# Definitions and statements—ODE

## • Ordinary differential equations.

**Definition.**
**Ordinary differential equation of order** $n$ (ODE) is any equation of the form $F(x,y,y',\dots,y^{(n)})=0$, where $F$ is a function of $n+2$ variables in which $y^{(n)}$ really appears.
Its **solution on an (open) interval** $I$ is any function $y=y(x)$ on interval $I$ that has all derivatives up to order $n$ on $I$ and $\forall x\in I$ satisfies $F(x,y(x),y'(x),\dots,y^{(n)}(x))=0$.
Let $y,v$ be some solutions of a given ODE on open intervals $I$, $J$ respectively. We say that $v$ is an **extension** of $y$ if $I\subset J$, $I\neq J$ and $v=y$ on $I$.
We say that a solution $y$ of a given ODR on some open interval is a **maximal solution** if it cannot be extended, that is, if there is no other solution that would be an extention of $y$.
**Definition.** If the set of all solutions of a given ODE on a certain open interval can be expressed using one formula with parameters, we say that it is a **general solution** of this ODE. One solution, obtained by a concrete choice of these parmeters, is then called a **particular solution**.
**Definition.** Consider an ODE of order $n$ $\quad F(x,y,y',\dots,y^{(n)})=0$.
**Cauchy problem** or **Initial Value Problem** for this equation is a problem of the form
(1) $F(x,y,y',\dots,y^{(n)})=0$;
(2) $y(x_0)=y_0$, $y'(x_0)=y_1$, $\dots$, $y^{(n-1)}(x_0)=y_{n-1}$, where $x_0$, $y_0$, $y_1$, $\dots$, $y_{n-1}$ are some fixed real numbers (**initial conditions**).
**Definition.**
Consider a Cauchy problem with initial conditions at some $x_0$.
We say that this problem is **uniquely solvable** if for every two solutions $u,v$ of this problem there exists a neighborhood $U$ of $x_0$ such that $u=v$ on $U$.
We say that this problem is **uniquely solvable on an interval** $I$ (containing $x_0$) if for any two solutions $u,v$ of this problem on intervals $I_u,I_v$ containing $I$ one has that $u=v$ on $I$.

## • ODE of order 1

**Theorem.** (on **existence and uniqueness** for ODE of order 1 with isolated $y$)
Consider an ODE that can be written in the form $y'=f(x,y)$. (O1)
Let $I,J$ be open intervals such that $f$ is continuous on the set $I\times J$. Then $\forall(x_0,y_0)\in I\times J$ there exists a solution of the Cauchy problem (O1), $y(x_0)=y_0$ on some neighborhood of $x_0$ and this solution can be extended to the boundary of $I\times J$.
If moreover $\frac{\partial f}{\partial y}$ is continuous or bounded on $I\times J$, then this solution is unique.
**Definition.**
By a **separable ODE of order 1** we mean any ODE that can be expressed in the form $y'=g(x)h(y)$ for some functions $g,h$.
**Fact.**
Consider a separable ODE of order 1, $y'=g(x)h(y)$. If $y_0$ satisfies $h(y_0)=0$ and $I$ is an open interval satisfying $I\subset D(g)$, then the function $y(x)=y_0$ is a solution to the given ODE on $I$ (so-called **stationary solution**).
**Theorem.**
Consider a separable ODE of order 1, $y'=g(x)h(y)$. Consider open intervals $I\subset D(g)$ and $J\subset D(h)$. If $g$ is continuous on $I$, $h$ is continuous on $J$ and $h\neq 0$ on $J$, then there is a solution to the given equation on $I$.
Let $G(x)$ be an antiderivative of $g(x)$ on $I$ and $H(y)$ be an antiderivative of $\frac{1}{h(y)}$ on $J$. If $H$ has an inverse function $H_{-1}$, then a general solution of the given equation on $I$ can be expressed as $y(x)=H_{-1}(G(x)+C)$.

## • Linear ODE of order 1

**Definition.**
By a **linear ODE of order 1** we mean any ODE that can be written in the form $y'+a(x)y=b(x)$, where $a,b$ are some functions.
This equation is called **homogeneous** if $b(x)=0$.
Given an ODE $y'+a(x)y=b(x)$, we define its **associated homogeneous equation** $y'+a(x)y=0$.
**Fact.** If $a,b$ are continuous on an open interval $I$, then $\forall x_0\in I$ and $\forall y_0\in I\!R$ there exists a solution of the Cauchy problem $y'+a(x)y=b(x)$, $y(x_0)=y_0$ on $I$ and it is unique there.
**Theorem.** (on **solution** of homogeneous linear ODE of order 1)
If $a$ is continuous on an open interval $I$, then the equation $y'+a(x)y=0$ has a solution on $I$ in the form $y(x)=C\,e^{-A(x)}$, where $A$ is some antiderivative of $a$ on $I$.
**Theorem.** (on **solution** of a linear ODE of order 1)
If $a,b$ are continuous on an open interval $I$, then the equation $y'+a(x)y=b(x)$ has a solution on $I$ in the form $\Bigl(\int b(x)e^{A(x)}dx\Bigr)e^{-A(x)}$, where $A$ is some antiderivative of $a$ on $I$.
If $B$ is some antiderivative of $b(x)e^{A(x)}$ on $I$, then a general solution of this equation on $I$ is

$$y(x)=(B(x)+C)\,e^{-A(x)}.$$

**Theorem.** (on **structure of solutions** of linear ODE of order 1)
Let $y_p$ be some particular solution of the equation $y' + a(x)y = b(x)$ on an open interval $I$. Then the set of all solutions of this equation on $I$ is

$$\{y_p + y_h;\ y_h \text{ is a solution of the associated homogeneous equation on } I\}.$$

**• Linear ODE of order $n$.**

**Definition.** By a **linear ODE of order** $n$ we mean any ODE that can be written in the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x),$$

where $a_{n-1}, \ldots, a_0, b$ are some functions.
This equation is called **homogeneous** if $b(x) = 0$.
Given a linear ODE $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y = b(x)$, we define its **associated homogeneous equation** as $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x) = 0$.

**Theorem.** (on **existence and uniqueness** for linear ODE)
Consider an equation $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$. (L)
If $a_{n-1}, \ldots, a_0, b$ are continuous on an open interval $I$, then $\forall x_0 \in I$ and $\forall y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$ there exists a solution to the Cauchy problem (L), $y(x_0) = y_0$, $y'(x_0) = y_1$, $\ldots$, $y^{(n-1)}(x_0) = y_{n-1}$ on $I$ and it is unique there.

**Theorem.** (on **structure of solutions** of linear ODE)
Let $y_p$ be some particular solution of a given linear ODE on an open interval $I$. Then the set of all solutions of this equation is

$$\{y_p + y_h;\ y_h \text{ is a solution of the associated homogeneous equation on } I\}.$$

**Theorem.** (on **structure of solutions** of homogeneous linear ODE)
Consider a homogeneous linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$. If $a_i$ are continuous on an open interval $I$, then the set of all solutions of this equation on $I$ is a linear space of dimension $n$.

**Definition.**
Consider a linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = b(x)$. Assume that $a_i$ are continuous on an open interval $I$. By a **fundamental system** of this equation on $I$ we mean an arbitrary basis of the space of solutions of its associated homogeneous equation.

**Definition.**
Let $y_1$, $y_2$, $\ldots$, $y_n$ be $(n-1)$-times differentiable functions. We define their **Wronskian** as

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}.$$

**Theorem.** Consider a homogeneous linear ODE $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0$.
Let $a_i$ be continuous on an open interval $I$. Let $y_1$, $y_2$, $\ldots$, $y_n$ be solutions of this equation on $I$, let $W$ be their Wronskian.
These functions form a linearly independent set (and thus a fundamental system) if and only if $W(x) \neq 0$ on $I$ if and only if $\exists x_0 \in I$: $W(x_0) \neq 0$.

**Theorem.** (**principle of superposition**)
Consider a linear ODE with left hand-side $L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y$. Let $y_1$ be a solution of $L(y) = b_1(x)$ on an open interval $I$ and $y_2$ be a solution of $L(y) = b_2(x)$ on $I$. Then $y_1 + y_2$ is a solution of $L(y) = b_1(x) + b_2(x)$ on $I$.

**Definition.** By a **linear ODE with constant coefficients** we mean any linear ODE for which $a_0(x) = a_0$, $a_1(x) = a_1$, $\ldots$, $a_{n-1}(x) = a_{n-1}$ are constant functions.

**Definition.**
Consider a linear ODE with constant coefficients $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = b(x)$.
We define its **characteristic polynomial** by $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0$.
We define its **characteristic equation** as $p(\lambda) = 0$. The solutions of this eqution are called **characteristic numbers** of the given ODE.

**Theorem.** (on fundamental system for linear ODE with constant coefficients)
Consider a linear ODE with constant coefficients $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = b(x)$. Let $\lambda$ be its characteristic number of multiplicity $m$.
(1) If $\lambda = \alpha \in \mathbb{R}$, then $e^{\alpha x}$, $x\,e^{\alpha x}$, $\ldots$, $x^{m-1}e^{\alpha x}$ are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.
(2) If $\lambda = \alpha \pm \beta j \in \mathbb{C}$, $\beta \neq 0$, then $e^{\alpha x}\sin(\beta x)$, $x\,e^{\alpha x}\sin(\beta x)$, $\ldots$, $x^{m-1}e^{\alpha x}\sin(\beta x)$, $e^{\alpha x}\cos(\beta x)$, $x\,e^{\alpha x}\cos(\beta x)$, $\ldots$, $x^{m-1}e^{\alpha x}\cos(\beta x)$ are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.
(3) The set of functions from (1) and (2) for all characteristic numbers is linearly independent and it forms a fundamental system of the given equation on $\mathbb{R}$.

**Theorem.** (on **guessing solution** for special right hand-side)
Consider a linear ODE with constant coefficients $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = b(x)$. Assume that $b(x) = e^{\alpha x}[P(x)\sin(\beta x) + Q(x)\cos(\beta x)]$ for some polynomials $P, Q$ with $m = \max(\deg(P), \deg(q))$. Let $k$ be the multiplicity of the number $\alpha \pm \beta j$ as a characteristic number of the given equation (we put $k = 0$ if it is not a char. no. at all).
Then there are polynomials $\widetilde{P}, \widetilde{Q}$ of order at most $m$ such that $y(x) = x^k e^{\alpha x}[\widetilde{P}(x)\sin(\beta x) + \widetilde{Q}(x)\cos(\beta x)]$ is a solution of the given equation on $I\!R$.

• **Systems of linear ODE's with constant coefficients.**

**Definition.** By a **system of linear ODE's of order 1 with constant coefficients** we mean a system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n + b_1(x)\\ y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n + b_2(x)\\ &\vdots\\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n + b_n(x) \end{aligned}$$

where $b_i(x)$ are right hand-sides, $a_{ij} \in I\!R$.
A Cauchy problem for such a system has initial conditions $y_1(x_0) = y_{10}$, $y_2(x_0) = y_{20}$, ... , $y_n(x_0) = y_{n0}$.
**Fact.**
Every system of $n$ linear ODE's of order 1 can be transformed via elimination to one linear ODE of order $n$, and vice versa.
**Theorem.** (on **existence and uniqueness** for systems)
Consider a system as in the definition above. If $b_i(x)$ are continuous on an open interval $I$, then for every $x_0 \in I$ and all $y_{10}, y_{20}, \ldots, y_{n0} \in I\!R$ there exists a solution of the corresponding Cauchy problem on $I$ and it is unique.

Matrix notation:

$$\vec{y}(x) = \begin{pmatrix} y_1(x)\\ \vdots\\ y_n(x)\end{pmatrix},\ \vec{y}\,' = \begin{pmatrix} y_1'\\ \vdots\\ y_n'\end{pmatrix},\ \textbf{matrix of the system } A = \begin{pmatrix} a_{11} & \ldots & a_{1n}\\ \vdots & & \vdots\\ a_{n1} & \ldots & a_{nn}\end{pmatrix},\ \text{we also use } \vec{0} = \begin{pmatrix} 0\\ \vdots\\ 0\end{pmatrix}_{n\times 1},$$

**vector of RHS** $\vec{b}(x) = \begin{pmatrix} b_1(x)\\ \vdots\\ b_n(x)\end{pmatrix}$, the equation is then $\vec{y}\,' = A\vec{y} + \vec{b}$, init. conditions are $\vec{y}(x_0) = \vec{y}_0$.

**Theorem.** (on **existence and uniqueness** for systems)
Let $A \in I\!R^{n\times n}$ be a matrix of the given system. If $\vec{b}(x)$ is continuous on an open interval $I$, then for all $x_0 \in I$, $\vec{y}_0 \in I\!R^n$ there exists a solution of the Cauchy problem $\vec{y}\,' = A\vec{y} + \vec{b}(x)$, $\vec{y}(x_0) = \vec{y}_0$ on $I$ and it is unique.
**Theorem.** (on **structure of solutions** for systems) Let $A \in I\!R^{n\times n}$ be a matrix.
The set of solutions of the system $\vec{y}\,' = A\vec{y}$ on some open interval $I$ is a linear space of dimension $n$.
If $\vec{y}_p$ is some solution of the system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$ on $I$, then the set of its solutions on $I$ is

$$\{\vec{y}_p + \vec{y}_h;\ \vec{y}_h \text{ is a solution of } \vec{y}\,' = A\vec{y} \text{ on } I\}.$$

**Definition.** Let $A \in I\!R^{n\times n}$ be a matrix.
By a **fundamental system** of solutions of a system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$ on an open interval $I$ we mean an arbitrary basis of the space of solutions of $\vec{y}\,' = A\vec{y}$ on $I$.
For a particular fundamental system $\{\vec{y}_1, \ldots, \vec{y}_n\}$ we define its **fundamental matrix solution** on $I$ by
$Y(x) = (\,\vec{y}_1(x) \ \cdots \ \vec{y}_n(x)\,)$ (matrix $n \times n$).
**Fact.** Let $A \in I\!R^{n\times n}$ be a matrix. If $Y(x)$ is a fundamental matrix solution on $I$ of a system $\vec{y}\,' = A\vec{y}$, then a general solution of this system on $I$ is $\vec{y}_h(x) = Y(x) \cdot \vec{c}$ for $\vec{c} \in I\!R^n$.
**Theorem.** Let $A \in I\!R^{n\times n}$ be a matrix. Let $\vec{y}_1, \ldots, \vec{y}_n$ be solutions of a system of $n$ linear ODE's $\vec{y}\,' = A\vec{y}$ on an open interval $I$. $\{\vec{y}_1, \ldots, \vec{y}_n\}$ is a fundamental system on $I$ if and only if $\det(Y(x)) \neq 0$ on $I$ if and only if $\exists x \in I$: $\det(Y(x)) \neq 0$.
**Definition.** Let $A \in I\!R^{n\times n}$ be a matrix.
We define its **characteristic polynomial** by $p(\lambda) = \det(A - \lambda E)$.
The roots of $p(\lambda)$ are called **eigenvalues** of matrix $A$.
If $\lambda$ is an eigenvalue, by an **eigenvector of** $A$ **associated with** $\lambda$ we mean an arbitrary vector $\vec{v} \in I\!R^n$, $\vec{v} \neq \vec{0}$ satisfying $(A - \lambda E)\vec{v} = \vec{0}$.
**Theorem.** Consider a system of linear ODE's $\vec{y}\,' = A\vec{y}$, where $A \in I\!R^{n\times n}$.
If $\vec{v}$ is an eigenvector corresponding to an eigenvalue $\lambda$ of matrix $A$, then $\vec{y} = \vec{v}\,e^{\lambda x}$ is a solution of the given system on $I\!R$.
If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of the matrix $A$, then the corresponding solutions form a linearly independent set.