EMA2: Lecture contents, week 5

1.4. Systems of linear ODE's with constant coefficients

Definition.
By a **system of linear ODE's of order 1 with constant coefficients** we mean a system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n + b_1(x) \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n + b_2(x) \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n + b_n(x) \end{aligned}$$

where $b_i(x)$ are right hand-sides, $a_{ij} \in I\!R$.
A Cauchy problem (or Initial value problem) for such a system has initial conditions
$y_1(x_0) = y_{10}$, $y_2(x_0) = y_{20}$, $\ldots$, $y_n(x_0) = y_{n0}$.

1. Elimination method

Fact.
Every system of n linear ODE's of order 1 can be transformed via elimination to one linear ODE of order n, and vice versa.

Example.

$$\begin{aligned} y_1' &= 2y_1 + y_2 - 3 \\ y_2' &= y_1 + 2y_2 + 3x - 4 \end{aligned}, \qquad y_1(0) = 3,\ y_2(0) = 1.$$

Elimination: from (#1) we get $y_2 = y_1' - 2y_1 + 3 \qquad (*)$, put it into (#2) and get
$y_1'' - 4y_1' + 3y_1 = 3x + 2$. Solution: First homogeneous $y_1'' - 4y_1' + 3y_1 = 0$, $\lambda = 1, 3$, hence $y_{1h}(x) = ae^x + be^{3x}$.
Guess $y_{1p}(x) = Ax + B$ gives $y_{1p}(x) = x + 2$, hence $y_1(x) = x + 2 + ae^x + be^{3x}$.
Substituting into $(*)$ we get $y_2(x) = -2x - ae^x + be^{3x}$, $x \in I\!R$. General solution of system is

$$\begin{aligned} y_1(x) &= x + 2 + ae^x + be^{3x} \\ y_2(x) &= -2x - ae^x + be^{3x} \end{aligned}, \ x \in I\!R.$$

Initial conditions: $\begin{aligned} 2 + a + b &= 3 \\ -a + b &= 1 \end{aligned}$, from this $\begin{aligned} a &= 0 \\ b &= 1 \end{aligned}$, hence solution is $\begin{aligned} y_1(x) &= x + 2 + e^{3x} \\ y_2(x) &= -2x + e^{3x} \end{aligned}$, $x \in I\!R$.

Theorem. (on **existence and uniqueness** for systems)
Consider a system as in the definition above.
If $b_i(x)$ are continuous on an open interval I, then for every $x_0 \in I$ and all $y_{10}, y_{20}, \ldots, y_{n0} \in I\!R$ there exists a solution of the corresponding Cauchy problem on I and it is unique.

Remark: For systems with more equations the elimination is not so simple any more. For instance in the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + y_3 \\ y_2' &= -y_1 + 2y_2 + 2y_3 \\ y_3' &= 2y_1 + y_2 + y_3 \end{aligned}$$

we can try to use the first equation to express $y_3 = y_1' - y_1 - 2y_2$, substitute into the other two and get $\begin{aligned} y_2' - 2y_1' &= -3y_1 - 2y_2 \\ y_1'' - 2y_1' - 2y_2' &= y_1 - y_2 \end{aligned}$. None of the two equations allows us to express directly y_1 or y_2. In fact elimination is possible, but it is not that easy.

Note: Systems of equations of higher order can be also transformed into a system of equations of orders 1 and into one equation of high order.

2. Matrix approach

Notation:

$\vec{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$, $\vec{y}\,' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$, **matrix of the system** $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$,

vector of RHS $\vec{b}(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$, we also use $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}$.

The equation is then $\vec{y}\,' = A\vec{y} + \vec{b}$, homogeneous if $\vec{b} = \vec{0}$, init. cond. are $\vec{y}(x_0) = \vec{y}_0$.

Example. We rewrite the previous example to matrix notation:
$\vec{y}\,' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \vec{y} + \begin{pmatrix} -3 \\ 3x-4 \end{pmatrix}$, the matrix of system is $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\vec{b}(x) = \begin{pmatrix} -3 \\ 3x-4 \end{pmatrix}$, initial condition is $\vec{y}(0) = \binom{3}{1}$.

General solution: $\vec{y}(x) = \begin{pmatrix} x+2+ae^x+be^{3x} \\ -2x-ae^x+be^{3x} \end{pmatrix} = \begin{pmatrix} x+2 \\ -2x \end{pmatrix} + \begin{pmatrix} ae^x+be^{3x} \\ ae^x+be^{3x} \end{pmatrix} = \vec{y}_p(x) + \vec{y}_h(x)$.

Note that $\vec{y}_h(x) = a\begin{pmatrix} e^x \\ -e^x \end{pmatrix} + b\begin{pmatrix} e^{3x} \\ e^{3x} \end{pmatrix}$. We see that all homogeneous solutions are given by vectors $\vec{y}_a(x) = \begin{pmatrix} e^x \\ -e^x \end{pmatrix}$, $\vec{y}_b(x) = \begin{pmatrix} e^{3x} \\ e^{3x} \end{pmatrix}$, they form a basis of the space of solutions. Note also that we can write $\vec{y}_h(x) = \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$.

Theorem. (on **existence and uniqueness** for systems)
Let $A \in I\!R^{n\times n}$ be a matrix of the given system. If $\vec{b}(x)$ is continuous on an open interval I, then for all $x_0 \in I$, $\vec{y}_0 \in I\!R^n$ there exists a solution of the Cauchy problem $\vec{y}\,' = A\vec{y} + \vec{b}(x)$, $\vec{y}(x_0) = \vec{y}_0$ on I and it is unique.

Theorem. (on **structure of solutions** for systems)
Let $A \in I\!R^{n\times n}$ be a matrix.
If $\vec{y}_p$ is some solution of the system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$ on I, then the set of its solutions on I is

$$\{\vec{y}_p + \vec{y}_h;\ \vec{y}_h \text{ is a solution of } \vec{y}\,' = A\vec{y} \text{ on } I\}.$$

Theorem. (on **structure of solutions** for homogeneous systems)
Let $A \in I\!R^{n\times n}$ be a matrix. The set of all solutions of the system $\vec{y}\,' = A\vec{y}$ on some open interval I is a linear space of dimension n.

Definition.
Let $A \in I\!R^{n\times n}$ be a matrix.
By a **fundamental system** of solutions of a system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$ on an open interval I we mean an arbitrary basis of the space of solutions of $\vec{y}\,' = A\vec{y}$ on I.
For a particular fundamental system $\{\vec{y}_1, \dots, \vec{y}_n\}$ we define its **fundamental matrix solution** on I by $Y(x) = (\,\vec{y}_1(x) \quad \cdots \quad \vec{y}_n(x)\,)$ (matrix $n \times n$).

Theorem.
Let $A \in I\!R^{n\times n}$ be a matrix. Let $\vec{y}_1, \dots, \vec{y}_n$ be solutions of a system of n linear ODE's $\vec{y}\,' = A\vec{y}$ on an open interval I. $\{\vec{y}_1, \dots, \vec{y}_n\}$ is a fundamental system on I if and only if $\det(Y(x)) \neq 0$ on I if and only if $\exists x \in I$: $\det(Y(x)) \neq 0$.

Fact.
Let $A \in I\!R^{n\times n}$ be a matrix. If $Y(x)$ is a fundamental matrix solution on I of a system $\vec{y}\,' = A\vec{y}$, then a general solution of this system on I is $\vec{y}_h(x) = Y(x)\cdot\vec{c}$ for $\vec{c} \in I\!R^n$.

In the example above we have fundamental matrix solution $Y(x) = \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix}$.
Can we directly solve systems, in particular find a fundamental system, withoug doing elimination?

1.4.1. Homogeneous systems of linear ODE's with constant coefficients

Method of eigenvalues

Definition.
Let $A \in I\!R^{n\times n}$ be a matrix.
We define its **characteristic polynomial** by $p(\lambda) = \det(A - \lambda E)$.
The roots of $p(\lambda)$ are called **eigenvalues** of matrix A.
If λ is an eigenvalue, by an **eigenvector of A associated with λ** we mean an arbitrary vector $\vec{v} \in I\!R^n$, $\vec{v} \neq \vec{0}$ satisfying $(A - \lambda E)\vec{v} = \vec{0}$.

Theorem.
Consider a system of linear ODE's $\vec{y}\,' = A\vec{y}$, where $A \in I\!R^{n\times n}$.
If $\vec{v}$ is an eigenvector corresponding to an eigenvalue λ of matrix A, then $\vec{y} = \vec{v}\,e^{\lambda x}$ is a solution of the given system on $I\!R$.
If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of the matrix A, then the corresponding solutions form a linearly independent set.

Example. We try homogeneous version of the previous example $\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= y_1 + 2y_2 \end{aligned}$.
Matrix of system $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, from here $p(\lambda) = |A - \lambda E| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$ gives eigenvalues $\lambda = 1, 3$. Eigenvectors:
$\underline{\lambda = 1}$: $(A - 1\cdot E)\vec{v} = \vec{0}$, by elimination $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, hence $v_1 + v_2 = 0$, choice $v_2 = 1$ gives $v_1 = -1$, hence $\vec{y}_a(x) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{1\cdot x} = \begin{pmatrix} -e^x \\ e^x \end{pmatrix}$.
$\underline{\lambda = 1}$: $(A - 3\cdot E)\vec{v} = \vec{0}$, by elimination $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, thus $-v_1 + v_2 = 0$, choice $v_2 = 1$ gives $v_1 = 1$, hence $\vec{y}_b(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3\cdot x} = \begin{pmatrix} e^{3x} \\ e^{3x} \end{pmatrix}$.
Fundamental matrix solution $Y(x) = \begin{pmatrix} -e^x & e^{3x} \\ e^x & e^{3x} \end{pmatrix}$ and general solution $\vec{y}(x) = a\vec{y}_a + b\vec{y}_b$, $x \in I\!R$.

Note on multiple eigenvalues: Then we need more solutions, we do a little chain:
$(A - \lambda E)\vec{v}_1 = \vec{0} \implies$ solution $\vec{y} = \vec{v}_1 e^{\lambda x}$;
$(A - \lambda E)\vec{v}_2 = \vec{v}_1 \implies$ solution $\vec{y} = [\int(\vec{v}_1)\,dx + \vec{v}_2]e^{\lambda x} = (\vec{v}_1 x + \vec{v}_2)e^{\lambda x}$;
$(A - \lambda E)\vec{v}_3 = \vec{v}_2 \implies$ solution $\vec{y} = [\int(\vec{v}_1 x + \vec{v}_2)\,dx + \vec{v}_3]e^{\lambda x} = (\frac{1}{2}\vec{v}_1 x^2 + \vec{v}_2 x + \vec{v}_3)e^{\lambda x}$; etc.

Note on a complex eigenvalue: Two solutions $\mathrm{Re}(\vec{v}e^{\lambda x})$ and $\mathrm{Im}(\vec{v}e^{\lambda x})$.

1.4.2. Non-homogeneous systems of linear ODE's with constant coefficients

Method of variation of parameters

First we find a general solution $\vec{y}_h$ of the associated homogeneous system.
There are two ways to do the variation.

Variation by rows

Homogeneous solution by rows:
$y_{1h}(x) = au_1(x) + bv_1(x) + \dots$, $y_{2h}(x) = \dots$, $y_{nh}(x) = au_n(x) + bv_n(x) + \dots$
Variation: $y_1(x) = a(x)u_1(x) + b(x)v_1(x) + \dots$, ... , $y_n(x) = a(x)u_n(x) + b(x)v_n(x) + \dots$, then we solve equations
$a'(x)u_1(x) + b'(x)v_1(x) + \dots = b_1(x)$, ... , $a'(x)u_n(x) + b'(x)v_n(x) + \dots = b_n(x)$. From this we find (by elimination or Cramer) $a'(x)$, $b'(x)$, etc., by integration we get $a(x)$, $b(x)$, etc, this we put into variated y_i and we get y_{1p}, ... , y_{np}.
General solution is $y_i = y_{ip} + y_{ih}$.

Example. The previous example again.

We already solved the associated homogeneous equation: $\begin{aligned} y_{1h}(x) &= ae^x + be^{3x} \\ y_{2h}(x) &= -ae^x + be^{3x} \end{aligned}$.

Variation: $\begin{aligned} y_1(x) &= a(x)e^x + b(x)e^{3x} \\ y_2(x) &= -a(x)e^x + b(x)e^{3x} \end{aligned}$ hence the equations $\begin{aligned} a'(x)e^x + b'(x)e^{3x} &= -3 \\ -a'(x)e^x + b'(x)e^{3x} &= 3x-4 \end{aligned}$.

From them we get $\begin{aligned} a'(x) &= \frac{1}{2}(1-3x)e^{-x} \\ b'(x) &= \frac{1}{2}(3x-7)e^{-3x} \end{aligned}$, by integration $\begin{aligned} a(x) &= \big(\frac{3}{2}x+1\big)e^{-x} \\ b(x) &= \big(1-\frac{1}{2}x\big)e^{-3x} \end{aligned}$, substituting we get $\begin{aligned} y_{1p}(x) &= x+2 \\ y_{2p}(x) &= -2x \end{aligned}$ and finally $y_1 = y_{1p} + y_{1h}$, $y_2 = y_{2p} + y_{2h}$.

Variation in vector form

Associated homogeneous system gives fund. matrix $Y(x)$ and general solution $\vec{y}_h = Y \cdot \vec{c}$.
Variation $\vec{y} = Y(x) \cdot \vec{c}(x)$, equation $Y(x) \cdot \vec{c}\,'(x) = \vec{b}(x)$, hence $\vec{c}\,'(x) = Y(x)^{-1}\vec{b}(x)$, we integrate by rows, substitute $\vec{c}(x)$ into $\vec{y}(x) = Y(x) \cdot \vec{c}(x)$, we get $\vec{y}_p$, then general solution $\vec{y} = \vec{y}_p + \vec{y}_h$.

Example. The previous example again.

We already had $\vec{y}_h(x) = \begin{pmatrix} -e^x & e^{3x} \\ e^x & e^{3x} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$.

Variation $\vec{y}_p(x) = \begin{pmatrix} -e^x & e^{3x} \\ e^x & e^{3x} \end{pmatrix} \begin{pmatrix} a(x) \\ b(x) \end{pmatrix}$, equation $\begin{pmatrix} -e^x & e^{3x} \\ e^x & e^{3x} \end{pmatrix} \begin{pmatrix} a'(x) \\ b'(x) \end{pmatrix} = \begin{pmatrix} -3 \\ 3x-4 \end{pmatrix}$,
$\begin{pmatrix} a'(x) \\ b'(x) \end{pmatrix} = Y^{-1}(x) \begin{pmatrix} -3 \\ 3x-4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-x} & -e^{-x} \\ e^{-3x} & e^{-3x} \end{pmatrix} \begin{pmatrix} -3 \\ 3x-4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1-3x)e^{-x} \\ (3x-7)e^{-3x} \end{pmatrix}$, by integration $\begin{pmatrix} a(x) \\ b(x) \end{pmatrix} = \begin{pmatrix} \big(\frac{3}{2}x+1\big)e^{-x} \\ \big(1-\frac{1}{2}x\big)e^{-3x} \end{pmatrix}$, thus $\vec{y}_p(x) = Y(x) \begin{pmatrix} \big(\frac{3}{2}x+1\big)e^{-x} \\ \big(1-\frac{1}{2}x\big)e^{-3x} \end{pmatrix} = \begin{pmatrix} x+2 \\ -2x \end{pmatrix}$.