Automata and Grammars (BIE-AAG)

7. Conversions between RG, RE and FA

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Relationship between RE and FA

Algorithm Construction of FA for a given regular expression – Glushkov's method

Input: Regular expression E.

Output: Finite automaton $M = (Q, T, \delta, q_0, F)$, $h(E) = L(M)$.

Method:

1. We assign numbers $1, 2, \dots, n$ to all occurrences of symbols from T in expression E. The resultant regular expression is denoted E'.
2. Set of symbols at the beginning $B = \{x_i : x \in T$, symbol x_i is the first symbol of some string in $h(E')\}$.
3. Set of neighbours $P = \{x_i y_j :$ symbols x_i and y_j occur next to each other in some string from $h(E')\}$.
4. Set of final symbols $F = \{x_i :$ symbol x_i is the last symbol of some string from $h(E)\} \cup \{q_0 : \varepsilon \in h(E)\}$.
5. $Q = \{q_0\} \cup \{x_i : x \in T, i \in \langle 1, n \rangle\}$.

Relationship between RE and FA

Algorithm (continued):

6. Mapping δ:
   (a) $\delta(q_0, x)$ contains x_i, $\forall x_i \in B$.
   (b) $\delta(x_i, y)$ contains y_j, $\forall x_i y_j \in P$
7. Set F is the set of final states. $\square$

Relationship between RE and FA

Example

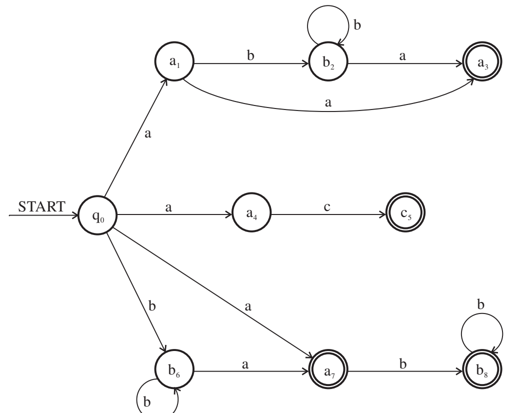
$E = ab^*a + ac + b^*ab^*$, $T = \{a, b, c\}$.

$E' = a_1 b_2^* a_3 + a_4 c_5 + b_6^* a_7 b_8^*$, $B = \{a_1, a_4, b_6, a_7\}$.

$P = \{a_1b_2, a_1a_3, b_2b_2, b_2a_3, a_4c_5, b_6b_6, b_6a_7, a_7b_8, b_8b_8\}$.

$F = \{a_3, c_5, a_7, b_8\}$.

$M = (\{q_0, a_1, b_2, a_3, a_4, c_5, b_6, a_7, b_8\}, \{a, b, c\}, \delta, q_0, \{a_3, c_5, a_7, b_8\})$



Relationship between RE and FA

Algorithm Construction of DFA for a given regular expression – method of derivatives, Janusz A. Brzozowski.
Input: Regular expression E over alphabet T.
Output: Finite automaton $M = (Q, T, \delta, q_0, F)$, $h(E) = L(M)$.
Method:

1. $Q = \{E\}, Q_0 = \{E\}, i := 1.$
2. $Q_i = \{\frac{dU}{da} : U \in Q_{i-1}, a \in T\} \setminus Q.$
3. If $Q_i \neq \emptyset$, $Q = Q \cup Q_i$, $i := i + 1$, go to step 2.
   If $Q_i = \emptyset$, we create automaton M thusly:
   $M = (Q, T, \delta, E, F)$,
   $\delta(\frac{dU}{dx}, a) = \frac{dU}{d(xa)}$.
   Set $F = \{\frac{dU}{dx} : \varepsilon \in h(\frac{dU}{dx})\}$. □

Relationship between RE and FA

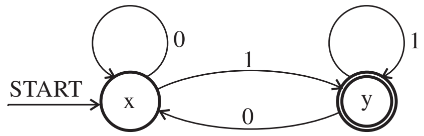
Example
$(0+1)^*1$

1. We set $Q = \{(0+1)^*1\}, Q_0 = \{(0+1)^*1\}$.
2. We compute Q_1:
   $\frac{d}{d1}((0+1)^*1) = (\emptyset + \varepsilon)(0+1)^*1 + \varepsilon = (0+1)^*1 + \varepsilon$
   $\frac{d}{d0}((0+1)^*1) = (\varepsilon + \emptyset)(0+1)^*1 + \emptyset = (0+1)^*1.$
   Since $\frac{d}{d0}((0+1)^*1) = (0+1)^*1$,
   $Q_1 = \{(0+1)^*1 + \varepsilon\}$ and $Q = \{(0+1)^*1, (0+1)^*1 + \varepsilon\}$.
3. We compute Q_2:
   $\frac{d}{d1}((0+1)^*1 + \varepsilon) = (\emptyset + \varepsilon)(0+1)^*1 + \varepsilon + \emptyset = (0+1)^*1 + \varepsilon$
   $\frac{d}{d0}((0+1)^*1 + \varepsilon) = (\varepsilon + \emptyset)(0+1)^*1 + \emptyset = (0+1)^*1.$
   Since both expressions are already in set Q, set Q_2 is empty.

Relationship between RE and FA

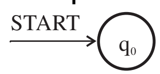
Example (continued)

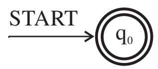
4. state $x = (0+1)^*1$,
   state $y = (0+1)^*1 + \varepsilon$,
   start state is x,
   final state is y, because $\varepsilon \in h((0+1)^*1 + \varepsilon)$
   $\frac{dx}{d1} = y$, $\frac{dx}{d0} = x$, $\frac{dy}{d1} = y$, $\frac{dy}{d0} = x$,

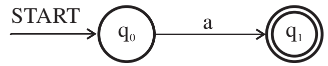


Relationship between RE and FA

Algorithm Construction of FA for a given regular expression – Thompson's method.
Input: Regular expression E over alphabet T.
Output: Finite automaton $M = (Q, T, \delta, q_0, F)$, $h(E) = L(M)$.
Method:

1. We create FA for elementary regular expressions:
   (a) For regular expression $E = \emptyset$:
   

   (b) For regular expression $E = \varepsilon$:
   

   (c) For regular expression $E = a$:
   

2. For parts of regular expression in form $E = E_1 + E_2,\ E = E_1E_2,\ E = E_1^*$ we create finite automata using appropriate algorithms.

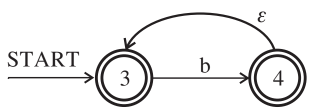
Relationship between RE and FA

Example

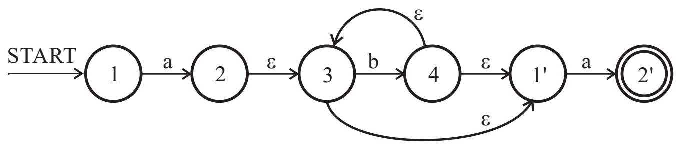
$E = ab^*a + ab$.

Necessary elementary expressions are:

We further create automaton for expression b^*:



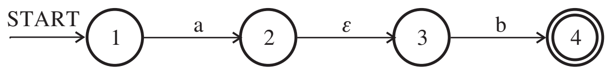
For expression ab^*a the FA is of the form:



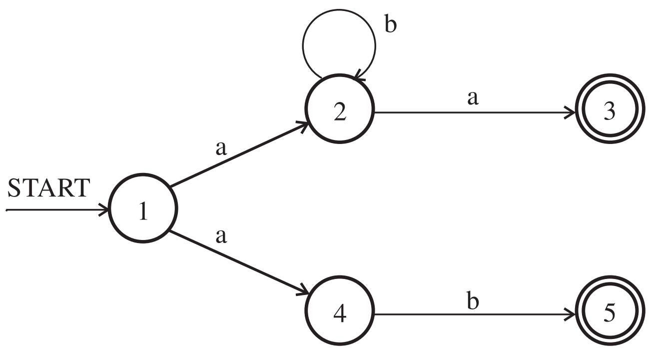
Relationship between RE and FA

Example (continued)

For expression ab the FA is of the form:



For expression $E = ab^*a + ab$:



Relationship between FA and RE

Theorem

For every finite automaton M a regular expression E can be constructed such that $L(M) = h(E)$. $\square$

Relationship between FA and RE

Definition

Extended finite automaton $M = (Q, T, \gamma, q_0, F)$, where
Q is a finite set of states,
T is a finite input alphabet,
γ is mapping from $Q \times Q$ into R_T,
$q_0 \in Q$ is the initial state,
$F \subseteq Q$ is a set of final states.
$\gamma(p, q) = \emptyset$, if transition from p to q is not defined.

Definition

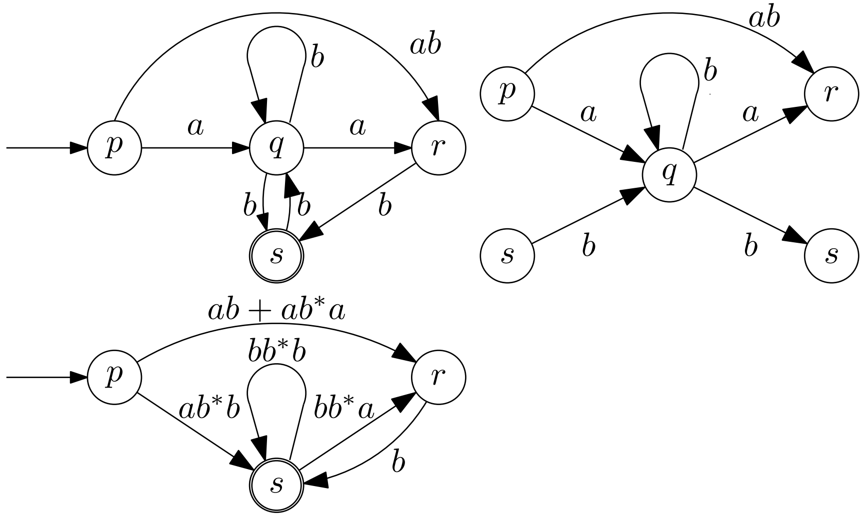
Language accepted by the extended FA M is $L(M) = \{x : x \in T^*,$
$x = x_1x_2 \dots x_n$ and there exists a sequence of states
$q_0, q_1, \dots, q_n,\ q_n \in F$ such that
$x_1 \in h(\gamma(q_0, q_1)),\ x_2 \in h(\gamma(q_1, q_2)), \dots,\ x_n \in h(\gamma(q_{n-1}, q_n))\}$.

Relationship between FA and RE

Theorem
Let $M = (Q, T, \gamma, q_0, F)$ be an extended finite automaton. Assume that q is neither a start state, nor a final state. Then the equivalent extended finite automaton is $M' = (Q \setminus \{q\}, T, \gamma', q_0, F)$, where mapping γ' is defined for every pair $p, r \in (Q \setminus \{q\})$ thusly:
$\gamma'(p, r) = \gamma(p, r) + \gamma(p, q)\gamma(q, q)^*\gamma(q, r)$. □

Relationship between FA and RE

Example



Relationship between FA and RE

Second term of expression
$\gamma'(p, r) = \gamma(p, r) + \gamma(p, q)\gamma(q, q)^*\gamma(q, r)$ will not apply if
$\gamma(p, q) = \emptyset$ or $\gamma(q, r) = \emptyset$.

Relationship between FA and RE

Algorithm Construction of regular expression for a given FA – elimination of states.
Input: Finite automaton $M = (Q, T, \delta, q_0, F)$.
Output: Regular expression E such that $h(E) = L(M)$.
Method:

1. We create $M_R = (Q, T, \gamma, q_0, F)$, where
   $\gamma(p, q) = \{a\},\ a \in T \cup \{\varepsilon\}$ if $\delta(p, a)$ contains q.
2. Automaton M_R will be extended:
   (a) If start state $q_0 \in F$ or $\gamma(q, q_0) \neq \emptyset$, then $Q = Q \cup \{q_0'\}$ a $\gamma(q_0', q_0) = \varepsilon$. q_0' is the start state.
   (b) If $|F| > 1$, then $Q = Q \cup \{f\}$ and $\gamma'(q, f) = \varepsilon, \forall q \in F$. Set of final states $F' = \{f\}$.

   After these modifications $M_R' = (Q', T, \gamma', q_0', F')$.

Relationship between FA and RE

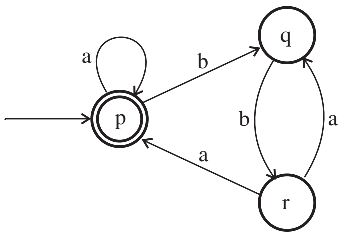
Algorithm (continued):

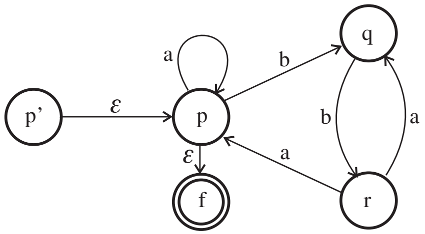
3. If $Q' = \{q'_0, f\}$, then regular expression is $\gamma'(q'_0, f)\gamma'(f, f)^*$. End.
   Else continue by step 4.
4. Choose $q \in Q', q \notin \{q'_0, f\}$. $Q' = Q' \setminus \{q\}$ and γ' are modified accordingly. Continue by step 3.

Relationship between FA and RE

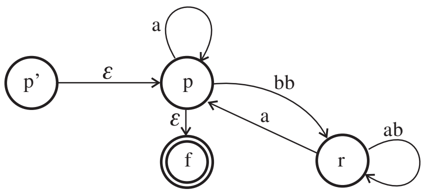
Example

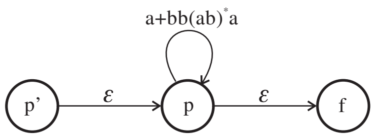
$M = (\{p, q, r\}, \{a, b\}, \delta, p, \{p\})$





We create the extended finite automaton and add a new initial and final state. In the next step we exclude states q and r.

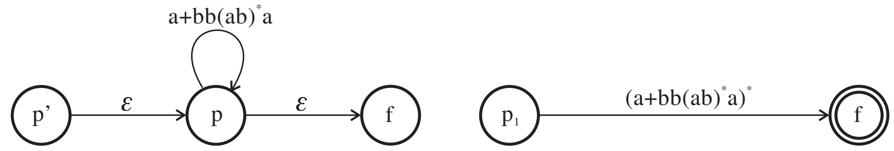




Relationship between FA and RE

Example (continued)

Eventually we exclude state p.



The resulting regular expression is $E = (a + bb(ab)^*a)^*$.

Relationship between FA and RE

Algorithm Construction of regular expression for a given finite automaton – solution of regular equations, incoming transitions

Input: Finite automaton $M = (Q, T, \delta, q_0, F)$.

Output: Regular expression E, $h(E) = L(M)$.

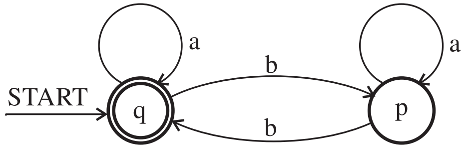
Method:

1. For every state q from Q:
   $X_q = X_{p_1}a_1 + X_{p_2}a_2 + \cdots + X_{p_n}a_n$, if $q \in \delta(p_i, a_i)$. In case that q is the start state, we add ε.
2. The system of left regular equations is solved through Gauss elimination.
3. Resulting regular expression:
   $E = X_{p_1} + X_{p_2} + ... + X_{p_n}$ if $p_i \in F,\ i = 1, 2, ..., n$.

Relationship between FA and RE

Example

$M = (\{q, p\}, \{a, b\}, \delta, q, \{q\})$



$$X_q = X_q a \ + X_p b + \varepsilon$$
$$X_p = X_p a \ + X_q b$$

$$X_p = X_q b a^*$$

$$X_q = X_q a \ + X_q b a^* b + \varepsilon = X_q(a + ba^*b) + \varepsilon$$

Resulting regular expression: $E = X_q = (a + ba^*b)^*$

Relationship between FA and RE

Algorithm Construction of reg. expression for a given finite automaton – solution of regular equations, outgoing transitions

Input: Finite automaton $M = (Q, T, \delta, q_0, F)$.

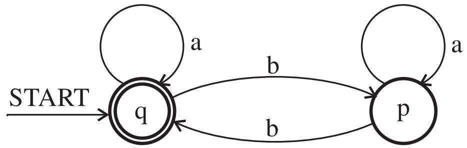
Output: Regular expression E, $h(E) = L(M)$

Method:

1. For each state q from Q:
   $X_q = a_1 X_{p_1} + a_2 X_{p_2} + ... + a_n X_{p_n}$, if $p_i \in \delta(q, a_i)$.
   In case that q is a final state, we add ε.
2. The system of right regular equations is solved through Gauss elimination.
3. Resulting regular expression is the expression for the initial state q_0 (for variable X_{q_0}).

Relationship between FA and RE

Example



$X_q = aX_q + bX_p + \varepsilon$

$X_p = aX_p + bX_q$

We express the variable X_p: $X_p = a^*bX_q$

We substitute for X_p into the first equation:

$X_q = aX_q + ba^*bX_q + \varepsilon$

$X_q = (a + ba^*b)X_q + \varepsilon$

We express the variable X_q

$X_q = (a + ba^*b)^* = E$

Relationship between FA and RE

Algorithm Construction of regular expression for a given finite automaton – integral method.

Input: Total DFA $M = (Q, T, \delta, q_0, F)$.

Output: Regular expression E, $h(E) = L(M)$.

Method:

1. For each state q we find the shortest string x_q for which there exists a path from state q_0 into state q; $x_{q_0} = \varepsilon$.
2. A sequence of expressions $\frac{dE}{dx_q} = Y_q$ ordered by the length of minimal strings x_q beginning with string x_{q_0}.
3. We create a tree for all derivatives by $x_q, q \in Q$:
   (a) $x_{q_o} = \varepsilon$: $E = \frac{dE}{d\varepsilon} = q_0$
   (b) $\forall a_i \in T, \forall x_q, q \in Q$, we add edges labeled by symbols a_i leading into nodes $Y_{qa_i} = \frac{dE}{dx_q a_i} = \delta(q, a_i)$
   If $\delta(q, a_i)$ is an error state, then $Y_{qa_i} = \emptyset$.

Relationship between FA and RE

Algorithm (continued):

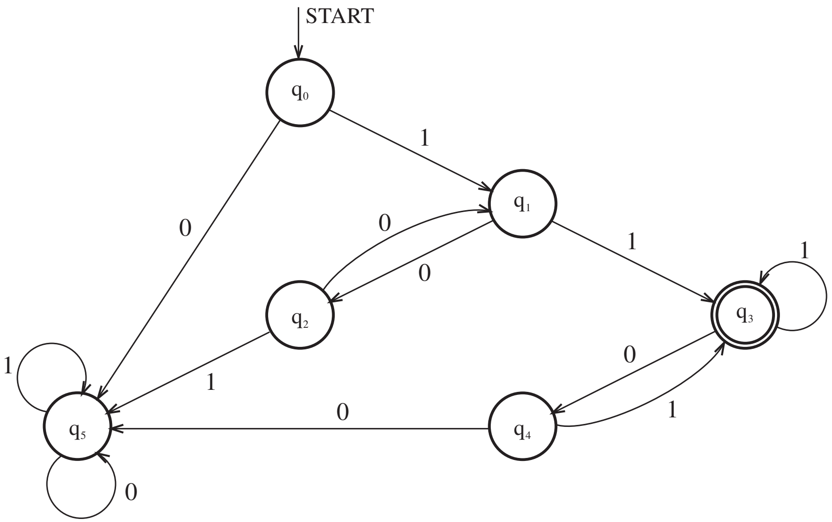
4. For the last minimal string in the ordering by the step 2 we perform integration of expressions by which the targets of the string are labeled.
5. If the node corresponding to state q_i has descendants for which we found in step 4 the values of integrals $p_1, p_2, \dots, p_k$ by $a_1, a_2, \dots, a_k$, then we construct the following equation:
   $q_i = a_1p_1 + a_2p_2 + \ldots + a_kp_k + \varepsilon$, if $q_i \in F$,
   $q_i = a_1p_1 + a_2p_2 + \ldots + a_kp_k$, otherwise.
   We solve this equation and remove nodes for which we performed the integration from the tree.
6. We repeat steps 4 and 5 for every minimal string by the ordering from step 2, until the integration is over.
7. The output is regular expression for the start state q_0. □

Relationship between FA and RE

Example

$M = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{0, 1\}, q_0, \delta, \{q_3\})$.

We add error state $q_\emptyset = q_5$.



Relationship between FA and RE

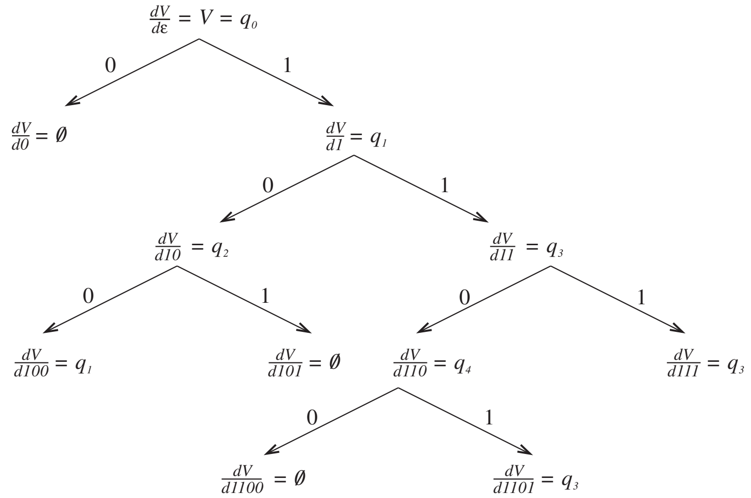
Example (continued)

1. Shortest strings for individual states are these:
   $x_{q_0} = \varepsilon$, $x_{q_1} = 1$,
   $x_{q_5} = 0$, $x_{q_2} = 10$,
   $x_{q_3} = 11$, $x_{q_4} = 110$.
2. This is the ordered sequence of expressions:
   $Y_{q_0} = \frac{dE}{d\varepsilon} = E$,
   $Y_{q_1} = \frac{dE}{d1}$,
   $Y_{q_5} = \frac{dE}{d0} = \emptyset$,
   $Y_{q_2} = \frac{dE}{d10}$,
   $Y_{q_3} = \frac{dE}{d11}$,
   $Y_{q_4} = \frac{dE}{d110}$.

Relationship between FA and RE

Example (continued) $Y_{q_0} = \frac{dE}{d\varepsilon} = E$, $Y_{q_1} = \frac{dE}{d1}$, $Y_{q_5} = \frac{dE}{d0} = \emptyset$, $Y_{q_2} = \frac{dE}{d10}$, $Y_{q_3} = \frac{dE}{d11}$, $Y_{q_4} = \frac{dE}{d110}$.

3. Tree constructed in step 3.



Relationship between FA and RE

Example (continued)

4. We integrate $\frac{dE}{d1100}$ a $\frac{dE}{d1101}$
   $\int \frac{dE}{d1100} d0 = \int \emptyset\, d0 = 0\emptyset = \emptyset,$
   $\int \frac{dE}{d1101} d1 = \int q_3 d1 = 1q_3.$
5. The equation has the form: $q_4 = 1q_3 + \emptyset$
   We further repeat steps 4 and 5:
   $\int \frac{dE}{d110} d0 = \int q_4 d0 = \int 1q_3 d0 = 01q_3,$
   $\int \frac{dE}{d111} d1 = \int q_3 d1 = 1q_3,$
   $q_3 = 01q_3 + 1q_3 + \varepsilon = (01+1)q_3 + \varepsilon = (01+1)^*,$
   $\int \frac{dE}{d100} d0 = \int q_1 d0 = 0q_1$
   $\int \frac{dE}{d101} d1 = \int \emptyset d1 = 1\emptyset = \emptyset,$
   $q_2 = 0q_1 + \emptyset$

Relationship between FA and RE

Example (continued)

$\int \frac{dE}{d10} d0 = \int q_2 d0 = \int 0q_1 d0 = 00q_1$
$\int \frac{dE}{d11} d1 = \int q_3 d1 = \int (01+1)^* d1 = 1(01+1)^*$
$q_1 = 00q_1 + 1(01+1)^* = (00)^*1(01+1)^*$
$\int \frac{dE}{d0} d0 = \int \emptyset d0 = 0\emptyset = \emptyset$
$\int \frac{dE}{d1} d1 = \int q_1 d1 = \int (00)^*(01+1)^* d1 = 1(00)^*1(01+1)^*$
$q_0 = 1(00)^*1(01+1)^* + \emptyset$

6. $L(M) = h(E) = 1(00)^*1(01+1)^*$

Relationship between RG and RE

Theorem
For every regular grammar $G = (N, T, P, S)$ a regular expression E such that $L(G) = h(E)$ can be constructed.

Definition
Extended (right) regular grammar is the quadruple $G = (N, T, P, S)$, where
N is the finite set of nonterminal symbols,
T is the finite set of terminal symbols,
P is the set of rules in the form $A \rightarrow \alpha B$ or
$A \rightarrow \alpha,\ A, B \in N,\ \alpha \in R_T,$
$S \in N$ is the start symbol.

Relationship between RG and RE

W.L.O.G. we assume that there are at most two production rules of the form $A \rightarrow \alpha B$ or $A \rightarrow \alpha$ in the extended regular grammar.
Modification: The pair of rules $A \rightarrow \alpha B,\ A \rightarrow \beta B$ is replaced by rule $A \rightarrow (\alpha + \beta)B$.

Definition
Language defined by the extended regular grammar
$L(G) = \{x : x \in T^*, x = x_1 x_2 \ldots x_n$ and there exists a derivation
$S \Rightarrow \alpha_1 A_1 \Rightarrow \alpha_2 A_2 \Rightarrow \ldots \Rightarrow \alpha_{n-1} A_{n-1} \Rightarrow \alpha_n$ such that
$x_i \in h(\alpha_i),\ 1 \leq i \leq n\}.$

Relationship between RG and RE

Theorem
Let $G = (N, T, P, S)$ be an extended regular grammar. Assume that the nonterminal symbol A is not a start symbol. Then the equivalent extended regular grammar is $G' = (N \setminus \{A\}, T, P', S)$, where rules in P' are formed thusly:
If there are rules in G:

$B \to \alpha_1 C$

$B \to \alpha_2 A$

$A \to \alpha_3 A$

$A \to \alpha_4 C$,

then the following rules are in P' of G':

$B \to (\alpha_1 + \alpha_2 \alpha_3^* \alpha_4)C$.

Relationship between RG and RE

Note:
If some of the given rules does not occur in grammar G, then it is replaced by rule $X \to \emptyset Y$, where $X \in \{A, B, C\},\ Y \in \{A, C\}$.

The corresponding rules will not occur in G' either.

For example if G misses a rule $A \to \alpha_4 C$, we replace it with a rule $A \to \emptyset C$ and then rule $B \to (\alpha_1 + \alpha_2 \alpha_3^* \emptyset)C$ degenerates to rule $B \to \alpha_1 C$.

Relationship between RG and RE

Algorithm Construction of regular expression for a given regular grammar – elimination of nonterminal symbols.
Input: Right regular grammar $G = (N, T, P, S)$.
Output: Regular expression E such that $h(E) = L(G)$.
Method:

1. Extended right regular grammar $G_R = (N, T, P_R, S)$ equivalent with G. All n-tuples of rules in the form $A \to \alpha_1 B, A \to \alpha_2 B, \ldots, A \to \alpha_n B$ are replaced by the rule
   $A \to (\alpha_1 + \alpha_2 + \ldots + \alpha_n)B$, where $A \in N,\ B \in N \cup \{\varepsilon\}$.

Relationship between RG and RE

Algorithm (continued):

2. Grammar G_R is further extended thusly:
   (a) new start symbol $S' \notin N$ and we add $S' \to S$,
   (b) new final nonterminal symbol $F \notin N$,
   all rules $A \to \alpha$ are replaced by rules $A \to \alpha F$ and we add $F \to \varepsilon$.
   $G'_R = (N', T, P'_R, S')$, where $N' = N \cup \{S', F\}$,
   $P'_R = \{A \to \alpha B : A \to \alpha B \in P_R\}$
   $\cup \{A \to \alpha F : A \to \alpha \in P_R\} \cup \{S' \to S,\ F \to \varepsilon\}$.
3. If $N' = \{S'\}$ and $P'_R = \{S' \to \alpha F, F \to \varepsilon\}$, then $E = \alpha$ and the algorithm ends. Otherwise it continues by step 4.
4. We choose $A \in N'$ such that $A \notin S'$. $N' = N' \setminus \{A\}$ and rules P_R are modified correspondingly.
   We continue by step 3. $\square$

Relationship between RG and RE

Example

$G = (\{S, A, B\}, \{a, b\}, P, S)$, where P:

$S \rightarrow bA \mid aS \mid \varepsilon$

$A \rightarrow bB$

$B \rightarrow aA \mid aS.$

$S' \rightarrow S$, S' is a new nonterminal symbol

$S \rightarrow \varepsilon$ is replaced by rule $S \rightarrow F$ and for F we add rule $F \rightarrow \varepsilon$.

$G'_R = (\{S', S, A, B, F\}, \{a, b\}, P', S')$:

$S' \rightarrow S$

$S \rightarrow bA \mid aS \mid F$

$A \rightarrow bB$

$B \rightarrow aA \mid aS$

$F \rightarrow \varepsilon.$

Relationship between RG and RE

Algorithm Construction of regular expression for the given regular grammar

Input: Regular grammar $G = (N, T, P, S)$.

Output: Regular expression E such that $h(E) = L(G)$.

Method:

1. For every nonterminal symbol from N we construct a regular equation.
2. The resulting system of regular equations is solved using Gauss elimination for the start symbol of the grammar S. □

Relationship between RG and RE

Example (continued)

We exclude symbol A. $G^1_R = (\{S', S, B, F\}, \{a, b\}, P^1, S')$:

$S' \rightarrow S$

$S \rightarrow bbB \mid aS \mid F$

$B \rightarrow abB \mid aS$

$F \rightarrow \varepsilon.$

We exclude symbol B. $G^2_R = (\{S', S, F\}, \{a, b\}, P^2, S')$:

$S' \rightarrow S$

$S \rightarrow (a + bb(ab)^*a)S \mid F$

$F \rightarrow \varepsilon.$

We exclude symbol S. $S' \rightarrow (a + bb(ab)^*a)^*F$
$F \rightarrow \varepsilon.$

$E = (a + bb(ab)^*a)^*$

Relationship between RG and RE

Example

$G = (\{S, A\}, \{0, 1\}, P, S)$, P:

$S \rightarrow 0S \mid 1A \mid 1$

$A \rightarrow 1S \mid 0A \mid 0$

The system of regular equations has the following form:

$S = 0S + 1A + 1$

$A = 1S + 0A + 0$

We solve the system by elimination:

$S = 0^*(1A + 1) = 0^*1(A + \varepsilon)$

$A = 10^*1A + 10^*1 + 0A + 0$

$A = (10^*1 + 0)A + 10^*1 + 0$

$A = (10^*1 + 0)^*(10^*1 + 0)$

$S = 0^*1((10^*1 + 0)^*(10^*1 + 0) + \varepsilon) = 0^*1(10^*1 + 0)^*$

Resulting regular expression describing language L(G) has the form:

$S = 0^*1(10^*1 + 0)^*$

Relationship between RE and RG

Theorem
For each regular expression E a regular grammar G such that $L(G) = h(E)$ can be constructed.

Relationship between RE and RG

Algorithm Construction of right regular grammar for a given regular expression – iterative construction.
Input: Regular expression E over alphabet T.
Output: $G = (N, T, P, S)$, $L(G) = h(E)$
Method:

1. $\forall a \in T$ we create grammars:
   $G_a = (\{A\}, \{a\}, \{A \to a\}, A)$.
2. We create grammar $G_\varepsilon = (\{E\}, \emptyset, \{E \to \varepsilon\}, E)$.
3. Grammars for subexpressions of the form $x_1 + x_2$, x_1x_2, x_1^*, if we already have grammars for subexpressions x_1, x_2.
4. We exclude ε-rules and simple rules.

Relationship between RE and RG

Example
$(ab + \varepsilon)^*$

1. $G_a = (\{A\}, \{a\}, \{A \to a\}, A)$
   $G_b = (\{B\}, \{b\}, \{B \to b\}, B)$.
2. $G_\varepsilon = (\{E\}, \emptyset, \{E \to \varepsilon\}, E)$.
3. Iteratively we construct grammars for languages that are values of expressions $ab, ab + \varepsilon, (ab + \varepsilon)^*$:
   $G(ab) = (\{A, B\}, \{a, b\}, \{A \to aB, B \to b\}, A)$
   $G(ab + \varepsilon) = (\{N_1, A, B, E\}, \{a, b\}, P_1, N_1)$, where P_1:
   $N_1 \to A \mid E$, $E \to \varepsilon$, $A \to aB$, $B \to b$.
   $G((ab + \varepsilon)^*) = (\{N_2, N_1, A, B, E\}, \{a, b\}, P_2, N_2)$, where P_2:
   $N_2 \to N_1 \mid \varepsilon$, $N_1 \to A \mid E$, $E \to N_2$, $A \to aB$, $B \to bN_2$.

Relationship between RE and RG

Example (continued)

4. We exclude ε-rules $\Big(N_\varepsilon = \{N_2, E, N_1\}\Big)$:
   $N_2' \to N_2 \mid \varepsilon$, $N_2 \to N_1$, $N_1 \to A \mid E$, $E \to N_2$, $A \to aB$,
   $B \to bN_2 \mid b$.
   Further we exclude simple rules:
   $N_{N2'} = \{N_2', N_2, N_1, A, E\}$, $N_{N2} = \{N_2, N_1, A, E\}$,
   $N_{N1} = \{A, E, N_2, N_1\}$, $N_E = \{E, N_2, N_1, A\}$, $N_A = \{A\}$,
   $N_B = \{B\}$ and we get rules:
   $N_2' \to \varepsilon \mid aB$, $N_2 \to aB$, $N_1 \to aB$, $E \to aB$, $A \to aB$,
   $B \to bN_2 \mid b$.

Symbols N_1, E, A are redundant:
$G = (\{N_2', N_2, B\}, \{a, b\}, P, N_2')$, where P:
$N_2' \to \varepsilon \mid aB$, $B \to bN_2 \mid b$, $N_2 \to aB$

Relationship between RE and RG

Algorithm Construction of right regular grammar for a given regular expression – method of derivatives.

Input: Regular expression E over alphabet T.

Output: $G = (N, T, P, S)$, $L(G) = h(E)$.

Method:

1. $N = \{E\}, N_0 = \{E\}, i = 1$.
2. We create derivatives of all expressions from N_{i-1} by all alphabet symbols from alphabet T. Into the set N_i we put all expressions created through derivatives of expressions from N_{i-1}.
3. If $N_i \neq \emptyset$, we add N_i into N, $i = i + 1$ and we go to st. 2. Otherwise we create gr. $G = (N, T, P, E)$, where P:

   We add $\frac{dE}{dx} \rightarrow a\frac{dE}{d(xa)}$ into P.

   We add $\frac{dE}{dx} \rightarrow a$ into P in case that $\varepsilon \in h\left(\frac{dE}{d(xa)}\right)$.

   We add $E \rightarrow \varepsilon$ into P in case that $\varepsilon \in h(E)$.

Relationship between RE and RG

Example

$(ab + \varepsilon)^*$

1. $N = \{(ab + \varepsilon)^*\}, N_0 = \{(ab + \varepsilon)^*\}, i = 1.$
2. $\frac{d(ab+\varepsilon)^*}{da}$ =$(b + \emptyset)(ab + \varepsilon)^* = b(ab + \varepsilon)^*$

   $\frac{d(ab+\varepsilon)^*}{db}$ = $(\emptyset + \emptyset)(ab + \varepsilon)^* = \emptyset$

   $N_1 = \{b(ab + \varepsilon)^*\}, N = \{(ab + \varepsilon)^*, b(ab + \varepsilon)^*\}.$
3. $\frac{d(b(ab+\varepsilon)^*)}{da}$ = $\emptyset$

   $\frac{d(b(ab+\varepsilon)^*)}{db}$ = $\varepsilon(ab + \varepsilon)^* = (ab + \varepsilon)^*$

   $N_2 = \emptyset, N = \{(ab + \varepsilon)^*, b(ab + \varepsilon)^*\}$

If we label $A = (ab + \varepsilon)^*$, $B = b(ab + \varepsilon)^*$, we get grammar $G = (\{A, B\}, \{a, b\}, P, A)$, where P contains rules:

$$A \rightarrow aB \mid \varepsilon$$
$$B \rightarrow bA \mid b$$