Applications of the Complete Solution to $Ax = b$

Rong Qin
Department of Mathematics

Overview

We will interpret the particular and special solutions to $Ax = b$ and consider an application.

Interpretation

The complete solution to $Ax = b$ is $x = x_p + x_s$, where $Ax_p = b$, and $Ax_s = 0$. What does this mean practically? Suppose, for example, we have $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T$, with x_1 and x_3 being pivot variables, and x_2 and x_4 being free variables. If we have the following solution

$$x_p = \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \qquad x_s = \begin{pmatrix} 4 & -5 \\ 1 & 0 \\ 1 & 6 \\ 0 & 1 \end{pmatrix},$$

then

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 4 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -5 \\ 0 \\ 6 \\ 1 \end{pmatrix}.$$

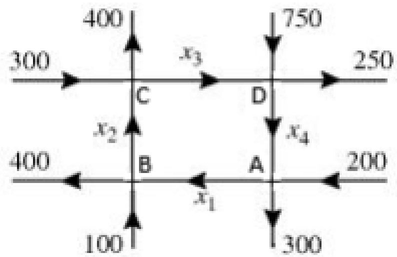
Writing this individually, we have

$$\begin{aligned} x_1 &= 2 + 4x_2 - 5x_4 \\ x_2 &= x_2 \\ x_3 &= -3 + 1x_2 + 6x_4 \\ x_4 &= x_4. \end{aligned}$$

Where x_2 and x_4 can be any value, and x_1 and x_3 are dependent upon them.

Example

Consider traffic moving along the one way streets as in the following diagram.



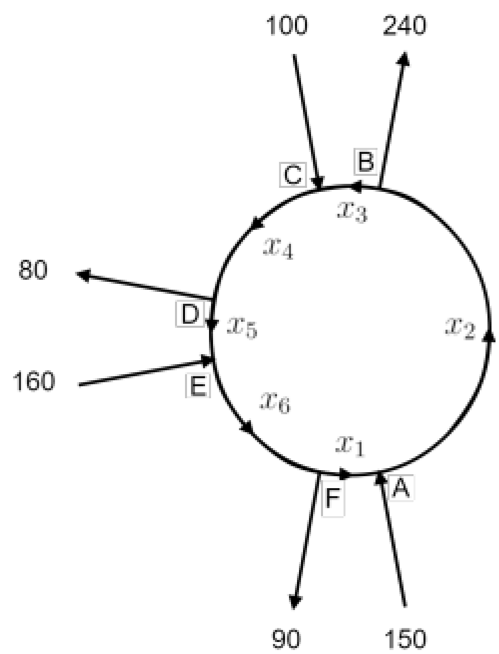
The direction of the traffic is indicated by the arrow. We have measurements for the traffic flow, in vehicles per hour (vph), for certain segments, as indicated by the numbers. The rule is that traffic flow into an intersection must equal the traffic flow out. The traffic flow for some segments are unknown, as indicated by the variables. So for example, one equation we can use is that $x_1 = x_4 + 200 - 300$.

What is the minimum traffic flow along each unknown segment?
The minimum traffic flow along each segment is as follows.

$$\begin{array}{lrr} AB & \text{min}: & 400 \quad vph \\ BC & \text{min}: & 100 \quad vph \\ CD & \text{min}: & 0 \quad vph \\ DA & \text{min}: & 500 \quad vph \end{array}$$

IN-CLASS EXERCISE

We wish to find the traffic flow in all unknown segments below.



1. Using conservation of traffic (the flow in is equal to the flow out), set up the equations that model this system. Write this in the form $Ax = b$.
2. Find the complete solution to this system.
3. What is the minimum traffic flow along each segment? (Use the fact that traffic flow is always nonnegative.)

Show your work. Submit via a .txt file. No .m file is needed.