# Toric degenerations of Bézier patches

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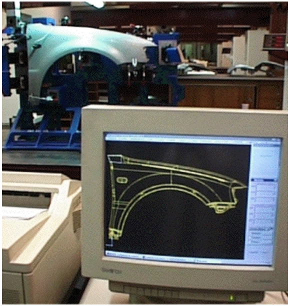
**Geometric modeling** uses polynomials to build computer models for industrial design and manufacture.



# Bézier curves

Bézier curves are **parametric curves** used in computer graphics to model smooth curves. Fundamental objects in geometric modeling.

- First introduced by Charles Hermite and Sergei Bernstein.
- Widely publicized in the 1960's by Pierre Bézier (Renault), and Paul De Casteljau (Citroën) in the design of automobile bodies.
- Used in animation software such as Adobe Flash to outline movement.
- Used also in the design of fonts:
  - Quadratic Bézier curves are used in True Type fonts,
  - cubic Bézier curves are used in Type 1 fonts,
  - cubic Bézier curves are also used in the TEX fonts.

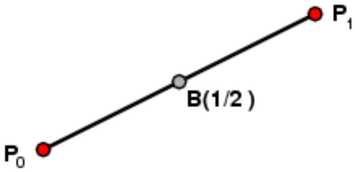
$$\mathbf{B}(\mathbf{x}) := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \mathbf{P}_i, \quad x \in [0,1]$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

$$\mathbf{B(x)} := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \mathbf{P}_i, \quad x \in [0,1]$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

$$B(x) = (1-x)\mathbf{P}_0 + x\mathbf{P}_1$$



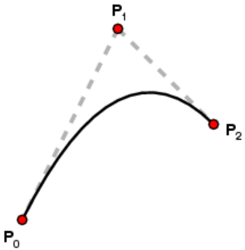

### Linear precision

$$\sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \frac{i}{d} = x$$

$$\mathbf{B}(\mathbf{x}) := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \mathbf{P}_i, \quad x \in [0,1]$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

$$B(x) = (1-x)^2\mathbf{P}_0 + 2x(1-x)\mathbf{P}_1 + x^2\mathbf{P}_2$$



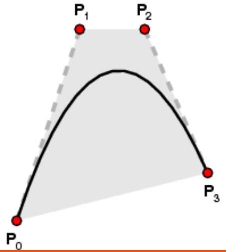

**Endpoint interpolation**

$$B(0) = \mathbf{P}_0, \quad B(1) = \mathbf{P}_2$$

$$\color{orange}{\mathbf{B}(\mathbf{x})} := \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i}\, \mathbf{P}_i, \quad x \in [0,1]$$

where $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_d$ are (control) points in $\mathbb{R}^n$ $(n = 2, 3)$.

$B(x) = (1-x)^3\mathbf{P}_0 + 3x(1-x)^2\mathbf{P}_1 + 3x^2(1-x)\mathbf{P}_2 + x^3\mathbf{P}_3$




### Convex hull

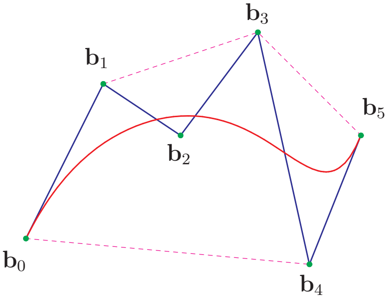
The curve $B([0,1])$ is contained in the convex hull of the control points.

# Control polygons

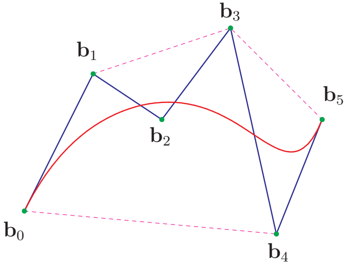
Let $B(x)$ be the Bézier curve given by

$$B(x) = \sum_{i=0}^{d} \binom{d}{i} x^i (1-x)^{d-i} \mathbf{b}_i, \qquad \text{with } x \in [0,1],$$

with $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_d$ control points in $\mathbb{R}^n$. The corresponding **control polygon** is the union of the line segments $\overline{\mathbf{b}_0, \mathbf{b}_1}, \overline{\mathbf{b}_1, \mathbf{b}_2}, \ldots, \overline{\mathbf{b}_{d-1}, \mathbf{b}_d}$.



$$B(x) = (1-x)^5\mathbf{b}_0 + 5x(1-x)^4\mathbf{b}_1 + 10x^2(1-x)^3\mathbf{b}_2 + 10x^3(1-x)^2\mathbf{b}_3 + 5x^4(1-x)\mathbf{b}_4 + x^5\mathbf{b}_5.$$




The number of points in which a Bézier curve meets a line is bounded by number of points in which its control polygon meets the same line.

Generalizing this property to surfaces is similar to the open problem of finding a satisfactory multivariate generalization of Descartes' rule of signs.

Rational Bézier curves add adjustable weights to provide closer approximations to arbitrary shapes.

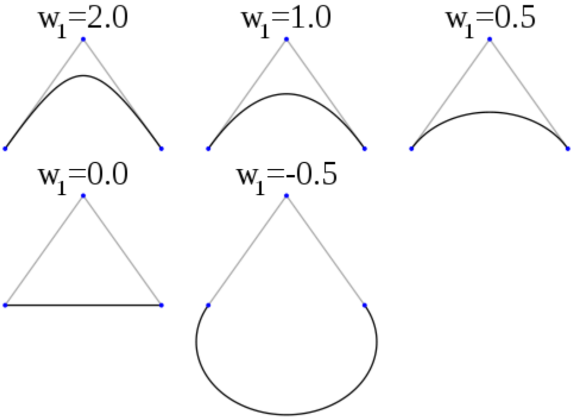
**Bernstein polynomials**

$$\beta_{\mathbf{i};\mathbf{d}}(\mathbf{x}) := \binom{d}{i} x^i (1-x)^{d-i}$$

Given **weights** $w_0, w_1, \ldots, w_d$ in $\mathbb{R}_>$ and control points $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_d$ in $\mathbb{R}^n$, the **rational Bézier curve** is

$$\mathbf{B}(\mathbf{x}) := \frac{\sum_{i=0}^{d} w_i \beta_{i;d}(x)\, \mathbf{b}_i}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)} \; : \; [0,1] \longrightarrow \mathbb{R}^n .$$

$$B(x) = \frac{\sum_{i=0}^{d} w_i \beta_{i;d}(x)\, \mathbf{b}_i}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)}$$



- For each $i = 0, \ldots, d$ redefine the **Bernstein polynomial** $\beta_{i;d}(x)$,

$$\beta_{\mathbf{i;d}}(\mathbf{x}) \ := \ x^i (d-x)^{d-i} \, .$$

  Substituting $x = dy$ and multiplying by $\binom{d}{i} d^{-d}$ for normalization, this becomes the usual Bernstein polynomial.

- Given weights $w_0, \ldots, w_d \in \mathbb{R}_>$ and control points $\mathbf{b}_0, \ldots, \mathbf{b}_d \in \mathbb{R}^n$ $(n = 2, 3)$, the parametrized **toric Bézier curve** is defined by

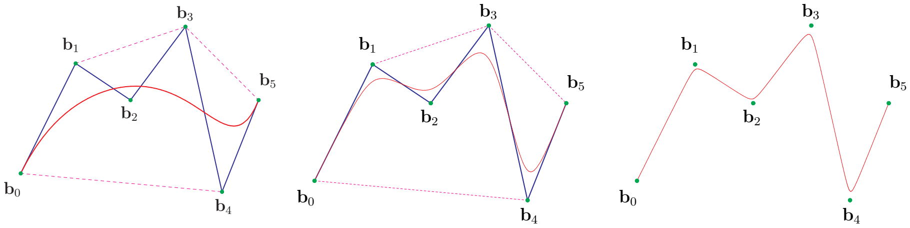
$$\mathbf{B}(\mathbf{x}) \ := \ \frac{\sum_{i=0}^{d} w_i \beta_{i;d}(x) \, \mathbf{b}_i}{\sum_{i=0}^{d} w_i \beta_{i;d}(x)} \ : \ [0, d] \ \longrightarrow \ \mathbb{R}^n \, .$$

- Differ from the rational Bézier curves in that the degree is encoded by the domain. This **linear reparametrization** does not affect the resulting curve.

# Toric Bézier curve deformations

**Theorem (Craciun-G-Sottile)**

*Given control points in $\mathbb{R}^n$ and $\epsilon > 0$, there is a choice of weights so that the toric Bézier curve lies within a distance $\epsilon$ of the control polygon.*



Let $\Delta^d \subset \mathbb{R}^{d+1}$ be the **standard simplex** of dimension $d$ with homogeneous coordinates

$$[z_0, z_1, \ldots, z_d] \ := \ \frac{1}{\sum_{i=0}^d z_i}(z_0, z_1, \ldots, z_d)\,.$$

Define $\beta : [0, d] \to \Delta^d, \quad x \longmapsto [\beta_0(x), \beta_1(x), \ldots, \beta_d(x)]$.

Define $\mathbf{w}\cdot : \Delta^d \to \Delta^d, \quad [z_0, z_1, \ldots, z_d] \longmapsto [w_0 z_0, w_1 z_1, \ldots, w_d z_d]\,.$

Define $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^n, \quad (z_0, \ldots, z_d) \longmapsto \sum_{i=0}^d z_i \, \mathbf{b}_i\,.$

The map $B(x) : [0, d] \longrightarrow \mathbb{R}^n$ admits the following factorization:

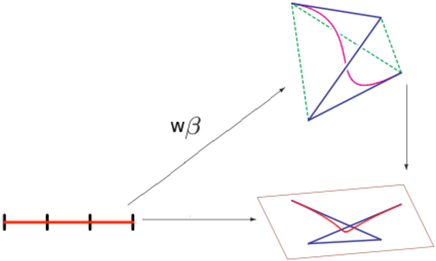
$$B(x) \ : \ [0, d] \xrightarrow{\beta} \Delta^d \xrightarrow{w\cdot} \Delta^d \xrightarrow{\pi} \mathbb{R}^n\,,$$

# Why toric?

The map $B(x) : [0,d] \longrightarrow \mathbb{R}^n$ admits the following factorization:

$$B(x) \ : \ [0,d] \xrightarrow{\beta} \Delta^d \xrightarrow{w\cdot} \Delta^d \xrightarrow{\pi} \mathbb{R}^n \, ,$$

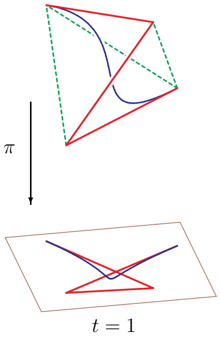
- $X = \beta([0,d])$ is the **positive real part** of the **rational normal curve**.
- Acting on $X$ by the map $w\cdot$ gives a **translated toric variety**.
- $\pi(\Delta^d)$ is the convex hull of the control points.

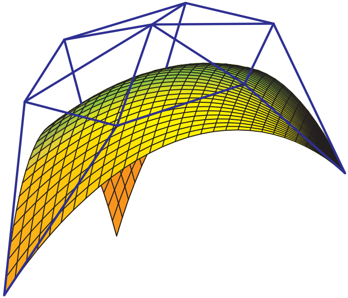


# Proof

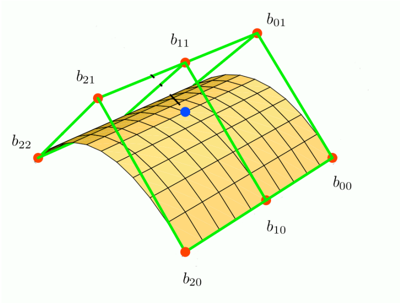
**Theorem (Craciun-G-Sottile)**

*Given control points in $\mathbb{R}^n$ and $\epsilon > 0$, there is a choice of weights so that the toric Bézier curve lies within a distance $\epsilon$ of the control polygon.*

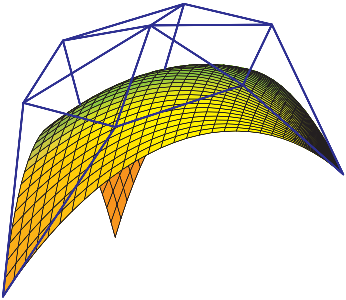




rational Bézier triangular patch




rational Bézier rectangular patch

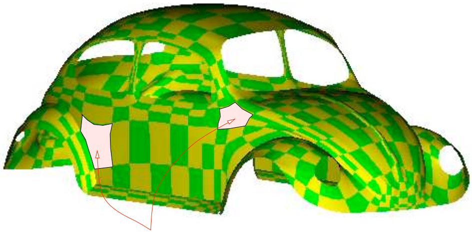


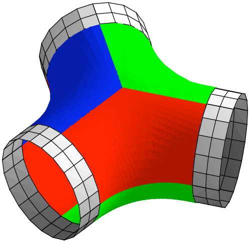
What is the significance for modeling of the **control net**?

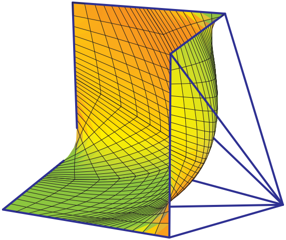
# Answer

These **control nets** encode certain $C^0$ spline surfaces called **regular control surfaces**. While not unique, regular control surfaces are exactly the possible limiting positions of a Bézier patch when the weights are allowed to vary.

# Multi-sided patches







# Toric Bézier patches (after Krasauskas)

A polytope $\Delta \subset R^m$ with integer vertices is given by **facet inequalities**

$$\Delta = \left\{x \in \mathbb{R}^m \mid h_e(x) \geq 0, \text{ for each facet } e \text{ of } \Delta\right\},$$

where $h_e(\mathbf{x}) = \mathbf{v}_e \cdot \mathbf{x} + c_e$ with inward pointing primitive normal vector $\mathbf{v}_e$.

For each $\mathbf{a} \in \mathscr{A} := \Delta \cap \mathbb{Z}^m$, define a **toric Bézier function**

$$\beta_{\mathbf{a},\mathscr{A}}(\mathbf{x}) := \prod_{e \text{ facet of } \Delta} h_e(\mathbf{x})^{h_e(\mathbf{a})} : \Delta \longrightarrow \mathbb{R}.$$

Given positive weights $w = \left\{w_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}\right\} \subset \mathbb{R}_{>}$ and control points $\mathscr{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}\} \subset \mathbb{R}^n$, the **toric Bézier patch of shape $\mathscr{A}$** is parametrized by

$$F_{\mathscr{A},w,\mathscr{B}}(\mathbf{x}) := \frac{\sum_{\mathbf{a}\in\mathscr{A}} w_{\mathbf{a}} \beta_{\mathbf{a},\mathscr{A}}(\mathbf{x})\, \mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathscr{A}} w_{\mathbf{a}} \beta_{\mathbf{a},\mathscr{A}}(\mathbf{x})} : \Delta \longrightarrow \mathbb{R}^n.$$

# Factorization

The map $F(\mathbf{x}) : \Delta_{\mathscr{A}} \longrightarrow \mathbb{R}^n$ admits the following **factorization**:

$$F(\mathbf{x}) \,:\, \Delta_{\mathscr{A}} \xrightarrow{\beta_{\mathscr{A}}} \triangle^{\mathscr{A}} \xrightarrow{w\cdot} \triangle^{\mathscr{A}} \xrightarrow{\pi_{\mathscr{B}}} \mathbb{R}^n\,,$$

where $\triangle^{\mathscr{A}} \subset \mathbb{R}^{\mathscr{A}}$ is the standard simplex of dimension $|\mathscr{A}|-1$ with homogeneous coordinates

$$[z_{\mathbf{a}} \,|\, \mathbf{a} \in \mathscr{A}] \;:=\; \frac{1}{\sum_{\mathbf{a}\in\mathscr{A}} z_{\mathbf{a}}}(z_{\mathbf{a}} \,|\, \mathbf{a} \in \mathscr{A}),$$

$$\beta_{\mathscr{A}}(\mathbf{x}) \;:=\; [\beta_{\mathbf{a},\mathscr{A}}(\mathbf{x}) \,|\, \mathbf{a} \in \mathscr{A}] : \Delta_{\mathscr{A}} \to \triangle^{\mathscr{A}}\,,$$

$$\mathbf{w}\cdot[z_{\mathbf{a}} \,|\, \mathbf{a} \in \mathscr{A}] \;:=\; [w_{\mathbf{a}} z_{\mathbf{a}} \,|\, \mathbf{a} \in \mathscr{A}] : \triangle^{\mathscr{A}} \to \triangle^{\mathscr{A}}\,.$$

$$\pi_{\mathscr{B}}(\mathbf{z}) \;:=\; \textstyle\sum_{\mathbf{a}\in\mathscr{A}} z_{\mathbf{a}}\, \mathbf{b}_{\mathbf{a}} : \mathbb{R}^{\mathscr{A}} \to \mathbb{R}^n.$$

# Factorization

The map $F(\mathbf{x}) : \Delta_{\mathscr{A}} \longrightarrow \mathbb{R}^n$ admits the following **factorization**:

$$F(\mathbf{x}) \,:\, \Delta_{\mathscr{A}} \xrightarrow{\beta_{\mathscr{A}}} \triangle^{\mathscr{A}} \xrightarrow{w\cdot} \triangle^{\mathscr{A}} \xrightarrow{\pi_{\mathscr{B}}} \mathbb{R}^n,$$

- The image $\mathbf{X}_{\mathscr{A}} := \beta_{\mathscr{A}}(\Delta_{\mathscr{A}}) \subset \triangle^{\mathscr{A}}$ is the **positive part of a toric variety**.
- Acting on $X_{\mathscr{A}}$ by the map $w\cdot$ gives a **translated toric variety** $X_{\mathscr{A},w}$.
- We call $X_{\mathscr{A},w}$ a **lift** of the toric Bézier patch $\mathbf{Y}_{\mathscr{A},\mathbf{w},\mathscr{B}} := \pi_{\mathscr{B}}(X_{\mathscr{A},w})$.

Elements $\mathbf{a}$ of $\mathbb{Z}^m$ are exponents of monomials,

$$\mathbf{a} \;=\; (a_1,\ldots,a_m) \;\longleftrightarrow\; \mathbf{x}^{\mathbf{a}} := x_1^{a_1}\cdots x_m^{a_m}\,,$$

The points of $\mathscr{A}$ define a map $\varphi_{\mathscr{A}} : \mathbb{R}^m_{>} \longrightarrow \triangle^{\mathscr{A}}$ by

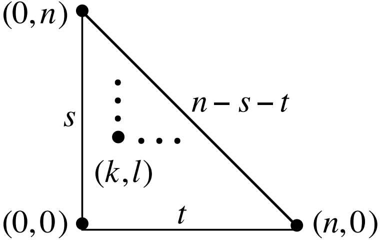
$$\varphi_{\mathscr{A}}(\mathbf{x}) \;:=\; [\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}]\,.$$

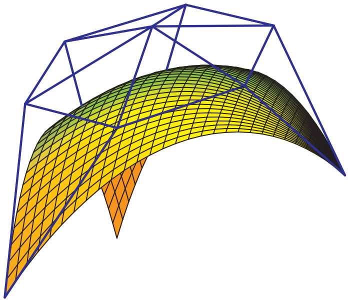
The **closure** in $\triangle^{\mathscr{A}}$ of the image of $\varphi_{\mathscr{A}}$ is the positive real part of a toric variety.

**PROPOSITION (KRASAUSKAS (2002))**

*The image of $\Delta_{\mathscr{A}}$ under the map $\beta_{\mathscr{A}}$ is the positive real part of the toric variety defined by $\varphi_{\mathscr{A}}$.*

Rimvydas Krasauskas, Toric surface patches, Adv. Comput. Math. **17** (2002), no. 1-2, 89–133.

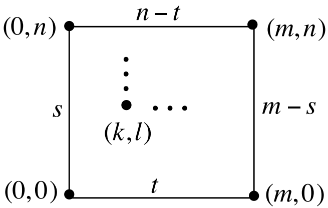


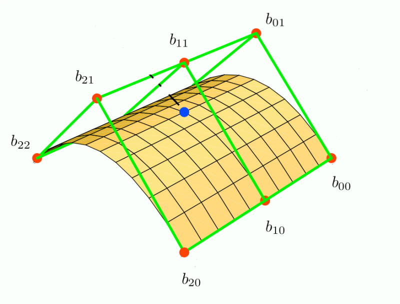



$$F(s,t) = \sum_{k,l} \frac{\binom{n}{k,l} s^k t^l (n-s-t)^{n-k-l}}{n^n} \mathbf{b}_{kl}$$

The corresponding toric variety is a Veronese surface of degree $n$.

# Toric Bézier rectangular surfaces




$$F(s,t) = \sum_{k,l} \frac{\binom{m}{k}\binom{n}{l} s^k (m-s)^{m-k} t^l (n-t)^{n-l}}{m^m n^n} \mathbf{b}_{kl}$$

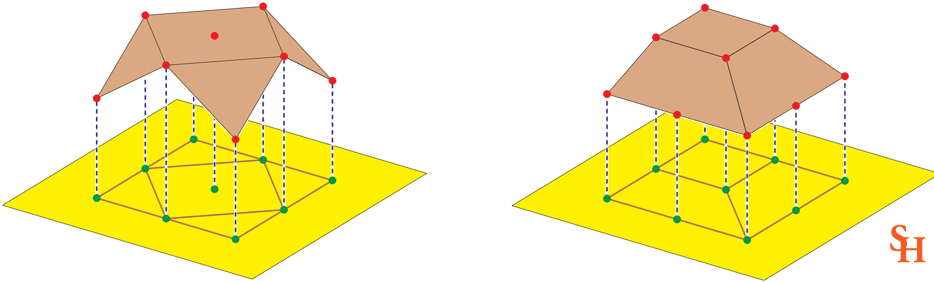
The corresponding toric variety is a is the Segre product of two rational normal curves of degrees $n$ and $m$.

Let $\mathscr{A} \subset \mathbb{R}^m$ and $\lambda\colon \mathscr{A} \to \mathbb{R}$ some function.

Let $\mathbf{P}_\lambda := \operatorname{conv}\{(\mathbf{a}, \lambda(\mathbf{a})) \mid \mathbf{a} \in \mathscr{A}\} \subset \mathbb{R}^{m+1}$.

Each face of $P_\lambda$ has an outward pointing normal vector, and its **upper facets** are those whose normal has positive last coordinate.

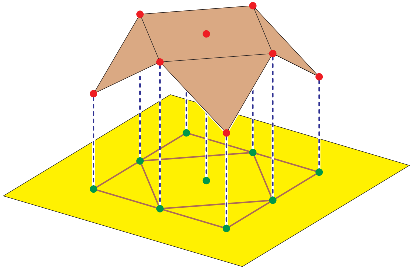
The **regular polyhedral decomposition** $\mathscr{T}_\lambda$ of $\Delta_{\mathscr{A}}$ induced by $\lambda$ is given by the projection of the upper facets back to $\mathbb{R}^m$.

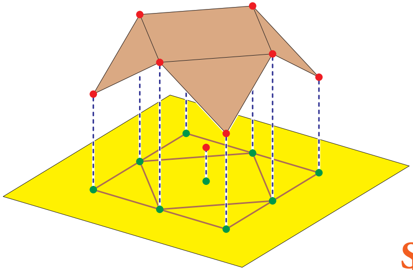


A **decomposition** $\mathscr{D}$ of the configuration $\mathscr{A}$ of points is a collection $\mathscr{D}$ of subsets of $\mathscr{A}$ called **faces**.

The **convex hulls** of these faces are required to be the faces of a **polyhedral subdivision** $\mathscr{T}(\mathscr{D})$ of $\Delta_{\mathscr{A}}$.

The decomposition $\mathscr{D}$ is **regular** if the polyhedral subdivision $\mathscr{T}(\mathscr{D})$ is regular.





Let $\mathscr{A} \subset \mathbb{Z}^m$ be a finite set, $w \in \mathbb{R}^{\mathscr{A}}_{>}$ be weights and $\mathscr{B} = \{\mathbf{b}_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}\}$ be control points for a toric patch $Y_{\mathscr{A},w,\mathscr{B}}$ of shape $\mathscr{A}$.

Let $\mathscr{D}$ be a **decomposition** of $\mathscr{A}$. The weights $w$ and control points $\mathscr{B}$ indexed by elements of a facet $\mathscr{F}$ can be used as weights and control points for a toric patch of shape $\mathscr{F}$, written $Y_{\mathscr{F},w|_{\mathscr{F}},\mathscr{B}|_{\mathscr{F}}}$.

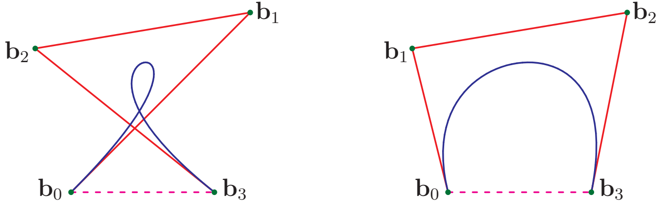
The **control surface** induced by $\mathscr{D}$ is the union

$$\mathbf{Y}_{\mathscr{A},\mathbf{w},\mathscr{B}}(\mathscr{D}) := \bigcup_{\mathscr{F}\in\mathscr{D}} Y_{\mathscr{F},w|_{\mathscr{F}},\mathscr{B}|_{\mathscr{F}}},$$

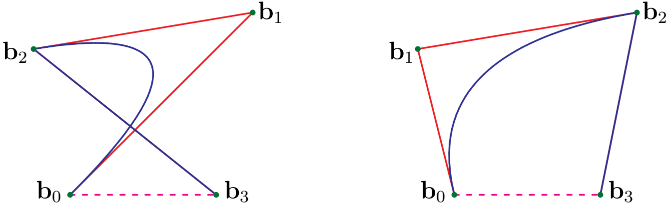
- Faces of toric patches are again toric patches,
- the control surface $Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D})$ is naturally a $C^0$ spline surface.
- $Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D})$ is **regular** if the decomposition $\mathscr{D}$ is regular.

# Control curves

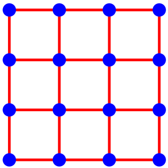
Given the regular decomposition $\{0,1,2\}$, $\{2,3\}$ of $\{0,1,2,3\}$ and the following two rational cubic Bézier planar curves.

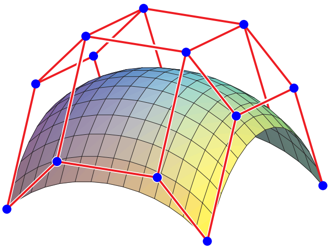


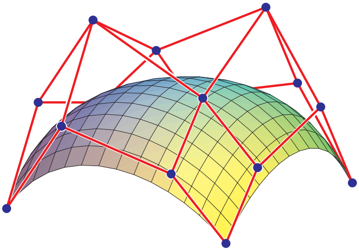

The regular control curves are



# Regular control surfaces for rational bicubic patches







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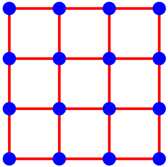
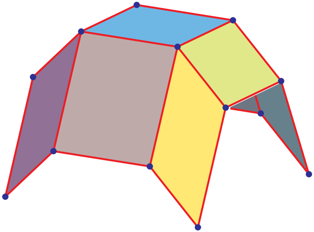
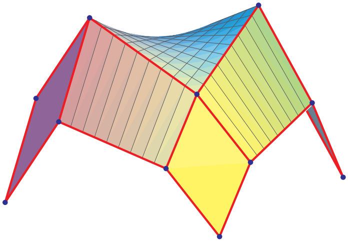
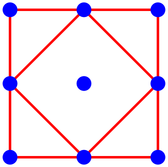
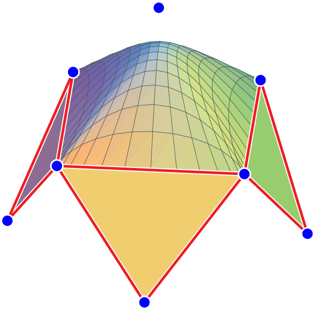


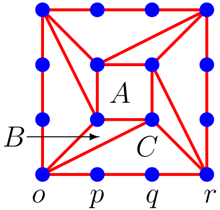




# Krasauskas's double pillow



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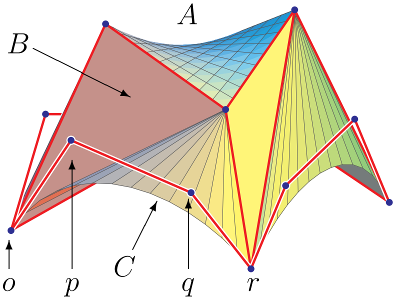
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Let $\lambda\colon \mathscr{A} \to \mathbb{R}$ be a **lifting function** and $w = \{w_{\mathbf{a}} \in \mathbb{R}_{>} \mid \mathbf{a} \in \mathscr{A}\}$ a set of weights.

Define $\mathbf{w}_{\lambda}(\mathbf{t}) := \{t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \mid \mathbf{a} \in \mathscr{A}\}$.

These weights are used to define a **toric degeneration** of the patch,

$$F_{\mathscr{A},w,\mathscr{B},\lambda}(\mathbf{x};t) := \frac{\sum_{\mathbf{a}\in\mathscr{A}} t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \beta_{\mathbf{a}}(\mathbf{x})\, \mathbf{b}_{\mathbf{a}}}{\sum_{\mathbf{a}\in\mathscr{A}} t^{\lambda(\mathbf{a})} w_{\mathbf{a}} \beta_{\mathbf{a}}(\mathbf{x})}\,.$$

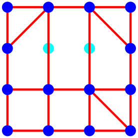
Let $\mathscr{D}_{\lambda}$ be the **regular decomposition** of $\mathscr{A}$ induced by $\lambda$.

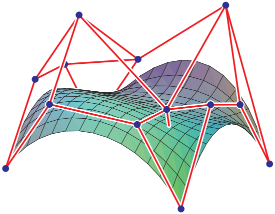
**THEOREM**

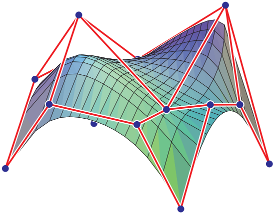
*Every regular control surface is the limit of the corresponding patch under a toric degeneration.*

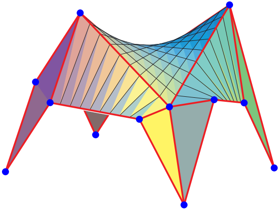
$$\lim_{t\to\infty} Y_{\mathscr{A},w,\mathscr{B},\lambda}(t) = Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D}_{\lambda}).$$

$$\begin{array}{cccc} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{array} \qquad \begin{array}{cccc} 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0.5 \end{array}$$









### Theorem

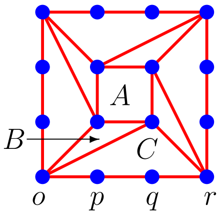
*Let $\mathscr{A} \subset \mathbb{Z}^m$ be a finite set and $\mathscr{B} = \{\mathbf{b_a} \mid \mathbf{a} \in \mathscr{A}\} \subset \mathbb{R}^n$ a set of control points. If $Y \subset \mathbb{R}^n$ is a set for which there is a sequence $w^1, w^2, \ldots$ of weights so that*

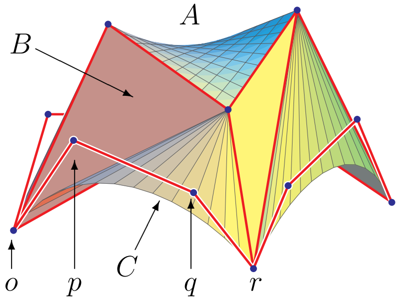
$$\lim_{i\to\infty} Y_{\mathscr{A},w^i,\mathscr{B}} = Y\,.$$

*then there is a lifting function $\lambda\colon \mathscr{A} \to \mathbb{R}$ and a weight $w \in \mathbb{R}^{\mathscr{A}}_{>}$ such that $Y = Y_{\mathscr{A},w,\mathscr{B}}(\mathscr{D}_\lambda)$, a regular control surface.*

**Regular control surfaces are exactly the possible limits of toric patches** when the control points $\mathscr{B}$ are fixed but the weights $w$ are allowed to vary.

The irregular control surface below cannot be the limit of toric Bézier patches.



# Proof of second main theorem

- To prove this theorem, we consider the sequence of translated toric varieties $X_{\mathscr{A},w^i} \subset \triangle^{\mathscr{A}}$.

- Work of Kapranov, Sturmfels, and Zelevinsky implies that the set of all translated toric varieties is **naturally compactified** by the set of all regular control surfaces in $\triangle^{\mathscr{A}}$.


M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, Quotients of toric varieties, Math. Ann. 290 (1991), no. 4, 643–655.

M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, Chow polytopes and general resultants, Duke Math. J. 67 (1992), no. 1, 189–218.


- Thus some subsequence of $\{X_{\mathscr{A},w^i}\}$ **converges** to a regular control surface in $\triangle^{\mathscr{A}}$, whose image must coincide with $Y$, implying that $Y$ is a regular control surface.

- This method of proof does not give a simple way to recover a lifting function $\lambda$ or the weight $w$ from the sequence of weights $w^1, w^2, \ldots$.

# Bibliography


- Gheorghe Craciun, Luis García-Puente, and Frank Sottile, Some geometrical aspects of control points for toric patches, Mathematical Methods for Curves and Surfaces, Lecture Notes in Computer Science, vol. 5862, Springer, 2010, pp. 111–135.
- Luis Garcia-Puente, Frank Sottile, and Chungang Zhu, Toric degenerations of Bézier patches, to appear in ACM Transactions on Graphics.