EMA2: Lecture contents, week 8

2. Laplace transform

Definition.
Let f, g be functions defined on $I\!R$. We define their **convolution** as the function $f * g$ on $I\!R$ given by $(f*g)(t) = \int\limits_{-\infty}^{\infty} f(t-s)g(s)\,ds = \int\limits_{-\infty}^{\infty} f(s)g(t-s)\,ds$.

If f, g are zero on $(-\infty, 0)$, for instance if $f, g \in \mathcal{L}_0$, then $(f*g)(t) = \int\limits_{0}^{t} f(t-s)g(s)\,ds$.

Fact.
$f*g = g*f$, $f*(g*h) = (f*g)*h$, $a(f*g) = (af)*g$, $f*(g+h) = f*g + f*h$.

Theorem.
Let $f, g \in \mathcal{L}_0$. Then $f*g \in \mathcal{L}_0$ and $\mathcal{L}\{f*g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$.
From this $\mathcal{L}^{-1}\{F \cdot G\} = \mathcal{L}^{-1}\{F\} * \mathcal{L}^{-1}\{G\}$.

Example.
$y' + \int\limits_0^t \cosh(t-u)y(u)\,du = e^{-t}$, $\qquad y(0^+) = 0$.
It is $y'(t) + \cosh(t) * y(t) = e^{-t}$, hence $pY - 0 + \mathcal{L}\{\cosh(t)\} \cdot Y = \frac{1}{p+1}$, here
$\mathcal{L}\{\cosh(t)\} = \frac{1}{2}\mathcal{L}\{e^t\} + \frac{1}{2}\mathcal{L}\{e^{-t}\} = \frac{1}{2}\left(\frac{1}{p-1} + \frac{1}{p+1}\right) = \frac{p}{p^2-1}$, thus
$pY + \frac{p}{p^2-1}Y = \frac{1}{p+1}$, $p^3 Y = p-1$, $Y(p) = \frac{1}{p^2} - \frac{1}{p^3}$, therefore $y(t) = t - \frac{1}{2}t^2$, $t \geq 0$.

3. Series of real numbers

Definition.
A **series** is a symbol $\sum\limits_{k=n_0}^{\infty} a_k = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \dots$,
where $n_0 \in Z\!\!\!Z$, $a_k \in I\!R$ (series of real numbers).

Definition.
Let $\sum\limits_{k=n_0}^{\infty} a_k$ be a series.

We define its **partial sums** by $s_N = \sum\limits_{k=n_0}^{N} a_k$ for $N \geq n_0$.
We say that the given series **converges** if $\{s_N\}_{N=n_0}^{\infty}$ converges.
We say that the given series **converges to** A, denoted $\sum\limits_{k=n_0}^{\infty} a_k = A$, if $\lim\limits_{N\to\infty}(s_N) = A$.
We say that the given series **diverges** if $\{s_N\}_{N=n_0}^{\infty}$ diverges.
We say that the given series **diverges to** ∞, denoted $\sum\limits_{k=n_0}^{\infty} a_k = \infty$, if $\lim\limits_{N\to\infty}(s_N) = \infty$.
We say that the given series **diverges to** $-\infty$, denoted $\sum\limits_{k=n_0}^{\infty} a_k = -\infty$, if $\lim\limits_{N\to\infty}(s_N) = -\infty$.

Example.
$\sum\limits_{k=1}^{\infty} \frac{1}{2^k}$: $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, induction: $s_N = 1 - \frac{1}{2^N}$, hence $s_N \to 1$ and $\sum\limits_{k=1}^{\infty} \frac{1}{2^k} = 1$ (series converges).

Example.
$\sum\limits_{k=1}^{\infty} 1$: $s_1 = 1$, $s_2 = 1+1 = 2$, $s_3 = 1+1+1 = 3$, induction: $s_N = N$, hence $s_N \to \infty$ and $\sum\limits_{k=1}^{\infty} 1 = \infty$ (series diverges).

Example.
$\sum\limits_{k=0}^{\infty} (-1)^k$: $s_0 = 1$, $s_1 = 1-1 = 0$, $s_2 = 1-1+1 = 1$, induction: $s_N = \begin{cases} 1, & N \text{ even;} \\ 0, & N \text{ odd,} \end{cases}$ thus $\lim\limits_{N\to\infty}(s_N)$ DNE and $\sum\limits_{k=0}^{\infty}(-1)^k$ diverges.

3.1. Summing up series

Definition.
Let $a, q \in I\!R$. The series $\sum\limits_{k=n_0}^{\infty} a\,q^k$ is called a **geometric series**.

Fact.
(i) For $N \in I\!N_0$ we have $\sum\limits_{k=0}^{N} q^k = \dfrac{1-q^{N+1}}{1-q}$;
for $N \in I\!N$, $N \geq n_0$ we have $\sum\limits_{k=n_0}^{N} q^k = q^{n_0}\dfrac{1-q^{N+1-n_0}}{1-q} = \dfrac{q^{n_0}-q^{N+1}}{1-q}$.
(ii) We have $\sum\limits_{k=0}^{\infty} q^k \begin{cases} = \frac{1}{1-q}, & |q| < 1; \\ = \infty \text{ (diverges)}, & q \geq 1; \\ \text{diverges}, & q \leq -1. \end{cases}$
More generally, $\sum\limits_{k=n_0}^{\infty} q^k = \dfrac{q^{n_0}}{1-q}$ for $|q| < 1$.

Definition.
Let $a, q \in I\!R$. The series $\sum\limits_{k=n_0}^{\infty} (a+qk)$ is called an **arithmetic series**.

Fact.
(i) For $N \in I\!N_0$ we have $\sum\limits_{k=0}^{N}(a+qk) = (N+1)a + \frac{1}{2}N(N+1)q$.
(ii) An arithmetic series converges only if $a = q = 0$.

Summing up a series: we can sum up directly only two kinds:
1) geometric series (might be in disguise):

Example.
$$\sum_{k=2}^{\infty}\frac{5\cdot 3^{k-1}}{2^{2k+1}} = \sum_{k=2}^{\infty}\frac{5\cdot 3^{-1}\cdot 3^k}{2^1\cdot(2^2)^k} = \frac{5}{6}\sum_{k=2}^{\infty}\frac{3^k}{4^k} = \frac{5}{6}\sum_{k=2}^{\infty}\left(\tfrac{3}{4}\right)^k = \frac{5}{6}\left(\tfrac{3}{4}\right)^2\sum_{k=0}^{\infty}\left(\tfrac{3}{4}\right)^k$$
$$= \left\langle\!\left\langle \left|\tfrac{3}{4}\right| < 1 \right\rangle\!\right\rangle = \frac{5}{6}\left(\tfrac{3}{4}\right)^2\frac{1}{1-\frac{3}{4}} = \frac{15}{8}.$$
Note: For a geometric series $\sum\limits_{k=n_0}^{\infty} q^k = q^{n_0}\sum\limits_{k=0}^{\infty} q^k$ is true in general.
Or substitution: $\sum\limits_{k=2}^{\infty}\left(\frac{3}{4}\right)^k = \left\langle\!\left\langle \begin{matrix} n = k-2 \implies k = n+2 \\ 2 \mapsto 0,\ \infty \mapsto \infty \end{matrix} \right\rangle\!\right\rangle = \sum\limits_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n+2} = \left(\frac{3}{4}\right)^2\sum\limits_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$.
This an be used for any series, sometimes I use notation $\langle\!\langle k-2 \mapsto k^* \rangle\!\rangle$.

2) telescopic series (might be in disguise):

Example.
$\sum\limits_{k=3}^{\infty} \dfrac{1}{k(k-1)} = \sum\limits_{k=3}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$.
Induction: $s_N = \frac{1}{2} - \frac{1}{N} \to \frac{1}{2}$, hence $\sum\limits_{k=3}^{\infty} \dfrac{1}{k(k-1)} = \frac{1}{2}$.

Remark: formulas for finite sums:
$\sum\limits_{k=1}^{n} 1 = n$, $\sum\limits_{k=1}^{n} k = \frac{1}{2}n(n+1)$, $\sum\limits_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, etc.

Theorem.
Let series $\sum\limits_{k=n_0}^{\infty} a_k$, $\sum\limits_{k=n_0}^{\infty} b_k$ converge.
Then also the series $\sum\limits_{k=n_0}^{\infty} (a_k + b_k)$ converges and $\sum\limits_{k=n_0}^{\infty} (a_k + b_k) = \sum\limits_{k=n_0}^{\infty} a_k + \sum\limits_{k=n_0}^{\infty} a_k$.
For $c \in I\!R$ also $\sum\limits_{k=n_0}^{\infty} (c\,a_k)$ converges and $\sum\limits_{k=n_0}^{\infty} (c\,a_k) = c\Big(\sum\limits_{k=n_0}^{\infty} a_k\Big)$.