EMA2: Lecture contents, week 12

4.2. Fourier series

Definition.
By a **trigonometric series** we mean any series of the form $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

By a **trigonometric polynomial** of degree N we mean $\frac{a_0}{2}+\sum\limits_{k=1}^{N}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

Remark: These are special cases of **Fourier series**, we will call them this way. Which functions can be expanded into Fourier series?

Fact.
Functions $\sin(k\omega t)$, $\cos(k\omega t)$ are periodic with period $T=\frac{2\pi}{\omega}$.
Thus also trigonometric polynomials and trigonometric series (if it converges) are T-periodic.

Hence only periodic functions can be sums of Fourier series. If we have such a function, which series is the best candidate for expansion?

Theorem.
Let $\omega>0$, $T=\frac{2\pi}{\omega}$. The following are true:
(i) $\int\limits_0^T\sin^2(k\omega t)\,dt=\int\limits_0^T\cos^2(k\omega t)\,dt=\frac{T}{2}$ for $k\in I\!N$,
(ii) $\int\limits_0^T\sin(k\omega t)\sin(m\omega t)\,dt=\int\limits_0^T\cos(k\omega t)\cos(m\omega t)\,dt=0$ for $k\neq m\in I\!N$,
(iii) $\int\limits_0^T\sin(k\omega t)\cos(m\omega t)\,dt=0$ for $k,m\in I\!N$.
Remark on (i): $\int\limits_0^T\sin^2(k\omega t)\,dt=0$ and $\int\limits_0^T\cos^2(k\omega t)\,dt=T$ for $k=0$.

Theorem.
Let f be a T-periodic function, denote $\omega=\frac{2\pi}{T}$.
If $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]\rightrightarrows f$ on $[0,T]$, then
$a_k=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt$ for $k\in I\!N_0$ and $b_k=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt$ for $k\in I\!N$.

Remark: $a_0=\frac{2}{T}\int\limits_0^T f(t)\,dt$.

Definition.
Let f be a function that is T-periodic and integrable on $[0,T]$.
We define its **Fourier series** as $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$, where
$a_k=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt$ for $k\in I\!N_0$ and $b_k=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt$ for $k\in I\!N$.
We denote $f\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

Remark: If a function g is T-periodic, then $\forall a \in I\!R$ we have $\int\limits_0^T g(t)\,dt = \int\limits_a^{a+T} g(t)\,dt$. This can be applied to functions and integrals in formulas for Fourier transform, popular versions are e.g. $a_k = \frac{2}{T} \int\limits_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt$ for $k \in I\!N_0$ a $b_k = \frac{2}{T} \int\limits_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt$ for $k \in I\!N$.

Definition.
Let f be a function defined on an interval $I = [a, a+T)$ for some $a \in I\!R$, $T > 0$.
We define its **periodic extension on $I\!R$** as the function
$f(t) = f(t - kT)$ for $t \in [a + kT, a + (k+1)T)$.
Remark: We obtain a T-periodic function on $I\!R$.

Definition.
Let f be a function defined on an interval $I = [a, a+T)$ for some $a \in I\!R$, $T > 0$.
We define its Fourier series as the Fourier series of its periodic extension.

Example.
Fourier series of function $f(t) = t^2$ on $[-1, 1)$. $T = 2$, $\omega = \pi$.
$a_0 = \frac{2}{2} \int\limits_0^2 f(t)\,dt = \int\limits_{-1}^1 f(t)\,dt = \int\limits_{-1}^1 t^2 dt = \frac{2}{3}$.
$a_k = \frac{2}{2} \int\limits_{-1}^1 t^2 \cos(k\pi t)\,dt = \frac{4\cos(k\pi)}{\pi^2 k^2} = \frac{4(-1)^k}{\pi^2 k^2}$.
$b_k = \frac{2}{2} \int\limits_{-1}^1 t^2 \sin(k\pi t)\,dt = 0$.
Thus $f \sim \frac{1}{3} + \sum\limits_{k=1}^{\infty} \frac{4(-1)^k}{\pi^2 k^2} \cos(k\pi t)$.

Theorem. (**Jordan criterion**)
Let f be a T-periodic function that is piecewise continuous on some interval I of length T, assume that it has a derivative f' piecewise continuous on I.
Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$. Then for every $t \in I\!R$ we have

$$\lim_{N\to\infty} \left(\frac{a_0}{2} + \sum_{k=1}^{N} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big] \right) = \frac{1}{2}[f(t^-) + f(t^+)].$$

If moreover f is continuous on $I\!R$, then $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big] \rightrightarrows f$.

Example.
For every $t \in [-1, 1]$ we have $t^2 = \frac{1}{3} + \sum\limits_{k=1}^{\infty} \frac{4(-1)^k}{\pi^2 k^2} \cos(k\pi t)$.
We use it for $t = 0$ to get $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$.
For $t = 1$ we get $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Definition.
Let f be a function defined and continuous on $[0, L)$.
We define its **sine series** as the Fourier series of its odd periodic extension.
We define its **cosine series** as the Fourier series of its even periodic extension.

Theorem.
Let f be a T-periodic function that is integrable on $[0,T)$, let $\omega=\frac{2\pi}{T}$.

(i) If f is odd, then $a_k=0$ and $b_k=\frac{4}{T}\int\limits_0^{T/2} f(t)\sin(k\omega t)\,dt$.

(ii) If f is even, then $b_k=0$ and $a_k=\frac{4}{T}\int\limits_0^{T/2} f(t)\cos(k\omega t)\,dt$.

Fact.
Let f be a function defined and continuous on $[0,L)$.

Its sine Fourier series can be obtained as a Fourier series with $a_k=0$, $b_k=\frac{2}{L}\int\limits_0^{L} f(t)\sin(k\omega t)\,dt$ and $\omega=\frac{\pi}{L}$.

Its cosine Fourier series can be obtained as a Fourier series with $b_k=0$, $a_k=\frac{2}{L}\int\limits_0^{L} f(t)\cos(k\omega t)\,dt$ and $\omega=\frac{\pi}{L}$.

Remark: The sum of the sine series is a $T-2L$-periodic extension of f into an odd function. The sum of the cosine series is a $T=2L$-periodic extension of f into an even function. Both sums must be also modified using the Jordan criterion.

Example.

$$f(t)=\begin{cases}1, & t\in[0,1);\\ 0, & t\in[1,2).\end{cases}$$

Fourier series: $T=2$, $\omega=\pi$.

$a_0=\frac{2}{2}\int\limits_0^2 f(t)\,dt=\int\limits_0^1 dt=1.$

$a_k=\frac{2}{2}\int\limits_0^1 \cos(k\pi t)\,dt=\big[\frac{1}{k\pi}\sin(k\pi t)\big]_0^1=0.$

$b_k=\frac{2}{2}\int\limits_0^1 \sin(k\pi t)\,dt=\big[-\frac{1}{k\pi}\cos(k\pi t)\big]_0^1=\frac{1}{k\pi}[1-\cos(k\pi)]=\frac{1}{k\pi}[1-(-1)^k]=\begin{cases}0, & k\text{ even};\\ \frac{2}{k\pi}, & k\text{ odd}.\end{cases}$

Thus $f\sim\frac12+\sum\limits_{k=1}^{\infty}\frac{1}{k\pi}[1-(-1)^k]\sin(k\pi t)=\frac12+\sum\limits_{k=0}^{\infty}\frac{2}{(2k+1)\pi}\sin((2k+1)\pi t).$

Sine Fourier series: $L=2$, $T=4$, $\omega=\frac{\pi}{2}$, $a_k=0$.

$b_k=\frac{2}{2}\int\limits_0^1 \sin\big(k\frac{\pi}{2}t\big)\,dt=\big[-\frac{2}{k\pi}\cos\big(k\frac{\pi}{2}t\big)\big]_0^1=\frac{2}{k\pi}[\cos(k\pi)-\cos\big(k\frac{\pi}{2}\big)].$

Thus $\sum\limits_{k=1}^{\infty}\frac{2}{k\pi}[(-1)^k-\cos\big(k\frac{\pi}{2}\big)]\sin\big(k\frac{\pi}{2}t\big).$

Cosine Fourier series: $L=2$, $T=4$, $\omega=\frac{\pi}{2}$, $b_k=0$.

$a_0=\frac{2}{2}\int\limits_0^1 dt=1.$

$a_k=\frac{2}{2}\int\limits_0^1 \cos\big(k\frac{\pi}{2}t\big)\,dt=\big[\frac{2}{k\pi}\sin\big(k\frac{\pi}{2}t\big)\big]_0^1=\frac{2}{k\pi}\sin\big(k\frac{\pi}{2}\big).$

Thus $\frac12+\sum\limits_{k=1}^{\infty}\frac{2}{k\pi}\sin\big(k\frac{\pi}{2}\big)\cos\big(k\frac{\pi}{2}t\big).$

Here $a_{2k}=0$, $a_{2k+1}=(-1)^{k+1}\frac{2}{(2k+1)\pi}$, so $\frac12+\sum\limits_{k=0}^{\infty}(-1)^{k+1}\frac{2}{(2k+1)\pi}\cos\big((2k+1)\frac{\pi}{2}t\big).$