For brevity we denote the category of topological spaces with continuous functions by . We begin with path connectedness and homotopies ([**?**, Section 2.1]).

Let $X \in$, $x, y \in X$ and $I = [0,1] \subseteq R$. A *path* in X from x to y is a continuous function $u\colon I \to X$ such that $u(0) = x$ and $u(1) = y$. In this case we say that x is *connectable by paths* with y

Being connected by paths is an equivalence relation: the constant function is a path, the inverse u^- of a path u is the precomposition with the function $t \mapsto 1 - t$ and reparametrizing we can compose paths. The equivalence classes of this relation in X are called *path components* of X.

The set of all path components of X will be denoted by $\pi_0(X)$. We say X is 0-*connected* or *path connected* if $\#\pi_0(X) = 1$.

Let X and Y be topological spaces. Two continuous functions $f, g\colon X \to Y$ are *homotopic* ($f \simeq g$) if there is a *homotopy* H from f to g, that is a continuous function $H\colon X \times I \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. We usually write $H_t = H|_{X\times\{t\}}$.

In the same way we proved that being connectable by paths is an equivalence relation, being homotopic is an equivalence relation.

A *homotopy inverse* of a continuous map $f : X \to Y$ is a continuous function $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. In such case f is a *homotopy equivalence* and X and Y are called *homotopy equivalent* or *have the same homotopy type*. When X is homotopy equivalent to a point is called *contractible*.

Let $X, Y \in$, $x \in X$ and $y \in Y$, a continuous function $f\colon X \to Y$ is called a pointed function if $f(x) = y$. The set of pointed functions from (X, x) to (Y, y) is denoted by $^0((X, x), (Y, y))$.

Let $f, g \in^0 ((X, x), (Y, y))$. A *pointed homotopy* from f to g is a homotopy H from f to g such that $H_t(x) = y$.

The above concepts and properties are generalized straightforward for pointed homotopies. So, $\simeq$ is an equivalence relation on $^0((X, x), (Y, y))$.

In what follows, the material has been taken from [**?**, Chapter 11]. When $X = S^n$ in (X, x) we assume that $x = (1, 0, \ldots, 0)$.

Let $n > 0$. The *n-th homotopy group* of (X, x) with $x \in X$ is $\pi_n(X, x) =^0 ((S^n, *), (X, x))/\simeq$.

We will not prove $\pi_n(X, x)$ is in fact a group, a complete proof can be found in [**?**, Section 6.1] and [**?**, Theorem 11.4 and Corollary 11.17].

Of course, if x and y are not in the same path component of X then it could be possible that $\pi_n(X, x)\pi_n(X, y)$. However, when x and y lie in the same path component $\pi_n(X, x) \cong \pi_n(X, y)$ ([**?**, Theorem 11.24]). Thus we write $\pi_n(X)$ instead of $\pi_n(X, x)$.

In this work the following is essential, the proof can be found in [**?**, Corollary 11.26]. If X and Y are homotopy equivalent, then $\pi_n(X) \cong \pi_n(Y)$

A topological space X is n-*connected* if $\#\pi_0(X) = 1$ and $\pi_i(X)$ is a trivial group for $i \leq n$. We say that $X \in$ is (-1)-*connected* if it is non-empty.

Theorem **??** implies the following.

If X is contractible, then it is n-connected for every $n \in N$.

Since a contractible space is homotopy equivalent to a point, from Theorem **??**, it is enough to note that $^0((S^n, *), (x, x))$ is a singleton.

The cone $X * a$ is contractible. Therefore an n-ball is contractible.

The homotopy between the constant map $X * a \to a$ and the inclusion $a \to X * a$ is given by the line segments joining a with $x \in X$.

We will prove that several spaces are n-connected for some n but our strategy needs to show explicitly that they are 1-connected. So, we recall a result that simplifies the calculation of $\pi_1(X)$. A simple proof of the next result can be found in [**?**, Theorem 2.6.2] [Seifert-van Kampen] Let $X \in$ and assume that $X_0^\circ \cup X_1^\circ = X$. If X_v and $X_0 \cap X_1$ are 0-connected, then

$$\pi_1(X_0 \cap X_1)[r, "(i_1)_*"][d, "(i_0)_*"']\pi_1(X_1)[d, "(j_1)_*"]$$
$$\pi_1(X_0)[r, "(j_0)_*"]\pi_1(X)$$

is a pushout in the category of groups. The pushout in the category of groups is the free product with amalgamation [**?**, Chapter 11]. However, we do not need its construction but the following two properties under the hypothesis of Theorem **??**:

1. If $\pi_1(X_1)$ and $\pi_1(X_0)$ are trivial then $\pi_1(X)$ is trivial trivial.
2. If $\pi_1(X_0 \cap X_1)$ is trivial, then $\pi_1(X)$ is the free product of $\pi_1(X_1)$ and $\pi_1(X_0)$.

For $n > 1$, the following equation holds: $\pi_1(S^n) = 0$

The sphere S^n is the union of two n-balls whose intersection is S^{n-1}. Since

We present the results from algebraic topology we need. We follow [?, ?, ?]. The most of the proofs are omitted inasmuch as they are classical results. With respect to homotopy, we offer calculations of the homotopy type of those spaces we use in the main text. Also, we recall the result about of cofibrations needed in Theorem **??**. We present the construction of homology groups of simplicial complexes. Of course we recall Mayer-Vietoris sequences and the relation between homology and homotopy due to Hurewicz. Finally, we offer a proof of the simplicial approximation theorem.

Homotopy groups