Linear Precision for Parametric Patches

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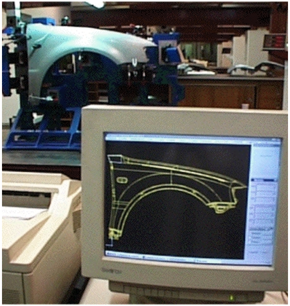
Algebraic Geometry and Geometric modeling

Geometric modeling uses polynomials to build computer models for industrial design and manufacture.

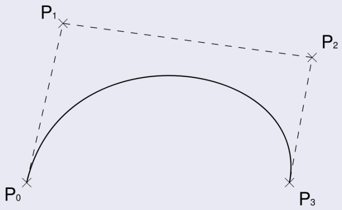


Algebraic geometry investigates the algebraic and geometric properties of polynomials.

Bézier curves and surfaces

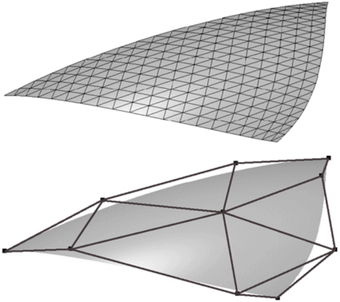
Bézier curves and surfaces are the fundamental units for geometric modeling of curves and surfaces

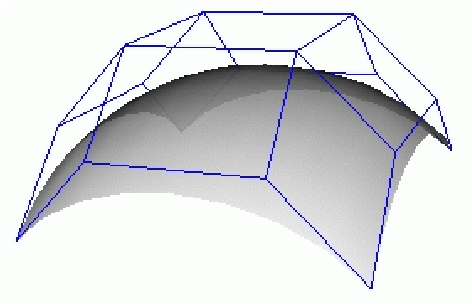
$$F(x) = P_0(1-x)^3 + 3P_1x(1-x)^2 + 3P_2x^2(1-x) + P_3x^3, \quad x \in [0,1]$$



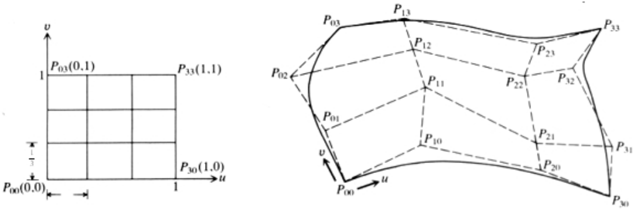
Bézier surfaces

Bézier surfaces come in two basic shapes – triangular Bézier patches and rectangular tensor product patches.

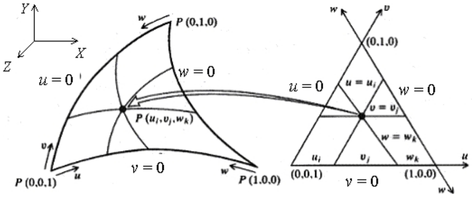




Bézier surfaces



(*a*) Parametric space (*b*) Cartesian space



Cartesian space Parametric space

Parametric representation of Bézier patches

Bézier curves of degree m

$\sum_{i=0}^{m} B_i^m(t)p_m, \; B_i^m(t) = \binom{m}{i} t^i (1-t)^{m-i}$

rectangular Bézier surfaces

$\sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) p_{ij}$

triangular Bézier surfaces

$\sum_{i+j+k=n} \frac{n!}{i!j!k!} u^i v^j w^k p_{ijk}$

rational Bézier patches

$$\frac{\sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) w_{ij} p_{ij}}{\sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) w_{ij}}$$

Toric patches

Toric patches . . .

- are a vast but well-controlled generalization of Bézier patches.
- are based on the geometry of toric varieties.
- depend on a polytope and some weights.

Toric patches

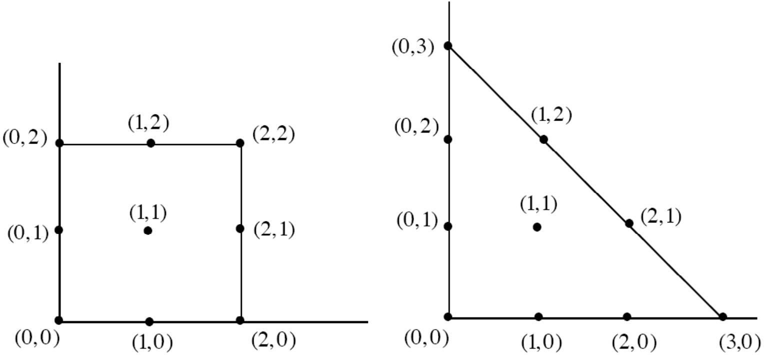
- Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon.
- Edges of Δ define lines $h_i(\mathbf{t}) = \langle \mathbf{n}_i, \mathbf{t} \rangle + a_i = 0$, with inward oriented normal primitive lattice vectors $\mathbf{n}_i$.

Let $\hat{\Delta} = \Delta \cap \mathbb{Z}^2$ be the set of lattice points of Δ. Note $h_i(\mathbf{m})$ is a non-negative integer for all $\mathbf{m} \in \hat{\Delta}$.

A toric patch associated to Δ is a rational patch with domain Δ and basis functions

$$h_1^{h_1(\mathbf{m})} h_2^{h_2(\mathbf{m})} \cdots h_r^{h_r(\mathbf{m})}.$$

Toric surface patches



Lattice Rectangle
(Biquadratic)

Lattice Triangle
(Cubic)

Linear precision

- Bézier patches have linear precision.
- It underlies numerical stability of Bézier patches.

Linear precision is the ability of a patch to replicate linear functions.

Rimvydas Krasauskas

Which toric Bézier patches have linear precision?

Parametric patch

Let $\mathcal{A}$ be a finite set of points in $\mathbb{R}^2$.

A control point scheme for parametric patches

A patch is a collection $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ of non-negative functions, called blending functions.

Partition of unity

$$\sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x) \;=\; 1$$

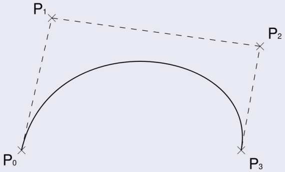
The common domain of the blending functions is the convex hull Δ of $\mathcal{A}$.

Parametric representation of a patch

Given a set $\{P_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\} \subset \mathbb{R}^3$ of control points, define a smooth map $F\colon \Delta \to \mathbb{R}^3$ by

$$F(x) \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x) P_{\mathbf{a}}\,.$$

$$F(x) = P_0(1-x)^3 + 3P_1x(1-x)^2 + 3P_2x^2(1-x) + P_3x^3, \quad x \in [0,1]$$



Linear precision

Parametric map

$$F(x) \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x) P_{\mathbf{a}}\,.$$

Tautological map

$$\tau(x) \;:=\; \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x)\mathbf{a}\,.$$

Definition

A patch has linear precision if and only if its tautological map is the identity map on Δ.

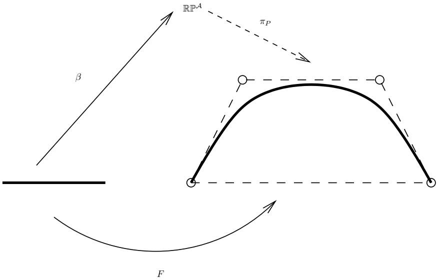
Bézier cubic in $\mathbb{R}^3$

- $\mathcal{A} = \{0, \frac{1}{3}, \frac{2}{3}, 1\} \subset [0, 1]$.
- Control points $P_0, P_1, P_2, P_3 \in \mathbb{R}^3$

$$F(x) = P_0(1-x)^3 + 3P_1x(1-x)^2 + 3P_2x^2(1-x) + P_3x^3$$

Linear precision

$$\tau(x) = x(1-x)^2 + 2x^2(1-x) + x^3 = x$$



$$F(x) = \sum_{\mathbf{a}\in\mathcal{A}} \beta_{\mathbf{a}}(x) P_{\mathbf{a}}$$

Map given by $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$

$$\beta\colon \Delta \to \mathbb{RP}^{\mathcal{A}}, \qquad \beta : x \longmapsto [\beta_{\mathbf{a}}(x) \mid \mathbf{a} \in \mathcal{A}]$$

Linear projection given by $P = \{P_{\mathbf{a}} \in \mathbb{R}^3 \mid \mathbf{a} \in \mathcal{A}\}$

$$\pi_P : \mathbb{RP}^{\mathcal{A}} \overset{\pi_P}{--\!\!\longrightarrow} \mathbb{RP}^3, \qquad y = [y_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}] \longmapsto \sum_{\mathbf{a}\in\mathcal{A}} y_{\mathbf{a}}(1, P_{\mathbf{a}})$$

Linear precision

The parametric map is the composition

$$\Delta \xrightarrow{\beta} X_\beta = \beta(\Delta) \subset \mathbb{RP}^{\mathcal{A}} \dashrightarrow^{\pi_P} \mathbb{RP}^3 .$$

$F(\Delta)$ is the image of X_β under the projection π_P.

The tautological map is the composition

$$\Delta \xrightarrow{\beta} X_\beta \subset \mathbb{RP}^{\mathcal{A}} \dashrightarrow^{\pi_{\mathcal{A}}} \Delta \subset \mathbb{RP}^2 .$$

Geometric criterion

The patch has linear precision if this composition is the identity.

Main result

Theorem

If a patch $\beta = \{\beta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{A}\}$ has linear precision, then

1. $Y_\beta = \overline{X_\beta}$ *is a rational variety,*
2. Y_β *meets the center* $E_{\mathcal{A}}$ *of the tautological projection in a maximally degenerate manner.*

In algebro-geometric terms, linear precision is a pathological situation

This result gives a very strong and precise tool to study linear precision

Results

Theorem

Bézier simploids (higher-dimensional generalization of Bézier curves and surfaces) are the only toric patches based on a product of standard simplices which have linear precision.

Theorem (Ranestad, Sottile)

Triangular Bézier patches and rectangular tensor product patches are the unique toric surface patches having linear precision.
There are no n-sided toric surface patches having linear precision for $n > 4$.