ON OPTIMAL ALGEBRAIC MULTIGRID METHODS

LUIS GARCÍA RAMOS* AND REINHARD NABBEN*

Abstract. In this note we present an alternative way to obtain optimal interpolation operators for two-grid methods applied to Hermitian positive definite linear systems. In [5, 10] the A-norm of the error propagation operator of algebraic multigrid methods is characterized. These results are just recently used in [3, 9] to determine optimal interpolation operators. Here we use a characterization not of the A-norm but of the spectrum of the error propagation operator of two-grid methods, which was proved in [6]. This characterization holds for arbitrary matrices. For Hermitian positive definite systems this result leads to optimal interpolation operators with respect to the A-norm in a short way, moreover, it also leads to optimal interpolation operators with respect to the spectral radius. For the symmetric two-grid method (with pre- and post-smoothing) the optimal interpolation operators are the same. But for a two-g rid method with only post-smoothing the optimal interpolations (and hence the optimal algebraic multigrid methods) are different. Moreover, using the characterization of the spectrum, we can show that the found optimal interpolation operators are also optimal with respect to the condition number of the multigrid preconditioned system.

Key words. multigrid, optimal interpolation operator, two-grid methods

AMS subject classifications. 65F10, 65F50, 65N22, 65N55.

1. Introduction.

Typical multigrid methods to solve the linear system

$$Ax = b,$$

where A is an $n \times n$ matrix, consist of two ingredients, the smoothing and the coarse grid correction. The smoothing is typically done by a few steps of a basic stationary iterative method, like the Jacobi or Gauss-Seidel method. For the coarse grid correction, a *prolongation* or *interpolation* operator $P \in \mathbb{C}^{n\times r}$ and a *restriction* operator $R \in \mathbb{C}^{r\times n}$ are needed. The coarse grid matrix is then defined as

$$A_C := RAP \in \mathbb{C}^{r\times r}. \tag{1.1}$$

Here we always assume that A and A_C are non-singular. The multigrid or algebraic multigrid (AMG) error propagation matrix is then given by

$$E_M = (I - M_2^{-1}A)^{\nu_2}(I - PA_C^{-1}RA)(I - M_1^{-1}A)^{\nu_1}, \tag{1.2}$$

where $M_1^{-1} \in \mathbb{C}^{n\times n}$ and $M_2^{-1} \in \mathbb{C}^{n\times n}$ are *smoothers*, ν_1 and ν_2 are the number of pre- and post-smoothing steps respectively, and $PA_C^{-1}R$ is the *coarse grid correction* matrix. The multigrid method is convergent if and only if the spectral radius of the error propagation matrix $\rho(E_m)$ is less than one. Alternatively, the norm of the error propagation matrix $\|E_M\|$ can be considered, where $\|\cdot\|$ is a consistent matrix norm, and in this case one has

$$\rho(E_M) \le \|E_M\|.$$

The aim of algebraic multigrid methods is to balance the interplay between smoothing and coarse grid correction steps. However, most of the existing AMG methods first fix a smoother and then optimize a certain quantity to choose the interpolation P and restriction R.

To simplify the analysis, we assume that there exists a non-singular matrix X such that

$$(I - X^{-1}A) = (I - M_1^{-1}A)^{\nu_1}(I - M_2^{-1}A)^{\nu_2}, \tag{1.3}$$

*Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, D-10623 Berlin, Germany ({garcia, nabben}@math.tu-berlin.de).

it can be shown that such a matrix X exists if the spectral radius of $(I-M_1^{-1}A)^{\nu_1}(I-M_2^{-1}A)^{\nu_2}$ is less than one, see e.g. [2]. Moreover, note that the matrix E_M can be written as

$$E_M = I - BA, \tag{1.4}$$

where the matrix B is known as the multigrid preconditioner, i.e., B is an approximation of A^{-1}. Therefore, eigenvalue estimates of BA are of interest, and they also lead to estimates for the eigenvalues of E_M.

The following theorem, proved by García Ramos, Kehl and Nabben in [6], gives a characterization of the spectrum of BA, and hence a characterization of the spectrum of the general error propagation matrix E_M.

THEOREM 1.1. *Let $A \in \mathbb{C}^{n\times n}$ be non-singular, and let $P \in \mathbb{C}^{n\times r}$ and $R \in \mathbb{C}^{r\times n}$ such that RAP is non-singular. Moreover, let $M_1 \in \mathbb{C}^{n\times n}$ and $M_2 \in \mathbb{C}^{n\times n}$ be such that that the matrices X in* (1.3) *and RXP are non-singular. Then the following statements hold:*

(a) The multigrid preconditioner B in (1.4) *is non-singular.*

(b) If $\tilde{P}, \tilde{R} \in \mathbb{C}^{n\times n-r}$ are matrices such that the columns of $\tilde{P}$ and $\tilde{R}$ form orthonormal bases of $(\mathcal{R}(P))^\perp$ and $(\mathcal{R}(R^H))^\perp$ respectively, then the matrices $\tilde{P}^H A^{-1}\tilde{R}$ and $P^H X^{-1}\tilde{R}$ are non-singular and the spectrum of BA is given by

$$\sigma(BA) = \{1\} \cup \sigma(\tilde{P}^H X^{-1}\tilde{R}(\tilde{P}^H A^{-1}\tilde{R})^{-1}).$$

We will apply this theorem to Hermitian positive definite (HPD) matrices to determine the optimal interpolation operators of AMG methods with respect to the spectral radius of the error propagation matrix. For HPD matrices, optimal interpolation operators with respect to the A-norm have been obtained recently in [3,9]. We will show that the optimal interpolation operators with respect to the spectral radius for the symmetric/symmetrized multigrid method (with pre- and post-smoothing) and the optimal interpolation operator with respect to the A-norm are the same. But for multigrid with only a post-smoothing step the optimal interpolation operators with respect to the spectral radius and A-norm (and hence the optimal algebraic multigrid methods) are different. Using Theorem 1.1 we can also show that the interpolation operators with respect to the spectral radius are also optimal with respect to the condition number of the multigrid preconditioned system.

2. Optimal interpolation for Hermitian positive definite matrices.

In this section we consider a HPD matrix A. Recall that he norm induced by A (or A-norm) is defined for $v \in \mathbb{C}^n$ and $S \in \mathbb{C}^{n\times n}$ by

$$\|v\|_A^2 = (v, v)_A = \|A^{\frac{1}{2}}v\|_2^2,$$

and

$$\|S\|_A = \|A^{\frac{1}{2}}SA^{-\frac{1}{2}}\|_2.$$

We will study the following two-grid methods given by the error propagation operators

$$E_{TG} = (I - M^{-H}A)(I - PA_C^{-1}P^H A) \tag{2.1}$$

and the symmetrized version

$$E_{STG} = (I - M^{-H}A)(I - PA_C^{-1}P^H A)(I - M^{-1}A). \tag{2.2}$$

Thus we are using $R = P^H$. The range of P, i.e. $\mathcal{R}(P)$, is called the coarse space V_c. Here we fix the smoother M^{-1} and consider E_{TG} and E_{STG} with respect to the choice of the interpolation operator P. So, in this note, E_{TG} and E_{STG} depend on P. In addition, we assume that the smoother M^{-1} satisfies

$$\|(I - M^{-1}A\|_A < 1,$$

which is equivalent to the condition

$$M + M^H - A \quad \text{is positive definite,} \tag{2.3}$$

see, e.g., [8].

It is proved by Falgout and Vassilevski [4] that

$$\|E_{STG}\|_A = \|E_{TG}\|_A^2. \tag{2.4}$$

Given a fixed smoother M^{-1} such that $\|I - M^{-1}A\|_A < 1$, many AMG methods are designed to minimizes $\|E_{TG}\|_A$ or a related quantity. If an operator P minimizes $\|E_{TG}\|_A$ directly, P is called optimal.

Zikatanov proved in [10] that

$$\|E_{TG}\|_A^2 = 1 - \frac{1}{K(V_c)},$$

where $K(V_c)$ is a value depending on the coarse space.

Although this equality is known for a long time, just recently it is used to determine optimal prolongation operators P which lead to a minimal value of $\|E_{TG}\|_A$ for a given smoother (see [3, 9]). Here we give an alternative proof of this result using the characterization of the eigenvalues of the multigrid iteration operator given in Theorem 1.1.

But before we consider the more general error propagation matrix E_M in (1.2) with $R = P^H$ and $E_M = I - BA$.

Let $\mathcal{U}$ be the subspace spanned by the columns of the interpolation operator P and let $\tilde{U}$ be a matrix whose columns span $\mathcal{U}^\perp$. Then Theorem 1.1 leads to

$$\sigma(BA) = \{1\} \cup \sigma(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}).$$

Next assume that X is Hermitian positive definite and that the largest eigenvalue of BA, i.e. $\lambda_{max}(BA)$, is at most one. Then we have $\rho(E_M) = 1 - \lambda_{min}(BA)$. In order to find an optimal interpolation operator for the error propagation matrix we need to first find

$$arg \max_{\tilde{U} \in \mathbb{C}^{n \times n-r}} \min \sigma(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}),$$

and then find vectors which are orthogonal to the found optimal subspace $\tilde{\mathcal{U}}$. The following Theorem solves the first problem.

Theorem 2.1. *Let $A, X \in \mathbb{C}^{n \times n}$ be Hermitian positive definite. Let*

$$\mu_1 \le \mu_2 \le \ldots \le \mu_n \tag{2.5}$$

be the eigenvalues of the generalized eigenvalue problem $X^{-1}w = \mu A^{-1}w$ and let w_i, $i = 1, \ldots, n$, be the eigenvectors corresponding to μ_i. Then

$$\max_{\tilde{U} \in \mathbb{C}^{n \times n-r}} \min \sigma(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}) = \mu_{r+1}$$

which is achieved by

$$\tilde{U} = [w_{r+1}, \ldots, w_n].$$

Proof. Let $\mathbf{V}$ be the set of subspaces of $\mathbb{C}^{n\times n}$ of dimension $n-r$. Using the Courant-Fischer theorem we obtain for $\tilde{U} \in \mathbb{C}^{n\times n-r}$

$$\begin{aligned}
&\min \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}) \\
&= \min_{z\in\mathbb{C}^{n-r}} (z^H\tilde{U}^H X^{-1}\tilde{U}z(z^H\tilde{U}^H A^{-1}\tilde{U}z)^{-1}) \\
&= \min_{\tilde{z}\in\mathcal{R}(\tilde{U})} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\max_{\tilde{U}\in\mathbb{C}^{n\times n-r}} \min \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}) \\
&= \max_{V\in\mathbf{V}} \min_{\tilde{z}\in V} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}) \\
&= \mu_{r+1}.
\end{aligned}$$

Moreover, the matrix $\tilde{U} = [w_{r+1}, \ldots, w_n]$ leads to μ_{r+1}. $\square$

We then have

THEOREM 2.2. *Let $A \in \mathbb{C}^{n\times n}$ and $X \in \mathbb{C}^{n\times n}$ as in (1.3) be Hermitian positive definite. Let $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ be the eigenvalues of $X^{-1}A$ and let u_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Let $\lambda_{max}(BA) \le 1$. Then*

$$\min_P \rho(E_M) = 1 - \min_P \lambda_{min}(BA) = 1 - \lambda_{r+1}. \tag{2.6}$$

An optimal interpolation operator is given by

$$P_{opt} = [u_1, \ldots, u_r].$$

Proof. Since $\lambda_{max}(BA) \le 1$, we have that

$$\rho(E_M) = 1 - \lambda_{min}(BA).$$

Note that the eigenvalues λ_i are the same as the μ_i in Theorem 2.2. With Theorem 2.2 we need to find vectors which are orthogonal to the eigenvectors $w_{r+1}, \ldots, w_n$ of the generalized eigenvalue problem $X^{-1}w = \mu A^{-1}w$. Now, consider the vectors u_i, $i = 1, \ldots, r$. The vectors are also eigenvectors of the generalized eigenvalue problem $Au = \lambda Xu$. All $Xu_i = w_i$ are eigenvectors of the generalized eigenvalue problem $X^{-1}w = \mu A^{-1}w$. But the w_i are X^{-1}-orthogonal (the $X^{-\frac{1}{2}}w_i$ are eigenvectors of the Hermitian matrix $X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}}$). Thus, the u_i, $i = 1, \ldots, r$ are orthogonal to the $w_{r+1}, \ldots, w_n$ and P_{opt} leads to the minimal value. $\square$

Now, we consider E_{TG} and E_{STG} defined in (2.1) and (2.2). Again E_{STG} and E_{TG} can be written as

$$\begin{aligned}
E_{STG} &= I - B_{STG}A, \\
E_{TG} &= I - B_{TG}A,
\end{aligned}$$

for some matrices B_{STG} and B_{TG} in $\mathbb{C}^{n\times n}$. A straightforward computation shows that B_{STG} is Hermitian. Lemma 2.11 of [1] gives

$$\|E_{STG}\|_A = \|I - B_{STG}A\|_A = \rho(I - B_{STG}A). \tag{2.7}$$

Moreover, the maximal eigenvalue of $B_{STG}A$ satisfies $\lambda_{max}(B_{STG}A) \leq 1$, see e.g. [8]. We then obtain

$$\|E_{TG}\|_A^2 = \|E_{STG}\|_A = \rho(I - B_{STG}A) = 1 - \lambda_{min}(B_{STG}A).$$

The matrix X in (1.3) is given by

$$X_{STG}^{-1} = M^{-H} + M^{-1} - M^{-H}AM^{-1} = M^{-H}(M + M^H - A)M^{-1}. \tag{2.8}$$

With (2.3) we have that X_{STG} is Hermitian positive definite. Thus we get

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ such $M + M^H - A$ is Hermitian positive definite. Let X_{STG}^{-1} be as in (2.8). Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $X_{STG}^{-1}A$ and let v_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \|E_{STG}\|_A = \min_P \rho(E_{STG}) = \min_P \|E_{TG}\|_A^2 = 1 - \lambda_{r+1}. \tag{2.9}$$

An optimal interpolation operator is given by

$$P_{opt} = [v_1, \ldots, v_r].$$

Proof. We have that X_{STG} is positive definite and $\lambda_{max}(B_{STG}A) \leq 1$. So Theorem 2.2 gives the desired result. □

Next let us consider the non symmetric multigrid. the A-norm and the spectral radius. Since $\sigma(B_{STG}A) \subset (0, 1]$ we have $\rho(E_{STG}) = 1 - \lambda_{min}(X_{STG}^{-1}A)$. This does not hold for the non symmetric multigrid We use a Hermitian positive definite smoother M^{-1}. The matrix X in (1.3) is given by

$$X_{TG}^{-1} = M^{-1}. \tag{2.10}$$

We have

$$\rho(E_{TG}) = 1 - \lambda_{min}(B_{TG}A) \quad \text{or} \quad \rho(E_{TG}) = -(1 - \lambda_{max}(B_{TG}A)).$$

Thus, here it is not clear, if $\lambda_{min}(B_{TG}A)$ or $\lambda_{max}(B_{TG}A)$ leads to the spectral radius. One way to overcome this problem is scaling. Note that we have for all Hermitian positive defnite matrices X and A and for all matrices $\tilde{U} \in \mathbb{C}^{n\times n-r}$

$$\begin{aligned}
&\max \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}) \\
&= \max_{z\in\mathbb{C}^{n-r}} (z^H\tilde{U}^H X^{-1}\tilde{U}z(z^H\tilde{U}^H A^{-1}\tilde{U}z)^{-1}) \\
&= \max_{\tilde{z}\in\mathcal{R}(\tilde{U})} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}) \\
&\leq \max_{\tilde{z}\in\mathbb{C}^n} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}) \\
&= \lambda_{max}(X^{-1}A).
\end{aligned}$$

Hence, the Hermitian smoother

$$\hat{M}^{-1} = \frac{1}{\lambda_{max}(M^{-1}A)} M^{-1}$$

satisfies

$$\lambda_{max}(\hat{M}^{-1}A) = 1. \tag{2.11}$$

With Theorem 1.1 and $X^{-1} = \hat{M}^{-1}$ we then have

$$\lambda_{max}((B_{TG}A) = 1,$$

thus

$$\rho(E_{TG}) = 1 - \lambda_{min}(B_{TG}A).$$

Note, that (2.11) is equivalent to $\hat{M} - A$ being positive semidefinite.

Thus we have

COROLLARY 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ such $M - A$ is Hermitian positive definite. Let $X_{TG}^{-1} = M^{-1}$. Let $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \ldots \le \tilde{\lambda}_n$ be the eigenvalues of $X_{TG}^{-1}A$ and let x_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \rho(E_{TG}) = 1 - \tilde{\lambda}_{r+1}. \tag{2.12}$$

An optimal interpolation operator is given by

$$P_{opt} = [x_1, \ldots, x_r]. \tag{2.13}$$

Proof. The matrix $X_{TG}^{-1} = M^{-1}$ is Hermitian positive definite. Moreover, since $M - A$ is also Hermitian positive definite the eigenvalues of $X_{TG}^{-1}A$ are less then one. Thus, with Theorem 1.1, $\lambda_{max}(B_{TG}A) = 1$. So, with Theorem 2.2 we obtain (2.12) and (2.13). □

Next let us compare the optimal interpolation with respect to the A-norm as given in Corollary 2.3, with the optimal interpolation with respect to the spectral radius as given in Corollary 2.4. Using $M = M^H$ and $M - A$ Hermitian positive definite, the vectors used in Corollary 2.3 are eigenvectors of

$$X_{STG}^{-1}A = 2M^{-1}A - M^{-1}AM^{-1}A,$$

while in Corollary 2.3 we use the eigenvectors of

$$X_{TG}^{-1}A = M^{-1}A.$$

But $X_{STG}^{-1}A$ is just a polynomial in $M^{-1}A$, where the polynomial is given by

$$p(t) = 2t - t^2. \tag{2.14}$$

Thus, the eigenvectors of both matrices are the same. Moreover, the eigenvalues are related by the above polynomial. Hence, the eigenvectors corresponding to the smallest eigenvalues of $X_{STG}^{-1}A$ are the same eigenvectors that correspond to the smallest eigenvalues of $X_{TG}^{-1}A$.

Hence, the optimal interpolation in Corollary 2.3 and Corollary 2.4 are the same, if we assume that $M - A$ is hermitian positive definite.

Next, let us have a closer look to the non symmetric multigrid and avoid scaling. We assume that the smoother M is Hermitian and leads to a convergent scheme, i.e.

$$\rho(I - M^{-1}A) < 1, \tag{2.15}$$

which implies $\sigma(M^{-1}A) \subset (0, 2)$. Thus, for the matrix E_{TG} we have as above

$$\rho(E_{TG}) = 1 - \lambda_{min}(B_{TG}^{-1}A) < 1 \quad \text{or} \quad \rho(E_{TG}) = -(1 - \lambda_{max}(B_{TG}^{-1}A)) < 1.$$

Let

$$Z = \tilde{U}^H X_{TG}^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}).$$

Then we have $\sigma(Z) \subset (0,2)$ and with Theorem 1.1

$$\sigma(E_{TG}) = \{0\} \cup \sigma(I - Z).$$

But $\sigma(I-Z) \subset (-1,1)$. To minimize the spectral radius of E_{TG} over all interpolation we consider the matrix $(I-Z)^2$. We obtain

THEOREM 2.5. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ be Hermitian such $\rho(I - M^{-1}A) < 1$. Let $X_{TG}^{-1} = M^{-1}$. Let $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \ldots \leq \hat{\lambda}_n$ be the eigenvalues of $(I - X_{TG}^{-1}A)^2$ and let y_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \rho(E_{TG}) = (\hat{\lambda}_{n-r})^{\frac{1}{2}}. \tag{2.16}$$

An optimal interpolation operator is given by

$$P_{opt} = [y_{n-r+1}, \ldots, y_n]. \tag{2.17}$$

Proof. Using the theorem of Courant and Fischer and Theorem 1.1 we have

$$\begin{aligned}
&\min_{\tilde{U}} \max \sigma((I-Z)^2) \\
&= \min_{\tilde{U}} \max \sigma(((\tilde{U}^H A^{-1} \tilde{U} - \tilde{U}^H X_{TG}^{-1} \tilde{U})(\tilde{U}^H A^{-1} \tilde{U})^{-1})^2) \\
&= \min_{\tilde{U}} \max_{z \in \mathbb{C}^{n-r}} ((z^H (\tilde{U}^H A^{-1} \tilde{U} - \tilde{U}^H X_{TG}^{-1} \tilde{U}) z)(z^H \tilde{U}^H A^{-1} \tilde{U} z)^{-1})^2) \\
&= \min_{\tilde{U}} \max_{y \in \mathcal{R}(\tilde{U})} ((y^H (A^{-1} - X_{TG}^{-1}) y)(y^H A^{-1} y)^{-1})^2) \\
&= \hat{\lambda}_{n-r}.
\end{aligned}$$

The optimal interpolation is then given by (2.17). □

Note, the above Theorem 2.5 and Corollary 2.3 lead to clear statements. The optimal interpolation operators are given by those eigenvectors for which the smoothing is slowest to converge.

3. The optimal interpolation with respect to the condition number. Note that for symmetric multigrid with $M + M^H - A$ Hermitian positive definite the largest eigenvalue of $B_{STG}A$ is one (see e.g. [7]). As seen in the proof of Corollary 2.4, the same holds for $B_{TG}A$ when we assume that $M - A$ is Hermitian positive definite. The later assumption can be obtained by scaling, however, this scaling effectes the spectral radius of the error propagation matrix. But for the condition number of the multigrid preconditioned system, this scaling has no effect.

Theorem 1.1 characterizes the spectrum of $B_{STG}A$ and $B_{TG}A$. Following the arguments above, where we found optimal interpolation operators, such that $\lambda_{min}(B_{STG}A)$ and $\lambda_{min}(B_{TG}A)$ are maximal, we obtain that the same interpolation operators are optimal with respect to the condition number κ of the preconditioned system. We then have

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ such $M + M^H - A$ is Hermitian positive definite. Let X_{STG}^{-1} be as in (2.8). Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $X_{STG}^{-1}A$ and let v_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \kappa(B_{STG}A) = \frac{1}{\lambda_{r+1}}. \tag{3.1}$$

An optimal interpolation operator is given by

$$P_{opt} = [v_1, \ldots, v_r].$$

For the non symmetric multigrid we obtain

THEOREM 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ be Hermitian positive definite such that $\rho(I - M^{-1}A) < 1$. Let $X_{TG}^{-1} = M^{-1}$. Let $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_n$ be the eigenvalues of $X_{TG}^{-1}A$ and let x_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \kappa(B_{TG}A) = \frac{1}{\tilde{\lambda}_{r+1}}$$

An optimal interpolation operator is given by

$$P_{opt} = [x_1, \ldots, x_r].$$

Note, that in all cases of the previous sections any other interpolation operator $\tilde{P}$ with $\mathcal{R}(\tilde{P}) = \mathcal{R}(P_{opt})$ is also optimal.

4. Conclusion. As mentioned in [9] the A in AMG methods can be seen as an A for Abstract Multigrid Methods. Here we contributed to the theory of abstract multigrid methods. Based on a characterization of the spectrum of the error propagation operator and the preconditioned system of two-grid methods we derived optimal interpolation operators with respect to the A-norm and the spectral radius of the error propagation operator matrix in a short way. For the symmetric multigrid method (pre- and post-smoothing) the optimal interpolation operators are the same. But for post-smoothing only multigrid the optimal interpolations and hence the optimal algebraic multigrid methods are different. We also showed that these interpolation operators are optimal with respect to the condition number of the preconditioned system.

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