Capitulo 10

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10.1) Supongase que X tiene fdp dado por

$f(x) = 2x, \; 0 \leq x \leq 1$

a) Determina la fgm de X

b) Usando la fgm calcular $\mathcal{E}(x)$ y $V(x)$ y verificar la respuesta.

a)

$$M_x(t) = \mathcal{E}(e^{tx}) = 2\int_{x=0}^{x=1} x e^{tx}\, dx = 2e^{tx}\frac{tx-1}{t^2}\Big|_0^1 = 2e^t\frac{t-1}{t^2} + \frac{2}{t}$$

$$= \frac{2}{t^2}\left[e^t(t-1)+1\right]$$

b) $$M'_x(t) = \frac{d}{dt}\left\{\frac{2}{t^2}\left[e^t(t-1)+1\right]\right\} = \frac{2e^t(t^2-2t+2)-4}{t^3}$$

$$M''_x(t) = \frac{d}{dt}\left\{\frac{2e^t(t^2-2t+2)-4}{t^3}\right\} = \frac{2\left[e^t(t^3-3t^2+6t-6)+6\right]}{t^4}$$

$$\mathcal{E}(x) = \lim_{t\to 0} M'_x(t) = \lim_{t\to 0}\left[\frac{2e^t(t^2-2t+2)-4}{t^3}\right]$$

$$= \lim_{t\to 0}\left[\frac{2t^2e^t}{3t^2}\right] = \frac{2}{3}$$

$$\mathcal{E}(x^2) = \lim_{t\to 0} M''_x(t) = \lim_{t\to 0}\left[\frac{2\left[e^t(t^3-3t^2+6t-6)+6\right]}{t^4}\right]$$

$$= 2\lim_{t\to 0}\left[\frac{t^3e^t}{4t^3}\right] = \frac{1}{2}$$

$$V(x) = \mathcal{E}(x^2) - \mathcal{E}^2(x) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$\mathcal{E}(x) = 2\int_{x=0}^{x=1} x^2\, dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$$

$$\mathcal{E}(x^2) = 2\int_{x=0}^{x=1} x^3\, dx = \frac{2}{4}x^4\Big|_0^1 = \frac{1}{2}$$

10.2

a) Encontrar la fgm (incluyendo el ruido del como se expuso)

b) Usando la fgm, obtener el valor esperado y la varianza de este voltaje

a) $M_V(t) = M_S(t) M_N(t) = E(e^{ts}) E(e^{tn})$

$$= \int_0^1 e^{ts}\, ds \, \frac{1}{2}\int_0^2 e^{tn}\, dn = \frac{e^{ts}}{t}\Big|_0^1 \frac{e^{tn}}{2t}\Big|_0^2$$

$$= \left(\frac{e^t}{t} - \frac{1}{t}\right)\left(\frac{e^{2t}}{2t} - \frac{1}{2t}\right) = \frac{1}{2t^2}(e^t - 1)(e^{2t} - 1)$$

b)

$$M'_X(t) = -\frac{1}{t^3}(e^t - 1)(e^{2t} - 1) + \frac{1}{2t^2}(e^t)(e^{2t} - 1) + \frac{1}{2t^2}(e^t - 1)(2e^{2t})$$

$$= \frac{1}{2t^3}(e^t - 1)[-2e^{2t} + 2 + te^t(e^t + 1) + 2te^{2t}]$$

$$= \frac{1}{2t^3}(e^t - 1)[te^t + e^{2t}(3t - 2) + 1]$$

$M_X''(t)$

$$= -\frac{3}{2t^4}(e^t - 1)[te^t + e^{2t}(3t - 2) + 1]$$

$$+ \frac{1}{2t^3}(e^t)[te^t + e^{2t}(3t - 2) + 1]$$

$$+ \frac{1}{2t^3}(e^t - 1)[e^t(t + 1) + 2e^{2t}(3t - 2) + e^{2t}(3)]$$

$$= \frac{1}{2t^4}\Big[[e^t(t - 3) + 3][e^t(t + e^t(3t - 2)) + 1]$$

$$+ te^t(e^t - 1)[t + 1 + e^t(6t - 1)]\Big]$$

$$E(X) = \lim_{t \to 0} M'_X(t) = \lim_{t \to 0}\left[\frac{(e^t - 1)[te^t + e^{2t}(3t - 2) + 1]}{2t^3}\right]$$

$$= -\infty$$

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10.3

Suponer que X tiene la fdp siguiente

$$f(x) = \lambda e^{-\lambda(x-a)} \quad , \quad x \geq a$$

a) f_g de x

b) $\mathcal{E}(x)$ y $V(x)$

a) $M_x(t) = \mathcal{E}(e^{xt}) = \lambda e^{\lambda a} \int_a^\infty e^{xt} e^{-\lambda x} dx = \lambda e^{\lambda a} \int_a^\infty e^{-x(t-\lambda)} dx$

$$= \frac{\lambda e^{\lambda a}}{-(t-\lambda)} \int_a^\infty e^{-x(t-\lambda)} [-(t-\lambda) dx] = \frac{\lambda e^{\lambda a}}{-(\lambda-\lambda)} e^{-x(t-\lambda)} \Big|_a^\infty$$

$$= \frac{\lambda e^{\lambda a} e^{-a(\lambda t)}}{\lambda - t} = \frac{\lambda e^{ta}}{\lambda - t}$$

b) $M'_x(t) = \frac{\lambda a e^{ta}(\lambda - t) + \lambda e^{ta}}{(\lambda - t)^2} = \frac{\lambda e^{ta}[a(\lambda - t) + 1]}{(\lambda - t)^2}$

$$= \frac{\lambda a e^{ta}}{(\lambda - t)} + \frac{\lambda e^{ta}}{(\lambda - t)^2}$$

$$M''_x = \frac{\lambda a^2 e^{ta}(\lambda - t) + \lambda a e^{ta}}{(\lambda - t)^2} + \frac{\lambda a e^{ta}(\lambda - t)^2 + 2\lambda e^{ta}(\lambda - t)}{(\lambda - t)^4}$$

$$= \frac{\lambda a^2 e^{ta}}{(\lambda - t)} + 2 \frac{\lambda a e^{ta}}{(\lambda - t)^2} + \frac{2\lambda e^{ta}}{(\lambda - t)^3}$$

$$= \frac{\lambda e^{ta}}{(\lambda - t)^3} (2 + a(\lambda - t)(a(\lambda - t) + 2))$$

$$\mathcal{E}(x) = \frac{\lambda a \lambda + \lambda}{\lambda^2} = \frac{\lambda a + 1}{\lambda}$$

$$\mathcal{E}(x^2) = \lim_{t \to 0} M''_x = \frac{2 + \lambda a(\lambda a + 2)}{\lambda^2} = \frac{\lambda^2 a^2 + 2\lambda a + 2}{\lambda^2}$$

$$V(x) = \mathcal{E}(x^2) - \mathcal{E}^2(x) = \frac{1}{\lambda^2}$$

10.4 Sea X el resultado cuando se lanza un dado regular

a) Encontrar la fgm de X

b) Usando la fgm encontrar $\mathcal{E}(X)$ y $V(X)$

a)

$P(X=x)=\frac{1}{2}$, $x=0,1$

$M_x(t)=\mathcal{E}(e^{xt})=\sum_{k=0}^{1} e^{kt}\, p(X=x)=\frac{1+e^t}{2}$

$P(X=x)=\frac{1}{6}$, $x=1,2,3,4,5,6$

$M_x(t)=\mathcal{E}(e^{xt})$

$\sum_{k=1}^{6} e^{kt}\, p(X=x)$

$=\frac{1}{6}(e^{t}+e^{2t}+e^{3t}+e^{4t}+e^{5t}+e^{6t})$

b)

$M'_x(t)=\frac{e^t}{2}$, $M''_x(t)=\frac{e^t}{2}$

$\mathcal{E}(x)=\lim_{t\to 0} M'_x(t)=\lim_{t\to 0}\frac{e^t}{2}=\frac{1}{2}$

$\mathcal{E}(x^2)=\lim_{t\to 0} M''_x(t)=\lim_{t\to 0}\frac{e^t}{2}=\frac{1}{2}$

$V(x)=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

Dado...

$M'_x(t)=\frac{1}{6}(e^t+2e^{2t}+3e^{3t}+4e^{4t}+5e^{5t}+6e^{6t})$

$M''_x(t)=\frac{1}{6}(e^t+4e^{2t}+9e^{3t}+16^{4t}+25e^t+36e^t)$

$\mathcal{E}(x)=\lim_{t\to 0} M'_x(t)=\lim_{t\to 0}\frac{1}{6}(e^t+2e^{2t}+3e^{3t}+4e^{4t}+5e^{5t}+6e^{6t})$

$=\frac{7}{2}$

$\mathcal{E}(x^2)=\lim_{t\to 0} M''_x(t)=\frac{91}{6}$

$V(x)=\frac{91}{6}-\frac{49}{8}=\frac{364-147}{24}=\frac{217}{24}$

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10.5) Encontrar la fgm de la variable aleatoria X. Usando la fgm encontrar $E(X)$ y $V(X)$

$$g(x) = \begin{cases} x-1 & 1 \le x \le 2 \\ -x+3 & 2 \le x \le 3 \\ 0 & c.c \end{cases}$$

$$M_x(t) = E(e^{xt}) = \int_1^2 e^{xt}(x-1)\,dx + \int_2^3 e^{xt}(3-x)\,dx$$

$$M'_x(t) = \frac{t^2e^t - 2te^t}{t^4}(e^t-1)^2 + \frac{e^t}{t^2}(e^t-1)e^t$$

$$= \frac{e^t(e^t-1)}{t^3}(e^t(3t-2)-t+2)$$

$$= \frac{e^t}{t^2}[1+e^{2t}-2e^t]$$

$$= \frac{e^t}{t^2}(e^t(e^t-2)+1)$$

$$= \frac{e^t}{t^2}(e^t-1)^2$$

$$M''_x(t) = \frac{(e^t(e^t-1)+e^t(e^t))t^3 - 3t^2e^t(e^t-1)}{t^6}(e^t(3t-2)-t+2)$$

$$= \frac{e^t}{t^4}(t^2+3e^{2t}(3t^2-4t+2)-4e^t(2t^2-4t+3)-4t+6)$$

$$E(x) = \lim_{t\to 0} M'_x(t) = \lim_{t\to 0} \frac{e^t(e^t-1)}{t^3}(e^t(3t-2)-t+2)$$

$$= \frac{27-16+1}{6}$$

$$= 2$$

$$E(x^2) = \lim_{t\to 0} M''_x(t)$$

$$= \lim_{t\to 0} \frac{e^t}{t^4}(t^2+3e^{2t}(3t^2-4t+2)-4e^t(2t^2-4t+3)-4t+6)$$

$$= 25/6$$

$$V(x) = \frac{25}{6} - 4 = \frac{1}{6}$$

10.6 - Supóngase que la variable aleatoria continua X tiene fdp

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

a) Obtener la fgm de X

b) Usando la fgm, encontrar $E(X)$ y $V(X)$

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < +\infty$$

a)

$$M_x(t) = E(e^{xt}) = \frac{1}{2} \int_{-\infty}^{\infty} e^{xt} e^{-|x|} dx = \frac{1}{2} \int_{-\infty}^{0} e^{-x(t-1)} dx + \frac{1}{2} \int_{0}^{\infty} e^{-x(x+1)} dx$$

$$= \frac{1}{1-t^2} \qquad -1 < t < 1$$

b) $$M'_x(t) = \frac{2t}{(t^2-1)^2}$$

$$M''_x(t) = -2 \frac{3t^2+1}{(t^2-1)^3}$$

$$E(x) = \lim_{t \to 0} M'_x(t) = \lim_{t \to 0} \frac{2t}{(t^2-1)^2} = 0$$

$$E(x^2) = \lim_{t \to 0} M''x(t) = \lim_{t \to 0} -2 \frac{3t^2+1}{(t^2-1)^3} = 2$$

$$V(x) = 2$$

10.7) Usando la fgm demostrar que si X y Y son variables aleatorias independientes con distribución $N(\mu_x, \sigma^2_x)$ y $N(\mu_y, \sigma^2_y)$ respectivamente, entonces $Z = aX + bY$ está de nuevo distribuida normalmente donde a y b son constantes

$$M_x(t) = \exp\left(t\mu_x + \frac{\sigma^2_x t^2}{2}\right), \quad M_y(t) = \exp\left(t\mu_y + \frac{\sigma^2_y t^2}{2}\right)$$

$$M_z(t) = M_x(t) M_y(t) = \exp\left(t\mu_x + \frac{\sigma^2_x t^2}{2}\right)\exp\left(t\mu_y + \frac{\sigma^2_y t^2}{2}\right)$$

$$= \exp\left[t(\mu_x + \mu_y) + \frac{t^2}{2}(\sigma^2_x + \sigma^2_x)\right]$$

$$E(z) = \mu_x + \mu_y, \quad V(z) = \sigma^2_x + \sigma^2_y \quad \therefore Z \sim N(\mu_x + \mu_y, \sigma^2_x + \sigma^2_y)$$

10.8) Suponer que la fgm de una variable aleatoria X es de la forma

$$M_x(t) = (0.4e^t + 0.6)^8$$

a) ¿Cuál es la fgm de la variable aleatoria $y = 3x + 2$?

b) Calcular $E(x)$

c) Reconocer otro método

a) $M_y(t) = e^{2t} M_x(3t) = e^{2t}(0.4e^{3t} + 0.6)^8$

b) $M'_x(t) = 3.2e^t(0.4e^t + 0.6)^7$

$$E(x) = \lim_{t \to 0} M'_x(t) = \lim_{t \to 0} 3.2e^t(0.4e^t + 0.6)^7$$

$$= 3.2$$

c) No reconozco otro :(

10.9. Varias resistencias R_i $i=1,2\ldots n$ se ponen en serie en un circuito. Suponer que cada una de las resistencias esta distribuida normalmente con $E(R_i)=10\Omega$ y $V(R_i)=0.16$

a) Si $n=5$ ¿Cuál es la probabilidad de que la resistencia del circ. sobrepase 49Ω?

b) ¿Cuál debe ser el valor de n de manera que la prob. de que la resistencia exceda los 100 ohms sea aprox. 0.05?

a) $R=\sum_{i=1}^{5} R_i$ $\quad E(R)=50 \quad V(R)=0.8$

$$P(R>49)=P\left(Y>\frac{49-50}{\sqrt{0.8}}\right)\simeq P(Y>-1.12)=\Phi(1.12)$$

$$\simeq 0.8686$$

b) $$P(R>100)=P\left(Y>\frac{100-10n}{\sqrt{0.16n}}\right)=\Phi\left(\frac{10n-100}{\sqrt{0.16n}}\right)\simeq 0.05$$

$$\frac{10n-100}{\sqrt{0.16n}}\simeq -1.645\equiv 10n+0.658\sqrt{n}-100\simeq 0$$

$$\sqrt{n}\simeq(3.12955)^2\cong 9.79$$

$$n=\begin{cases} 9\to P(R>100)=P\left(Y>\frac{100-90}{1.2}\right)=\Phi(-8.33)\simeq 0 \\ 10\to P(R>100)=P\left(Y>\frac{100-100}{\sqrt{1.6}}\right)=\Phi\simeq 0.5 \end{cases}$$

$n=9$

10.10. En un circuito se ponen n resistencias en serie. Supongase que cada una de las resistencias está distribuida uniformemente en $[0,1]$ y supongase además que todas las resistencias son independientes. Sea R la resistencia total

a) Encontrar la fgm de R

b) Usando la fgm, obtener $E(R)$ y $V(R)$. Verifique la respuesta con calculo directo

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a) $f(r_i)=1 \quad , \quad 0 \le r_i \le 1 \quad i=1,2,\dots n$

$$M_{R_i}(t)=\int_0^1 e^{r_i t}\,dr_i=\frac{e^t-1}{t}$$

$$M_R(t)=\prod_{i=1}^{n} M_{R_i}(t)=\left(\frac{e^t-1}{t}\right)^n$$

b) $$M'_R(t)=n\left(\frac{e^t-1}{t}\right)^{n-1}\left(\frac{te^t-e^t+1}{t^2}\right)$$

$$M''_R(t)=n\left(\frac{e^t-1}{t}\right)^n\left(\frac{e^{2t}(n(t-1)^2+1-e^t(2n(1-t)+t^2+2)+n+1}{(e^t-1)^2t^2}\right)$$

$$E(R)=\lim_{t\to 0} M'_R(t)=\lim_{t\to 0} n\left(\frac{e^t-1}{t}\right)^{n-1}\left(\frac{te^t-e^t+1}{t}\right)=\frac{n}{2}$$

$$E(R^2)=\lim_{t\to 0} M''_R(t)$$

$$=\lim_{t\to 0} n\left(\frac{e^t-1}{t}\right)^n\left(\frac{e^{2t}(n(t-1)^2+1)-e^t(2n(1-t)+t^2+2)+n+1}{(e^t-1)^2t^2}\right)$$

$$=\frac{3n^2+n}{12}=\frac{n^2}{4}=\frac{n}{12}$$

$$V(R)=\frac{3n^2+n}{12}-\frac{n^2}{4}=\frac{n}{12}$$

$$E(R)=E\left(\sum_{i=1}^{n} R_i\right)=nE(R_i)=\frac{n}{2}$$

$$V(R)=V\left(\sum_{i=1}^{n} R_i\right)=n^2V(R_i)=\frac{n^2}{12}$$

$$E(R_i^2)=\int_0^1 r_i^2\,dr_i=\frac{1}{3}$$