

ON OPTIMAL ALGEBRAIC MULTIGRID METHODS

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Abstract. In this note we present an alternative way to obtain optimal interpolation operators for two-grid methods applied to Hermitian positive definite linear systems. In [5, 10] the A -norm of the error propagation operator of algebraic multigrid methods is characterized. These results are just recently used in [3, 9] to determine optimal interpolation operators. Here we use a characterization not of the A -norm but of the spectrum of the error propagation operator of two-grid methods, which was proved in [6]. This characterization holds for arbitrary matrices. For Hermitian positive definite systems this result leads to optimal interpolation operators with respect to the A -norm in a short way, moreover, it also leads to optimal interpolation operators with respect to the spectral radius. For the symmetric two-grid method (with pre- and post-smoothing) the optimal interpolation operators are the same. But for a two-grid method with only post-smoothing the optimal interpolations (and hence the optimal algebraic multigrid methods) are different. Moreover, using the characterization of the spectrum, we can show that the found optimal interpolation operators are also optimal with respect to the condition number of the multigrid preconditioned system.

Key words. multigrid, optimal interpolation operator, two-grid methods

AMS subject classifications. 65F10, 65F50, 65N22, 65N55.

1. Introduction. Typical multigrid methods to solve the linear system

$$Ax = b,$$

where A is an $n \times n$ matrix, consist of two ingredients, the smoothing and the coarse grid correction. The smoothing is typically done by a few steps of a basic stationary iterative method, like the Jacobi or Gauss-Seidel method. For the coarse grid correction, a *prolongation* or *interpolation* operator $P \in \mathbb{C}^{n \times r}$ and a *restriction* operator $R \in \mathbb{C}^{r \times n}$ are needed. The coarse grid matrix is then defined as

$$A_C := RAP \in \mathbb{C}^{r \times r}. \quad (1.1)$$

Here we always assume that A and A_C are non-singular. The multigrid or algebraic multigrid (AMG) error propagation matrix is then given by

$$E_M = (I - M_2^{-1}A)^{\nu_2}(I - PA_C^{-1}RA)(I - M_1^{-1}A)^{\nu_1}, \quad (1.2)$$

where $M_1^{-1} \in \mathbb{C}^{n \times n}$ and $M_2^{-1} \in \mathbb{C}^{n \times n}$ are *smoothers*, ν_1 and ν_2 are the number of pre- and post-smoothing steps respectively, and $PA_C^{-1}R$ is the *coarse grid correction* matrix. The multigrid method is convergent if and only if the spectral radius of the error propagation matrix $\rho(E_M)$ is less than one. Alternatively, the norm of the error propagation matrix $\|E_M\|$ can be considered, where $\|\cdot\|$ is a consistent matrix norm, and in this case one has

$$\rho(E_M) \leq \|E_M\|.$$

The aim of algebraic multigrid methods is to balance the interplay between smoothing and coarse grid correction steps. However, most of the existing AMG methods first fix a smoother and then optimize a certain quantity to choose the interpolation P and restriction R .

To simplify the analysis, we assume that there exists a non-singular matrix X such that

$$(I - X^{-1}A) = (I - M_1^{-1}A)^{\nu_1}(I - M_2^{-1}A)^{\nu_2}, \quad (1.3)$$

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39 it can be shown that such a matrix X exists if the spectral radius of $(I - M_1^{-1}A)^{\nu_1}(I -$
 40 $M_2^{-1}A)^{\nu_2}$ is less than one, see e.g. [2]. Moreover, note that the matrix E_M can be
 41 written as

$$E_M = I - BA, \quad (1.4)$$

42 where the matrix B is known as the multigrid preconditioner, i.e., B is an approxi-
 43 mation of A^{-1} . Therefore, eigenvalue estimates of BA are of interest, and they also
 44 lead to estimates for the eigenvalues of E_M .

45 The following theorem, proved by García Ramos, Kehl and Nabben in [6], gives a
 46 characterization of the spectrum of BA , and hence a characterization of the spec-
 47 trum of the general error propagation matrix E_M .

48 **THEOREM 1.1.** *Let $A \in \mathbb{C}^{n \times n}$ be non-singular, and let $P \in \mathbb{C}^{n \times r}$ and $R \in \mathbb{C}^{r \times n}$*
 49 *such that RAP is non-singular. Moreover, let $M_1 \in \mathbb{C}^{n \times n}$ and $M_2 \in \mathbb{C}^{n \times n}$ be such*
 50 *that that the matrices X in (1.3) and RXP are non-singular. Then the following*
 51 *statements hold:*

- 52 (a) *The multigrid preconditioner B in (1.4) is non-singular.*
- 53 (b) *If $\tilde{P}, \tilde{R} \in \mathbb{C}^{n \times n-r}$ are matrices such that the columns of \tilde{P} and \tilde{R} form*
 54 *orthonormal bases of $(\mathcal{R}(P))^\perp$ and $(\mathcal{R}(R^H))^\perp$ (the orthogonal complements*
 55 *of $\mathcal{R}(P)$ and $\mathcal{R}(R^H)$ in the Euclidean inner product) respectively, then the*
 56 *matrices $\tilde{P}^H A^{-1} \tilde{R}$ and $P^H X^{-1} \tilde{R}$ are non-singular and the spectrum of BA*
 57 *is given by*

$$\sigma(BA) = \{1\} \cup \sigma(\tilde{P}^H X^{-1} \tilde{R}(\tilde{P}^H A^{-1} \tilde{R})^{-1}).$$

58 We will apply this theorem to Hermitian positive definite (HPD) matrices to de-
 59 termine the optimal interpolation operators of AMG methods with respect to the
 60 spectral radius of the error propagation matrix. For HPD matrices, optimal interpo-
 61 lation operators with respect to the A -norm have been obtained recently in [3,9]. We
 62 will show that the optimal interpolation operators with respect to the spectral radius
 63 for the symmetric/symmetrized multigrid method (with pre- and post-smoothing)
 64 and the optimal interpolation operator with respect to the A -norm are the same.
 65 But for multigrid with only a post-smoothing step the optimal interpolation op-
 66 erators with respect to the spectral radius and A -norm (and hence the optimal
 67 algebraic multigrid methods) are different. Using Theorem 1.1 we can also show
 68 that the interpolation operators with respect to the spectral radius are also optimal
 69 with respect to the condition number of the multigrid preconditioned system.

70 **2. Optimal interpolation for Hermitian positive definite matrices.** In
 71 this section we consider a HPD matrix A . Recall that the norm induced by A (or
 72 A -norm) is defined for $v \in \mathbb{C}^n$ and $S \in \mathbb{C}^{n \times n}$ by

$$\|v\|_A^2 = (v, v)_A = \|A^{\frac{1}{2}}v\|_2^2,$$

73 and

$$\|S\|_A = \|A^{\frac{1}{2}}SA^{-\frac{1}{2}}\|_2.$$

74 We will study the following two-grid methods given by the error propagation oper-
 75 ators

$$E_{TG} = (I - M^{-H}A)(I - PA_C^{-1}P^HA) \quad (2.1)$$

76 and the symmetrized version

$$E_{STG} = (I - M^{-H}A)(I - PA_C^{-1}P^HA)(I - M^{-1}A). \quad (2.2)$$

77 Thus we are using $R = P^H$. The range of P , i.e. $\mathcal{R}(P)$, is called the coarse space
 78 V_c . Here we fix the smoother M^{-1} and let E_{TG} and E_{STG} vary with respect to the
 79 choice of the interpolation operator P . In addition, we assume that the smoother
 80 M^{-1} satisfies

$$\|(I - M^{-1}A)\|_A < 1,$$

81 which is equivalent to the condition

$$M + M^H - A \text{ is positive definite,} \quad (2.3)$$

82 see, e.g., [8]. Given a fixed smoother M^{-1} such that $\|I - M^{-1}A\|_A < 1$, many
 83 AMG methods are designed to minimize $\|E_{TG}\|_A$ or a related quantity. We say
 84 an interpolation operator P^* is optimal if it minimizes $\|E_{TG}\|_A$. In view of the
 85 equality

$$\|E_{STG}\|_A = \|E_{TG}\|_A^2, \quad (2.4)$$

86 proved by Falgout and Vassilevski in [4], we can conclude that an optimal interpol-
 87 ation operator P^* also minimizes $\|E_{STG}\|_A$. Zikatanov proved in [10, Lemma 2.3]
 88 that

$$\|E_{TG}\|_A^2 = 1 - \frac{1}{K(V_c)},$$

89 where $K(V_c)$ is a quantity depending on the coarse space, defined by (check this)

$$K(V_c) = \sup_{v \in \mathbb{C}^n} \frac{\|(I - Q)v\|_{M^{-1}}^2}{\|v\|_A}$$

90 Although this equality has been known for a long time, only recently it was used
 91 to determine optimal prolongation operators P which lead to a minimal value of
 92 $\|E_{TG}\|_A$ for a given smoother (see [3, 9]). We now recall this result.

93 We will give an alternative proof of this result using the characterization of the
 94 eigenvalues of the multigrid iteration operator given in Theorem 1.1.

95 We consider first the more general error propagation matrix E_M in (1.2) with $R =$
 96 P^H and $E_M = I - BA$. We let $\mathcal{U} = \mathcal{R}(P)$ be the range of the interpolation
 97 operator $P \in \mathbb{C}^{n \times r}$, and $\tilde{U} \in \mathbb{C}^{n-r \times n-r}$ be a matrix with orthonormal columns
 98 that span \mathcal{U}^\perp (the orthogonal complement of \mathcal{U} with respect to the Euclidean inner
 99 product). Then Theorem 1.1 leads to

$$\sigma(BA) = \{1\} \cup \sigma(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}).$$

100 In what follows, given a matrix $C \in \mathbb{C}^{n \times n}$ with real eigenvalues we will denote by
 101 $\lambda_{\max}(C)$ and $\lambda_{\min}(C)$ the maximum and minimum eigenvalues of C respectively.
 102 Assuming that X is Hermitian positive definite and that $\lambda_{\max}(BA)$ is at most one,
 103 we have $\rho(E_M) = 1 - \lambda_{\min}(BA)$. In order to find an optimal interpolation operator
 104 for the error propagation matrix, we need to first find

$$\tilde{U}^* \in \underset{\tilde{U} \in \mathbb{C}^{n \times n-r}, \tilde{U}^H \tilde{U} = I}{\operatorname{argmax}} \lambda_{\min}(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}),$$

105 and then find an interpolation operator $P^* \in \mathbb{C}^{n \times r}$ such that $\mathcal{R}(P^*) = \mathcal{R}(\tilde{U}^*)^\perp$.
 106 The following lemma solves the first problem.

107 **LEMMA 2.1.** *Let $A, X \in \mathbb{C}^{n \times n}$ be Hermitian positive definite and let $\{(\mu_i, w_i)\}_{i=1}^n$*
 108 *be the eigenpairs of the generalized eigenvalue problem*

$$X^{-1}w = \mu A^{-1}w,$$

109 where

$$0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n. \quad (2.5)$$

110 Then

$$\max_{\tilde{U} \in \mathbb{C}^{n \times n-r}, \tilde{U}^H \tilde{U} = I} \lambda_{\min}(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}) = \mu_{r+1}$$

111 which is achieved by

$$\tilde{W} = [\tilde{w}_{r+1}, \dots, \tilde{w}_n], \in \mathbb{C}^{n-r}$$

112 where the columns of \tilde{W} are orthogonal in the Euclidean inner product and satisfy
 113 $\text{span}\{\tilde{w}_i\}_{i=1}^n = \text{span}\{w_i\}_{i=1}^n$.

Proof. Let $\tilde{U} \in \mathbb{C}^{n \times (n-r)}$ with $\tilde{U}^H \tilde{U} = I$. By the Courant-Fischer theorem we obtain

$$\begin{aligned} \lambda_{\min}(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}) &= \min_{z \in \mathbb{C}^{n \times n-r}} \frac{z^H \tilde{U} X^{-1} \tilde{U} z}{z^H (\tilde{U}^H A^{-1} \tilde{U})^{-1} z} \\ &= \min_{z \in \mathcal{R}(\tilde{U})} \frac{z^H X^{-1} z}{z^H A^{-1} z}, \end{aligned}$$

114 Thus, if \mathbf{V} is the set of subspaces of $\mathbb{C}^{n \times n}$ of dimension $n \times (n-r)$, we have

$$\max_{\tilde{U} \in \mathbb{C}^{n \times n-r}, \tilde{U}^H \tilde{U} = I} \lambda_{\min}(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}) = \max_{\tilde{U} \in \mathbf{V}} \min_{z \in \tilde{U}} \frac{z^H X^{-1} z}{(z^H A^{-1} z)^{-1}} = \mu_{r+1},$$

115 and the maximum is attained by choosing a matrix $\tilde{W} = [\tilde{w}_{r+1}, \dots, \tilde{w}_n]$ such
 116 that the columns of \tilde{W} are orthogonal in the Euclidean inner product and satisfy
 117 $\text{span}\{\tilde{w}_i\}_{i=1}^n = \text{span}\{w_i\}_{i=1}^n$. \square

118 The previous lemma is the main tool to obtain the optimal interpolation operators.

119 **THEOREM 2.2.** Let $A \in \mathbb{C}^{n \times n}$ and $X \in \mathbb{C}^{n \times n}$ as in (1.3) be Hermitian positive
 120 definite. Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of $X^{-1}A$ and let $u_i, i = 1, \dots, n$,
 121 be the corresponding eigenvectors. Let $\{(\lambda_i, u_i)\}_{i=1}^n$ be the eigenpairs of $X^{-1}A$,
 122 where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$, and suppose that $\lambda_{\max}(BA) \leq 1$. Then

$$\min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \rho(E_M) = 1 - \min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \lambda_{\min}(BA) = 1 - \lambda_{r+1}. \quad (2.6)$$

123 An optimal interpolation operator is given by

$$P_{\text{opt}} = [u_1, \dots, u_r]. \quad (2.7)$$

124

125 *Proof.* Since $\lambda_{\max}(BA) \leq 1$, we have that

$$\rho(E_M) = 1 - \lambda_{\min}(BA).$$

126 Note that the eigenvalues λ_i are the same as the μ_i in Theorem 2.2. According
 127 to theorem 2.2, we need to find vectors which are orthogonal to the eigenvectors
 128 w_{r+1}, \dots, w_n of the generalized eigenvalue problem $X^{-1}w = \mu A^{-1}w$. Now, consider
 129 the vectors $u_i, i = 1, \dots, r$. The vectors are also eigenvectors of the generalized
 130 eigenvalue problem $Au = \lambda Xu$. All $Xu_i = w_i$ are eigenvectors of the generalized
 131 eigenvalue problem $X^{-1}w = \mu A^{-1}w$. But the w_i are X^{-1} -orthogonal (the $X^{-\frac{1}{2}}w_i$
 132 are eigenvectors of the Hermitian matrix $X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}}$). Thus, the $u_i, i = 1, \dots, r$ are

orthogonal to the w_{r+1}, \dots, w_n in the Euclidean inner product and the interpolation operator P_{opt} given by (2.7) is the corresponding minimizer. \square
 Now, we consider E_{TG} and E_{STG} defined in (2.1) and (2.2). Again E_{STG} and E_{TG} can be written as

$$\begin{aligned} E_{STG} &= I - B_{STG}A, \\ E_{TG} &= I - B_{TG}A, \end{aligned}$$

for some matrices B_{STG} and B_{TG} in $\mathbb{C}^{n \times n}$. A straightforward computation shows that B_{STG} is Hermitian, and by [1, Lemma 2.11] we have

$$\|E_{STG}\|_A = \|I - B_{STG}A\|_A = \rho(I - B_{STG}A). \quad (2.8)$$

Moreover, the maximal eigenvalue of $B_{STG}A$ satisfies $\lambda_{\max}(B_{STG}A) \leq 1$, see e.g. [8, Theorem 3.16]. We then obtain

$$\|E_{TG}\|_A^2 = \|E_{STG}\|_A = \rho(I - B_{STG}A) = 1 - \lambda_{\min}(B_{STG}A).$$

The matrix X in (1.3) is given by

$$X_{STG}^{-1} = M^{-H} + M^{-1} - M^{-H}AM^{-1} = M^{-H}(M + M^H - A)M^{-1}. \quad (2.9)$$

With (2.3) we have that X_{STG} is Hermitian positive definite. We have thus the following corollary.

COROLLARY 2.3. *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n \times n}$ such $M + M^H - A$ is Hermitian positive definite, and let X_{STG}^{-1} be as in (2.9), and let $\{(\lambda_i, u_i)\}_{i=1}^n$ be the eigenpairs of $X_{STG}^{-1}A$, where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then*

$$\min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \|E_{STG}\|_A = \min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \rho(E_{STG}) = \min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \|E_{TG}\|_A^2 = 1 - \lambda_{r+1} \quad (2.10)$$

An optimal interpolation operator is given by

$$P_{\text{opt}} = [v_1, \dots, v_r].$$

148

Proof. We have that X_{STG} is positive definite and $\lambda_{\max}(B_{STG}A) \leq 1$. By Theorem 2.2 we obtain the desired result. \square

Next, let us consider the non-symmetric multigrid method defined implicitly by E_{TG} , in (2.1). We use a Hermitian positive definite smoother M^{-1} . The matrix X in (1.3) is given by

$$X_{TG}^{-1} = M^{-1}. \quad (2.11)$$

We have

$$\rho(E_{TG}) = 1 - \lambda_{\min}(B_{TG}A) \quad \text{or} \quad \rho(E_{TG}) = -(1 - \lambda_{\max}(B_{TG}A)).$$

Therefore, it is not clear which of $\lambda_{\min}(B_{TG}A)$ $\lambda_{\max}(B_{TG}A)$ leads to the spectral radius. One way to overcome this problem is scaling. Note that we have for all Hermitian positive definite matrices X and A and for all matrices $\tilde{U} \in \mathbb{C}^{n \times n-r}$

$$\begin{aligned} \lambda_{\max}(\tilde{U}^H X^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}) &= \max_{z \in \mathbb{C}^{n-r}} \frac{z^H \tilde{U}^H X^{-1} \tilde{U} z}{z^H \tilde{U}^H A^{-1} \tilde{U} z} \\ &= \max_{\tilde{z} \in \mathcal{R}(\tilde{U})} \frac{\tilde{z}^H X^{-1} \tilde{z}}{\tilde{z}^H A^{-1} \tilde{z}} \\ &\leq \max_{\tilde{z} \in \mathbb{C}^n} \frac{\tilde{z}^H X^{-1} \tilde{z}}{\tilde{z}^H A^{-1} \tilde{z}} \\ &= \lambda_{\max}(X^{-1}A). \end{aligned}$$

158 Hence, the Hermitian smoother

$$\hat{M}^{-1} = \frac{1}{\lambda_{\max}(M^{-1}A)} M^{-1}$$

159 satisfies

$$\lambda_{\max}(\hat{M}^{-1}A) = 1. \quad (2.12)$$

160 With Theorem 1.1 and $X^{-1} = \hat{M}^{-1}$ we then have

$$\lambda_{\max}(B_{TG}A) = 1,$$

161 thus

$$\rho(E_{TG}) = 1 - \lambda_{\min}(B_{TG}A).$$

162 Note that (2.12) is equivalent to $\hat{M} - A$ being positive semidefinite. This discussion
163 leads to the following corollary.

164 **COROLLARY 2.4.** *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n \times n}$ such
165 $M - A$ is Hermitian positive definite. Let $X_{TG}^{-1} = M^{-1}$. Let $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_n$ be
166 the eigenvalues of $X_{TG}^{-1}A$ and let $x_i, i = 1, \dots, n$, be the corresponding eigenvectors.
167 Then*

$$\min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \rho(E_{TG}) = 1 - \tilde{\lambda}_{r+1}. \quad (2.13)$$

168 An optimal interpolation operator is given by

$$P_{\text{opt}} = [x_1, \dots, x_r]. \quad (2.14)$$

169

170 *Proof.* The matrix $X_{TG}^{-1} = M^{-1}$ is Hermitian positive definite. Moreover, since
171 $M - A$ is also Hermitian positive definite the eigenvalues of $X_{TG}^{-1}A$ are less than
172 one. Thus, with Theorem 1.1, $\lambda_{\max}(B_{TG}A) = 1$. So, with Theorem 2.2 we obtain
173 (2.13) and (2.14). \square

174 Now we will compare the optimal interpolation with respect to the A -norm as given
175 in Corollary 2.3, with the optimal interpolation with respect to the spectral radius
176 as given in Corollary 2.4. Using $M = M^H$ and $M - A$ Hermitian positive definite,
177 the vectors used in Corollary 2.3 are eigenvectors of

$$X_{STG}^{-1}A = 2M^{-1}A - M^{-1}AM^{-1}A,$$

178 while in Corollary 2.3 we use the eigenvectors of

$$X_{TG}^{-1}A = M^{-1}A.$$

179 But $X_{STG}^{-1}A$ is just a polynomial in $M^{-1}A$, where the polynomial is given by

$$p(t) = 2t - t^2. \quad (2.15)$$

180 Thus, the eigenvectors of both matrices are the same. Moreover, the eigenvalues
181 are related by the above polynomial. Hence, the eigenvectors corresponding to the
182 smallest eigenvalues of $X_{STG}^{-1}A$ are the same eigenvectors that correspond to the
183 smallest eigenvalues of $X_{TG}^{-1}A$. Hence, the optimal interpolation in Corollary 2.3

184 and Corollary 2.4 are the same, if we assume that $M - A$ is Hermitian positive
185 definite.

186 Next, let us have a closer look to the non-symmetric two grid method and avoid
187 scaling. We assume that the smoother M is Hermitian and leads to a convergent
188 scheme, i.e.

$$\rho(I - M^{-1}A) < 1, \quad (2.16)$$

189 which implies $\sigma(M^{-1}A) \subset (0, 2)$. Thus, for the matrix E_{TG} we have as above

$$\rho(E_{TG}) = 1 - \lambda_{\min}(B_{TG}^{-1}A) < 1 \quad \text{or} \quad \rho(E_{TG}) = -(1 - \lambda_{\max}(B_{TG}^{-1}A)) < 1.$$

190 Let

$$Z = \tilde{U}^H X_{TG}^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}.$$

191 Then we have $\sigma(Z) \subset (0, 2)$ and with Theorem 1.1

$$\sigma(E_{TG}) = \{0\} \cup \sigma(I - Z).$$

192 But $\sigma(I - Z) \subset (-1, 1)$. To minimize the spectral radius of E_{TG} over all interpola-
193 tion we consider the matrix $(I - Z)^2$. Our next theorem deals with this case.

194 **THEOREM 2.5.** *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian positive definite, and let $M \in \mathbb{C}^{n \times n}$ be*
195 *Hermitian such $\rho(I - M^{-1}A) < 1$. Let $X_{TG}^{-1} = M^{-1}$, and let $\{(\lambda_i, y_i)\}_{i=1}^n$ be the*
196 *eigenpairs of $(I - X_{TG}^{-1}A)^2$ with $\hat{\lambda}_1 \leq \hat{\lambda}_2 \leq \dots \leq \hat{\lambda}_n$. Then*

$$\min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \rho(E_{TG}) = (\hat{\lambda}_{n-r})^{\frac{1}{2}}. \quad (2.17)$$

197 *An optimal interpolation operator is given by*

$$P_{\text{opt}} = [y_{n-r+1}, \dots, y_n]. \quad (2.18)$$

198

199 *Proof.* Using the theorem of Courant and Fischer and Theorem 1.1 we have

$$\begin{aligned} & \min_{\tilde{U} \in \mathbb{C}^{n \times n-r}, \tilde{U}^H \tilde{U} = I} \lambda_{\max}((I - Z)^2) \\ &= \min_{\tilde{U} \in \mathbb{C}^{n \times n-r}, \tilde{U}^H \tilde{U} = I} \lambda_{\max}((\tilde{U}^H A^{-1} \tilde{U} - \tilde{U}^H X_{TG}^{-1} \tilde{U})(\tilde{U}^H A^{-1} \tilde{U})^{-1})^2) \\ &= \min_{\tilde{U} \in \mathbb{C}^{n \times n-r}, \tilde{U}^H \tilde{U} = I} \max_{z \in \mathbb{C}^{n-r}} ((z^H (\tilde{U}^H A^{-1} \tilde{U} - \tilde{U}^H X_{TG}^{-1} \tilde{U}) z)(z^H \tilde{U}^H A^{-1} \tilde{U} z)^{-1})^2) \\ &= \min_{\tilde{U} \in \mathbb{C}^{n \times n-r}, \tilde{U}^H \tilde{U} = I} \max_{y \in \mathcal{R}(\tilde{U})} ((y^H (A^{-1} - X_{TG}^{-1}) y)(y^H A^{-1} y)^{-1})^2) \\ &= \hat{\lambda}_{n-r}. \end{aligned}$$

200 The optimal interpolation is then given by (2.18). \square

201 Note that the above Theorem 2.5 and Corollary 2.3 correspond to clear statements:
202 the optimal interpolation operators are given by those eigenvectors for which the
203 smoothing is slowest to converge.

204 3. The optimal interpolation with respect to the condition number.

205 Note that for symmetric multigrid where $M + M^H - A$ is Hermitian positive definite
206 the largest eigenvalue of $B_{STG}A$ is one (see e.g. [7]). As seen in the proof of Corollary
207 2.4, the same holds for $B_{TG}A$ when we assume that $M - A$ is Hermitian positive
208 definite. The later assumption can be obtained by scaling, however, this scaling

affects the spectral radius of the error propagation matrix. But for the condition number of the multigrid preconditioned system, this scaling has no effect.

Theorem 1.1 characterizes the spectrum of $B_{STG}A$ and $B_{TG}A$. Following the arguments above, where we found optimal interpolation operators, such that $\lambda_{\min}(B_{STG}A)$ and $\lambda_{\min}(B_{TG}A)$ are maximal, we obtain that the same interpolation operators are optimal with respect to the condition number κ of the preconditioned system. This leads to the next result.

THEOREM 3.1. *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n \times n}$ such $M + M^H - A$ is Hermitian positive definite. Let X_{STG}^{-1} be as in (2.9). Let $\{(\lambda_i, v_i)\}_{i=1}^n$ be the eigenpairs of $X_{STG}^{-1}A$, where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then*

$$\min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \kappa(B_{STG}A) = \frac{1}{\lambda_{r+1}}. \quad (3.1)$$

An optimal interpolation operator is given by

$$P_{\text{opt}} = [v_1, \dots, v_r].$$

Our final result gives the optimal interpolation operator for the non-symmetric two-grid method with respect to the condition number κ .

THEOREM 3.2. *Let $A \in \mathbb{C}^{n \times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n \times n}$ be Hermitian positive definite such that $\rho(I - M^{-1}A) < 1$. Let $X_{TG}^{-1} = M^{-1}$, and let $\{(\lambda_i, x_i)\}_{i=1}^n$ be the eigenpairs of $X_{TG}^{-1}A$ where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Then*

$$\min_{\substack{P \in \mathbb{C}^{n \times r} \\ \text{rank}(P)=r}} \kappa(B_{TG}A) = \frac{1}{\lambda_{r+1}}$$

An optimal interpolation operator is given by

$$P_{\text{opt}} = [x_1, \dots, x_r].$$

Note, that in all cases of the previous sections any other interpolation operator \tilde{P} with $\mathcal{R}(\tilde{P}) = \mathcal{R}(P_{\text{opt}})$ is also optimal.

4. Conclusion. As mentioned in [9], the A in AMG methods can also be understood as an A for Abstract Multigrid Methods. Here we contributed to the theory of abstract multigrid methods by presenting alternate derivations of previously known results. Building on a result from [6] which gives a characterization of the spectrum of the error propagation operator and the preconditioned system of two-grid methods, we derived optimal interpolation operators with respect to the A -norm and the spectral radius of the error propagation operator matrix in a short way. For the symmetric two-grid method (pre- and post-smoothing) the optimal interpolation operators are the same. But for a two-grid method with only post-smoothing the optimal interpolations and hence the optimal algebraic multigrid methods are different. We also showed that these interpolation operators are optimal with respect to the condition number of the preconditioned system.

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