# EMA2: Lecture contents, week 9

## 3.2. Convergence of series

**Theorem.**
Let $n_0 < n_1$, consider a series $\sum\limits_{k=n_0}^{\infty} a_k$. $\sum\limits_{k=n_0}^{\infty} a_k$ converges if and only if $\sum\limits_{k=n_1}^{\infty} a_k$ converges.

Then we also have $\sum\limits_{k=n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{n_1-1} a_k + \sum\limits_{k=n_1}^{\infty} a_k$.

If we are only interested in convergence of a series and not its sum (if it exists), then we leave out the index specification.

**Theorem.** (**necessary condition** for convergence)
If a series $\sum a_k$ converges, then necessarily $\lim\limits_{k\to\infty}(a_k) = 0$.
Equivalently: If $\lim\limits_{k\to\infty}(a_k) = 0$ is not true, then the series $\sum a_k$ necessarily diverges.

**Theorem.**
Consider a series $\sum a_k$. If $a_k \geq 0$ for all $k$, then either $\sum a_k$ converges, or $\sum a_k = \infty$.

### 3.2.1. Tests for series with non-negative numbers

**Theorem.** (**integral test**)
Let $f \geq 0$ be a non-increasing function on $[n_0, \infty)$ for $n_0 \in \mathbb{Z}$.
The series $\sum\limits_{k=n_0}^{\infty} f(k)$ converges if and only if $\int\limits_{n_0}^{\infty} f(x)\,dx$ converges.
Moreover we then have $\int\limits_{n_0}^{\infty} f(x)\,dx \leq \sum\limits_{k=n_0}^{\infty} f(k) \leq f(n_0) + \int\limits_{n_0}^{\infty} f(x)\,dx$.

**Example.**
$\sum\limits_{k=3}^{\infty} \dfrac{1}{k\ln^2(k)}$: $\int\limits_{x=3}^{\infty} \dfrac{dx}{x\ln^2(x)} = \left| \begin{matrix} y = \ln(x) \\ dy = \frac{dx}{x} \end{matrix} \right| = \int\limits_{x=\ln(3)}^{\infty} \dfrac{dy}{y^2} < \infty$. Therefore the series converges.

Morevoer, $\sum\limits_{k=3}^{\infty} \frac{1}{k\ln^2(k)} \in \left[\frac{1}{\ln(3)}, \frac{1}{3\ln^2(3)} + \frac{1}{\ln(3)}\right] \sim [0.91, 1.19]$.

Trick: $\sum\limits_{k=3}^{\infty} \frac{1}{k\ln^2(k)} = \sum\limits_{k=3}^{9} \frac{1}{k\ln^2(k)} + \sum\limits_{k=10}^{\infty} \frac{1}{k\ln^2(k)}$

$\in \left[\sum\limits_{k=3}^{9} \frac{1}{k\ln^2(k)} + \int\limits_{10}^{\infty} \frac{dx}{x\ln^2(x)}, \sum\limits_{k=3}^{9} \frac{1}{k\ln^2(k)} + \frac{1}{10\ln^2(10)} + \int\limits_{10}^{\infty} \frac{dx}{x\ln^2(x)}\right] \sim [1.059, 1.078]$.

**Corollary.** ($p$**-test**)
$\sum \frac{1}{k^p}$ converges if and only if $p > 1$.

**Example.**
$\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2} = \dfrac{\pi^2}{6}$.
$\sum\limits_{k=1}^{\infty} \dfrac{1}{k} = \infty$ (**harmonic series**).

**Theorem.** (**comparison test**)
Let $\exists n_0$ so that $0 \leq a_k \leq b_k$ for $k \geq n_0$.
If $\sum b_k$ converges, then also $\sum a_k$ converges.
If $\sum a_k$ diverges, then also $\sum b_k$ diverges (i.e. $\sum a_k = \infty \implies \sum b_k = \infty$).
Remark: Symbolically (and roughly) $a_k \leq b_k \implies \sum a_k \leq \sum b_k$.

**Theorem.** (**limit comparison test**) Let $\exists n_0 \in \mathbb{Z}$ so that $a_k \geq 0, b_k \geq 0$ for $k \geq n_0$. If $a_k \sim b_k$, i.e. $\lim\limits_{k\to\infty}\left(\frac{a_k}{b_k}\right) = A > 0$, then $\sum a_k \sim \sum b_k$, i.e. $\sum a_k$ converges if and only if $\sum b_k$ converges.

**Example.**
$\sum \frac{1}{k^2+1}$: $0 \leq \frac{1}{k^2+1} \leq \frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ converges, therefore by CT also $\sum \frac{1}{k^2+1}$ converges.
Remark: Also IT and LCT would work here.

**Example.**
$\sum \frac{1}{2k^2-1}$: $\frac{1}{2k^2-1} \geq \frac{1}{2k^2} \geq 0$, $\sum \frac{1}{2k^2} = \frac{1}{2}\sum \frac{1}{k^2}$ converges, but the inequality goes the wrong way, so no conclusion possible.
Guess $\frac{1}{2k^2-1} \sim \frac{1}{2k^2}$ for large $k$, confirm: $\lim\limits_{k\to\infty}\left(\frac{\frac{1}{2k^2-1}}{\frac{1}{2k^2}}\right) = 1 > 0$,
$\sum \frac{1}{2k^2} = \frac{1}{2}\sum \frac{1}{k^2}$ converges, hence by LCT also $\sum \frac{1}{2k^2-1}$ converges.

**Example.**
$\sum \frac{1}{k\ln^2(k)}$: Two comparisons seem reasonable, $\frac{1}{k^2} \leq \frac{1}{k\ln^2(k)} \leq \frac{1}{k}$, but both are in the wrong direction, so nothing here.
Limit comparison: No candidate, $\frac{1}{k\ln^2(k)} \sim \frac{1}{k}$ or $\frac{1}{k\ln^2(k)} \sim \frac{1}{k^2}$ definitely not true.
Thus comparison tests won't help.

**Theorem.**
Let $a_k \geq 0$ for all $k$.
**ratio test**:
(i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\frac{a_{k+1}}{a_k} \leq q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\frac{a_{k+1}}{a_k} \geq 1$, then $\sum a_k$ diverges $(=\infty)$.
**limit ratio test**: Let $\lambda = \lim\limits_{k\to\infty}\left(\frac{a_{k+1}}{a_k}\right)$, assuming that the limit converges.
(i) If $\lambda < 1$, then $\sum a_k$ converges.
(ii) If $\lambda > 1$, then $\sum a_k$ diverges $(=\infty)$.
**root test**:
(i) If $\exists q < 1$ and $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\sqrt[k]{a_k} \leq q$, then $\sum a_k$ converges.
(ii) If $\exists n_0 \in \mathbb{Z}$ such that $\forall k \geq n_0$: $\sqrt[k]{a_k} \geq 1$, then $\sum a_k$ diverges $(=\infty)$.
**limit root test**: Let $\varrho = \lim\limits_{k\to\infty}\left(\sqrt[k]{a_k}\right)$, assuming that the limit converges.
(i) If $\varrho < 1$, then $\sum a_k$ converges.
(ii) If $\varrho > 1$, then $\sum a_k$ diverges $(=\infty)$.

**Example.**
$\sum \frac{k!}{2^k}$: Limit ratio test $\lambda = \lim\limits_{k\to\infty}\left(\frac{a_{k+1}}{a_k}\right) = \lim\limits_{k\to\infty}\left(\frac{(k+1)!}{k!}\frac{2^k}{2^{k+1}}\right) = \lim\limits_{k\to\infty}\left(\frac{1}{2}(k+1)\right) = \infty > 1$.
Thus $\sum \frac{k!}{2^k}$ diverges.

**Example.**
$\sum \frac{2}{\ln^k(k+1)}$: Limit root test $\varrho = \lim\limits_{k\to\infty}\left(\sqrt[k]{a_k}\right) = \lim\limits_{k\to\infty}\left(\frac{\sqrt[k]{2}}{\ln(k+1)}\right) = \frac{1}{\infty} = 0 < 1$.
Thus $\sum \frac{2}{\ln^k(k+1)}$ converges.

**Example.**
$\sum\left(\frac{k}{k+1}\right)^k$: Limit root test $\varrho = \lim\limits_{k\to\infty}\left(\sqrt[k]{a_k}\right) = \lim\limits_{k\to\infty}\left(\frac{k}{k+1}\right) = 1$, no conclusion.
Similarly limit ratio fails. Integral test without chance, comparison as well.
But: $a_k = \left(1 - \frac{1}{k+1}\right)^k \to e^{-1} \neq 0$, hence $\sum\left(\frac{k}{k+1}\right)^k$ diverges.

## 3.2.2. Tests for alternating series

**Theorem.** (**Alternating series test** or **Leibniz test**)
Let $b_k \geq 0$ for all $k$ and let $\{b_k\}$ be non-increasing.
The series $\sum(-1)^k b_k$ converges if and only if $\lim\limits_{k\to\infty}(b_k) = 0$.

**Example.**
$\sum \frac{(-1)^k}{k}$: $b_k = \frac{1}{k} \geq 0$ is decreasing and $\to 0$, hence $\sum \frac{(-1)^k}{k}$ converges (compare with harmonic series).

## 3.3. Absolute convergence of series

**Definition.**
We say that a series $\sum a_k$ **converges absolutely** if the series $\sum |a_k|$ converges.

**Theorem.**
If a series $\sum a_k$ converges absolutely, then it also converges and we have $\Big|\sum\limits_{k=n_0}^{\infty} a_k\Big| \leq \sum\limits_{k=n_0}^{\infty} |a_k|$.
But not the other way around! Recall that $\sum \frac{(-1)^k}{k}$ converges, but $\sum\big|\frac{(-1)^k}{k}\big| = \sum \frac{1}{k} = \infty$.

**Definition.**
We say that a series **converges conditionally** if it converges, but not absolutely.
Thus there are three possibilities now:
—$\sum a_k$ converges, $\sum |a_k|$ converges: absolute convergence (the second implies the first here)
—$\sum a_k$ converges, $\sum |a_k|$ diverges: conditional convergence
—$\sum a_k$ diverges, $\sum |a_k|$ diverges (the first implies the second)

**Example.**
conditional convergence: $\sum \frac{(-1)^k}{k}$; absolute convergence: $\sum \frac{(-1)^k}{k^2}$; divergence: $\sum(-1)^k$.

**Example.**
$\sum \frac{\sin(k)}{2^k}$: We do not know how to investigate this series directly. Its terms are not non-negative, therefore the tests won't work. We can't use AST, since the series is not alternating. The necessary condition won't help either, since $a_k \to 0$.
Thus we try the absolute convegence and hope that it will come out true, so that we have some conclusion:
$\sum\big|\frac{\sin(k)}{2^k}\big| = \sum \frac{|\sin(k)|}{2^k} \leq \sum \frac{1}{2^k}$, this converges, therefore by comparison test also $\sum\big|\frac{\sin(k)}{2^k}\big|$ converges, hence $\sum \frac{\sin(k)}{2^k}$ converges absolutely.

**Example.**
$\sum(-1)^k \frac{2^k}{k^3}$: absolute: $\sum\big|(-1)^k \frac{2^k}{k^3}\big| = \sum \frac{2^k}{k^3}$, ratio test: $\frac{a_{k+1}}{a_k} = 2\big(\frac{k}{k+1}\big)^3 \to 2 = \lambda > 1$,
thus $\sum\big|(-1)^k \frac{2^k}{k^3}\big|$ diverges, hence $\sum(-1)^k \frac{2^k}{k^3}$ does not converge absolutely. But we do not know whether it by itself does converge (then it would be conditional convergence) or not.
However, $\frac{2^k}{k^3} \to \infty$, thus $a_k = (-1)^k \frac{2^k}{k^3} \not\to 0$, so the series diverges.