Control Systems Engineering (EYAG-1005): **Unit 05:** Frequency Response Techniques

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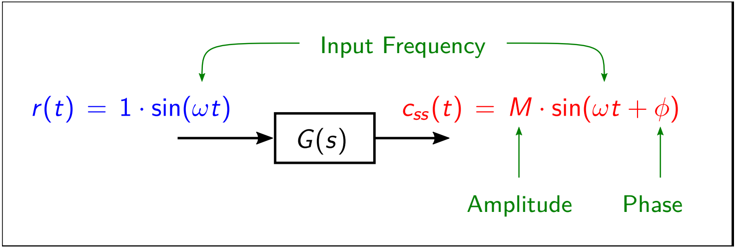
1 Bodе Plots

- Introduction
- Elementary Systems

Bode Plots

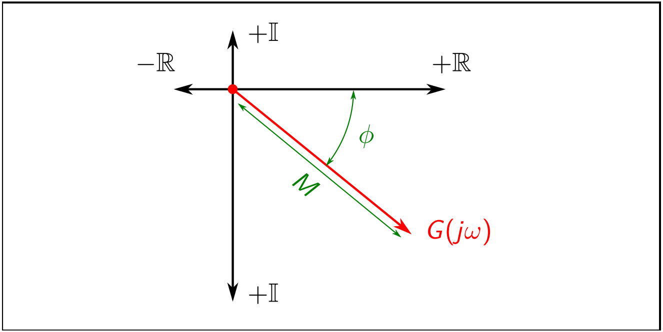
Consider a stable system with transfer function $G(s)$.

- Suppose the input $r(t)$ is a sinusoid with frequency ω and unit amplitude.
- Then the steady-state output $c_{ss}(t)$ must be a a sinusoid with the same frequency ω but with a particular amplitude M and phase ϕ which depend on the transfer function $G(s)$ and on the input frequency ω.



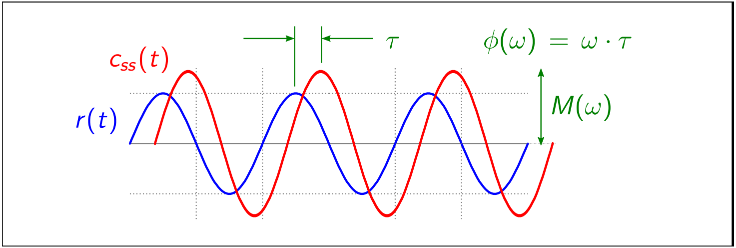
Bode Plots

- Given $G(s)$ and a frequency ω we can evaluate the amplitude and phase of the steady-state output by computing the phasor $G(j\omega)$. In particular:
  - The phasor's magnitude yields the amplitude $M(\omega)$.
  - The phasor's angle with $+\mathbb{R}$ yields the phase $\phi(\omega)$.



Bode Plots

- We can also estimate $M(\omega)$ and $\phi(\omega)$ experimentally:



Bode Plots

- Furthermore, notice that:
  - Amplitude $M(\omega)$ is always positive.
    - If $M(\omega) \in (0, 1)$ we get attenuation.
    - If $M(\omega) = 1$ we get amplitude matching.
    - If $M(\omega) > 1$ we get amplification.
  - Phase $\phi(\omega)$ may be negative, zero or postive.
    - If $\phi(\omega) < 0$ then the output lags the input.
    - If $\phi(\omega) = 0$ then the output matches the input.
    - If $\phi(\omega) > 0$ then the output leads the input.

Bode Plots

Bode Plots show amplitude $M(\omega)$ and phase $\phi(\omega)$ as functions of frequency ω. More precisely, they consist of the following two plots.

- Magnitude Plot
  - Amplitude $M(\omega)$ versus frequency ω.
  - The x-axis is frequency ω in decades, *i.e.*, $x = \log_{10}(\omega)$.
  - The y-axis is amplitude $M(\omega)$ in decibels, *i.e.*, $y = 20 \cdot \log_{10}(M(\omega))$.
- Phase Plot
  - Phase $\phi(\omega)$ versus frequency ω.
  - The x-axis is frequency ω in decades, *i.e.*, $x = \log_{10}(\omega)$.
  - The y-axis is phase in degrees, *i.e.*, $y = \phi(\omega)$.

Notice that when sketching Bode Plots by hand, we usually don't draw exactly the functions $M(\omega)$ and $\phi(\omega)$ but instead sketch *asymptotic approximations*.

1. Bode Plots
   - Introduction
   - Elementary Systems

Bode Plots: Elementary Systems

Bode plot for a simple amplifier: $G(s) = K$

- Magnitude plot is constant at $y = 20 \cdot \log_{10}(K)$ decibels.
- Phase plot is constant at $y = 0$ degree.

Bode Plots: Elementary Systems

Bode plot for an integrator: $G(s) = \dfrac{1}{s}$

- Phasor as a function of ω :

$$G(j\omega) \;=\; \frac{1}{j\omega} \;=\; -\frac{j}{\omega} \qquad \Longrightarrow \qquad M(\omega) \;=\; \frac{1}{\omega} \quad \& \quad \phi(\omega) \;=\; -90^\circ$$

- Magnitude plot is $y = -20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of -20 decibels per decade which hits zero decibels at $\omega = 1$ rad/s.
- Phase plot is constant at $y = -90$ degree.

Bode Plots: Elementary Systems

Bode plot for a differentiator: $G(s) = s$

- Phasor as a function of ω :

$$G(j\omega) \;=\; j\omega \qquad \Longrightarrow \qquad M(\omega) \;=\; \omega \quad \& \quad \phi(\omega) \;=\; +90^\circ$$

- Magnitude plot is $y = +20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of $+20$ decibels per decade which hits zero decibels at $\omega = 1$ rad/s.

- Phase plot is constant at $y = +90$ degree.

Bode Plots: Elementary Systems

Bode plot for a single-pole system: $G(s) = \dfrac{a}{s+a}$

- Phasor as a function of ω :

$$G(j\omega) \;=\; \frac{a}{j\omega + a} \cdot \frac{(-j\omega + a)}{(-j\omega + a)} \;=\; \frac{a^2 - j\omega a}{\omega^2 + a^2}$$

- Behavior at the break-away frequency, *i.e.*, $\omega = a$:

$$G(j\omega) \;=\; \frac{a^2 - ja^2}{2a^2} \;=\; \frac{1}{2} - j\left(\frac{1}{2}\right)$$

$$\implies \quad M(\omega) \;=\; \frac{1}{\sqrt{2}} \quad \& \quad \phi(\omega) = -45^\circ$$

  - In addition, phase $\phi(\omega)$ decreases at -45° per decade on the range of frequencies from $\omega = 0.1a$ to $\omega = 10a$.

Bodes Plots: Elementary Systems

- Behavior at low frequencies, *i.e.*, $\omega << a$:

$$G(j\omega) \;=\; \frac{a^2 - j\omega a}{\omega^2 + a^2} \cdot \frac{(1/a^2)}{(1/a^2)} \;=\; \frac{1 - j(\omega/a)}{(\omega/a)^2 + 1} \;\longrightarrow\; 1$$

  - Magnitude plot is constant at $y = 0$ decibels.
  - Phase plot is constant at $y = 0$ degree.

- Behavior at high frequencies, *i.e.*, $\omega >> a$:

$$G(j\omega) \;=\; \frac{a^2 - j\omega a}{\omega^2 + a^2} \cdot \frac{(1/\omega^2)}{(1/\omega^2)} \;=\; \frac{(a/\omega)^2 - j(1/\omega)}{1 + (a/\omega)^2} \;\longrightarrow\; -\frac{j}{\omega}$$

  - Magnitude plot is $y = -20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of -20 decibels per decade.
  - Phase plot is constant at $y = -90$ degree.

Bode Plots: Elementary Systems

Bode plot for a single-zero system: $G(s) = \dfrac{s+a}{a}$

- Phasor as a function of ω :

$$G(j\omega) \;=\; \frac{j\omega + a}{a} \;=\; 1 + j\left(\frac{\omega}{a}\right)$$

- Behavior at the break-away frequency, *i.e.*, $\omega = a$:

$$G(j\omega) \;=\; 1 + j \quad \implies \quad M(\omega) \;=\; \sqrt{2} \quad \& \quad \phi(\omega) = -45^\circ$$

  - In addition, phase $\phi(\omega)$ increases at $+45^\circ$ per decade on the range of frequencies from $\omega = 0.1a$ to $\omega = 10a$.

Bode Plots: Elementary Systems

- Behavior at low frequencies, *i.e.*, $\omega << a$:

$$G(j\omega) \;=\; 1 + j\left(\frac{\omega}{a}\right) \;\longrightarrow\; 1$$

  - Magnitude plot is constant at $y = 0$ decibels.
  - Phase plot is constant at $y = 0$ degree.

- Behavior at high frequencies, *i.e.*, $\omega >> a$:

$$G(j\omega) \;=\; 1 + j\left(\frac{\omega}{a}\right) \;\longrightarrow\; j\omega$$

  - Magnitude plot is $y = +20 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of $+20$ decibels per decade.
  - Phase plot is constant at $y = +90$ degree.

Bode Plots: Elementary Systems

Bode plot for a second-order underdamped system: $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

- Phasor as a function of ω :

$$G(j\omega) \;=\; \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + 2\zeta\omega_n\omega\, j} \;=\; \frac{\omega_n^2\,[\,(\omega_n^2 - \omega^2) - 2\zeta\omega_n\omega\, j\,]}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}$$

- Behavior at the break-away frequency, *i.e.*, $\omega = a$:

$$G(j\omega) \;=\; \cdots$$

$$\Longrightarrow \quad M(\omega) \;= ? \quad \& \quad \phi(\omega) = -90^\circ$$

  - In addition, phase $\phi(\omega)$ decreases at -90° per decade on the range of frequencies from $\omega = 0.1a$ to $\omega = 10a$.

Bode Plots: Elementary Systems

- Behavior at low frequencies, *i.e.*, $\omega << \omega_n$:

$$G(j\omega) \; = \; \cdots \; \longrightarrow \; 1$$

  - Magnitude plot is constant at $y = 0$ decibels.
  - Phase plot is constant at $y = 0$ degree.

- Behavior at high frequencies, *i.e.*, $\omega >> \omega_n$:

$$G(j\omega) \; = \; \cdots \; \longrightarrow \; -\frac{1}{\omega^2}$$

  - Magnitude plot is $y = -40 \cdot \log_{10}(\omega)$ decibels, *i.e.*, it is a line with slope of -40 decibels per decade.
  - Phase plot is constant at $y = -180$ degree.