Control Systems Engineering (EYAG-1005): **Unit 03**

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First-order Systems

Definition:

We say that a system with transfer function $G(s)$ is of first-order if:

$$G(s) = \frac{K}{s + \sigma} \qquad \text{for some } K, \sigma > 0$$

First-order Systems

Step response, *i.e.*, $r(t) = u(t)$:

$$
\begin{aligned}
R(s) = \frac{1}{s} \qquad \implies \quad c(t) &= \mathcal{L}^{-1}[\, R(s)\, G(s)\,] \\
\implies \quad c(t) &= \mathcal{L}^{-1}\left[\, \frac{K}{s(s+\sigma)} \,\right] \\
\implies \quad c(t) &= \mathcal{L}^{-1}\left[\, \frac{K/\sigma}{s} - \frac{K/\sigma}{s+\sigma} \,\right] \\
\implies \quad c(t) &= \left[\, \frac{K}{\sigma} - \frac{K}{\sigma}\, e^{-\sigma t} \,\right] u(t)\,, \quad \forall t \in \mathbb{R} \\
\implies \quad c(t) &= \frac{K}{\sigma} - \frac{K}{\sigma}\, e^{-\sigma t}, \quad \forall t \geq 0
\end{aligned}
$$

Second-order Systems

Recall the following results from elementary algebra:

- For any $\alpha, \beta, \gamma \in \mathbb{R}$ the roots of the second-order polinomial

$$D(s) = \alpha\, s^2 + \beta\, s + \gamma$$

are as follows:

$$s_1 = -\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 - 4\,\alpha\,\gamma}}{2\,\alpha} \qquad s_2 = -\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 - 4\,\alpha\,\gamma}}{2\,\alpha}$$

Therefore:

  - If $\beta^2 > 4\,\alpha\,\gamma$ then the roots are repeated and different.
  - If $\beta^2 = 4\,\alpha\,\gamma$ then the roots are repeated and identical.
  - If $\beta^2 < 4\,\alpha\,\gamma$ then the roots are complex conjugates.

Second-order Systems

- For any $a, b \in \mathbb{R}$ the roots of the second-order polinomial

$$D(s) = s^2 + a\,s + b$$

are as follows:

$$s_1 = -\frac{a}{2} + \frac{\sqrt{a^2 - 4b}}{2} \qquad s_2 = -\frac{a}{2} - \frac{\sqrt{a^2 - 4b}}{2}$$

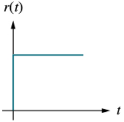
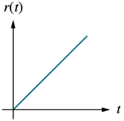
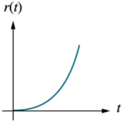
Therefore:

  - If $a^2 > 4b$ then the roots are repeated and different.
  - If $a^2 = 4b$ then the roots are repeated and identical.
  - If $a^2 < 4b$ then the roots are complex conjugates.

Steady-state Errors

Recall the reference signals:

TABLE 7.1 Test waveforms for evaluating steady-state errors of position control systems

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Steady-state Errors

Recall the following property of the Laplace Transform:

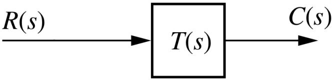
Final Value Theorem

For any function $f(t)$ defined for $t \geq 0$ with Laplace Transform $F(s)$ we have that:

$$\lim_{t \to \infty} f(t) \quad = \quad \lim_{s \to 0} s\, F(s)$$

Steady-state Errors

Steady-state errors for simple feedthrough systems:



$$E(s) \;=\; R(s) - C(s) \;=\; R(s) - T(s)\,R(s)$$

$$\implies E(s) \;=\; R(s)\,[\,1 - T(s)\,]$$

$$\implies e(\infty) \;=\; \lim_{t\to\infty} e(t) \;=\; \lim_{s\to 0} s\,E(t) \;=\; \lim_{s\to 0}\; s\,R(s)\,[\,1 - T(s)\,]$$

- Step input $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \qquad \Longrightarrow \qquad e_{step}(\infty) = \lim_{s \to 0} 1 - T(s)$$

- Ramp input $r(t) = t\,u(t)$:

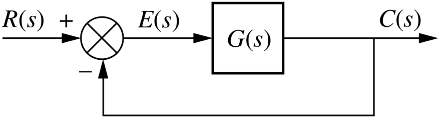
$$R(s) = \frac{1}{s^2} \qquad \Longrightarrow \qquad e_{ramp}(\infty) = \lim_{s \to 0} \frac{1 - T(s)}{s}$$

- Parabolic input $r(t) = (1/2)\,t^2\,u(t)$:

$$R(s) = \frac{1}{s^3} \qquad \Longrightarrow \qquad e_{parabolic}(\infty) = \lim_{s \to 0} \frac{1 - T(s)}{s^2}$$

Steady-state Errors

Steady-state errors for unity-feedback systems:



$$E(s) \;=\; R(s) - C(s) \;=\; R(s) - G(s)\,E(s)$$

$$\implies E(s)\,[\,1 + G(s)\,] \;=\; R(s)$$

$$\implies E(s) \;=\; \frac{R(s)}{1 + G(s)}$$

$$\implies e(\infty) \;=\; \lim_{t \to \infty} e(t) \;=\; \lim_{s \to 0} s\,E(t) \;=\; \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)}$$

Steady-state Errors

- Step input $r(t) = u(t)$:

$$R(s) = \frac{1}{s} \qquad \Longrightarrow \qquad e_{step}(\infty) = \lim_{s \to 0} \frac{1}{1 + G(s)}$$

- Ramp input $r(t) = t\, u(t)$:

$$R(s) = \frac{1}{s^2} \qquad \Longrightarrow \qquad e_{ramp}(\infty) = \lim_{s \to 0} \frac{1}{s\, G(s)}$$

- Parabolic input $r(t) = (1/2)\, t^2\, u(t)$:

$$R(s) = \frac{1}{s^3} \qquad \Longrightarrow \qquad e_{parabolic}(\infty) = \lim_{s \to 0} \frac{1}{s^2\, G(s)}$$

Steady-state Errors

Furthermore, for unity-feedback systems:

- Position constant K_p :

$$K_p = \lim_{s \to 0} G(s) \qquad \Longrightarrow \qquad e_{step}(\infty) = \frac{1}{1 + K_p}$$

- Velocity constant K_v :

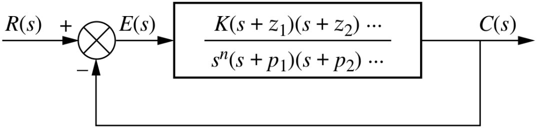
$$K_v = \lim_{s \to 0} s\, G(s) \qquad \Longrightarrow \qquad e_{ramp}(\infty) = \frac{1}{K_v}$$

- Acceleration constant K_a :

$$K_a = \lim_{s \to 0} s^2\, G(s) \qquad \Longrightarrow \qquad e_{parabolic}(\infty) = \frac{1}{K_a}$$

Steady-state Errors

System Type: Number of integrators (poles at $s = 0$) on the forward path.

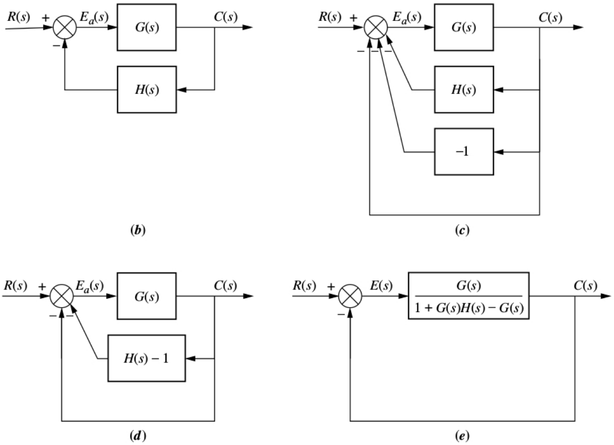


Steady-state error as a function of system type:

		Type 0		Type 1		Type 2	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

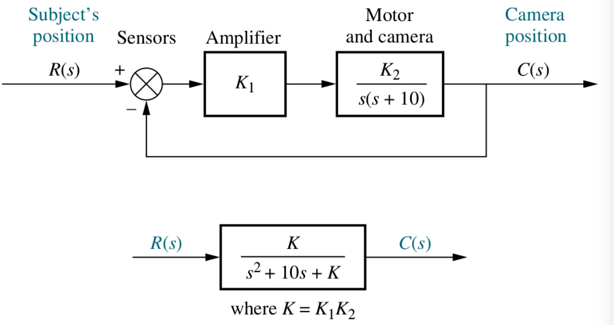
Steady-state Errors

For non-unity non-unity-feedback systems, simply re-write the system in unity-feedback form:



Evan's Root Locus

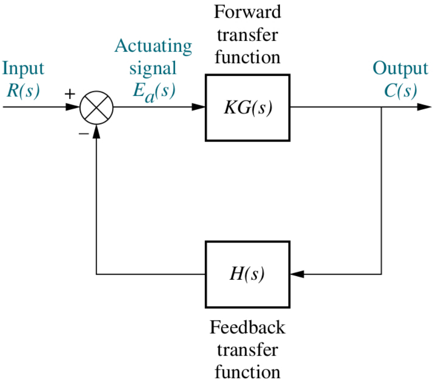
Motivating example:



K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$

Evan's Root Locus

The root locus concerns the design of closed-loop control systems with the following architecture:



Evan's Root Locus

Definitions:

- Loop gain: K
- Open-loop transfer function: $G(s)\,H(s)$

Objective:

- Sketch the roots of the closed-loop transfer function as the loop gain K ranges from near zero (*i.e.*, $K \to 0^+$) to infinity (*i.e.*, $K \to +\infty$).

Evan's Root Locus

Root locus sketching rules:

- The root locus is symmetric about the real axis.
- The number of branches, *i.e.*, pole trayectories, equals the number of poles of the open-loop transfer function.
- Each branch begins at an open-loop pole and ends either:
  - At an open-loop zero.
  - At infinity along an asymptote.
- Along the real line, root locus branches can be found to the left of any odd number of real open-loop poles or open-loop zeros.

Evan's Root Locus

- If the root locus has asymptotes, then the number of asymptotes is:

$$(\text{ number of open-loop poles }) - (\text{ number of open-loop zeros })$$

- If the root locus has asymptotes, then the centroid of the asymptotes is located along the real axis at the point:

$$\sigma_a = \frac{\sum(\text{open-loop pole locations}) - \sum(\text{open-loop zero locations})}{\text{number of asymptotes}}$$

- If the root locus has asymptotes, then their angles in radians are:

$$\theta_a = \frac{(2k+1)\,\pi}{\text{number of asymptotes}} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$