EMA2: Lecture contents, week 3

1.3.1. Homogeneous linear ODE of order n

Theorem. (on **structure of solutions** of homogeneous linear ODE)
Consider a homogeneous linear ODE $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=0$. If a_i are continuous on an open interval I, then the set of all solutions of this equation on I is a linear space of dimension n.

Consider a homogeneous linear ODE $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=0$. Define the transformation $L(y)=y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y$ between spaces of functions on an interval I. Then the set of all solutions of the given equation on I corresponds to the set $\{y;\ L(y)=0\}=\mathrm{Ker}(L)$. This transformation is linear, hence $\mathrm{Ker}(L)$ is a linear space.

Definition.
Consider a linear ODE $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=b(x)$. Assume that a_i are continuous on an open interval I. By a **fundamental system** of this equation on I we mean an arbitrary basis of the space of solutions of its associated homogeneous equation.

Definition.
Let $y_1, y_2, \ldots, y_n$ be $(n-1)$-times differentiable functions. We define their **Wronskian** as

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1'(x) & y_2'(x) & \ldots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem.
Consider a homogeneous linear ODE $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=0$.
Let a_i be continuous on an open interval I. Let $y_1, y_2, \ldots, y_n$ be solutions of this equation on I, let W be their Wronskian.
These functions form a linearly independent set (and thus a fundamental system) if and only if $W(x)\neq 0$ on I if and only if $\exists x_0\in I$: $W(x_0)\neq 0$.

Definition.
By a **linear ODE with constant coefficients** we mean any linear ODE for which $a_0(x)=a_0$, $a_1(x)=a_1, \ldots, a_{n-1}(x)=a_{n-1}$ are constant functions.

Definition.
Consider a linear ODE with constant coefficients

$$y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1y'+a_0y=b(x).$$

We define its **characteristic polynomial** by $p(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\ldots+a_1\lambda+a_0$.
We define its **characteristic equation** as $p(\lambda)=0$. The solutions of this eqution are called **characteristic numbers** of the given ODE.

Theorem. (on **fundamental system** for linear ODE with constant coefficients)
Consider a linear ODE with constant coefficients $y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1y'+a_0y=b(x)$.
Let λ be its characteristic number of multiplicity m.
(1) If $\lambda=\alpha\in\mathbb{R}$, then $e^{\alpha x}$, $x\,e^{\alpha x}, \ldots, x^{m-1}e^{\alpha x}$ are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.
(2) If $\lambda=\alpha\pm\beta j\in\mathbb{C}$, $\beta\neq 0$, then $e^{\alpha x}\sin(\beta x)$, $x\,e^{\alpha x}\sin(\beta x), \ldots, x^{m-1}e^{\alpha x}\sin(\beta x)$, $e^{\alpha x}\cos(\beta x)$, $x\,e^{\alpha x}\cos(\beta x), \ldots, x^{m-1}e^{\alpha x}\cos(\beta x)$ are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.

(3) The set of functions from (1) and (2) for all characteristic numbers is linearly independent and it forms a fundamental system of the given equation on $I\!R$.

Example.
Cauchy problem: $y'''' - 3y'' + 2y' = 0$, $y(0) = 3$, $y'(0) = -6$, $y''(0) = 13$, $y'''(0) = -22$.
a) Fundamental system:
$p(\lambda) = \lambda^4 - 3\lambda^2 + 2\lambda = 0$ gives $\lambda = 0, 1\ (2\times), -2$, hence we get fundamental system $\{e^{0\cdot x} = 1, e^x, x\,e^x, e^{-2x}\}$.
b) General solution: $y(x) = a + be^x + cx\,e^x + de^{-2x}$, $x \in I\!R$.
c) Initial conditions:

$y(x) = a + be^x + cx\,e^x + de^{-2x}$	$3 = a + b + d$
$y'(x) = be^x + c(x+1)e^x - 2de^{-2x}$	$-6 = b + c - 2d$
$y''(x) = be^x + c(x+2)e^x + 4de^{-2x}$	$13 = b + c2 + 4d$
$y'''(x) = be^x + c(x+3)e^x - 8de^{-2x}$	$-22 = b + c3 - 8d$

Hence $a = 1$, $b = -1$, $c = 1$, $d = 3$, $y(x) = 1 - e^x + x\,e^x + 3e^{-2x}$, $x \in I\!R$.