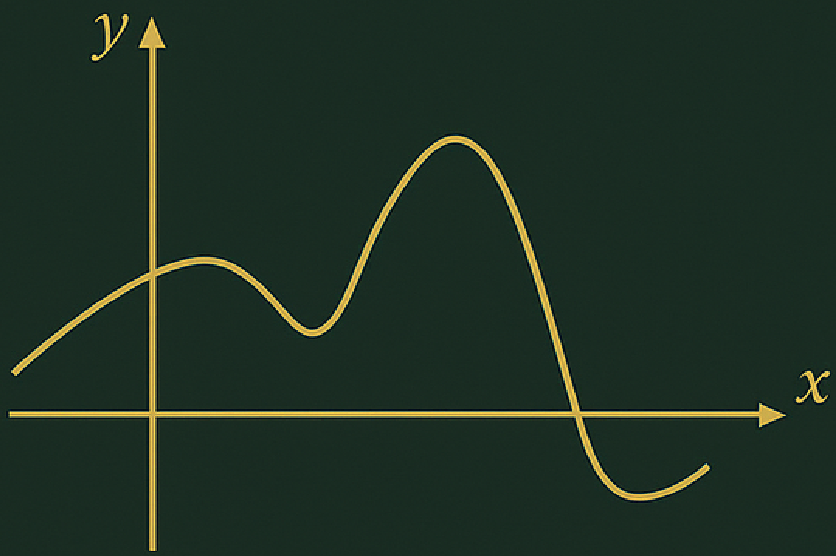
REAL ANALYSIS

A SELF-TAUGHT APPROACH



Luis Vasquez

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1 Introduction

1.1 Sources

The following notes are taken from the compilation of a few sources

- **MIT 18.100B Real Analysis, Spring 2025**, available at the MIT OCW youtube channel

1.2 Purpose

This notes are taken in a way that is easy to understan math. No obscure proof or incomplete idea will be included, avoiding partial understanding of a certain topic. It also covers the need of having an easy-to-follow apporach to *Real Analysis*, making it possible to read this through to revisit known topics without the need of looking at other sources and rabbithole-ing into old books with russian lastnames on its cover. Since I will also be studying the course while taking this notes, the document as a whole will be written by hand without any AI slob nor blind copy/pasting, and since English is not my native language, typos may happen.

2 Base knowledge

2.1 What is a Field

"We can verify that a set is a field by checking that multiplication is a well defined operation i.e. it is independent of the representative."

For example, for arbitrary rational numbers Q:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1}$$

And evaluate an equivalent expresion with different representatives of the same numbers:

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2}$$

Given that:

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \quad \wedge \quad \frac{p_1}{q_1} = \frac{p_2}{q_2}$$

We want to verify that both numbers (both multiplication results) are the same. To review this, we start from the tautology (intuitive truth):

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \Longleftrightarrow m_1 \times n_2 = m_2 \times n_1 \tag{A}$$

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \Longleftrightarrow p_1 \times q_2 = p_2 \times q_1 \tag{B}$$

Then, operating the multiplication using both representatives:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_1.p_1}{n_1.q_1}$$

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2} = \frac{m_2.p_2}{n_2.q_2}$$

Conveniently, we want to form $m_1 \times n_2$ to form the first ground truth.

$$\begin{aligned}
\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 &= \frac{m_1.n_2.p_1}{n_1.q_1} \\
&= \frac{m_2.n_1.p_1}{n_1.q_1} && \text{(replacing } A\text{)} \\
&= \frac{m_2.p_1}{q_1} && \text{(simplifying } n_1\text{)}
\end{aligned}$$

Applying the same logic for $p_1 \times q_2$ to form the second ground truth.

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 = \frac{m_2.p_1.q_2}{q_1}$$

$$= \frac{m_2.p_2.q_1}{q_1} \qquad \text{(replacing } B\text{)}$$

$$= m_2.p_2 \qquad \text{(simplifying } q_1\text{)}$$

Finally, rearranging:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 = m_2.p_2$$

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_2.p_2}{n_2.q_2}$$

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_2}{n_2} \times \frac{p_2}{q_2} \;\blacksquare$$

This is not a regurous demonstration, but gives us a first step to go from the intuiton of a solution (particularly for Q), to a more formal procedure based on the real definition of a field.

2.2 Formal definition

DEFINITION Field 2.1 A Field $\mathbb{F}$ is a set A with two operations: addtion ($\bigoplus$) and multiplication ($\bigotimes$), with the following properties:

- $x, y \in \mathbb{F} \rightarrow x \bigoplus y \in \mathbb{F}$
- $x, y \in \mathbb{F} \rightarrow x \bigoplus y = y \bigoplus x$
- $x, y, z \in \mathbb{F} \rightarrow (x \bigoplus y) \bigoplus z = x \bigoplus (y \bigoplus z)$
- $\exists 0 \in \mathbb{F} / \forall x \in \mathbb{F} \rightarrow x \bigoplus 0 = x$
- $\forall x \in \mathbb{F}, \exists -x \in \mathbb{F} : x \bigoplus -x = 0$
- $x, y \in \mathbb{F} \rightarrow x \bigotimes y \in \mathbb{F}$
- $x, y \in \mathbb{F} \rightarrow x \bigotimes y = y \bigotimes x$
- $x, y, z \in \mathbb{F} \rightarrow (x \bigotimes y) \bigotimes z = x \bigotimes (y \bigotimes z)$
- $\exists 1 \in \mathbb{F} / \forall x \in \mathbb{F} \rightarrow x \bigotimes 1 = x$
- $\forall x \in \mathbb{F} - \{0\}, \exists x^{-1} \in \mathbb{F} : x \bigotimes x^{-1} = 1$

First five correspond to the addion operation, and the last five to the multiplication operation. In order to relate both sets of properties, the following axiom is stated:

AXIOM
Distributive law
2.1

Let $x, y, z \in \mathbb{F}$

$$x \bigotimes (y \bigoplus z) = x \bigotimes y \bigoplus x \bigotimes z$$

THEOREM
Zero uniqueness
2.1

For any Field $\mathbb{F}$, there exists only one zero element.

PROOF

Assume 0_1 and 0_2 are zeros for a field $\mathbb{F}$.

$$\forall x \in \mathbb{F} : \begin{cases} 0_1 + x = x & \text{(A)} \\ 0_2 + x = x & \text{(B)} \end{cases}$$

For both cases, let x be $0_1, 0_2$ respectively

$$x = 0_2 \Rightarrow (A) \quad 0_1 + 0_2 = 0_2 \quad \text{(C)}$$

$$x = 0_1 \Rightarrow (B) \quad 0_2 + 0_1 = 0_1 \quad \text{(D)}$$

$$\Rightarrow (C) \quad 0_2 + 0_1 = 0_2 \text{ (Commutativity)}$$

Comparing this result with D

$$\Rightarrow 0_2 + 0_1 = 0_2 \text{ and } 0_2 + 0_1 = 0_1$$

$$\Rightarrow 0_1 = 0_2$$

□

2.3 Order

DEFINITION
Ordered sets
2.2

A set S will be ordered when having an ordering ”$<$”, so that $\forall x, y \in S$ one of the following properties should hold:

1. $x = y$
2. $x < y$
3. $y < x$

DEFINITION Ordered fields 2.3 A field $\mathbb{F}$ will be ordered if it is also an ordered set. As consequence, the following properties apply:

- $x, y \in \mathbb{F}, x < y \rightarrow \forall z \in \mathbb{F}, x \bigoplus z < y \bigoplus z$
- $x, y \in \mathbb{F}, 0 \ < \ x, y \rightarrow 0 < x \bigotimes y$

THEOREM 2.2 Given $\mathbb{F}$: ordered field. $x < y \wedge 0 < z \Rightarrow x \bigotimes z < y \bigotimes z$

PROOF Proving by contradiction, lets assume the opposite:

$$x \bigotimes z \geq y \bigotimes z$$

Then adding $(-x \bigotimes z)$ on both sides, we would maintain the ordering of the expression

$$\begin{aligned} x \bigotimes z \bigoplus (-x \bigotimes z) &\geq y \bigotimes z \bigoplus (-x \bigotimes z) \\ 0 &\geq y \bigotimes z \bigoplus (-x \bigotimes z) \end{aligned}$$

Now, using **??** 2.1, we get:

$$0 \geq z \bigotimes (y \bigoplus (-x))$$

Considering the first initial considitions for $x < y \Rightarrow y \bigoplus (-x) > 0$ and having $z > 0$, we would expect that the product these two to be > 0 by Definition 2.3 (second property). Hence:

$$0 \geq z \bigotimes (y \bigoplus (-x)) \wedge 0 < z \bigotimes (y \bigoplus (-x)) \nrightarrow \text{(Contradiction found)}$$

$\square$

From this point and for a better readability, + and . (or $\times$) will be used instead of $\bigoplus$ or $\bigotimes$. They will still represent the abstraction of a field's addition and product operations, without necessarily being the known addition and multiplication we could expect them to be.

2.4 Completeness