EMA2: Lecture contents, week 13

4.2. Fourier series

Remark: If f is T-periodic, then also its derivative f' is T-periodic, but this is not true for its antiderivative $F(t) = \int\limits_0^t f(u)\,du$. This one is T-periodic if $\int\limits_0^T f(u)\,du = 0$, i.e. $a_0 = 0$.

Theorem.
Let f be a T-periodic function that is piecewise continuous on $[0,T)$ and it has a piecewise continuous derivative on $[0,T)$. Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$.

(i) Then $f' \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[-a_k \sin(k\omega t)k\omega + b_k \cos(k\omega t)k\omega\big] = \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[b_k k\omega \cos(k\omega t) - a_k k\omega \sin(k\omega t)\big]$.

(ii) If $\int\limits_0^T f(u)\,du = 0$ (that is, $a_0 = 0$), then

$$F(t) = \int\limits_0^t f(u)\,du \sim \sum_{k=1}^{\infty} \big[a_k \sin(k\omega t)\tfrac{1}{k\omega} - b_k \cos(k\omega t)\tfrac{1}{k\omega}\big] = \sum_{k=1}^{\infty} \big[\tfrac{-b_k}{k\omega} \cos(k\omega t) + \tfrac{a_k}{k\omega} \sin(k\omega t)\big].$$

Remark: If we know that Fourier series converges to f on some interval $[a,b]$, then we unfortunately cannot claim that the convergence is uniform there. At the ends of the interval we have the so-called Gibbs problem.

Definition.
Let $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} \big[a_k \cos(k\omega t) + b_k \sin(k\omega t)\big]$. Denote $A_k = \sqrt{a_k^2 + b_k^2}$, find φ_k so that $b_k = A_k \cos(\varphi_k)$ and $a_k = A_k \sin(\varphi_k)$. Then $f \sim \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} A_k \sin(k\omega t + \varphi_k)$.
This series is called the **Fourier series in amplitude-phase form**.
Denote $c_0 = \frac{a_0}{2}$ a $c_k = \frac{1}{2}(a_k - j\, b_k)$, $c_{-k} = \frac{1}{2}(a_k + j\, b_k)$ for $k \in \mathbb{N}$. Then $f \sim \sum\limits_{k=-\infty}^{\infty} c_k e^{jk\omega t}$.
This series is called the **Fourier series in complex form**.

Remark $\varphi_k = \operatorname{arctg}\big(\frac{a_k}{b_k}\big)$, or $\varphi_k = \operatorname{arccotg}\big(\frac{b_k}{a_k}\big)$, or some shifts, see transformation of Cartesian coordinates to polar.

Fact.
We have $c_k = \frac{1}{T}\int\limits_0^T f(t) e^{-jk\omega t} dt$.

4.3. Application of series

Example.

$$y'' + y = \begin{cases} 1, & t \in [2k, 2k+1); \\ 0, & t \in [2k-1, 2k). \end{cases}$$

Expand the right hand-side $f = \frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi} \sin(k\pi t)$.

We assume that a solution can be found of the form $y = \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} [a_k \cos(k\pi t) + b_k \sin(k\pi t)]$.
We substitute into the equation and obtain

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k(1 - k^2\pi^2)\cos(k\pi t) + b_k(1 - k^2\pi^2)\sin(k\pi t)] = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi} \sin(k\pi t).$$

Comparing both sides we get $a_0 = 1$, $a_k = 0$ for $k \geq 1$ and $b_k = \frac{1-(-1)^k}{k\pi(1-k^2\pi^2)}$

and thus $y = \frac{1}{2} + \sum\limits_{k=1}^{\infty} \frac{1-(-1)^k}{k\pi(1-k^2\pi^2)} \sin(k\pi t)] = \frac{1}{2} + \sum\limits_{k=0}^{\infty} \frac{2}{(2k+1)\pi(1-(2k+1)^2\pi^2)} \sin((2k+1)\pi t)]$.

Example.
$y'' - x^3 y = 24x^2$ around $x_0 = 0$.
On the right we have a power series with $x_0 = 0$, we will try to find a solution of the form $y = \sum\limits_{k=0}^{\infty} a_k x^k$.
We substitute into the equation and get $\sum\limits_{k=2}^{\infty} a_k k(k-1)x^{k-2} - \sum\limits_{k=0}^{\infty} a_k x^{k+3} = 24x^2$, hence
$2a_2 + 6a_3 x_1 2a_4 x^2 + \sum\limits_{k=3}^{\infty} [a_{k+2}(k+1)(k+2) - a_{k-3}]x^k = 24x^2$.
Comparing both sides we get $a_2 = 0$, $a_3 = 0$, $a_4 = 2$, and equations $3\cdot 5a_5 - a_0 = 0$, $5\cdot 6a_6 - a_1 = 0$, $6\cdot 7a_7 - a_2 = 0$, $7\cdot 8a_8 - a_3 = 0$, $8\cdot 9a_9 - a_4 = 0$, $9\cdot 10a_{10} - a_5 = 0$, $10\cdot 11a_{11} - a_6 = 0$, etc.
We choose $a_0 = a$, $a_1 = b$, then $a_5 = \frac{a}{20}$, $a_6 = \frac{b}{30}$, $a_7 = a_8 = 0$, $a_9 = \frac{1}{36}$, $a_{10} = \frac{a}{1800}$, $a_{11} = \frac{b}{3300}$, $a_{12} = a_{13} = 0$, etc.
and thus $y(x) = a + bx + 2x^4 + \frac{a}{20}x^5 + \frac{b}{30}x^6 + \frac{1}{36}x^9 + \frac{a}{1800}x^{10} + \frac{b}{3300}x^{11} + \dots$

Example.
$\lim\limits_{x\to 0}\left(\frac{1-\cos(x)}{x^2}\right) = \lim\limits_{x\to 0}\left(\frac{1-[1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots]}{x^2}\right) = \lim\limits_{x\to 0}\left(\frac{\frac{x^2}{2}-\frac{x^4}{4!}+\dots}{x^2}\right) = \lim\limits_{x\to 0}\left(\frac{1}{2} - \frac{x^2}{4!} + \dots\right) = \frac{1}{2}$.

Example.
$\int \frac{1}{x}\sin(x)\,dx = \int \frac{1}{x} \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}\,dx = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int x^{2k}\,dx = \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{2k+1}}{2k+1} + C$
$= \sum\limits_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!(2k+1)} x^{2k+1} + C$.