1) 10.2

$$\max \ \xi^*(b) = c^T x$$
$$\text{s.t} \quad Ax \leq b$$
$$x \geq 0$$

since $\xi^*(b) < \infty$ for all b $\Rightarrow$ Primal is always feasible and bounded $\Rightarrow$ strong duality holds.

So, (D)

$$\min \ \xi^*(b) = \min \ b^T y$$
$$\text{s.t} \quad A^T y \geq c$$
$$y \geq 0$$

Note that for each $\vec{y} \in Y$, the function

$b \mapsto b^T y$ is linear, and hence both convex and concave

So, $\xi^*(b)$ is the minimum of many linear functions:

$$\xi^*(b) = \min_{y \in Y} b^T y$$

so, suppose that $b_1, b_2 \in \mathbb{R}^m$ and $0 < t < 1$

and $y^* \in Y$ be the optimal dual vector for

$$\begin{aligned}
\xi^*(t b_1 + (1-t) b_2) &= (t b_1 + (1-t) b_2)^T y^* \\
&= t b_1^T y^* + (1-t) b_2^T y^* \\
&\geq t \min b_1^T y^* + (1-t) \min b_2^T y^* \\
&= t\, \xi^*(b_1) + (1-t)\, \xi^*(b_2) \Rightarrow \text{concave by defn.}
\end{aligned}$$

23.4 Initial dictionary

Hint Undo

Number format: Fraction Zero Visibility: Visible

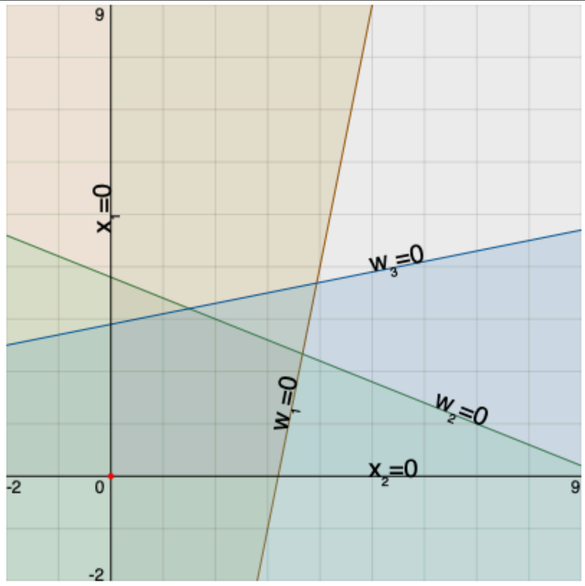
Current Dictionary

$$\begin{array}{lrcl} \text{maximize} & \zeta & = & 0 + 8\,x_1 + 13\,x_2 \\ \text{subject to:} & w_1 & = & 32 - 10\,x_1 - (-2)\,x_2 \\ & w_2 & = & 38 - 4\,x_1 - 10\,x_2 \\ & w_3 & = & 29 - (-2)\,x_1 - 10\,x_2 \end{array}$$

$x_1 \;\; x_2 \;\; w_1 \;\; w_2 \;\; w_3 \geq 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson



if we solve the relaxed problem we get:

Hint Undo

Number format: Fraction Zero Visibility: Visible

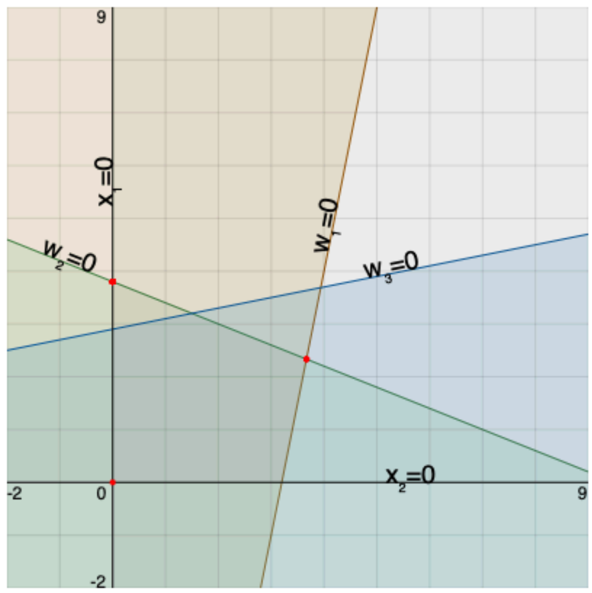
Current Dictionary

$$\begin{array}{lrcl} \text{maximize} & \zeta & = & 179/3 + (-7/27)\,w_1 + (-73/54)\,w_2 \\ \text{subject to:} & x_1 & = & 11/3 - 5/54\,w_1 - 1/54\,w_2 \\ & x_2 & = & 7/3 - (-1/27)\,w_1 - 5/54\,w_2 \\ & w_3 & = & 13 - 5/9\,w_1 - (-8/9)\,w_2 \end{array}$$

$x_1 \;\; x_2 \;\; w_1 \;\; w_2 \;\; w_3 \geq 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson



$(11/3, 7/3, 0, 0, 13) \leftarrow$ Relaxed optimal

$$\boxed{\begin{aligned} P_0 : x_1 = 11/3,\ x_2 = 7/3 \\ \zeta = \frac{179}{3} \end{aligned}}$$

$x_1 \le 3$ $\qquad$ $x_1 \ge 4$

$\boxed{\text{infeasible}}$

we prune this because it is infeasible, fails some constraints

$x_1 \le 3 \Rightarrow$

$$w_4 = 3 - \frac{11}{3} + \frac{5}{54} w_1 + \frac{1}{54} w_2$$
$$= -\frac{2}{3} + \frac{5}{54} w_1 + \frac{1}{54} w_2$$

New dictionary

Hint Undo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

maximize	ζ	=	179/3	+	-7/27	w_1	+	-73/54	w_2
subject to:	x_1	=	11/3	-	5/54	w_1	-	1/54	w_2
	x_2	=	7/3	-	-1/27	w_1	-	5/54	w_2
	w_3	=	13	-	-5/9	w_1	-	-8/9	w_2
	w_4	=	-2/3	-	-5/54	w_1	-	-1/54	w_2

$w_1\ w_2\ x_1\ x_2\ w_3\ w_4 \ge 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson

optimal dictionary

Hint Undo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

maximize	ζ	=	289/5	+	-14/5	w_4	+	-13/10	w_2
subject to:	x_1	=	3	-	1	w_4	-	0	w_2
	x_2	=	13/5	-	-2/5	w_4	-	1/10	w_2
	w_3	=	17	-	-6	w_4	-	-7/9	w_2
	w_1	=	36/5	-	-54/5	w_4	-	1/5	w_2

$w_1\ w_2\ x_1\ x_2\ w_3\ w_4 \ge 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson

$$\boxed{\begin{aligned} P_1 : x_1 = 3,\ x_2 = \frac{13}{5} \\ \zeta = \frac{289}{5} \end{aligned}}$$

$\downarrow\ x_2 \le 2 \Rightarrow w_5 = -\frac{3}{5} - \frac{2}{5} w_4 + \frac{1}{10} w_2$

Current Dictionary

← new dictionary

maximize	ζ	=	289/5	+	-14/5	w₄	+	-13/10	w₂
subject to:	x_1	=	3	-	1	w_4	-	0	w_2
	x_2	=	13/5	-	-2/5	w_4	-	1/10	w_2
	w_3	=	17	-	-6	w_4	-	-7/9	w_2
	w_1	=	36/5	-	-54/5	w_4	-	1/5	w_2
	w_5	=	-3/5	-	2/5	w_4	-	-1/10	w_2

$w_4\ w_2\ x_1\ x_2\ w_3\ w_1\ w_5 \ge 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson

Hint Undo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

← optimal

maximize	ζ	=	50	+	-8	w_4	+	-13	w_5
subject to:	x_1	=	3	-	1	w_4	-	0	w_5
	x_2	=	2	-	0	w_4	-	1	w_5
	w_3	=	65/3	-	-82/9	w_4	-	-70/9	w_5
	w_1	=	6	-	-10	w_4	-	2	w_5
	w_2	=	6	-	-4	w_4	-	-10	w_5

$w_4\ w_2\ x_1\ x_2\ w_3\ w_1\ w_5 \ge 0$

Optimal Infeasible Unbounded

$x_2 \ge 3 \Rightarrow g_1 = x_2 - 3 = -\frac{2}{5} + \frac{2}{5} w_4 - \frac{1}{10} w_2$

P_2: $x_1 = 3$, $x_2 = 2$

$\zeta = 50$

initial →

Hint Undo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

maximize	ζ	=	289/5	+	-14/5	w_4	+	-13/10	w_2
subject to:	x_1	=	3	-	1	w_4	-	0	w_2
	x_2	=	13/5	-	-2/5	w_4	-	1/10	w_2
	w_3	=	17	-	-6	w_4	-	-7/9	w_2
	w_1	=	36/5	-	-54/5	w_4	-	1/5	w_2
	g_1	=	-2/5	-	-2/5	w_4	-	1/10	w_2

$w_4\ w_2\ x_1\ x_2\ w_3\ w_1\ w_5 \ge 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson

Hint Undo

Number format: Fraction Zero Visibility: Visible

optimal →

Current Dictionary

maximize	ζ	=	55	+	-7	g_1	+	-2	w
subject to:	x_1	=	2	-	5/2	g_1	-	1/4	w
	x_2	=	3	-	-1	g_1	-	0	w
	w_3	=	23	-	-15	g_1	-	-41/18	w
	w_1	=	18	-	-27	g_1	-	-5/2	w
	w_4	=	1	-	-5/2	g_1	-	-1/4	w

$w_4\ w_2\ x_1\ x_2\ w_3\ w_1\ w_5 \ge 0$

Thus we get:

$P_0: x_1 = \frac{11}{3}, \quad x_2 = \frac{7}{3}$

$z = \frac{179}{3}$

$x_1 \leq 3$

$P_1: x_1 = 3, \quad x_2 = \frac{13}{5}$

$z = \frac{289}{5}$

$x_2 \leq 2$

$P_2: x_1 = 3, \quad x_2 = 2$

$z = 50$

$x_2 \geq 3$

$P_3: x_1 = 2, \quad x_2 = 3$

$z = 55$ ← optimal.

23.5

Initial dictionary

Hint Undo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

$$
\begin{array}{lrcl}
\text{maximize} & \zeta & = & 17x_1 + 12x_2 \\
\text{subject to:} & w_1 & = & 40 - 10x_1 - 7x_2 \\
 & w_2 & = & 5 - 1x_1 - 1x_2
\end{array}
$$

$x_1 \quad x_2 \quad w_1 \quad w_2 \geq 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson

Optimal dictionary

Hint Undo

Number format: Fraction Zero Visibility: Visible

Current Dictionary

$$
\begin{array}{lrcl}
\text{maximize} & \zeta & = & 205/3 + -5/3\, w_1 + -1/3\, w_2 \\
\text{subject to:} & x_1 & = & 5/3 - 1/3\, w_1 - -7/3\, w_2 \\
 & x_2 & = & 10/3 - -1/3\, w_1 - 10/3\, w_2
\end{array}
$$

$w_1 \quad w_2 \quad x_1 \quad x_2 \geq 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson

$$x_1 + \frac{1}{3}w_1 - \frac{7}{3}w_2 = \frac{5}{3} \Rightarrow x_1 + 0w_1 - 2w_2 \le 1$$

$$\frac{5}{3} - \frac{1}{3}w_1 + \frac{7}{3}w_2 - 2w_2 \le 1$$

$$\Rightarrow w_3 = -\frac{2}{3} + \frac{1}{3}w_1 + \frac{2}{3}w_2$$

Initial

Current Dictionary

maximize	ζ	=	205/3	+	-5/3	w_1	+	-1/3	w_2
subject to:	x_1	=	5/3	-	1/3	w_1	-	-7/3	w_2
	x_2	=	10/3	-	-1/3	w_1	-	10/3	w_2
	w_3	=	-2/3	-	-1/3	w_1	-	-2/3	w_2

$w_1 \ w_2 \quad x_1 \ x_2 \ w_3 \ge 0$

Optimal

maximize	ζ	=	65	+	-5	w_1	+	-2	w_2
subject to:	x_1	=	1	-	1	w_3	-	-2	w_2
	x_2	=	4	-	-1	w_3	-	3	w_2
	w_1	=	2	-	-3	w_3	-	-1	w_2

$w_1 \ w_2 \quad x_1 \ x_2 \ w_3 \ge 0$

$$x_2 - \frac{1}{3}w_1 + \frac{10}{3}w_2 = \frac{10}{3} \Rightarrow x_2 - 0w_1 + 3w_2 \le 3$$

$$\frac{10}{3} + \frac{1}{3}w_1 - \frac{10}{3}w_2 + 3w_2 \le 3$$

$$w_3 = -\frac{1}{3} - \frac{1}{3}w_1 + \frac{1}{3}w_2$$

we get the following optimal dictionary

$$
\begin{array}{llcl}
\text{maximize} & \zeta & = & 68 + (-2)\, w_1 + (-1)\, w_3 \\
\text{subject to:} & x_1 & = & 4 - (-2)\, w_1 - (-7)\, w_3 \\
& x_2 & = & 0 - 3\, w_1 - 10\, w_3 \\
& w_2 & = & 1 - (-1)\, w_1 - (-3)\, w_3 \\
\end{array}
$$

$$w_1 \quad w_2 \quad x_1 \quad x_2 \quad w_3 \geq 0$$

soln: $x_1 = 4$, $x_2 = 0$, $\zeta = 68$

2. A is $m \times n$, $x \in \mathbb{R}^n$ (column) $y \in \mathbb{R}^m$ (row) $(y(Ax) = (yA)x$

(i) $\Leftarrow$ Suppose that $Ax < 0$ has a soln: x

then each $(Ax)_i < 0$ and $y_i \geq 0$ with at least one $y_i > 0$.

so, $$y(Ax) = \sum_{i=1}^{m} y_i (Ax)_i < 0$$

every term is ≤ 0 and at least one is < 0

but $yA = 0$, so

$$y(Ax) = (yA)x = 0x = 0 \ \lightning \text{ contradiction}$$

so $Ax < 0$ must be unsolvable.

(ii) $\Leftarrow$ Suppose there is x with $Ax \leq 0$ and $bx > 0$

so $y \geq 0 \Rightarrow y(Ax) = \sum_i y_i (Ax)_i \leq 0$

but $y(Ax) = (yA)x = bx > 0$

and $yA = b$, so $\lightning$ contradiction

$Ax \leq 0$, $bx \geq 0$ is unsolvable

(iii) $\Leftarrow$ Suppose $Ax = 0$, $x > 0$ has soln x.

then $0 = y(Ax) = (yA)x = cx$

but $c \geq 0$ and $c \neq 0$ so some $c_i > 0$ and all $x_i > 0 \Longrightarrow$ $cx = \sum_i c_i x_i > 0$

hmm.

which is a contradiction. ↯

(iv) $\Leftarrow$ Suppose that $Ax < 0$, $x \geq 0$ has soln x

so $Ax < 0$ and $y \geq 0$, $y \neq 0$

$$y(Ax) = \sum_i y_i (Ax)_i < 0$$

but $yA \geq 0$ and $x \geq 0$

$$y(Ax) = (yA)x = \sum_i (yA)_i x_i \geq 0 \text{ ↯ contradiction}$$

then is no such x

(v) Suppose there is $x \geq 0$ w/ $Ax \geq 0$ and $x_k > 0$

so, $Ax > 0$ and $y \geq 0$

$\Rightarrow y(Ax) \geq 0$ (*)

and $(yA)x = \sum_{j}^{n} x_j \left(\sum_{i}^{m} y_i a_{ij} \right)$

for each j: $\sum_{i} y_i a_{ij} \leq 0$ bc $yA \leq 0$ and $x_j \geq 0$

for $j \neq k$: ≤ 0

for $j = k$: $\sum y_i a_{ij} < 0$, $x_k > 0$, $j = k$ term < 0

thus we get $(yA)x < 0$ (**)

but $y(Ax) = (yA)x$

so (*) and (**) contradict each other $\lightning$

thus there is no soln of $Ax \geq 0$, $x \geq 0$ w/ $x_k > 0$.

3. $P_1 = \{ x_1 \geq 0;\ x_2 \geq \frac{x_1}{2} + 5;\ x_1 + x_2 \geq 10 \}$

$P_2 = \{ x_1 = -1,\ x_2 = 7 \}$

$P_2 = \{ x_1 \leq -1;\ -x_1 \leq 1;\ x_2 \leq 7,\ -x_2 \leq -7 \}$

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ -1/2 & 1 \\ 1 & 1 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \geq \overset{b}{\begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix}} \quad \Bigg| \quad \overset{G}{\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \leq \overset{h}{\begin{bmatrix} -1 \\ 1 \\ 7 \\ -7 \end{bmatrix}}$$

$$\max \quad b^T y - h^T z = 0y_1 + 5y_2 + 10y_3 + 1z_1 - 1z_2 - 7z_3 + 7z_4$$

s.t

$$A^T y - G^T z = \begin{bmatrix} 1 & -1/2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} - \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = 0$$

$$1y_1 - \tfrac{1}{2}y_2 + y_3 - 1z_1 + 1z_2 - 0z_3 - 0z_4 = 0$$

$$0y_1 + y_2 + y_3 - 0z_1 - 0z_2 - 1z_3 + 1z_4 = 0$$

$$1y^T + 1^T z = y_1 + y_2 + y_3 + z_1 + z_2 + z_3 + z_4 = 1$$

$$y_i \geq 0, \quad z_i \geq 0$$

Initial dictionary:

Basic	y_1	y_2	y_3	z_1	z_2	z_3	z_4	Y_1	Y_2	Y_3	Solution
Z	0	0	0	0	0	0	0	$-M$	$-M$	$-M$	0
Y_1	1	$-\frac{1}{2}$	1	-1	1	0	0	1	0	0	0
Y_2	0	1	1	0	0	-1	1	0	1	0	0
Y_3	1	1	1	1	1	1	1	0	0	1	1

optimal dictionary:

Basic	y_1	y_2	y_3	z_1	z_2	z_3	z_4	Solution
Z	$\frac{5}{3}$	$\frac{9}{2}$	0	0	$\frac{8}{3}$	0	$\frac{8}{3}$	$\frac{4}{3}$
y_3	$\frac{2}{3}$	$\frac{1}{2}$	1	0	$\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$
z_1	$-\frac{1}{3}$	1	0	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$
z_3	$\frac{2}{3}$	$-\frac{1}{2}$	0	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$

soln: $(\underbrace{0,0,\tfrac{1}{3}}_{y},\underbrace{\tfrac{1}{3},0,\tfrac{1}{3},0}_{z})$

$$a = A^T y = \begin{bmatrix} 1 & -1/2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$$

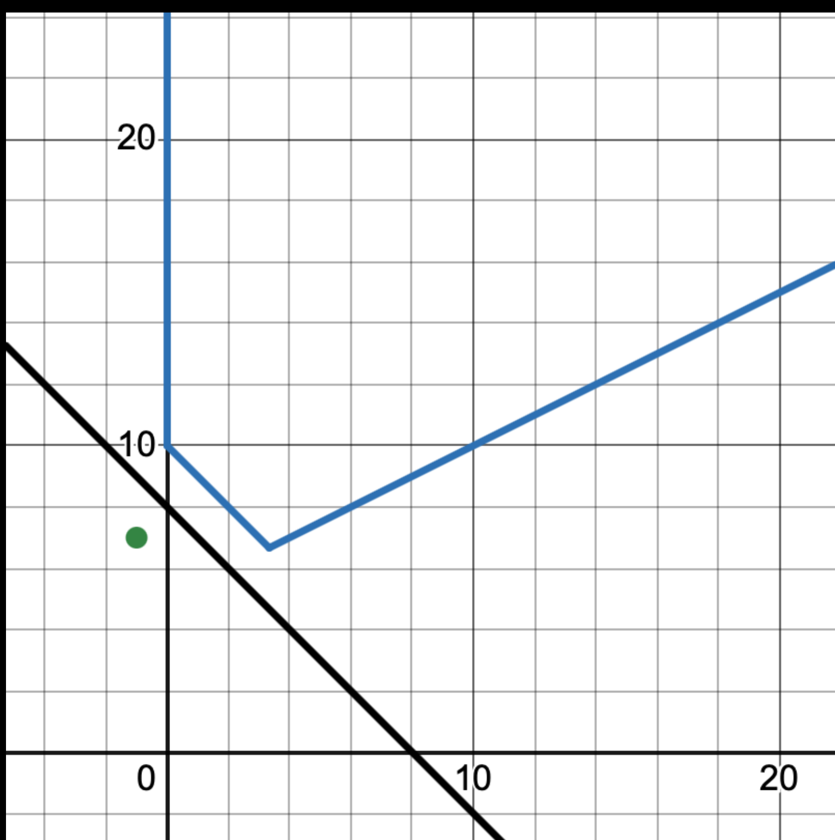
$$\beta = \frac{1}{2}\left(b^T y + h^T z\right) = \frac{1}{2}\left((0\ 5\ 10)\begin{pmatrix}0\\0\\1/3\end{pmatrix} + (-1\ 1\ 7\ -7)\begin{pmatrix}1/3\\0\\1/3\\0\end{pmatrix}\right)$$

$$= \frac{1}{2}\left(10/3 + (-1/3 + 7/3)\right)$$

$$= \frac{1}{2}\left(10/3 + (6/3)\right) = \frac{1}{\cancel{2}}\left(\frac{\cancel{16}^{\,8}}{3}\right) = \frac{8}{3}$$

$$\begin{pmatrix}1/3 & 1/3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \frac{8}{3}$$

$$1/3\, x_1 + 1/3\, x_2 = \frac{8}{3} \leftarrow \text{hyperplane}$$



4. a) $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$

s.t $g_1 = \frac{1}{5}x_2 - x_1^2 \leq 0$

$g_2 \quad 5x_1 + \frac{1}{2}x_2 - 5 \leq 0$

$(-1+\sqrt{3}, 20-10\sqrt{3})$ ✓

$(-\sqrt{3}-1, 10\sqrt{3}+20)$ ✓

$\Big\}$ Solns

$L(x, y, \mu_1, \mu_2) = f(x, y) + \mu_1 g_1 + \mu_2 g_2 \quad , \mu_1, \mu_2 \geq 0$

$= x^2 + y^2 - 2x - 10y + 26 + \mu_1\left(\frac{1}{5}y - x^2\right) + \mu_2\left(5x + \frac{1}{2}y - 5\right)$

$= x^2 + y^2 - 2x - 10y + 26 + \frac{1}{5}\mu_1 y - \mu_1 x^2 + 5\mu_2 x + \frac{1}{2}\mu_2 y - 5\mu_2$

$$\nabla L = 0 \Rightarrow \begin{cases} \frac{\partial}{\partial x} = 2x - 2 - 2\mu_1 x + 5\mu_2 = 0 & (1) \\ \frac{\partial}{\partial y} = 2y - 10 + \frac{1}{5}\mu_1 + \frac{1}{2}\mu_2 = 0 & (2) \\ \frac{\partial}{\partial \mu_1} = \frac{1}{5}y - x^2 = 0 & (3) \\ \frac{\partial}{\partial \mu_2} = 5x + \frac{1}{2}y - 5 = 0 & (4) \end{cases}$$

① $\mu_1 = \mu_2 = 0$

$$\left.\begin{array}{l} 2x_1 - 2 = 0 \Rightarrow x_1 = 1 \\ 2x_2 - 10 = 0 \Rightarrow x_2 = 5 \end{array}\right\} g_2 > 0 \quad \lightning \text{ conch.}$$

② $g_1 = 0$, $\mu_1 > 0$, g_2 inactive

$g_1 = 0 \Rightarrow x_2 = 5x_1^2$

$$\begin{cases} 2x_1 - 2 - 2\mu_1 x_1 = 0 \\ 2x_2 - 10 - \frac{1}{5}\mu_1 = 0 \end{cases}$$

$\Rightarrow 100x_1^3 - 98x_1 - 2 = 0 \Rightarrow x_1 = 1, -\frac{1}{2} \pm \frac{\sqrt{23}}{10}$

$x_1 = 1 \Rightarrow$ same as (1)

$x_1 = -\frac{1}{2} - \frac{\sqrt{23}}{10} \Rightarrow x_2 = \frac{12}{5} + \frac{\sqrt{23}}{2}$,

$\mu_1 = 26 - 5\sqrt{23} > 0$ ✓

and $g_2 < 0$ ✓ $\left(-\frac{1}{2} - \frac{\sqrt{23}}{10}, \frac{12}{5} + \frac{\sqrt{23}}{2}\right)$ (*) (local min)

$x_1 = -\frac{1}{2} + \frac{\sqrt{23}}{10} \Rightarrow x_2 = \frac{12}{5} - \frac{\sqrt{23}}{2}$

$\mu_1 = 26 + 5\sqrt{23} > 0$ ✓

and $g_2 < 0$ ✓ $\left(-\frac{1}{2} + \frac{\sqrt{23}}{10}, \frac{12}{5} - \frac{\sqrt{23}}{2}\right)$ (**)

③ Both active $g_1 = g_2 = 0$

$$\begin{cases} \frac{1}{5}x_2 - x_1^2 = 0 \\ 5x_1 + \frac{1}{2}x_2 - 5 = 0 \end{cases} \Rightarrow x_1 = -1 + \sqrt{3}, \quad x_2 = 20 - 10\sqrt{3}$$

$\mu_1 > 0$ ✓, $\mu_2 > 0$ ✓

So we get the following local extremizers:

$$\underbrace{\left(-\frac{1}{2}-\frac{\sqrt{23}}{10}, \frac{12}{5}+\frac{\sqrt{23}}{2}\right)}_{\approx 3.96}, \underbrace{\left(-1+\sqrt{3}, 20-10\sqrt{3}\right)}_{\approx 5.46}$$ both min, no max

b) $\min f(x_1, x_2) = x_1^2 + x_2^2$

s.t $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \geq 5$

$$L(x_1, x_2, \mu) = x_1^2 + x_2^2 + \mu_1(-x_1) + \mu_2(-x_2) + \mu_3(5 - x_1 - x_2)$$

$$\nabla L = 0 \Rightarrow \begin{cases} 2x_1 - \mu_1 - \mu_3 = 0 \\ 2x_2 - \mu_2 - \mu_3 = 0 \\ \mu_1 x_1 = 0 \\ \mu_2 x_2 = 0 \\ \mu_3(5 - x_1 - x_2) = 0 \end{cases}$$

① $x_1 > 0$, $x_2 > 0 \Rightarrow \mu_1 = \mu_2 = 0$

$\Rightarrow x_1 + x_2 = 5$ w/ $\mu_3 > 0$

$$\begin{matrix} 2x_1 - \mu_3 = 0 \\ 2x_2 - \mu_3 = 0 \end{matrix} \Rightarrow x_1 = x_2$$

feasible ✓

$x_1 + x_2 = 5 \Rightarrow x_1 = x_2 = \frac{5}{2}$, $\mu_3 = 5$

② $\mu_1 = \mu_2 = \mu_3 = 0$

$2x_1 = 0 \Rightarrow x_1 = 0$, $2x_2 = 0 \Rightarrow x_2 = 0$

but $x_1 + x_2 \geq 5$ ↯ ouch

③ $\mu_1 = 0$, $\mu_2 = 0$, $x_2 = 0$

$2x_1 - \mu_1 - \mu_3 = 2x_1 = 0 \Rightarrow x_1 = 0$

$x_1 > 5$ ouch ↯

④ $x_1 = 0$, $x_2 > 0$ $(x_2 \geq 5)$

$\mu_2 = 0$, $\mu_3 = 0$, $\mu_1 \geq 0$

$2x_2 - \mu_2 - \mu_3 = 2x_2 = 0 \Rightarrow x_2 = 0$ ↯ $x_2 \geq 5$

only extremizer $\underbrace{\left(\frac{5}{2}, \frac{5}{2}\right)}_{\frac{50}{4}}$

5.

$a_1 = (0,0)$, $a_2 = (0,1)$, $a_3 = (1,2)$, $a_4 = (2,-1)$

$\|a_1\|^2 = 0$, $\langle a_1, x\rangle = 0$

$\|a_2\|^2 = 1$, $\langle a_2, x\rangle = x_2$

$\|a_3\|^2 = 5$, $\langle a_3, x\rangle = x_1 + 2x_2$

$\|a_4\|^2 = 5$, $\langle a_4, x\rangle = 2x_1 - x_2$

$2\langle a_i, x\rangle + \tilde{\lambda} \geq \|a_i\|^2$ obj. $\tilde{\lambda} + \|x\|^2$

a_1: $\tilde{\lambda} \geq 0$ $\Longrightarrow \tilde{\lambda} + x_1^2 + x_2^2$

a_2: $2x_2 + \tilde{\lambda} \geq 1$

a_3: $2x_1 + 4x_2 + \tilde{\lambda} \geq 5$

a_4: $4x_1 - 2x_2 + \tilde{\lambda} \geq 5$

center $x = (x_1, x_2)$

QP : $\min \; x_1^2 + x_2^2 + \tilde{\lambda}$ $\left(r = \sqrt{\tilde{\lambda} + \|x\|^2}\right)$

s.t.
$$\tilde{\lambda} \geq 0$$
$$2x_2 + \tilde{\lambda} \geq 1$$
$$2x_1 + 4x_2 + \tilde{\lambda} \geq 5$$
$$4x_1 - 2x_2 + \tilde{\lambda} \geq 5$$

$\tilde{\lambda} = 0 \Longrightarrow 2x_2 + \tilde{\lambda} \geq 1 \Longrightarrow x_2 = \frac{1}{2}$

$2x_1 + 4\left(\frac{1}{2}\right) + 0 \geq 5 \Longrightarrow x_1 = \frac{3}{2}$

$4\left(\frac{3}{2}\right) - 2\left(\frac{1}{2}\right) + 0 = 6 - 1 \geq 5$ ✓

$r^* = \sqrt{0 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{9}{4}} = \sqrt{\frac{10}{4}} = \sqrt{\frac{5}{2}}$

So we get a circle of radius $r = \sqrt{\frac{5}{2}}$

in the x,y plane: $(x - \frac{3}{2})^2 + (y - \frac{1}{2})^2 = (\sqrt{\frac{5}{2}})^2$