EMA2: Lecture contents, week 11

4.1. Power series

Remark: If $f(x)=\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$, then $f^{(n)}(x_0)=n!\,a_n$ for $n\in I\!N_0$.

Definition.
Let f have derivatives of all orders at x_0.
We define the **Taylor series** of f with center at x_0 by the formula $\sum\limits_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Corollary. (**uniqueness of expansion**)
If $f(x)=\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$, then this series is necessarily the Taylor series.
The finding of this series is called **expanding a given function into a power/Taylor series (with center x_0)**. Partial sum is a Taylor polynomial, so we know all that we need, even what is the difference between T_n and f by Lagrange.

Example.
$e^x=\sum\limits_{k=0}^{\infty}\frac{x^k}{k!}$ for $x\in I\!R$.

Proof: Denote $T_N=\sum\limits_{k=0}^{N}\frac{x^k}{k!}$. Then by the Lagrange form of remainder $e^x-T_N(x)=R_N(x)$, where $R_N(x)=\frac{[e^x]^{(N+1)}(c)}{(N+1)!}x^{N+1}$. Fix some x and estimate $|R_N(x)|\le\frac{1}{(N+1)!}\max\limits_{x\in\overline{0x}}|e^x||x|^{N+1}$, thus for $x\ge 0$ we have $|R_N(x)|\le\frac{e^x|x|^{N+1}}{(N+1)!}$, for $x\le 0$ we have $|R_N(x)|\le\frac{|x|^{N+1}}{(N+1)!}$. In any case $R_N(x)\to 0$, so $T_N(x)\to e^x$.

Theorem.
Let a function f have derivatives of all orders on some $U_r(x_0)$ and let $\exists M>0$ such that $|f^{(k)}(x_0)|\le M$ for all $k\in I\!N_0$ and $x\in U_r(x_0)$. Then for $x\in U_r(x_0)$ we have $f(x)=\sum\limits_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Fact.

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+x^4+\dots,\qquad x\in(-1,1);$$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots,\qquad x\in I\!R;$$

$$\sin(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots,\qquad x\in I\!R;$$

$$\cos(x)=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots,\qquad x\in I\!R;$$

$$\ln(1+x)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}=1-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\dots,\qquad x\in(-1,1];$$

$$(1+x)^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+\frac{\alpha(\alpha-1)(\alpha-3)}{3!}x^3+\dots,\qquad x\in(-1,1).$$

Here $\binom{\alpha}{k} = \dfrac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-k+2)\cdot(\alpha-k+1)}{k!}$.

Other functions are done using these and algebra, substitution etc.

Example.
Expand $f(x)=(x+3)e^{4x}$ into a series with center $x_0=0$:
$(x+3)e^{4x} = xe^{4x}+3e^{4x} = \langle\!\langle y=4x \rangle\!\rangle = x\sum\limits_{k=0}^{\infty}\frac{(4x)^k}{k!} + 3\sum\limits_{k=0}^{\infty}\frac{(4x)^k}{k!} = \sum\limits_{k=0}^{\infty}\frac{4^k}{k!}x^{k+1} + \sum\limits_{k=0}^{\infty}\frac{3\cdot 4^k}{k!}x^k$
$= \langle\!\langle k \mapsto k^* \rangle\!\rangle = \sum\limits_{k=1}^{\infty}\frac{4^{k-1}}{(k-1)!}x^k + \sum\limits_{k=0}^{\infty}\frac{3\cdot 4^k}{k!}x^k = \sum\limits_{k=1}^{\infty}\frac{k4^{k-1}}{k!}x^k + 3 + \sum\limits_{k=1}^{\infty}\frac{3\cdot 4^k}{k!}x^k$
$= 3+\sum\limits_{k=1}^{\infty}\frac{(k+12)4^{k-1}}{k!}x^k,\ x\in I\!R.$

Example.
Expand $f(x)=\frac{1}{1+3x^2}$ into a series with center $x_0=0$:
$\frac{1}{1+3x^2} = \frac{1}{1-(-3x^2)} = \langle\!\langle y=-3x^2,\ |y|<1 \rangle\!\rangle = \sum\limits_{k=0}^{\infty}(-3x^2)^k = \sum\limits_{k=0}^{\infty}(-1)^k 3^k x^{2k},\ |x|<\frac{1}{\sqrt{3}}.$

Example.
Expand $f(x)=(x-1)\sin(\pi x)$ into a series with center $x_0=-1$:
$(x-1)\sin(\pi x) = (x-(-1)-2)\sin(\pi(x-(-1)-1))$
$= (x-(-1))\sin(\pi(x-(-1))-\pi) - 2\sin(\pi(x-(-1))-\pi)$
$= -(x+1)\sin(\pi(x+1)) + 2\sin(\pi(x+1)) = \langle\!\langle y=\pi(x+1) \rangle\!\rangle$
$= -(x+1)\sum\limits_{k=0}^{\infty}\frac{[\pi(x+1)]^{2k+1}}{(2k+1)!} + 2\sum\limits_{k=0}^{\infty}\frac{[\pi(x+1)]^{2k+1}}{(2k+1)!}$
$= \sum\limits_{k=0}^{\infty}\frac{2\pi^{2k+1}}{(2k+1)!}(x-(-1))^{2k+1} - \sum\limits_{k=0}^{\infty}\frac{\pi^{2k+1}}{(2k+1)!}(x-(-1))^{2k+2},\ x\in I\!R.$

Example.
Expand $f(x)=\ln(1+x)$ into a series with center $x_0=0$:
$\ln(1+x) = \int\frac{dx}{x+1} = \int\frac{1}{1-(-x)}\,dx = \langle\!\langle y=-x,\ |y|<1 \rangle\!\rangle = \int\sum\limits_{k=0}^{\infty}(-x)^k\,dx = \sum\limits_{k=0}^{\infty}\int(-1)^k x^k\,dx$
$= \sum\limits_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1} + C = \langle\!\langle k \mapsto k^* \rangle\!\rangle = \sum\limits_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k} + C;$
How much is C? Substitute $x=0$: $\ln(1+0)=\sum 0 + C$, hence $C=0$.
Thus $\ln(1+x) = \sum\limits_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k},\ |x|<1.$
This works also for $x=1$, but not for $x=-1$.

Example.
Expand $f(x)=\frac{1}{(2x-5)^2}$ into a series with center $x_0=1$:
$\frac{1}{(2x-5)^2} = \left[-\frac12\,\frac{1}{2x-5}\right]' = -\frac12\left[\frac{1}{2(x-1)-3}\right]' = \frac16\left[\frac{1}{1-\frac23(x-1)}\right]' = \langle\!\langle y=\frac23(x-1),\ |y|<1 \rangle\!\rangle$
$= \frac16\Big[\sum\limits_{k=0}^{\infty}\big(\frac23(x-1)\big)^k\Big]' = \frac16\sum\limits_{k=0}^{\infty}\big(\frac23\big)^k[(x-1)^k]' = \sum\limits_{k=1}^{\infty}\frac16\big(\frac23\big)^k k(x-1)^{k-1} = \langle\!\langle k-1 \mapsto k^* \rangle\!\rangle$
$= \sum\limits_{k=0}^{\infty}\frac{2^k}{3^{k+2}}(k+1)(x-1)^k,\ |x-1|<\frac32.$

Series of complex numbers.
Complex ∞, $\frac{1}{\infty}=0$, $\frac{1}{0}=\infty$, e^∞ DNE!
Neighborhood in complex plane: $U_\varepsilon(z_0)=\{z\in\mathbb{C};\ |z-z_0|<\varepsilon\}$, $U_\varepsilon(\infty)=\{z\in\mathbb{C};\ |z|>1/\varepsilon\}$.
Definitions that are done using neighborhoods are the same in $\mathbb{R}$ and in $\mathbb{C}$ (limit, sum of a series). Therefore also theorems that use neighborhoods and do not use comparison between terms (but they can use comparison of absolute values of terms) are valid in complex case, for instance the ratio and root tests of convergence when applied to absolute convergence.
Power series then work the same, including the fact that if r is the ratio of convergence of a power series, then the series converges absolutely on $U_r(z_0)=\{z\in\mathbb{C};\ |z-z_0|<r\}$.
Besides using absolute value we can also use another approach.

Theorem.
Let a_k be complex numbers.
Then $a_k\to A$ if and only if $\mathrm{Re}(a_k)\to\mathrm{Re}(A)$ and $\mathrm{Im}(a_k)\to\mathrm{Im}(A)$.
Similarly a series $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if $\sum\limits_{k=n_0}^{\infty}\mathrm{Re}(a_k)$ and $\sum\limits_{k=n_0}^{\infty}\mathrm{Im}(a_k)$ converge.
Then also $\sum\limits_{k=n_0}^{\infty} a_k=\Big(\sum\limits_{k=n_0}^{\infty}\mathrm{Re}(a_k)\Big)+j\Big(\sum\limits_{k=n_0}^{\infty}\mathrm{Im}(a_k)\Big)$.
Thus we can do a lot of things using this decomposition.