Definitions and statements—series

Definition. A **series** is a symbol $\sum\limits_{k=n_0}^{\infty} a_k = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \dots$,
where $n_0 \in \mathbb{Z}$, $a_k \in \mathbb{R}$ (series of real numbers).

Definition. Let $\sum\limits_{k=n_0}^{\infty} a_k$ be a series.

We define its **partial sums** by $s_N = \sum\limits_{k=n_0}^{N} a_k$ for $N \geq n_0$.

We say that the given series **converges** if $\{s_N\}_{N=n_0}^{\infty}$ converges.

We say that the given series **converges to** A, denoted $\sum\limits_{k=n_0}^{\infty} a_k = A$, if $\lim\limits_{N\to\infty}(s_N) = A$.

We say that the given series **diverges** if $\{s_N\}_{N=n_0}^{\infty}$ diverges.

We say that the given series **diverges to** ∞, denoted $\sum\limits_{k=n_0}^{\infty} a_k = \infty$, if $\lim\limits_{N\to\infty}(s_N) = \infty$.

We say that the given series **diverges to** $-\infty$, denoted $\sum\limits_{k=n_0}^{\infty} a_k = -\infty$, if $\lim\limits_{N\to\infty}(s_N) = -\infty$.

Definition.
Let $a, q \in \mathbb{R}$. The series $\sum\limits_{k=n_0}^{\infty} a\,q^k$ is called a **geometric series**.

Fact.
(i) For $N \in \mathbb{N}_0$ we have $\sum\limits_{k=0}^{N} q^k = \dfrac{1-q^{N+1}}{1-q}$;
for $N \in \mathbb{N}$, $N \geq n_0$ we have $\sum\limits_{k=n_0}^{N} q^k = q^{n_0}\dfrac{1-q^{N+1-n_0}}{1-q} = \dfrac{q^{n_0}-q^{N+1}}{1-q}$.

(ii) We have $\sum\limits_{k=0}^{\infty} q^k \begin{cases} = \frac{1}{1-q}, & |q|<1; \\ = \infty \text{ (diverges)}, & q \geq 1; \\ \text{diverges}, & q \leq -1. \end{cases}$ More generally, $\sum\limits_{k=n_0}^{\infty} q^k = \dfrac{q^{n_0}}{1-q}$ for $|q|<1$.

Theorem. Let series $\sum\limits_{k=n_0}^{\infty} a_k$, $\sum\limits_{k=n_0}^{\infty} b_k$ converge.

Then also the series $\sum\limits_{k=n_0}^{\infty} (a_k+b_k)$ converges and $\sum\limits_{k=n_0}^{\infty} (a_k+b_k) = \sum\limits_{k=n_0}^{\infty} a_k + \sum\limits_{k=n_0}^{\infty} a_k$.

For $c \in \mathbb{R}$ also $\sum\limits_{k=n_0}^{\infty} (c\,a_k)$ converges and $\sum\limits_{k=n_0}^{\infty} (c\,a_k) = c\Big(\sum\limits_{k=n_0}^{\infty} a_k\Big)$.

• Convergence of series.

Theorem. Let $n_0 < n_1$, consider a series $\sum\limits_{k=n_0}^{\infty} a_k$. $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if $\sum\limits_{k=n_1}^{\infty} a_k$ converges.

Then we also have $\sum\limits_{k=n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{n_1-1} a_k + \sum\limits_{k=n_1}^{\infty} a_k$.

Theorem. (**necessary condition** for convergence)
If a series $\sum a_k$ converges, then necessarily $\lim\limits_{k\to\infty}(a_k) = 0$.
Equivalently: If $\lim\limits_{k\to\infty}(a_k) = 0$ is not true, then the series $\sum a_k$ necessarily diverges.

Theorem. (**integral test**) Let $f \geq 0$ be a non-increasing function on $[n_0,\infty)$ for $n_0 \in \mathbb{Z}$.
The series $\sum\limits_{k=n_0}^{\infty} f(k)$ converges if and only if $\int\limits_{n_0}^{\infty} f(x)\,dx$ converges.

Moreover we then have $\int\limits_{n_0}^{\infty} f(x)\,dx \leq \sum\limits_{k=n_0}^{\infty} f(k) \leq f(n_0) + \int\limits_{n_0}^{\infty} f(x)\,dx$.

Corollary. (p-test) $\sum \frac{1}{k^p}$ converges if and only if $p > 1$.

Theorem. (**comparison test**) Let $\exists n_0$ so that $0 \leq a_k \leq b_k$ for $k \geq n_0$.
If $\sum b_k$ converges, then also $\sum a_k$ converges.
If $\sum a_k$ diverges, then also $\sum b_k$ diverges (i.e. $\sum a_k = \infty \implies \sum b_k = \infty$).

Theorem. (**limit comparison test**) Let $\exists n_0 \in \mathbb{Z}$ so that $a_k \geq 0, b_k \geq 0$ for $k \geq n_0$.
If $a_k \sim b_k$, i.e. $\lim\limits_{k\to\infty}\big(\frac{a_k}{b_k}\big) = A > 0$, then $\sum a_k \sim \sum b_k$, i.e. $\sum a_k$ converges if and only if $\sum b_k$ converges.

Theorem. Let $a_k \geq 0$ for all k.
ratio test:
(i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\frac{a_{k+1}}{a_k} \leq q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\frac{a_{k+1}}{a_k} \geq 1$, then $\sum a_k$ diverges $(= \infty)$.
limit ratio test: Let $\lambda = \lim\limits_{k\to\infty} \left(\frac{a_{k+1}}{a_k}\right)$, assuming that the limit converges.
(i) If $\lambda < 1$, then $\sum a_k$ converges.
(ii) If $\lambda > 1$, then $\sum a_k$ diverges $(= \infty)$.
root test:
(i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\sqrt[k]{a_k} \leq q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\sqrt[k]{a_k} \geq 1$, then $\sum a_k$ diverges $(= \infty)$.
limit root test: Let $\varrho = \lim\limits_{k\to\infty} \left(\sqrt[k]{a_k}\right)$, assuming that the limit converges.
(i) If $\varrho < 1$, then $\sum a_k$ converges.
(ii) If $\varrho > 1$, then $\sum a_k$ diverges $(= \infty)$.
Theorem. (**Alternating series test** or **Leibniz test**)
Let $b_k \geq 0$ for all k and let $\{b_k\}$ be non-increasing.
The series $\sum (-1)^k b_k$ converges if and only if $\lim\limits_{k\to\infty} (b_k) = 0$.

Definition. We say that a series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges.

Theorem. If a series $\sum a_k$ converges absolutely, then it also converges and we have $\left|\sum\limits_{k=n_0}^{\infty} a_k\right| \leq \sum\limits_{k=n_0}^{\infty} |a_k|$.

Definition. We say that a series **converges conditionally** if it converges, but not absolutely.

Theorem. Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then for every choice of signs $\varepsilon_k = \pm 1$ also $\sum \varepsilon_k a_k$ converges.
If $\sum a_k$ converges conditionally, then there exists a choice of signs $\varepsilon_k = \pm 1$ such that $\sum \varepsilon_k a_k = \infty$.

Definition. Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
By a **rearrangement** of $\sum\limits_{k=n_0}^{\infty} a_k$ we mean any series $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)}$, where π is an arbitrary bijective mapping of $\{n_0, n_0+1, n_0+2, \dots\} \subset \mathbb{Z}$ onto $\{n_0, n_0+1, n_0+2, \dots\}$, i.e. π is a permutation of $\{n_0, n_0+1, n_0+2, \dots\}$.

Theorem. Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then also all its rearrangements $\sum a_{\pi(k)}$ converge and we have $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = \sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum\limits_{k=n_0}^{\infty} a_k$ converges conditionally, then $\forall c \in \mathbb{R} \cup \{\pm\infty\}$ there exists its rearrangement such that $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = c$.