Definitions and statements—series of functions and Fourier series

• Series of functions.

Definition. By a **sequence of functions** we mean an ordered set $\{f_k\}_{k=n_0}^{\infty} = \{f_{n_0}, f_{n_0+1}, f_{n_0+2}, \dots\}$, where f_k are functions.

Remark: Given a sequence of functions $\{f_k\}_{k=n_0}^{\infty}$ and $x \in \bigcap D(f_k)$, then $\{f_k(x)\}$ is a standard sequence of real (complex) numbers.

Definition. Let $\{f_k\}_{k\geq n_0}$, f be functions on a set M.
We say that $\{f_k\}$ **converges (pointwise)** to f on M, denoted $f_k \to f$ or $f = \lim\limits_{k\to\infty}(f_k)$,
if $\forall x \in M$: $\lim\limits_{k\to\infty}\big(f_k(x)\big) = f(x)$.
We say that $\{f_k\}$ **converges uniformly** to f on M, denoted $f_k \rightrightarrows f$,
if $\forall \varepsilon > 0 \exists N_0 \in \mathbb{N}$ such that $\forall k \geq N_0 \forall x \in M : |f(x) - f_k(x)| < \varepsilon$.

Theorem. Let $f_k \rightrightarrows f$ on M.
(i) If all f_k are continuous on M, then also f is continuous there.
(ii) If all f_k have a derivative on M, then also f has it there and $f' = \lim\limits_{k\to\infty}(f_k')$ on M.
(iii) If all f_k have antiderivative on M, then also f has it there and $\int_{x_0}^{x} f\,dx = \lim\limits_{k\to\infty}\big(\int_{x_0}^{x} f_k\,dx\big)$ for $\overline{x_0, x} \subseteq M$.

Definition. A **series of functions** is a symbol $\sum\limits_{k=n_0}^{\infty} f_k = f_{n_0} + f_{n_0+1} + f_{n_0+2} + \dots$, where f_k are functions.

Remark: Given a series of functions $\sum f_k$ and $x \in \bigcap D(f_k)$, then $\sum f_k(x)$ is a standard series of real (complex) numbers.

Definition. Consider a series of functions $\sum\limits_{k=n_0}^{\infty} f_k$.
The **region of convergence** of this series is the set $\big\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges}\big\}$. By defining $f(x) = \sum\limits_{k=n_0}^{\infty} f_k(x)$ we then obtain a function f on this set called the **sum of the series**, denoted $\sum\limits_{k=n_0}^{\infty} f_k = f$.
The **region of absolute convergence** of this series is the set $\big\{x \in \bigcap D(f_k);\ \sum f_k(x) \text{ converges absolutely}\big\}$.
We say that this series **converges uniformly** to f on M, denoted $\sum f_k \rightrightarrows f$ on M, if the sequence of partial sums $\Big\{\sum\limits_{k=n_0}^{N} f_k(x)\Big\}$ converges uniformly to f on M.

Theorem. Consider series of functions $\sum f_k$ and $\sum g_k$.
If $\sum\limits_{k=n_0}^{\infty} f_k = f$ on M and $\sum\limits_{k=n_0}^{\infty} g_k = g$ on M, then $\forall a, b \in \mathbb{R}$: $\sum\limits_{k=n_0}^{\infty} (af_k + bg_k) = af + bg$ on M.

Theorem. (Weierstrass criterion)
Let f_k for $k \geq n_0$ be functions on M. Let $a_k \geq 0$ satisfy $\forall x \in M \forall k \geq n_0$: $|f_k(x)| \leq a_k$.
If $\sum a_k$ converges, then $\sum f_k$ converges uniformly on M.

Theorem. Let $\sum f_k \rightrightarrows f$ on M.
(i) If all f_k are continuous on M, then also f is continuous there.
(ii) If all f_k have a derivative on M, then also f has it there and $f' = \sum\limits_{k=n_0}^{\infty} f_k'$ on M.
(iii) If all f_k have an antideriative on M, then also f has it there and $\int_{x_0}^{x} f\,dx = \sum\limits_{k=n_0}^{\infty} \int_{x_0}^{x} f_k\,dx$ for $\overline{x_0, x} \subseteq M$.

• Power series.

Definition. Let $z_0 \in \mathbb{R}$.
By a **power series with center** x_0 we mean any series of functions of the form $\sum\limits_{k=0}^{\infty} a_k(x - x_0)^k$, where $a_k \in \mathbb{R}$.

Theorem. Consider a power series $\sum\limits_{k=0}^{\infty} a_k(x - x_0)^k$.
There exists $r \in \mathbb{R}_0^+ \cup \{\infty\}$ such that $\sum\limits_{k=0}^{\infty} a_k(x - x_0)^k$ converges absolutely on $U_r(x_0) = (x_0 - r, x_0 + r)$ and diverges for $|x - x_0| > r$.
Moreover, $r = \dfrac{1}{\limsup\limits_{k\to\infty}(\sqrt[k]{|a_k|})}$.

Remark: We also have $r = \dfrac{1}{\lim\limits_{k\to\infty}\left(\frac{|a_{k+1}|}{|a_k|}\right)}$, assuming that this limit exists.

Remark: A power series always converges (absolutely) at $x = x_0$.

Definition. Consider a power series $\sum\limits_{k=0}^{\infty} a_k(x - x_0)^k$.
The number r with properties as in the previous theorem is called the **radius of convergence** of this series.

Theorem. Let $x_0 \in I\!R$, assume that $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k = f$, $\sum\limits_{k=0}^{\infty} b_k(x-x_0)^k = g$ have radii of convergence r_f and r_g.

(i) Then $\forall a,b \in I\!R$: $\sum\limits_{k=0}^{\infty} (aa_k + bb_k)(x-x_0)^k = af + bg$ has radius of convergence $r = \min(r_f, r_g)$.

(ii) The series $\sum\limits_{k=0}^{\infty} \Big(\sum\limits_{i=0}^{k} a_i b_{k-i}\Big)(x-x_0)^k = \Big(\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k\Big) \cdot \Big(\sum\limits_{k=0}^{\infty} b_k(x-x_0)^k\Big) = f \cdot g$ has radius of convergence $r = \min(r_f, r_g)$.

Theorem. Let $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k = f$ have radius of convergence $r > 0$.

(i) For any $\varrho \in (0,r)$: $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k \rightrightarrows f$ on $U_\varrho(x_0)$.

(ii) f is continuous, it has the derivative $f'(x) = \sum\limits_{k=1}^{\infty} k\, a_k(x-x_0)^{k-1}$ with radius of convergence r and an antiderivative $F(x) = \sum\limits_{k=0}^{\infty} \frac{a_k}{k+1}(x-x_0)^{k+1}$ with radius of convergence r.

Corollary. (uniqueness of expansion) Let $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$.

Then on $U_r(x_0)$ we have for $n \in I\!N$ also derivatives $f^{(n)}(x) = \sum\limits_{k=n}^{\infty} k(k-1)\cdot\ldots\cdot(k-n+1)\, a_k(x-x_0)^{k-n}$.

From this we get $a_n = \frac{f^{(n)}(x_0)}{n!}$ for $n \in I\!N_0$.

Definition. Let f have derivatives of all orders at x_0.

We define the **Taylor series** of f with center at x_0 by the formula $\sum\limits_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

The process of finding this series is called **expanding a given function into a power/Taylor series (with center x_0)**.

Corollary. If $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$, then this series is necessarily the Taylor series.

Theorem. Let a function f have derivatives of all orders on some $U_r(x_0)$ and let $\exists M > 0$ such that $|f^{(k)}(x_0)| \leq M$ for all $k \in I\!N_0$ and $x \in U_r(x_0)$. Then for $x \in U_r(x_0)$ we have $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$.

Fact.

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + x^4 + \ldots, \qquad x \in (-1,1);$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots, \qquad x \in I\!R;$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \qquad x \in I\!R;$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \qquad x \in I\!R;$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \ldots, \qquad x \in (-1,1];$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-3)}{3!}x^3 + \ldots, \qquad x \in (-1,1).$$

Here $\binom{\alpha}{k} = \frac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-k+2)\cdot(\alpha-k+1)}{k!}$.

Theorem. Let a_k be complex numbers. A series $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if the series $\sum\limits_{k=n_0}^{\infty} \mathrm{Re}(a_k)$ and $\sum\limits_{k=n_0}^{\infty} \mathrm{Im}(a_k)$ converge. Then also $\sum\limits_{k=n_0}^{\infty} a_k = \Big(\sum\limits_{k=n_0}^{\infty} \mathrm{Re}(a_k)\Big) + j\Big(\sum\limits_{k=n_0}^{\infty} \mathrm{Im}(a_k)\Big)$.

Corollary. All theorems that do not use comparison by inequality between terms of series are also true for complex series.

Remark: Power series are also investigated in the setting of complex numbers and everything works the same. For instance, if r is the radius of convergence of a complex power series, then the series converges absolutely on $U_r(z_0) = \{z \in C\!\!\!\!I;\ |z-z_0| < r\}$.

• **Fourier series.**

Definition. By a **trigonometric series** we mean any series of the form $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

By a **trigonometric polynomial** of degree N we mean $\frac{a_0}{2}+\sum\limits_{k=1}^{N}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

Remark: These are special cases of **Fourier series**.

Fact. Functions $\sin(k\omega t)$, $\cos(k\omega t)$ are periodic with period $T=\frac{2\pi}{\omega}$.
Therefore also trigonometric polynomials and trigonometric series (if it converges) are T-periodic.

Theorem. Let $\omega>0$, $T=\frac{2\pi}{\omega}$. The following are true:

(i) $\int\limits_0^T \sin^2(k\omega t)\,dt=\int\limits_0^T\cos^2(k\omega t)\,dt=\frac{T}{2}$ for $k\in\mathbb{N}$,

(ii) $\int\limits_0^T \sin(k\omega t)\sin(m\omega t)\,dt=\int\limits_0^T\cos(k\omega t)\cos(m\omega t)\,dt=0$ for $k\neq m\in\mathbb{N}$,

(iii) $\int\limits_0^T \sin(k\omega t)\cos(m\omega t)\,dt=0$ for $k,m\in\mathbb{N}$.

Theorem. Let f be a T-periodic function, denote $\omega=\frac{2\pi}{T}$.
If $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]\rightrightarrows f$ on $[0,T]$, then $a_k=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt$ for $k\in\mathbb{N}_0$ and $b_k=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt$ for $k\in\mathbb{N}$.

Definition. Let f be a function that is T-periodic and integrable on $[0,T]$.
We define its **Fourier series** as $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$, where $a_k=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt$ for $k\in\mathbb{N}_0$ and $b_k=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt$ for $k\in\mathbb{N}$.

We denote $f\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$.

Remark: Also $a_k=\frac{2}{T}\int\limits_{-T/2}^{T/2} f(t)\cos(k\omega t)\,dt$ for $k\in\mathbb{N}_0$ and $b_k=\frac{2}{T}\int\limits_{-T/2}^{T/2} f(t)\sin(k\omega t)\,dt$ for $k\in\mathbb{N}$.

Definition. Let f be a function defined on an interval $I=[a,a+T)$ for some $a\in\mathbb{R}$, $T>0$.
We define its **periodic extension on $\mathbb{R}$** as the function $f(t)=f(t-kT)$ for $t\in[a+kT,a+(k+1)T)$.

Definition. Let f be a function defined on an interval $I=[a,a+T)$ for some $a\in\mathbb{R}$, $T>0$.
We define its Fourier series as the Fourier series of its periodic extension.

Theorem. (**Jordan criterion**) Let f be a T-periodic function that is piecewise continuous on some interval I of length T, assume that it has a derivative f' piecewise continuous on I.
Let $f\sim\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]$. Then for every $t\in\mathbb{R}$ we have

$$\lim_{N\to\infty}\Bigl(\frac{a_0}{2}+\sum_{k=1}^{N}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]\Bigr)=\tfrac{1}{2}[f(t^-)+f(t^+)].$$

If moreover f is continuous on $\mathbb{R}$, then $\frac{a_0}{2}+\sum\limits_{k=1}^{\infty}\bigl[a_k\cos(k\omega t)+b_k\sin(k\omega t)\bigr]\rightrightarrows f$.

Theorem. Let f be a T-periodic function that is integrable on $[0,T)$, let $\omega=\frac{2\pi}{T}$.

(i) If f is odd, then $a_k=0$ and $b_k=\frac{4}{T}\int\limits_0^{T/2} f(t)\sin(k\omega t)\,dt$.

(ii) If f is even, then $b_k=0$ and $a_k=\frac{4}{T}\int\limits_0^{T/2} f(t)\cos(k\omega t)\,dt$.

Definition. Let f be a function defined and continuous on $[0,T)$.
We define its **sine series** by $a_k=0$, $b_k=\frac{2}{T}\int\limits_0^T f(t)\sin(k\omega t)\,dt$ for $\omega=\frac{\pi}{T}$.

We define its **cosine series** by $b_k=0$, $a_k=\frac{2}{T}\int\limits_0^T f(t)\cos(k\omega t)\,dt$ for $\omega=\frac{\pi}{T}$.

Remark: The sum of the sine series is a $2T$-periodic extension of f into an odd function. The sum of the cosine series is a $2T$-periodic extension of f into an even function.

Theorem. Let f be a T-periodic function that is piecewise continuous on $[0,T)$ and it has a piecewise continuous derivative on $[0,T)$. Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$.

(i) Then $f' \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[-a_k \sin(k\omega t)k\omega + b_k \cos(k\omega t)k\omega\big] = \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[b_k k\omega \cos(k\omega t) - a_k k\omega \sin(k\omega t)\big]$.

(ii) If $\int\limits_0^T f(u)\,du = 0$ (that is, $a_0 = 0$), then

$$F(t) = \int\limits_0^t f(u)\,du \sim \sum_{k=1}^{\infty} \big[a_k \sin(k\omega t)\tfrac{1}{k\omega} - b_k \cos(k\omega t)\tfrac{1}{k\omega}\big] = \sum_{k=1}^{\infty} \big[\tfrac{-b_k}{k\omega} \cos(k\omega t) + \tfrac{a_k}{k\omega} \sin(k\omega t)\big].$$

Definition. Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$. Denote $A_k = \sqrt{a_k^2 + b_k^2}$, find φ_k so that $b_k = A_k \cos(\varphi_k)$ and $a_k = A_k \sin(\varphi_k)$.

The series $\frac{a_0}{2} + \sum\limits_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k)$ is called the **Fourier series in amplitude-phase form**.

Denote $c_0 = \frac{a_0}{2}$ and $c_k = \frac{1}{2}(a_k - j\,b_k)$, $c_{-k} = \frac{1}{2}(a_k + j\,b_k)$ for $k \in \mathbb{N}$.

The series $\sum\limits_{k=-\infty}^{\infty} c_k e^{jk\omega t}$ is called the **Fourier series in complex form**.

Fact. We have $c_k = \frac{1}{T}\int\limits_0^T f(t)e^{-jk\omega t}\,dt$.