ON OPTIMAL ALGEBRAIC MULTIGRID METHODS

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Abstract. In this note we give an alternative approach to establish optimal interpolation operators of two-grid methods applied to Hermitian positive definite systems. In [5, 10] the A-norm of the error propagation operator of algebraic multigrid methods is characterized. These results are just recently used in [3, 9] to determine optimal interpolation operators. Here we use a characterization not of the A-norm but of the spectrum of the error propagation operator of two-grid methods which was proved in [6]. This characterization holds for arbitrary matrices. For Hermitian positive definite systems this result leads to optimal interpolation operators with respect to the A-norm in a short way. But it also leads to optimal interpolation operators with respect to the spectral radius. For the symmetric multigrid method (pre- and post-smoothing) the optimal interpolation operators are the same. But for post-smoothing only multigrid the optimal interpolations and hence the optimal algebraic multigrid methods are different. Moreover, using the characterization of the spectrum, we can show that the found optimal interpolation operators are also optimal with respect to the condition number of the multigrid preconditioned system.

Key words. multigrid, optimal interpolation operator, two-grid methods

AMS subject classifications. 65F10, 65F50, 65N22, 65N55.

1. Introduction.

Typical multigrid methods to solve the linear system

$$Ax = b,$$

where A is an $n \times n$ matrix, consist of two ingredients, the smoothing and the coarse grid correction. The smoothing is mostly done by one or just a few steps of a basic stationary iterative method like the Jacobi or Gauss-Seidel method. For the coarse grid correction a *prolongation or interpolation operator* $P \in \mathbb{C}^{n\times r}$ and a *restriction operator* $R \in \mathbb{C}^{r\times n}$ are needed. The coarse grid matrix is then defined as

$$A_C := RAP \in \mathbb{C}^{r\times r}. \tag{1.1}$$

Here we always assume that A and A_C are non-singular. The typical multigrid or algebraic multigrid (AMG) iteration or error propagation matrix is then given by

$$E_M = (I - M_2^{-1}A)^{\nu_2}(I - PA_C^{-1}RA)(I - M_1^{-1}A)^{\nu_1}, \tag{1.2}$$

where $M_1^{-1} \in \mathbb{C}^{n\times n}$ and $M_2^{-1} \in \mathbb{C}^{n\times n}$ are *smoothers*, ν_1 and ν_2 are the number of smoothing steps and $PA_C^{-1}R$ is the *coarse grid correction* matrix.

The multigrid method is convergent if and only if the spectral radius of E_M, i.e. $\rho(E_m)$, is less than one. Often not the spectral radius but a consistent norm $\|.\|$ of E_M is considered. Note that

$$\rho(E_M) \le \|E_M\|.$$

The aim of algebraic multigrid methods is to balance the interplay between smoothing and coarse grid correction steps. However, most of the existing AMG methods first fix a smoother and then optimize a certain quantity to choose the interpolation P and restriction R.

The matrix E_M can be written as

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$$E_M = I - BA, \tag{1.3}$$

where the matrix B also acts as the multigrid preconditioner. Therefore eigenvalue estimates of BA are of interest. Note that the eigenvalues of E_M are just one minus the eigenvalues of BA.

We assume that there is a non-singular matrix X such that

$$(I - X^{-1}A) = (I - M_1^{-1}A)^{\nu_1}(I - M_2^{-1}A)^{\nu_2}. \tag{1.4}$$

If the spectral radius of $(I - M_1^{-1}A)^{\nu_1}(I - M_2^{-1}A)^{\nu_2}$ is less than one, then there exists such a matrix X, see e.g. [2].

The following theorem, proved by Garcia, Kehl and Nabben in [6], gives a characterization of the spectrum $\sigma(BA)$ of BA and hence a characterization of the spectrum of the general error propagation matrix E_M. It holds for arbitrary matrices A.

THEOREM 1.1. *Let $A \in \mathbb{C}^{n\times n}$ be non-singular. Let $P \in \mathbb{C}^{n\times r}$ and $R \in \mathbb{C}^{r\times n}$ such that RAP is non-singular. Moreover, let $M_1 \in \mathbb{C}^{n\times n}$ and $M_2 \in \mathbb{C}^{n\times n}$ such that that the matrices X in* (1.4) *and RXP are non-singular. Then B in* (1.3) *is non-singular. Let $\tilde{P}, \tilde{R} \in \mathbb{C}^{n\times n-r}$ such that the columns of $\tilde{P}$ and $\tilde{R}$ build orthonormal bases of $(\mathcal{R}(P))^\perp$ and $(\mathcal{R}(R^H))^\perp$. Then $\tilde{P}^H A^{-1}\tilde{R}$ is non-singular and BA has the eigenvalue one with multiplicity r, the other eigenvalues are nonzero and are the eigenvalues of*

$$\tilde{P}^H X^{-1}\tilde{R}(\tilde{P}^H A^{-1}\tilde{R})^{-1},$$

i.e.

$$\sigma(BA) = \{1\} \cup \sigma(\tilde{P}^H X^{-1}\tilde{R}(\tilde{P}^H A^{-1}\tilde{R})^{-1}).$$

We apply this theorem to Hermitian positive matrices and will determine the optimal interpolation operators of AMGs with respect to the spectral radius of the error propagation matrix. For Hermitian positive definite matrices just recently optimal interpolation operators with respect to the A-norm are established in [3, 9]. For the symmetric multigrid method (pre- and post-smoothing) the optimal interpolation operators are the same. But for post-smoothing only multigrid the optimal interpolations and hence the optimal algebraic multigrid methods are different.

Using Theorem 1.1 we can also show that the found optimal interpolation operators are also optimal with respect to the condition number of the multigrid preconditioned system.

2. Optimal interpolation for Hermitian positive definite matrices. Here we consider Hermitian positive definite matrices matrices A. The matrix A then induces the so-called A norm, defined for $v \in \mathbb{C}^n$ and $S \in \mathbb{C}^{n\times n}$ by

$$\|v\|_A^2 = (v, v)_A = \|A^{\frac{1}{2}}v\|_2^2,$$

and

$$\|S\|_A = \|A^{\frac{1}{2}}SA^{-\frac{1}{2}}\|_2.$$

We consider the following two-grid methods given by the error propagation operator

$$E_{TG} = (I - M^{-H}A)(I - PA_C^{-1}P^H A) \tag{2.1}$$

and the symmetrized version

$$E_{STG} = (I - M^{-H}A)(I - PA_C^{-1}P^HA)(I - M^{-1}A). \tag{2.2}$$

Thus we use $R = P^H$. The range of P, i.e. $\mathcal{R}(P)$, is called the coarse space V_c. Here we fix the smoother M^{-1} and consider E_{TG} and E_{STG} with respect to the choice of the interpolation operator P. So, in this note, E_{TG} and E_{STG} depend on P.

We assume that the smoother M^{-1} satisfies

$$\|(I - M^{-1}A\|_A < 1,$$

which is equivalent to

$$M + M^H - A \quad \text{is positive definite,} \tag{2.3}$$

see e.g. [8].

It is proved by Falgout and Vassilewski [4] that

$$\|E_{STG}\|_A = \|E_{TG}\|_A^2. \tag{2.4}$$

Given a fixed smoother M^{-1} such that $\|I - M^{-1}A\|_A < 1$, many AMG methods are designed to minimizes $\|E_{TG}\|_A$ or a related quantity. If an operator P minimizes $\|E_{TG}\|_A$ directly, P is called optimal.

Zikatanov proved in [10] that

$$\|E_{TG}\|_A^2 = 1 - \frac{1}{K(V_c)},$$

where $K(V_c)$ is a value depending on the coarse space.

Although this equality is known for a long time, just recently it is used to determine optimal prolongation operators P which lead to a minimal value of $\|E_{TG}\|_A$ for a given smoother (see [3, 9]). Here we give an alternative proof of this result using the characterization of the eigenvalues of the multigrid iteration operator given in Theorem 1.1.

But before we consider the more general error propagation matrix E_M in (1.2) with $R = P^H$ and $E_M = I - BA$.

Let $\mathcal{U}$ be the subspace spanned by the columns of the interpolation operator P and let $\tilde{U}$ be a matrix whose columns span $\mathcal{U}^\perp$. Then Theorem 1.1 leads to

$$\sigma(BA) = \{1\} \cup \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}).$$

Next assume that X is Hermitian positive definite and that the largest eigenvalue of BA, i.e. $\lambda_{max}(BA)$, is at most one. Then we have $\rho(E_M) = 1 - \lambda_{min}(BA)$. In order to find an optimal interpolation operator for the error propagation matrix we need to first find

$$arg \max_{\tilde{U} \in \mathbb{C}^{n \times n-r}} \min \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}),$$

and then find vectors which are orthogonal to the found optimal subspace $\tilde{\mathcal{U}}$. The following Theorem solves the first problem.

Theorem 2.1. *Let $A, X \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let*

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \tag{2.5}$$

be the eigenvalues of the generalized eigenvalue problem $X^{-1}w = \mu A^{-1}w$ *and let* w_i, $i = 1, \ldots, n$, *be the eigenvectors corresponding to* μ_i. *Then*

$$\max_{\tilde{U}\in\mathbb{C}^{n\times n-r}} \min \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}) = \mu_{r+1}$$

which is achieved by

$$\tilde{U} = [w_{r+1}, \ldots, w_n].$$

Proof. Let $\mathbf{V}$ be the set of subspaces of $\mathbb{C}^{n\times n}$ of dimension $n-r$. Using the Courant-Fischer theorem we obtain for $\tilde{U} \in \mathbb{C}^{n\times n-r}$

$$\begin{aligned} & \min \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}) \\ = & \min_{z\in\mathbb{C}^{n-r}} (z^H\tilde{U}^H X^{-1}\tilde{U}z(z^H\tilde{U}^H A^{-1}\tilde{U}z)^{-1}) \\ = & \min_{\tilde{z}\in\mathcal{R}(\tilde{U})} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}). \end{aligned}$$

Thus

$$\begin{aligned} & \max_{\tilde{U}\in\mathbb{C}^{n\times n-r}} \min \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}) \\ = & \max_{V\in\mathbf{V}} \min_{\tilde{z}\in V} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}) \\ = & \mu_{r+1}. \end{aligned}$$

Moreover, the matrix $\tilde{U} = [w_{r+1}, \ldots, w_n]$ leads to μ_{r+1}. □

We then have

THEOREM 2.2. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $X \in \mathbb{C}^{n\times n}$ *as in* (1.4) *be Hermitian positive definite. Let* $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$ *be the eigenvalues of* $X^{-1}A$ *and let* u_i, $i = 1, \ldots, n$, *be the corresponding eigenvectors. Let* $\lambda_{max}(BA) \le 1$. *Then*

$$\min_P \rho(E_M) = 1 - \min_P \lambda_{min}(BA) = 1 - \lambda_{r+1}. \tag{2.6}$$

An optimal interpolation operator is given by

$$P_{opt} = [u_1, \ldots, u_r].$$

Proof. Since $\lambda_{max}(BA) \le 1$, we have that

$$\rho(E_M) = 1 - \lambda_{min}(BA).$$

Note that the eigenvalues λ_i are the same as the μ_i in Theorem 2.2. With Theorem 2.2 we need to find vectors which are orthogonal to the eigenvectors $w_{r+1}, \ldots, w_n$ of the generalized eigenvalue problem $X^{-1}w = \mu A^{-1}w$. Now, consider the vectors u_i, $i = 1, \ldots, r$. The vectors are also eigenvectors of the generalized eigenvalue problem $Au = \lambda Xu$. All $Xu_i = w_i$ are eigenvectors of the generalized eigenvalue problem $X^{-1}w = \mu A^{-1}w$. But the w_i are X^{-1}-orthogonal (the $X^{-\frac{1}{2}}w_i$ are eigenvectors of the Hermitian matrix $X^{\frac{1}{2}}A^{-1}X^{\frac{1}{2}}$). Thus, the u_i, $i = 1, \ldots, r$ are orthogonal to the $w_{r+1}, \ldots, w_n$ and P_{opt} leads to the minimal value. □

Now, we consider E_{TG} and E_{STG} defined in (2.1) and (2.2). Again E_{STG} and E_{TG} can be written as

$$\begin{aligned} E_{STG} &= I - B_{STG}A, \\ E_{TG} &= I - B_{TG}A, \end{aligned}$$

for some matrices B_{STG} and B_{TG} in $\mathbb{C}^{n\times n}$. A straightforward computation shows that B_{STG} is Hermitian. Lemma 2.11 of [1] gives

$$\|E_{STG}\|_A = \|I - B_{STG}A\|_A = \rho(I - B_{STG}A). \tag{2.7}$$

Moreover, the maximal eigenvalue of $B_{STG}A$ satisfies $\lambda_{max}(B_{STG}A) \leq 1$, see e.g. [8]. We then obtain

$$\|E_{TG}\|_A^2 = \|E_{STG}\|_A = \rho(I - B_{STG}A) = 1 - \lambda_{min}(B_{STG}A).$$

The matrix X in (1.4) is given by

$$X_{STG}^{-1} = M^{-H} + M^{-1} - M^{-H}AM^{-1} = M^{-H}(M + M^H - A)M^{-1}. \tag{2.8}$$

With (2.3) we have that X_{STG} is Hermitian positive definite. Thus we get

Corollary 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ such $M + M^H - A$ is Hermitian positive definite. Let X_{STG}^{-1} be as in (2.8). Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $X_{STG}^{-1}A$ and let v_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \|E_{STG}\|_A = \min_P \rho(E_{STG}) = \min_P \|E_{TG}\|_A^2 = 1 - \lambda_{r+1}. \tag{2.9}$$

An optimal interpolation operator is given by

$$P_{opt} = [v_1, \ldots, v_r].$$

Proof. We have that X_{STG} is positive definite and $\lambda_{max}(B_{STG}A) \leq 1$. So Theorem 2.2 gives the desired result. ◻

Next let us consider the non symmetric multigrid. We use a Hermitian positive definite smoother M^{-1}. The matrix X in (1.4) is given by

$$X_{TG}^{-1} = M^{-1}. \tag{2.10}$$

We have

$$\rho(E_{TG}) = 1 - \lambda_{min}(B_{TG}A) \quad \text{or} \quad \rho(E_{TG}) = -(1 - \lambda_{max}(B_{TG}A)).$$

Thus, here it is not clear, if $\lambda_{min}(B_{TG}A)$ or $\lambda_{max}(B_{TG}A)$ leads to the spectral radius. One way to overcome this problem is scaling. Note that we have for all Hermitian positive defnite matrices X and A and for all matrices $\tilde{U} \in \mathbb{C}^{n\times n-r}$

$$\begin{aligned}
&\max \sigma(\tilde{U}^H X^{-1}\tilde{U}(\tilde{U}^H A^{-1}\tilde{U})^{-1}) \\
&= \max_{z\in\mathbb{C}^{n-r}} (z^H\tilde{U}^H X^{-1}\tilde{U}z(z^H\tilde{U}^H A^{-1}\tilde{U}z)^{-1}) \\
&= \max_{\tilde{z}\in\mathcal{R}(\tilde{U})} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}) \\
&\leq \max_{\tilde{z}\in\mathbb{C}^n} (\tilde{z}^H X^{-1}\tilde{z}(\tilde{z}^H A^{-1}\tilde{z})^{-1}) \\
&= \lambda_{max}(X^{-1}A).
\end{aligned}$$

Hence, the Hermitian smoother

$$\hat{M}^{-1} = \frac{1}{\lambda_{max}(M^{-1}A)} M^{-1}$$

satisfies

$$\lambda_{max}(\hat{M}^{-1}A) = 1. \tag{2.11}$$

With Theorem 1.1 and $X^{-1} = \hat{M}^{-1}$ we then have

$$\lambda_{max}((B_{TG}A) = 1,$$

thus

$$\rho(E_{TG}) = 1 - \lambda_{min}(B_{TG}A).$$

Note, that (2.11) is equivalent to $\hat{M} - A$ being positive semidefinite.

Thus we have

Corollary 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ such $M - A$ is Hermitian positive definite. Let $X_{TG}^{-1} = M^{-1}$. Let $\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \ldots \le \tilde{\lambda}_n$ be the eigenvalues of $X_{TG}^{-1}A$ and let x_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \rho(E_{TG}) = 1 - \tilde{\lambda}_{r+1}. \tag{2.12}$$

An optimal interpolation operator is given by

$$P_{opt} = [x_1, \ldots, x_r]. \tag{2.13}$$

Proof. The matrix $X_{TG}^{-1} = M^{-1}$ is Hermitian positive definite. Moreover, since $M - A$ is also Hermitian positive definite the eigenvalues of $X_{TG}^{-1}A$ are less then one. Thus, with Theorem 1.1, $\lambda_{max}(B_{TG}A) = 1$. So, with Theorem 2.2 we obtain (2.12) and (2.13). $\square$

Next let us compare the optimal interpolation with respect to the A-norm as given in Corollary 2.3, with the optimal interpolation with respect to the spectral radius as given in Corollary 2.4. Using $M = M^H$ and $M - A$ Hermitian positive definite, the vectors used in Corollary 2.3 are eigenvectors of

$$X_{STG}^{-1}A = 2M^{-1}A - M^{-1}AM^{-1}A,$$

while in Corollary 2.3 we use the eigenvectors of

$$X_{TG}^{-1}A = M^{-1}A.$$

But $X_{STG}^{-1}A$ is just a polynomial in $M^{-1}A$, where the polynomial is given by

$$p(t) = 2t - t^2. \tag{2.14}$$

Thus, the eigenvectors of both matrices are the same. Moreover, the eigenvalues are related by the above polynomial. Hence, the eigenvectors corresponding to the smallest eigenvalues of $X_{STG}^{-1}A$ are the same eigenvectors that correspond to the smallest eigenvalues of $X_{TG}^{-1}A$.

Hence, the optimal interpolation in Corollary 2.3 and Corollary 2.4 are the same, if we assume that $M - A$ is hermitian positive definite.

Next, let us have a closer look to the non symmetric multigrid and avoid scaling. We assume that the smoother M is Hermitain and leads to a convergent scheme, i.e.

$$\rho(I - M^{-1}A) < 1, \tag{2.15}$$

which implies $\sigma(M^{-1}A) \subset (0,2)$. Thus, for the matrix E_{TG} we have as above

$$\rho(E_{TG}) = 1 - \lambda_{min}(B_{TG}^{-1}A) < 1 \quad \text{or} \quad \rho(E_{TG}) = -(1 - \lambda_{max}(B_{TG}^{-1}A)) < 1.$$

Let

$$Z = \tilde{U}^H X_{TG}^{-1} \tilde{U} (\tilde{U}^H A^{-1} \tilde{U})^{-1}).$$

Then we have $\sigma(Z) \subset (0,2)$ and with Theorem 1.1

$$\sigma(E_{TG}) = \{0\} \cup \sigma(I - Z).$$

But $\sigma(I - Z) \subset (-1,1)$. To minimize the spectral radius of E_{TG} over all interpolation we consider the matrix $(I - Z)^2$. We obtain

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ be Hermitian such $\rho(I - M^{-1}A) < 1$. Let $X_{TG}^{-1} = M^{-1}$. Let $\hat{\lambda}_1 \le \hat{\lambda}_2 \le \ldots \le \hat{\lambda}_n$ be the eigenvalues of $(I - X_{TG}^{-1}A)^2$ and let y_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \rho(E_{TG}) = (\hat{\lambda}_{n-r})^{\frac{1}{2}}. \tag{2.16}$$

An optimal interpolation operator is given by

$$P_{opt} = [y_{n-r+1}, \ldots, y_n]. \tag{2.17}$$

Proof. Using the theorem of Courant and Fischer and Theorem 1.1 we have

$$\begin{aligned}
&\min_{\tilde{U}} \max \sigma((I - Z)^2) \\
&= \min_{\tilde{U}} \max \sigma(((\tilde{U}^H A^{-1} \tilde{U} - \tilde{U}^H X_{TG}^{-1} \tilde{U})(\tilde{U}^H A^{-1} \tilde{U})^{-1})^2) \\
&= \min_{\tilde{U}} \max_{z \in \mathbb{C}^{n-r}} ((z^H (\tilde{U}^H A^{-1} \tilde{U} - \tilde{U}^H X_{TG}^{-1} \tilde{U}) z)(z^H \tilde{U}^H A^{-1} \tilde{U} z)^{-1})^2) \\
&= \min_{\tilde{U}} \max_{y \in \mathcal{R}(\tilde{U})} ((y^H (A^{-1} - X_{TG}^{-1}) y)(y^H A^{-1} y)^{-1})^2) \\
&= \hat{\lambda}_{n-r}.
\end{aligned}$$

The optimal interpolation is then given by (2.17). □

Note, the above Theorem 2.5 and Corollary 2.3 lead to clear statements. The optimal interpolation operators are given by those eigenvectors for which the smoothing is slowest to converge.

3. The optimal interpolation with respect to the condition number. Note that for symmetric multigrid with $M + M^H - A$ Hermitian positive definite the largest eigenvalue of $B_{STG}A$ is one (see e.g. [7]). As seen in the proof of Corollary 2.4, the same holds for $B_{TG}A$ when we assume that $M - A$ is Hermitian positive definite. The later assumption can be obtained by scaling, however, this scaling effectes the spectral radius of the error propagation matrix. But for the condition number of the multigrid preconditioned system, this scaling has no effect.

Theorem 1.1 characterizes the spectrum of $B_{STG}A$ and $B_{TG}A$. Following the arguments above, where we found optimal interpolation operators, such that $\lambda_{min}(B_{STG}A)$ and $\lambda_{min}(B_{TG}A)$ are maximal, we obtain that the same interpolation operators are optimal with respect to the condition number κ of the preconditioned system. We then have

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ such $M + M^H - A$ is Hermitian positive definite. Let X_{STG}^{-1} be as in (2.8). Let*

$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of $X_{STG}^{-1}A$ and let v_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then

$$\min_P \kappa(B_{STG}A) = \frac{1}{\lambda_{r+1}}. \tag{3.1}$$

An optimal interpolation operator is given by

$$P_{opt} = [v_1, \ldots, v_r].$$

For the non symmetric multigrid we obtain

THEOREM 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian positive definite. Let $M \in \mathbb{C}^{n\times n}$ be Hermitian positive definite such that $\rho(I - M^{-1}A) < 1$. Let $X_{TG}^{-1} = M^{-1}$. Let $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_n$ be the eigenvalues of $X_{TG}^{-1}A$ and let x_i, $i = 1, \ldots, n$, be the corresponding eigenvectors. Then*

$$\min_P \kappa(B_{TG}A) = \frac{1}{\tilde{\lambda}_{r+1}}$$

An optimal interpolation operator is given by

$$P_{opt} = [x_1, \ldots, x_r].$$

Note, that in all cases of the previous sections any other interpolation operator $\tilde{P}$ with $\mathcal{R}(\tilde{P}) = \mathcal{R}(P_{opt})$ is also optimal.

4. Conclusion. As mentioned in [9] the A in AMG methods can be seen as an A for Abstract Multigrid Methods. Here we contributed to the theory of abstract multigrid methods. Based on a characterization of the spectrum of the error propagation operator and the preconditioned system of two-grid methods we derived optimal interpolation operators with respect to the A-norm and the spectral radius of the error propagation operator matrix in a short way. For the symmetric multigrid method (pre- and post-smoothing) the optimal interpolation operators are the same. But for post-smoothing only multigrid the optimal interpolations and hence the optimal algebraic multigrid methods are different. We also showed that these interpolation operators are optimal with respect to the condition number of the preconditioned system.

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