Resolutions of matroid ideals

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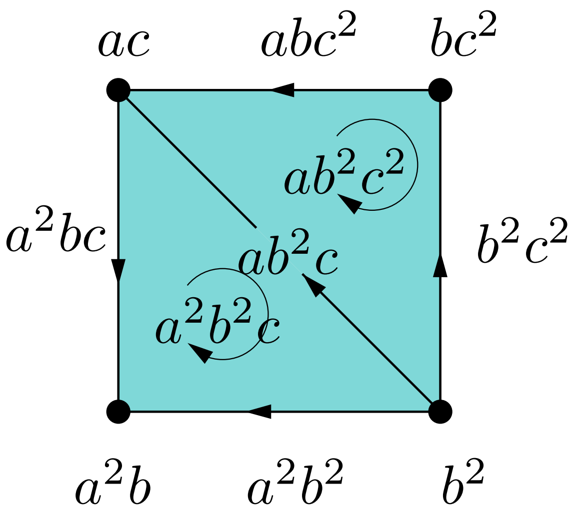
Virginia Tech

- Let $M = \langle a^2b, ac, b^2, bc^2 \rangle$ monomial ideal in $S = \mathbf{k}[a, b, c]$.
- Let X be the finite regular CW-complex:



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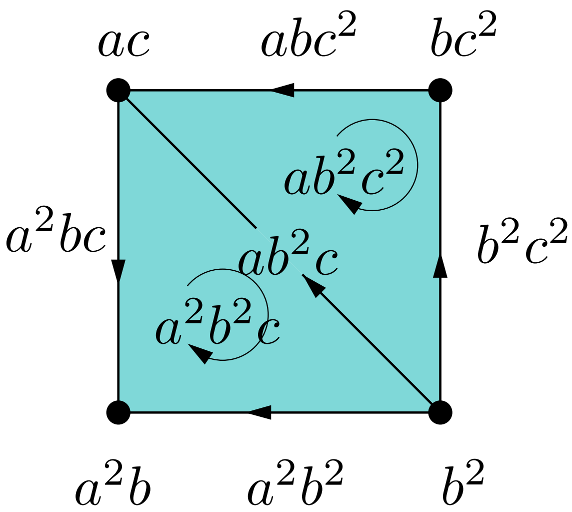

- Let $\mathbb{F}_X$ be the complex of $\mathbb{Z}^n$-graded free S-modules

$$0 \to S[-(2,2,1)] \oplus S[-(1,2,2)] \xrightarrow{\partial_2} S[-(2,1,1)] \oplus S[-(2,2,0)] \oplus$$

$$S[-(1,2,1)] \oplus S[-(1,1,2)] \oplus S[-(0,2,2)] \xrightarrow{\partial_1} S[-(2,1,0)] \oplus$$

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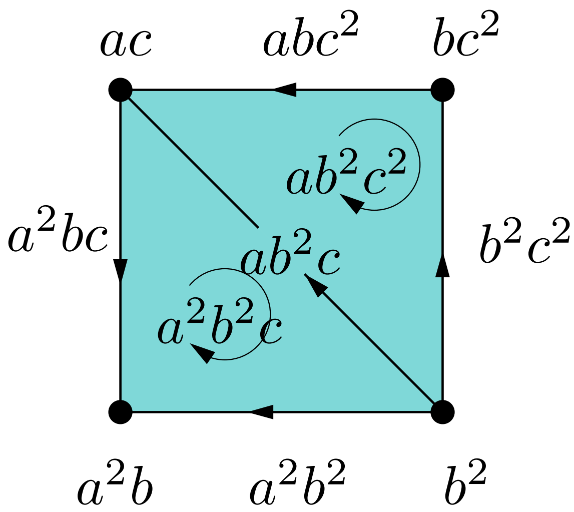
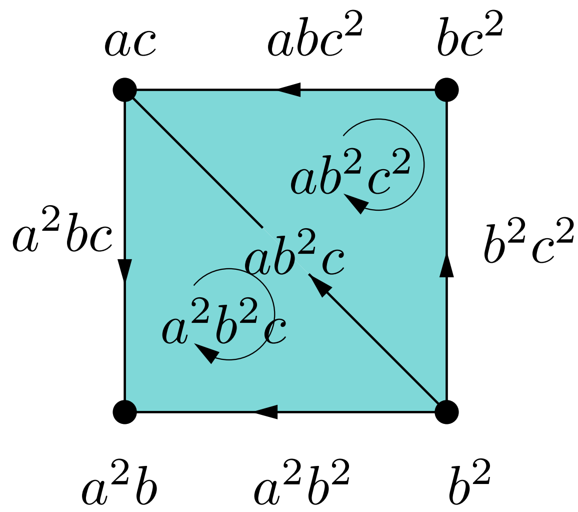
Cellular Resolutions (2)



- The differential ∂ acts on basis vectors

$$\partial(a^2b^2c) = -b \cdot a^2bc + c \cdot a^2b^2 - a \cdot ab^2c,$$

$$\partial(F) = \sum_{\text{facets } G \text{ of } F} \operatorname{sign}(G, F) \frac{\mathbf{x}^{\mathbf{a}_F}}{\mathbf{x}^{\mathbf{a}_G}} G.$$



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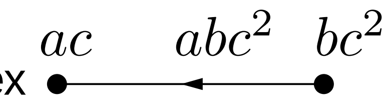
Observe $\partial_0 = \begin{bmatrix} a^2b & ac & bc^2 & b^2 \end{bmatrix}$. Thus $\mathrm{Coker}(\partial_0) = S/M$.

For $\mathbf{b} \in \mathbb{N}^n$, let $X_{\preceq \mathbf{b}}$ be the subcomplex of X consisting of all faces whose degrees are coordinatewise at most $\mathbf{b}$.

Example. Let $\mathbf{b} = (1, 1, 2)$, then $X_{\preceq \mathbf{b}}$ is the subcomplex 

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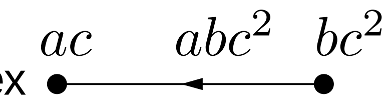
Example. Let $\mathbf{b} = (1, 1, 2)$, then $X_{\preceq \mathbf{b}}$ is the subcomplex 

Theorem. $\mathbb{F}_X$ *is exact if and only if* $X_{\preceq \mathbf{b}}$ *is acyclic over* $\mathbf{k}$ *for all* $\mathbf{b} \in \mathbb{N}^n$*.*

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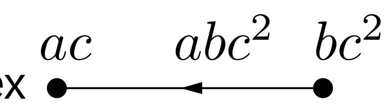
Example. Let $\mathbf{b} = (1, 1, 2)$, then $X_{\preceq \mathbf{b}}$ is the subcomplex 

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- The $\mathbb{Z}^n$-graded degree $\mathbf{b}$ part of $\mathbb{F}_X$ is the chain complex

$$0 \to \mathbf{k} \cdot 1 \to \mathbf{k} \cdot bc \oplus \mathbf{k} \cdot a \to \mathbf{k} \cdot abc^2$$

- ∂ is a degree 0 map. Then$(\mathbb{F}_X)_{\mathbf{b}}$ is the reduced chain complex

$$\widetilde{C}(X_{\preceq \mathbf{b}}; \mathbf{k}) = 0 \to \mathbf{k} \to \mathbf{k}^2 \to \mathbf{k}$$

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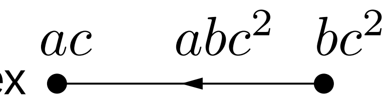
Example. Let $\mathbf{b} = (1, 1, 2)$, then $X_{\preceq \mathbf{b}}$ is the subcomplex 

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- $X_{\preceq \mathbf{b}}$ is contractible, so it has no reduced homology.

Matroid Ideals

An ideal M is a matroid ideal if and only if the following conditions hold

- M is a square-free monomial ideal,
- for every pair of monomials $m_1, m_2 \in M$ and any $i \in \{1, \ldots, n\}$ such that x_i divides both m_1 and m_2, the monomial $\mathrm{lcm}(m_1, m_2)/x_i$ is in M as well.

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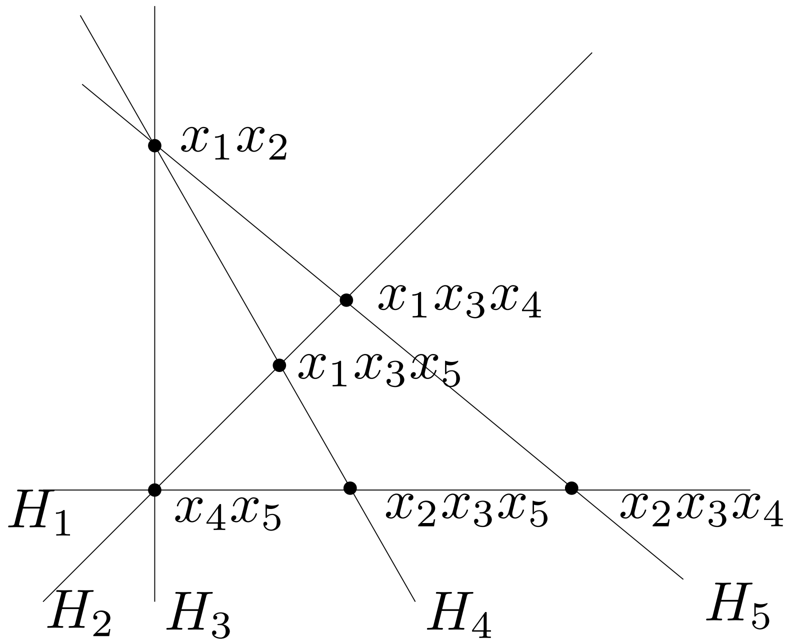
$$M = \langle x_1x_2, x_1x_3x_4, x_1x_3x_5, x_2x_3x_4, x_2x_3x_5, x_4x_5 \rangle.$$

Let $\mathcal{A}$ be the hyperplane arrangement in $\mathbb{R}^2$



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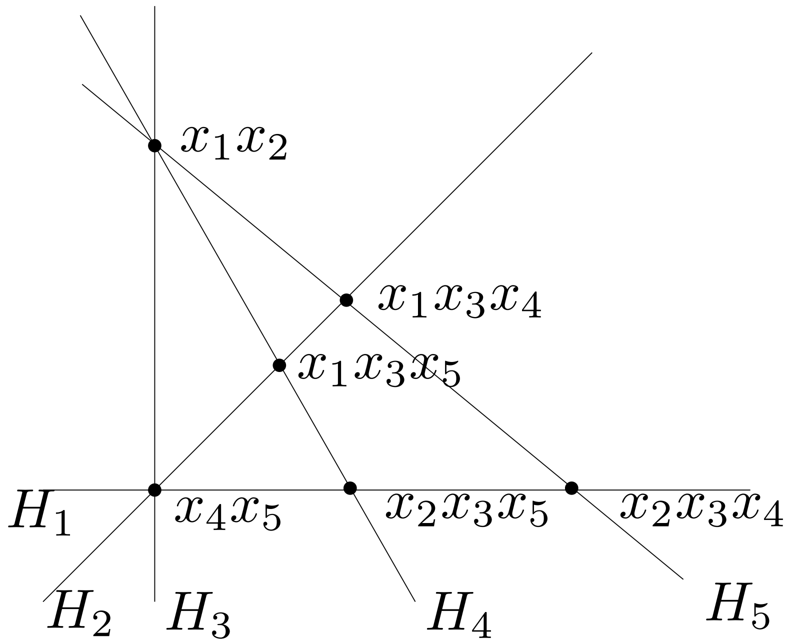

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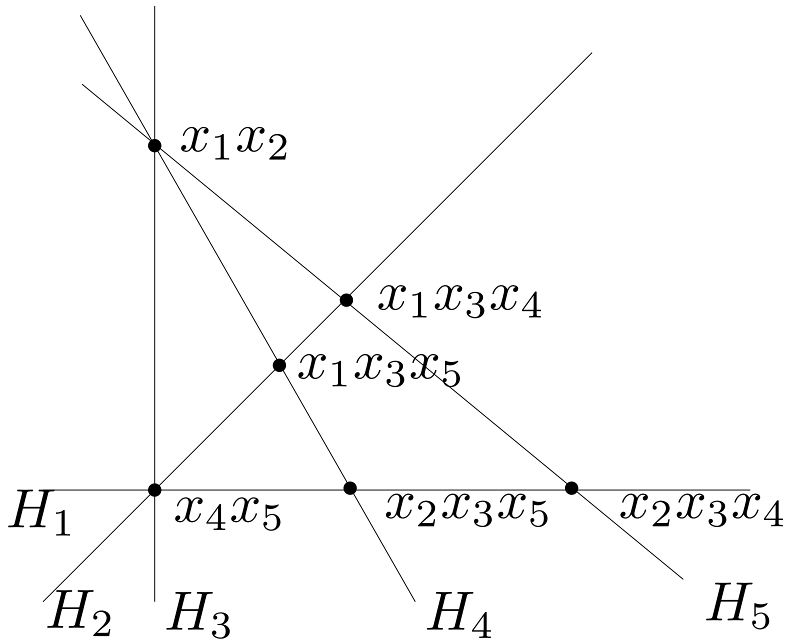

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- Hence $0 \longrightarrow S^4 \longrightarrow S^9 \longrightarrow S^6 \longrightarrow S$ is a minimal free resolution for $S/M_{\mathcal{A}}$.

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Generic Monomial Ideals

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- For $I \subseteq \{1, \ldots, r\}$, let $m_I = \mathrm{lcm}(m_i, i \in I)$.
- The Scarf complex of M consists of the following subsets:

$$\Delta_M := \{I \subseteq \{1, \ldots, r\} \mid m_I = m_J \Rightarrow I = J\}.$$

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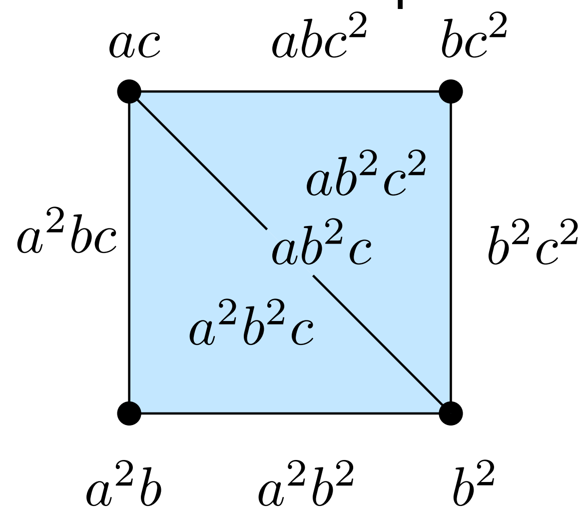

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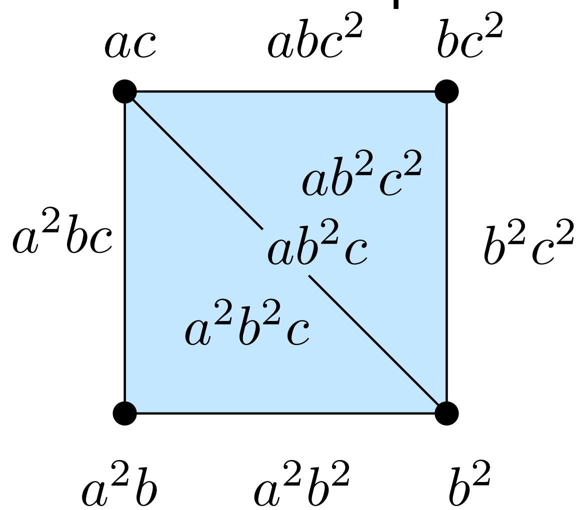

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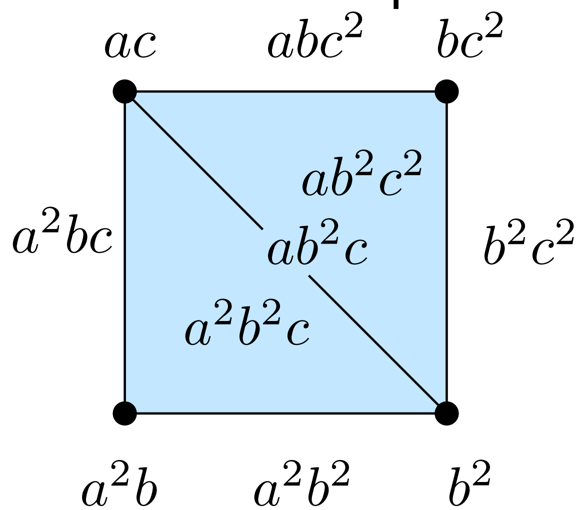

Theorem. *For M generic, the cellular complex $\mathbb{F}_{\Delta_M}$ is a minimal free resolution of S/M.*

- The minimal free resolution of S/M is

$$\boxed{0 \to S^2 \to S^5 \to S^4 \to S}$$

- From a monomial ideal M we can get a generic monomial ideal $\widetilde{M}$ by deforming the exponents of the minimal generators of M.
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CM Generic Deformations of Matroid Ideals (2)

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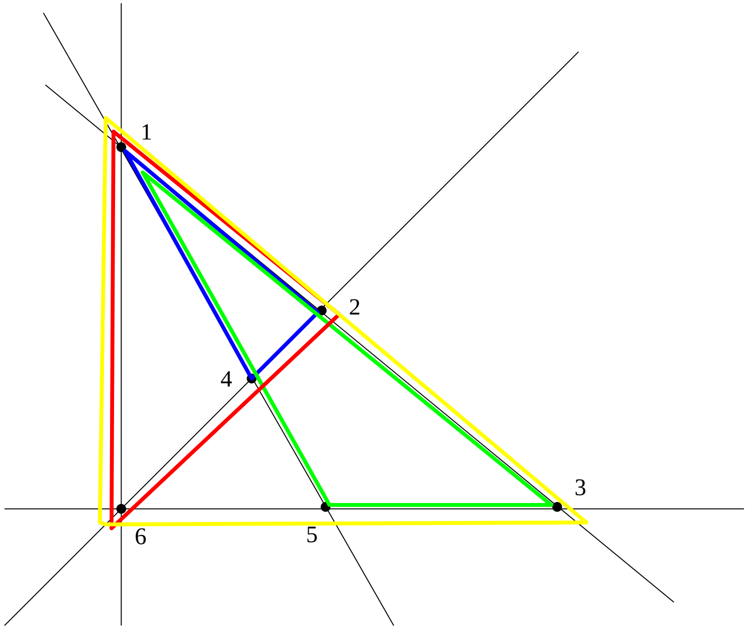
- Δ_{M^π} has facets $\{1,2,4\}$, $\{1,3,5\}$, $\{1,2,6\}$, $\{1,3,6\}$



- The **cellular complex** $\mathbb{F}_{\Delta_{M^\pi}}$ is equal to

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Non-minimal resolution

- Fix the order $\sigma = 1, 3, 4, 2, 5$. Then

$$M^\sigma = \langle x_1x_3x_4, x_1^2x_2^2, x_2^2x_3^2x_4^2, x_1^3x_3^3x_5^3, x_2^3x_3^3x_5^3, x_4^3x_5^3 \rangle$$

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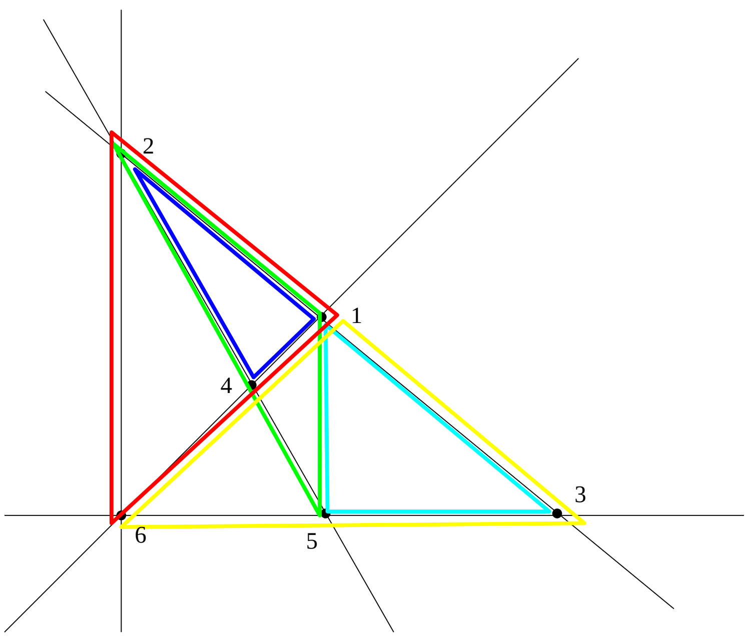

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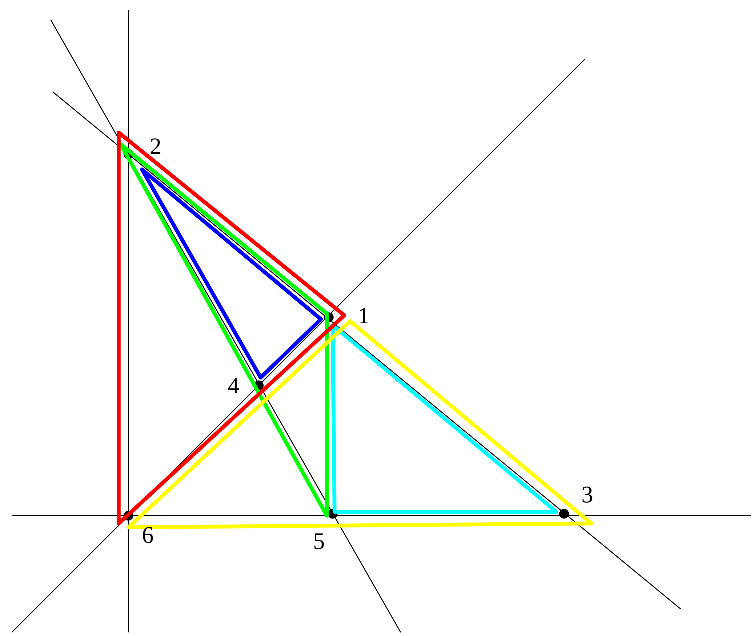
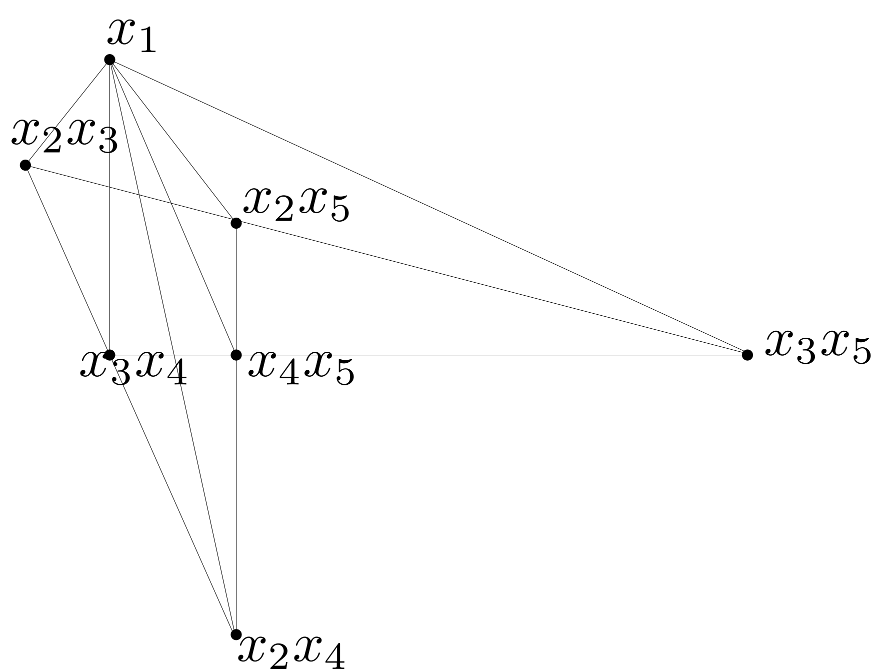


- $\mathbb{F}_{\Delta_{M^\sigma}}$ is equal to $0 \longrightarrow S^5 \longrightarrow S^{10} \longrightarrow S^6 \longrightarrow S$

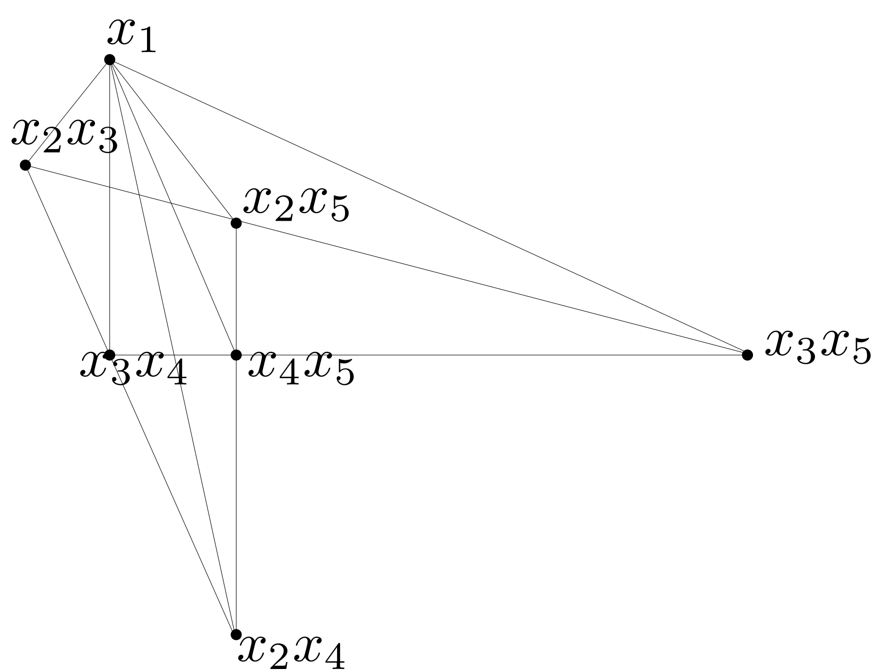
Theorem. *Let* $\dim(\Delta_{M^\pi}) = n$*, and let* $h(\pi)$ *be the unique monomial generator of* M_1*. Then* $\mathbb{F}_{\Delta_{M^\pi}}$ *is a minimal free resolution if and only if any facet* F*, with* $h(\pi) \in F$*, lies in a hyperplane.*



- $\langle x_1, x_2x_3, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_5 \rangle$
- Minimal free resolution

$$0 \longrightarrow S^3 \longrightarrow S^{11} \longrightarrow S^{14} \longrightarrow S^7 \longrightarrow S$$

Example



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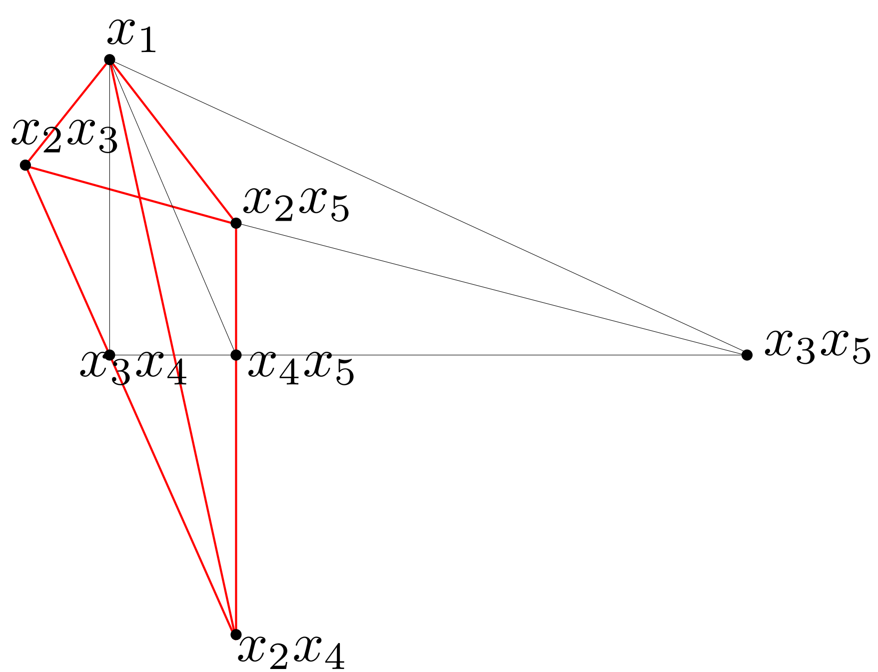

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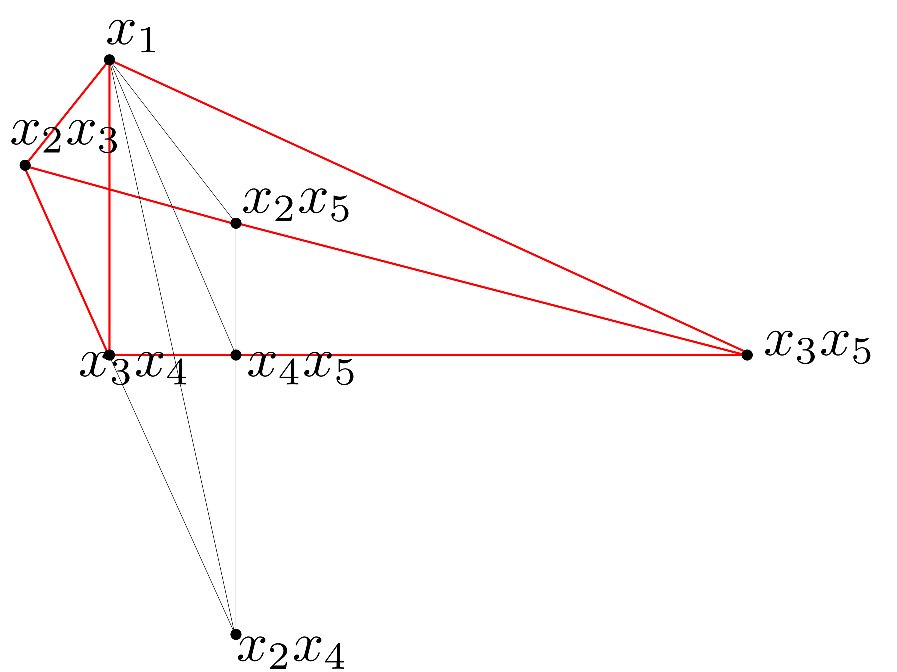

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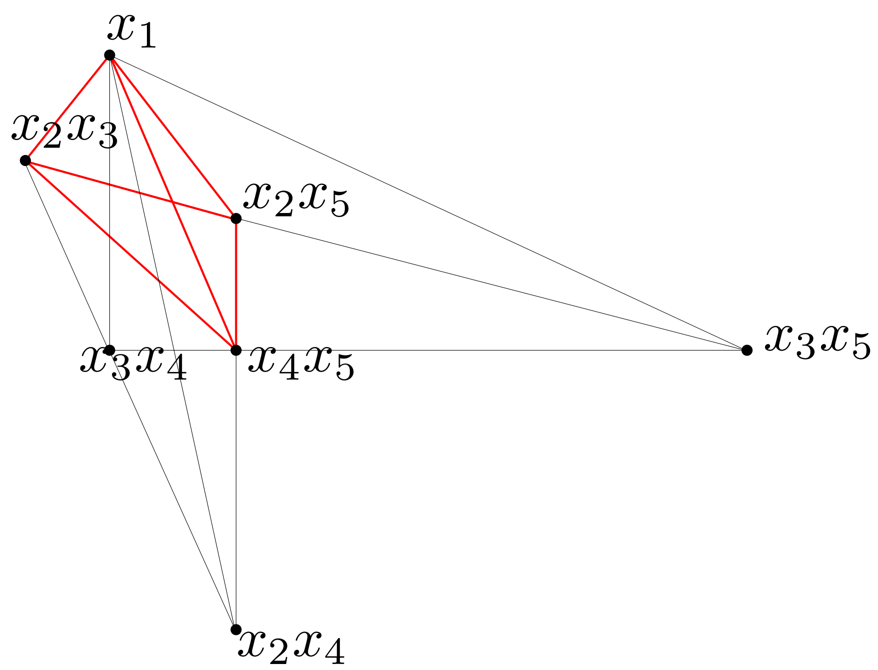

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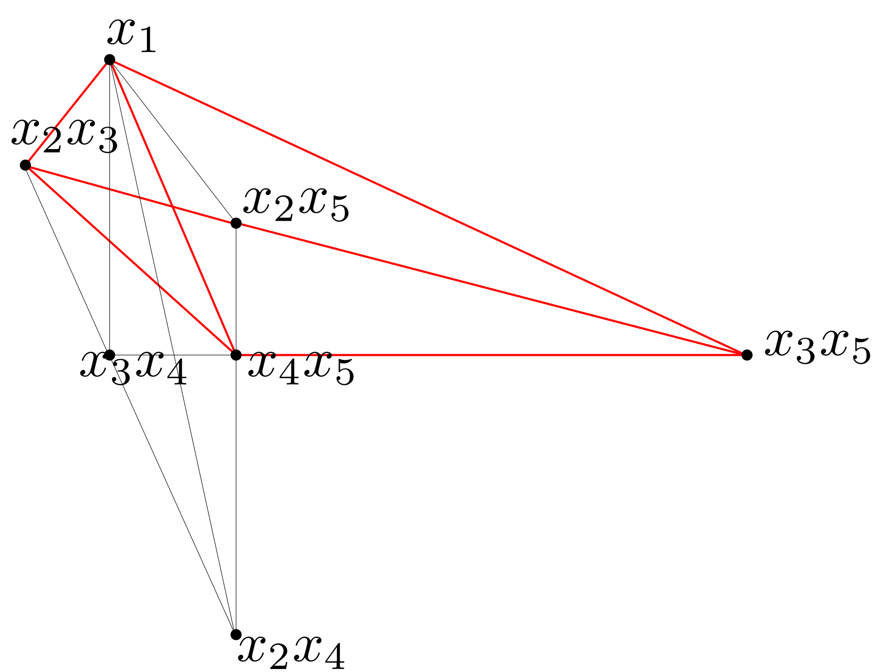

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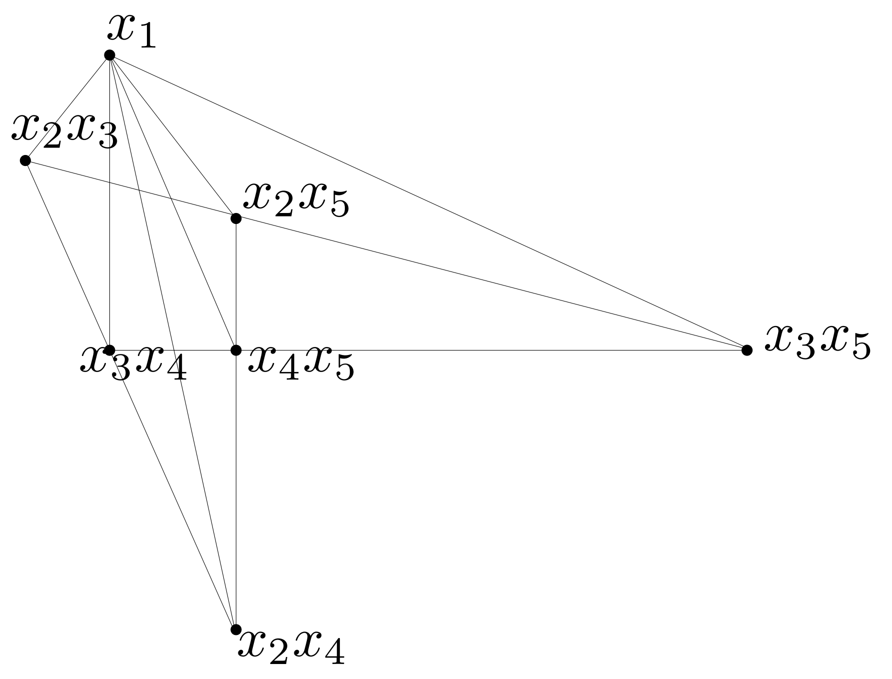
$$\{x_1, x_2x_3, x_3x_5, x_3x_4\},$$

$$\{x_1, x_2x_3, x_2x_5, x_4x_5\},$$

$$\{x_1, x_2x_3, x_3x_5, x_4x_5\}$$

Example



- There does not exist a minimal $\mathcal{A}$-deformation.
- Generic deformation which is not CM

$$\langle x_1, x_2x_4^3, x_3x_5^3, x_2^2x_3^2, x_4^2x_5^2, x_2^3x_5, x_3^3x_4 \rangle$$

- Scarf complex has dimension 4.