A linear-algebraic method to compute polynomial PDE conservation laws*

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Abstract

We present a method to compute polynomial conservation laws for systems of partial differential equations (PDEs). The method only relies on linear algebraic computations and is complete, in the sense it can find a basis for all polynomial fluxes that yield conservation laws, up to a specified order of derivatives and degree. We compare our method to a state-of-the-art algorithm based on the direct approach on a few PDE systems drawn from mathematical physics.

Keywords: PDEs, conservation laws, linear algebra, polynomials, Gröbner bases

1 Introduction

Let Σ be a system of partial differential equations (PDEs) in n independent variables. A conservation law of Σ is vector of n expressions, called *fluxes*, whose divergence vanishes on the solutions of the system [12, Ch.4]. In the case of ODEs ($n = 1$), this is the same as a first integral of the system. Conservation laws often express physical principles, such as conservation of mass, energy, momentum and so on, and as such are of fundamental importance to gain a qualitative insight into the phenomenon being studied. Moreover, the existence of many conservation laws points to complete integrability of Σ [1]. Conservation laws are also crucial in applications, in particular numerical methods: knowledge of conservation laws of Σ makes it possible to apply effective numerical schemes, including finite volume and finite elements methods; see e.g., [10, Ch.12].

Methods to systematically search conservation laws have traditionally been linked to the existence of symmetries, on account of a celebrated theorem by Emmy Noether [12, Ch.4]. A variety of algorithms based on a *direct approach*, which does not presuppose the existence of symmetries and is more widely applicable, have also been developed: see [12, Ch.4] and e.g. [2, 16, 11, 7, 5] and references therein. In the direct approach,

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one first finds *multipliers*, whose linear combinations with the system's equations yield divergence expressions vanishing on solutions, then inverts the divergences to obtain the corresponding fluxes.

In this paper, we put forward a new method to automatically find conservation laws of polynomial form. The method can be applied to any polynomial PDE system, independently of the existence of symmetries, and as such, it might be classified as 'direct' itself. Its distinctive features are:

(a) its completeness is given in terms of *fluxes*: differently from most direct methods, the user directly specifies an ansatz (order and degree) of the searched polynomial fluxes, rather than that of multipliers;

(b) computationally, it only relies on equational rewriting and linear algebraic operations — at least when it is applied to leading linear systems.

An additional benefit of this method is, in our opinion, its conceptual simplicity: it just takes some elementary algebra to give a rather detailed description of its functioning. This we do below.

Let Σ be a polynomial system of PDEs. Under a certain nondegeneracy condition, the set of invariant polynomials of Σ up to a given differential order coincides with the ideal generated by Σ, denoted $\langle \Sigma \rangle$, in a sub-ring of the differential polynomials. Now, a polynomial conservation law is a vector of polynomials, the fluxes, whose divergence is an invariant, that is belongs to $\langle \Sigma \rangle$. Equivalently, upon performing polynomial division of such a divergence by a basis of $\langle \Sigma \rangle$, the obtained remainder is zero. Symbolically, we represent a *set* of polynomials as a formal linear combination of monomials with unknown coefficients, or *template*; and a set of candidate conservation laws as a single vector of user-specified flux templates. We then seek necessary and sufficient conditions on the coefficients for the remainder of the resulting divergence to vanish upon division by $\langle \Sigma \rangle$. This results in a homogeneous linear algebraic system for the unknown coefficients. When the solutions of this system are substituted back into the original flux templates, all polynomial conservation laws fitting the specified templates are obtained. In the case of PDEs whose leading term is linear, a basis of $\langle \Sigma \rangle$ is Σ itself, under an appropriate monomial ordering; and taking the remainder of a polynomial effectively corresponds to rewriting it into a normal form. This justifies our claim that for leading linear systems, computationally the method requires no more than equational rewriting and linear algebra. Additionally, trivial laws can be easily filtered out from the output.

In this context, an important goal will be to identify easy to check syntactic conditions on Σ that guarantee nondegeneracy, which is essential for completeness. We will show that the Riquier's format [14, 15], a generalization of Cauchy-Kovaleskvaya's one [12, Ch.4], does imply the considered notion of nondegeneracy. We will cover the cases of both formal and real analytic solutions of PDEs.

On the negative side, it may be observed that the completeness of the algorithm holds relative to polynomial fluxes only up to a specified degree. We will comment on this limitation when we will compare our algorithm to the direct method.

Structure of the paper The rest of the paper is organized as follows. Background on PDEs and their solutions is given in Section 2. The method is then presented in Section 3. A few experiments and a comparison with a state-of-the-art algorithm based on the direct approach are discussed in Section 4. For most part of the paper we will be concerned with formal power series solutions of PDEs; the obtained results carry over to real analytic solutions with very minor modifications, which are dealt with in Section 5. Concluding remarks and further comparison with related work are in Section 6. A few technical proofs have been confined to Appendix A.

2 Preliminaries

We introduce some standard terminology and notation on PDEs and their solutions. Let $X = \{x_1, ..., x_n\}$ and $\mathcal{U} = \{u^1, ..., u^m\}$ be disjoint, nonempty sets of independent and dependent variables, respectively. We let $t, x, y, ...$ range over X and $u, v, ...,$ possibly with superscripts $u^i, ...,$ range over $\mathcal{U}$. Let $X^{\otimes}$, ranged over by $\tau, \xi, ...$ be the set of monomials that can be formed from variables in X — that is, the free commutative monoid generated by X — with ϵ denoting the identity monomial. For $\tau = x_1^{k_1} \cdots x_n^{k_n}$, we let $|\tau| \triangleq k_1 + \cdots + k_n$. We let $\mathcal{D} \triangleq \{u_\tau : u \in \mathcal{U}, \tau \in X^{\otimes}\}$ denote the set of *derivatives*; here u_ϵ will be identified with u. We let $\mathbb{R}[X \cup \mathcal{D}]$, ranged over by $f, g, ..., p, q, ...,$ be the set of *(differential) polynomials* with coefficients in $\mathbb{R}$ and indeterminates in $X \cup \mathcal{D}$. The *order* of a polynomial p is $\max\{|\tau| : u_\tau \text{ occurs in } p\}$. For example, $f = xv_z u_{xy} + v_y^2 + u + 5x$ is a differential polynomial of degree 3 and order 2.

A system of polynomial PDEs is a finite set $\Sigma \subseteq \mathbb{R}[X \cup \mathcal{D}]$. The order of Σ is the maximum order of polynomials in Σ. In what follows, we shall consider an arbitrarily fixed, *finite* $D \subseteq \mathcal{D}$ that is a superset of the derivatives occurring in Σ and let $\mathcal{P} \triangleq \mathbb{R}[X \cup D]$. The set $\mathcal{P}$ will act as our 'universe' of differential polynomials, in the sense that we will be interested in finding conservation laws whose divergence (see Section 3) lies in $\mathcal{P}$. For example, in the case of the wave equation, with $X = \{x, y\}, \mathcal{U} = \{u\}$ and $\Sigma = \{u_{xx} - u_{yy}\}$, we might fix $D = \{u, u_x, u_y, u_{xy}, u_{xx}, u_{yy}\}$ if we are interested in laws with fluxes built out of $\{u, u_x, u_y\}$. Elements of $\mathcal{P}$ are multivariate polynomials in a finite number of indeterminates in the usual algebraic sense: in particular, they can be evaluated at any point $(x, u_D) \in \mathbb{R}^{X \cup D} \cong \mathbb{R}^k$, with $k \triangleq |X \cup D|$, where we are implicitly fixing an arbitrary total order on $X \cup D$. Therefore, any subset $P \subseteq \mathcal{P}$ induces an *algebraic variety* $\mathcal{V}(P) \subseteq \mathbb{R}^k$, defined as $\mathcal{V}(P) \triangleq \{(x, u_D) \in \mathbb{R}^k : p(x, u_D) = 0 \text{ for each } p \in P\}$. In particular, as $\Sigma \subseteq \mathcal{P}$, we can consider $\mathcal{V}(\Sigma)$, the algebraic variety induced by Σ.

We will be mostly concerned with formal power series solutions of Σ — but see Section 5 on analyticity. For a monomial $\tau = x_1^{k_1} \cdots x_n^{k_n}$ and $x^0 = (x_1^0, ..., x_n^0) \in \mathbb{R}^n$, we let $(x - x^0)^\tau \triangleq (x_1 - x_1^0)^{k_1} \cdots (x_n - x_n^0)^{k_n}$ be a monomial in the terms $x_i - x_i^0$. A formal power series centered at $x^0 \in \mathbb{R}^n$ is a formal sum of monomials[1] $F = \sum_{\tau \in X^{\otimes}} c_\tau (x - x^0)^\tau$, with $c_\tau \in \mathbb{R}$. The value of F at x^0, denoted $F(x^0)$, is the constant coefficient c_ϵ. Sum, product and partial derivative $\partial F / \partial x_i$ ($x_i \in X$, extension to monomials denoted by $\partial^{|\tau|} F / \partial \tau$) of formal power series are defined as usual and satisfy the expected properties. Let $U = (U^1, ..., U^m)$ be a tuple of formal power series centered at x^0. We let $U_D \triangleq (\frac{\partial^{|\tau|}}{\partial \tau} U^i)_{u^i_\tau \in D}$ denote the tuple of formal power series corresponding to the derivatives in D, and let

[1] Rigorously, a formal power series centered at x^0 is a function $F : \{x_1 - x_1^0, ..., x_n - x_n^0\}^{\otimes} \longrightarrow \mathbb{R}$. Accordingly, when writing polynomial expressions of such series, each $x_i \in X$ is to be interpreted as the formal power series $F = x_i^0 + 1 \cdot (x_i - x_i^0)$, that is: $F(\epsilon) = x_i^0$, $F(x_i - x_i^0) = 1$ and $F((x - x^0)^\tau) = 0$ for any $\tau \neq \epsilon, x_i$.

$U_D(x^0)$ be the tuple $(\frac{\partial^{|\tau|}}{\partial \tau} U^i(x^0))_{u^i_\tau \in D}$. We say U is a formal *solution* of Σ centered at x^0 if for each $p \in \Sigma$ the formal power series $p(x, U_D(x))$ centered at x^0 is zero. The formal *D-solution variety* of Σ is

$$\begin{aligned}\mathcal{S}(\Sigma) \stackrel{\triangle}{=} \{(x^0, u^0_D) \in \mathcal{V}(\Sigma) \, : \, & \text{there is a formal solution } U \text{ of } \Sigma \text{ centered at } x^0 \\ & \text{such that } U_D(x^0) = u^0_D \, \} \, .\end{aligned}$$

We say Σ is formally *D-locally solvable* if its algebraic and D-solution varieties coincide, $\mathcal{V}(\Sigma) = \mathcal{S}(\Sigma)$. In what follows, we shall omit the qualification "D-" if the set D is clear from the context.

Remark 1 (on local solvability). Ignoring for the time being the distinction between analytic and formal solutions, we see the above definition of local solvability is more flexible, when compared to the usual one [12, Ch.2,Def.2.70] that requires D to be the set of *all* derivatives up to the order of Σ. For instance, the following system of order 2

$$\Sigma = \{u_x - v \; , \; v_{xx} - u_y\}$$

is not locally solvable in the sense of [12, Ch.2,Def.2.70], simply because there are 2nd order differential consequences, like $u_{xy} - v_y$, that are not algebraic consequences. However Σ is D-locally solvable for, say, $D = \{u, v, u_x, u_y, v_{xx}\}$. Generally speaking, our definition of D-local solvability appears to be a sensible extension of the usual one, when considering systems that are not in Cauchy-Kovalevskaya form, but satisfy more general forms, like Riquier's format [14]. This point will be further discussed in the next section.

3 The method

Our main object of interest is a notion invariant, a sort of logical consequence of Σ.

Definition 1 (invariant polynomials). *We say $p \in \mathcal{P}$ is an* invariant polynomial *of Σ if, for each $x^0 \in \mathbb{R}^n$, the systems Σ and $\Sigma \cup \{p\}$ have the same solutions centered at x^0. We let $\mathcal{I}nv(\Sigma)$ denote the set of polynomial invariants of Σ that are in $\mathcal{P}$.*

We will rely on a simple algebraic-geometric characterization of $\mathcal{I}nv(\Sigma)$. Let us introduce the necessary terminology; see [6, Ch.1-2] for a more comprehensive treatment. For any $W \subseteq \mathbb{R}^k$, we let $\mathcal{I}(W) \subseteq \mathcal{P}$ be the ideal of polynomials that vanish on W, that is $\mathcal{I}(W) \stackrel{\triangle}{=} \{p \in \mathcal{P} \, : \, p(w) = 0 \text{ for each } w \in W\}$. Moreover, for any $P \subseteq \mathcal{P}$, we let $\langle \, P \, \rangle \subseteq \mathcal{P}$ be the ideal generated by P. The ideal $\mathcal{I}(\mathcal{V}(P))$ is called the *real radical* of P. In the following definition, we consider the real radical of Σ.

Definition 2 (D-nondegeneracy). *We say Σ is D-*nondegenerate *if Σ is D-locally solvable and $\langle \, \Sigma \, \rangle = \mathcal{I}(\mathcal{V}(\Sigma))$.*

Lemma 1. *$\langle \, \Sigma \, \rangle \subseteq \mathcal{I}nv(\Sigma)$, with equality if Σ is D-nondegenerate.*

Proof. First suppose $p \in \langle \, \Sigma \, \rangle$, that is $p = \sum_j q_j f_j$ for some $q_j \in \mathcal{P}$ and $f_j \in \Sigma$. By definition, any solution of Σ, however centered, makes each f_j, hence p, identically zero.

Suppose now Σ is D-locally solvable and $\langle \, \Sigma \, \rangle = \mathcal{I}(\mathcal{V}(\Sigma))$, and consider any $p \in \mathcal{I}nv(\Sigma)$. Consider any $(x^0, u^0_D) \in \mathcal{V}(\Sigma) = \mathcal{S}(\Sigma)$. By definition of local solvability, there is a solution U of Σ centered at x^0 such that $U_D(x^0) = u^0_D$. Then we have: $p(x^0, u^0_D) = p(x^0, U_D(x^0)) = p(x, U_D(x))_{|x=x^0} = 0$, where the last equality stems from p being an invariant. Since $(x^0, u^0_D) \in \mathcal{V}(\Sigma)$ is arbitrary, we have shown that $p \in \mathcal{I}(\mathcal{V}(\Sigma)) = \langle\ \Sigma\ \rangle$. $\square$

Consider now any Gröbner basis Δ of $\langle\ \Sigma\ \rangle$. For each $p \in \mathcal{P}$, we can then consider the unique remainder of the polynomial division of p by Δ, that is

$$S_\Delta\, p \stackrel{\triangle}{=} p \bmod \Delta\,.$$

In what follows, we shall write Sp, leaving Δ implicit. By Lemma 1, $Sp = 0$ ensures that p is an invariant. The converse as well can be stated if Σ is a nondegenerate. We introduce below an important class of nondegenerate systems.

We recall that a *ranking* [15] of the derivatives is a total ordering on $\mathcal{D}$ such that, for all $u, v \in U$, $\tau, \xi \in X^{\otimes}$ and $x_i \in X$:

1. $u_\tau < u_{\tau x_i}$;
2. $u_\tau < v_\xi$ implies $u_{\tau x_i} < v_{\xi x_i}$.

Let us say Σ is *leading linear* if its elements are of the form $u_\tau + f$, where $u_\tau > v_\xi$ for each v_ξ occurring in f; in this case, we let $\mathrm{dom}(\Sigma)$ be the set of such leading derivatives u_τ. In what follows, $\mathrm{D}_{x_i}p$ denotes the formal total derivative of p along $x_i \in X$: this is computed like the usual derivative of p along x_i, just taking into account that $\mathrm{D}_{x_i}u_\tau = u_{\tau x_i}$. As $\mathrm{D}_{x_i}\mathrm{D}_{x_j}p = \mathrm{D}_{x_j}\mathrm{D}_{x_i}p$, the notation $\mathrm{D}_\tau p$ for $\tau \in X^{\otimes}$ is well defined. For $k \geq 0$, let us denote the k-th prolongation of Σ and D as $\Sigma^{(k)} \stackrel{\triangle}{=} \{\mathrm{D}_\tau p\ :\ p \in \Sigma, |\tau| \leq k\}$ and $D^{(k)} \stackrel{\triangle}{=} \{u_{\xi\tau}\ :\ u_\xi \in D \text{ and } |\tau| \leq k\}$, respectively. A leading linear Σ is *passive* if it implies all its integrability conditions, that is: whenever $u_\tau + f \in \Sigma$ and $u_\xi + g \in \Sigma$ and $\tau\xi' = \xi\tau'$ then, for some k, $(\mathrm{D}_{\xi'}f - \mathrm{D}_{\tau'}g) \in \langle\ \Sigma^{(k)}\ \rangle$. In fact, for passivity it is sufficient to check the finitely many integrability conditions with $\xi' = \sigma/\tau$ and $\tau' = \sigma/\xi$, where σ is the least common multiple of τ and ξ; see [15, Cor.1]. A leading linear, passive system is also called a *Riquier basis* in [15]: from now on we shall adopt this terminology[2], with the further specification that any two distinct elements in a Riquier basis must have distinct leading derivatives. This implies no loss of expressiveness. For a leading linear system Σ, we can define the sets of *principal* derivatives $\mathcal{P}\mathrm{r}(\Sigma) \stackrel{\triangle}{=} \{u_{\tau\xi}\ :\ u_\tau \in \mathrm{dom}(\Sigma) \text{ and } \xi \in X^{\otimes}\}$, and *parametric* derivatives $\mathcal{P}\mathrm{a}(\Sigma) \stackrel{\triangle}{=} \mathcal{D} \setminus \mathcal{P}\mathrm{r}(\Sigma)$. The following proposition ensures D-nondegeneracy for Riquier bases, whenever the principal derivatives occurring in Σ coincide with its leading derivatives, as well as with the principal derivatives in D.

Proposition 1. *Let Σ be a Riquier basis and assume $D \cap \mathcal{P}\mathrm{r}(\Sigma) = \mathrm{dom}(\Sigma)$. Then Σ is D-nondegenerate. Moreover, Σ is a Gröbner basis of $\mathcal{I}(\mathcal{V}(\Sigma))$ w.r.t. a suitable monomial order.*

PROOF. We first show that Σ is D-locally solvable. An initial data specification for Σ is a function $\rho : \mathcal{P}\mathrm{a}(\Sigma) \to \mathbb{R}$. A solution of the initial value problem (Σ, ρ) at $x^0 \in \mathbb{R}^n$ is a solution U of Σ centered at x^0 such that, for each $u^i_\tau \in \mathcal{P}\mathrm{a}(\Sigma)$, $\frac{\partial^{|\tau|}}{\partial\tau}U^i_\tau(x^0) = \rho(u^i_\tau)$.

[2] The *coherent* systems in [4] are an equivalent notion.

The formal Riquier existence theorem [15, Th.2] ensures that for each x^0 and ρ there is a unique solution U of the problem (Σ, ρ).

Now, let $(x^0, u^0_D) \in \mathcal{V}(\Sigma)$. Below, we will denote the component of u^0_D corresponding to $u^i_\tau \in D$ by $u^{0,i}_\tau$. Let us define ρ as follows: $\rho(u^i_\tau) = u^{0,i}_\tau$ for each $u^i_\tau \in D \cap \mathcal{P}\mathrm{a}(\Sigma)$, and arbitrarily for any other element of $\mathcal{P}\mathrm{a}(\Sigma)$. According to the formal existence theorem, there is a solution U of (Σ, ρ). In particular, we have that for each $u^i_\tau \in D \cap \mathcal{P}\mathrm{a}(\Sigma)$, $\frac{\partial^{|\tau|}}{\partial \tau} U^i_\tau(x^0) = \rho(u^i_\tau) = u^{0,i}_\tau$. We now proceed to show the same for each $u^i_\tau \in \mathrm{dom}(\Sigma)$. We proceed by induction on the ranking of u^i_τ. Let $u^i_\tau + f \in \Sigma$, where $f = f(x, u_{D'})$, for some $D' \subseteq D$ that only contains elements smaller than u^i_τ in the ranking. In the base case, D' only contains parametric derivatives. In any case, we have

$$\begin{aligned}\frac{\partial^{|\tau|}}{\partial \tau} U^i_\tau(x^0) &= -f(x^0, U_{D'}(x_0)) \\ &= -f(x^0, u^0_{D'}) \\ &= u^{0,i}_\tau\end{aligned}$$

where the first equality follows from U being a solution of Σ, the second one from either the result on parametric derivatives above (base case) or the induction hypothesis (inductive step), and the third one from $(x^0, u^0_D) \in \mathcal{V}(\Sigma)$ and $u^i_\tau + f \in \Sigma$. Overall, we have shown that $u^0_D = U_D(x^0)$, hence $(x^0, u^0_D) \in \mathcal{S}(\Sigma)$.

We now show that $\langle\ \Sigma\ \rangle = \mathcal{I}(\mathcal{V}(\Sigma))$. First, we show that Σ is a Gröbner basis for $\langle\ \Sigma\ \rangle$, once we consider the lexicographic monomial order on $(X \cup D)^{\otimes}$ induced by the ranking $<$ on $\mathcal{D}$, augmented by the rules $x_i < x_j < u_\tau$ for every $i < j$ and $u_\tau \in \mathcal{D}$. To see this, take any $0 \neq p \in \langle\ \Sigma\ \rangle$, and assume by contradiction that the leading monomial of p is not divisible by the leading derivative of any element in Σ. By the chosen monomial order and the condition $D \cap \mathcal{P}\mathrm{r}(\Sigma) = \mathrm{dom}(\Sigma)$, p has no occurrences of principal derivatives. Now partition D as $D = D_0 \cup D_1$, where D_0 are parametric derivatives and $D_1 = \mathrm{dom}(\Sigma)$ are principal. Consider any tuple $(x^0, u^0_{D_0}) \in \mathbb{R}^{X \cup D_0}$. For each $u_\tau \in \mathrm{dom}(\Sigma)$, we define $u^0_\tau \in \mathbb{R}$, by induction on the ranking and using the same notation as above, as follows: $u^0_\tau \stackrel{\triangle}{=} -f(x^0, u^0_{D'})$, where $u_\tau + f \in \Sigma$. By construction, the tuple $(x^0, u^0_D) = (x^0, u^0_{D_0}, u^0_{D_1})$ so defined is in $\mathcal{V}(\Sigma)$. Hence, recalling that $p \in \langle\ \Sigma\ \rangle \subseteq \mathcal{I}(\mathcal{V}(\Sigma))$, we have: $0 = p(x^0, u^0_D) = p(x^0, u^0_{D_0})$. Since this holds for arbitrary $(x^0, u^0_{D_0})$ and $p \in \mathbb{R}[X \cup D_0]$, we have shown that $p = 0$, contradicting the initial assumption.

We now show that $\langle\ \Sigma\ \rangle = \mathcal{I}(\mathcal{V}(\Sigma))$. The inclusion $\langle\ \Sigma\ \rangle \subseteq \mathcal{I}(\mathcal{V}(\Sigma))$ is obvious, so consider any $f \in \mathcal{I}(\mathcal{V}(\Sigma))$, and write $f = f_0 + p$, where $f_0 \in \langle\ \Sigma\ \rangle$ and $p = f \bmod \Sigma$, hence $p \in \mathbb{R}[X \cup D_0]$. For arbitrary $(x^0, u^0_{D_0})$, let us build $(x^0, u^0_D) = (x^0, u^0_{D_0}, u^0_{D_1})$ exactly as above. As $(x^0, u^0_D) \in \mathcal{V}(\Sigma)$, we have $0 = f(x^0, u^0_D) = f_0(x^0, u^0_D) + p(x^0, u^0_D) = p(x^0, u^0_{D_0})$. Since this holds for arbitrary $(x^0, u^0_{D_0})$, we deduce $p = 0$, hence $f \in \langle\ \Sigma\ \rangle$. $\square$

The above result implies that, for Riquier bases, provided the set of leading derivatives coincides with the set of principal derivatives occurring in Σ, D can be taken to consist of all the derivatives occurring in Σ, plus an arbitrary finite set of parametric derivatives.

Example 1. The system Σ in Remark 1 is a Riquier basis w.r.t., for instance, a graded lexicographic ranking with $u < v$ and $x < y$. In particular, no integrability conditions arise. Take D as specified in the remark. As $D \cap \mathcal{P}\mathrm{r}(\Sigma) = \{u_x, v_{xx}\} = \mathrm{dom}(\Sigma)$, Proposition 1 ensures that Σ is D-locally solvable.

It may also be the case that the condition $D \cap \mathcal{P}\mathrm{r}(\Sigma) = \mathrm{dom}(\Sigma)$ is not satisfied, because there are principal derivatives in D that are not in $\mathrm{dom}(\Sigma)$, so one cannot apply Proposition 1 directly to Σ. In this case, it is enough to expand Σ by adding the corresponding equations. Formally, one applies the following lemma, possibly several times, until a Σ is obtained that satisfies $D \cap \mathcal{P}\mathrm{r}(\Sigma) = \mathrm{dom}(\Sigma)$. The proof of the lemma is an immediate consequence of the definition of Riquier basis and of invariant polynomial.

Lemma 2. *Let Σ be a Riquier basis, $u_\tau \in \mathcal{P}\mathrm{r}(\Sigma) \setminus \mathrm{dom}(\Sigma)$ and $u_\xi + f \in \Sigma$ with $\tau = \xi\zeta$. Then $\Sigma' \stackrel{\triangle}{=} \Sigma \cup \{\mathrm{D}_\zeta(u_\xi + f)\}$ is still a Riquier basis. If $\mathrm{D}_\zeta(u_\xi + f) \in \mathcal{P}$, $\mathcal{I}nv(\Sigma) = \mathcal{I}nv(\Sigma')$.*

Let us now introduce conservation laws.

Definition 3 (conservation laws). *A n-tuple of polynomials $\Phi = (p_1, ..., p_n)$ is a (polynomial)* conservation law *for Σ if its divergence $\mathbf{div}\,\Phi \stackrel{\triangle}{=} \sum_{j=1}^{n} \mathrm{D}_{x_j} p_j$ is an invariant polynomial for Σ. The components p_i in Φ are called* fluxes.

For any finite $Z \subseteq (X \cup D)^{\otimes}$, let $\mathcal{P}_Z$ be the set of polynomials that can be formed from monomials in Z. We let $\mathrm{CL}(\Sigma, Z) \subseteq \mathcal{P}_Z^n$ denote the set of conservation laws with fluxes in $\mathcal{P}_Z$. A typical choice will be, $Z = Y^{\leq d}$, for some $Y \subseteq X \cup D$ and $d \geq 1$, that is the monomials in $Y^{\otimes}$ of degree $\leq d$: this will give all the conservation laws with fluxes of degree $\leq d$ that can be built out Y. In what follows, we will refer to Z as to the chosen ansatz. We shall assume that D is chosen large enough[3] to ensure that $\mathbf{div}\,\Phi \in \mathcal{P}$ whenever $\Phi \in \mathcal{P}_Z^n$. $\mathrm{CL}(\Sigma, Z)$ is clearly a finite-dimensional vector space: our goal is to give a method to compute a basis for this space.

With this goal in mind, we introduce a way of representing succinctly sets of polynomials by means of *templates*. Fix a set of $s \geq 1$ distinct symbols, the *parameters* $a = \{a_1, ..., a_s\}$. A linear expression $\ell = \sum_j \lambda_j a_j$ is linear combination those s parameters with real coefficients $\lambda_j \in \mathbb{R}$. A template is a polynomial expression with linear expressions as coefficients, $\pi = \sum_{j=1}^{h} \ell_j \cdot \alpha_j$, for monomials $\alpha_1, ..., \alpha_h \in (X \cup D)^{\otimes}$. We say π is *complete* for the ansatz Z if $\ell_j = a_{i_j}$ are pairwise distinct parameters, and $Z = \{\alpha_1, ..., \alpha_h\}$. For $v = (v_1, ..., v_s) \in \mathbb{R}^s$, we let $\ell[v] \in \mathbb{R}$ and $\pi[v] \in \mathcal{P}$ be the real value and the polynomial, respectively, obtained by replacing each a_i with v_i, for $i = 1, ..., s$. For any $A \subseteq \mathbb{R}^s$, we let $\pi[A] = \{\pi[v] : v \in A\} \subseteq \mathcal{P}$. For a set of linear expressions $L = \{\ell_1, ..., \ell_t\}$, we let $\mathrm{span}(L) \stackrel{\triangle}{=} \{v \in \mathbb{R}^s : \ell[v] = 0 \text{ for each } \ell \in L\} \subseteq \mathbb{R}^s$ denote the vector space of parameter evaluations that annihilate all expressions in L.

It is now convenient to regard both polynomials and templates as elements of a larger polynomial ring, $\mathbb{R}[a \cup X \cup D]$. Let Δ be a Gröbner basis of $\langle\ \Sigma\ \rangle$ in $\mathcal{P}$ w.r.t. some monomial order. Δ is still a Gröbner basis in the larger polynomial ring, with respect to any monomial on $(a \cup X \cup D)^{\otimes}$ that conservatively extends the original one on $(X \cup D)^{\otimes}$ (see Appendix A.1 for additional details about this technical point). Then we can conservatively extend the function S to templates

$$S\pi \stackrel{\triangle}{=} \pi \bmod \Delta\,.$$

In the special case of $\Sigma = \Delta$ a Riquier basis, this means to rewrite π applying repeatedly the equations in Σ from left to right, treating the parameters a_is as arbitrary constants,

[3] This may imply expanding Σ via Lemma 2 in order to apply Proposition 1 and ensure nondegeneracy.

until no leading derivative remains. The following lemma gives a simple substitution property for templates; for a proof, see [3, Lemma 3].

Lemma 3. *Let π be a template. Then $S\pi$ is still a template. Moreover, for each $v \in \mathbb{R}^s$, $S(\pi[v]) = (S\pi)[v]$.*

We extend the total derivative operator D_{x_i} to templates as expected, by setting $\mathrm{D}_{x_i}(\sum_j \ell_j \alpha_j) \triangleq \sum_j \ell_j \mathrm{D}_{x_i}\alpha_j$. Then, for a tuple of templates $\Pi = (\pi_1, ..., \pi_n)$, we let $\mathbf{div}\,\Pi \triangleq \sum_{i=1}^n \mathrm{D}_{x_i}\pi_i$ denote its divergence. A *C.L. template for Z* is a tuple $\Pi = (\pi_1, ..., \pi_n)$ of complete templates for Z, such that the components π_is are formed from pairwise disjoint subsets of parameters in a.

Given a C.L. template Π, we are interested in those $v \in \mathbb{R}^s$ such that $\Pi[v] \triangleq (\pi_1[v], ..., \pi_n[v])$ is a conservation law. That is, those v's such that $\mathbf{div}\,(\Pi[v])$ is an invariant for Σ. That is, according to Lemma 1 and Lemma 3, those v's such that $S(\mathbf{div}\,\Pi[v]) = (S\,(\mathbf{div}\,\Pi))[v] = 0$. In other words, we are interested in the v's that annihilate all the linear expressions (coefficients) of $S(\mathbf{div}\,\Pi)$. This reasoning leads to the following result. A detailed proof is reported in Appendix A.1.

Corollary 1 (completeness for CL). *Let Π be a C.L. template for the ansatz Z. Let L be the set of linear expressions (coefficients) of $S(\mathbf{div}\,\Pi)$ and $V = \mathrm{span}(L)$. Then $\Pi[V] \triangleq \{\Pi[v] \,:\, v \in V\} \subseteq \mathrm{CL}(\Sigma, Z)$, with equality if Σ is D-nondegenerate.*

Note, from the above result, that once we have a basis B of V, then $\Pi[B]$ is a basis for $\Pi[V]$. To sum up, our method conceptually consists of the following steps, assuming a Gröbner basis Δ of $\langle\, \Sigma \,\rangle$ has been pre-computed. Given an ansatz Z:

1. using n disjoint sets of parameters, build a C.L. template for Z, $\Pi = (\pi_1, ..., \pi_n)$;
2. compute the divergence template $\mathbf{div}\,\Pi$ and its normal form $r = S(\mathbf{div}\,\Pi)$;
3. extract from r its coefficients (linear expressions), $L = \{\ell_1, ..., \ell_h\}$;
4. compute a basis B for $V = \mathrm{span}(L)$;
5. return $\Pi[B]$, a basis of $\Pi[V]$.

Remark 2 (trivial, equivalent and independent laws). As an optional final step in the above method, one might want to remove trivial conservation laws from $\Pi[B]$. Recall that a conservation law $\Phi = (p_1, ..., p_n)$ is *trivial* if it is a linear combination of trivial laws of the first kind (each flux p_i in Φ is an invariant for Σ) and of the second kind ($\mathbf{div}\,\Phi$ is zero as a polynomial[4]). Two laws Φ_1 and Φ_2 are *equivalent* if their difference $\Phi_1 - \Phi_2$ is a trivial law.

Note any law $\Phi = (p_1, ..., p_n)$ is equivalent to $\tilde{\Phi} = (Sp_1, ..., Sp_n)$, as $\Phi - \tilde{\Phi} \in \langle\, \Sigma \,\rangle^n$ is a trivial law of the first kind by Lemma 1. Therefore, without loss of generality, that is up to equivalence, it is always possible to choose an ansatz Z such that $Sp = p$ for each $p \in \mathcal{P}_Z$. Syntactically, this means that no monomial in Z is divisible by the leading monomial of any element in Δ. For leading linear systems, this amounts to making sure

[4] The general definition requires $(\mathbf{div}\,\Phi)(x, U_D(x))$ to be identically 0 for any smooth function U. In the polynomial case, this is equivalent to $\mathbf{div}\,\Phi = 0$.

that *no principal derivative occurs in* Z. Now further assume that Σ is D-nondegenerate. Then, by virtue of Lemma 1, the only trivial law of the first kind in $\mathrm{CL}(\Sigma, Z)$ is $(0, ..., 0)$ (n times). In this case, it is therefore sufficient to search and remove from $\Pi[B]$ trivial laws of the second kind, which is computationally easy.

An even less redundant representation of the space $\Pi[V]$ can be obtained by requiring that the set of returned laws, say $C \subseteq \Pi[V]$, satisfies the following property of *independence* up to triviality: if a linear combination of the laws in C, $\Psi = \sum_{\lambda_\Phi \in C} \lambda_\Phi \cdot \Phi$ ($\lambda_\Phi \in \mathbb{R}$), is trivial, then for each $\Phi \in C$, $\lambda_\Phi = 0$. If Σ is nondegenerate and the ansatz Z is chosen as specified above, again triviality of Ψ is equivalent to $\mathbf{div}\,\Psi = 0$ as a polynomial. Hence independence up to triviality of C is equivalent to the usual linear independence of the set of divergences $\{\mathbf{div}\,\Phi : \Phi \in C\}$ in the vector space of polynomials $\mathcal{P}$. Therefore, given $\Pi[B]$, computing a set C with the desired independence property is a matter of applying familiar linear algebraic techniques.

We illustrate the above method with a simple example.

Example 2 (Wave equation). Let Σ consists of the wave equation $u_{xx} - u_{yy}$. We can fix $D = \{u, u_x, u_y, u_{xy}, u_{xx}, u_{yy}\}$. We consider the ansatz $Z = Y^{\leq 2}$ where $Y = \{u_x, u_y\}$, and compute a basis for $\mathrm{CL}(\Sigma, Z)$. Σ is a Riquier basis w.r.t. any ranking where $u_{xx} > u_{yy}$, and $D \cap \mathcal{P}\mathrm{r}(\Sigma) = \{u_{xx}\} = \mathrm{dom}(\Sigma)$: hence Σ is D-nondegenerate by Proposition 1. Now we have the following steps.

1. CL template:

$$\Pi \;=\; (\pi_1, \pi_2) \quad \text{where} \quad \begin{aligned} \pi_1 &= a_1 + a_2 u_x + a_3 u_x u_y + a_4 u_y^2 + a_5 u_x^2 + a_6 u_y \\ \pi_2 &= a_7 + a_8 u_x + a_9 u_x u_y + a_{10} u_y^2 + a_{11} u_x^2 + a_{12} u_y\,. \end{aligned}$$

2. Divergence template and its normal form:

$$\begin{aligned} \mathbf{div}\,\Pi = {} & 2a_{10} u_y u_{yy} + a_{12} u_{yy} + a_2 u_{xx} + a_3 u_y u_{xx} + 2a_5 u_x u_{xx} + a_9 u_{yy} u_x + \\ & u_y u_{xy}(2a_4 + a_9) + u_x u_{xy}(2a_{11} + a_3) + u_{xy}(a_6 + a_8) \\ S(\mathbf{div}\,\Pi) = {} & u_y u_{yy}(2a_{10} + a_3) + u_y u_{xy}(2a_4 + a_9) + u_{yy} u_x (2a_5 + a_9) + \\ & + u_{yy}(a_{12} + a_2) + u_x u_{xy}(2a_{11} + a_3) + u_{xy}(a_6 + a_8)\,. \end{aligned}$$

3. Linear expressions in $S(\mathbf{div}\,\Pi)$:

$$L = \{2a_4 + a_9, 2a_{10} + a_3, 2a_{11} + a_3, 2a_5 + a_9, a_6 + a_8, a_{12} + a_2\}\,.$$

4. Basis in $\mathbb{R}^{12}$ for $V = \mathrm{span}(L)$:

$$B = \{e_1, e_2 - e_{12}, e_3 - \frac{1}{2}(e_{10} + e_{11}), e_7, -e_6 + e_8, -\frac{1}{2}(e_4 + e_5) + e_9\}\,.$$

5. Basis of $\Pi[V]$, with trivial laws TL removed:

$$\Pi[B] \setminus TL = \{(-2u_y u_x,\ u_y^2 + u_x^2),\ (-u_y^2/2 - u_x^2/2,\ u_y u_x),\ (u_x,\ -u_y)\}\,.$$

For instance, the second law above is usually interpreted as conservation of mechanical (potential+kinetic) energy; see [12, Ch4]. As Σ is nondegenerate, $\Pi[V] = \mathrm{CL}(\Sigma, Z)$.

In principle, the method can be applied 'as is' also to non leading-linear systems. However, since Proposition 1 cannot be invoked, D-nondegeneracy hence completeness (equality in Corollary 1) may not be guaranteed. Also, identification of trivial laws may be somewhat more laborious.

Example 3. Consider the single PDE system $\Sigma = \{u_y^2 + u_x^2 - 1\}$, a special case of the Eikonal equation. We can fix $D = \{u, u_x, u_y, u_{xy}, u_{xx}, u_{yy}\}$ and consider the ansatz $Z = Y^{\leq 3}$, where $Y = \{u, u_x, u_y\}$. We fix a complete flux template $\Pi = (\pi_1, \pi_2)$ and apply the algorithm with Σ as a Gröbner basis; to this purpose, we fix the lexicographic monomial order induced by $y > x$, so that u_y^2 is the leading term of $u_y^2 + u_x^2 - 1$. Proceeding like in the previous example, we obtain as a result a vector space $\Pi[V] \subseteq \mathrm{CL}(\Sigma, Z)$ (Corollary 1); or better, a concrete basis $\Pi[B]$ of it. In order to identify trivial laws more easily, we normalize the fluxes in $\Pi[B]$, that is, we consider $S\Pi[B] \triangleq \{(Sp, Sq) : (p, q) \in \Pi[B]\}$. Clearly, $\Pi[B]$ and $S\Pi[B]$ are equivalent up to triviality, in particular $S\Pi[B]$ is still a basis of $\Pi[V]$. Removing zeros and trivial laws of the second kind from $S\Pi[B]$, we are left with two laws

$$\{\ (-u_y u_x^2 - 2u_y\ ,\ u_x^3)\ \ ,\ \ (-u_x^3\ ,\ -u_y u_x^2 + u_y)\ \}\ .$$

We remark that Σ is not D-locally solvable.

4 Experiments

We present experimental results obtained from a proof-of-concept Python implementation[5] of the algorithm described in Section 3. We shall refer to this implementation of the algorithm as PolyCons.

We apply PolyCons to some classic PDEs drawn from mathematical physics and presented, in equational form, in Table 1. Some of the original equations have been transformed into lower order equivalent systems: this is beneficial for efficiency, but not strictly necessary for our method to work. An exception is the sine-Gordon (s-G) equation, whose original form is: $u_{tt} - u_{xx} + \sin u = 0$. In this case, the transformation is necessary to remove the transcendent nonlinearity[6] $\sin u$. For a suitable choice of a ranking on $\mathcal{D}$, each of the considered systems is a Riquier basis where all occurring principal derivatives are also leading. Hence Proposition 1, possibly after expansion of Σ via Lemma 2, implies D-nondegeneracy, for appropriate choices of D. For each system, the leading derivatives are those on the left-hand side of the equalities. In particular, for the sine-Gordon equation we consider the graded lexicographic order on $\mathcal{D}$ induced by $x < t$ and $u < c < s$, in which derivatives are first graded by total degree and then lexicographically:

$$u < c < s < u_x < u_t < c_x < c_t < s_x < s_t < \cdots$$

For Euler equations, the (plain or graded) lexicographic order induced by $u > v > w > p$ and $t > x > y > z$ suffices. For the remaining three systems, we consider the following

[5]Code and examples available at `https://github.com/luisacollodi-stud/conservationLaws.git`. Experiments run on a 2.5 GHz Intel Core i5 machine under Windows.

[6]Strictly speaking, the resulting system is not equivalent to the original sine-Gordon equation, as we have replaced $\sin u$ with a generic sinusoid $v = A \sin u + B \cos u$, for A, B arbitrary constants.

Korteweg-de Vries (KdV)	$u_t = -uu_x - u_{xxx}$
Drinfel'd-Sokolov-Wilson (DSW)	$u_t = -3vv_x$ $v_t = -2uv_x - u_x v - 2v_{xxx}$
Boussinesq (Bou)	$u_t = -v_x$ $v_t = -u_x + 3uu_x + u_{xxx}$
sine-Gordon (s-G)	$u_t = -v$ $v_t = -u_{xx} - s$ $s_x = cu_x$ $c_x = -su_x$ $s_t = cu_t$ $c_t = -su_t$
Euler's incompressible fluid (Eul)	$u_t = -(uu_x + vu_y + wu_z + p_x)$ $v_t = -(uv_x + vv_y + wv_z + p_y)$ $w_t = -(uw_x + vw_y + ww_z + p_z)$ $u_x = -(v_y + w_z)$

Table 1: PDEs considered in the experiments.

total order on $\mathcal{D}$:

$$\begin{array}{ccccccccccccccc} & v & < & v_x & < & v_{xx} & < & \cdots & < & u & < & u_x & < & u_{xx} & < & \cdots \\ < & v_t & < & v_{tx} & < & v_{txx} & < & \cdots & < & u_t & < & u_{tx} & < & u_{txx} & < & \cdots \\ < & v_{tt} & < & v_{ttx} & < & v_{ttxx} & < & \cdots & < & u_{tt} & < & u_{ttx} & < & u_{ttxx} & < & \cdots \end{array}$$

This is equivalent to view each derivative ω_τ, with $\omega \in \{u, v\}$ and $\tau \in \{t, x\}^\otimes$, as a monomial $\omega\tau$ and then to consider the lexicographic order induced by $t > u > v > x$ on $\{u, v, t, x\}^\otimes$. This is easily checked to be a ranking.

To frame our experiments in the general context of conservation laws methods, we also provide a comparison with the results obtained on the same examples by applying the direct approach [12, Ch.4]. Generally speaking, methods that follow this approach comprise the following two steps: (1) once a set of indeterminates — independent and differential variables — has been fixed, find all multipliers, that is functions depending on those indeterminates, which, when linearly combined with the equations of the system, yield divergence expressions vanishing on its solutions; (2) invert those divergence expressions to find the corresponding flows. Step (1) is carried out by first applying the variational derivative (a.k.a Euler) operator, which gives conditions on the searched multipliers in terms of linear PDEs, and then solving the resulting PDEs. Step (2) typically relies on homotopy operators, which reduce the inversion problem to computing some 1-dimensional integral. See e.g. [12, 2, 7, 5] for details. Here, we have chosen GeM [5], a state-of-art algorithm also implemented in a Maple package, as a representative of methods based on the direct approach.

Both for GeM and PolyCons the returned laws depend, of course, on the initially chosen set of indeterminates, say Y. This set is used in very different ways by the two algorithms, though: it serves to build multipliers in GeM, and fluxes in PolyCons. Moreover, in the case of PolyCons, one has also to specify a maximum degree of monomials. These differences complicate a direct comparison between the two methods. We prefer to divide the experiments, and the ensuing comparisons, into two parts, depending on whether the same or different sets of indeterminates are used with the two considered algorithms:

1. always use the same set of indeterminates Y with GeM and PolyCons;
2. when Y is used with PolyCons, use $Y \cup Y'$ with GeM, where Y' is obtained from the first derivatives of the indeterminates in Y.

The rationale behind the second form of comparison is the following: if a differential indeterminate occurs in the fluxes, it is likely that some of its first derivatives will occur in the resulting divergence expression, hence in its multipliers. This way, the two algorithms will hopefully be compared on a more equal footing.

4.1 Using the same set of indeterminates

For each experiment, we fix a finite set of indeterminates $Y \subseteq X \cup \mathcal{P}\mathrm{a}(\Sigma)$ and use Y as an ansatz for GeM, and each of $Z = Y^{\leq d}$, for $d = 2, 3, 4$, as possible ansätze for PolyCons. The results of these experiments are reported in Table 2. In the case of PolyCons, the returned sets of laws have been checked to be independent up to triviality, in the sense of Remark 2. In addition to the execution time $\mathbf{t}$ and to the overall number $\mathbf{n}$ of independent laws found, the table also displays the number $\mathbf{n}^*$ of extra laws Φ returned by the considered algorithm but not by the other. Here, Φ must be a *genuinely* new law, that is we require that: (a) Φ is not equivalent, up to triviality, to a linear combination of the laws returned by the other algorithm; and (b) that this holds even after S-normalization of the fluxes, that is rewriting Φ to eliminate the principal derivatives.

We see that $\mathbf{n}^*$ for PolyCons is generally quite large, with a maximum value of 101. On the contrary, $\mathbf{n}^*$ for GeM never exceeds 1. The only reason for a polynomial law not to be found by PolyCons is that one of its fluxes either has a degree higher than d or contains an indeterminate not in Y. For example, in the case of the Boussinesq equation with $Y = \{u, u_x, u_{xx}, v, v_x\}$ and $d = 3$, the only law returned by GeM and not by PolyCons is:

$$\left(-\frac{u^3}{2} + \frac{u^2}{2} - \frac{1}{2}uu_{xx} + \frac{v^2}{2}\ ,\quad -\frac{3}{2}u^2v + \frac{u(2v + u_{tx})}{2} - vu_{xx} - \frac{1}{2}u_xu_t\right)$$

which contains two *principal* derivatives not in the chosen Y, that is u_t and u_{tx}. One can eliminate principal derivatives by normalization, that is by applying the function $S(\cdot)$ to both fluxes, thus getting the equivalent law:

$$\left(-\frac{u^3}{2} + \frac{u^2}{2} - \frac{1}{2}uu_{xx} + \frac{v^2}{2}\ ,\quad -\frac{3}{2}u^2v + \frac{1}{2}u(2v - v_{xx}) + \frac{u_xv_x}{2} - u_{xx}v\right).$$

But this law is not among those returned by PolyCons either, nor is a linear combination of them. In fact, it contains v_{xx}, a parametric derivative which is not in the fixed set Y. A similar case occurs for the sine-Gordon equation. In the case of the Korteweg-de Vries (KdV) equation, the only law returned by GeM and not by PolyCons has degree 5, higher than the maximum $d = 4$ considered for PolyCons.

As an example of law found by PolyCons and not by GeM, consider the case of the KdV equation and $Y = \{t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}\}$. With $d = 4$, exactly 23 extra laws are found, including for example

$$\left(u^3 - 3uu_{xx} - 6u_x^2\ ,\quad \frac{3}{4}u^4 - 12uu_x^2 - 3uu_{xxxx} - 9u_xu_{xxx} + 3u_{xx}^2\right).$$

$\mathbf{\Sigma}$	**Indeterminates** Y	GeM			PolyCons								
					$d = 2$			$d = 3$			$d = 4$		
		$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$	$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$	$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$	$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$
KdV	$\{u, u_x, u_{xx}, u_{xxx}\}$	1.8	3	1	0.4	2	1	1.5	4	2	3.9	6	4
	$\{t, x, u, u_x, u_{xx}, u_{xxx}\}$	3.7	5	1	0.7	2	1	3.4	6	4	17.6	14	10
	$\{t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}\}$	4.1	7	1	1.1	3	2	6.9	11	9	40.1	29	23
DSW	$\{u, u_x, v, v_x, v_{xx}\}$	2.7	2	0	0.7	2	1	2.7	4	2	8.4	5	3
	$\{t, x, u, u_x, v, v_x, v_{xx}, v_{xxx}\}$	3.3	3	0	1.5	3	2	12.3	11	9	95.0	29	26
	$\{t, x, u, u_x, u_{xx}, v, v_x, v_{xx}, v_{xxx}, v_{xxxx}\}$	18.1	4	0	2.9	5	4	33.1	24	21	458.2	78	74
Bou	$\{u, u_x, u_{xx}, v, v_x\}$	3.3	4	1	0.6	4	2	2.6	7	3	7.8	8	5
	$\{t, x, u, u_x, u_{xx}, u_{xxx}, v, v_x\}$	3.7	7	1	1.4	7	5	10.6	20	16	81.6	40	34
	$\{t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, v, v_x, v_{xx}\}$	20.7	10	1	2.8	13	11	32.4	43	36	441.7	110	101
s-G	$\{u, u_x, v, c\}$	3.9	2	0	0.4	5	5	1.3	8	8	3.4	12	10
	$\{u, u_x, u_{xx}, v, v_x, c\}$	4.7	4	1	1.1	8	8	6.4	14	14	20.7	24	21
	$\{t, x, u, u_x, u_{xx}, v, v_x, c\}$	9.0	4	1	1.9	12	12	17.4	32	32	89.2	69	66
Eul	$\{u, v, w, p\}$	2.1	5	0	0.04	4	0	2.8	5	0	8.7	5	0

Table 2: Comparison between GeM and PolyCons using the same set of indeterminates Y. Legenda: $\mathbf{t}$ = execution time in seconds, $\mathbf{n}$ = number of independent laws found, $\mathbf{n}^*$ = number of independent laws found by one algorithm but not by the other (see text), d = degree of the complete polynomial template for PolyCons.

For another example, consider the Boussinesq equation with $Y = \{t, x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, v, v_x, v_{xx}\}$ and $d = 4$. One of the 101 laws found by PolyCons and not by GeM is

$$\left(u_x^4 + 3uu_x^2 u_{xx} \;\; , \;\; 3uu_x v_{xx} + u_x^3 v_x\right) .$$

Euler's equations (Eul) govern the flow of an inviscid, incompressible fluid in three spatial dimensions x, y, z and time t: here, $\mathbf{u} = (u, v, w)$ is the fluid's velocity vector and p is its pressure. With $Y = \{u, v, w, p\}$, among the laws found by both GeM and PolyCons — in the latter case already with $d = 3$ — we have

$$\left(u^2 + v^2 + w^2,\ 2pu + u^3 + uv^2 + uw^2 \;\; , \;\; 2pv + u^2 v + v^3 + vw^2,\ 2pw + u^2 w + v^2 w + w^3\right) .$$

This is the conservation of kinetic energy, more commonly expressed in divergence form, vectorially:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{u}|^2\right) + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\mathbf{u} + p\mathbf{u}\right) = 0$$

where $\nabla\cdot$ and $|\cdot|$ are the gradient and 2-norm of a vector, respectively.

$\mathbf{\Sigma}$	GeM				PolyCons	$d=2$			$d=3$			$d=4$		
	Indet. Y_{GeM}	$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$	**Indet.** Y	$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$	$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$	$\mathbf{t}$(s)	$\mathbf{n}$	$\mathbf{n}^*$
KdV	$\{u, u_x, u_{xx}, u_{xxx}, u_{xxxx}\}$	2.6	3	1	$\{u, u_x, u_{xx}, u_{xxx}\}$	0.5	2	1	1.3	4	2	3.1	6	4
DSW	$\{u, u_x, u_{xx}, v, v_x, v_{xx}, v_{xxx}, v_{xxxx}\}$	12.3	3	1	$\{u, u_x, v, v_x, v_{xx}, v_{xxx}\}$	1.0	3	2	5.3	7	5	24.0	12	10
Bous	$\{u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, v, v_x, v_{xx}\}$	13.0	4	1	$\{u, u_x, u_{xx}, u_{xxx}, v, v_x\}$	0.8	5	3	4.3	10	7	19.5	14	11
s-G	$\{u, u_x, u_{xx}, v, v_x, c\}$	4.0	4	1	$\{u, u_x, v, c\}$	0.2	5	5	1.1	8	8	3.2	12	11
Eul	$\{u, v, w, p, u_y, u_z, v_x, v_y, v_z, w_x, w_y, w_z, p_t, p_x, p_y, p_z\}$	3.5	6	1	$\{u, v, w, p\}$	0.04	4	0	2.8	5	0	8.7	5	0

Table 3: Comparison between GeM and PolyCons using different sets of indeterminates, Y and Y_{GeM}. Legenda: see Table 2.

Overall, execution times grow as expected with the cardinality of Y and, in the case of PolyCons, also with the template degree d. Execution times of PolyCons are generally higher than GeM's. This is compensated by the large number of extra laws $\mathbf{n}^*$ found by PolyCons. The large values of $\mathbf{n}^*$ also suggest that a more fair comparison between GeM and PolyCons should take into account the different roles played by the indeterminates set Y in the two approaches. This is explored in the next subsection.

4.2 Using different sets of indeterminates

Given a set of indeterminates Y for PolyCons, we will let the ansatz Y_{GeM} for GeM consist of Y plus all the first derivatives of the indeterminates in Y, normalized. Explicitly

$$Y_{\text{GeM}} \stackrel{\triangle}{=} Y \cup \{u_\tau : u_\tau \ \text{ occurs in } S(v_{\xi x_i}), \text{ for some } v_\xi \in Y,\ x_i \in X\}\,.$$

The results of the new experiments are reported in Table 3. The comparison gives now more equilibrate results in terms of execution times, which are comparable. Yet, PolyCons still consistently finds more conservation laws than GeM, although $\mathbf{n}^*$ never exceeds 11 now. In each case, there is precisely one law returned by GeM and not by PolyCons: for KdV this is a degree 5 law, in the other four cases the fluxes found by GeM, even after normalization, contain some derivative not belonging to Y.

Overall, the above comparison indicates that, when confining to polynomial laws, GeM and PolyCons somehow complement with each other. In principle, GeM finds all conservation laws with multipliers built out of a given ansatz of indeterminates. PolyCons finds all conservation laws with fluxes built out of a given ansatz of indeterminates, up to a given degree. The completeness result of PolyCons is seemingly less strong, because of the upper bound on the degree, but more direct, because expressed in terms of fluxes. More generally, it is unclear how the completeness of the direct approach in terms of multipliers translates into completeness in terms of fluxes. In other words, it is unclear what is a finite ansatz for multipliers, yielding all laws with polynomial fluxes built out of a given Y. Our attempt with Y_{GEM} falls short of achieving this goal, as seen from the nonzero values of $\mathbf{n}^*$ for PolyCons. Note that, at least at an initial stage of exploration of a PDE system, one might want to reason more naturally in terms of fluxes, rather than in terms of multipliers. In this respect, PolyCons's higher value of $\mathbf{n}^*$ points at least to a practical advantage over the multiplier-based direct approach.

A profiling of the code of PolyCons shows that the most time-consuming phase of its execution is by far the application of the substitution that solves the linear system to the original flux templates Π (step 4). This suggests that there is much room for improvement, by devising data structures for polynomials that support efficient linear substitution operations.

5 Analyticity

The definitions, results and algorithm of sections 2 and 3 carry over to real analytic solutions of PDEs: essentially, all we have to do is to replace the word "formal" with the word "analytic" throughout the sections. The only difference is in Proposition 1, where we need to strengthen the conditions on Σ to ensure existence of analytic solutions of the involved initial value problems. To this purpose, we shall rely on Riquier's analyticity theorem. We introduce the necessary definitions and the analytic counterpart of Proposition 1 below.

A ranking $>$ over $\mathcal{D}$ is *weakly orderly* if whenever $|\tau| > |\xi|$ then $u_\tau > u_\xi$ for each u; it is a *Riquier ranking* if whenever $u_\tau > u_\xi$ for some u then $v_\tau > v_\xi$ for all v. A *passive orthonomic* system is a Riquier basis with the following two additional properties: (a) for each element $u_\tau + f \in \Sigma$ (u_τ leading), f does not contain principal derivatives; (b) whenever u_τ and u_ξ are distinct leading derivatives, neither $\tau \leq \xi$ nor $\xi \leq \tau$ hold true. The following is the analytic counterpart of Proposition 1; the straightforward proof is reported in Appendix A.2.

Proposition 2 (analytic D-nondegeneracy). *Let Σ be passive orthonomic w.r.t. a ranking which is Riquier and weakly orderly. Assume $D \cap \mathcal{P}\mathrm{r}(\Sigma) = \mathrm{dom}(\Sigma)$. Then Σ is analytically D-nondegenerate. Moreover, Σ is a Gröbner basis of $\mathcal{I}(\mathcal{V}(\Sigma))$ w.r.t. a suitable monomial order.*

All the examples considered in the previous section are seen to be passive orthonomic for some suitable ranking which is Riquier and weakly orderly. More precisely, for the sine-Gordon and Euler systems, one considers the same graded lexicographic rankings introduced in the previous section. On the contrary, the already introduced ranking for the KdV, DSW and Boussinesq fails to be weakly orderly: one can instead consider the graded lexicographic ranking induced by $u > v$ and $t > x$. Obviously, a change in the ranking also leads to a change in the leading derivatives of these systems.

Note that, for a system that is both formally and analytically D-nondegenerate, the sets of formal and analytic invariant polynomials in the ring $\mathcal{P}$ coincide with $\langle\, \Sigma \,\rangle$ (Lemma 1). Hence the set $\Pi[V] = CL(\Sigma, Z)$ (Corollary 1), considered in a formal or analytic sense, is just the same. This gives one some freedom in the choice of the ranking when it comes to actually computing $\Pi[V]$. Of course, the concrete representation of $\Pi[V]$, that is the the basis $\Pi[B]$ concretely returned by the algorithm, does depend on the chosen ranking. Moreover, identifying and filtering equivalent and trivial laws out from $\Pi[B]$ may be non obvious, in case principal derivatives w.r.t. the chosen ranking occur in Z (see Remark 2).

6 Conclusion

We have put forward a method to compute PDE polynomial conservation laws. Under a certain nondegeneracy condition, the method is complete, relatively to a user specified polynomial template for fluxes. Computationally, the proposed method is based entirely on equational rewriting and linear algebraic operations. This should be contrasted with the direct approach, that heavily relies on variational tools (Euler operator), coupled with linear PDE solving and and symbolic integration (homotopy). The simplicity of our method's underlying principles is, we believe, an additional benefit in terms of the audience that can be reached.

In Section 4, we have discussed at length the differences between our work and the direct approach. Somewhat halfway between our method and the direct approach, one might place the work of W. Hereman and collaborators based on scaling symmetries [8, 13]. Like in our case, their starting point is a polynomial template. In their case, though, the template represents a candidate density, that is a flux corresponding to time. Moreover, only monomials invariant under the same scaling symmetry of the PDE are involved in the template. Taking the time total derivative of the candidate density, they seek conditions on the unknown coefficients for the resulting expression to be a (spatial) divergence: this is done by equating its variational derivative to zero and forming a linear system for the unknown coefficients. The solution of the system is then substituted into the spatial divergence expression, and homotopy operators are used to recover the spatial fluxes. The methods has limitations, in that only applies to evolution equations and requires the existence of scaling symmetries.

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A Proofs

A.1 Proof of Corollary 1

We can conservatively extend any given monomial order $\prec$ on $(X \cup D)^{\otimes}$ by introducing an *elimination order* $\prec_{\text{el}}$ on $(a \cup X \cup D)^{\otimes}$, as follows. First, order the elements in a as $a_1 < \cdots < a_s$. Then, for $\alpha, \alpha' \in a^{\otimes}$ and $\beta, \beta' \in (X \cup D)^{\otimes}$, we let $\alpha\beta \prec_{\text{el}} \alpha'\beta'$ if and only if either $\alpha \prec_{\rm lex} \alpha'$ or ($\alpha = \alpha'$ and $\beta \prec \beta'$). In other words, monomials are first compared lexicographically in their a-part and in case of ties, in their $(X \cup D)$-part according the the original order. Any $\Delta \subseteq \mathbb{R}[X \cup D]$ that is a Gröbner basis in the ring $\mathcal{P} = \mathbb{R}[X \cup D]$ w.r.t. the original order is also a Gröbner basis in the larger ring $\mathbb{R}[a \cup X \cup D]$ w.r.t. the elimination order: this is an immediate consequence of Buchberger's Criterion [6, Ch.2,§6,Th.6], as the S-polynomials of Δ w.r.t. the original order and the elimination order coincide.

Proof of Corollary 1. First, by the linearity of total derivative, it is easily seen that for each $x_i \in X$, template π and $v \in \mathbb{R}^s$: $\mathrm{D}_{x_i}(\pi[v]) = (\mathrm{D}_{x_i}\pi)[v]$. Now, in order to proof the inclusion in the statement, consider any $v \in V$. We have:

$$
\begin{aligned}
S(\mathbf{div}\,(\Pi[v])) &= S(\sum_i \mathrm{D}_{x_i}(\pi_i[v]))\\
&= S((\sum_i \mathrm{D}_{x_i}\pi_i)[v])\\
&= S((\mathbf{div}\,\Pi)[v])\\
&= (S(\mathbf{div}\,\Pi))[v] \qquad (1)\\
&= 0 \qquad (2)
\end{aligned}
$$

where (1) follows from Lemma 3 and (2) from the definition of V. Therefore $\mathbf{div}\,(\Pi[v]) \in \langle\, \Sigma \,\rangle$ and, by Lemma 1, $\mathbf{div}\,(\Pi[v]) \in \mathcal{I}nv(\Sigma)$. Assume now that $\langle\, \Sigma \,\rangle = \mathcal{I}(\mathcal{V}(\Sigma))$, we prove the reverse inclusion. Let $(p_1, ..., p_n) \in \mathrm{CL}(\Sigma, Z)$. By assumption, there is $v \in \mathbb{R}^s$ such that $(p_1, ..., p_n) = (\pi_1[v], ..., \pi_n[v]) = \Pi[v]$. We prove that $v \in V$, that is $(S(\mathbf{div}\,\Pi))[v] = 0$. Indeed

$$
\begin{aligned}
(S(\mathbf{div}\,\Pi))[v] &= S((\mathbf{div}\,\Pi)[v]) \qquad (3)\\
&= S((\sum_i \mathrm{D}_{x_i}\pi_i)[v])\\
&= S(\sum_i \mathrm{D}_{x_i}(\pi_i[v]))\\
&= S(\mathbf{div}\,(\Pi[v]))\\
&= 0 \qquad (4)
\end{aligned}
$$

where (3) follows again from Lemma 3 and (4) from the fact that by hypothesis $\mathbf{div}\,(\Pi[v]) \in \mathcal{I}nv(\Sigma)$, hence $\mathbf{div}\,(\Pi[v]) \in \langle\, \Sigma \,\rangle$ by Lemma 1. $\square$

A.2 Proof of Proposition 2

We first introduce Riquier's analyticity theorem. An *initial data specification* $\rho : \mathcal{P}\mathrm{a}(\Sigma) \longrightarrow \mathbb{R}$ is said to be *analytic at* $x^0 \in \mathbb{R}^n$ if, for each $u \in \mathcal{U}$, the following power series[7] defines a real analytic function of x in a neighborhood of x^0

$$u^\rho(x) \triangleq \sum_{u_\zeta \in \mathcal{P}\mathrm{a}(\Sigma)} \frac{\rho(u_\zeta)}{\zeta!}(x - x^0)^\zeta\,. \qquad (5)$$

[7] Here for $\zeta = x_1^{k_1} \cdots x_n^{k_n}$ we let $\zeta! \triangleq k_1! \cdots k_n!$.

We report below Riquier's analyticity theorem, seen as a special case of the version given by Lemaire [9, Th.1].

Theorem A.1 (Riquier's analyticity theorem). *Let Σ be passive orthonomic w.r.t. a ranking which is Riquier and weakly orderly. For any initial data specification ρ analytic at x^0 there is a unique analytic solution U of Σ around x^0 such that $\frac{\partial^{|\tau|}}{\partial\tau}U^i(x^0) = \rho(u^i_\tau)$ for each $u^i_\tau \in \mathcal{P}\mathrm{a}(\Sigma)$.*

Proof of Proposition 2. We show that Σ is D-locally solvable, analytically. Partition D as $D = D_0 \cup D_1$, where D_1 are the leading derivatives of Σ, and D_0 are the remaining derivatives, which must therefore be parametric. Let $(x^0, u^0_D) = (x^0, u^0_{D_0}, u^0_{D_1}) \in \mathcal{V}(\Sigma)$. Below, we will denote the component of u^0_D corresponding to $u^i_\tau \in D$ by $u^{0,i}_\tau$. Define an initial data specification $\rho : \mathcal{P}\mathrm{a}(\Sigma) \longrightarrow \mathbb{R}$ as follows:

$$\rho(u^i_\tau) \triangleq \begin{cases} u^{0,i}_\tau & \text{if } u^i_\tau \in D_0 \\ 0 & \text{otherwise.} \end{cases}$$

For each $u \in \mathcal{U}$, the corresponding function $u^\rho(x)$ defined in (5) is clearly analytic around x^0 — in fact, a polynomial — hence ρ is analytic at x^0. Riquier's Theorem A.1 ensures therefore the existence of an analytic solution of the corresponding initial value problem, that is an analytic function $U = (U^1, ..., U^m)$ around x^0 such that $\frac{\partial^{|\tau|}}{\partial\tau}U^i(x^0) = \rho(u^i_\tau)$ for each $u^i_\tau \in \mathcal{P}\mathrm{a}(\Sigma)$. In particular, for each $u^i_\tau \in D_0$, we have $\frac{\partial^{|\tau|}}{\partial\tau}U^i(x^0) = u^{0,i}_\tau$. In other words, $U_{D_0}(x^0) = u^0_{D_0}$.

Consider now any leading derivative $u^i_\tau \in D_1$, with $u^i_\tau + f \in \Sigma$. By assumption, $f = f(x, u_{D_0})$, that is f does not depend on u_{D_1}. Then we have

$$\begin{aligned} \frac{\partial^{|\tau|}}{\partial\tau}U^i(x^0) &= -f(x^0, U_{D_0}(x^0)) \\ &= -f(x^0, u^0_{D_0}) \\ &= u^{0,i}_\tau \end{aligned}$$

where the first equality follows because U is a solution of Σ, the second one from $U_{D_0}(x^0) = u^0_{D_0}$ and the last one because $(x^0, u^0_D) \in \mathcal{V}(\Sigma)$, hence $u^{0,i}_\tau + f(x^0, u^0_D) = 0$. Overall, we have shown that $U_D(x^0) = u^0_D$. Since $(x^0, u^0_D) \in \mathcal{V}(\Sigma)$ is arbitrary, this proves that Σ is D-locally solvable, analytically.

The rest of the proof is exactly like that of Proposition 1. □