THE SECANT CONJECTURE IN THE REAL SCHUBERT CALCULUS

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Abstract. We formulate the Secant Conjecture, which is a generalization of the Shapiro Conjecture for Grassmannians. It asserts that an intersection of Schubert varieties in a Grassmannian is transverse with all points real, if the flags defining the Schubert varieties are secant along disjoint intervals of a rational normal curve. We present both theoretical evidence for its validity and computational evidence obtained in an experiment using over one terahertz-year of computing, and we discuss some phenomena we observed in our data.

1. Introduction

Some solutions to a system of real polynomial equations are real and the rest occur in complex conjugate pairs. The total number of solutions is determined by the structure of the equations, but the distribution between the two types depends rather subtly on the coefficients. Sometimes the structure of the equations leads to finer information in terms of upper bounds [20, 2] or lower bounds [8, 35] on the number of real solutions. The Shapiro and Secant Conjectures assert the extreme situation of having only real solutions.

The Shapiro Conjecture for Grassmannians asserts that if the Wronskian of a vector space of univariate *complex* polynomials has only real roots, then that space is spanned by *real* polynomials. This striking instance of unexpected reality was proven by Eremenko and Gabrielov for two-dimensional spaces of polynomials [9, 7], and the general case was established by Mukhin, Tarasov, and Varchenko [25, 27]. While the statement concerns spaces of polynomials, or more generally the Schubert calculus on Grassmannians, these proofs of the Shapiro Conjecture used complex analysis [9, 7] and mathematical physics [25, 27]. This story was described in the AMS Bulletin [42].

The Shapiro Conjecture for Grassmannians was first popularized by partial results and computations reported in [39, 45], where a counterexample to the conjecture for flag manifolds was found. Further study [40] led to the Monotone Conjecture, which is an extension of the Shapiro Conjecture that appears to hold for flag manifolds and is supported by significant computational evidence and partial results, reported in [31]. Using the results of [9],

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Eremenko, Gabrielov, Shapiro, and Vainshtein [10] proved the Monotone Conjecture for manifolds of two-step flags consisting of a codimension-two plane lying on a hyperplane. These conjectures—Shapiro and Monotone—concern intersections of Schubert varieties defined by flags that osculate (are tangent to) a rational normal curve.

The result of [10] was in fact a proof of reality for intersections of Schubert varieties in the Grassmannian of codimension-two planes defined with respect to certain *disjoint secant flags*. We formulate the *Secant Conjecture*, which postulates an extension of this result of [10] concerning disjoint secant flags to all Grassmannians. We state here the simplest open instance of the Secant Conjecture. Let $x_1,\dots,x_6$ be indeterminates and consider the polynomial

$$f(s,t,u;x) \ := \ \det\begin{pmatrix} 1 & 0 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 & x_6 \\ 1 & s & s^2 & s^3 & s^4 \\ 1 & t & t^2 & t^3 & t^4 \\ 1 & u & u^2 & u^3 & u^4 \end{pmatrix}, \tag{1.1}$$

which depends upon parameters s,t,u.

Conjecture 1.1. *Let $s_1 < t_1 < u_1 < s_2 < t_2 < \cdots < u_5 < s_6 < t_6 < u_6$ be real numbers. Then the system of polynomials*

$$f(s_i,t_i,u_i;x) \ = \ 0 \qquad i = 1,\dots,6 \tag{1.2}$$

has five solutions, and all of them are real.

Geometrically, the equation $f(s,t,u;x) = 0$ says that the 2-plane (spanned by the first two rows of the matrix in (1.1)) meets the 3-plane which is secant to the rational curve $\gamma\colon y \mapsto (1,y,y^2,y^3,y^4)$ at the points $\gamma(s),\gamma(t),\gamma(u)$. The point of the hypothesis is that each of the six 3-planes is secant to γ along an interval $[s_i,u_i]$, and these six intervals are pairwise disjoint. That is, it asserts that all of the 2-planes meeting six 3-planes are real, when the 3-planes are secant to the rational normal curve along disjoint intervals. This conjecture held in each of the 285,502 instances we tested. We remark that if we relax the disjointness condition slightly, so that we have $\cdots t_1 < s_2 < u_1 < t_2 < \cdots$ with the other points in order, then in each of the 13,605 cases we tested, all five solutions were still real. However, in cases of larger overlap among the intervals of secancy $[s_i, u_i]$, we did discover instances with some nonreal solutions.

We also present extensive evidence for the validity of the Secant Conjecture and describe related phenomena that we observed in our data. These data come mostly from an experiment we conducted that used 1.065 terahertz-years of computing, mostly on computers in instructional labs at Texas A&M University which moonlight as a supercomputer outside of teaching hours. The experiment determined the number of real solutions to 1,855,810,000 instances of 703 Schubert problems on 13 different Grassmannians, verifying the Secant Conjecture in all 448,381,157 instances it computed. These data may be viewed online [51], and our MySQL database and the Perl code for conducting the experiment are freely available from our website [51]. The design and execution of this computational experiment was described in [15], including information on how to run similar large-scale computations.

In addition to the symbolic computation of this experiment, the Secant Conjecture was also verified in 25,000 instances of a Schubert problem with 126 solutions on the Grassmannian of 4-planes in 8-space. In contrast to the symbolic computation in the experiment, this verification was numerical, using the Bertini package [3] (based on numerical homotopy continuation [34]) to compute the solutions, whose reality was softly certified using Smale's α-theory [33] as implemented in the package alphaCertified [14].

Mathematics includes other examples of large-scale computation, computer-generated data to support conjectures, and published data sets. The largest computation that we know of is the Great Internet Mersenne Prime Search (GIMPS) [23], which uses distributed computing conducted by computers controlled by volunteers, and has found the largest known primes since it started in 1996. Daily, it uses approximately 50 gigahertz-years of computation. Another distributed computation is the search for Golomb rulers [48]. A notable computation whose data set stimulated much research [6] is Odlyzko's study [28] (using Cray supercomputers) of the zeroes of Riemann's ζ-function on the critical line $\frac{1}{2}+\mathbb{R}\sqrt{-1}$. There are several large computer-generated databases which have been useful for mathematical research, such as the Atlas of Finite Group Representations [47], the Atlas of Lie Groups and Representations [50], and the Knot Atlas [49].

This paper is organized as follows. In Section 2, we describe the classical problem of four lines and use it to motivate the Shapiro, Monotone, and Secant Conjectures, giving a detailed history of their formulation and evolution. In Section 3, we present some theoretical justification for the Secant Conjecture and explain the data from our experiment. Almost 3/4 of our 1.8 billion computations did not directly test the Secant Conjecture, but rather tested geometric configurations that were close to those of the conjecture. As a result, our data contains much more information than that in support of the Secant Conjecture. We explore this additional information in our last three sections, first describing completely the problem of lines meeting all possible configurations of four secant lines in Section 4—often both solutions are real, even when the secant lines are not disjoint. In general, if the secant lines are only mildly disjoint, at least some of the solutions will be real. Such lower bounds on the numbers of real solutions produce a striking *inner border* in the tabulation of our data, which we discuss in Section 5. Finally, in Section 6, we discuss Schubert problems with provable lower bounds and gaps in their numbers of real solutions, a phenomenon we first noticed while trying to understand our data.

2. The Shapiro Conjecture

While the Shapiro and Secant Conjectures are compellingly conjectures about real solutions to systems of equations, they are also fundamentally conjectures about configurations of subspaces in relation to rational normal curves. We motivate this point of view by beginning with the classical problem of four lines and the meaning of the Shapiro and Secant

Conjectures in this problem, which are vividly about configurations of tangent and secant lines. We then state both conjectures and give a short history of their development.

2.1. **The problem of four lines.** The classical *problem of four lines* asks for the finitely many lines that meet four given lines in 3-space. It turns out that there are two, if the four lines are general. In this case, the first three lines lie in one ruling of a unique doubly-ruled quadric surface Q, with the other ruling consisting of all the lines that meet the first three lines. The fourth line meets Q in two points, and through each of these points is a line in the second ruling. These two lines are the solutions to our problem of four lines. If the four given lines are real then so is Q, but the intersection of Q with the fourth line may consist either of two real points or of a complex conjugate pair of points. In the first case, the problem of four lines has two real solutions, while in the second case it has two complex conjugate (and no real) solutions.

The Shapiro Conjecture asserts that if the four given lines are tangents to a real rational normal curve then both solutions are real. We illustrate this. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ be the curve $\gamma(t) = (6t^2 - 1, \frac{7}{2}t^3 + \frac{3}{2}t, -\frac{1}{2}t^3 + \frac{3}{2}t)$, which is a rational normal curve. We ask for the finitely many lines that meet four tangent lines to γ, which we take to be the tangents at the points $\gamma(t)$ for $t = -1,0,1$, and s for some $s \in (0,1)$. The first three tangent lines lie on the quadric Q defined by $x^2 - y^2 + z^2 = 1$. This is illustrated in Figure 1, where $\ell(t)$ is the tangent line at the point $\gamma(t)$. As we see in Figure 2, $\ell(s)$ meets the quadric in two real points, giving

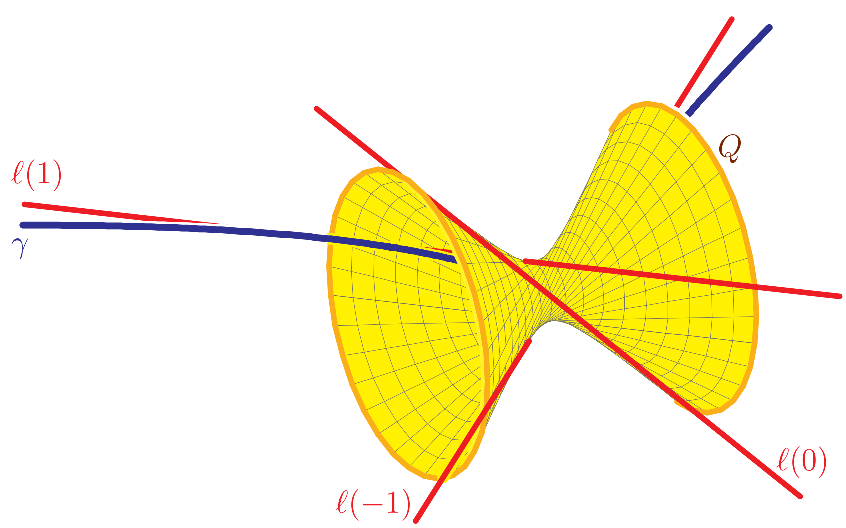


FIGURE 1. Quadric containing three lines tangent to the curve γ.

two real solutions to this instance of the problem of four lines.

We perturb the line tangent at $\gamma(s)$ to a secant line. In Figure 3 the fourth line is secant to γ at two points $\gamma(u), \gamma(v)$ for $0 < u < v < 1$. As illustrated, the two solutions to the problem of four lines are real. In contrast, the fourth line in Figure 4 is secant at two points

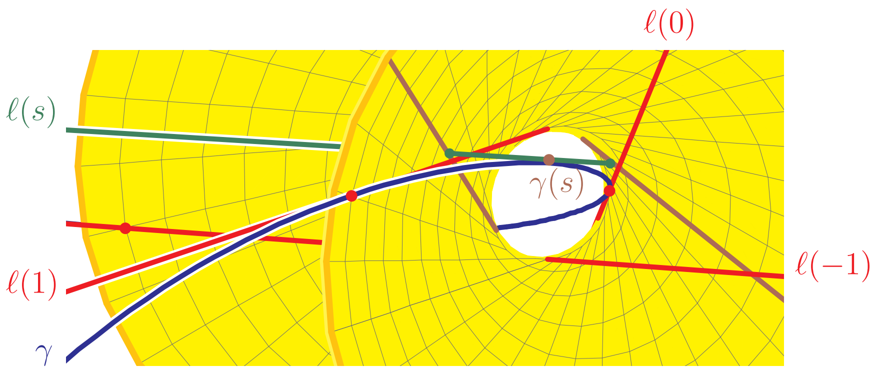


FIGURE 2. All tangent lines give two real solutions.

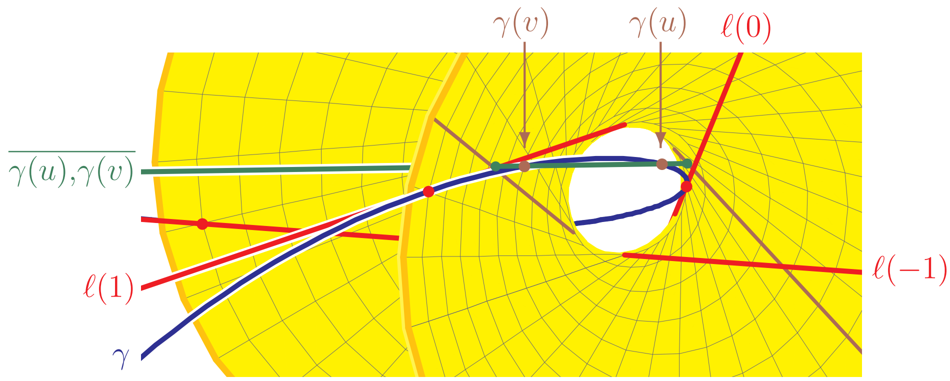


FIGURE 3. Secant line with two real solutions.

$\gamma(u), \gamma(v)$ for $-1 < u < 0 < v < 1$, and the solutions to the problem of four lines are not real because the secant line does not meet Q at real points. In Figure 3, the interval of secancy $\gamma([u,v])$ is disjoint from the points of tangency $-1,0,1$, while in Figure 4, neither of the two arcs in $\gamma \cong \mathbb{RP}^1 \cong S^1$ determined by u,v is disjoint from $-1,0,1$.

This example shows that the combinatorics of the points of secancy affect the number of real solutions, and leads to the Secant Conjecture.

2.2. **The Shapiro Conjecture.** The Schubert calculus [12, 13] involves problems of determining the linear spaces (or flags of linear spaces) that have specified positions with respect to other, fixed (flags of) linear spaces. For example, what are the 3-planes in $\mathbb{C}^7$ meeting 12 given 4-planes non-trivially? (There are 462 [32].) The specified positions are a *Schubert problem*, an example being the Schubert problem of lines meeting four lines in 3-space. The actual linear spaces imposing the conditions give an *instance* of the Schubert problem, so that the given four lines $\ell(-1)$, $\ell(0)$, $\ell(1)$, and $\ell(s)$ with which we began determine an

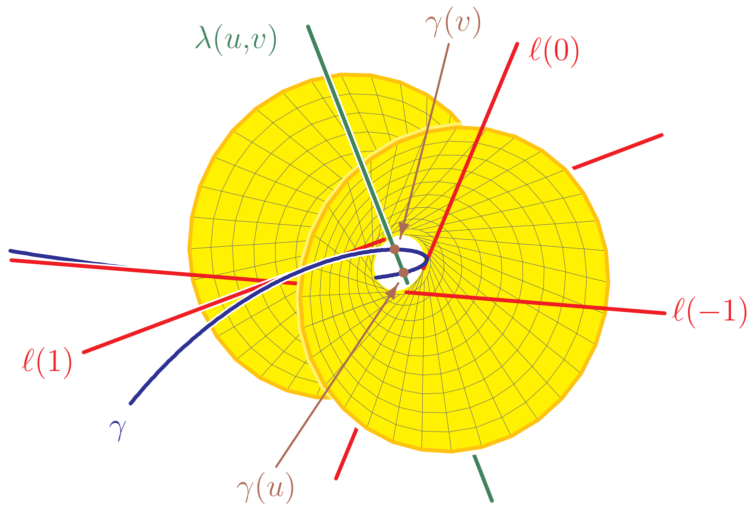


FIGURE 4. Secant line with no real solutions.

instance of the problem of four lines. The number of solutions depends upon the Schubert problem, while the solutions depend upon the instance.

A rational normal curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is affinely equivalent to the moment curve

$$t \longmapsto (1, t, t^2, \ldots, t^{n-1})\,. \tag{2.1}$$

Around 1993, Boris Shapiro and Michael Shapiro conjectured that if the fixed linear spaces osculate a rational normal curve, then all solutions to the Schubert problem will be real. When it was communicated to Sottile in 1995 it seemed too strong to be true, even for Grassmannians, which are the simplest of flag manifolds.

This perception changed dramatically after a few computations [30, 36]. This led to a systematic study of the conjecture for Grassmannians, both theoretical and experimental [39], in which about 40,000 instances of 11 different Schubert problems were computed. Several extremely large instances were also verified by others [11, 45].

This early study led to a proof of the Shapiro Conjecture in a limiting sense for Grassmannians [37] and a related result in the quantum cohomology of Grassmannians [38], and the publicity drew others to it. Eremenko and Gabrielov [9] proved the Shapiro Conjecture for Grassmannians of codimension-two subspaces where it is the statement that a univariate rational function with only real critical points is (equivalent to) a quotient of real polynomials.

Later, Mukhin, Tarasov, and Varchenko [25] used ideas from integrable systems and representation theory to prove the Shapiro Conjecture for all Grassmannians. Since then, a second proof was given by Mukhin, Tarasov, and Varchenko [27] which revealed deep connections between geometry and representation theory. This story was the subject of an article in the AMS Bulletin [42].

We state the theorem of Mukhin, Tarasov, and Varchenko more precisely and fix some notation. Let $1 \leq k \leq n$ be integers. The *Grassmannian* $G(k,n)$ is the set of all k-dimensional linear subspaces of $\mathbb{C}^n$, which is an algebraic manifold of dimension $k(n-k)$. A *flag* $F_\bullet$ is a sequence of linear subspaces

$$F_\bullet \ :\ F_1 \subset F_2 \subset \cdots \subset F_n\,,$$

where $\dim F_i = i$. A *partition* λ: $(n-k) \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ is a weakly decreasing sequence of integers. A fixed flag $F_\bullet$ and a partition λ together define a *Schubert variety* $X_\lambda F_\bullet$,

$$X_\lambda F_\bullet \ :=\ \{H \in G(k,n) \mid \dim H \cap F_{n-k+i-\lambda_i} \geq i \quad \text{for } i = 1,\ldots,k\}\,,$$

which is a subvariety of $G(k,n)$ of codimension $|\lambda| := \lambda_1 + \cdots + \lambda_k$. Note that not every element of the flag is used to define the Schubert variety.

A *Schubert problem* is a list $\lambda^1,\ldots,\lambda^m$ of partitions with $|\lambda^1| + \cdots + |\lambda^m| = k(n-k)$. Given general flags $F^1_\bullet,\ldots,F^m_\bullet$, the intersection

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^m}F^m_\bullet$$

is transverse [21] and consists of a certain number of $d = d(\lambda^1,\ldots,\lambda^m)$ points, which may be computed using algorithms in the Schubert Calculus (see [12, 22]). We may abbreviate a Schubert problem by writing $\lambda^1 \cdot \cdots \cdot \lambda^m = d(\lambda^1,\ldots,\lambda^m)$. For example, writing $\square$ for the partition (1,0) with $|\square| = 1$, we have $\square\cdot\square\cdot\square\cdot\square\cdot\square\cdot\square = \square^6 = 5$ for the Schubert problem on $G(2,5)$ involving six partitions, each equal to $\square$.

Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a rational normal curve (2.1). The *osculating flag* $F_\bullet(t)$ is the flag whose i-dimensional subspace is the span of the first i derivatives $\gamma(t),\gamma'(t),\ldots,\gamma^{(i-1)}(t)$ of γ at t.

Theorem of Mukhin-Tarasov-Varchenko.([25, 27]) *For any Schubert problem $\lambda^1,\ldots,\lambda^m$ on a Grassmannian $G(k,n)$ and any distinct real numbers $t_1,\ldots,t_m$, the intersection*

$$X_{\lambda^1}F_\bullet(t_1) \cap X_{\lambda^2}F_\bullet(t_2) \cap \cdots \cap X_{\lambda^m}F_\bullet(t_m)$$

is transverse and consists of $d(\lambda^1,\ldots,\lambda^m)$ real points.

The transversality in this theorem is unexpected as osculating flags are not general; in fact, Schubert varieties in the flag manifold given by osculating flags may not meet in the expected dimension [31, § 3.3.6].

We relate this to the problem of four lines. A line in projective 3-space $\mathbb{P}^3$ corresponds to a 2-dimensional linear subspace in $\mathbb{C}^4$, so that the set of lines in $\mathbb{P}^3$ is $G(2,4)$. The Schubert variety $X_\square F_\bullet$ is

$$\begin{aligned} X_\square F_\bullet \ &=\ \{H \in G(2,4) \mid \dim H \cap F_{4-2+1-1} \geq 1 \ \text{ and } \ \dim H \cap F_{4-2+2-0} \geq 2\} \\ &=\ \{H \in G(2,4) \mid \dim H \cap F_2 \geq 1\}. \end{aligned}$$

If $\ell \subset \mathbb{P}^3$ is the line corresponding to the 2-plane F_2 in $F_\bullet$ and h denotes a line in $\mathbb{P}^3$, then

$$X_\square F_\bullet \ =\ \{h \mid h \cap \ell \neq \emptyset\}\,,$$

which we write in shorthand as $X_{\square}\ell$. Thus the problem of four lines is to determine the intersection $X_{\square}\ell_1 \cap X_{\square}\ell_2 \cap X_{\square}\ell_3 \cap X_{\square}\ell_4$. As there are two solutions, we have $\square^4 = 2$.

The Shapiro Conjecture makes sense for any flag manifold (compact homogeneous space). Early calculations [40] showed that it could hold for orthogonal Grassmannians, and found counterexamples for general SL_n-flag manifolds and the Lagrangian Grassmannian. Calculations suggested modifications in these last two cases [41] and limiting versions of these (modified) conjectures were proven [40]. Recently, Purbhoo used the results of [25, 27] to prove the Shapiro Conjecture for the orthogonal Grassmannians [29].

The modification for SL_n-flag manifolds, the *Monotone Conjecture*, is, in fact, an extension of the Shapiro Conjecture for Grassmannians. The Monotone Conjecture was tested in an experimental project involving two of the present authors and two others [31]. This ran on computers at the University of Massachusetts, the Mathematical Sciences Research Institute, and Texas A&M University. It used 15.76 gigahertz-years of computing to study over 520 million instances of 1126 different Schubert problems on 29 flag manifolds. Over 165 million instances of the Monotone Conjecture were verified, but the investigation did much more, discovering many interesting and unexplained phenomena discussed in [31].

2.3. **The Secant Conjecture.** Eremenko, et al. [10] proved a generalization of the Monotone Conjecture, but only for flags consisting of a codimension-two plane lying on a hyperplane, where it becomes a statement about real rational functions. In terms of the Grassmannian of codimension-two planes, the generalization asserts that a Schubert problem has only real solutions if the flags are disjoint secant flags. More specifically, a flag $F_\bullet$ of linear subspaces is *secant along an interval I* of a rational normal curve γ if every subspace in the flag is spanned by its intersections with I. Since γ is a rational normal curve, this means that there are distinct points $t_1,\ldots,t_n \in I$ such that for each $i = 1,\ldots,n$, the subspace F_i of the flag $F_\bullet$ is spanned by $\gamma(t_1),\ldots,\gamma(t_i)$.

Secant Conjecture 2.1. *For any Schubert problem $\lambda^1,\ldots,\lambda^m$ on a Grassmannian $G(k,n)$ and any flags $F^1_\bullet,\ldots,F^m_\bullet$ that are secant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^m}F^m_\bullet$$

is transverse and consists of $d(\lambda^1,\ldots,\lambda^m)$ real points.

Conjecture 1.1 from the Introduction is the case of this Secant Conjecture for the Schubert problem $\square^6 = 5$ on $G(2,5)$. The Schubert variety $X_{\square}F_\bullet$ is

$$X_{\square}F_\bullet \ = \ \{H \in G(2,5) \mid \dim H \cap F_3 \geq 1\}\,,$$

that is, the set of 2-planes meeting a fixed 3-plane non-trivially. Since F_4 and F_5 are irrelevant we drop them from the flag and refer to F_3 and $X_{\square}F_3$. For every Schubert variety there is a largest element of the flag imposing a relevant condition; call this the *relevant subspace*. For the Schubert condition $\square$ on $G(2,5)$, the relevant subspace is F_3.

For $s,t,u \in \mathbb{R}$, let $F_3(s,t,u)$ be the linear span of $\gamma(s)$, $\gamma(t)$, and $\gamma(u)$, a secant 3-plane to the rational normal curve with points $\gamma(s)$, $\gamma(t)$, and $\gamma(u)$ of secancy. The condition

$f(s,t,u;x) = 0$ of Conjecture 1.1 implies that the linear span H of the first two rows of the matrix in (1.1)—a general 2-plane in 5-space—meets the linear span of the last three rows, which is $F_3(s,t,u)$. Thus

$$f(s,t,u;x) \;=\; 0 \quad\Longleftrightarrow\quad H \in X_{\square} F_3(s,t,u)\,.$$

Lastly, the condition on the ordering of the points s_i,t_i,u_i in Conjecture 1.1 implies that the six flags $F_3(s_i,t_i,u_i)$ are secant along disjoint intervals.

By the results of Eremenko, et al. [10], the Secant Conjecture holds for the Grassmannians $G(n{-}2,n)$, as well as (trivially) for $G(1,n)$ and $G(n{-}1,n)$, which are projective spaces. Nevertheless, it is instructive to consider the Secant Conjecture in the case of the problem of four lines in $G(2,4)$. Figure 5 shows a hyperboloid containing three lines that are secant to a rational normal curve γ along disjoint intervals. Any line secant along the indicated arc

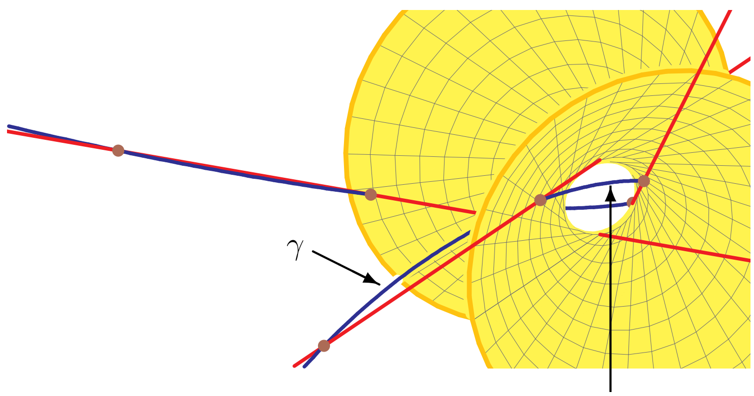


Figure 5. The problem of four secant lines.

(which is disjoint from the intervals of the other three lines) meets the hyperboloid in two points, giving two real solutions to this instance of the Secant Conjecture.

3. Evidence for the Secant Conjecture

The historical development of the Secant Conjecture presented in Section 2 provides a degree of justification for posing it, as this history shows its connection to proven results and established conjectures. We present some more concrete justifications, both proofs of the Secant Conjecture in some special cases and overwhelming computational evidence. We begin with the special cases.

3.1. **Some special cases of the Secant Conjecture.**

The work of Eremenko, et al. [10] verifies the Secant Conjecture for Schubert problems on Grassmannians $G(n{-}2,n)$ of codimension-two subspaces. But more is true: the Secant Conjecture holds in an asymptotic sense that we explain below, and work of Mukhin, Tarasov, and Varchenko establishes it for certain Schubert problems when the points of secancy form an arithmetic sequence.

3.1.1. *Arithmetic sequences of secancy.* The notion of secant points forming an arithmetic sequence is in reference to a fixed parametrization $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ of a rational normal curve. For $t \in \mathbb{R}$ and $h > 0$, let $F_\bullet^h(t)$ be the flag whose i-dimensional subspace is

$$F_i^h(t) \ := \ \mathrm{Span}\{\gamma(t), \gamma(t{+}h),\, \dots, \gamma(t{+}(i{-}1)h)\}\,,$$

the span of the image under γ of the arithmetic sequence of length i starting at t with step size h. The work of Mukhin, Tarasov, and Varchenko implies the Secant Conjecture for the Schubert problem

$$\square^{k(n-k)} \ = \ [k(n{-}k)]!\frac{1!2!\cdots(k{-}1)!}{(n{-}k)!\cdots(n{-}2)!(n{-}1)!} \tag{3.1}$$

with such secant flags. This number $d(\square^{k(n-k)})$ was determined by Schubert [32].

Let $\mathbb{C}_{n-1}[t]$ be the space of polynomials of degree at most $n{-}1$. The discrete Wronskian with step h of polynomials $f_1, \dots, f_k$ is the determinant

$$\mathrm{W}_h(f_1, \dots, f_k) \ := \ \det \begin{pmatrix} f_1(t) & f_1(t+h) & \cdots & f_1(t+(k{-}1)h) \\ f_2(t) & f_2(t+h) & \cdots & f_2(t+(k{-}1)h) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(t) & f_k(t+h) & \cdots & f_k(t+(k{-}1)h) \end{pmatrix} . \tag{3.2}$$

When $f_1, \dots, f_k$ are general, this is a polynomial of degree $k(n{-}k)$. If we consider this Wronskian up to a scalar, then it depends only on the linear span V of the polynomials $f_1, \dots, f_k$, therefore giving a map

$$\mathrm{W}_h \ : \ G(k, \mathbb{C}_{n-1}[t]) \ \longrightarrow \ \mathbb{P}^{k(n-k)}\,,$$

in which $\mathbb{P}^{k(n-k)}$ is the projective space of polynomials of degree at most $k(n-k)$. Mukhin, Tarasov, and Varchenko [26] show this is a finite map. In fact, it is a linear projection of the Grassmannian in its Plücker embedding; therefore, the fiber over a general polynomial $w(t) \in \mathbb{P}^{k(n-k)}$ consists of $d(\square^{k(n-k)})$ reduced points, each of which is a space V of polynomials with discrete Wronskian $w(t)$. As a special case of Theorem 2.1 in [26], we have the following statement.

Proposition 3.1. *Let $V \subset \mathbb{C}_{n-1}[t]$ be a k-dimensional space of polynomials whose discrete Wronskian $\mathrm{W}_h(V)$ has distinct real roots $z_1, \dots, z_N$, each of multiplicity 1. If for all $i \neq j$, we have $|z_i - z_j| \geq h$, then the space V has a basis of real polynomials.*

Corollary 3.2. *Set $N := k(n{-}k)$ and suppose that $F_\bullet^h(z_1), \dots, F_\bullet^h(z_N)$ are disjoint secant flags, so that $z_i + (n{-}1)h < z_{i+1}$ for each $i = 1, \dots, N-1$. Then the intersection*

$$\bigcap_{i=1}^{k(n-k)} X_{\square} F_\bullet^h(z_i)$$

in $G(n{-}k,n)$ is transverse with all points real.

Proof. We will identify the points in the intersection with the fibers of the discrete Wronski map W_h over the polynomial $(t-z_1)\cdots(t-z_{k(n-k)})$, which will prove the reality portion of the statement. Transversality follows by an argument of Eremenko and Gabrielov given in [43, Ch. 13]: a finite analytic map between complex manifolds that has only real points in its fibers above an open set of real points is necessarily unramified over those points.

Note that a polynomial $f(t)$ of degree $n-1$ is the composition of the parametrization $\gamma\colon \mathbb{C}\to\mathbb{C}^n$ of the rational normal curve with a linear form $\mathbb{C}^n\to\mathbb{C}$. In this way, a subspace V of polynomials of dimension k corresponds to a surjective map $V\colon\mathbb{C}^n\to\mathbb{C}^k$. We will identify such a map with its kernel H, which is a point in $G(n-k,n)$.

The column space of the matrix in (3.2) is the image under V of the linearly independent vectors $\gamma(t),\gamma(t+h),\ldots,\gamma(t+(k-1)h)$. These vectors span $F^h_k(t)$. Thus the determinant $W_h(V)$ vanishes at a point t exactly when the map

$$V\ :\ F^h_k(t)\ \longrightarrow\ \mathbb{C}^k$$

does not have full rank; that is, when

$$\dim H\cap\ F^h_k(t)\ \geq\ 1\,,$$

which is equivalent to $H\in X_{\square}F^h_{\bullet}(t)\subset G(n-k,n)$.

Thus the points in the intersection

$$\bigcap_{i=1}^{k(n-k)} X_{\square}F^h_{\bullet}(z_i)$$

correspond exactly to k-dimensional spaces of polynomials V whose discrete Wronskian $W_h(V)$ is is equal to $(t-z_1)\cdots(t-z_{k(n-k)})$, and each of these are real, by Proposition 3.1. $\square$

3.1.2. *The Shapiro Conjecture is the limit of the Secant Conjecture.* The osculating plane $F_i(s)$ is the unique i-dimensional plane having maximal order of contact with the rational normal curve γ at the point $\gamma(s)$. This and compactness of $\mathbb{RP}^1$ (or a direct calculation) implies that it is a limit of secant flags.

Lemma 3.3. *Let $\{s^{(j)}_1,\ldots,s^{(j)}_i\}$ for $j=1,2,\ldots$ be a sequence of lists of i distinct complex numbers with the property that for each $p=1,\ldots,i$, we have*

$$\lim_{j\to\infty} s^{(j)}_p\ =\ s\,,$$

for some number s. Then

$$\lim_{j\to\infty}\mathrm{Span}\{\gamma(s^{(j)}_1),\ldots,\gamma(s^{(j)}_i)\}\ =\ F_i(s)\,.$$

We can deduce that the Shapiro Conjecture is the limiting case of the Secant Conjecture by a standard limiting argument.

Theorem 3.4. *Let $\lambda^1,\ldots,\lambda^m$ be a Schubert problem and $t_1,\ldots,t_m$ be distinct points of the rational normal curve γ. Then there exists a number $\epsilon > 0$ such that if for each $i = 1,\ldots,m$, $F^i_\bullet$ is a flag secant to γ along an interval of length ϵ containing t_i, then the intersection*

$$X_{\lambda^1}F^1_\bullet \ \cap\ X_{\lambda^2}F^2_\bullet \ \cap\ \cdots\ \cap\ X_{\lambda^m}F^m_\bullet$$

is transverse with all points real.

3.1.3. *Generalized Secant Conjecture.* Section 3.1.2 and in particular Theorem 3.4 suggest a more general version of the Secant Conjecture involving flags that are intermediate between secant and osculating, and which includes the Secant Conjecture and the theorem of Mukhin, Tarasov, and Varchenko as special cases.

A linear subspace is a *generalized secant subspace* to the rational normal curve γ if it is spanned by osculating subspaces of γ. This notion includes secant subspaces, for a one-dimensional subspace that osculates γ is simply one that is spanned by a point of γ. A flag $F_\bullet$ is generalized secant to γ if each of the linear spaces in $F_\bullet$ are generalized secant subspaces. A generalized secant flag is secant along an interval I of γ if the osculating subspaces that span its linear spaces osculate γ at points of I.

Conjecture 3.5 (Generalized Secant Conjecture)**.** *For any Schubert problem $\lambda^1,\ldots,\lambda^m$ on a Grassmannian $G(k,n)$ and any generalized secant flags $F^1_\bullet,\ldots,F^m_\bullet$ that are secant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1}F^1_\bullet \cap X_{\lambda^2}F^2_\bullet \cap \cdots \cap X_{\lambda^m}F^m_\bullet$$

is transverse and consists of $d(\lambda^1,\ldots,\lambda^m)$ real points.

This conjecture includes the Secant Conjecture as the general case when all of the flags are secant flags, but it also includes the theorem of Mukhin, Tarasov, and Varchenko, which is the case when all of the flags are osculating. Besides the utility of stating the Secant Conjecture in a general form, it interests us because we actually computed many instances of the Generalized Secant Conjecture, rather than of the Secant Conjecture.

In particular, to facilitate computation, we often studied this general secant conjecture where one or two of the flags were osculating while the rest were ordinary secant flags. We explain this below.

3.2. **Experimental evidence for the Secant Conjecture.** While the historical context and precedents as presented in Section 2 and the special cases of Subsection 3.1 provide compelling reasons to pose the Secant Conjecture, we believe it is the immense weight of empirical evidence which provides the strongest evidence in its favor.

For example, 25,000 instances of the Shapiro Conjecture for the Schubert problem $\square\!\square^8 = 126$ were computed in [14], and for each instance the software alphaCertified *softly* certified that all solutions were real. Here, $\square\!\square$ is the Schubert condition that a 4-plane in 8-space meets a fixed 3-plane non-trivially. The solutions were computed numerically using the software package Bertini [3], which is based on numerical homotopy continuation [34]. Given a (square) system of n polynomial equations in n unknowns, Smale's α-theory [33] gives algorithms for certifying that Newton iterations applied to a numerically computed approximate solution will converge to a nearby solution, and also may be used to certify that the solution is real. These algorithms are implemented in the software alphaCertified [14]. Unfortunately, this Schubert problem $\square\!\square^{8} = 126$ does not have a formulation as a complete intersection (square polynomial system), so alphaCertified can only provide a soft certificate that the computed solutions are real solutions to a random square subsystem. This is described in the paper accompanying the software.

In addition, we conducted a massive experiment testing the Secant Conjecture in several hundred million instances. We describe this below and invite you to peruse the online data at [51]. These computations are possible because Schubert problems are readily modeled on a computer, and for those of moderate size (as explained below), we may determine their number of real solutions in specific instances with software tools. If the software is reliably implemented, which we believe, then this computation provides a proof that the given instance has the computed number of real solutions. This procedure may be semi-automated and run on supercomputers, which allows us to amass the considerable evidence we have collected in support of the Secant Conjecture. We will not describe how this automation is done, for that is the subject of the paper [15], but instead discuss our data and the computations that were performed.

Our experiment not only tested this Secant Conjecture but also studied the relationship between combinatorics of the points of secancy and the number of real solutions of the Schubert problem for many Schubert problems on small Grassmannians. Table 1 shows how many problems on each Grassmannian of k-planes in n-space had been computed when we halted the experiment on 26 May 2010. We computed 1,855,810,000 instances of 703 Schubert

$k\backslash n-k$	2	3	4	5	6
2	1	5	22	81	55
3	5	64	114	79	
4	22	107	67		
5	81				

Table 1. Schubert problems studied

problems. About one-fourth of these (448,381,157) were instances of the (Generalized) Secant Conjecture, and the rest involved non-disjoint secant flags. We found that the generalized Secant Conjecture held true for every instance involving disjoint secant flags. The remaining 1,407,428,843 instances involved flags that were not secant along disjoint intervals, but had some overlap in their intervals of secancy. We measured this overlap with a statistic that we call the *overlap number* (described in § 3.3). It is zero if and only if the flags are secant along disjoint intervals.

Table 2 shows part of the data we obtained for a Schubert problem (written in shorthand as $X_{\square}^6 X_{\boxplus} = 16$) with 16 solutions on the Grassmannian of 3-planes in 6-space. We computed

Overlap Number

Real Solutions \	0	1	2	3	4	5	6	7	$\cdots$	Total
0								3	$\cdots$	566
2							10	32	$\cdots$	7452
4				406	1699	176	191	411	$\cdots$	51416
6				1926	5233	958	662	1184	$\cdots$	160629
8				2821	7382	1691	1130	1975	$\cdots$	321827
10				2484	6500	2591	1116	2026	$\cdots$	430179
12				3288	6185	3296	1320	2250	$\cdots$	417358
14				1610	2832	2346	767	1376	$\cdots$	244259
16	560827		19741	61429	50832	17096	8527	9674	$\cdots$	866314
Total	560827		19741	73964	80663	28154	13723	18931	$\cdots$	2500000

TABLE 2. Real solutions vs. overlap number for $X_{\square}^6 X_{\boxplus} = 16$.

2,500,000 instances of the Schubert problem, all involving flags that were secant to the rational normal curve. This took 16.327 gigahertz-years. The rows are labeled with the even integers from 0 to 16, as the number of real solutions has the same parity as the number of complex solutions. The first column with overlap number 0 represents tests of the Secant Conjecture. Since the only entries are in the row for 16 real solutions, the Secant Conjecture was verified in 560,827 instances. The column labeled overlap number 1 is empty because flags for this problem cannot have overlap number 1. Perhaps the most interesting feature is that for overlap number 2, all computed solutions were real, while for overlap numbers 3, 4, and 5, at least four of the 16 solutions were always real. This inner border, which indicates that some solutions are forced to be real when there is low overlap, is found on many of the other problems that we investigated and is a new phenomenon that we do not understand.

We also computed 20,000,000 instances of this same Schubert problem with six secant flags and one osculating flag (for the condition $X_{\boxplus}$). These data are compiled in Table 3. This computation took 4.473 gigahertz-years—eight times as many instances in just over one-fourth of the time. This speed-up is possible because the one osculating flag allows us to formulate the problem with only six variables instead of nine. As one flag was osculating, this computation tested the Generalized Secant Conjecture; its computed instances form the first column. As the only entries of that column are in the row for 16 real solutions, the Generalized Secant Conjecture was verified in 4,568,553 instances. As with Table 2 there is visibly an inner border to these data.

3.3. **Overlap number.** The number of possible different configurations of points of secancy can be astronomical. For the problem of Table 2, the condition $X_{\square}F_{\bullet}$ has relevant subspace F_3 and the condition $X_{\boxplus}F_{\bullet}$ has relevant subspace F_4. The resulting 22 points of secancy

Overlap Number

Real Solutions \	0	1	2	3	4	5	6	$\cdots$	Total
0							20	$\cdots$	7977
2							116	$\cdots$	88578
4				6154	23561	526	3011	$\cdots$	542521
6				25526	63265	2040	9460	$\cdots$	1571582
8				33736	78559	2995	13650	$\cdots$	2834459
10				25953	39252	2540	11179	$\cdots$	3351159
12				36657	44840	3271	14160	$\cdots$	2944091
14				17367	17180	1705	7821	$\cdots$	1602251
16	4568553		182668	583007	468506	36983	83169	$\cdots$	7057382
Total	4568553		182668	727321	735163	50060	142586	$\cdots$	20000000

Table 3. Real solutions vs. overlap number for $X_{\square}^{6} X_{\boxplus} = 16$.

have at least

$$\binom{22}{3,3,3,3,3,3,4} \cdot \frac{1}{6!} \cdot \frac{1}{22} \cdot \frac{1}{2} \;=\; 31{,}685{,}654{,}000$$

combinatorially different configurations. It is impossible to meaningfully investigate each different configuration. Instead, we introduce a statistic on these configurations—the *overlap number*—which is zero if and only if the flags are disjoint, and we tabulate the results of our experiment using this overlap number, which we now describe.

Given a Schubert problem $\lambda^1,\ldots,\lambda^m$, suppose that the relevant subspaces have dimensions $i_1,\ldots,i_m$, respectively. In an instance of this Schubert problems given by flags that are secant to the rational normal curve γ, the relevant subspaces of the jth flag,

$$F_1^j \;\subsetneq\; F_2^j \;\subsetneq\; \cdots \;\subsetneq\; F_{i_j}^j\,,$$

require a choice of an ordered set T_j of i_j points of γ. The overlap number measures how much these sets of points $T_1,\ldots,T_m \subset \gamma$ overlap. Let T be their union.

Since γ is isomorphic to $\mathbb{RP}^1$, which is a topological circle, removing a point p of γ not in T, we may assume that $T_1,\ldots,T_m \subset \mathbb{R}$. Each set T_j defines an interval I_j of $\mathbb{R}$ and we let o_j be the number of points of $T \setminus T_j$ lying in I_j. This sum $o_1 + o_2 + \cdots + o_m$ depends upon our choice of point $p \in \gamma \setminus T$, and the overlap number is the minimum of these sums as the point p varies.

For example, consider a Schubert problem with relevant subspaces of dimensions 3, 2, and 2. Suppose that we have chosen seven points on γ in groups of 3, 2, and 2. This is represented schematically on the left in Figure 6, in which γ is a circle, and the points in the sets T_1, T_2, and T_3 are represented by the circles (●), squares (■), and triangles (▲), respectively. For each of three points p_1, p_2, and p_3 of γ, we compute the number o_i and their sum Σ, displaying the results in the table on the right in Figure 6. The minimum of the sum Σ for all choices of points is achieved by p_3.

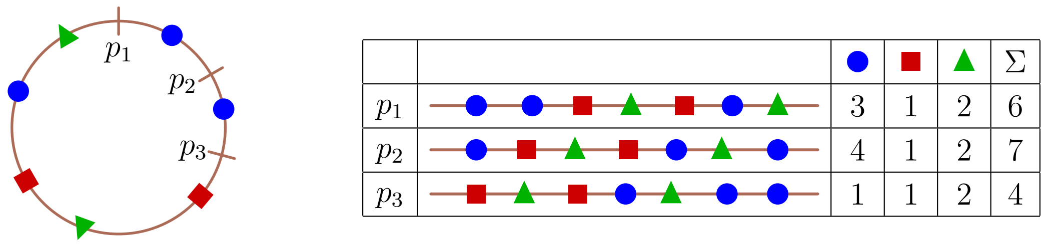


FIGURE 6. Computation of overlap number.

In Section 4 we study the problem of four lines, not from the perspective of overlap number, but by considering all 17 configurations of four secant lines.

3.4. **Computing Schubert problems.** We model Schubert problems as solutions to systems of polynomial equations, and we determine their number of real solutions by symbolically computing an eliminant whose number of real roots is also computed symbolically. Computation of eliminants is sensitive to the number of variables, so we work locally in a coordinate patch on the Grassmannian, rather than in the global Plücker coordinates.

A $k \times (n-k)$ matrix $X \in \mathbb{C}^{k\times(n-k)}$ determines a point on the Grassmannian, namely the row space H of the $k \times n$ matrix (also written H),

$$H \;:=\; \bigl(I_k \;:\; X\bigr)\,. \tag{3.3}$$

If we represent an i-plane F_i as the row space of a $i \times n$ matrix F_i of full rank, then

$$\dim H \cap F_i \;\geq\; j \iff \operatorname{rank}\begin{pmatrix} H \\ F_i \end{pmatrix} \;\leq\; k+i-j\,, \tag{3.4}$$

which is given by the vanishing of all $(k{+}i{-}j{+}1) \times (k{+}i{-}j{+}1)$ subdeterminants. We represent a flag $F_\bullet$ by a full rank $n{\times}n$ matrix where F_i is the row span of the first i rows. Then (3.4) leads to a set of equations for the Schubert variety $X_\lambda F_\bullet$ in the coordinate patch (3.3). In practice, we only need a $(n{-}k{+}i{-}\lambda_i) \times n$ matrix, where λ_i is the relevant condition.

Given numbers $t_1, \ldots, t_i$, construct a $i \times n$ matrix $F_i(t_1, \ldots, t_i)$ whose jth row is the vector $\gamma(t_j)$, where $\gamma(t) = (1, t, \ldots, t^{n-1})$ is the rational normal curve. This allows us to model Schubert problems given by secant flags. Similarly, the osculating i-plane $F_i(t_0)$ to γ at the point $\gamma(t)$ is represented by the $i \times n$ matrix whose jth row is $\gamma^{(j-1)}(t)$. This allows us to model Schubert problems given by osculating flags, or a mixture of osculating and secant flags.

For example, Conjecture 1.1 involves the Schubert problem $\square^6 = 5$ on $G(2{,}5)$ where $\square$ is the Schubert condition of a 2-plane meeting a 3-plane. The solutions are 2-planes spanned by the first two rows of the matrix in (1.1). The last three rows in the matrix are the points $\gamma(s_i)$, $\gamma(t_i)$, $\gamma(u_i)$, spanning the 3-plane of a secant flag.

We use the computer algebra system Singular [5] to compute an *eliminant* of the polynomial system modeling a given instance of the Schubert problem $\lambda^1, \ldots, \lambda^m$. This is a univariate polynomial $f(x)$ whose roots are all the x-coordinates of solutions to the Schubert problem in the patch (3.3). When the eliminant $f(x)$ has degree equal to $d(\lambda^1,\ldots,\lambda^m)$ and is square-free, then the Shape Lemma [4] guarantees that its number of real roots is equal to the number of real solutions to the Schubert problem. We use Maple's `realroot` command to compute the number of real roots of the eliminant $f(x)$.

If the eliminant does not satisfy the hypotheses of the Shape Lemma, we compute an eliminant with respect to a different coordinate. When no eliminant works, we repeat this elimination procedure after deterministically perturbing the points of secancy, which has always worked.

Working in a different set of local coordinates enables us to efficiently compute instances of the Generalized Secant Conjecture for one (and sometimes two) osculating flags. If $e_1,e_2,\ldots,e_n$ are the standard basis vectors corresponding to columns of our matrices, then the flag $F_\bullet(\infty)$ osculating the rational normal curve γ at $\gamma(\infty)=e_n$ has

$$F_i(\infty) \;=\; \langle e_{n+1-i},\ldots,e_{n-1},e_n\rangle\,.$$

The Schubert variety $X_\lambda F_\bullet(\infty)$ has an open subset consisting of those k-planes H which are the row space of an echelon matrix of the form

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \ddots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & * & \cdots & * \end{pmatrix}\ ,$$

where each entry $*$ represents an arbitrary complex number (a variable) and the columns of the pivots (1s) are

$$\lambda_k+1\,,\ \lambda_{k-1}{+}2\,,\ \ldots\,,\ \lambda_2{+}k{-}1\,,\ \lambda_1{+}k\,,$$

which form a $k\times k$ identity matrix I_k.

This reduction, computing instances of the Generalized Secant Conjecture 3.5 on $X_{\lambda^1}F_\bullet(\infty)$, rather than the Secant Conjecture enables us to study many more instances of these conjectures. We already saw this for the Schubert problem $\square^6\cdot\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}=16$ on $G(3,6)$.

3.5. **Grassmann Duality and the Cosecant Conjecture.** The notion of a flag osculating a rational normal curve is preserved under linear algebra duality, but the notion of a secant flag is not. Thus, under the Grassmann duality isomorphism between $G(k,n)$ and $G(n{-}k,n)$, the Secant Conjecture becomes a new conjecture which we call the Cosecant Conjecture.

If we identify $\mathbb{C}^n$ with its linear dual, then the association

$$\delta\ :\ H\ \longmapsto\ H^\perp\ :=\ \text{annihilator of } H\,,$$

where H is a linear space, gives an isomorphism $\delta\colon G(k,n)\xrightarrow{\sim}G(n{-}k,n)$. This *Grassmann duality* respects Schubert varieties. Given a flag $F_\bullet\subset\mathbb{C}^n$, let $F_\bullet^\perp$ be the flag whose i-dimensional subspace is defined by $F_i^\perp:=(F_{n-i})^\perp$. Then,

$$\delta(X_\lambda F_\bullet)\ =\ X_{\lambda^T}F_\bullet^\perp\,,$$

where λ^T is what is called the the conjugate partition to λ. For example,

$$\ydiagram{2}^T \;=\; \ydiagram{1,1}\,, \qquad \ydiagram{2,1}^T \;=\; \ydiagram{2,1}\,, \qquad \text{and} \qquad \ydiagram{3,1}^T \;=\; \ydiagram{2,1,1}\,.$$

That is, if we represent λ by its Young diagram—a left-justified array of boxes with λ_i boxes in row i—then the diagram of λ^T is the matrix-transpose of the diagram of λ.

If $\gamma(t) = (1,t,t^2,\ldots,t^{n-1})$ is the rational normal curve, then, under duality, the family $F_{n-1}(t)$ of its osculating $(n-1)$-planes becomes a curve $(F_{n-1}(t))^\perp$, which we invite the reader to check is

$$\gamma^\perp(t) \;=\; \left((-t)^{n-1}\tbinom{n-1}{n-1}\,,\;\ldots\;,\;-\,t^3\tbinom{n-1}{3}\,,t^2\tbinom{n-1}{2}\,,\;-\,t(n-1)\,,1\,\right),$$

in the dual basis to the standard basis. Moreover, $(F_{n-k}(t))^\perp$ is the osculating k-plane to this dual rational normal curve $\gamma^\perp$ at the point $\gamma^\perp(t)$. Thus Grassmann duality preserves Schubert varieties given by flags osculating the rational normal curve. We deduce that the dual statement to the Mukhin-Tarasov-Varchenko Theorem (Shapiro Conjecture for Grassmannians) is simply itself.

This is however not the case for secant flags. Consider the general secant $(n-1)$-plane

$$F_{n-1}(s_1,\ldots,s_{n-1}) = \;\operatorname{span}\{\gamma(s_1)\,,\gamma(s_2)\,,\;\ldots\;,\gamma(s_{n-1})\}\,,$$

which is secant to γ at the points $\gamma(s_1),\ldots,\gamma(s_{n-1})$. Its dual space is spanned by the vector

$$\left((-1)^{n-1}e_{n-1}\,,e_{n-2}\,,\;\ldots\;,\;-\,e_3\,,e_2\,,\;-\,e_1\,,1\right),$$

where e_i is the ith elementary symmetric function in the parameters $s_1,\ldots,s_{n-1}$. This will be secant to the dual rational normal curve $\gamma^\perp$ only in the limit as the points $s_1,\ldots,s_{n-1}$ converge to a point t.

In general, a *cosecant subspace* is a subspace that is dual to a secant subspace. If

$$F_k(s_1,\ldots,s_k) \;=\; \operatorname{span}\{\gamma(s_1)\,,\gamma(s_2)\,,\;\ldots\;,\gamma(s_k)\}\,,$$

then the corresponding cosecant subspace is

$$F^\perp_{n-1}(s_1)\,\cap\,F^\perp_{n-1}(s_2)\,\cap\,\cdots\,\cap\,F^\perp_{n-1}(s_k)\,,$$

the intersection of k hyperplanes osculating the rational normal curve $\gamma^\perp$. A *cosecant flag* is a flag whose subspaces are cut out by hyperplanes osculating the rational normal curve. It is cosecant along an interval of γ if these hyperplanes osculate γ at points of the interval.

Thus, under Grassmann duality the Secant Conjecture for $G(n-k,n)$ becomes the following *Cosecant Conjecture* for $G(k,n)$.

Conjecture 3.6 (Cosecant Conjecture)**.** *For any Schubert problem $\lambda^1,\ldots,\lambda^m$ on a Grassmannian $G(k,n)$ and any flags $F^1_\bullet,\ldots,F^m_\bullet$ that are cosecant to a rational normal curve γ along disjoint intervals, the intersection*

$$X_{\lambda^1}F^1_\bullet\,\cap\,X_{\lambda^1}F^1_\bullet\,\cap\,\cdots\,\cap\,X_{\lambda^1}F^1_\bullet$$

is transverse and consists of $d(\lambda^1,\ldots,\lambda^m)$ real points.

There is also a Generalized Cosecant Conjecture, which we do not formulate.

4. Problem of four secant lines

We take a closer look at the Schubert problem $\square^4 = 2$ on $G(2{,}4)$ where $\square$ denotes the Schubert condition that a two-plane in $\mathbb{C}^4$ meets a fixed two-plane nontrivially. Equivalently, $\square$ is the condition that a line in $\mathbb{P}^3$ meets a fixed line. Then $\square^4 = 2$ denotes the Schubert problem of lines in $\mathbb{P}^3$ that meet four fixed lines. Let $\gamma : \mathbb{R} \to \mathbb{P}^3$ be a rational normal curve and $\ell_1, \ldots, \ell_4$ be four secant lines. We consider the lines in $\mathbb{P}^3$ that meet these four secant lines.

We computed 7,500,000 instances of this Schubert problem with four secant lines. This included 1,942,362 instances with overlap number zero, all of which had both solutions real in accordance with the Secant Conjecture. (In this case of $\square^4 = 2$ on $G(2{,}4)$, the Secant Conjecture is a theorem of Eremenko, Gabrielov, Shapiro, and Vainshtein [10].) Our results for the 7,500,000 instances are displayed in Table 4. Observe that every instance tested

	0	2	4	6	8	10	12	Total
0		319239	136713	156702		41529		654183
2	1942362	1576591	1576452	1345659	237699	129668	37386	6845817
Total	1942362	1895830	1713165	1502361	237699	171197	37386	7500000

Table 4. Real solutions vs. overlap number for $\square^4 = 2$.

with overlap number 8 or 12 had only real solutions, which is not covered by the Secant Conjecture or by [10]. We will give a simple explanation of this phenomenon.

In Section 2.1 we saw that there is a unique doubly-ruled quadric surface Q that contains the lines ℓ_1, ℓ_2, and ℓ_3 in one ruling. The two lines of the second ruling of Q through the two points of intersection of ℓ_4 with Q are the solutions of the Schubert problem $\square^4 = 2$ for these lines $\ell_1, \ldots, \ell_4$.

For two numbers s_1, s_2 we let $\ell(s_1{,}s_2)$ denote the secant line to γ through $\gamma(s_1)$, $\gamma(s_2)$. Given eight numbers $s_1 < \cdots < s_8$, the Secant Conjecture asserts that both lines meeting the four fixed lines

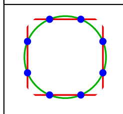
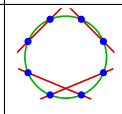
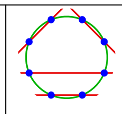
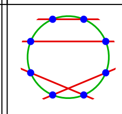
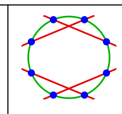
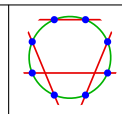
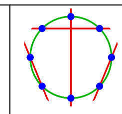
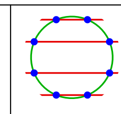
$$\ell(s_1{,}s_2)\,,\ \ell(s_3{,}s_4)\,,\ \ell(s_5{,}s_6)\,,\ \ell(s_7{,}s_8)$$

are real. We investigate phenomena beyond the border of the Secant Conjecture by choosing a permutation ρ of $\{1, \ldots, 8\}$ and taking the four lines $\ell(s_{\rho(1)}, s_{\rho(2)}), \ldots, \ell(s_{\rho(7)}, s_{\rho(8)})$.

In general, a given value of the overlap number may be realized by many combinatorial arrangements of the points of secancy. Table 5 illustrates every possible combinatorial configuration of the four secant lines along with their overlap numbers. We computed 100,000 instances of each configuration, with results as shown in Table 5. Most of these combinatorial configurations lead to Schubert problems with only real solutions; only four configurations (with overlap numbers 2, 4, 6, and 10) can lead to Schubert problems with solutions not necessarily real. Proving this will explain the observation in Table 4 that instances with overlap number 8 or 12 have only real solutions.

The quadric Q divides its complement in $\mathbb{RP}^3$ into two connected components (the domains where the quadratic form Q is positive or negative), which we will call the *sides* of Q. Three

Overlap number

		0	2		4				
									
Real roots	0			23723					29398
	2	100000	100000	76277	100000	100000	100000	100000	70602

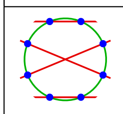
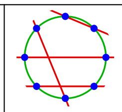
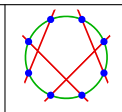
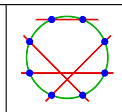
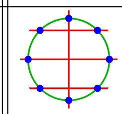
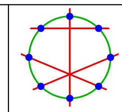
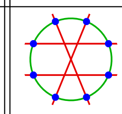
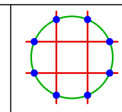
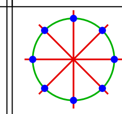
	6				8		10		12
									
0				52395				65783	
2	100000	100000	100000	47605	100000	100000	100000	34217	100000

TABLE 5. Configurations of four secant lines with overlap numbers and results of an experiment.

lines ℓ_1, ℓ_2, ℓ_3 give six points of secancy which are the intersections of γ with Q and which divide γ into six segments. Those segments alternate between the two sides of Q. If the fourth secant line ℓ_4 has its two points of secancy lying on opposite sides of Q, then by the Intermediate Value Theorem ℓ_4 has a real intersection with Q, meaning the Schubert problem has one (and hence two) real solutions. The points of secancy of ℓ_4 lie on opposite sides of Q if in the interval between the two points of secancy, the curve γ crosses Q an odd number of times. That is, the interval contains an odd number of points of secancy of the lines ℓ_1, ℓ_2, and ℓ_3.

This simple topological argument shows that if at least one of the four secant lines has an interval of secancy that contains an odd number of points of secancy of the other three lines, then the Schubert problem will have only real solutions, independent of the actual positions of the secant lines. This condition is met by at least one of the four lines in each of the combinatorial configurations shown in Table 5, with exactly five exceptions: The four configurations which have non-real solutions in Table 4 fail this condition (each interval of secancy of each of the four lines contains an even number of points of secancy of other lines), as does the configuration with overlap number zero.

Configurations satisfying this condition for at least one line will have real solutions. Out of the five configurations that fail the condition, four of them can have zero or two real roots, depending on the positions of the secant lines. Only the configuration with overlap number 0 fails the condition and yet has only real solutions. This is a deeper fact proven in [10].

5. Lower bounds and inner borders

Perhaps the most ubiquitous and enigmatic phenomenon that we have observed in our data is the apparent "inner border" in many of the tables. Typically we do not observe instances with no or few real solutions when the overlap number is small. This manifests itself by a prominent staircase separating observed pairs of (real solutions, overlap number) from unobserved pairs. This feature is clearly visible in Tables 2 and 3 for the problem $X_{\square}^{6} X_{(2,1)} = 16$, and in Table 6 for the problem $W_{\square}^{8} = 14$ in $G(2,6)$. There, it is only with

\	0	1	2	3	4	5	6	···	Total
0								···	4272
2								···	127217
4					693	1481	6660	···	879658
6					224	510	2541	···	2304233
8					526	939	3561	···	2914837
10					1052	2074	6985	···	2205198
12					1556	2595	7300	···	1224667
14	3328772		60860	120625	310819	246910	237704	···	5339918
Total	3328772		60860	120625	305870	254509	264751	···	15000000

Table 6. Real solutions vs. overlap number for $W_{\square}^{8} = 14$.

overlap number 8 or larger that we observe instances with two real solutions, and with overlap number 16 or larger instances with no real solutions. (These columns are not displayed for reasons of space.)

This problem involves eight secant four-planes having 32 points of secancy in all. The number of possible configurations is at least

$$\binom{32}{4,4,4,4,4,4,4,4} \cdot \frac{1}{8!} \cdot \frac{1}{32} \cdot \frac{1}{2} \;=\; \frac{32!}{(4!)^8 \cdot 8! \cdot 32 \cdot 2} \;\sim\; 10^{18}\,,$$

and hence it is impossible to systematically study all configurations as in Section 4. This is the case for most of the problems we studied. Because of the coarseness of our measure of overlap, we doubt it is possible to formulate a meaningful conjecture about this inner border based on our data. Nevertheless, we believe that this problem, like the problem of four lines, contains rich geometry, with certain configurations having a lower bound on the number of real solutions.

Such a phenomenon of a polynomial system having a non-trivial lower bound on its number of real solutions is well-established. Such systems include rational curves interpolating points on toric del Pezzo surfaces [16, 17, 18, 24, 46], sparse polynomial systems from posets [19, 35], as well as some lower bounds in the Schubert calculus [1, 8] which we describe.

Lower bounds and inner borders were observed in the computations studying the Monotone Conjecture [31, § 3.2.2]. The original example of lower bounds was due to Eremenko and Gabrielov [8] concerning the Wronski map. The Wronskian of linearly independent polynomials $f_1(t), f_2(t), \ldots, f_k(t)$ of degree $n-1$,

$$\mathrm{W}(f_1,\ldots,f_k) \ := \ \det \begin{pmatrix} f_1(t) & f_1'(t) & f_1''(t) & \cdots & f_1^{(k-1)}(t) \\ f_2(t) & f_2'(t) & f_2''(t) & \cdots & f_2^{(k-1)}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_k(t) & f_k'(t) & f_k''(t) & \cdots & f_k^{(k-1)}(t) \end{pmatrix} ,$$

is a polynomial of degree $k(n-k)$, and this gives a finite map $\mathrm{W}\colon G(k,\mathbb{C}_{n-1}[t]) \longrightarrow \mathbb{P}^{k(n-k)}$ with the general fiber consisting of $d(\square^{k(n-k)})$ (see (3.1)) linear spaces of polynomials. The theorem of Mukhin, Tarasov, and Varchenko [25, 27] implies that, if $w(t)$ is a polynomial with $k(n-k)$ distinct real roots, then each of the $d(\square^{k(n-k)})$ points in the fiber of W over $w(t)$ is real. Eremenko and Gabrielov showed that if n is odd, then there is a non-trivial lower bound on the number of real spaces of polynomials in the fiber of W over *any* polynomial $w(t)$ with real coefficients.

More recently, Azar and Gabrielov [1] studied the problem $\square^{2n-4}$ in $G(n-2,n)$ of $(n-2)$-planes in $\mathbb{C}^n$ which meet one secant line and $2n-5$ tangent lines. When the interval of secancy contains no tangent points, this is an instance of the Generalized Secant Conjecture 3.5. They establish lower bounds on the number of real solutions which depend upon the configuration of the points of secancy and tangency. This gives proofs for some of the observations in [31] concerning partial flags consisting of a codimension-two plane lying on a hyperplane.

6. Gaps

Consider the Schubert problem $W_{\boxplus}^4 = 6$ on $G(4,8)$, counting four-planes which have at least a two-dimensional intersection with four general four-planes. We computed 1,000,000 instances of this problem, obtaining the results in Table 7. A system of real polynomial

	0	1	2	3	4	5	6	$\cdots$	Total
0									0
2				1441	7730	14277	16636	$\cdots$	147326
4									0
6	280304		13131	25708	62833	55919	57719	$\cdots$	852674
Total	280304	0	13131	27149	70563	70196	74355	$\cdots$	1000000

Table 7. Real solutions vs. overlap number for $W_{\boxplus}^4 = 6$.

equations with 6 solutions can, *a priori*, have 0, 2, 4, or 6 real solutions, yet, strikingly, this Schubert problem only has 2 or 6 real solutions, never 0 or 4.

In fact, although our observations involved only secant flags, this phenomenon holds for any real flags. This follows from the ideas in [44, §5.13] where Vakil discusses this Schubert problem, focusing on Derksen's observation that it has deficient Galois group.

Following Vakil, we consider the *auxiliary Schubert problem* $U^4_{\square\square\square} = 4$ on $G(2{,}8)$, counting 2-planes which meet four general 4-planes. Given 4-planes $W_1, \ldots, W_4$, let $P_1, \ldots, P_4$ be the 2-planes which meet them. Then the solutions to the original Schubert problem $W^4_{\boxplus} = 6$ are precisely the 6 sums of the form $P_i + P_j$. Such sum is real if and only if P_i and P_j are each real or if P_i and P_j are a pair of complex conjugate subspaces.

If the W_i are real, then some of the the P_i are real and the rest are in complex conjugate pairs; there can be 0, 1, or 2 conjugate pairs among the P_i. Then the number of solutions $P_i + P_j$ which are real is, respectively, 6, 2, and 2. This explains the observations in Table 7.

This is the first in a family of Schubert problems in $G(4{,}2n)$ for $n \geq 4$ with such gaps in their numbers of real solutions. These problems involve enumerating the 4-planes which meet four general n-planes in $\mathbb{C}^{2n}$. For each, there is an auxiliary Schubert problem on $G(2{,}2n)$ of two-planes meeting four general n-planes. This will have n solutions, and the solutions to the original problem are four planes spanned by pairs of solutions to the original problem. Thus the original problem will have $\binom{n}{2}$ solutions, and as before, there are restrictions on the number of these which may be real. These restrictions are identical to restrictions on the number of real factorizations of a real polynomial as in [35, Theorem 7.8].

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