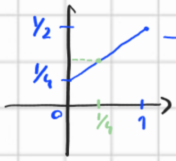
Exercícios

$\boxed{1}$

$P[X < -\frac{1}{2}] = 0$

$P[X < 0] = F_X(0^-) = 0$

$P[X \leq 0] = F_X(0) = 1/4$



$\longrightarrow y = mx + 1/4$

$(1, 1/2) \rightarrow 1/2 = m + \frac{1}{4} \Rightarrow m = \frac{1}{4}$

$\hookrightarrow y = \frac{1}{4}x + \frac{1}{4}$

Logo, para $x = \frac{1}{4} \Rightarrow y = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$ //

$x = \frac{1}{2} \Rightarrow y = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

$P[1/4 \leq X < 1] = F_X(1^-) - F_X(\frac{1}{4})$
$= \frac{1}{2} - \frac{5}{16} = \frac{3}{16}$

$P[1/4 \leq X \leq 1] = F_X(1) - F_X(\frac{1}{4})$
$= 1 - \frac{5}{16}$
$= \frac{11}{16}$

$P[X > \frac{1}{2}] = 1 - P[X \leq \frac{1}{2}]$
$= 1 - F_X(\frac{1}{2})$
$= 1 - \frac{3}{8}$
$= \frac{5}{8}$

$P[X > 5] = 1 - P[X \leq 5]$
$= 1 - 1 = 0$

$\boxed{2}$

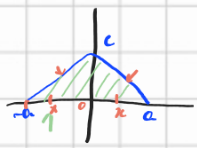
a) Sabemos que $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Leftrightarrow \frac{\cancel{2} a \times c}{\cancel{2}} = 1 \Leftrightarrow a \times c = 1 \Leftrightarrow c = \frac{1}{a}$

b)

$y = -\frac{c}{a}x + c$;

$\Leftrightarrow y = -\frac{x}{a^2} + \frac{1}{a}$; $y = \frac{x}{a^2} + \frac{1}{a}$

$$F_X(x) = \begin{cases} 0, & \text{se } x < -a \\ 1, & \text{se } x \geq a \\ \int_{-a}^{x} -\frac{t}{a^2} + \frac{1}{a}\,dt, & -a \leq x < 0 \\ \frac{a \times c}{2} + \int_{0}^{x} -\frac{t}{a^2} + \frac{1}{a}\,dt, & 0 \leq x < a \end{cases}$$



$$\int_{-a}^{x} \frac{t}{a^2} + \frac{1}{a}\,dt = \frac{1}{a^2}\frac{(x+a)^2}{2} + \frac{1}{a}(x+a) = \frac{(x+a)^2}{2a^2} + \frac{x}{a} + 1$$

$p_X(x)$ Σ

$F_X(x)$ $\int$

$f_X(x)$

$$\underbrace{\frac{a \times c}{2}}_{=\frac{1}{2}} + -\frac{1}{a^2}\frac{(x-0)^2}{2} + \frac{1}{a}(x-0) = \frac{1}{2} - \frac{x^2}{2a^2} + \frac{x}{a}$$

$$F_X(x) = \begin{cases} 0, & \text{se } x < -a \\ 1 & \text{se } x \geq a \\ \frac{(x+a)^2}{2a^2} + \frac{x}{a} + 1, & \text{se } -a \leq x < 0 \\ \frac{1}{2} - \frac{x^2}{2a^2} + \frac{x}{2}, & \text{se } 0 \leq x < a \end{cases}$$

c)

Assumimos $b>0$

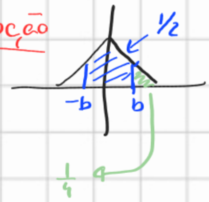
$$P[\,|X|<b\,]=\tfrac{1}{2} \Leftrightarrow P[-b<X<b]=\tfrac{1}{2}$$

$$\Leftrightarrow F_X(b)-F_X(-b)=\tfrac{1}{2}$$

$$\Leftrightarrow \frac{1}{2}-\frac{b^2}{2a^2}+\frac{b}{2}-\frac{(b+a)^2}{2a^2}+\frac{b}{a}+1=\tfrac{1}{2}$$

$\Leftrightarrow (\dots)$

Outra opção



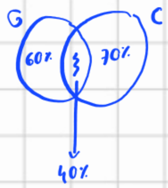
$$\Rightarrow P(b<x\leq a)=\frac{1}{4}$$

$$\Leftrightarrow F_x(a)-F_x(b)=\frac{1}{4}$$

$$\Leftrightarrow 1-\frac{1}{2}-\frac{b^2}{2a^2}+\frac{x}{2}=\frac{1}{4}$$

$=(\dots)$

$\boxed{3}$



Lei de Morgan

$$P(\bar{G}\cap\bar{C})=P(\overline{G\cup C})=1-P(G\cup C)$$

$$=1-P(G)+P(C)-P(G\cap C)$$

$$=1-0.6+0.7-0.4$$

$$=0.1$$

$\boxed{4}$

$P_A=0.001$

$P_B=0.005$

$P_C=0.01$

a)

D - "chip tem defeito"

ser um chip de A

$$P(A)=P(B)=P(C)=\frac{1}{3}$$

$$P(D)=\frac{1}{3}(0.001+0.005+0.01)=\frac{0.016}{3}=\frac{\frac{16}{1000}}{3}=\frac{16}{3000}=\frac{8}{1500}=\frac{4}{750}=\frac{2}{375}$$

Fabricante A:

$$P(A\mid D)=\frac{P(A\cap D)}{P(D)}=\frac{P(D\mid A)\,P(A)}{P(D)}=\frac{\frac{1}{3}\times\frac{1}{1000}}{\frac{16}{3000}}=\frac{\frac{1}{3000}}{\frac{16}{3000}}=\frac{1}{16}$$

B

$$P(B\mid D)=\frac{P(D\mid B)\,P(B)}{P(D)}=\frac{\frac{5}{1000}\times\frac{1}{3}}{\frac{16}{3000}}=\frac{5}{16}$$

C

$$P(C\mid D)=\frac{P(D\mid C)\,P(C)}{P(D)}=\frac{\frac{10}{1000}\times\frac{1}{3}}{\frac{16}{3000}}=\frac{10}{16}=\frac{5}{8}$$

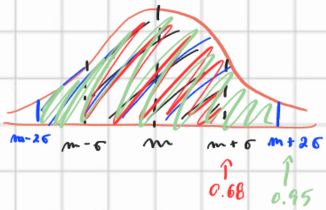
b)

$P(C)=\frac{1}{2}$, assumimos que $P(A)=\frac{1}{4}=P(B)$

(...) Repetimos os cálculos

5

$\Rightarrow P_x(x)$



$N(m, \sigma^2)$

$$P[X < m-5\sigma \vee X > m+5\sigma] = P[X < m-5\sigma] + P[X > m+5\sigma]$$
$$= 1 - P[m-5\sigma \leq X \leq m+5\sigma]$$

$F_x(x) = \Phi\left(\frac{x-m}{\sigma}\right) = \Phi(U_x)$

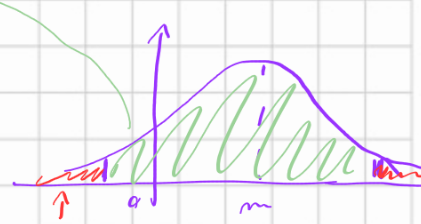
$$= 1 - P\left[\frac{(m-5\sigma)-m}{\sigma} \leq U_x \leq \frac{(m+5\sigma)-m}{\sigma}\right]$$

$Q(x) = 1 - \Phi(x)$

$$= 1 - P[-5 \leq U_x \leq 5]$$
$$= 1 - (Q(-5) - Q(5))$$
$$= 1 - Q(-5) + Q(5)$$

$Q(-x) = 1 - Q(x)$

$$= 1 - (1 - Q(5)) + Q(5)$$
$$= 2\underbrace{Q(5)}_{\approx 0}$$
$$\simeq 0$$



6

$$m = 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.125 \times 3$$
$$= 1 + 0.375 + 0.375$$
$$= 1 + 0.75$$
$$= 1.75$$

7

C - "resultado correto"   D - "pessoa tem doença"

$P(P \mid D) = 0.95$; $P(\bar{P} \mid D) = 0.05$

$P(\bar{P} \mid \bar{D}) = 0.95$; $P(P \mid \bar{D}) = 0.05$

$P(D) = 0.001$

Teoricamente: $P(P) = P(P \cap D) + P(P \cap \bar{D})$
$= P(P \mid D)P(D) + P(P \mid \bar{D})P(\bar{D})$
$= 0.95 \times 0.001 + 0.05 \times 0.999$

$$P(P) = 0.001 \times 0.95 + 0.999 \times 0.05 = 0.00095 + 0.04995 = 0.0509$$

$$P(D \mid P) = \frac{P(P \mid D)P(D)}{P(B)} = \frac{0.95 \times 0.001}{0.0509} =$$

8

$p(N1) = 0.5$; $P(V \mid N1) = 0.3$

$p(N2) = 0.25$; $P(V \mid N2) = 0.4$

$P_3(N3) = 0.25$; $P(V \mid N3) = 0.5$

a)

$$P(V) = P(V \wedge N1) + P(V \wedge N2) + P(V \wedge N3)$$
$$= P(V \mid N1)P(N1) + (\ldots)$$
$$= 0.5 \times 0.3 + 0.25 \times 0.4 + 0.25 \times 0.5$$
$$= 0.15 + 0.1 + 0.125$$
$$= 0.25 + 0.125$$
$$= 0.375$$

b)

$$P(N1 \mid V) = \frac{P(V \mid N_1)P(N_1)}{P(V)} = \frac{0.3 \times 0.5}{0.375} = \frac{0.15}{0.375} = \frac{150}{375} = \frac{30}{75} = \frac{6}{15} = \frac{2}{5} = 0.4$$

$\boxed{9}$

$S = \{0,1,2\} \times \{0,1,2\}$

a) "Sair 0 no primeiro"
"Sair 0 no segundo"

b) Sim, "Sair n no primeiro" $\Big\}$ $n = 0, 1, 2$
"Sair n no segundo"

$\boxed{10}$

a) Probabilidade de sair o "0" $\Rightarrow p$

C: "ser recebido corretamente"

$P(\text{"0"}) = p$
$p(C \mid \text{"0"}) = 1 - \varepsilon_0$
$p(C \mid \text{"1"}) = 1 - \varepsilon_1$

$$
\begin{aligned}
P(C) &= P(C \cap \text{"0"}) + P(C \cap \text{"1"}) \\
&= P(C \mid \text{"0"})\, p(\text{"0"}) + p(C \mid \text{"1"})\, p(\text{"1"}) \\
&= (1 - \varepsilon_0)\, p + (1 - \varepsilon_1)(1 - p)
\end{aligned}
$$

b)

$$
\begin{aligned}
P &= P(C \mid \text{"1"}) \times P(C \mid \text{"0"}) \times P(C \mid \text{"1"}) \times P(C \mid \text{"1"}) \\
&= (1 - \varepsilon_1)^3 (1 - \varepsilon_0)
\end{aligned}
$$

c)

Para ser correto: 2 ou 3 símbolos têm de ser zero

$P(C \mid \text{"0"}) = (1 - \varepsilon_0)$

Segue uma Distribuição Binomial: $p_x(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Onde, $n = 3$ $\Rightarrow p_x(k) = \binom{3}{k} (1 - \varepsilon_0)^k \varepsilon_0^{3-k}$
$p = (1 - \varepsilon_0)$

$$
\begin{aligned}
\text{Assim, } P_x(2) + P_x(3) &= \binom{3}{2} (1 - \varepsilon_0)^2 \varepsilon_0 + \binom{3}{3} (1 - \varepsilon_0)^3 \\
&= 3 (1 - \varepsilon_0)^2 \varepsilon_0 + (1 - \varepsilon_0)^3
\end{aligned}
$$

d) "101"

$$
\begin{aligned}
P_T &= P(\bar{C} \mid \text{"1"}) + P(C \mid \text{"0"}) + P(\bar{C} \mid \text{"1"}) \\
&= 2\varepsilon_1 + (1 - \varepsilon_0) \\
&= 1 + 2\varepsilon_1 - \varepsilon_0
\end{aligned}
$$

$\boxed{11}$

$P(X=6)=0.4 \qquad p(X=1)=p(X=2)=(\dots)=p(X=5)=\frac{3}{25}$

$E[X]=\sum_{x_i=1}^{6} x_i \times p(x_i) = \frac{3}{25}+\frac{2\times3}{25}+\frac{3\times3}{25}+\frac{4\times3}{25}+\frac{5\times3}{25}+\frac{6\times10}{25}$

$=\frac{105}{25}=\frac{21}{5}\simeq 4.2_{//}$

$E[X^2]=\sum_{x_i=1}^{6} x_i^2\, p(x_i) = \frac{3}{25}+\frac{2^2\times3}{25}+\frac{3^2\times3}{25}+\frac{4^2\times3}{25}+\frac{5^2\times3}{25}\ \frac{6^2\times10}{25}$

$=\frac{165+360}{25}=\frac{525}{25}=21$

$E^2[X]=17{,}64$

$var(X)=E[X^2]-E^2[X]=21-17{,}64=3{,}36_{//}$

$\boxed{12}$

$U(-1,3)$

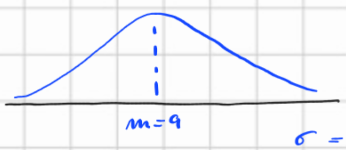
$E[X]=\int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-1}^{3} x f_X(x)\,dx = \int_{-1}^{3} x\times\frac{1}{4}dx = \frac{1}{4}\times\int_{-1}^{3} x\,dx = \frac{1}{4}\times\frac{x^2}{2}\Big|_{-1}^{3} = \frac{1}{4}\ \frac{3^2-(-1)^2}{2}$

$=\frac{9-1}{8}=1_{//}$

$E^2[X]=1_{//}\ ;\ E[X^2]=\frac{1}{4}\times\int_{-1}^{3} x^2\,dx = \frac{1}{4}\times\frac{x^3}{3}\Big|_{-1}^{3} = \frac{3^3+1}{12}=\frac{28}{12}=\frac{14}{6}=\frac{7}{3}$

$var(X)=\frac{7}{3}-1=\frac{4}{3}_{//}$

$\boxed{13}$



$\sigma = 2 \qquad \Rightarrow N(9,4)$

$\begin{cases} Y=aX+b \\ E[Y]=10 \\ var(Y)=6 \end{cases} \quad \begin{cases} \overline{} \\ E[aX+b]=10 \\ var(aX+b)=6 \end{cases} \quad \begin{cases} \overline{} \\ aE[X]+b=10 \\ a^2\,var(X)=6 \end{cases}$

E[X] é um operador linear

$\hookrightarrow \begin{cases} var(aX)=a^2 var(X) \\ var(X+b)=var(X) \end{cases}$

Como em $N(9,4)$, $E[X]=m=9$, $var(X)=\sigma^2=4$ $\Rightarrow \begin{cases} \overline{} \\ 9a+b=10 \\ 4a^2=6 \end{cases} \quad \begin{cases} \overline{} \\ b=10-9a \\ a=\pm\sqrt{\frac{3}{2}} \end{cases}$

$\begin{cases} \overline{} \\ b=10+9\sqrt{\frac{3}{2}} \\ a=-\sqrt{\frac{3}{2}} \end{cases} \vee \begin{cases} \overline{} \\ b=10-9\sqrt{\frac{3}{2}} \\ a=\sqrt{\frac{3}{2}} \end{cases}$

$\boxed{14}$

a) $m = 1$

$\sigma = 2$

$F_x(x) = \Phi\left(\frac{x-m}{\sigma}\right) = \Phi(U_x)$

$P[X < 1] = P\left[U_x < \frac{1-1}{2}\right] = P_x[U_x < 0] = 1 - P[U_x \geq 0]$

$= 1 - Q(0) = 1 - 0.5 = 0.5$

$\boxed{P[X < m] = \frac{1}{2}}$

$\Phi(x) = 1 - Q(x)$

$F_x(x) = \Phi\left(\frac{x-m}{\sigma}\right) = \Phi(U_x)$

c) $P[-2 < X < 1] = P\left[-\frac{2-1}{2} < U_x < \frac{1-1}{2}\right] = P\left[-\frac{3}{2} < U_x < 0\right]$

$\Phi(x) = 1 - Q(x)$

$= Q\left(-\frac{3}{2}\right) - Q(0)$

$Q(-x) = 1 - Q(x)$

$= 1 - Q(1.5) - Q(0)$

$= 0.456$

$\boxed{15}$

X \ Y	-1	0	1	$P_x(x)$
-1	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{1}{24}$	$\frac{7}{24}$
0	$\frac{3}{24}$	$\frac{6}{24}$	$\frac{3}{24}$	$\frac{12}{24}$
1	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{1}{24}$	$\frac{5}{24}$
$P_Y(y)$	$\frac{7}{24}$	$\frac{12}{24}$	$\frac{5}{24}$	1

a) $E[X] = \sum_{x_i=-1}^{1} x_i \, p_x(x_i) = -1 \times \frac{7}{24} + 0 \times \frac{12}{24} + 1 \times \frac{5}{24} = -\frac{2}{24} = -\frac{1}{12}$

$E[X^2] = \sum_{x_i=-1}^{1} x_i^2 \, p_x(x_i) = 1 \times \frac{7}{24} + 0 \times \frac{12}{24} + 1 \times \frac{5}{24} = \frac{12}{24} = \frac{1}{2}$

$E^2[X] = \frac{1}{12^2} = \frac{1}{144}$

$\mathrm{var}(X) = E[X^2] - E^2[X] = \frac{1}{2} - \frac{1}{144} = \frac{71}{144}$

b)

$\mathrm{cov}(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = \sum_{x_i=-1}^{1} \sum_{y_i=-1}^{1} x_i y_i \, p_{XY}(x_i, y_i) = (-1)(-1)\left(\frac{3}{24}\right) + (-1)(0)\left(\frac{3}{24}\right) + (-1)(1)\left(\frac{1}{24}\right)$

$= \frac{3}{24} - \frac{1}{24} = \frac{1}{12}$

$E[Y] = \sum_{y_i=-1}^{1} y_i \, P_Y(y_i) = -1 \times \frac{7}{24} + 0 \times \frac{12}{24} + 1 \times \frac{5}{24} = -\frac{1}{12}$

$\mathrm{cov}(X,Y) = \frac{1}{12} - \left(-\frac{1}{12}\right)\left(-\frac{1}{12}\right) = \frac{12}{144} - \frac{1}{144} = \frac{11}{144}$

c) Não! Pois como se X e Y fossem independentes $E[XY] = E[X]\,E[Y]$

$\Rightarrow cov(x, y) = E[XY] - E[XY] = 0$

$\Rightarrow$ Como $cov(x, Y) \neq 0 \Rightarrow X$ e Y não são independentes!

d)

$W = X^2$ \ $Z = Y^2$	1	0	1	$P_x(x)$
1	$\frac{3}{24}$	$\frac{3}{24}$	$\frac{1}{24}$	$\frac{7}{24}$
0	$\frac{3}{24}$	$\frac{6}{24}$	$\frac{3}{24}$	$\frac{12}{24}$
1	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{1}{24}$	$\frac{5}{24}$
$P_Y(y)$	$\frac{7}{24}$	$\frac{12}{24}$	$\frac{5}{24}$	1

$\Longrightarrow$

$W = X^2$ \ $Z = Y^2$	0	1	$P_w(w)$
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$P_z(z)$	$\frac{1}{2}$	$\frac{1}{2}$	1

Como $P_z(z) = \frac{1}{2}$ e $P_w(w) = \frac{1}{2}$ e $P_{wz}(w,z) = \frac{1}{4}$

$\Rightarrow P_{wz}(w,z) = P_w(w)\,P_z(z)$

$\Rightarrow W$ e Z são independentes

$\boxed{16}$

$$E[(X + Y)^2] = E[X^2 + 2XY + Y^2] = E[X^2] + 2E[XY] + E[Y^2]$$