

Definiton and statements—Laplace transform

Definition. For $f\colon [0,\infty)\mapsto \mathbb{R}$ we define its **Laplace transform** $\mathcal{L}\{f(t)\}$ by

$$\mathcal{L}\{f(t)\}\colon p\mapsto \int\limits_0^\infty f(t)e^{-pt}dt,$$

assuming that the integral converges for at least one p.
Notation: $\mathcal{L}\{f(t)\}$, $\mathcal{L}\{f\}$, F, alternative $f(t)\ \widehat{=}\ F(p)$.

Definition. **Heaviside function** is defined $H(t)=\begin{cases}1, & t\geq 0;\\ 0, & t<0.\end{cases}$

Definition. We say that a function f is **piecewise continuous** on an interval I if there are $x_0<x_1<\ldots\in\overline{I}$ such that $\{x_k\}$ is either finite or a sequence going to infinity as $k\to\infty$, $\overline{I}=\bigcup[x_{k-1},x_k]$ and for every $k=1,2,\ldots$ the function f is continuous on (x_{k-1},x_k) and it has one-sided limits $f({x_{k-1}}^+)$, $f({x_k}^-)$.
We say that a function f is of **at most exponential growth** if $\exists\alpha,M>0$ such that $\forall t$: $|f(t)|\leq Me^{\alpha t}$.

Definition. We define the space $\mathcal{L}_0$ by

$\mathcal{L}_0=\{f\colon[0,\infty)\mapsto\mathbb{R};\ f$ is of at most exponential growth and piecewise continuous on $[0,\infty)\}$.

Theorem. If $f\in\mathcal{L}_0$ then $\mathcal{L}\{f\}$ exists on some (p_f,∞).
Moreover, $\lim\limits_{p\to\infty}\big(\mathcal{L}\{f\}(p)\big)=0$.

Theorem. (**dictionary**)
(i) $\forall\alpha\in\mathbb{R}$: $e^{\alpha t}\in\mathcal{L}_0$ and $\mathcal{L}\{e^{\alpha t}\}=\frac{1}{p-\alpha}$, $p>\alpha$;
(ii) $\forall n\in\mathbb{N}_0$: $t^n\in\mathcal{L}_0$ and $\mathcal{L}\{t^n\}=\frac{n!}{p^{n+1}}$, $p>0$;
(iii) $\forall\omega\in\mathbb{R}$: $\sin(\omega t)\in\mathcal{L}_0$ and $\mathcal{L}\{\sin(\omega t)\}=\frac{\omega}{p^2+\omega^2}$, $p\in\mathbb{R}$;
(iv) $\forall\omega\in\mathbb{R}$: $\cos(\omega t)\in\mathcal{L}_0$ and $\mathcal{L}\{\cos(\omega t)\}=\frac{p}{p^2+\omega^2}$, $p\in\mathbb{R}$.

Theorem. (**linearity**)
Let $f,g\in\mathcal{L}_0$. Then $\forall a,b\in\mathbb{R}$: $af+bg\in\mathcal{L}_0$ and $\mathcal{L}\{af(t)+bg(t)\}=a\mathcal{L}\{f(t)\}+b\mathcal{L}\{g(t)\}$.

Theorem. (**grammar**) Let $f\in\mathcal{L}_0$. Then the following are true:
(i) (change of scale) $\forall a>0$: $f(at)\in\mathcal{L}_0$ and $\mathcal{L}\{f(at)\}=\frac{1}{a}\mathcal{L}\{f(t)\}\big|_{p/a}$, $p_{f(at)}=ap_f$;
(ii) (shift in image) $\forall a\in\mathbb{R}$: $e^{at}f(t)\in\mathcal{L}_0$ and $\mathcal{L}\{e^{at}f(t)\}=\mathcal{L}\{f(t)\}\big|_{p-a}$, $p_{e^{at}f(t)}=a+p_f$;
(iii) (shift in preimage) $\forall a>0$: $f(t-a)H(t-a)\in\mathcal{L}_0$ and $\mathcal{L}\{f(t-a)H(t-a)\}=e^{-ap}\mathcal{L}\{f(t)H(t)\}$, $p_{f(t-a)H(t-a)}=p_f$;
(iv) (derivative of image) $\forall n\in\mathbb{N}$: $t^nf(t)\in\mathcal{L}_0$ and $\mathcal{L}\{t^nf(t)\}=(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}p^n}\mathcal{L}\{f(t)\}$;
(v) (integration of image) If $\lim\limits_{t\to0^+}\big(\frac{f(t)}{t}\big)$ converges, then $\frac{f(t)}{t}\in\mathcal{L}_0$ and $\mathcal{L}\big\{\frac{1}{t}f(t)\big\}=\int\limits_p^\infty\mathcal{L}\{f(t)\}(q)\,dq$.
(vi) If $f^{(n)}\in\mathcal{L}_0$, then $\mathcal{L}\{f^{(n)}(t)\}=p^n\mathcal{L}\{f(t)\}(p)-p^{n-1}f(0^+)-p^{n-2}f'(0^+)-\ldots-pf^{(n-2)}(0^+)-f^{(n-1)}(0^+)$;
(vii) $\int\limits_0^t f(s)\,ds\in\mathcal{L}_0$ and $\mathcal{L}\big\{\int_0^t f(s)\,ds\big\}=\frac{1}{p}\mathcal{L}\{f(t)\}$.
Remark: Instead of (iii) we usually prefer $\mathcal{L}\{f(t)H(t-a)\}=e^{-ap}\mathcal{L}\{f(t+a)H(t)\}$.

Definition. By a **finite impuls** we mean any function defined on $[0,\infty)$ that is non-zero only on some bounded closed interval.
Let M be a subset of $\mathbb{R}$. We define its **characteristic function** $\chi_M=\begin{cases}1, & x\in M;\\ 0, & x\notin M.\end{cases}$

Theorem. (on **periodic** function)
Let f be a function that is T-periodic on $[0,\infty)$. We mark one period by $f_T=f\cdot\chi_{[0,T)}$.
Then $\mathcal{L}\{f(t)\}=\dfrac{\mathcal{L}\{f_T(t)\}}{1-e^{-pT}}$.

Theorem. If $f,g\in\mathcal{L}_0$ have $\mathcal{L}\{f\}=\mathcal{L}\{g\}$ on some $[p_0,\infty)$, then $f=g$ with exception of a countable set of isolated points.
If moreover f and g are continuous from the right everywhere, then $f=g$.

Theorem. (**dictionary** for $\mathcal{L}^{-1}$)
$\mathcal{L}^{-1}\big\{\frac{1}{p-\alpha}\big\}=e^{\alpha t}$, $\quad\mathcal{L}^{-1}\big\{\frac{1}{p^n}\big\}=\frac{1}{(n-1)!}t^{n-1}$, $\quad\mathcal{L}^{-1}\big\{\frac{\omega}{p^2+\omega^2}\big\}=\sin(\omega t)$, $\quad\mathcal{L}^{-1}\big\{\frac{p}{p^2+\omega^2}\big\}=\cos(\omega t)$.

Theorem. (**grammar** for $\mathcal{L}^{-1}$)
(0) $\mathcal{L}^{-1}$ is linear;
(1) $\mathcal{L}^{-1}\{e^{-ap}F(p)\}=\mathcal{L}^{-1}\{F(p)\}\big|_{t-a}\cdot H(t-a)$;
(2) $\mathcal{L}^{-1}\{F(p-a)\}=e^{at}\mathcal{L}^{-1}\{F(p)\}$;
(3) $\mathcal{L}^{-1}\{F(ap)\}=\frac{1}{a}\mathcal{L}^{-1}\{F(p)\}\big|_{t/a}$;
(4) $\mathcal{L}^{-1}\{F'(p)\}=-t\mathcal{L}^{-1}\{F(p)\}$;
(5) $\mathcal{L}^{-1}\{pF(p)\}=\big[\mathcal{L}^{-1}\{F(p)\}\big]'+\mathcal{L}^{-1}\{F(p)\}(0^+)$.

Theorem.
If $F(p)$ is a proper rational function, then $\mathcal{L}^{-1}\{F(p)\}$ exists and it can be found using partial fractions decomposition.

Definition. Let f, g be functions defined on $\mathbb{R}$. We define their **convolution** as the function $f * g$ on $\mathbb{R}$ given by $(f * g)(t) = \int\limits_{-\infty}^{\infty} f(t-s)g(s)\,ds = \int\limits_{-\infty}^{\infty} f(s)g(t-s)\,ds$.

If f, g are zero on $(-\infty, 0)$, for instance if $f, g \in \mathcal{L}_0$, then $(f * g)(t) = \int\limits_0^t f(t-s)g(s)\,ds$.

Fact. $f * g = g * f$, $f * (g * h) = (f * g) * h$, $a(f * g) = (af) * g$, $f * (g + h) = f * g + f * h$.
Theorem. Let $f, g \in \mathcal{L}_0$. Then $f * g \in \mathcal{L}_0$ and $\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$.
From this $\mathcal{L}^{-1}\{F \cdot G\} = \mathcal{L}^{-1}\{F\} * \mathcal{L}^{-1}\{G\}$.