) = \frac{1}{s^2} \qquad \Longrightarrow \qquad e_{ramp}(\infty) = \lim_{s \to 0} \frac{1}{s\, G(s)}$$

- Parabolic input $r(t) = (1/2)\, t^2\, u(t)$:

$$R(s) = \frac{1}{s^3} \qquad \Longrightarrow \qquad e_{parabolic}(\infty) = \lim_{s \to 0} \frac{1}{s^2\, G(s)}$$

Steady-state Errors

Furthermore, for unity-feedback systems:

- Position constant K_p :

$$K_p \;=\; \lim_{s\to 0} \; G(s) \qquad \Longrightarrow \qquad e_{step}(\infty) \;=\; \frac{1}{1+K_p}$$

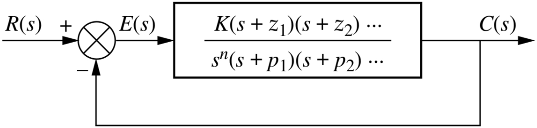
- Velocity constant K_v :

$$K_v \;=\; \lim_{s\to 0} \; s\,G(s) \qquad \Longrightarrow \qquad e_{ramp}(\infty) \;=\; \frac{1}{K_v}$$

- Acceleration constant K_a :

$$K_a \;=\; \lim_{s\to 0} \; s^2\,G(s) \qquad \Longrightarrow \qquad e_{parabolic}(\infty) \;=\; \frac{1}{K_a}$$

System Type: Number of integrators (poles at $s = 0$) on the forward path.

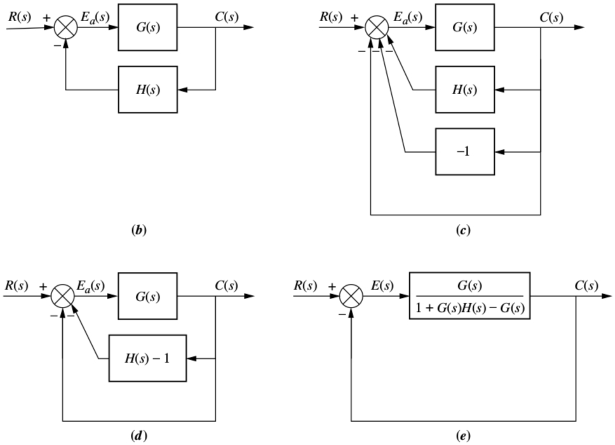


Steady-state error as a function of system type:

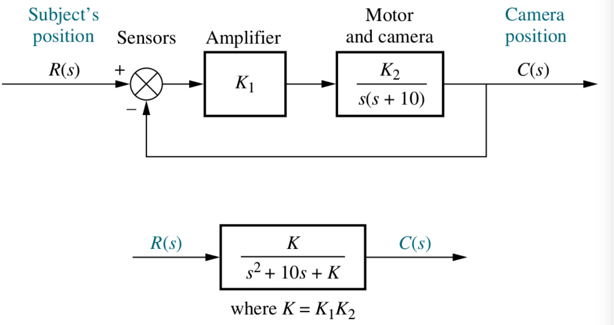
		Type 0		Type 1		Type 2	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\dfrac{1}{1+K_p}$	$K_p = \text{Constant}$	$\dfrac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\dfrac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\dfrac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\dfrac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\dfrac{1}{K_a}$

Steady-state Errors

For non-unity non-unity-feedback systems, simply re-write the system in unity-feedback form:



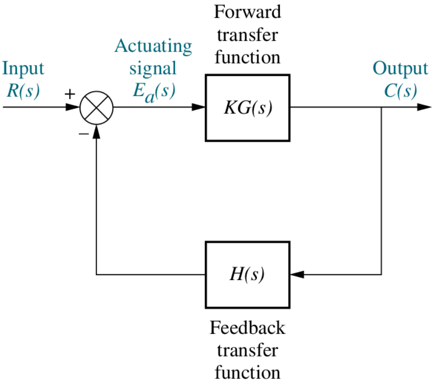
Motivating example:



Evan's Root Locus

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

The root locus concerns the design of closed-loop control systems with the following architecture:



Evan's Root Locus

Definitions:

- Loop gain: K
- Open-loop transfer function: $G(s)\,H(s)$

Objective:

- Sketch the roots of the closed-loop transfer function as the loop gain K ranges from near zero (*i.e.*, $K \to 0^+$) to infinity (*i.e.*, $K \to +\infty$).

Evan's Root Locus

Root locus sketching rules:

- The root locus is symmetric about the real axis.
- The number of branches, *i.e.*, pole trayectories, equals the number of poles of the open-loop transfer function.
- Each branch begins at an open-loop pole and ends either:
  - At an open-loop zero.
  - At infinity along an asymptote.
- Along the real line, root locus branches can be found to the left of any odd number of real open-loop poles or open-loop zeros.

Evan's Root Locus

- If the root locus has asymptotes, then the number of asymptotes is:

$$\text{(number of open-loop poles)} \; - \; \text{(number of open-loop zeros)}$$

- If the root locus has asymptotes, then the centroid of the asymptotes is located along the real axis at the point:

$$\sigma_a \; = \; \frac{\sum \text{(open-loop pole locations)} - \sum \text{(open-loop zero locations)}}{\text{number of asymptotes}}$$

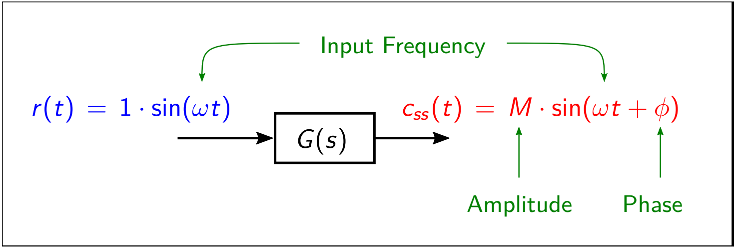
- If the root locus has asymptotes, then their angles in radians are:

$$\theta_a \; = \; \frac{(2k+1)\,\pi}{\text{number of asymptotes}} \qquad \text{for } k \; = \; 0, \, \pm 1, \, \pm 2, \, \ldots$$

Bode Plots

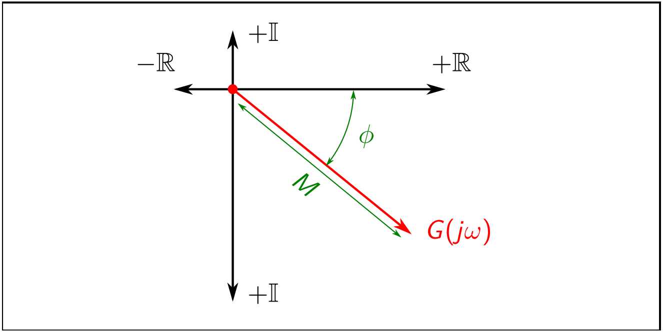
Consider a stable system with transfer function $G(s)$.

- Suppose the input $r(t)$ is a sinusoid with frequency ω and unit amplitude.
- Then the steady-state output $c_{ss}(t)$ must be a a sinusoid with the same frequency ω but with a particular amplitude M and phase ϕ which depend on the transfer function $G(s)$ and on the input frequency ω.

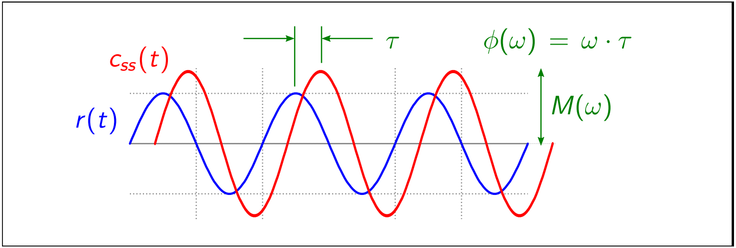


Bodes Plots

- Given $G(s)$ and a frequency ω we can evaluate the amplitude and phase of the steady-state output by computing the phasor $G(j\omega)$. In particular:
  - The phasor's magnitude yields the amplitude $M(\omega)$.
  - The phasor's angle with $+\mathbb{R}$ yields the phase $\phi(\omega)$.



- We can also estimate $M(\omega)$ and $\phi(\omega)$ experimentally:



Bode Plots

- Furthermore, notice that:
  - Amplitude M is always positive.
    - If $M \in (0, 1)$ we get attenuation.
    - If $M = 1$ we get amplitude matching.
    - If $M > 1$ we get amplification.
  - Phase may be negative, zero or postive.
    - If $\phi < 0$ then the output lags the input.
    - If $\phi = 0$ then the output matches the input.
    - If $\phi > 0$ then the output leads the input.

Bode Plots

Bode Plots are diagrams of $M(\omega)$ and $\phi(\omega)$. More precisely, they consist of the following two plots:

- **Magnitude Plot**: Amplitude $M(\omega)$ versus frequency ω.
  - The x-axis is frequency ω in decades, *i.e.*, $x = \log_{10}(\omega)$.
  - The y-axis is amplitude $M(\omega)$ in decibels, *i.e.*, $y = 20 \cdot \log(M(\omega))$.
- **Phase Plot**: Phase $\phi(\omega)$ versus frequency ω.
  - The x-axis is frequency ω in decades, *i.e.*, $x = \log_{10}(\omega)$.
  - The y-axis is phase in degrees, *i.e.*, $y = \phi(\omega)$.

Notice that when sketching Bode Plots by hand, we usually don't draw exactly the functions $M(\omega)$ and $\phi(\omega)$ but instead sketch *asymptotic approximations*.

Bode Plots

Bode plot for a simple amplifier: $G(s) = K$

- Magnitude plot is constant at $y = 20 \cdot \log_{10}(K)$ decibels.
- Phase plot is constant at $y = 0$ degree.

Bode Plots

Bode plot for an integrator: $G(s) = \dfrac{1}{s}$

- Phasor as a function of ω :

$$G(j\omega) \;=\; \frac{1}{j\omega} \;=\; -\frac{j}{\omega} \qquad \Longrightarrow \qquad M(\omega) \;=\; \frac{1}{\omega} \quad \& \quad \phi(\omega) \;=\; -90^\circ$$

- Magnitude plot is $y = -20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of -20 decibels per decade which hits zero decibels at $\omega = 1$ rad/s.
- Phase plot is constant at $y = -90$ degree.

Bode Plots

Bode plot for a differentiator: $G(s) = s$

- Phasor as a function of ω :

$$G(j\omega) \;=\; j\omega \qquad \Longrightarrow \qquad M(\omega) \;=\; \omega \quad \& \quad \phi(\omega) \;=\; +90^\circ$$

- Magnitude plot is $y = +20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of $+20$ decibels per decade which hits zero decibels at $\omega = 1$ rad/s.

- Phase plot is constant at $y = +90$ degree.