EMA2: Lecture contents, week 2

1.1. ODE of order 1

Example.
$y' = 3y^{2/3}$, $y(0) = 0$.
Stationary solution $y(x) = 0$, $x \in I\!R$.
Separation: $\frac{dy}{dx} = 3y^{2/3} \iff \int \frac{1}{3}y^{-2/3}dy = \int dx \iff y^{1/3} = x + C$,
hence $y(x) = (x + C)^3$, $x \in I\!R$.
Initial conditions give $y(x) = x^3$.
The stationary solution also fits, thus there are two solutions to this Cauchy problem.

Theorem. (on **existence and uniqueness** for ODE of order 1 with isolated y)
Consider an ODE that can be written in the form $y' = f(x, y)$. (O1)
Let I, J be open intervals such that f is continuous on the set $I \times J$. Then $\forall (x_0, y_0) \in I \times J$ there exists a solution of the Cauchy problem (O1), $y(x_0) = y_0$ on some neighborhood of x_0 and this solution can be extended to the boundary of $I \times J$.
If moreover $\frac{\partial f}{\partial y}$ is continuous or bounded on $I \times J$, then this solution is unique.

We already had a theorem on existence for separable ODE, now if we also know that h' is on I continuou or bounded, we get uniqueness as well.

1.2. Linear ODE of order 1

Definition.
By a **linear ODE of order 1** we mean any ODE that can be written in the form
$y' + a(x)y = b(x)$, where a, b are some functions.
This equation is called **homogeneous** if $b(x) = 0$.
Given an ODE $y' + a(x)y = b(x)$, we define its **associated homogeneous equation**
$y' + a(x)y = 0$.

Fact.
If a, b are continuous on an open interval I, then $\forall x_0 \in I$ and $\forall y_0 \in I\!R$ there exists a solution of the Cauchy problem $y' + a(x)y = b(x)$, $y(x_0) = y_0$ on I and it is unique there.

This follows from the theorem on E and U for ODE of order 1.

Theorem. (on **solution** of homogeneous linear ODE of order 1)
If a is continuous on an open interval I, then the equation $y' + a(x)y = 0$ has a solution on I in the form $y(x) = C\, e^{-A(x)}$, where A is some antiderivative of a on I.

The set of all solutions $\{C \cdot e^{-A(x)};\ C \in C\}$ is a linear space of dimension 1, its basis is $\{e^{-A(x)}\}$.

Theorem. (on **solution** of a linear ODE of order 1)
If a, b are continuous on an open interval I, then the equation $y' + a(x)y = b(x)$ has a solution on I in the form $\Big(\int b(x)e^{A(x)}dx\Big)e^{-A(x)}$, where A is some antiderivative of a on I.
If B is some antiderivative of $b(x)e^{A(x)}$ on I, then a general solution of this equation on I is

$$y(x) = (B(x) + C)\, e^{-A(x)}.$$

Multiplying this formula out we get $y(x) = B(x)\, e^{-A(x)} + C\, e^{-A(x)}$, that is, $y = y_p + y_h$, where y_p is one particular solution of the equation $y' + a(x)y = b(x)$ and y_h is a general solution of the associated homogeneous equation $y' + a(x)y = 0$.
Thus we get:

Theorem. (on **structure of solutions** of linear ODE of order 1)
Let y_p be some particular solution of the equation $y'+a(x)y=b(x)$ on an open interval I. Then the set of all solutions of this equation on I is
$$\{y_p+y_h;\ y_h \text{ is a solution of the associated homogeneous equation on } I\}.$$

Method of variation of parameter for linear ODE with non-zero RHS.
Given an equation $y'+a(x)y=b(x)$.
Step 1. By separation find a general solution y_h of the associated homogeneous equation $y'+a(x)y=0$. It has the form $y_h(x)=C\cdot u(x)$, it also includes the stationary solution.
Step 2. Do the variation of parameter trick: We are looking for a solution of the form $y(x)=C(x)\cdot u(x)$. Such a solution exists for a certain $C(x)$, we find it by substituting $y(x)$ into the given equation $y'+a(x)y=b(x)$. We get an equation $C'(x)u(x)=b(x)$.
Then $C(x)=\int\frac{b(x)}{u(x)}\,dx$ and we put this $C(x)$ into $y=Cu$. If we take for $C(x)$ one concrete antiderivative, we get one particular solution $y_p(x)$, a general solution is then $y=y_p+y_h$.
If we use $+C$ when integrating to get $C(x)$, then after substituting it into $y=Cu$ we get a general solution right away.

Example.
Cauchy problem $y'-y\,\mathrm{cotg}(x)=2x\sin(x)$, $\quad y\left(\frac{3\pi}{2}\right)=1-\left(\frac{3\pi}{2}\right)^2$.
1) Associated homogeneous equation: $y'-y\,\mathrm{cotg}(x)=0$. Separation:
$$\frac{dy}{dx}=y\,\mathrm{cotg}(x)\iff\int\frac{dy}{y}=\int\frac{\cos(x)}{\sin(x)}\,dx\iff \ln|y|=\ln|\sin(x)|+C,$$
hence $y_h(x)=C\sin(x)$, $x\neq k\pi$.
2) Variation: $y(x)=C(x)\sin(x)$, after substituting into the given equation we get $C'(x)\sin(x)=2x\sin(x)$, $C'(x)=2x$.
a) $C(x)=x^2$, $y_p(x)=x^2\sin(x)$, general solution $y(x)=y_p(x)+y_h(x)=x^2\sin(x)+C\sin(x)$, $x\neq k\pi$.
b) $C(x)=x^2+C$, general solution $y(x)=(x^2+C)\sin(x)$, $x\neq k\pi$.
3) Initial conditions: $1-\left(\frac{3\pi}{2}\right)^2=\left[\left(\frac{3\pi}{2}\right)^2+C\right]\cdot(-1)$, hence $C=-1$, solution
$y(x)=(x^2-1)\sin(x)$, $x\in(\pi,2\pi)$.

1.3. Linear ODE of order n

Definition.
By a **linear ODE of order** n we mean any ODE that can be written in the form
$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=b(x),$$
where $a_{n-1},\ldots,a_0,b$ are some functions.
This equation is called **homogeneous** if $b(x)=0$.
Given a linear ODE $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=b(x)$, we define its **associated homogeneous equation** as $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)=0$.

Theorem. (on **existence and uniqueness** for linear ODE)
Consider an equation $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=b(x)$. (L)
If $a_{n-1},\ldots,a_0,b$ are continuous on an open interval I, then $\forall x_0\in I$ and $\forall y_0,y_1,\ldots,y_{n-1}\in\mathbb{R}$ there exists a solution to the Cauchy problem (L), $y(x_0)=y_0$, $y'(x_0)=y_1$, $\ldots$, $y^{(n-1)}(x_0)=y_{n-1}$ on I and it is unique there.

Theorem. (on **structure of solutions** of linear ODE)
Let y_p be some particular solution of a given linear ODE on an open interval I. Then the set of all solutions of this equation is
$$\{y_p+y_h;\ y_h \text{ is a solution of the associated homogeneous equation on } I\}.$$