The Gram-Schmidt Process

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Overview

We implement the Gram-Schmidt process for a set of linearly independent vectors. The Gram-Schmidt process takes a set of vectors and finds a new set which spans the same space, the vectors

Algorithm

- The input is a matrix $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ consisting of n linearly independent column vectors, and the output is a matrix $Q = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n]$, where Q consists of n orthonormal column vectors generated from A using the following process:

$$
\begin{aligned}
\mathbf{q}_1 &= \mathbf{a}_1 \\
\mathbf{q}_2 &= \mathbf{a}_2 - \frac{\mathbf{q}_1^T \mathbf{a}_2}{\mathbf{q}_1^T \mathbf{q}_1} \mathbf{q}_1 \\
\mathbf{q}_3 &= \mathbf{a}_3 - \frac{\mathbf{q}_1^T \mathbf{a}_3}{\mathbf{q}_1^T \mathbf{q}_1} \mathbf{q}_1 - \frac{\mathbf{q}_2^T \mathbf{a}_3}{\mathbf{q}_2^T \mathbf{q}_2} \mathbf{q}_2 \\
&\vdots \\
\mathbf{q}_j &= \mathbf{a}_j - \frac{\mathbf{q}_1^T \mathbf{a}_j}{\mathbf{q}_1^T \mathbf{q}_1} \mathbf{q}_1 - \frac{\mathbf{q}_2^T \mathbf{a}_j}{\mathbf{q}_2^T \mathbf{q}_2} \mathbf{q}_2 - \cdots - \frac{\mathbf{q}_{j-1}^T \mathbf{a}_j}{\mathbf{q}_{j-1}^T \mathbf{q}_{j-1}} \mathbf{q}_{j-1} \\
&= \mathbf{a}_j - \sum_{k=1}^{j-1} \frac{\mathbf{q}_k^T \mathbf{a}_j}{\mathbf{q}_k^T \mathbf{q}_k} \mathbf{q}_k \\
&\vdots \\
\mathbf{q}_n &= \mathbf{a}_n - \frac{\mathbf{q}_1^T \mathbf{a}_n}{\mathbf{q}_1^T \mathbf{q}_1} \mathbf{q}_1 - \frac{\mathbf{q}_2^T \mathbf{a}_n}{\mathbf{q}_2^T \mathbf{q}_2} \mathbf{q}_2 - \cdots - \frac{\mathbf{q}_{n-1}^T \mathbf{a}_n}{\mathbf{q}_{n-1}^T \mathbf{q}_{n-1}} \mathbf{q}_{n-1} \\
&= \mathbf{a}_n - \sum_{k=1}^{n-1} \frac{\mathbf{q}_k^T \mathbf{a}_n}{\mathbf{q}_k^T \mathbf{q}_k} \mathbf{q}_k
\end{aligned}
$$

Note: The $\mathbf{q_i}$ are not unit vectors yet! It is not wrong to use the normalized (or unitized) version first to make computation easier, but for writing a program, it is best to save the normalizing (or unitizing) for the next step.

After these steps, we have obtained n orthogonal column vectors. We then normalize each of them (dividing by their respective norms).

$$\mathbf{q}_1 = \frac{\mathbf{q}_1}{|\mathbf{q}_1|}, \quad \mathbf{q}_2 = \frac{\mathbf{q}_2}{|\mathbf{q}_2|}, \quad \ldots, \quad \mathbf{q}_n = \frac{\mathbf{q}_n}{|\mathbf{q}_n|}$$

Note: The column spaces of A and Q are the same.

- Command Window:

```
>> A = [3 9 0; -9 -6 2; 0 -1 -5]
>> Q = A;
>> Q(:,2) = A(:,2) - (Q(:,1)'*A(:,2))/(Q(:,1)'*Q(:,1)) * Q(:,1)
>> Q(:,3) = A(:,3) - (Q(:,1)'*A(:,3))/(Q(:,1)'*Q(:,1)) * Q(:,1)
                   - (Q(:,2)'*A(:,3))/(Q(:,2)'*Q(:,2)) * Q(:,2)
>> Q(:,1) = Q(:,1)/norm(Q(:,1))
>> Q(:,2) = Q(:,2)/norm(Q(:,2))
>> Q(:,3) = Q(:,3)/norm(Q(:,3))
```

Activities

Write a function `grams.m` that performs the Gram-Schmidt process on the columns of an arbitrary square matrix A and returns a matrix Q whose columns are the resulting orthonormal vectors. Test your code on the following matrix,

$$A = \begin{pmatrix} 1 & -1 & 7 & 1 \\ 0 & 6 & -3 & 3 \\ -7 & -7 & -7 & 4 \\ -9 & 6 & 0 & -1 \end{pmatrix}$$

Check that your answer is orthonormal. Think about what it means to be orthonormal in terms of a matrix and what operation can verify this property.