Linear Precision for Toric Patches

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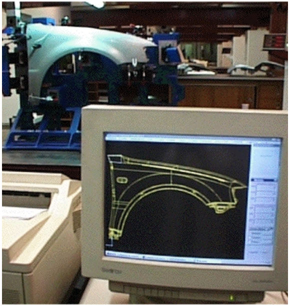
Constructive Function Theory

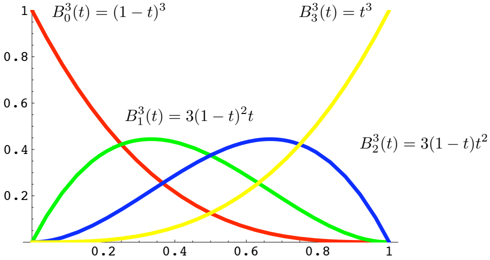
Geometric modeling and algebraic geometry

Geometric modeling uses polynomials to build computer models for industrial design and manufacture.

Algebraic geometry investigates the algebraic and geometric properties of polynomials.



Bézier curves



Bernstein polynomials

$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$

Parametric definition

$$C(t) = \sum_{i=0}^{n} p_i B_i^n(t), \quad t \in [0,1]$$

where $p_0, p_1, \ldots, p_n$ are control points in some affine space.

Properties of Bézier curves

Affine invariance

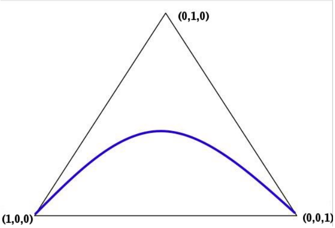
Let $T : \mathbb{A}^m \longrightarrow \mathbb{A}^m$ be an Affine map. Then

$$T(C(t)) = T(\sum_{i=0}^{n} p_i B_i^n(t)) = \sum_{i=0}^{n} T(p_i) B_i^n(t)$$

Convex hull property

The curve $C([0,1])$ is contained in the convex hull of the control points

Endpoint interpolation



Properties of Bézier curves

More properties

- Symmetry
- Pseudo-local control
- Subdivision
- Recursive evaluation

Linear precision

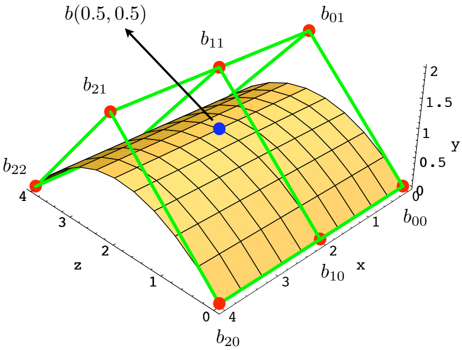
$$\frac{1}{n}\left(\sum_{i=0}^{n} iB_i^n(t)\right) = t$$

$$\frac{1}{3}\left[0(1-t)^3 + 3t(1-t)^2 + 6t^2(1-t) + 3t^3\right] = t$$

Rectangular Bézier surfaces

Parametric representation

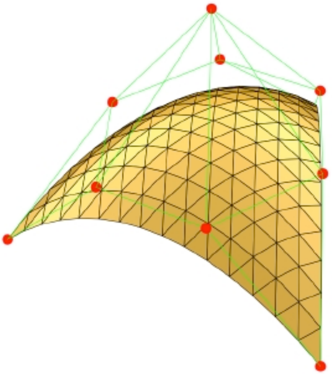
$$\sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(s)B_j^n(t)p_{ij}, \quad 0 \le s,t \le 1$$



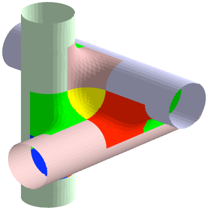
Triangular Bézier surfaces

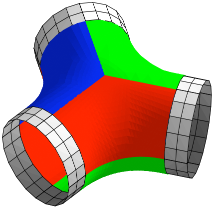
Parametric representation

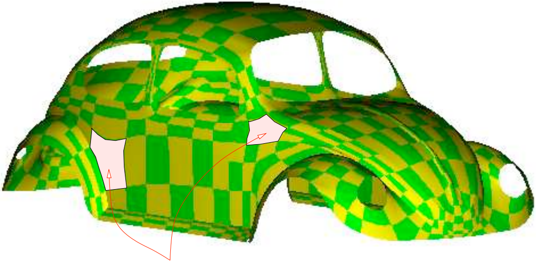
$$\sum_{i+j+k=n} \frac{n!}{i!j!k!} u^i v^j w^k p_{ijk}, \quad 0 \leq u, v, w \leq 1 \text{ and } u + v + w = 1$$



Multi-sided patches

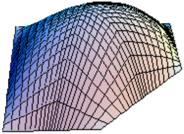




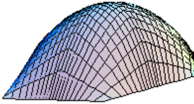


Toric patches

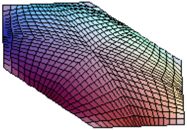
- well-controlled generalization of Bézier patches.
- based on the geometry of toric varieties.
- depend on a polytope and some weights.



top view



side view



Properties toric Bézier patches

- Affine invariance
- Convex hull
- Boundaries are rational Bézier curves determined by boundary control points
- surfaces interpolate corner control points

Rimvydas Krasauskas

Which toric Bézier patches have linear precision?

Patches

Data

- $\mathcal{A} \subset \mathbb{Z}^d$ finite subset
- $\Delta = \text{conv}(\mathcal{A})$ polytope of dimension d in $\mathbb{R}^d$

Patch

A patch $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ is a collection of non-negative (blending or basis) functions indexed by $\mathcal{A}$ with common domain Δ and no base points in their domain.

Parametric patch

A set $\{\mathbf{b}_{\mathbf{a}} \in \mathbb{R}^\ell \mid \mathbf{a} \in \mathcal{A}\}$ of control points indexed by $\mathcal{A}$ gives a parametric map

$$\frac{\sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x)\mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x)}$$

Toric varieties

Monomials in d indeterminates

$$x^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d}, \quad (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$$

Data

- $\mathcal{A} \subset \mathbb{Z}^d$ finite subset
- $\Delta = \mathrm{conv}(\mathcal{A})$ polytope of dimension d in $\mathbb{R}^d$
- $w = \{w_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathcal{A}\}$ set of positive weights indexed by $\mathcal{A}$

Monomial map $\varphi_{\mathcal{A},w} : (\mathbb{C}^*)^d \longrightarrow \mathbb{P}^{\mathcal{A}}$

$$\varphi_{\mathcal{A},w} : \ x \longrightarrow [w_{\mathbf{a}} x^{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}]$$

Translated toric variety

$Y_{\mathcal{A},w}$ the Zariski closure of $\varphi_{\mathcal{A},w}((\mathbb{C}^*)^d)$

Toric patches

Non-negative part of a toric variety

$X_{\mathcal{A},w}$ the Zariski closure of $\varphi_{\mathcal{A},w}(\mathbb{R}^d_>)$

Definition

A Toric patch of shape $(\mathcal{A}, w)$ is any patch β such that the closure X_β of the image of

$$\beta : \Delta \longrightarrow \mathbb{RP}^{\mathcal{A}}, \quad x \longmapsto [\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}]$$

equals $X_{\mathcal{A},w}$.

Linear precision

Definition

A patch $\{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has linear precision if the tautological map τ

$$\tau \ := \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x)\mathbf{a}$$

is the identity function on Δ.

Theorem

A toric patch has a unique reparametrization which has linear precision.

Definition

A toric patch has rational linear precision if its reparametrization having linear precision has blending functions that are rational functions.

Quadratic (rescaled) Bézier curve

Let $\mathcal{A} = \{0, 1, 2\}$ and $w = (1, 2, 1)$, then $X_{[0,2],w}$ is the image of

$$t \mapsto [1, 2t, t^2], \quad t > 0.$$

Let $\beta : [0, 2] \to X_{[0,2],w}$ be given by

$$t \longmapsto [(2-t)^2, 2t(2-t), t^2], \quad t \in [0, 2].$$

The tautological map $\tau : [0, 2] \to [0, 2]$ is given by

$$\frac{0 \cdot (2-t)^2 + 1 \cdot 2t(2-t) + 2t^2}{(2-t)^2 + 2t(2-t) + t^2} = \frac{4t}{4} = t\,.$$

Main result

Laurent polynomial

Let $\mathcal{A} \subset \mathbb{Z}^d$ be a finite subset and $w \in \mathbb{R}^{\mathcal{A}}_{>}$ be a system of weights, the the Laurent polynomial $f = f_{\mathcal{A},w}$ is defined by

$$f = f_{\mathcal{A},w} := \sum_{\mathbf{a}\in\mathcal{A}} w_{\mathbf{a}} x^{\mathbf{a}}$$

Theorem

A toric patch of shape $(\mathcal{A}, w)$ has rational linear precision if and only if the rational function $\psi_{\mathcal{A},w} : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ defined by

$$\frac{1}{f}\left(x_1\frac{\partial}{\partial x_1}f, x_2\frac{\partial}{\partial x_2}f, \ldots, x_d\frac{\partial}{\partial x_d}f\right)$$

is a birational isomorphism.

Tautological projection

Definition

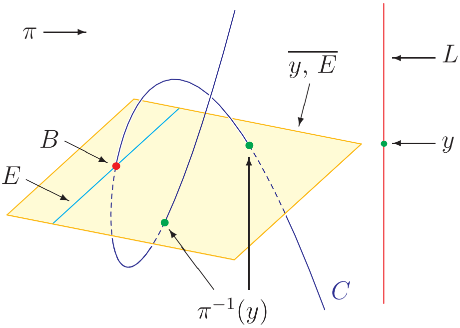
Let $\mathcal{A} \subset \mathbb{R}^d$. Given a point $y = [y_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \in \mathbb{RP}^{\mathcal{A}}$, if the sum

$$\sum_{\mathbf{a} \in \mathcal{A}} y_{\mathbf{a}} \cdot (1, \mathbf{a}) \in \mathbb{R}^{d+1}$$

is non-zero then it represents a point in $\mathbb{RP}^d$.

This map is the tautological projection

$$\pi : \mathbb{RP}^{\mathcal{A}} \dashrightarrow \mathbb{RP}^d.$$



Inverse of the tautological projection

Theorem

The blending functions for the toric patch X_β which have linear precision are given by the coordinates of the inverse of $\pi : X_\beta \longrightarrow \Delta$.

$X_{\mathcal{A},w}$ the Zariski closure of $\varphi_{\mathcal{A},w}(\mathbb{R}^d_>)$

Algebraic statistics

In statistics $\varphi_{\mathcal{A},w}(\mathbb{R}^d_>)$ is known as as a log-linear model or discrete exponential family.

Theorem (Darroch and Ratcliff)

The inverse image of the tautological projection can be numerically obtained by the method know as iterative proportional fitting.

Maximum likelihood

Identify the non-negative orthant $\mathbb{RP}^{\mathcal{A}}_{\geq}$ with the probability simplex

$$\Delta^{\mathcal{A}} := \{y \in \mathbb{R}^{\mathcal{A}}_{\geq 0} \mid \sum_{\mathbf{a}\in\mathcal{A}} y_{\mathbf{a}} = 1\}$$

$$X_{\mathcal{A},w} = Y_{\mathcal{A},w} \cap \mathbb{RP}^{\mathcal{A}}_{\geq} = Y_{\mathcal{A},w} \cap \Delta^{\mathcal{A}} \subset \Delta^{\mathcal{A}}$$

Lemma (Maximum likelihood estimation in $X_{\mathcal{A},w}$)

Given (normalized) data $q \in \Delta^{\mathcal{A}}$, the maximum likelihood estimate is the unique point $p \in X_{\mathcal{A},w}$ such that

$$\pi(p) = \pi(q)$$

Iterative proportional fitting

Theorem

Suppose that $\mathcal{A} \subset \Delta$ and $q \in \Delta^{\mathcal{A}}$. Then the sequence

$$\{p^{(n)} \mid n = 0, 1, 2, \dots\}$$

whose **a***-coordinates are defined by $p^0_{\mathbf{a}} := w_{\mathbf{a}}$, and, for $n > 0$,*

$$p^{(n+1)}_{\mathbf{a}} := p^{(n)}_{\mathbf{a}} \cdot \frac{\pi(q)^{\mathbf{a}}}{\pi(p^{(n)})^{\mathbf{a}}},$$

converges to the unique point $p \in X_{\mathcal{A},w}$ such that $\pi(p) = \pi(q)$.

Corollary

Iterative proportional fitting computes the unique parametrization of a toric patch having linear precision.