EMA2: Lecture contents, week 4

1.3.2. Non-homogeneous linear ODE of order n

1. Method of variation of parameter.

Given: an equation $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = b(x)$.
Step 1. Using the method of characteristic numbers find a general solution y_h of associated homogeneous equation. It has the form $y_h(x) = c_1 \cdot u_1(x) + \ldots + c_n \cdot u_n(x)$.
Step 2. Variation of parameter: We look for solution of the form
$y(x) = c_1(x) \cdot u_1(x) + \ldots + c_n(x) \cdot u_n(x)$. We find $c_i(x)$ by solving equations:

$$\begin{aligned} c_1'(x)u_1(x) + \ldots + c_n'(x)u_n(x) &= 0 \\ c_1'(x)u_1'(x) + \ldots + c_n'(x)u_n'(x) &= 0 \\ &\vdots \\ c_1'(x)u_1^{(n-2)}(x) + \ldots + c_n'(x)u_n^{(n-2)}(x) &= 0 \\ c_1'(x)u_1^{(n-1)}(x) + \ldots + c_n'(x)u_n^{(n-1)}(x) &= b(x) \end{aligned}$$

From this we get $c_i'(x)$ and then by integrating $c_i(x)$, these we sub into $y(x) = \sum c_i(x)u_i(x)$. If we take for $c_i(x)$ particular antiderivatives, we get one particular solution $y_p(x)$, then a general solution is $y = y_p + y_h$.
If when looking for $c_i(x)$ we use $+C_i$ while integrating, then after substituting into $y(x) = \sum c_i(x)u_i(x)$ we get a general solution right away.

Example.
General solution of $y'' - \frac{x}{2(x-2)}y' + \frac{1}{2(x-2)}y = \frac{x-2}{x}$.
1) Associated homogeneous equation: general solution $y_h(x) = ax + be^{x/2}$, $x \neq 2$ (see labs in week 3).
2) Variation: $y(x) = a(x)x + b(x)e^{x/2}$, equations

$$\left[\begin{array}{c} a'(x)x + b'(x)e^{x/2} = 0 \\ a'(x)[x]' + b'(x)[e^{x/2}]' = \frac{x-2}{x} \end{array}\right\} \Longrightarrow \left[\begin{array}{l} a'(x)x + b'(x)e^{x/2} = 0 \\ a'(x) + b'(x)\frac{1}{2}e^{x/2} = \frac{x-2}{x} \end{array}\right\} \Longrightarrow \left[\begin{array}{l} a'(x) = -\frac{2}{x} \\ b'(x) = 2e^{-x/2} \end{array}\right.$$

a) $a(x) = -2\ln|x|$, $b(x) = -4e^{-x/2}$, hence $y_p(x) = -2\ln|x|x - 4e^{-x/2}e^{x/2}$ and
$y(x) = -2x\ln|x| - 4 + ax + be^{x/2}$, $x \neq 2$.
b) $a(x) = -2\ln|x| + a$, $b(x) = -4e^{-x/2} + b$, hence
$y(x) = (a - 2\ln|x|)x + (b - 4e^{-x/2})e^{x/2}$, $x \neq 2$.

Example.
Cauchy problem: $\ddot{x} + x = \frac{1}{\cos(t)}$, $\quad x(0) = 2$, $\dot{x}(0) = 0$.
1) Associated homogeneous equation: $\ddot{x} + x = 0$. $\lambda^2 + 1 = 0$, hence $\lambda = \pm j$, general solution $x_h(t) = a\sin(t) + b\cos(t)$, $t \in I\!R$.
2) Variation: $x(t) = a(t)\sin(t) + b(t)\cos(t)$, equations

$$\left[\begin{array}{c} a'(t)\sin(t) + b'(t)\cos(t) = 0 \\ a'(t)[\sin(t)]' + b'(t)[\cos(t)]' = \frac{1}{\cos(t)} \end{array}\right\} \Longrightarrow \left[\begin{array}{l} a'(t)\sin(t) + b'(t)\cos(t) = 0 \\ a'(t)\cos(t) - b'(t)\sin(t) = \frac{1}{\cos(t)} \end{array}\right.$$

From this (by elimination or Cramer) $a'(t) = 1$, $b'(t) = -\frac{\sin(t)}{\cos(t)}$.
a) $a(x) = t$, $b(x) = \ln|\cos(t)|$, hence $x_p(t) = t\sin(t) + \ln|\cos(t)|\cos(t)$ and
$x(t) = t\sin(t) + \ln|\cos(t)|\cos(t) + a\sin(t) + b\cos(t)$, $t \neq \frac{\pi}{2} + k\pi$.
b) $a(x) = t + a$, $b(x) = \ln|\cos(t)| + b$, hence
$x(t) = (t + a)\sin(t) + (\ln|\cos(t)| + b)\cos(t)$, $t \neq \frac{\pi}{2} + k\pi$.
3) $x(t) = t\sin(t) + \ln|\cos(t)|\cos(t) + a\sin(t) + b\cos(t)$ $\qquad 2 = b$
$\quad \dot{x}(t) = \sin(t) + t\cos(t) - \sin(t) - \ln|\cos(t)|\sin(t) + a\cos(t) - b\sin(t)$ $\qquad 0 = a$
Hence $x(t) = t\sin(t) + \ln|\cos(t)|\cos(t) + 2\cos(t)$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

2. Guessing method, method of unknown coefficients

Theorem. (on **guessing solution** for special right hand-side)

Consider a linear ODE with constant coefficients $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = b(x)$. Assume that $b(x) = e^{\alpha x}[P(x)\sin(\beta x) + Q(x)\cos(\beta x)]$ for some polynomials P, Q with $m = \max(\deg(P), \deg(q))$. Let k be the multiplicity of the number $\alpha \pm \beta j$ as a characteristic number of the given equation (we put $k = 0$ if it is not a char. no. at all).
Then there are polynomials $\widetilde{P}, \widetilde{Q}$ of order at most m such that
$y(x) = x^k e^{\alpha x}[\widetilde{P}(x)\sin(\beta x) + \widetilde{Q}(x)\cos(\beta x)]$ is a solution of the given equation on $I\!R$.

Example.
General solution of $y'' + y = e^{-x}[4\sin(x) + 5x\cos(x)]$.
1) Associated homogeneous equation: $y'' + y = 0$. $\lambda^2 + 1 = 0$, hence $\lambda = \pm j$, general solution $y_h(x) = a\sin(x) + b\cos(x)$, $x \in I\!R$.
2) Right hand-side is special: $b(x) = e^{(-1)\cdot x}[4\sin(1\cdot x) + (5x)\cos(1\cdot x)]$.
Thus $\alpha = -1$, $\beta = 1$, hence $\alpha + \beta j = -1 + j$, then $k = 0$.
$P(x) = 4$, $\deg(P) = 0$; $Q(x) = 5x$, $\deg(Q) = 1$, $m = \max(0,1) = 1$. Thus there is a solution of the type $y(x) = x^0 e^{-x}[(Ax + B)\sin(x) + (Cx + D)\cos(x)]$. We find A, B, C, D by substituting into the given equation:

$$[e^{-x}[(Ax+B)\sin(x) + (Cx+D)\cos(x)]]'' + e^{-x}[(Ax+B)\sin(x) + (Cx+D)\cos(x)] = e^{-x}[4\sin(x) + 5x\cos(x)]$$

$$[(A+2C)x + (-2A+B-2C+2D)]e^{-x}\sin(x) + [(-2A+C)x + (2A-2B-2C+D)]e^{-x}\cos(x) = [0x+4]e^{-x}\sin(x) + [5x+0]e^{-x}\cos(x)$$ and by comparison

$$\left.\begin{aligned} A + 2C &= 0\\ -2A + B - 2C + 2D &= 4\\ -2A + C &= 5\\ 2A - 2B - 2C + D &= 0 \end{aligned}\right\} \implies A = -2,\ B = -2,\ C = 1,\ D = 2.$$

Hence $y_p(x) = -(2x+2)e^{-x}\sin(x) + (x+2)e^{-x}\cos(x)$ and general solution is
$y(x) = -(2x+2)e^{-x}\sin(x) + (x+2)e^{-x}\cos(x) + a\sin(x) + b\cos(x)$, $x \in I\!R$.

Simpler forms for partial right hand-sides:
- $b(x) = P(x) \implies y(x) = x^k\widetilde{P}(x)$, where k is multiplicity of 0.
- $b(x) = P(x)e^{\alpha x} \implies y(x) = x^k\widetilde{P}(x)e^{\alpha x}$, where k is multiplicity α;
- $b(x) = P(x)\sin(\beta x) + Q(x)\cos(\beta x) \implies y(x) = x^k[\widetilde{P}(x)\sin(\beta x) + \widetilde{Q}(x)\cos(\beta x)]$, where k is multiplicity of $0 \pm \beta j$;

Theorem. (**principle of superposition**)
Consider a linear ODE with left hand-side $L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$. Let y_1 be a solution of $L(y) = b_1(x)$ on an open interval I and y_2 be a solution of $L(y) = b_2(x)$ on I.
Then $y_1 + y_2$ is a solution of $L(y) = b_1(x) + b_2(x)$ on I.

Example.
General solution of $y'' - 2y' + y = 2e^x + 27e^{-2x} + 1$.
1) Associated homogeneous equation: $y'' - 2y' + y = 0$. $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$, hence $\lambda = 1$ $(2\times)$, general solution $y_h(x) = ae^x + bx\,e^{-x}$, $x \in I\!R$.
2) Right hand-side is a combination of three special ones:
a) $2e^x$: $\alpha + \beta j = 1$, $k = 2$; $\deg(P) = 0$, hence x^2Ae^x.
b) $27e^{-2x}$: $\alpha + \beta j = -2$, $k = 0$; $\deg(P) = 0$, hence Be^{-2x}.
c) 1: $\alpha + \beta j = 0$, $k = 0$; $\deg(P) = 0$, hence C.
There is a solution of the type $y(x) = Ax^2e^x + Be^{-2x} + C$. We find A, B, C by substituting into the given equation:

$$[Ax^2e^x + Be^{-2x} + C]'' - 2[Ax^2e^x + Be^{-2x} + C]' + [Ax^2e^x + Be^{-2x} + C] = 2e^x + 27e^{-2x} + 1$$

$[2A]e^x + [9B]e^{-2x} + C = [2]e^x + [27]e^{-2x} + [1]$, hence $A = 1,\ B = 3,\ C = 1$,
general solution $y(x) = y_p + y_h = x^2 e^x + 3e^{-2x} + 1 + ae^x + bx\, e^{-x}$, $x \in \mathbb{R}$.