# EMA2: Lecture contents, week 10

## 3.3. Absolute convergence of series

**Theorem.**
Consider a series $\sum\limits_{k=2n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then also $\sum a_{2k}$ and $\sum a_{2k+1}$ converge and
$\sum\limits_{k=2n_0}^{\infty} a_k = \sum\limits_{k=n_0}^{\infty} a_{2k} + \sum\limits_{k=n_0}^{\infty} a_{2k+1}$.

Not true for conditional convergence, see $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$.

**Theorem.**
Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then for every choice of signs $\varepsilon_k = \pm 1$ also $\sum \varepsilon_k a_k$ converges.
If $\sum a_k$ converges conditionally, then there is a choice of signs $\varepsilon_k = \pm 1$ such that $\sum \varepsilon_k a_k = \infty$.

**Definition.**
Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
By a **rearrangement** of $\sum\limits_{k=n_0}^{\infty} a_k$ we mean any series $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)}$, where $\pi$ is an arbitrary bijective mapping of $\{n_0, n_0+1, n_0+2, \dots\} \subset \mathbb{Z}$ onto $\{n_0, n_0+1, n_0+2, \dots\}$, i.e. $\pi$ is a permutation of $\{n_0, n_0+1, n_0+2, \dots\}$.

**Theorem.**
Consider a series $\sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum a_k$ converges absolutely, then also all its rearrangements $\sum a_{\pi(k)}$ converge and we have $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = \sum\limits_{k=n_0}^{\infty} a_k$.
If $\sum\limits_{k=n_0}^{\infty} a_k$ converges conditionally, then $\forall c \in \mathbb{R} \cup \{\pm\infty\}$ there exists its rearrangement such that $\sum\limits_{k=n_0}^{\infty} a_{\pi(k)} = c$.

## 4. Sequences and series of functions

**Definition.**
By a **sequence of functions** we mean an ordered set $\{f_k\}_{k=n_0}^{\infty} = \{f_{n_0}, f_{n_0+1}, f_{n_0+2}, \dots\}$, where $f_k$ are functions.
Remark: Given a sequence of functions $\{f_k\}_{k=n_0}^{\infty}$ and $x \in \bigcap D(f_k)$, then $\{f_k(x)\}$ is a standard sequence of real (complex) numbers.

**Definition.**
Let $\{f_k\}_{k \geq n_0}$, $f$ be functions on a set $M$.
We say that $\{f_k\}$ **converges (pointwise)** to $f$ on $M$, denoted $f_k \to f$ or $f = \lim\limits_{k\to\infty}(f_k)$, if $\forall x \in M$: $\lim\limits_{k\to\infty}\big(f_k(x)\big) = f(x)$.

**Example.**

Consider $f_k(x)=\arctan(kx)$. Then $\lim\limits_{k\to\infty}\big(f_k(x)\big)=\begin{cases} 0, & x=0;\\ \frac{\pi}{2}, & x>0;\\ -\frac{\pi}{2}, & x<0.\end{cases}$

**Definition.**
Let $\{f_k\}_{k\geq n_0}$, $f$ be functions on a set $M$.
We say that $\{f_k\}$ **converges uniformly** to $f$ on $M$, denoted $f_k\rightrightarrows f$,
if $\forall\varepsilon>0\exists N_0\in I\!N$ such that $\forall k\geq N_0\forall x\in M: |f(x)-f_k(x)|<\varepsilon$.

**Theorem.**
Let $f_k\rightrightarrows f$ on $M$.
(i) If all $f_k$ are continuous on $M$, then also $f$ is continuous there.
(ii) If all $f_k$ have a derivative on $M$, then also $f$ has it there and $f'=\lim\limits_{k\to\infty}(f_k')$ on $M$.
(iii) If all $f_k$ have antiderivative on $M$, then also $f$ has it there and $\int_{x_0}^x f\,dx=\lim\limits_{k\to\infty}\big(\int_{x_0}^x f_k\,dx\big)$ for $\overline{x_0,x}\subseteq M$.

**Definition.**
A **series of functions** is a symbol $\sum\limits_{k=n_0}^{\infty} f_k=f_{n_0}+f_{n_0+1}+f_{n_0+2}+\dots$, where $f_k$ are functions.
Remark: Given a series of functions $\sum f_k$ and $x\in\bigcap D(f_k)$, then $\sum f_k(x)$ is a standard series of real (complex) numbers.

**Definition.**
Consider a series of functions $\sum\limits_{k=n_0}^{\infty} f_k$.
The **region of convergence** of this series is the set $\big\{x\in\bigcap D(f_k);\ \sum f_k(x)\text{ converges}\big\}$. By defining $f(x)=\sum\limits_{k=n_0}^{\infty} f_k(x)$ we then obtain a function $f$ on this set called the **sum of the series**, denoted $\sum\limits_{k=n_0}^{\infty} f_k=f$.
The **region of absolute convergence** of this series is the set
$\big\{x\in\bigcap D(f_k);\ \sum f_k(x)\text{ converges absolutely}\big\}$.
We say that this series **converges uniformly** to $f$ on $M$, denoted $\sum f_k\rightrightarrows f$ on $M$, if the sequence of partial sums $\Big\{\sum\limits_{k=n_0}^{N} f_k(x)\Big\}$ converges uniformly to $f$ on $M$.

**Theorem.**
Consider series of functions $\sum f_k$ and $\sum g_k$.
If $\sum\limits_{k=n_0}^{\infty} f_k=f$ on $M$ and $\sum\limits_{k=n_0}^{\infty} g_k=g$ on $M$, then $\forall a,b\in I\!R$: $\sum\limits_{k=n_0}^{\infty}(af_k+bg_k)=af+bg$ on $M$.

**Theorem.** (Weierstrass criterion)
Let $f_k$ for $k\geq n_0$ be functions on $M$. Let $a_k\geq 0$ satisfy $\forall x\in M\forall k\geq n_0$: $|f_k(x)|\leq a_k$.
If $\sum a_k$ converges, then $\sum f_k$ converges uniformly on $M$.

**Example.**
$\sum x^k=\frac{1}{1-x}$ on $(-1,1)$, but the convergence is not uniform. It will be uniform if we restrict our attention to $[-\varrho,\varrho]$ for $\varrho\in(0,1)$.

**Theorem.**
Let $\sum f_k \rightrightarrows f$ on $M$.
(i) If all $f_k$ are continuous on $M$, then also $f$ is continuous there.
(ii) If all $f_k$ have a derivative on $M$, then also $f$ has it there and $f' = \sum\limits_{k=n_0}^{\infty} f_k'$ on $M$.
(iii) If all $f_k$ have an antideriative on $M$, then also $f$ has it there and $\int_{x_0}^{x} f\,dx = \sum\limits_{k=n_0}^{\infty} \int_{x_0}^{x} f_k\,dx$ for $\overline{x_0, x} \subseteq M$.

None of this is true in general for ordinary (pointwise) convergence.

## 4.1. Power series

**Definition.**
Let $z_0 \in I\!R$.
By a **power series with center** $x_0$ we mean any series of functions of the form $\sum\limits_{k=0}^{\infty} a_k (x-x_0)^k$, where $a_k \in I\!R$.

**Theorem.**
Consider a power series $\sum\limits_{k=0}^{\infty} a_k (x-x_0)^k$.
There exists $r \in I\!R_0^+ \cup \{\infty\}$ such that $\sum\limits_{k=0}^{\infty} a_k (x-x_0)^k$ converges absolutely on $U_r(x_0) = (x_0 - r, x_0 + r)$ and diverges for $|x - x_0| > r$. Moreover, $r = \dfrac{1}{\limsup\limits_{k\to\infty}(\sqrt[k]{|a_k|})}$.
Remark: We also have $r = \dfrac{1}{\lim\limits_{k\to\infty}\left(\frac{|a_{k+1}|}{|a_k|}\right)}$, assuming that this limit exists.
Remark: A power series always converges (absolutely) at $x = x_0$.

**Definition.**
Consider a power series $\sum\limits_{k=0}^{\infty} a_k (x-x_0)^k$.
The number $r$ with properties as in the previous theorem is called the **radius of convergence** of this series.

**Example.**
$\sum \frac{(2x)^k}{k\,3^k} = \sum \frac{2^k}{k\,3^k}(x-0)^k$, hence $x_0 = 0$.
Absolute convergence by limit root test: $\sqrt[k]{|a_k|} = \frac{2|x|}{3\sqrt[k]{k}} \to \frac{2|x|}{3} = \varrho$.
$\varrho < 1 \iff \frac{2|x|}{3} < 1 \iff |x| < \frac{3}{2}$, thus the radius of convergence $r = \frac{3}{2}$. Endpoints $x_0 \pm r = \pm\frac{3}{2}$:
$x = \frac{3}{2}$: $\sum \frac{1}{k} = \infty$.
$x = -\frac{3}{2}$: $\sum \frac{(-1)^k}{k}$ converges.
Region of convergence $\left[-\frac{3}{2}, \frac{3}{2}\right)$, region of absolute convergence $\left(-\frac{3}{2}, \frac{3}{2}\right)$.

**Example.**
$\sum \frac{(2x-4)^k}{k!} = \sum \frac{2^k}{k!}(x-2)^k$, hence $x_0 = 2$.
Absolute convergence by limit ratio test: $\frac{|a_{k+1}|}{|a_k|} = \frac{2}{k+1}|x-2| \to 0 = \lambda$.
$\lambda < 1$ is true $\forall x$, hence radius of convergence $r = \infty$.
Region of convergence and region of absolute convergence $I\!R$.

**Example.**
$\sum k^k(2x+3)^k = \sum k^k 2^k\big(x-\big(-\frac{3}{2}\big)\big)^k$, hence $x_0 = -\frac{3}{2}$.
Absolute convergence by limit root test:
$\sqrt[k]{|a_k|} = 2k\big|x+\frac{3}{2}\big| \to \begin{cases} \infty, & x \neq -\frac{3}{2}; \\ 0, & x = -\frac{3}{2} \end{cases} = \varrho.$
$\varrho < 1 \iff x = -\frac{3}{2}$, hence radius of convergence $r = 0$.
Region of convergence ans radius of absolute convergence $\big\{-\frac{3}{2}\big\}$.

**Theorem.**
Let $x_0 \in I\!R$, assume that $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k = f$, $\sum\limits_{k=0}^{\infty} b_k(x-x_0)^k = g$ have radii of convergence $r_f$ and $r_g$.
(i) Then $\forall a, b \in I\!R$: $\sum\limits_{k=0}^{\infty}(aa_k + bb_k)(x-x_0)^k = af + bg$ has radius of convergence $r = \min(r_f, r_g)$.
(ii) The series $\sum\limits_{k=0}^{\infty}\Big(\sum\limits_{i=0}^{k} a_i b_{k-i}\Big)(x-x_0)^k = \Big(\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k\Big)\cdot\Big(\sum\limits_{k=0}^{\infty} b_k(x-x_0)^k\Big) = f\cdot g$ has radius of convergence $r = \min(r_f, r_g)$.

**Theorem.**
Let $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k = f$ have radius of convergence $r > 0$.
(i) For any $\varrho \in (0, r)$: $\sum\limits_{k=0}^{\infty} a_k(x-x_0)^k \rightrightarrows f$ on $U_\varrho(x_0)$.
(ii) $f$ is continuous, it has the derivative $f'(x) = \sum\limits_{k=1}^{\infty} k\,a_k(x-x_0)^{k-1}$ with radius of convergence $r$ and an antiderivative $F(x) = \sum\limits_{k=0}^{\infty}\dfrac{a_k}{k+1}(x-x_0)^{k+1}$ with radius of convergence $r$.

**Corollary.**
Let $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-x_0)^k$ on $U_r(x_0)$.
Then on $U_r(x_0)$ we have for $n \in I\!N$ also derivatives
$f^{(n)}(x) = \sum\limits_{k=n}^{\infty} k(k-1)\cdot\ldots\cdot(k-n+1)\,a_k(x-x_0)^{k-n}$.

Remark: At endpoints $x_0 \pm r$ anything can happen, there is no theorem that would also include behaviour there, so we can lose properties there (convergence for instance).

**Example.**
$f(x) = -\ln(1-x) = \sum\limits_{k=1}^{\infty}\frac{1}{k+1}x^{k+1}$ converges on $[-1, 1)$, but $f'(x) = \frac{1}{1-x} = \sum\limits_{k=0}^{\infty} x^k$ converges only on $(-1, 1)$.