# REAL ANALYISIS

A Self-Taught Approach

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# Contents


**1 Introduction** **3**
1.1 Sources . . . 3
1.2 Purpose . . . 3

**2 Base knowledge** **4**
2.1 What is a Field . . . 4
2.2 Formal definition . . . 5
2.3 Order . . . 6
2.4 Completeness . . . 7

# 1 Introduction

## 1.1 Sources

The following notes are taken from the compilation of a few sources

- **MIT 18.100B Real Analysis, Spring 2025**, available at the MIT OCW youtube channel

## 1.2 Purpose

This notes are taken in a way that is easy to understan math. No obscure proof or incomplete idea will be included, avoiding partial understanding of a certain topic. It also covers the need of having an easy-to-follow apporach to *Real Analysis*, making it possible to read this through to revisit known topics without the need of looking at other sources and rabbithole-ing into old books with russian lastnames on its cover. Since I will also be studying the course while taking this notes, the document as a whole will be written by hand without any AI slob nor blind copy/pasting, and since English is not my native language, typos may happen.

# 2 Base knowledge

## 2.1 What is a Field

*"We can verify that a set is a field by checking that multiplication is a well defined operation i.e. it is independent of the representative."*

For example, for arbitrary rational numbers $Q$:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1}$$

And evaluate an equivalent expresion with different representatives of the same numbers:

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2}$$

Given that:

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \quad \wedge \quad \frac{p_1}{q_1} = \frac{p_2}{q_2}$$

We want to verify that both numbers (both multiplication results) are the same. To review this, we start from the tautology (intuitive truth):

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \iff m_1 \times n_2 = m_2 \times n_1 \tag{$A$}$$

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \iff p_1 \times q_2 = p_2 \times q_1 \tag{$B$}$$

Then, operating the multiplication using both representatives:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_1.p_1}{n_1.q_1}$$

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2} = \frac{m_2.p_2}{n_2.q_2}$$

Conveniently, we want to form $m_1 \times n_2$ to form the first ground truth.

$$\begin{aligned}
\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 &= \frac{m_1.n_2.p_1}{n_1.q_1} \\
&= \frac{m_2.n_1.p_1}{n_1.q_1} && \text{(replacing } A\text{)} \\
&= \frac{m_2.p_1}{q_1} && \text{(simplifying } n_1\text{)}
\end{aligned}$$

Applying the same logic for $p_1 \times q_2$ to form the second ground truth.

$$\begin{aligned}
\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 &= \frac{m_2.p_1.q_2}{q_1} \\
&= \frac{m_2.p_2.q_1}{q_1} && \text{(replacing } B\text{)} \\
&= m_2.p_2 && \text{(simplifying } q_1\text{)}
\end{aligned}$$

Finally, rearranging:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 = m_2.p_2$$

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_2.p_2}{n_2.q_2}$$

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_2}{n_2} \times \frac{p_2}{q_2} \quad \blacksquare$$

This is not a regurous demonstration, but gives us a first step to go from the intuiton of a solution (particularly for $Q$), to a more formal procedure based on the real definition of a field.

## 2.2 Formal definition

**Definition 2.1** (Field)**.** A Field $\mathbb{F}$ is a set $A$ with two operations: addtion ($\bigoplus$) and multiplication ($\bigotimes$), with the following properties:

- $x, y \in \mathbb{F} \to x \bigoplus y \in \mathbb{F}$
- $x, y \in \mathbb{F} \to x \bigoplus y = y \bigoplus x$
- $x, y, z \in \mathbb{F} \to (x \bigoplus y) \bigoplus z = x \bigoplus (y \bigoplus z)$
- $\exists 0 \in \mathbb{F} / \forall x \in \mathbb{F} \to x \bigoplus 0 = x$
- $\forall x \in \mathbb{F}, \exists - x \in \mathbb{F} : x \bigoplus -x = 0$
- $x, y \in \mathbb{F} \to x \bigotimes y \in \mathbb{F}$
- $x, y \in \mathbb{F} \to x \bigotimes y = y \bigotimes x$
- $x, y, z \in \mathbb{F} \to (x \bigotimes y) \bigotimes z = x \bigotimes (y \bigotimes z)$
- $\exists 1 \in \mathbb{F} / \forall x \in \mathbb{F} \to x \bigotimes 1 = x$
- $\forall x \in \mathbb{F} - \{0\}, \exists x^{-1} \in \mathbb{F} : x \bigotimes x^{-1} = 1$

First five correspond to the addion operation, and the last five to the multiplication operation. In order to relate both sets of properties, the following axiom is stated:

**Axiom 2.1** (Distributive law)**.** Let $x, y, z \in \mathbb{F}$

$$x \bigotimes (y \bigoplus z) = x \bigotimes y \bigoplus x \bigotimes z$$

**Theorem 2.1** (Zero uniqueness)**.** For any Field $\mathbb{F}$, there exists only one zero element.

*Proof.* Assume $0_1$ and $0_2$ are zeros for a field $\mathbb{F}$.

$$\forall x \in \mathbb{F} : \begin{cases} 0_1 + x = x & \text{(A)} \\ 0_2 + x = x & \text{(B)} \end{cases}$$

For both cases, let $x$ be $0_1, 0_2$ respectively

$$x = 0_2 \Rightarrow (A) \quad 0_1 + 0_2 = 0_2 \quad \text{(C)}$$

$$x = 0_1 \Rightarrow (B) \quad 0_2 + 0_1 = 0_1 \quad \text{(D)}$$

$$\Rightarrow (C) \quad 0_2 + 0_1 = 0_2 \text{ (Commutativity)}$$

Comparing this result with $D$

$$\Rightarrow 0_2 + 0_1 = 0_2 \text{ and } 0_2 + 0_1 = 0_1$$

$$\Rightarrow 0_1 = 0_2$$

□

## 2.3 Order

**Definition 2.2** (Ordered sets)**.** A set $S$ will be ordered when having an ordering "$<$", so that $\forall x, y \in S$ one of the following properties should hold:

1. $x = y$
2. $x < y$
3. $y < x$

**Definition 2.3** (Ordered fields)**.** A field $\mathbb{F}$ will be ordered if it is also an ordered set. As consequence, the following properties apply:

- $x, y \in \mathbb{F}, x < y \rightarrow \forall z \in \mathbb{F}, x \bigoplus z < y \bigoplus z$
- $x, y \in \mathbb{F}, 0 \ < \ x, y \rightarrow 0 < x \bigotimes y$

**Theorem 2.2.** Given $\mathbb{F}$: ordered field. $x < y \wedge 0 < z \Rightarrow x \bigotimes z < y \bigotimes z$

*Proof.* Proving by contradiction, lets assume the opposite:

$$x \bigotimes z \geq y \bigotimes z$$

Then adding $(-x \otimes z)$ on both sides, we would maintain the ordering of the expression

$$\begin{aligned} x \bigotimes z \bigoplus (-x \bigotimes z) &\geq y \bigotimes z \bigoplus (-x \bigotimes z) \\ 0 &\geq y \bigotimes z \bigoplus (-x \bigotimes z) \end{aligned}$$

Now, using Axiom 2.1, we get:

$$0 \geq z \bigotimes (y \bigoplus (-x))$$

Considering the first initial considitions for $x < y \Rightarrow y \oplus (-x) > 0$ and having $z > 0$, we would expect that the product these two to be $> 0$ by Definition 2.3 (second property). Hence:

$$0 \geq z \bigotimes (y \bigoplus (-x)) \wedge 0 < z \bigotimes (y \bigoplus (-x)) \not\leftrightarrow \text{(Contradiction found)}$$

□

From this point and for a better readability, + and . (or $\times$) will be used instead of $\oplus$ or $\otimes$. They will still represent the abstraction of a field's addition and product operations, without necessarily being the known addition and multiplication we could expect them to be.

## 2.4 Completeness