EMA2: Lecture contents, week 1

1. Ordinary differential equations

Definition.
Ordinary differential equation of order n (ODE) is any equation of the form $F(x,y,y',\dots,y^{(n)})=0$, where F is a function of $n+2$ variables in which $y^{(n)}$ really appears.
Its **solution on an (open) interval** I is any function $y=y(x)$ on interval I that has all derivatives up to order n on I and $\forall x\in I$ satisfies $F(x,y(x),y'(x),\dots,y^{(n)}(x))=0$.
Let y,v be some solutions of a given ODE on open intervals I, J respectively. We say that v is an **extension** of y if $I\subset J$, $I\neq J$ and $v=y$ on I.
We say that a solution y of a given ODR on some open interval is a **maximal solution** if it cannot be extended, that is, if there is no other solution that would be an extention of y.

Example.
The equation $\frac{\sin(y''')+xy'}{(y'')^2+1}-e^{x+y'}=\frac{y}{x^2+1}$ is an ODE of order 3.

Example.
The equation $\quad(*)\ \frac{y'}{x}=3x$ is an ODE of order 1.
For instance the function $y(x)=x^2+1$ is a solution of $(*)$ on an interval $(13,\infty)$.
For any $C\in\mathbb{R}$, the function $y(x)=x^2+C$ is a maximal solution of $(*)$ on intervals $(-\infty,0)$ and on $(0,\infty)$.
If we want a solution that satisfies the coindition $y(-1)=3$, we get $y(x)=x^2+2$, $x\in(-\infty,0)$.

Definition.
If the set of all solutions of a given ODE on a certain open interval can be expressed using one formula with parameters, we say that it is a **general solution** of this ODE. One solution, obtained by a concrete choice of these parameters, is then called a **particular solution**.

Definition.
Consider an ODE of order $n \qquad F(x,y,y',\dots,y^{(n)})=0$.
Cauchy problem or **Initial Value Problem** for this equation is a problem of the form
(1) $F(x,y,y',\dots,y^{(n)})=0$;
(2) $y(x_0)=y_0$, $y'(x_0)=y_1$, $\dots$, $y^{(n-1)}(x_0)=y_{n-1}$, where x_0, y_0, y_1, $\dots$, y_{n-1} are some fixed real numbers (**initial conditions**).

Definition.
Consider a Cauchy problem with initial conditions at some x_0.
We say that this problem is **uniquely solvable** if for every two solutions u,v of this problem there exists a neighborhood U of x_0 such that $u=v$ on U.
We say that this problem is **uniquely solvable on an interval** I (containing x_0) if for any two solutions u,v of this problem on intervals I_u,I_v containing I one has that $u=v$ on I.

Informally: A "nice" ODE has as many parameters in its solution as its order. The Cauchy conditions then make sense, we get n equations with n unknown parameters.

General algorithm for solving a Cauchy problem: 1. find a general solution, 2. apply initial conditions to that general solution and determine constants.
Integration is actually a way to solve ODE $F'=f$, but in general finding a general solution is a big problem, we restrict our attention to some very nice equations.

1.1. ODE of order 1

Definition.
By a **separable ODE of order 1** we mean any ODE that can be expressed in the form $y' = g(x)h(y)$ for some functions g, h.

Fact.
Consider a separable ODE of order 1, $y' = g(x)h(y)$. If y_0 satisfies $h(y_0) = 0$ and I is an open interval satisfying $I \subset D(g)$, then the function $y(x) = y_0$ is a solution to the given ODE on I (so-called **stationary solution**).

Theorem. (existence)
Consider a separable ODE of order 1, $y' = g(x)h(y)$. Consider open intervals $I \subset D(g)$ and $J \subset D(h)$. If g is continuous on I, h is continuous on J and $h \neq 0$ on J, then there is a solution to the given equation on I.
Let $G(x)$ be an antiderivative of $g(x)$ on I and $H(y)$ be an antiderivative of $\frac{1}{h(y)}$ on J. If H has an inverse function H_{-1}, then a general solution of the given equation on I can be expressed as $y(x) = H_{-1}(G(x) + C)$.

Practical way of solving: If an equation can be separated, then we do so moving x and y to opposite sides of the equation and then integrate:

$$\frac{dy}{dx} = g(x)h(y) \implies \frac{dy}{h(y)} = g(x)\,dx \implies \int \frac{dy}{h(y)} = \int g(x)\,dx \implies H(y) = G(x) + C,$$

then $y(x) = H_{-1}(G(x) + C)$ if it is possible.

Example.
Equation $x^5 y' = -\frac{2}{y}$ with conditions a) $y(1) = 3$; b) $y(-1) = -1$; c) $y(0) = 2$; d) $y(1) = \frac{\sqrt{15}}{4}$.
1) General solution: Condition $y \neq 0$.
$x^5 \frac{dy}{dx} = -\frac{2}{y} \iff y\,dy = -\frac{2}{x^5} \iff \int y\,dy = -\int \frac{2}{x^5} \iff \frac{1}{2}y^2 = \frac{1}{2}\frac{1}{x^4} + C \iff y^2 = \frac{1}{x^4} + C.$
Thus $y(x) = \pm\sqrt{\frac{1}{x^4} + C}$, $x \neq 0$.
Note: Formally $y' = -\frac{1}{x^5}\frac{2}{y}$, hence for instance $g(x) = -\frac{1}{x^5}$, $h(y) = \frac{2}{y}$. Both functions are continuous on intervals $(-\infty, 0)$ and $(0, \infty)$, therefore there will be solutions there and Cauchy problems will be solvable uniquely. The equation $h(y) = \frac{2}{y} = 0$ does not have a solution, hence no stationary solution.
2) Initial conditions.
a) In order to get $y(1) = 3 > 0$ we choose the version $y(x) = +\sqrt{\frac{1}{x^4} + C}$.
We substitute: $3 = \sqrt{\frac{1}{1} + C} = \sqrt{1 + C}$, hence $C = 8$. What interval?
Condition: $x \neq 0$, two possible intervals, but we want the one that contains $x_0 = 1$.
Solution: $y(x) = \sqrt{\frac{1}{x^4} + 8}$, $x \in (0, \infty)$.
b) In order to get $y(-1) = -1 < 0$ we choose the version $y(x) = -\sqrt{\frac{1}{x^4} + C}$.
We substitute: $-1 = -\sqrt{\frac{1}{1} + C} = \sqrt{1 + C}$, hence $C = 0$. What interval?
Condition: $x \neq 0$, two possible intervals, but we want the one that contains $x_0 = -1$.
Solution: $y(x) = -\sqrt{\frac{1}{x^4}} = -\frac{1}{x^2}$, $x \in (-\infty, 0)$.
c) The initial condition $y(0) = 2$ has $x_0 = 0$, this is not possible.
There is no solution.

d) In order to get $y(1) = \frac{\sqrt{15}}{4} > 0$ we choose the version $y(x) = +\sqrt{\frac{1}{x^4} + C}$
We substitute: $\frac{\sqrt{15}}{4} = \sqrt{\frac{1}{1} + C} = \sqrt{1+C}$, hence $C = -\frac{1}{16}$. What interval?
Conditions: $x \neq 0$ and $|x| < 2$, two possible intervals but we want the one containing $x_0 = 1$.
Solution: $y(x) = \sqrt{\frac{1}{x^4} - \frac{1}{16}}$, $x \in (0,2)$.

ODE come from applications, thus we also use different variables, for instance t for time, then the derivative is also denoted as $\dot{y}$, $\ddot{y}$, often we have x as a function!

Example.
General solution of $\dot{x} = 2tx^2$. Here t is variable, x function. We separate:

$$\frac{dx}{dt} = 2tx^2 \iff \frac{dx}{x^2} = 2t\,dt \iff \int \frac{dx}{x^2} = \int 2t\,dt \iff -\frac{1}{x} = t^2 - C,$$

hence $x(t) = \frac{1}{C-t^2}$, $x \neq \pm C$. For values $C > 0$ possible solutions exist on $(-\infty, C)$, on $(-C, C)$ and on (C, ∞). For $C = 0$ solutions exist on $(-\infty, 0)$ and on $(0, \infty)$. If one gets $C < 0$, then solutions exist on $\mathbb{R}$.
Since one can have $h(x) = x^2 = 0$ for $x = 0$, we also have stationary solution $x(t) = 0$, $t \in \mathbb{R}$.