EMA2: Lecture contents, week 6

2. Laplace transform

Definition.
For $f\colon [0,\infty) \mapsto I\!R$ we define its **Laplace transform** $\mathcal{L}\{f(t)\}$ by

$$\mathcal{L}\{f(t)\}\colon p \mapsto \int\limits_0^\infty f(t)e^{-pt}dt,$$

assuming that the integral converges for at least one p.
Notation: $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{f\}$, F, alternative $f(t) \,\widehat{=}\, F(p)$.

Example.

$\mathcal{L}$ of $f(t) = e^{\alpha t}$ for $t \ge 0$ is the function F given by formula $F(p) = \int\limits_0^\infty e^{\alpha t}e^{-pt}dt = \dfrac{1}{p-\alpha}$ for $p > \alpha$.

If we want to apply $\mathcal{L}$ to functions f defined on a larger set, for instance on $I\!R$, then we will consider them equal to zero for $t < 0$.

Definition.
Heaviside function is defined $H(t) = \begin{cases} 1, & t \ge 0; \\ 0, & t < 0. \end{cases}$

Fact.
Let f be a function on $I\!R$, $a \in I\!R$. Then $f(t)H(t-a) = \begin{cases} f(t), & t \ge a; \\ 0, & t < a. \end{cases}$

Notation: If f is a function defined by a formula and we write $\mathcal{L}\{f(t)\}$, then by this we automatically understand $\mathcal{L}\{f(t)H(t)\}$.

Example.
The previous example can be written as $\mathcal{L}\{e^{\alpha t}\} = \frac{1}{p-\alpha}$ or for instance $e^{\alpha t} \,\widehat{=}\, \frac{1}{p-\alpha}$.

Example.
$\mathcal{L}\{e^{t^2}\} = \mathcal{L}\{e^{t^2}H(t)\}$ DNE.

Definition.
We say that a function f is **piecewise continuous** on an interval I if there are $x_0 < x_1 < \ldots \in \overline{I}$ such that $\{x_k\}$ is either finite or a sequence going to infinity as $k \to \infty$, $\overline{I} = \bigcup [x_{k-1}, x_k]$ and for every $k = 1, 2, \ldots$ the function f is continuous on (x_{k-1}, x_k) and it has one-sided limits $f({x_{k-1}}^+)$, $f({x_k}^-)$.
We say that a function f is of **at most exponential growth** if $\exists \alpha, M > 0$ such that $\forall t$: $|f(t)| \le Me^{\alpha t}$.

Definition.
We define the space $\mathcal{L}_0$ by
$\mathcal{L}_0 = \{f\colon [0,\infty) \mapsto I\!R;\ f$ is of at most exponential growth and piecewise continuous on $[0,\infty)\}$.

Theorem.
If $f \in \mathcal{L}_0$ then $\mathcal{L}\{f\}$ exists on some (p_f, ∞).
Moreover, $\lim\limits_{p\to\infty} \big(\mathcal{L}\{f\}(p)\big) = 0$.

The space $\mathcal{L}_0$ contains for instance $e^{\alpha t}$, t^n for $n \geq 0$ and all (piecewise) continuous functions are there as well. For most functions we find their Laplace transform algoritmically.

2.1. Calculating Laplace transform

Theorem. (**dictionary**)
(i) $\forall \alpha \in I\!R$: $e^{\alpha t} \in \mathcal{L}_0$ and $\mathcal{L}\{e^{\alpha t}\} = \frac{1}{p-\alpha}$, $p > \alpha$;
(ii) $\forall n \in I\!N_0$: $t^n \in \mathcal{L}_0$ and $\mathcal{L}\{t^n\} = \frac{n!}{p^{n+1}}$, $p > 0$;
(iii) $\forall \omega \in I\!R$: $\sin(\omega t) \in \mathcal{L}_0$ and $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{p^2+\omega^2}$, $p \in I\!R$;
(iv) $\forall \omega \in I\!R$: $\cos(\omega t) \in \mathcal{L}_0$ and $\mathcal{L}\{\cos(\omega t)\} = \frac{p}{p^2+\omega^2}$, $p \in I\!R$.

Theorem. (**linearity**)
Let $f, g \in \mathcal{L}_0$. Then $\forall a, b \in I\!R$: $af + bg \in \mathcal{L}_0$ and $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$.

Theorem. (**grammar**)
Let $f \in \mathcal{L}_0$. Then the following are true:
(i) (change of scale) $\forall a > 0$: $f(at) \in \mathcal{L}_0$ and
$$\mathcal{L}\{f(at)\} = \tfrac{1}{a}\mathcal{L}\{f(t)\}\big|_{p/a};$$
(ii) (shift in image) $\forall a \in I\!R$: $e^{at}f(t) \in \mathcal{L}_0$ and
$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}\big|_{p-a};$$
(iii) (shift in preimage) $\forall a > 0$: $f(t-a)H(t-a) \in \mathcal{L}_0$ and
$$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-ap}\mathcal{L}\{f(t)H(t)\};$$
(iv) (derivative of image) $\forall n \in I\!N$: $t^n f(t) \in \mathcal{L}_0$ and
$$\mathcal{L}\{t^n f(t)\} = (-1)^n \tfrac{\mathrm{d}^n}{\mathrm{d}p^n}\mathcal{L}\{f(t)\};$$
(v) (integration of image) If $\lim\limits_{t\to 0^+}\left(\frac{f(t)}{t}\right)$ converges, then $\frac{f(t)}{t} \in \mathcal{L}_0$ and
$$\mathcal{L}\left\{\tfrac{1}{t}f(t)\right\} = \int\limits_p^\infty \mathcal{L}\{f(t)\}(q)\,dq.$$
(vi) (derivative of preimage) If $f^{(n)} \in \mathcal{L}_0$, then
$$\mathcal{L}\{f^{(n)}(t)\} = p^n\mathcal{L}\{f(t)\}(p) - p^{n-1}f(0^+) - p^{n-2}f'(0^+) - \ldots - pf^{(n-2)}(0^+) - f^{(n-1)}(0^+);$$
(vii) (integration of preimage) $\int\limits_0^t f(s)\,ds \in \mathcal{L}_0$ and
$$\mathcal{L}\left\{\int_0^t f(s)\,ds\right\} = \tfrac{1}{p}\mathcal{L}\{f(t)\}.$$

Remark: Instead of (iii) we usually prefer $\mathcal{L}\{f(t)H(t-a)\} = e^{-ap}\mathcal{L}\{f(t+a)H(t)\}$.

Example.
$\mathcal{L}\{t\,e^{3t}\} = -[\mathcal{L}\{e^{3t}\}]' = -\left[\frac{1}{p-3}\right]' = \frac{1}{(p-3)^2}$.
$\mathcal{L}\{t\,e^{3t}\} = \mathcal{L}\{e^{3t}t\} = \mathcal{L}\{t\}\big|_{p-3} = \frac{1}{p^2}\big|_{p-3} = \frac{1}{(p-3)^2}$.
$\mathcal{L}\left\{\frac{\sin(t)}{t}\right\} = \int_p^\infty \mathcal{L}\{\sin(t)\}(q)\,dq = \int_p^\infty \frac{1}{q^2+1}\,dq = \big[\arctan(q)\big]_p^\infty = \frac{\pi}{2} - \arctan(p)$.
Remark: $\frac{\cos(t)}{t} \notin \mathcal{L}_0$.
$\mathcal{L}\left\{\sin(2t)H\left(t-\frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}p}\mathcal{L}\left\{\sin\left(2\left(t+\frac{\pi}{2}\right)\right)\right\} = e^{-\frac{\pi}{2}p}\mathcal{L}\{\sin(2t+\pi)\}$
$\quad = e^{-\frac{\pi}{2}p}\mathcal{L}\{-\sin(2t)\} = -\frac{2e^{-\frac{\pi}{2}p}}{p^2+4}$.

Definition.
By a **finite impuls** we mean any function defined on $[0, \infty)$ that is non-zero only on some bounded closed interval.

Definition.
Let M be a subset of $\mathbb{R}$. We define its **characteristic function** $\chi_M = \begin{cases} 1, & x \in M; \\ 0, & x \notin M. \end{cases}$

Fact.
Let M be a subset of $\mathbb{R}$, f a function on $\mathbb{R}$. Then $f(t)\chi_M = \begin{cases} f(t), & t \in M; \\ 0, & t \notin M. \end{cases}$

Fact.
Let $a < b \in \mathbb{R}$. Then $\chi_{[a,b)} = H(t-a) - H(t-b)$.

Example.
Laplace transform of one hill of sine of $2t$:
$\mathcal{L}\big\{\sin(2t)\big[H(t) - H\big(t - \frac{\pi}{2}\big)\big]\big\} = \mathcal{L}\{\sin(2t)\} - \mathcal{L}\big\{\sin(2t)H\big(t - \frac{\pi}{2}\big)\big]\big\} = \frac{2}{p^2+4} + \frac{2e^{-\frac{\pi}{2}p}}{p^2+4}.$