EMA2: Lecture contents, week 7

2.1. Laplace transform

Theorem. (on **periodic** function)
Let f be a function that is T-periodic on $[0,\infty)$. We mark one period by $f_T = f\cdot\chi_{[0,T)}$. Then $\mathcal{L}\{f(t)\} = \dfrac{\mathcal{L}\{f_T(t)\}}{1-e^{-pT}}$.

Example.

$$\mathcal{L}\{|\sin(2t)|\} = \frac{\mathcal{L}\{f_T(t)\}}{1-e^{-p\pi}} = \frac{\mathcal{L}\{\sin(2t)[H(t)-H(t-\frac{\pi}{2})]\}}{1-e^{-\pi p}} = \frac{\frac{2}{p^2+4}+\frac{2e^{-\frac{\pi}{2}p}}{p^2+4}}{1-e^{-\pi p}} = \frac{2}{p^2+4}\,\frac{1+e^{-\frac{\pi}{2}p}}{1-e^{-\pi p}}$$
$$= \frac{2}{p^2+4}\,\frac{1+e^{-\frac{\pi}{2}p}}{(1-e^{-\frac{\pi}{2}p})(1+e^{-\frac{\pi}{2}p})} = \frac{2}{p^2+4}\,\frac{1}{1-e^{-\frac{\pi}{2}p}}.$$

2.2. Inverse Laplace transform

There is a problem with Laplace transform not being one-to-one.

Theorem.
If $f,g\in\mathcal{L}_0$ have $\mathcal{L}\{f\}=\mathcal{L}\{g\}$ on some $[p_0,\infty)$, then $f=g$ with exception of a countable set of isolated points.
If moreover f and g are continuous from the right everywhere, then $f=g$.

Corollary.
Consider the linear space $V=\{f\in\mathcal{L}_0;\ f \text{ continuous from the right on } \mathbb{R}_0^+\}$. On this space the Laplace transform is one-to-one, therefore we can consider its inverse $\mathcal{L}^{-1}$.

Theorem. (**dictionary** for $\mathcal{L}^{-1}$)
$\mathcal{L}^{-1}\{\frac{1}{p-\alpha}\}=e^{\alpha t}$, $\mathcal{L}^{-1}\{\frac{1}{p^n}\}=\frac{1}{(n-1)!}t^{n-1}$, $\mathcal{L}^{-1}\{\frac{\omega}{p^2+\omega^2}\}=\sin(\omega t)$, $\mathcal{L}^{-1}\{\frac{p}{p^2+\omega^2}\}=\cos(\omega t)$.

Theorem. (**grammar** for $\mathcal{L}^{-1}$)
(0) $\mathcal{L}^{-1}$ is linear;
(1) $\mathcal{L}^{-1}\{e^{-ap}F(p)\}=\mathcal{L}^{-1}\{F(p)\}\big|_{t-a}\cdot H(t-a)$;
(2) $\mathcal{L}^{-1}\{F(p-a)\}=e^{at}\mathcal{L}^{-1}\{F(p)\}$;
(3) $\mathcal{L}^{-1}\{F(ap)\}=\frac{1}{a}\mathcal{L}^{-1}\{F(p)\}\big|_{t/a}$;
(4) $\mathcal{L}^{-1}\{F'(p)\}=-t\mathcal{L}^{-1}\{F(p)\}$;
(5) $\mathcal{L}^{-1}\{pF(p)\}=\big[\mathcal{L}^{-1}\{F(p)\}\big]'+\mathcal{L}^{-1}\{F(p)\}(0^+)$.

Example.

$$\mathcal{L}^{-1}\{\tfrac{pe^{-\pi p}}{p^2+1}\}=\mathcal{L}^{-1}\{\tfrac{p}{p^2+1}\}\big|_{t-\pi}H(t-\pi)=\cos(t)\big|_{t-\pi}H(t-\pi)=\cos(t-\pi)H(t-\pi)$$
$$=-\cos(t)H(t-\pi)=\begin{cases}0, & t\in[0,\pi);\\ -\cos(t), & t\geq\pi.\end{cases}$$

Theorem.
If $F(p)$ is a proper rational function, then $\mathcal{L}^{-1}\{F(p)\}$ exists and it can be found using partial fractions decomposition.

2.3. Laplace transform and differential equations

Solving differential equations (Cauchy problems) using LT; Laplace the equation, solve the resulting algebraic equation, unlaplace it.

Example.

$$\ddot{x} - x = \begin{cases} 2, & t \in [0,1); \\ 0, & \text{elsewhere} \end{cases} = 2\chi_{[0,1)}, \qquad x(0^+) = \dot{x}(0^+) = 0.$$

We denote $\mathcal{L}\{x\} = X$, then $[p^2X - 0p - 0] - X = \mathcal{L}\{2[H(t) - H(t-1)]\}$, $(p^2-1)X = \frac{2}{p} - e^{-p}\frac{2}{p}$, so $X(p) = \frac{2}{(p^2-1)p} - e^{-p}\frac{2}{(p^2-1)p} = \left(\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right) - e^{-p}\left(\frac{1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right)$, hence $x(t) = e^t + e^{-t} - 2 - (e^t + e^{-t} - 2)\big|_{t-1} H(t-1) = e^t + e^{-t} - 2 - (e^{t-1} + e^{1-t} - 2)H(t-1)$

$$= \begin{cases} e^t + e^{-t} - 2, & t \in [0,1); \\ e^t(1 - e^{-1}) + e^{-t}(1 - e), & t \geq 1 \end{cases}$$

or $x(t) = 2\cosh(t) - 2 - (2\cosh(t-1) - 2)H(t-1) = \begin{cases} 2\cosh(t) - 2, & t \in [0,1); \\ 2\cosh(t) - 2\cosh(t-1), & t \geq 1. \end{cases}$

Finding a general solution using LT: Two possibilities.
1) Choose null initial conditions, find one particular solution using LT, then add to it a general homogeneous solution (most likely found via characteristic numbers).
2) Choose general initial conditions $y(0^+) = a$ etc., solve the problem using LT, we get a solution with parameters, that is, a general one.

Example.

General solution of $\dot{x} + 9\int\limits_0^t x(u)\,du = 0$.

Choice $x(0^+) = a$, then $pX - a + 9\frac{1}{p}X = 0$, $X(p) = \frac{ap}{p^2+9}$, $x(t) = a\cos(3t)$, $t \geq 0$.

Using LT one can also solve systems of equations.

Example.

$$\begin{array}{l} y_1' = 2y_1 + y_2 \\ y_2' = y_1 + 2y_2 \end{array}, \qquad y_1(0) = 1,\ y_2(0) = 1.$$

Denote $\begin{array}{l} \mathcal{L}\{y_1\} = Y_1 \\ \mathcal{L}\{y_1\} = Y_1 \end{array}$, then $\begin{array}{l} pY_1 - 1 = 2Y_1 + Y_2 \\ pY_2 - 1 = Y_1 + 2Y_2 \end{array}$, hence $\begin{array}{r} (p-2)Y_1 - Y_2 = 1 \\ -Y_1 + (p-2)Y_2 = 1 \end{array}$, from this (by elimination or Cramer) $Y_1(p) = \frac{1}{p-3}$, $Y_2(p) = \frac{1}{p-3}$, thus $y_1(x) = y_2(x) = e^{3x}$, $x \in I\!R$.