Kerion Framework: A Novel Approach to Describe Physical Phenomena with Fractal Dynamics

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Abstract

We develop a Kerion framework that captures physical processes as fractal dynamics embedded in a twelve–dimensional vector space. Each classical quaternion coefficient $a, b, c, d \in \mathbb{R}$ is upgraded to a three–vector, yielding the coefficient ring $\mathbb{K} \cong \mathbb{R}^{12}$. The twelfth dimension records the entire evolutionary history of the first eleven coordinates, endowing the model with intrinsic memory. Coefficients live in a Banach algebra of absolutely convergent series; their Cauchy product guarantees well–defined multiplication. A sequence of iterated–function systems (IFS) along the persistence axis supplies algorithmic richness, while set–theoretic union—*under mild overlap conditions*—conserves both self–similarity and perfection and acts as the evolutionary operator. Upgrading the Banach framework to a Hilbert space equips the model with an inner–product structure and a rich operator algebra, features that *permit*—but do not by themselves entail—a quantum mechanics. The resulting model exhibits unbounded memory, self–execution, and norm–preserving dynamics, offering a concise mathematical foundation for natural fractal phenomena.

Contents

1 Introduction

Geometric patterns that repeat across scales—*fractals*—pervade the natural world. Snowflakes, coastlines, and galactic filaments all display self-similar structure. High-energy experiments challenge the notion of indivisible particles, hinting at an ever finer internal structure and motivating a mathematical framework that eschews strict atomicity. M-theory's eleven coordinates already unify disparate physical interactions; we posit an additional *persistence* dimension that records every state, endowing the universe with an intrinsic memory.

2 Kerion Algebra

2.1 Definition

Let

$$\mathfrak{g} := \mathfrak{su}(2)_{(1)} \oplus \mathfrak{su}(2)_{(2)} \oplus \mathfrak{su}(2)_{(3)}$$

and choose the dominant integral weight $\lambda = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ (one fundamental of each $\mathfrak{su}(2)$ factor). Attach a "grey node" carrying λ to the Dynkin diagram of $\mathfrak{g}$ to obtain the **tensor-hierarchy algebra**

$$T(\mathfrak{g}, \lambda) = \bigoplus_{p \in \mathbb{Z}} T_p, \qquad [T_p, T_q] \subseteq T_{p+q}.$$

Degree +1. The level $p = +1$ carries the 12-dimensional representation

$$T_{+1} \cong R_\lambda = \bigoplus_{r=1}^{3} \mathbf{4}_{(r)} = \mathrm{Span}_{\mathbb{R}}\{E_\mu^{(r)} \mid \mu = 0, 1, 2, 3;\ r = 1, 2, 3\},$$

where, for each block r, $E_0^{(r)}$ is a singlet and $\{E_1^{(r)}, E_2^{(r)}, E_3^{(r)}\}$ transform as the adjoint of $\mathfrak{su}(2)_{(r)}$.

Kerion space. We *define the Kerion algebra*[1] as the degree-+1 component of the THA,

$$\boxed{\mathbb{K} \;:=\; T_{+1} \;\cong\; \mathbb{R}^{12}}$$

with the basis above identified, for convenience, with the physical coordinates

$$(x, y, z,\ t, \pi, \rho,\ c_1, c_2, c_3,\ \tilde{c}_1, \tilde{c}_2, \tilde{c}_3) \;\longleftrightarrow\; \left(E^{(1)}_{0,1,2,3}, E^{(2)}_{0,1,2,3}, E^{(3)}_{0,1,2,3}\right).$$

Product via the embedding tensor. The grey-node relations of the THA provide a canonical bilinear map

$$\Theta :\ \mathbb{K} \otimes \mathbb{K} \;\longrightarrow\; T_0 \cong \mathfrak{g}, \qquad \Theta(E_\mu^{(r)}, E_\nu^{(s)}) = \delta_{rs} \begin{cases} 0, & \mu = 0 \text{ or } \nu = 0, \\ 2\,\varepsilon_{\mu\nu\kappa}\, J_\kappa^{(r)}, & \mu, \nu \in \{1, 2, 3\}, \end{cases}$$

with $J_\kappa^{(r)}$ the generators of $\mathfrak{su}(2)_{(r)}$.

(H1) Graded Jacobi identity. The tensor-hierarchy algebra $\mathfrak{T} = \bigoplus_{k \in \mathbb{Z}} E^{(k)}$ equipped with the bracket $[\,,\,]$ satisfies the graded Jacobi relation for all homogeneous $x, y, z \in \mathfrak{T}$:

$$(-1)^{|x||z|}[x, [y, z]] \;+\; (-1)^{|y||x|}[y, [z, x]] \;+\; (-1)^{|z||y|}[z, [x, y]] \;=\; 0.$$

(H2) Quadratic (closure) constraint on Θ. We denote the nondegenerate invariant form on $\mathfrak{g}$ by k_{PQ} with inverse k^{PQ}. Then the fully covariant and mixed-index versions of the embedding tensor are given by

$$\Theta_{MN,P} \;=\; k_{PQ}\, \Theta_{MN}{}^{Q}, \qquad \Theta_{M}{}^{N}{}_{P} \;=\; k^{NQ}\, \Theta_{MQ}{}^{P}.$$

Here M, N, Q are covariant (lower) indices labeling the two copies of T, and P, R are contravariant (upper) indices labeling the output in $\mathfrak{g}$.

Then your quadratic constraint reads

$$\Theta_{MN}{}^{P}\, \Theta_{PQ}{}^{R} \;=\; 0, \qquad M, N, Q \text{ covariant}, \quad P, R \text{ contravariant}.$$

[1]From now on we write $\mathbb{K}$ instead of $\mathbb{H}^{(12)}$.

with the understanding that M, N, Q are subscripts (covariant) and P, R are superscripts (contravariant). Let

$$\Theta \,:\, T_{+1} \otimes T_{+1} \longrightarrow \mathfrak{g}$$

be the (symmetric, $\mathfrak{g}$-equivariant) embedding tensor. We require:

$$\Theta_{MN}{}^{P}\,\Theta_{PQ}{}^{R} \;=\; 0, \quad\Longleftrightarrow\quad [\,\Theta(u,v),\,w\,] \;+\; \Theta\big(u,\,v{\cdot}w\big) \;=\; 0, \quad \forall\, u,v,w \in T_{+1}, \tag{2}$$

plus

$$[\,\Im\Theta,\;\Im\Theta\,] \;=\; 0 \quad : \big(\Im\Theta \subset \mathfrak{g} \text{ is an Abelian ideal}\big).$$

With **(H1)–(H2)** in place, The proof of associativity follows from Palmkvist [20, Prop. 2.5] and Lavau–Samtleben–Trentin [21, §4]. implies that the Kerion product is associative and unital. All results that follow in Sections 2.2–2.7 therefore remain valid.

Under hypotheses **(H1)**–**(H2)**, define

$$u{\cdot}v \;:=\; [\,\Theta(u,v),\,v\,] \;\in\; T_{+1}. \tag{3}$$

Then this product is associative:

$$u{\cdot}\big(v{\cdot}w\big) \;=\; \big(u{\cdot}v\big){\cdot}w, \quad \forall\, u,v,w \in T_{+1}.$$

Closure is immediate because T_{+1} is a $\mathfrak{g}$-module. For associativity compute

$$u \cdot (v \cdot w) \;=\; \big[\Theta\big(u, [\Theta(v,w),w]\big),\, [\Theta(v,w),w]\big].$$

Insert (1) with $u \leftrightarrow v$ and use the graded Jacobi identity:

$$\Big[[\Theta(u,v),v],w\Big] \;=\; (\,u \cdot v\,) \cdot w.$$

Hence $u \cdot (v \cdot w) = (u \cdot v) \cdot w$.

This lemma is a direct corollary of Palmkvist [20, Prop. 2.5] and was later reviewed in Lavau–Samtleben–Trentin [21, §4]. Either reference may be cited in lieu of the explicit proof above.

Associativity and unit. Because the THA satisfies the graded Jacobi identities, (3) obeys the graded–Leibniz identity, and—with the chosen block-diagonal Θ—reduces to the familiar quaternion multiplication inside each block and annihilates cross-block products. Explicitly,

$$E_i^{(r)} \cdot E_j^{(r)} = -\delta_{ij} E_0^{(r)} + \varepsilon_{ijk} E_k^{(r)}, \qquad E_0^{(r)} \cdot E_\mu^{(r)} = E_\mu^{(r)}, \qquad E_\mu^{(r)} \cdot E_\nu^{(s)} = 0 \; (r \neq s).$$

Hence the Leibniz identity collapses to associativity on the $\mathbb{K}$-ideal generated by $E_\mu^{(r)}$.

Cross–block vanishing of Θ. The embedding tensor $\Theta : T_1 \otimes T_1 \longrightarrow T_0$ is $\mathfrak{g}$-equivariant and of weight 0. Decomposing the level-1 space as $T_1 = T^{(1)} \oplus T^{(2)} \oplus T^{(3)}$, Schur's lemma forces Θ to be block-diagonal:

$$\Theta\Big(E_\mu^{(r)}, E_\nu^{(s)}\Big) = \delta_{rs}\, \Theta_{\mu\nu}^{(r)} \in T_0, \qquad 1 \le r, s \le 3.$$

Hence, for $r \neq s$,

$$E_\mu^{(r)} \cdot E_\nu^{(s)} = \big[\Theta(E_\mu^{(r)}, E_\nu^{(s)}),\, E_\nu^{(s)}\big] = 0,$$

so all cross-block products vanish. It follows immediately that $1_{\mathbb{K}} := \sum_{r=1}^{3} E_0^{(r)}$ acts as a two-sided identity in $\mathbb{K}$.

$$1_{\mathbb{K}} \cdot E_\mu^{(s)} = E_\mu^{(s)} = E_\mu^{(s)} \cdot 1_{\mathbb{K}} \quad \text{for all } s, \mu.$$

This tensor-hierarchy realisation reproduces the classical block-diagonal quaternion product *inside* a graded-Lie framework, allowing seamless coupling to extended-geometry or Kac–Moody constructions that are already formulated in THA language.

Persistence-Axis Basis

Recall that the level +1 component of the tensor-hierarchy algebra T_{+1} carries a 12-dimensional basis

$$\{E_\mu^{(r)}\}_{r=1,2,3,\ \mu=0,1,2,3} \;=\; \{e_1,\, e_2,\, \ldots,\, e_{12}\},$$

where the vector

$$e_{12} \;=\; E_3^{(3)}$$

plays the role of the "persistence" direction. In particular, the two-sided identity in the algebra can be written as the sum of the three quaternion-block units,

$$1_T \;=\; \sum_{r=1}^{3} E_0^{(r)} \;=\; e_4 \;+\; e_8 \;+\; e_{12}\,.$$

2.2 Series–Valued Coefficients in a Banach Setting

Replace each scalar coefficient by an absolutely convergent series $s = (a_n)_{n\geq 0} \in \ell_1(\mathbb{R})$ with $\|s\|_1 = \sum_{n\geq 0} |a_n| < \infty$. Because $\ell_1(\mathbb{R})$ is a Banach algebra under the Cauchy product,

$$\|s * t\|_1 \leq \|s\|_1 \|t\|_1$$

quaternion multiplication extends continuously.

Projective tensor norm. For normed spaces E, F and $u = \sum_{k=1}^{m} e_k \otimes f_k \in E \otimes F$ set $\|u\|_\pi = \inf \sum_k \|e_k\|_E \|f_k\|_F$; its completion is $E \widehat{\otimes}_\pi F$. If A is a Banach algebra then $\ell_1 \widehat{\otimes}_\pi A$ is a Banach algebra.

The Euclidean norm on quaternions is multiplicative:

$$|pq|_2 = |p|_2\, |q|_2 \forall p, q \in \mathbb{H} \tag{2.2}$$

Write $p = a + \mathbf{b},\ q = c + \mathbf{d}$ with $\mathbf{b}, \mathbf{d} \in \mathbb{R}^3$. A direct calculation gives

$$|pq|_2^2 = (a^2 + |\mathbf{b}|^2)(c^2 + |\mathbf{d}|^2) = |p|_2^2 |q|_2^2.$$

Hence $\ell_1 \widehat{\otimes}_\pi \mathbb{H}$ inherits a sub-multiplicative norm and is complete.

2.3 Perfect Fractal Series $\mathcal{F}_{\text{perf}}$

$\mathcal{F}_{\text{perf}}$ consists of all non-empty, compact, perfect, self-similar sets $F \subset \mathbb{R}^n$ that are attractors of a *finite* IFS $S = \{S_i\}_{i=1}^{N} \subset \mathrm{Sim}(\mathbb{R}^n)$ satisfying the OSC. If $\{F_i\}_{i\in I} \subset \mathcal{F}_{\text{perf}}$ are generated by the *same* IFS S and have uniformly bounded diameters, then $K := \bigcup_{i\in I} F_i$ is compact, perfect and self-similar, so $K \in \mathcal{F}_{\text{perf}}$. (The uniform bound prevents loss of compactness.) We define $\mathcal{F}_{\text{perf}}$ to be the set of all non–empty, compact, *perfect*, self–similar subsets of $\mathbb{R}^n$ that arise as the attractor of a finite iterated–function system (IFS)

$$S = \{S_i\}_{i=1}^{N} \subset \mathrm{Sim}(\mathbb{R}^n)$$

satisfying the *open set condition* (OSC). Recall that the OSC requires the existence of a non–empty open set $O \subset \mathbb{R}^n$ such that $S_i(O) \subset O$ and $S_i(O) \cap S_j(O) = \varnothing$ whenever $i \neq j$. The triple of properties

- **Self–similarity:** $F = \bigcup_{i=1}^{N} S_i(F)$,
- **Perfection:** every point of F is a limit point of F,
- **OSC:** separation control of the pieces,

guarantees that F possesses a well–defined Hausdorff dimension determined by the Moran equation and that this dimension is preserved under rigid motions.

The family $\mathcal{F}_{\text{perf}}$ is *closed* under rigid motions and under finite (and countable) unions of members generated by the *same* IFS, because the OSC extends to such unions. Standard codings of compact subsets of $[0, 1]$ by infinite binary trees show that $|\mathcal{F}_{\text{perf}}| = \mathfrak{c}$, the cardinality of the continuum.

To avoid loss of compactness we restrict henceforth to unions $\bigcup_{i\in I} F_i$ whose diameters are *uniformly bounded*. Under this assumption the union of a countable OSC family remains compact, hence stays inside $\mathcal{F}_{\text{perf}}$. See Falconer [22, Thm. 3.3] for the required dimension estimates.

2.4 Mathematical Notation

Write any $q \in \mathbb{K}$ as

$$q = \sum_{r=1}^{3} \left(a_r + b_r i + c_r j + d_r k\right)_r \equiv \underbrace{(x, y, z)}_{\text{macroscopic space}} + \underbrace{(t, \pi, \rho)\, i}_{\text{time / persistence / radius}} + \underbrace{(c_1, c_2, c_3)\, j}_{\text{Calabi–Yau real}} + \underbrace{(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)\, k}_{\text{Calabi–Yau imaginary}}$$

Multiplication reduces to three simultaneous quaternion products.

2.5 Algorithmic Perspective

All contractions are taken with respect to the projective norm on H^{ℓ^1} so that the Banach fixed–point theorem guarantees a unique attractor for every contractive IFS
T_i acting on that space. Let $H^{\ell^1} = \ell^1(\mathbb{R})\widehat{\otimes}_\pi \mathbb{H}$. Define a sequence $(F_n)_{n\in\mathbb{N}}$ of attractors by $F_{n+1} = \Phi_n(F_n)$, where each Φ_n is a finite family of contracting similarities on H^{ℓ^1}. Classic fractals occupy fixed indices; e.g. F_j is the quaternionic Mandelbrot set $M_{\mathbb{H}}$.

2.6 Self–Execution Mechanism

Fix an *injective* map

$$\kappa : \mathcal{F}_{\text{perf}} \to (0,1), \qquad \kappa\big((i_k)_{k\geq 1}\big) = 0.i_1 i_2 i_3 \ldots_2\,,$$

where the binary expansion is always chosen in its *terminating* form (no infinite tail of 1's), eliminating the usual ambiguity. Group the twelve coefficients at epoch n as before and set $q_n = \varphi_0(n) + \varphi_1(n)i + \varphi_2(n)j + \varphi_3(n)k$. Each unary operator u lifts to $\hat{u}(q) := (u(q), q)$, appending immutable metadata.

2.7 Upgrading to a Hilbert Space

Let $G = \mathbb{Z}$ and $H = \ell_2(G, \mathbb{C})$ with convolution operators L_f. Form $\widehat{H} = \mathbb{K}\widehat{\otimes} H$ and

$$\mathcal{A} = \left\{h \otimes L_f : h \in \mathbb{H}^{(12)},\ f \in \ell_1(G) \cap \ell_2(G)\right\} \subset B(\widehat{H}).$$

For every $h \in \mathbb{K}$, $L_h(q) = hq$ is bounded with $\|L_h\| = |h|_2$. Proposition 2.2 gives $|hq|_2 = |h|_2 |q|_2$. Thus $\mathcal{A}$ is a $*$-subalgebra of bounded operators, and every $h \otimes L_{\delta_k}$ with $|h|_2 = 1$ is unitary.

Unitary elements. Left multiplication by a *unit* quaternion h preserves the Euclidean norm on $\mathbb{K} \cong \mathbb{R}^{12}$ because $|hq| = |q|$ for every $q \in \mathbb{K}$. After complexification this map is therefore *unitary*. On the sequence side, L_{δ_k} is the bilateral shift on $\ell^2(\mathbb{Z})$ and is unitary. Hence each simple tensor $h \otimes L_{\delta_k}$ with $|h| = 1$ is unitary on $\widehat{\mathcal{H}}$, furnishing the model with norm–preserving dynamics.

3 Union as an Evolutionary Operator

3.1 Union and Fractal Stability

Let S be a finite OSC–IFS with attractor F. For any countable family $\{w_i\}_{i\in I}$ of words in S with uniformly bounded $\operatorname{diam} w_i(F)$,

$$\dim_H\Big(\bigcup_{i\in I} w_i(F)\Big) = \dim_H F.$$

3.2 Requirements for Self–Similarity

Because each w_i is a composition of maps in S, $K = \bigcup_{i\in I} w_i(F)$ satisfies $K = \bigcup_{j=1}^{N} S_j(K)$, so K is an OSC attractor and self-similar.

3.3 Requirements for Perfection

A union of compact perfect sets is compact and perfect: for any $x \in K$ the component containing x accumulates at x.

3.4 Union as the Model's Evolutionary Feature

For $F \in \mathcal{F}_{\text{perf}}$ let T_τ translate by τ along the persistence axis and define

$$G(F) := \begin{cases} F, & F \in \mathcal{F}_{\text{perf}}, \\ \operatorname{proj}_{\mathcal{F}_{\text{perf}}}(F), & \text{otherwise}, \end{cases}$$

where the projection uses the Hausdorff metric and breaks ties lexicographically on binary addresses. Set

$$U_\tau(a) = \kappa\big(G(\kappa^{-1}(a)) \cup T_\tau G(\kappa^{-1}(a))\big), \qquad a \in \operatorname{im}\kappa,$$

and extend U_τ to all of $\mathbb{R}^{12}$ by first projecting onto $\operatorname{im}\kappa$. Iterating $q_{n+1} = U_\tau(q_n)$ yields the model's dynamics.

3.5 Continuity of the Evolution Operator

Translations in $\mathbb{R}^d$ are isometries for the Hausdorff distance:

$$d_{\mathrm{H}}\big(T_t A, T_t B\big) = d_{\mathrm{H}}(A, B) \qquad \forall\, A, B \subset \mathbb{R}^d,\ t \in \mathbb{R}^d.$$

Moreover, the canonical projection

$$\pi : \mathcal{K} \longrightarrow \mathcal{A}, \qquad \mathcal{A} := \big\{ F \subset \mathbb{R}^d \,\big|\, F = \textstyle\bigcup_{i=1}^m w_i(F) \big\},$$

from the space $\mathcal{K}$ of non-empty compact sets onto the compact set of OSC-attractors is 1-Lipschitz:

$$d_{\mathrm{H}}\big(\pi A, \pi B\big) \le d_{\mathrm{H}}(A, B) \qquad \forall\, A, B \in \mathcal{K}.$$

Hence the evolution operator defined by

$$U_\tau(a) \;=\; \kappa\big(G(\kappa^{-1}(a)) \cup T_\tau G(\kappa^{-1}(a))\big),$$

where κ and G are as in Section 3, is continuous in the Hausdorff metric:

$$d_{\mathrm{H}}\big(U_\tau(A), U_\tau(B)\big) \le d_{\mathrm{H}}(A, B) \qquad \forall\, A, B \in \mathcal{K}.$$

4 Persistence Series

A *persistence series* is an ordered family $\{F_k\}_{k \in \mathbb{N}}$ of perfect, self–similar fractals. Unary operators σ with $\sigma(F_k) = F_{k+1}$ generate the evolutionary sequence of every 12–D quaternion; no higher–arity operators are required.

5 Approximation of Physical Equations by Perfect-Fractal Dynamics

Let $B := \ell^1() \,\widehat{\otimes}_\pi K$ be the Banach algebra introduced in Sect. **??**, endowed with the Cauchy product $\star$ and norm $\|\cdot\|_B$. Recall the *union-evolution operator* $U_h\colon \;\to$ (Sect. **??**) acting on compact sets by first translating each block by h in the e_{12}-direction and then taking the union.

[Discrete derivative] For $f \in B$ and step size $h > 0$ define the forward difference

$$_h f \;:=\; \frac{U_h f - f}{h} \;\; \in B,$$

and write ^m_h for its m-fold iterate. If the limit $\partial_{e_{12}} f := \lim_{h\to 0} {}_h f$ exists in B, call it the e_{12}-*derivative* of f.

[Consistency] For every $f \in B$ that is (Fréchet) differentiable in the e_{12}-direction one has $\big\|{}_h f - \partial_{e_{12}} f\big\|_B = (h)$ as $h \to 0$.

The proof is standard: write $U_h = \exp(h\,\partial_{e_{12}}) + (h^2)$ using the Baker–Campbell–Hausdorff formula inside B and expand.

[Fractal-dynamics approximation of analytic PDEs] Let $E\big(f, \partial^\alpha f\big) = 0$ be a polynomial partial differential equation in one time-like variable e_{12} and finitely many spatial variables, with real-analytic coefficients on a compact domain $D \subset^{d+1}$. For every $\varepsilon > 0$ there exist

- a step size $h \in (0, 1)$,
- a *finite* sequence of perfect fractals $F_0, \ldots, F_{N(h)} \in F_{\text{perf}}$,
- an element $f_h := \sum_{k=0}^{N(h)} \chi_{F_k} T^k \in B$,

such that the discrete equation obtained by replacing each $\partial^m_{e_{12}}$ with ^m_h holds up to ε in the $\|\cdot\|_B$-norm:

$$\big\|\, E\big(f_h, ^\alpha_h f_h\big)\,\big\|_B \;<\; \varepsilon.$$

If, in addition, E contains *no* derivatives, the equality holds *exactly* (i.e. with $\varepsilon = 0$).

[Idea of proof] *(i) Lifting of coefficients.* Because real-analytic functions on D admit uniformly convergent power-series expansions, each scalar coefficient can be approximated in ℓ^1 by a truncated series; use the Cantor-coding map κ (Sect. **??**) to lift those coefficients to finite unions of perfect fractals.

(ii) Replacement of derivatives. Lemma 5 shows $^\alpha_h f \to \partial^\alpha f$ as $h \to 0$ for every multi-index α. Choose h small enough that the (h) truncation error in each term of E is below $\varepsilon/(\#\text{terms})$.

(iii) Algebraic closure. Because B is a Banach algebra, the polynomial combination $E\big(f_h, ^\alpha_h f_h\big)$ is well defined and its norm is controlled by the individual truncation errors; sum the estimates to reach the claimed bound. If E is derivative-free, Step (ii) is unnecessary and equality holds exactly.

[Density] The set of Banach-valued functions realised by finite unions of perfect fractals is dense in the space of solutions to analytic PDEs on D.

[Vector and tensor fields] Multi-component fields can be treated block-wise: take $f = (f^{(1)}, \ldots, f^{(q)})$ with each component in B and apply Theorem 5 to every $f^{(i)}$ separately. Coupled systems (e.g. Maxwell's equations) require working in the product Banach algebra $B^{\times q}$, which is again complete.

[Limit to exact dynamics] If the analytic solution f is known *a priori* to be entire in the e_{12}-direction, then letting $h \to 0$ in Theorem 5 recovers $E(f, \partial^\alpha f) = 0$ exactly, so the conjecture of Sect. 5.3 in the previous version holds for this subclass of equations.

6 Photon Generation via Unknown Fractal Evolution

Let $\{F_n\}_{n\in\mathbb{N}} \subset F_{\text{perf}}$ be the (a priori unknown) sequence of perfect fractals, and let U_τ be the union-based evolution operator advancing the system by a time step τ. We define the emergent electromagnetic potential $A^\mu(x)$ by projecting the infinite union–dynamics back to the scalar fields:

$$A^\mu(x) \;=\; \lim_{N\to\infty} \Pi_{\text{scalar}}\Big(\bigcup_{n=1}^{N} U_{\tau_n}\big[F_n(x)\big]\Big) \quad , \quad \tau_n \to 0^+.$$

Here Π_{scalar} denotes the projection from the fractal Banach algebra back to the usual function space.

Next, introduce the canonical conjugate momentum

$$\pi_\mu(x) \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\mu)} \;=\; F_{0\mu}(x),$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor.

The photon creation operator in the usual Fock-space representation then emerges as

$$a^\dagger(\mathbf{k}, \lambda) \;=\; \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \int d^3x \,\big[\,\epsilon_\mu(\mathbf{k}, \lambda)\,\pi^\mu(x) + \,i\,\omega_{\mathbf{k}}\,\epsilon_\mu(\mathbf{k}, \lambda)\,A^\mu(x)\big] e^{-\,i\,\mathbf{k}\cdot\mathbf{x}}\,,$$

with $\omega_{\mathbf{k}} = |\mathbf{k}|$ and $\epsilon^\mu(\mathbf{k}, \lambda)$ the polarization vectors.

Finally, the n-photon state is generated by repeated application of these union-evolution–induced operators on the vacuum:

$$\big|\gamma(\mathbf{k}_1, \lambda_1; \ldots; \mathbf{k}_n, \lambda_n)\big\rangle \;=\; \prod_{j=1}^{n} a^\dagger(\mathbf{k}_j, \lambda_j)\,\big|0\big\rangle.$$

6.1 Photon-Generation Limit and Convergence

In this section we make precise the convergence in the photon-generation limit and verify compactness so that the scalar projection is well-defined.

Let

$$A_\mu^{(N)}(x) \;:=\; \Pi_{\text{scalar}}\Big(\bigcup_{n=1}^{N} U_{\tau_n}\big(F_n(x)\big)\Big), \qquad \tau_n \to 0^+,$$

and consider the Banach-algebra $(\mathcal{B}, \|\cdot\|_B)$ in which each $A_\mu^{(N)}(x)$ takes values. Since each finite union $\bigcup_{n=1}^{N} U_{\tau_n}(F_n(x))$ is an OSC-attractor and hence compact in the Hausdorff metric, the projection $\Pi_{\text{scalar}}\colon \big(\mathcal{K}(\mathbb{R}^{12}), d_H\big) \to (\mathcal{B}, \|\cdot\|_B)$ is well-defined and 1-Lipschitz continuous.

We then define the photon-field by the norm-limit

$$A_\mu(x) \;=\; \lim_{N\to\infty} A_\mu^{(N)}(x),$$

where the limit is taken in the Banach-algebra norm $\|\cdot\|_B$. Equivalently, the sequence $\{A_\mu^{(N)}(x)\}_{N\in}$ is Cauchy in $\|\cdot\|_B$ because

$$\big\|A_\mu^{(N+M)}(x) - A_\mu^{(N)}(x)\big\|_B \;\le\; \big\|\Pi_{\text{scalar}}\big\|_{\text{Lip}}\, d_H\Big(\bigcup_{n=1}^{N+M} U_{\tau_n}(F_n(x)),\, \bigcup_{n=1}^{N} U_{\tau_n}(F_n(x))\Big),$$

and $d_H \to 0$ as $N, M \to \infty$. Completeness of $(\mathcal{B}, \|\cdot\|_B)$ then guarantees existence of the limit $A_\mu(x)$.

Remark. One may alternatively formulate convergence in the weak-$*$ topology of distributions; the same compactness argument ensures consistency of Π_{scalar} under that mode of convergence.

7 Generation of a Continuous Electromagnetic Field via Fractal–Evolution Dynamics

Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_{\text{perf}}$ be a (a priori unknown) sequence of perfect fractals in $\mathbb{R}^{12}$, and let

$$U_\tau : \mathcal{K}(\mathbb{R}^{12}) \to \mathcal{K}(\mathbb{R}^{12})$$

be the evolutionary union operator advancing any compact fractal union by a time-step $\tau > 0$. Denote by $\Pi_{\text{scalar}} : \big(\mathcal{K}(\mathbb{R}^{12}), d_H\big) \to (B, \|\cdot\|_B)$ the 1–Lipschitz projection onto the Banach space $B = C(\mathbb{R}^3; \mathbb{R})$ of scalar fields.

Define the fractal-generated electric and magnetic scalar potentials at $x \in \mathbb{R}^3$ and $t \ge 0$ by

$$\Phi_n(x,t) = \Pi_{\text{scalar}}\big(U_{t/n}^{n}(F_n)\big)(x)\,, \tag{4}$$

$$\Psi_n(x,t) = \Pi_{\text{scalar}}\big(U_{t/n}^{n}(F_{n+1})\big)(x)\,. \tag{5}$$

Assuming uniform convergence as $n \to \infty$, we set

$$\Phi(x,t) \;=\; \lim_{n\to\infty} \Phi_n(x,t), \qquad \Psi(x,t) \;=\; \lim_{n\to\infty} \Psi_n(x,t). \tag{6}$$

The resulting continuous electromagnetic fields are then given by

$$\mathbf{E}(x,t) \;=\; -\nabla\Phi(x,t) - \frac{\partial}{\partial t}\mathbf{A}(x,t), \qquad \mathbf{B}(x,t) \;=\; \nabla\times\mathbf{A}(x,t), \tag{7}$$

where the vector potential $\mathbf{A}(x,t)$ arises from an analogous fractal-evolution sequence $\{G_n\} \subset \mathcal{F}_{\text{perf}}$ and

$$\mathbf{A}(x,t) = \lim_{n\to\infty} \nabla\times\Big[\Pi_{\text{scalar}}\big(U_{t/n}^{n}(G_n)\big)\Big](x).$$

Finally, one verifies that $\mathbf{E}, \mathbf{B} \in C^1(\mathbb{R}^3 \times [0,\infty); \mathbb{R}^3)$ satisfy Maxwell's equations in vacuum:

$$\nabla\cdot\mathbf{E} = 0, \qquad \nabla\times\mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = 0, \tag{8a}$$

$$\nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0. \tag{8b}$$

8 Maxwell Equations in the Kerion Algebra

We recall from Section 2.1 that the *Kerion algebra*

$$\mathbb{K} \;=\; \mathbb{H}^{(12)} \;=\; H\,\widehat{\otimes}_{\mathbb{R}}\mathrm{Span}_{\mathbb{R}}\{e_1,e_2,e_3\}, \qquad (h_1\otimes e_r)(h_2\otimes e_s)=\delta_{rs}\,(h_1h_2)\otimes e_r,$$

is a direct sum of three mutually-orthogonal quaternion blocks. Throughout this section the summation index $r=1,2,3$ selects a single block; because the product is block-diagonal the dynamics in each block are completely independent.

8.1 Kerionic differential operator

Let the spatial gradient in a fixed block be $\nabla := i\,\partial_x + j\,\partial_y + k\,\partial_z$ and write $1_{\mathbb{K}} := 1_{\mathbb{H}}\otimes\big(e_1+e_2+e_3\big)$ for the algebraic unit in $\mathbb{K}$. The *Kerionic four-gradient* and its conjugate are

$$\partial := \partial_t\,1_{\mathbb{K}} \;+\; \nabla\,1_{\mathbb{K}}, \qquad \bar{\partial} := \partial_t\,1_{\mathbb{K}} \;-\; \nabla\,1_{\mathbb{K}}. \tag{9}$$

8.2 Potential and field Kerions

A Kerion-valued four-potential is

$$A(t,\mathbf{x}) \;=\; \varphi\,1_{\mathbb{K}} \;+\; A_x\,i + A_y\,j + A_z\,k, \qquad \varphi, A_\mu\colon \mathbb{R}^{1+3}\to\mathbb{R}.$$

Its associated *field Kerion*

$$F \;:=\; \partial A \;-\; A\partial \tag{10}$$

splits automatically as $F=\sum_{r=1}^{3}\big(\mathbf{E}^{(r)}+i\,\mathbf{B}^{(r)}\big)\otimes e_r$ with

$$\mathbf{E}^{(r)} = -\nabla\varphi^{(r)} - \partial_t\mathbf{A}^{(r)}, \qquad \mathbf{B}^{(r)} = \nabla\times\mathbf{A}^{(r)}.$$

8.3 Compact Kerionic form of Maxwell's equations

Introduce the *current Kerion*

$$J \;=\; \rho\,1_{\mathbb{K}} \;-\; J_x\,i - J_y\,j - J_z\,k, \qquad \rho, J_\mu\colon \mathbb{R}^{1+3}\to\mathbb{R}.$$

Classical electrodynamics is then the single Kerion pair

$$\boxed{\partial F = J, \qquad \bar{\partial}F = 0} \tag{11}$$

which expands, for each block r, to the standard component form

$$\begin{aligned} \nabla\cdot\mathbf{E}^{(r)} &= \rho, & \nabla\times\mathbf{B}^{(r)} - \partial_t\mathbf{E}^{(r)} &= \mathbf{J},\\ \nabla\cdot\mathbf{B}^{(r)} &= 0, & \nabla\times\mathbf{E}^{(r)} + \partial_t\mathbf{B}^{(r)} &= 0. \end{aligned}$$

8.4 Physical interpretation

1. Redundant memory. The three quaternion blocks yield three identical copies of electromagnetism; coefficients along the "persistence" axis (Section 2.6) store the entire time-series of each block without affecting local dynamics.

2. Gauge freedom. Lorenz ($\partial\cdot A = 0$) or Coulomb ($\nabla\cdot\mathbf{A}^{(r)} = 0$) gauges may be imposed independently in every block.

3. Standard limit. If only block $r=1$ is populated, Eqs. (9)–(11) reduce exactly to Maxwell's equations in $\mathbb{R}^{1+3}$; the extra Kerion structure simply augments the theory with intrinsic fractal memory.

9 Quantum Physics Equations in the Kerion Algebra

9.1 Planck–Einstein Relation

In the Kerion framework every real scalar is promoted to the algebraic unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$. Accordingly we define the *energy Kerion* and the *angular-frequency Kerion*

$$E_{\mathbb{K}} \;:=\; E\,1_{\mathbb{K}}, \qquad \Omega_{\mathbb{K}} \;:=\; \omega\,1_{\mathbb{K}}, \qquad E, \omega \in \mathbb{R}.$$

The **Planck–Einstein relation** lifts unchanged to

$$\boxed{E_{\mathbb{K}} \;=\; \hbar\,\Omega_{\mathbb{K}}} \qquad \Longleftrightarrow \qquad E \;=\; \hbar\,\omega. \tag{12}$$

Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (12) reduces block-wise to the standard complex (or quaternionic) formula $E^{(r)} = \hbar\omega^{(r)}$ for each $r = 1, 2, 3$. Thus the Kerion algebra introduces no change to the numerical content of the Planck–Einstein relation; rather, it packages three independent copies of the relation into a single algebraic identity.

9.2 Heisenberg Picture

Let $\widehat{H}$ be the Kerion Hamiltonian introduced in Section 9.20 and fix the global imaginary unit $I_{\mathbb{K}} := i\,1_{\mathbb{K}} \in \mathbb{K}$. For any Schrödinger-picture observable $\widehat{O}_S\colon L^2(\mathbb{R}^3, T) \to L^2(\mathbb{R}^3, T)$ define its *Heisenberg evolution* by

$$\widehat{O}_H(t) \;:=\; e^{\,I_T \widehat{H} t/\hbar}\,\widehat{O}_S\,e^{-I_T \widehat{H} t/\hbar}.$$

Because I_T commutes with $\widehat{H}$ the usual BCH expansion applies block-wise, so the exponential is well-defined on the Kerion Hilbert space.

Kerionic Heisenberg equation. Differentiating with respect to $\mathbb{K}$ and using $[\widehat{H}, \widehat{H}] = 0$ gives

$$\boxed{I_T \hbar\,\partial_t \widehat{O}_H(t) \;=\; \big[\widehat{O}_H(t), \widehat{H}\big] \;+\; I_T \hbar \Big(\partial_t \widehat{O}_S\Big)_H.} \tag{13}$$

Block decomposition yields, for each $r = 1, 2, 3$,

$$i\hbar\,\partial_t \widehat{O}_H^{(r)} \;=\; \big[\widehat{O}_H^{(r)}, \widehat{H}\big] \;+\; i\hbar\Big(\partial_t \widehat{O}_S^{(r)}\Big)_H,$$

which is precisely the standard complex Heisenberg equation in block r.

Remarks.

1. Operator algebra. The commutator $[\widehat{A}, \widehat{B}] := \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$ is computed in the full operator algebra $\mathcal{B}\big(L^2(\mathbb{R}^3, \mathbb{K})\big)$; block-orthogonality ensures that cross-terms $\big[\widehat{A}^{(r)}, \widehat{B}^{(s)}\big]$ vanish whenever $r \neq s$.

2. Expectation values. For any normalized Kerion state $\Psi(t)$ the Ehrenfest theorem follows as usual: $\partial_t \langle \widehat{O} \rangle = \frac{1}{I_T \hbar} \langle [\widehat{O}, \widehat{H}] \rangle + \langle (\partial_t \widehat{O})_H \rangle$.

3. Complex limit. Restricting to a single block and to complex-valued states collapses (13) to the familiar complex Heisenberg equation.

9.3 Ehrenfest Theorem

Let $\Psi(t) \in L^2(\mathbb{R}^3, \mathbb{K})$ evolve according to the Kerionic Schrödinger equation $I_{\mathbb{K}} \hbar\,\partial_{\mathbb{K}} \Psi = \widehat{H}\Psi$ (Section 9.20), and let $\widehat{O}_H(t)$ be an observable in the Heisenberg picture (Section 9.2). The *expectation value* of $\widehat{O}_H(t)$ is

$$\langle \widehat{O} \rangle_{\mathbb{K}} \;:=\; \big\langle \Psi(t), \widehat{O}_H(t)\Psi(t) \big\rangle \;=\; \sum_{r=1}^{3} \int_{\mathbb{R}^3} \overline{\psi^{(r)}(t, \mathbf{x})}\, \widehat{O}_H^{(r)}(t) \psi^{(r)}(t, \mathbf{x})\, d^3\mathbf{x}.$$

Ehrenfest relation. Differentiating with respect to $\mathbb{K}$ and using the Kerionic Heisenberg equation $I_{\mathbb{K}}\hbar\,\partial_{\mathbb{K}}\widehat{O}_H = [\widehat{O}_H, \widehat{H}] + I_{\mathbb{K}}\hbar(\partial_{\mathbb{K}}\widehat{O}_S)_H$ yields

$$\boxed{\partial_{\mathbb{K}}\langle\widehat{O}\rangle_{\mathbb{K}} \;=\; \frac{1}{I_{\mathbb{K}}\hbar}\big\langle\,[\widehat{O}_H(t),\widehat{H}]\,\big\rangle_{\mathbb{K}} \;+\; \big\langle\,(\partial_{\mathbb{K}}\widehat{O}_S)_H\big\rangle_{\mathbb{K}}.} \tag{14}$$

Component form. Because $I_{\mathbb{K}} = i\,1_{\mathbb{K}}$ commutes with $\widehat{H}$ and block-orthogonality annihilates mixed commutators, Eq. (14) decomposes into three independent complex identities

$$\partial_{\mathbb{K}}\langle\widehat{O}\rangle_{\mathbb{K}}^{(r)} \;=\; \frac{1}{i\hbar}\big\langle\,[\widehat{O}_H^{(r)}(t),\widehat{H}]\,\big\rangle_{\mathbb{K}} \;+\; \big\langle\,(\partial_{\mathbb{K}}\widehat{O}_S^{(r)})_H\big\rangle_{\mathbb{K}}, \qquad r = 1,2,3,$$

which is exactly the standard Ehrenfest theorem in each block.

Interpretation. Equation (14) shows that the Kerion framework preserves the correspondence between quantum expectation values and classical equations of motion: the rate of change of any observable equals the expectation of its commutator with the Hamiltonian, plus explicit time dependence, replicated identically in every quaternion block and augmented—if desired—by the "persistence" coefficients along the twelfth axis.

9.4 Photoelectric Effect

Promote every real scalar to the algebraic unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3) \subset \mathbb{K} = \mathbb{H}^{(12)}$. Define the corresponding Kerion quantities

$$E_{\mathbb{K}} := E\,1_{\mathbb{K}}, \qquad \Omega_{\mathbb{K}} := \omega\,1_{\mathbb{K}}, \qquad \Phi_{\mathbb{K}} := \phi\,1_{\mathbb{K}}, \qquad K_{\mathbb{K}} := K_{\max}\,1_{\mathbb{K}}, \qquad E,\omega,\phi,K_{\max} \in \mathbb{R}.$$

Kerion photoelectric equation. Combining the Planck–Einstein relation $E_{\mathbb{K}} = \hbar\Omega_{\mathbb{K}}$ (Section 9.1) with energy conservation gives the single Kerion identity

$$\boxed{K_{\mathbb{K}} \;=\; E_{\mathbb{K}} \;-\; \Phi_{\mathbb{K}} \;=\; \hbar\,\Omega_{\mathbb{K}} \;-\; \Phi_{\mathbb{K}}\,.} \tag{15}$$

Component form. Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (15) decomposes into three identical scalar relations

$$K_{\max}^{(r)} = E^{(r)} - \phi^{(r)} = \hbar\,\omega^{(r)} - \phi^{(r)}, \qquad r = 1,2,3,$$

which is the usual Einstein photoelectric equation in each quaternion block.

Threshold frequency. The threshold angular frequency for block r is $\omega_0^{(r)} = \frac{\phi^{(r)}}{\hbar}$, so that $K_{\max}^{(r)} \geq 0$ iff $\omega^{(r)} \geq \omega_0^{(r)}$. In the full Kerion algebra this yields three independent threshold conditions, each of which can be monitored or recorded along the "persistence" coefficients of the twelfth axis without affecting local dynamics.

9.5 Threshold Frequency and Work Function

Within the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ every real scalar multiplies the algebraic unit $1_{\mathbb{K}} := 1_{\mathbb{H}}\otimes(e_1 + e_2 + e_3)$. Hence the **work-function Kerion** is defined by

$$\boxed{\Phi_{\mathbb{K}} := \phi\,1_{\mathbb{K}}}, \qquad \phi \in \mathbb{R},$$

so that $\Phi_{\mathbb{K}}$ acts identically in all three quaternion blocks.

Threshold (angular) frequency. The minimum photon energy needed to liberate an electron satisfies $E_{\mathbb{K}} = \Phi_{\mathbb{K}}$. Using the Planck–Einstein relation $E_{\mathbb{K}} = \hbar\Omega_{\mathbb{K}}$ (Section 9.1) we obtain the *threshold angular-frequency Kerion*

$$\boxed{\Omega_{0,\mathbb{K}} := \omega_0 \, 1_{\mathbb{K}} = \frac{\Phi_{\mathbb{K}}}{\hbar}}.$$

Dividing by 2π yields the ordinary frequency form

$$\boxed{\nu_{0,\mathbb{K}} := \frac{\Omega_{0,T}}{2\pi} = \frac{\Phi_{\mathbb{K}}}{h}}, \qquad h = 2\pi\hbar.$$

Component form. Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, each quaternion block $r = 1, 2, 3$ inherits the familiar scalar relations

$$\phi^{(r)} = \hbar\,\omega_0^{(r)} \quad\Longleftrightarrow\quad \omega_0^{(r)} = \frac{\phi^{(r)}}{\hbar}, \qquad \nu_0^{(r)} = \frac{\phi^{(r)}}{h}.$$

Thus the Kerion formalism packages three independent threshold-frequency conditions into a single algebraic identity while leaving their numerical content unchanged.

9.6 Photon Momentum

All real scalars are absorbed into the algebraic unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3) \subset \mathbb{K} = \mathbb{H}^{(12)}$. The *energy Kerion* has already been defined as $E_{\mathbb{K}} = E\,1_{\mathbb{K}}$ (Section 9.1). Introduce the **wave-vector Kerion**

$$\boxed{\mathbf{K}_{\mathbb{K}} := (k_x\, i + k_y\, j + k_z\, k)\, 1_{\mathbb{K}}}, \qquad \mathbf{k} = (k_x, k_y, k_z) \in \mathbb{R}^3,$$

and the **momentum Kerion**

$$\boxed{\mathbf{P}_{\mathbb{K}} := (p_x\, i + p_y\, j + p_z\, k)\, 1_{\mathbb{K}}}, \qquad \mathbf{p} = (p_x, p_y, p_z) \in \mathbb{R}^3.$$

Photon momentum relations. For a massless photon the standard identities $E = \hbar\omega$, $|\mathbf{p}| = \frac{E}{c}$, $\mathbf{p} = \hbar\mathbf{k}$ lift block-wise to the compact Kerion equations

$$\boxed{\mathbf{P}_{\mathbb{K}} \;=\; \frac{E_{\mathbb{K}}}{c}\,\hat{\mathbf{n}} \;=\; \hbar\,\mathbf{K}_{\mathbb{K}}}, \qquad \hat{\mathbf{n}} = \frac{\mathbf{k}}{|\mathbf{k}|} = \frac{\mathbf{p}}{|\mathbf{p}|} \text{ (unit propagation vector).} \tag{16}$$

Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (16) decomposes into three independent complex relations

$$\mathbf{p}^{(r)} = \frac{E^{(r)}}{c}\,\hat{\mathbf{n}} = \hbar\,\mathbf{k}^{(r)}, \qquad r = 1, 2, 3,$$

leaving the physical content of photon momentum unchanged while encoding all three copies within a single Kerion identity.

9.7 Heisenberg Uncertainty Principle

Let $\widehat{A}, \widehat{B}$ be Hermitian observables acting on the Kerion Hilbert space $L^2(\mathbb{R}^3, \mathbb{K})$ and let $\Psi(t)$ be a normalized state vector that evolves according to the Kerionic Schrödinger equation (Section 9.20). Introduce the centred operators

$$\widetilde{A} := \widehat{A} - \langle \widehat{A} \rangle_{\mathbb{K}}, \qquad \widetilde{B} := \widehat{B} - \langle \widehat{B} \rangle_{\mathbb{K}},$$

where the expectation value is $\langle \widehat{O} \rangle_{\mathbb{K}} := \langle \Psi(t), \widehat{O}\,\Psi(t) \rangle$ (cf. Section 9.3). Define the *standard deviations*

$$(\Delta A)^2 := \langle \widetilde{A}^2 \rangle_{\mathbb{K}}, \qquad (\Delta B)^2 := \langle \widetilde{B}^2 \rangle_{\mathbb{K}}.$$

General uncertainty relation. Applying the Cauchy–Schwarz inequality to $\widetilde{A}\Psi$ and $\widetilde{B}\Psi$ and using the block-commutativity of the global imaginary unit $I_{\mathbb{K}} := i\,1_{\mathbb{K}}$ yields

$$\boxed{\Delta A\,\Delta B \;\geq\; \frac{1}{2}\left|\big\langle[\widehat{A},\widehat{B}]\big\rangle_{\mathbb{K}}\right|}. \tag{17}$$

Because $[\widehat{A},\widehat{B}]$ is proportional to $I_{\mathbb{K}}$ for canonical pairs, the right–hand side is a real scalar and Eq. (17) decomposes into three identical complex inequalities—one per quaternion block.

Canonical position–momentum pair

Choose the Cartesian position operator $\widehat{x}_j := x_j\,1$

9.8 Dispersion of an Observable

For any Hermitian operator $\widehat{O}\colon L^2(\mathbb{R}^3,\mathbb{K}) \to L^2(\mathbb{R}^3,\mathbb{K})$ and normalized state $\Psi(t) \in L^2(\mathbb{R}^3,\mathbb{K})$ define the *expectation value*

$$\langle\widehat{O}\rangle_{\mathbb{K}} := \big\langle\Psi(t), \widehat{O}\,\Psi(t)\big\rangle.$$

Introduce the **centred operator** $\widetilde{O} := \widehat{O} - \langle\widehat{O}\rangle_{\mathbb{K}}$.

Dispersion (variance). The *dispersion* (or variance) of $\widehat{O}$ in state $\Psi(t)$ is

$$\boxed{(\Delta O)^2 := \langle\widetilde{O}^2\rangle_{\mathbb{K}} = \big\langle\Psi(t),\, \widetilde{O}^2\Psi(t)\big\rangle}. \tag{18}$$

Block decomposition. Because $\langle\cdot,\cdot\rangle$ and $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$ factor block-wise, (18) splits into three independent complex variances,

$$(\Delta O^{(r)})^2 = \langle\psi^{(r)},\, \big(\widehat{O}^{(r)} - \langle\widehat{O}^{(r)}\rangle\big)^2\psi^{(r)}\rangle, \qquad r = 1,2,3,$$

where $\psi^{(r)}$ is the r-th quaternion block of Ψ (cf. Section 9.20).

Standard deviation. The *standard deviation* (root-mean-square spread) is

$$\boxed{\Delta O := \sqrt{(\Delta O)^2}},$$

and likewise decomposes as $\Delta O = \Delta O^{(1)} \oplus \Delta O^{(2)} \oplus \Delta O^{(3)}$.

Role in uncertainty relations. The quantities ΔA and ΔB defined by (18) enter directly into the Heisenberg inequalities (Section 9.7), showing that all familiar statistical measures of quantum observables carry over unchanged to the Kerion framework, merely replicated across its three quaternion blocks.

9.9 General Uncertainty Relation

Let $\widehat{A}, \widehat{B}$ be Hermitian observables on the Kerion Hilbert space $L^2\big(\mathbb{R}^3,\mathbb{K}\big)$ and let $\Psi(t)$ be a normalized state. Define the centred operators

$$\widetilde{A} := \widehat{A} - \langle\widehat{A}\rangle_{\mathbb{K}}, \qquad \widetilde{B} := \widehat{B} - \langle\widehat{B}\rangle_{\mathbb{K}},$$

and introduce the expectation of the commutator and anticommutator

$$C := \big\langle[\widehat{A},\widehat{B}]\big\rangle_{\mathbb{K}}, \qquad D := \big\langle\{\widehat{A},\widehat{B}\}\big\rangle_{\mathbb{K}} - 2\langle\widehat{A}\rangle_{\mathbb{K}}\langle\widehat{B}\rangle_{\mathbb{K}}.$$

Because the global imaginary unit $I_{\mathbb{K}} = i\,1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, both C and D are real scalars.

Robertson–Schrödinger inequality in $\mathbb{K} = \mathbb{H}^{(12)}$. Applying the Cauchy–Schwarz inequality to the vectors $\widetilde{A}\,\Psi$ and $\widetilde{B}\,\Psi$ gives the *general uncertainty relation*

$$\boxed{(\Delta A)^2\,(\Delta B)^2 \;\geq\; \frac{1}{4}\,|C|^2 \;+\; \frac{1}{4}\,|D|^2}, \tag{19}$$

where ΔA and ΔB are the standard deviations defined in Eq. (18).

Block decomposition. Since C and D are block-independent, Eq. (19) decomposes into three identical complex inequalities

$$(\Delta A^{(r)})^2\,(\Delta B^{(r)})^2 \;\geq\; \frac{1}{4}\,|C|^2 \;+\; \frac{1}{4}\,|D|^2, \qquad r = 1, 2, 3,$$

reproducing the usual Robertson–Schrödinger bound in each quaternion block.

Special cases.

1. For canonical pairs $[\widehat{A}, \widehat{B}] = I_{\mathbb{K}}\hbar$, the anticommutator term D vanishes and Eq. (19) reduces to the familiar Heisenberg inequality $\Delta A\,\Delta B \geq \frac{\hbar}{2}$ (Section 9.7).

2. If $C = 0$ but $D \neq 0$ (e.g. for two commuting but statistically correlated observables), the bound is set entirely by the covariance term D, exactly as in complex quantum mechanics.

Hence the Kerion framework retains the full structure of the Robertson–Schrödinger uncertainty principle: all quantum statistical bounds remain intact, merely replicated across the three quaternion blocks and optionally recorded along the twelfth "persistence" coordinate.

9.10 Density of States

The *density of states* (DOS) counts the number of single–particle energy levels per unit energy interval. Because all real scalars in the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ multiply the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$, we define the **DOS Kerion**

$$\boxed{g_{\mathbb{K}}(E_{\mathbb{K}}) := g(E)\,1_{\mathbb{K}}}, \qquad E_{\mathbb{K}} = E\,1_{\mathbb{K}}, \quad E \in \mathbb{R}_{\geq 0},$$

so that $g_{\mathbb{K}}$ acts identically in all three quaternion blocks.

Free, non-relativistic particle in a box. For a particle of mass m confined to a volume V in three dimensions the usual DOS is

$$g(E) = \frac{V}{2\pi^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{E}\,\Theta(E),$$

where $\Theta(E)$ is the Heaviside step function. Promoting $E \mapsto E_{\mathbb{K}}$ gives

$$\boxed{g_{\mathbb{K}}(E_{\mathbb{K}}) = \frac{V}{2\pi^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{E}\;1_{\mathbb{K}}\,\Theta(E)}.$$

Component form. Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, $g_{\mathbb{K}}(E_{\mathbb{K}})$ decomposes into three identical scalar DOS functions,

$$g^{(r)}(E) = \frac{V}{2\pi^2}\left(\tfrac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{E}\,\Theta(E), \qquad r = 1, 2, 3,$$

which is exactly the standard result for each quaternion block.

Photon (bosonic) density of states. Setting $E = \hbar\omega$ for massless photons in a cavity of volume V yields

$$g(\omega) = \frac{V}{\pi^2 c^3}\,\omega^2\,\Theta(\omega),$$

and hence the Kerion form

$$\boxed{g_{\mathbb{K}}(\Omega_{\mathbb{K}}) = \frac{V}{\pi^2 c^3}\,\omega^2\,1_{\mathbb{K}}\,\Theta(\omega), \qquad \Omega_{\mathbb{K}} = \omega\,1_{\mathbb{K}}.}$$

Summary. The Kerion formalism leaves the numerical content of the density of states unchanged; it simply wraps three identical copies—one per quaternion block—into a single algebraic object $g_{\mathbb{K}}$. Any integrals over $g(E)$ (e.g. for partition functions or Bose/Fermi-Dirac statistics) carry over block-wise without modification, while the twelfth "persistence" coordinate may be used to store the DOS history if desired.

9.11 Fermi–Dirac Distribution

All real scalars in the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ multiply the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$. Promote the energy, chemical potential, Boltzmann factor and temperature to

$$E_{\mathbb{K}} = E\,1_{\mathbb{K}}, \qquad \mu_{\mathbb{K}} = \mu\,1_{\mathbb{K}}, \qquad \beta_{\mathbb{K}} = \beta\,1_{\mathbb{K}} = \frac{1_{\mathbb{K}}}{k_B T}, \qquad (E, \mu, T \in \mathbb{R},\ T > 0).$$

Occupation probability. The **Fermi–Dirac distribution Kerion** is defined by

$$\boxed{f_{\mathbb{K}}(E_{\mathbb{K}}) \ := \ \Big[\exp\big(\beta_{\mathbb{K}}(E_{\mathbb{K}} - \mu_{\mathbb{K}})\big) + 1_{\mathbb{K}}\Big]^{-1}}, \tag{20}$$
$$\iff \quad f_{\mathbb{K}}(E_{\mathbb{K}}) = \frac{1_{\mathbb{K}}}{\exp\big[\beta(E - \mu)\big] + 1}.$$

Component form. Because the unit $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (20) factorises into three identical complex occupancy probabilities,

$$f^{(r)}(E) = \frac{1}{\exp\big[\beta\big(E - \mu\big)\big] + 1}, \qquad r = 1, 2, 3,$$

leaving the numerical content of the Fermi–Dirac statistics unchanged.

Density–of–states integration. Given the Kerion density of states $g_{\mathbb{K}}(E_{\mathbb{K}}) = g(E)\,1_{\mathbb{K}}$ (Section 9.10), the particle–number Kerion at temperature $\mathbb{K}$ is

$$N_{\mathbb{K}} \ = \ \int_0^\infty g_{\mathbb{K}}(E_{\mathbb{K}})\,f_{\mathbb{K}}(E_{\mathbb{K}})\,dE \ = \ 1_{\mathbb{K}} \int_0^\infty g(E)\,\frac{dE}{\exp\big[\beta(E - \mu)\big] + 1},$$

so each quaternion block satisfies the standard integral for fermionic systems.

Zero–temperature limit. As $T \to 0^+$ we recover the Kerion step function

$$\boxed{f_{T\to 0}(E_{\mathbb{K}}) \ = \ \Theta(\mu - E)\,1_{\mathbb{K}},}$$

again reproducing three identical Fermi seas—one per quaternion block— while the twelfth "persistence" coordinate may record their evolution without altering the local fermionic statistics.

9.12 Bose–Einstein Distribution

In the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ every real scalar multiplies the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$. Promote the thermodynamic scalars to Kerions

$$E_{\mathbb{K}} = E\,1_{\mathbb{K}}, \qquad \mu_{\mathbb{K}} = \mu\,1_{\mathbb{K}}, \qquad \beta_{\mathbb{K}} = \beta\,1_{\mathbb{K}} = \frac{1_{\mathbb{K}}}{k_B \mathbb{K}}, \qquad (E, \mu, T \in \mathbb{R},\ \mathbb{K} > 0).$$

For bosons the chemical potential satisfies $\mu \le 0$; for photons one sets $\mu = 0$.

Occupation probability. The **Bose–Einstein distribution Kerion** is defined by

$$\boxed{n_{\mathbb{K}}(E_{\mathbb{K}}) := \Big[\exp\big(\beta_{\mathbb{K}}(E_{\mathbb{K}} - \mu_{\mathbb{K}})\big) - 1_{\mathbb{K}}\Big]^{-1}}, \tag{21}$$
$$\iff \quad n_{\mathbb{K}}(E_{\mathbb{K}}) = \frac{1_{\mathbb{K}}}{\exp\big[\beta(E-\mu)\big] - 1}.$$

Component form. Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (21) splits into three identical scalar occupancies,

$$n^{(r)}(E) = \frac{1}{\exp\big[\beta(E-\mu)\big] - 1}, \qquad r = 1, 2, 3,$$

thus preserving the usual Bose–Einstein statistics in each quaternion block.

Density–of–states integration. With the Kerion density of states $g_{\mathbb{K}}(E_{\mathbb{K}}) = g(E)\,1_{\mathbb{K}}$ (Section 9.10) the total boson number is

$$N_{\mathbb{K}} = \int_0^\infty g_{\mathbb{K}}(E_{\mathbb{K}})\,n_{\mathbb{K}}(E_{\mathbb{K}})\,dE = 1_{\mathbb{K}} \int_0^\infty g(E)\,\frac{dE}{\exp\big[\beta(E-\mu)\big] - 1}.$$

Each quaternion block therefore satisfies the standard integral for bosonic systems.

Bose–Einstein condensation. When $\mu \to 0^-$ a macroscopic population may accumulate in the ground state $E = 0$. In the Kerion picture this phenomenon occurs *simultaneously* in all three quaternion blocks, while the twelfth "persistence" coordinate can record the condensate fraction's temporal evolution without altering the local bosonic statistics.

9.13 Orbital, Spin, and Total Angular Momentum

All real scalars in the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ multiply the unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$. Promote the usual orbital ($\widehat{\mathbf{L}}$), spin ($\widehat{\mathbf{S}}$), and total ($\widehat{\mathbf{J}} = \widehat{\mathbf{L}} + \widehat{\mathbf{S}}$) operators to

$$\widehat{L}_j := -I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,x_k\partial_{x_l}, \qquad \widehat{S}_j := \frac{\hbar}{2}\,\sigma_j\,1_{\mathbb{K}}, \qquad \widehat{J}_j := \widehat{L}_j + \widehat{S}_j, \quad j \in \{x, y, z\},$$

where $I_{\mathbb{K}} = i\,1_{\mathbb{K}}$ and σ_j are the Pauli matrices acting on the two-component spinor part of the Kerion wave-function (cf. Section 9.20).

Commutation relations

Because $I_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, the canonical $\mathfrak{su}(2)$ relations hold block-wise:

$$[\widehat{L}_j, \widehat{L}_k] = I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,\widehat{L}_l, \qquad [\widehat{S}_j, \widehat{S}_k] = I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,\widehat{S}_l, \qquad [\widehat{J}_j, \widehat{J}_k] = I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,\widehat{J}_l.$$

Hence each quaternion block inherits an independent copy of the usual angular-momentum algebra.

Quantum numbers

Define the Casimir operators

$$\widehat{L}^2 := \sum_j \widehat{L}_j^2, \quad \widehat{S}^2 := \sum_j \widehat{S}_j^2, \quad \widehat{J}^2 := \sum_j \widehat{J}_j^2.$$

For any block $r \in \{1, 2, 3\}$ the simultaneous eigenstates satisfy

$$\widehat{L}^2\,\psi^{(r)} = \hbar^2\,l(l+1)\,\psi^{(r)}, \qquad \widehat{L}_z\,\psi^{(r)} = \hbar\,m_l\,\psi^{(r)},$$

$$\widehat{S}^2\,\psi^{(r)} = \hbar^2\,s(s+1)\,\psi^{(r)}, \qquad \widehat{S}_z\,\psi^{(r)} = \hbar\,m_s\,\psi^{(r)},$$

$$\widehat{J}^2\,\psi^{(r)} = \hbar^2\,j(j+1)\,\psi^{(r)}, \qquad \widehat{J}_z\,\psi^{(r)} = \hbar\,m_j\,\psi^{(r)}.$$

Allowed values (per block).

$$\begin{array}{lcl} l & \in & \{0, 1, 2, \dots\}, \\ m_l & \in & \{-l, \, -l+1, \dots, \, l\}, \\ s & \in & \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots\right\}, \\ m_s & \in & \{-s, \, -s+1, \dots, \, s\}, \\ j & \in & \{|l-s|, \, |l-s|+1, \dots, \, l+s\}, \\ m_j & \in & \{-j, \, -j+1, \dots, \, j\}. \end{array}$$

Because $1_{\mathbb{K}}$ commutes with all operators, these spectra are *identical* in each quaternion block; the Kerion formalism merely packages the triple $\left(l^{(r)}, m_l^{(r)}, s^{(r)}, m_s^{(r)}, j^{(r)}, m_j^{(r)}\right)$ into a single algebraic object while preserving their usual numerical ranges.

Raising and lowering operators

Define

$$\widehat{J}_{\pm} := \widehat{J}_x \pm I_{\mathbb{K}} \, \widehat{J}_y, \qquad [\widehat{J}_z, \widehat{J}_{\pm}] = \pm I_{\mathbb{K}} \hbar \, \widehat{J}_{\pm}, \qquad [\widehat{J}_+, \widehat{J}_-] = 2 \, I_{\mathbb{K}} \hbar \, \widehat{J}_z.$$

Acting on an eigenstate in block r,

$$\widehat{J}_{\pm} \, \psi_{j,m_j}^{(r)} = \hbar \sqrt{j(j+1) - m_j(m_j \pm 1)} \; \psi_{j,m_j \pm 1}^{(r)},$$

exactly as in complex quantum mechanics.

Summary. Angular-momentum quantum numbers and their algebraic structure are preserved unchanged in the Kerion setting: each quaternion block carries its own $\mathfrak{su}(2)$ representation, while the twelfth "persistence" coordinate offers optional archival of the state history without perturbing local rotational dynamics.

9.14 Angular-Momentum Magnitudes

Recall the Casimir operators introduced in Section 9.13,

$$\widehat{L}^2 = \sum_j \widehat{L}_j^2, \qquad \widehat{S}^2 = \sum_j \widehat{S}_j^2, \qquad \widehat{J}^2 = \sum_j \widehat{J}_j^2,$$

with $I_{\mathbb{K}} = i \, 1_{\mathbb{K}}$ the global imaginary unit in $\mathbb{K} = \mathbb{H}^{(12)}$. Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, each eigenvalue problem factorises block-wise.

Eigenvalues of the squared operators

For an eigenstate in quaternion block r we have

$$\widehat{L}^2 \, \psi^{(r)} = \hbar^2 \, l(l+1) \, \psi^{(r)}, \qquad \widehat{S}^2 \, \psi^{(r)} = \hbar^2 \, s(s+1) \, \psi^{(r)}, \qquad \widehat{J}^2 \, \psi^{(r)} = \hbar^2 \, j(j+1) \, \psi^{(r)}.$$

Exactly the same spectra obtain in the full Kerion space because $\widehat{L}^2, \widehat{S}^2, \widehat{J}^2$ are diagonal in the block index and simply multiply $1_{\mathbb{K}}$.

Magnitude operators

Define the (positive) magnitude operators

$$|\widehat{\mathbf{L}}| := \sqrt{\widehat{L}^2}, \quad |\widehat{\mathbf{S}}| := \sqrt{\widehat{S}^2}, \quad |\widehat{\mathbf{J}}| := \sqrt{\widehat{J}^2}.$$

Acting on an eigenstate,

$$|\widehat{\mathbf{L}}| \, \psi^{(r)} = \hbar \sqrt{l(l+1)} \; \psi^{(r)}, \qquad |\widehat{\mathbf{S}}| \, \psi^{(r)} = \hbar \sqrt{s(s+1)} \; \psi^{(r)}, \qquad |\widehat{\mathbf{J}}| \, \psi^{(r)} = \hbar \sqrt{j(j+1)} \; \psi^{(r)}.$$

Kerion-level summary

Collecting the three identical block results into a single algebraic statement, the magnitude Kerions are

$$\boxed{|\widehat{\mathbf{L}}| = \hbar\sqrt{l(l+1)}\,1_{\mathbb{K}}, \quad |\widehat{\mathbf{S}}| = \hbar\sqrt{s(s+1)}\,1_{\mathbb{K}}, \quad |\widehat{\mathbf{J}}| = \hbar\sqrt{j(j+1)}\,1_{\mathbb{K}}.}$$

Thus the familiar angular-momentum magnitudes carry over unchanged, merely replicated across the three quaternion blocks, while the twelfth "persistence" coordinate may archive their historical values without affecting local rotational dynamics.

9.15 Cartesian Components

We work in the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ where every real scalar multiplies the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$ and $I_{\mathbb{K}} := i\,1_{\mathbb{K}}$ is the distinguished imaginary unit.

Orbital components

Promoting the position $\mathbf{x} = (x, y, z)$ and momentum $\widehat{\mathbf{p}} = -I_{\mathbb{K}}\hbar\,\nabla$ operators gives the orbital angular-momentum components

$$\boxed{\widehat{L}_x = y\,\widehat{p}_z - z\,\widehat{p}_y, \quad \widehat{L}_y = z\,\widehat{p}_x - x\,\widehat{p}_z, \quad \widehat{L}_z = x\,\widehat{p}_y - y\,\widehat{p}_x.}$$

Equivalently, $\widehat{L}_j = -I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,x_k\partial_{x_l}$ $(j, k, l \in \{x, y, z\}$; Einstein summation).

Spin components

Using Pauli matrices σ_j that act on the two-component spinor part of the wave-function,

$$\boxed{\widehat{S}_x = \frac{\hbar}{2}\,\sigma_x\,1_{\mathbb{K}}, \quad \widehat{S}_y = \frac{\hbar}{2}\,\sigma_y\,1_{\mathbb{K}}, \quad \widehat{S}_z = \frac{\hbar}{2}\,\sigma_z\,1_{\mathbb{K}}.}$$

Total components

The total angular momentum is the block-wise sum

$$\boxed{\widehat{J}_j = \widehat{L}_j + \widehat{S}_j, \qquad j \in \{x, y, z\}.}$$

Commutation relations

Because $I_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, the usual $\mathfrak{su}(2)$ algebra holds in each quaternion block and hence in $\mathbb{K}$:

$$[\widehat{L}_j, \widehat{L}_k] = I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,\widehat{L}_l, \qquad [\widehat{S}_j, \widehat{S}_k] = I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,\widehat{S}_l, \qquad [\widehat{J}_j, \widehat{J}_k] = I_{\mathbb{K}}\hbar\,\varepsilon_{jkl}\,\widehat{J}_l.$$

z-component eigenvalues

For an eigenstate $\psi^{(r)}_{j,m_j}$ in block r,

$$\widehat{J}_z\,\psi^{(r)}_{j,m_j} = \hbar\,m_j\,\psi^{(r)}_{j,m_j}, \qquad m_j \in \{-j, -j+1, \ldots, j\}.$$

Raising and lowering operators

Define

$$\boxed{\widehat{J}_\pm := \widehat{J}_x \pm I_{\mathbb{K}}\,\widehat{J}_y}, \qquad [\widehat{J}_z, \widehat{J}_\pm] = \pm I_{\mathbb{K}}\hbar\,\widehat{J}_\pm, \qquad [\widehat{J}_+, \widehat{J}_-] = 2\,I_{\mathbb{K}}\hbar\,\widehat{J}_z.$$

Their action on $\psi^{(r)}_{j,m_j}$ is

$$\widehat{J}_\pm\,\psi^{(r)}_{j,m_j} = \hbar\sqrt{j(j+1) - m_j(m_j \pm 1)}\,\psi^{(r)}_{j,m_j\pm 1}.$$

Summary

The Cartesian components $(\widehat{J}_x, \widehat{J}_y, \widehat{J}_z)$ retain their familiar algebra and spectra within the Kerion framework. They are replicated—unchanged—across the three quaternion blocks, while the twelfth "persistence" coordinate may archive their time-dependent expectation values without influencing local rotational dynamics.

9.16 Orbital Magnetic Dipole Moment

All real scalars multiply the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3) \subset \mathbb{K} = \mathbb{H}^{(12)}$, and we fix the distinguished imaginary $I_{\mathbb{K}} := i\, 1_{\mathbb{K}}$. Let

$$q_{\mathbb{K}} = q\, 1_{\mathbb{K}}, \qquad m_{\mathbb{K}} = m\, 1_{\mathbb{K}}, \qquad \widehat{\mathbf{L}} = (\widehat{L}_x, \widehat{L}_y, \widehat{L}_z)$$

be the particle charge, mass, and orbital–angular–momentum operator already promoted to Kerions (cf. Section 9.15).

Dipole–moment operator. The **orbital magnetic dipole moment Kerion operator** is

$$\boxed{\widehat{\boldsymbol{\mu}}_{L,\mathbb{K}} \;:=\; -\frac{q_{\mathbb{K}}}{2m_{\mathbb{K}}}\,\widehat{\mathbf{L}} \;=\; -\frac{q}{2m}\,\widehat{\mathbf{L}}\,1_{\mathbb{K}}}. \tag{22}$$

Cartesian components. Using the components of $\widehat{\mathbf{L}}$ from Section 9.15,

$$\widehat{\mu}_{L,x} = -\frac{q}{2m}\,\widehat{L}_x, \qquad \widehat{\mu}_{L,y} = -\frac{q}{2m}\,\widehat{L}_y, \qquad \widehat{\mu}_{L,z} = -\frac{q}{2m}\,\widehat{L}_z,$$

each multiplied by $1_{\mathbb{K}}$.

Bohr magneton (electron). For an electron ($q = -e,\ m = m_e$) the *Bohr magneton Kerion* is

$$\boxed{\mu_{B,\mathbb{K}} := \frac{e\hbar}{2m_e}\,1_{\mathbb{K}}}, \qquad \widehat{\boldsymbol{\mu}}_{L,\mathbb{K}} = \mu_{B,\mathbb{K}}\,\frac{\widehat{\mathbf{L}}}{\hbar}.$$

Block decomposition. Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (22) factorises into three identical complex dipole–moment operators in the quaternion blocks:

$$\widehat{\boldsymbol{\mu}}_L^{(r)} = -\frac{q}{2m}\,\widehat{\mathbf{L}}^{(r)}, \qquad r = 1, 2, 3.$$

Expectation values. For an eigenstate $\psi^{(r)}_{l,m_l}$ of $\widehat{L}^2, \widehat{L}_z$ in block r,

$$\langle \widehat{\mu}_{L,z} \rangle^{(r)} = -\frac{q\hbar}{2m}\,m_l, \qquad \left|\langle \widehat{\boldsymbol{\mu}}_L \rangle^{(r)}\right| = \frac{|q|\hbar}{2m}\,\sqrt{l(l+1)}.$$

Hence the familiar orbital g-factor $g_L = 1$ and all numerical predictions remain unchanged—merely replicated across the three quaternion blocks—while the twelfth "persistence" coordinate can record their dynamical history without influencing local magnetic behaviour.

9.17 Spin Magnetic Dipole Moment

As before, all real scalars in the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ multiply the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$; the distinguished imaginary unit is $I_{\mathbb{K}} := i\, 1_{\mathbb{K}}$. Denote the particle charge and mass by the Kerions $q_{\mathbb{K}} = q\, 1_{\mathbb{K}},\ m_{\mathbb{K}} = m\, 1_{\mathbb{K}}$, and recall the spin operator components $\widehat{S}_j = \frac{\hbar}{2}\,\sigma_j\, 1_{\mathbb{K}}$ from Section 9.15.

Dipole–moment operator

For a Dirac particle with (dimension-less) spin g-factor g_S ($g_S \approx 2$ for an electron) the **spin magnetic dipole moment Kerion operator** is

$$\boxed{\widehat{\boldsymbol{\mu}}_{S,T} \;:=\; -\,g_S\,\frac{q_{\mathbb{K}}}{2m_{\mathbb{K}}}\,\widehat{\mathbf{S}} \;=\; -\,g_S\,\frac{q}{2m}\,\widehat{\mathbf{S}}\,1_{\mathbb{K}}}. \tag{23}$$

Cartesian components.

$$\widehat{\mu}_{S,x} = -\,g_S\,\frac{q}{2m}\,\widehat{S}_x, \qquad \widehat{\mu}_{S,y} = -\,g_S\,\frac{q}{2m}\,\widehat{S}_y, \qquad \widehat{\mu}_{S,z} = -\,g_S\,\frac{q}{2m}\,\widehat{S}_z,$$

each multiplied by $1_{\mathbb{K}}$.

Bohr magneton (electron)

For an electron ($q = -e,\ m = m_e,\ g_S \approx 2$) the *Bohr magneton Kerion* introduced in Section 9.16 becomes

$$\mu_{B,\mathbb{K}} = \frac{e\hbar}{2m_e}\,1_{\mathbb{K}}, \qquad \widehat{\boldsymbol{\mu}}_{S,\mathbb{K}} = -\,g_S\,\mu_{B,T}\,\frac{\widehat{\mathbf{S}}}{\hbar}.$$

Block decomposition

Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (23) factorises into three identical complex operators acting in the quaternion blocks:

$$\widehat{\boldsymbol{\mu}}_S^{(r)} = -\,g_S\,\frac{q}{2m}\,\widehat{\mathbf{S}}^{(r)}, \qquad r = 1,2,3.$$

Expectation values for a spin-$\frac{1}{2}$ particle

For a spin eigenstate $\chi_{m_s}^{(r)}$ in block r ($s = \frac{1}{2},\ m_s = \pm\frac{1}{2}$),

$$\langle \widehat{\mu}_{S,z} \rangle^{(r)} = -\,g_S\,\frac{q\hbar}{4m}\,m_s, \qquad \left|\langle \widehat{\boldsymbol{\mu}}_S \rangle^{(r)}\right| = g_S\,\frac{|q|\hbar}{4m}\,\sqrt{3}\,.$$

Summary

The spin magnetic dipole moment retains its familiar form inside the Kerion framework: each quaternion block exhibits the standard operator $\widehat{\boldsymbol{\mu}}_S = -\frac{g_S q}{2m}\,\widehat{\mathbf{S}}$, while the twelfth "persistence" dimension can chronicle its temporal behaviour without perturbing local magnetic dynamics.

9.18 Dipole–Moment Potential Energy

All real scalars in the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ multiply the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$; we write the distinguished imaginary unit as $I_{\mathbb{K}} := i\,1_{\mathbb{K}}$.

Magnetic dipole in an external field

Let

$$\widehat{\boldsymbol{\mu}}_{\mathbb{K}} = \widehat{\boldsymbol{\mu}}_{L,\mathbb{K}} + \widehat{\boldsymbol{\mu}}_{S,\mathbb{K}}, \qquad \mathbf{B}_{\mathbb{K}} = B_x\,i\,1_{\mathbb{K}} + B_y\,j\,1_{\mathbb{K}} + B_z\,k\,1_{\mathbb{K}}$$

denote, respectively, the total magnetic–dipole–moment operator (orbital + spin; cf. Sections 9.16–9.17) and a classical, static magnetic field promoted to $\mathbb{K}$. The **magnetic dipole potential–energy operator** is

$$\boxed{\widehat{U}_{B,\mathbb{K}} \;:=\; -\,\widehat{\boldsymbol{\mu}}_{\mathbb{K}}\!\cdot\!\mathbf{B}_{\mathbb{K}} \;=\; -\sum_{j\in\{x,y,z\}} \widehat{\mu}_j\,B_j\;1_{\mathbb{K}}}. \tag{24}$$

Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (24) factorises into three identical complex operators,

$$\widehat{U}_B^{(r)} = -\,\widehat{\boldsymbol{\mu}}^{(r)}\cdot\mathbf{B}, \qquad r = 1,2,3.$$

Electric dipole in an external field (optional)

If a permanent electric dipole $\boldsymbol{d}_{\mathbb{K}} = d_x\, i\, 1_{\mathbb{K}} + d_y\, j\, 1_{\mathbb{K}} + d_z\, k\, 1_{\mathbb{K}}$ interacts with a static electric field $\mathbf{E}_{\mathbb{K}} = E_x\, i\, 1_{\mathbb{K}} + E_y\, j\, 1_{\mathbb{K}} + E_z\, k\, 1_{\mathbb{K}}$, the corresponding **electric dipole potential energy** is

$$\boxed{U_{E,T} \;:=\; -\,\boldsymbol{d}_{\mathbb{K}}\cdot\mathbf{E}_{\mathbb{K}} \;=\; -\sum_{j\in\{x,y,z\}} d_j\, E_j\, 1_{\mathbb{K}}.} \tag{25}$$

Block decomposition and summary

Both $\widehat{U}_{B,\mathbb{K}}$ (quantum operator) and $U_{E,\mathbb{K}}$ (classical scalar) decompose into three identical complex expressions in the quaternion blocks,

$$\widehat{U}_B^{(r)} = -\,\widehat{\boldsymbol{\mu}}^{(r)}\cdot\mathbf{B}, \qquad U_E^{(r)} = -\,\boldsymbol{d}^{(r)}\cdot\mathbf{E}, \qquad r = 1,2,3.$$

Thus the familiar dipole–field interaction energies are preserved unchanged; the Kerion formalism merely packages three parallel copies into a single algebraic identity, while the twelfth "persistence" coordinate can log their temporal evolution without affecting local electromagnetic dynamics.

9.19 Hydrogen-Atom Energy Spectrum

In the Kerion algebra $\mathbb{K} = \mathbb{H}^{(12)}$ every real scalar multiplies the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$. Promote the Bohr constants to Kerions

$$\alpha_{\mathbb{K}} = \alpha\, 1_{\mathbb{K}}, \quad m_{e,T} = m_e\, 1_{\mathbb{K}}, \quad \hbar_{\mathbb{K}} = \hbar\, 1_{\mathbb{K}}.$$

Non-relativistic spectrum. Solving the Schrödinger equation in the Coulomb potential (Section 9.20) yields the **energy-level Kerion**

$$\boxed{E_{n,\mathbb{K}} \;=\; -\,\frac{m_{e,\mathbb{K}}\,\alpha_{\mathbb{K}}^2\, c^2}{2\,n^2} \;=\; \left(-\frac{m_e\,\alpha^2\, c^2}{2\,n^2}\right) 1_{\mathbb{K}}, \qquad n = 1,2,3,\ldots} \tag{26}$$

where $\alpha = e^2/(4\pi\varepsilon_0\hbar c)$ is the fine-structure constant and c is the speed of light.

Component form. Because $1_{\mathbb{K}}$ commutes with every element of $\mathbb{K}$, Eq. (26) decomposes into three identical scalar energy ladders,

$$E_n^{(r)} = -\,\frac{m_e\,\alpha^2\,c^2}{2\,n^2}, \qquad r = 1,2,3,$$

exactly reproducing the standard Balmer series in each quaternion block.

Relativistic (Dirac) correction. Including fine-structure via the Dirac equation introduces the usual quantum numbers (n, j) and modifies the spectrum to

$$E_{n,j,\mathbb{K}} \;=\; m_{e,\mathbb{K}}\, c^2 \left[1 + \frac{\alpha_{\mathbb{K}}^2}{\left(n - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - \alpha_{\mathbb{K}}^2}\right)^2}\right]^{-\frac{1}{2}},$$

which again factorises into three identical Dirac energy series—one per quaternion block—while the twelfth "persistence" coordinate may archive level populations or transition histories without influencing local atomic physics.

9.20 Schrödinger Dynamics in the Kerion Algebra

The *Kerion* algebra $T = \mathbb{H}^{(12)} = H \widehat{\otimes}_{\mathbb{R}} \operatorname{Span}_{\mathbb{R}}\{e_1, e_2, e_3\}$ is the orthogonal direct sum of three quaternion blocks $H \otimes e_r$ ($r = 1, 2, 3$). All constructions below therefore act *block-wise* and reduce, for each r, to the standard quaternionic (or complex) theory.

9.20.1 Kerion wave-functions and inner product

Let

$$\Psi : \mathbb{R}^{1+3} \longrightarrow T, \qquad \Psi(t, \mathbf{x}) = \sum_{r=1}^{3} \psi^{(r)}(t, \mathbf{x}) \otimes e_r,$$

where every component $\psi^{(r)} : \mathbb{R}^{1+3} \to H$ is square-integrable. Equip the space $L^2(\mathbb{R}^3, \mathbb{K})$ with the real-valued inner product

$$\langle \Psi_1, \Psi_2 \rangle := \sum_{r=1}^{3} \int_{\mathbb{R}^3} \big(\overline{\psi_1^{(r)}(t, \mathbf{x})}\, \psi_2^{(r)}(t, \mathbf{x})\big)\, d^3\mathbf{x},$$

so that each block inherits the usual quaternionic Hilbert structure. (Complex quantum mechanics is recovered by restricting $\psi^{(r)}(t, \mathbf{x})$ to the subalgebra $\mathbb{C} \subset H$.)

9.20.2 Hamiltonian operator

For a scalar potential $V : \mathbb{R}^3 \to \mathbb{R}$, define the Kerion Hamiltonian

$$\widehat{H} \; := \; -\frac{\hbar^2}{2m} \nabla^2\, 1_{\mathbb{K}} \; + \; V(\mathbf{x})\, 1_{\mathbb{K}},$$

where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$. Because $1_{\mathbb{K}} = \sum_r 1_H \otimes e_r$ commutes with the potential, $\widehat{H}$ acts identically in every block.

9.20.3 Kerionic Schrödinger equation

Choose the quaternion unit $i \in H$ as the distinguished imaginary unit and promote it to $I_{\mathbb{K}} := i\, 1_{\mathbb{K}} \in \mathbb{K}$. The time evolution of a Kerion wave-function is then governed by the single equation

$$\boxed{I_{\mathbb{K}} \hbar\, \partial_t \Psi(t, \mathbf{x}) \; = \; \widehat{H}\, \Psi(t, \mathbf{x}).} \tag{27}$$

Expanding (27) in blocks gives three independent copies

$$i\hbar\, \partial_t \psi^{(r)} \; = \; \Big(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x})\Big) \psi^{(r)}, \qquad r = 1, 2, 3,$$

which is precisely the usual (complex) Schrödinger equation for each $\psi^{(r)}$.

9.20.4 Remarks

1. Intrinsic multiplicity. A single Kerion wave-function Ψ simultaneously encodes three independent quantum systems; the extra "persistence" coefficients introduced in Section 2.6 can record the entire time-history of each block without disturbing the local dynamics.

2. Super-selection of the imaginary unit. Choosing i as the global imaginary ensures that I_T commutes with $\widehat{H}$, so probability conservation follows exactly as in complex quantum mechanics. One may equally pick j or k; all choices are related by Kerion automorphisms.

3. Coupling to gauge fields. Minimal coupling $\nabla \mapsto \nabla + \frac{q}{\hbar}\mathbf{A}$ and the scalar potential shift $V \mapsto V - q\varphi$ lift unchanged to (27), reproducing the Kerion form of the Pauli–Schrödinger equation in an external electromagnetic field (cf. Section 8).

4. Complex limit. Restricting to block $r = 1$ and $\psi^{(1)} \in \mathbb{C} \subset H$ collapses (27) to the traditional Schrödinger equation in $\mathbb{C}$.

10 Conclusion

The twelve–dimensional *Kerion* algebra $\mathbb{K} = \mathbb{H}^{(12)} = H\widehat{\otimes}_{\mathbb{R}}E$ offers a conservative extension of conventional formalisms rather than a radical departure.

Block-wise compatibility. Because the product in $\mathbb{K}$ is block-diagonal, every equation built from real scalars or the global unit $1_{\mathbb{K}} := 1_H \otimes (e_1 + e_2 + e_3)$ factorises into three independent quaternion (hence complex) copies. Maxwell's compact pair $\partial F = J$, $\bar{\partial} F = 0$ therefore reproduces the four standard component equations in each block without altering gauge structure or Lorentz covariance. Likewise, the Kerionic Schrödinger (20), Heisenberg, and Planck–Einstein relations disentangle into textbook expressions for every block, ensuring that spectra, cross-sections, and statistical distributions (Bose–Einstein, Fermi–Dirac) remain numerically unchanged.

Intrinsic memory layer. The extra basis vector e_{12} stores a *persistence coordinate* that can archive the full time-evolution of any observable (e.g. field strengths, wave-functions, occupation numbers) without feeding back into local dynamics. This built-in "black-box recorder" is invisible to experiments restricted to four-space-time but becomes available for coarse-graining histories, defining path-dependent actions, or coupling to computational models of emergence.

Redundancy and multiplexing. Packaging three identical dynamical copies inside $\mathbb{K}$ admits novel interpretations: parallel simulations of stochastic branches, error-correcting redundancy, or simultaneous modelling of isospin-like degrees of freedom. In each case the physical predictions per block match the orthodox theory, so consistency with experiment is automatically inherited.

Minimal cost of adoption. Because the Kerion upgrade is effected by the replacement $a \mapsto a\,1_{\mathbb{K}}$ for every real scalar, existing analytic and numerical codes require only a trivial wrapper to lift scalars to $1_{\mathbb{K}}$ and to vectorise over three blocks. No additional renormalisation, gauge fixing, or regularisation is introduced.

Outlook. Future work should investigate whether interactions *between* quaternion blocks—suppressed here by the chosen multiplication—can model phenomena such as flavour oscillations or entangled histories, and whether the persistence axis provides a natural arena for entropy production in non-equilibrium systems.

In short, Kerions supply an algebraic envelope that *preserves* all established predictions while endowing them with fractal memory and latent parallelism, making the framework immediately applicable to traditional electrodynamics, quantum mechanics, and statistical physics.

11 Gentle Primer on Key Concepts

Purpose. This appendix distills the minimum background needed to follow the main text without assuming prior training in abstract algebra or functional analysis. Each topic is introduced through everyday intuition first, then linked—only as far as necessary—to the precise language used in the paper.

11.1 Fractals and Self-Similarity

- **Fractal**,: a shape that looks similar at many different zoom levels. Classic examples include snowflakes, coastlines, and the branching of trees.
- **Self-similarity**,: the property that each part of the shape is a smaller copy of the whole. Precisely, a set \$F\$ is self-similar if it can be written as a union of scaled-down copies of itself.
- **Why it matters here**,: physical phenomena—crack patterns, turbulence, even galaxy clusters—often display scale-free structure. The paper models them with mathematics that is natively self-similar.

11.2 Iterated Function Systems (IFS)

- An *IFS* is just a small collection of simple geometric moves ("functions"). Starting from any shape and repeatedly applying those moves produces a unique fractal "attractor".
- Think of pressing a stamp over and over, but shrinking and moving it each time—you eventually fill out a complex picture.
- In the paper, an IFS builds each time–slice of the fractal dynamics.

11.3 Quaternions and Beyond: Twelve Dimensions in a Nutshell

- **Quaternions**,: a four-number upgrade of ordinary complex numbers, ideal for describing 3-D rotations (widely used in computer graphics).
- **Kerion algebra**,: the paper takes three independent quaternion copies (for space, an internal "Calabi–Yau" space, and a memory axis) and stitches them side-by-side, giving 12 coordinates total.
- **Why 12?** Eleven coordinates already appear in M-theory; the twelfth records the full history of the other eleven—like a built-in black-box recorder.

11.4 Banach and Hilbert Spaces—Only the Idea

- A *space* is just a collection of objects that can be added and scaled.
- A **Banach space** is one where you can measure the size ("norm") of every object and guarantee that infinite processes converge sensibly.
- A **Hilbert space** adds an inner product—a way to measure angles and projections—letting you talk about orthogonality and energy.
- The upgrade from Banach to Hilbert supplies the tools of quantum mechanics without changing the underlying dynamics.

11.5 Union as an Evolution Rule

- Instead of adding numbers or multiplying matrices, evolution here is just *taking the union* of two fractals (after sliding one forward in the memory direction).
- Union preserves the key geometric properties (compact, perfect, self-similar) while letting the pattern grow.

11.6 Embedding Tensors and Tensor-Hierarchy Algebras (THA)

- These intimidating names hide a simple goal: *systematically glue* lower-dimensional objects into higher-dimensional ones while keeping track of symmetries.
- The **embedding tensor** is a recipe for turning two Kerions into something that lives in an internal symmetry algebra (here three copies of $su(2)$).
- The **THA** provides the bookkeeping rules so that algebraic operations stay associative and well behaved.

11.7 Maxwell and Quantum Equations in Kerions

- Maxwell's four equations collapse to a single compact Kerion equation. Each of the three quaternion blocks carries an ordinary copy of electromagnetism.
- Likewise, the Schrödinger equation evolves three parallel quantum systems at once—offering redundancy or "parallel universes" in simulation—without changing familiar predictions.

11.8 The Persistence Axis—Built-In Memory

- One coordinate (labelled e_12 in the paper) is singled out to store the *entire time history.* Sliding a fractal along this axis is like adding a new frame to a filmstrip.
- Crucially, this memory does not feed back into the local physics, so standard results remain intact while long-term patterns can be logged.

11.9 Quick Reading Tips

1. Skim Section 2 first—its Table **??** lists every symbol and how to pronounce it.
2. Keep this appendix open; flip back whenever a new term appears.
3. Remember: each equation secretly contains *three* familiar copies (one per quaternion block). Focus on a single block to recover standard physics.

11.10 Glossary of Symbols and Operators

Notation follows the conventions of Sections 2–6 of the main text. Greek indices μ, ν, κ *run over* $0, 1, 2, 3$*; Latin indices* i, j, k *run over* $1, 2, 3$*.*

K Kerion algebra, a 12-dimensional space built from three orthogonal quaternion blocks.

$F_{\mathbf{perf}}$ Family of perfect, compact, self-similar (OSC-attractor) fractals.

U_τ Union-evolution operator that adjoins a fractal to its translate by the time-step τ.

T_τ Translation by τ along the persistence (memory) axis.

e_{12} Basis vector $E_3^{(3)}$ pointing along the persistence axis; $\{e_n\}_{n=1}^{12}$ is the full coordinate basis of K.

Θ Embedding tensor $K \otimes K \to \mathfrak{g}$ that defines (and associates) the Kerion product.

∂ Kerionic four-gradient $\partial_t 1_K + \nabla 1_K$; its conjugate is $\bar{\partial} = \partial_t 1_K - \nabla 1_K$.

1_K Global algebraic unit $E_0^{(1)} + E_0^{(2)} + E_0^{(3)}$ acting as the identity in K.

I_K Distinguished imaginary unit $i\, 1_K$ (also denoted I_T in the full THA context).

g Internal symmetry algebra $\mathfrak{su}(2)_{(1)} \oplus \mathfrak{su}(2)_{(2)} \oplus \mathfrak{su}(2)_{(3)}$.

λ Dominant integral weight $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ that seeds the THA.

$T(g, \lambda)$ Tensor–hierarchy algebra obtained by attaching a grey node of weight λ to the Dynkin diagram of g.

T_p p-th graded component of the THA; T_{+1} is canonically K and $T_0 \cong g$.

T_{+1} Degree $+1$ component of the THA, identified with the Kerion space K.

$E_\mu^{(r)}$ Basis of T_{+1} ($r = 1, 2, 3$ labels the quaternion block, $\mu = 0, 1, 2, 3$ the component).

κ Injective binary-encoding map $F_{\text{perf}} \to (0, 1)$ that turns fractals into real numbers.

H_{ℓ^1} Banach tensor product $\ell^1(\mathbb{R}) \widehat{\otimes}_\pi \mathbb{H}$ supporting quaternion series.

E_K, Ω_K Energy and angular-frequency Kerions ($E\, 1_K$ and $\omega\, 1_K$).

β_K, μ_K Thermodynamic inverse-temperature and chemical-potential Kerions.

$g_K(E_K)$ Density-of-states Kerion as a function of E_K.

f_K, n_K Kerionic Fermi–Dirac and Bose–Einstein occupation probabilities.

K_K Wave-vector Kerion $(k_x i + k_y j + k_z k)\, 1_K$.

P_K Momentum Kerion $(p_x i + p_y j + p_z k)\, 1_K$.

L_b, S_b, J_b Orbital, spin, and total angular-momentum Kerions.

H_b Kerion Hamiltonian acting on $\mathcal{H}_b = K \otimes_b \ell^2(\mathbb{Z})$.

$\mu_{L,K}, \mu_{S,K}$ Orbital and spin magnetic-dipole-moment Kerions.

δ_{ij} Kronecker delta ($\delta_{ij} = 1$ if $i = j$, else 0).

ε_{ijk} Levi-Civita symbol with $\varepsilon_{123} = +1$.

∇ Spatial gradient $i\, \partial_x + j\, \partial_y + k\, \partial_z$ inside a fixed quaternion block.

$\times$ Vector (cross) product in $\mathbb{R}^3$, applied block-wise.

$\otimes$ Tensor (outer) product of vectors or linear maps.

$\oplus$ Direct sum of vector spaces or Lie algebras.

$\cdot$ Kerion multiplication induced by Θ (coincides with quaternion product inside each block).

$\cup, \cap$ Set-theoretic union and intersection.

J Current Kerion $\rho\, 1_K - \mathbf{J} \cdot (i, j, k)$.

F Field Kerion $F = \partial A - A\partial$, packaging $\mathbf{E}$ and $\mathbf{B}$.

A Kerion four-potential $\phi\, 1_K + \mathbf{A} \cdot (i, j, k)$.

$*$ Cauchy (convolution) product of series in ℓ^1.

ℓ^1, ℓ^2 Sequence spaces of absolutely-summable and square-summable series, respectively.

d_H Hausdorff distance on non-empty compact subsets of $\mathbb{R}^d$.

$\dim_H$ Hausdorff (fractal) dimension.

Take-away: If you can picture a snowflake, recall that quaternions help rotate 3-D models in video games, and accept that a big vector can record its own past, you already grasp 90

Declaration of generative AI and AI-assisted technologies in the writing process During the preparation of this work the author used OpenAI O3 model in order to check correctness. After using this tool, the author reviewed and edited the content as needed and take full responsibility for the content of the publication.

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