

REAL ANALYSIS

A Self-Taught Approach

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1 Introduction

1.1 Sources

The following notes are taken from the compilation of a few sources

- **MIT 18.100B Real Analysis, Spring 2025**, available at the MIT OCW youtube channel
- **Real Analysis**, available at The Bright Side of Mathematics youtube channel

1.2 Purpose

This notes are taken in a way that is easy to understand math. No obscure proof or incomplete idea will be included, avoiding partial understanding of a certain topic. It also covers the need of having an easy-to-follow approach to *Real Analysis*, making it possible to read this through to revisit known topics without the need of looking at other sources and rabbit-hole-ing into old books with russian lastnames on its cover. Since I will also be studying the course while taking this notes, the document as a whole will be written by hand without any AI slob nor blind copy/pasting, and since English is not my native language, typos may happen.

2 Base knowledge

2.1 What is a Field \mathbb{F}

“We can verify that a set is a field by checking that multiplication is a well defined operation i.e. it is independent of the representative.”

For example, for arbitrary rational numbers Q :

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1}$$

And evaluate an equivalent expresion with different representatives of the same numbers:

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2}$$

Given that:

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \quad \wedge \quad \frac{p_1}{q_1} = \frac{p_2}{q_2}$$

We want to verify that both numbers (both multiplication results) are the same. To review this, we start from the tautology (intuitive truth):

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \iff m_1 \times n_2 = m_2 \times n_1 \tag{A}$$

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \iff p_1 \times q_2 = p_2 \times q_1 \tag{B}$$

Then, operating the multiplication using both representatives:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_1 \cdot p_1}{n_1 \cdot q_1}$$

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2} = \frac{m_2 \cdot p_2}{n_2 \cdot q_2}$$

Conveniently, we want to form $m_1 \times n_2$ to form the first ground truth.

$$\begin{aligned} \frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 &= \frac{m_1 \cdot n_2 \cdot p_1}{n_1 \cdot q_1} \\ &= \frac{m_2 \cdot n_1 \cdot p_1}{n_1 \cdot q_1} && \text{(replacing A)} \\ &= \frac{m_2 \cdot p_1}{q_1} && \text{(simplifying } n_1) \end{aligned}$$

Applying the same logic for $p_1 \times q_2$ to form the second ground truth.

$$\begin{aligned}
\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 &= \frac{m_2 \cdot p_1 \cdot q_2}{q_1} \\
&= \frac{m_2 \cdot p_2 \cdot q_1}{q_1} && \text{(replacing } B) \\
&= m_2 \cdot p_2 && \text{(simplifying } q_1)
\end{aligned}$$

Finally, rearranging:

$$\begin{aligned}
\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 &= m_2 \cdot p_2 \\
\frac{m_1}{n_1} \times \frac{p_1}{q_1} &= \frac{m_2 \cdot p_2}{n_2 \cdot q_2} \\
\frac{m_1}{n_1} \times \frac{p_1}{q_1} &= \frac{m_2}{n_2} \times \frac{p_2}{q_2} \quad \blacksquare
\end{aligned}$$

This is not a rigorous demonstration, but gives us a first step to go from the intuition of a solution (particularly for Q), to a more formal procedure based on the real definition of a field.

2.2 Formal definition

Definition 2.1 (Field). A Field \mathbb{F} is a set A with two operations: addition (\oplus) and multiplication (\otimes), with the following properties:

- $x, y \in \mathbb{F} \rightarrow x \oplus y \in \mathbb{F}$
- $x, y \in \mathbb{F} \rightarrow x \oplus y = y \oplus x$
- $x, y, z \in \mathbb{F} \rightarrow (x \oplus y) \oplus z = x \oplus (y \oplus z)$
- $\exists 0 \in \mathbb{F} / \forall x \in \mathbb{F} \rightarrow x \oplus 0 = x$
- $\forall x \in \mathbb{F}, \exists -x \in \mathbb{F} : x \oplus -x = 0$
- $x, y \in \mathbb{F} \rightarrow x \otimes y \in \mathbb{F}$
- $x, y \in \mathbb{F} \rightarrow x \otimes y = y \otimes x$
- $x, y, z \in \mathbb{F} \rightarrow (x \otimes y) \otimes z = x \otimes (y \otimes z)$
- $\exists 1 \in \mathbb{F} / \forall x \in \mathbb{F} \rightarrow x \otimes 1 = x$
- $\forall x \in \mathbb{F} - \{0\}, \exists x^{-1} \in \mathbb{F} : x \otimes x^{-1} = 1$

First five correspond to the addition operation, and the last five to the multiplication operation. In order to relate both sets of properties, the following axiom is stated:

Axiom 2.1 (Distributive law). Let $x, y, z \in \mathbb{F}$

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$$

Theorem 2.1 (Zero uniqueness). *For any Field \mathbb{F} , there exists only one zero element.*

Proof. Assume 0_1 and 0_2 are zeros for a field \mathbb{F} .

$$\forall x \in \mathbb{F} : \begin{cases} 0_1 + x = x & (A) \\ 0_2 + x = x & (B) \end{cases}$$

For both cases, let x be $0_1, 0_2$ respectively

$$x = 0_2 \Rightarrow (A) \quad 0_1 + 0_2 = 0_2 \quad (C)$$

$$x = 0_1 \Rightarrow (B) \quad 0_2 + 0_1 = 0_1 \quad (D)$$

$$\Rightarrow (C) \quad 0_2 + 0_1 = 0_2 \quad (\text{Commutativity})$$

Comparing this result with D

$$\Rightarrow 0_2 + 0_1 = 0_2 \text{ and } 0_2 + 0_1 = 0_1$$

$$\Rightarrow 0_1 = 0_2$$

□

Definition 2.2 (Ordered sets). A set S will be ordered when having an ordering " $<$ ", so that $\forall x, y \in S$ one of the following properties should hold:

1. $x = y$
2. $x < y$
3. $y < x$

Definition 2.3 (Ordered fields). A field \mathbb{F} will be ordered if it is also an ordered set. As consequence, the following properties apply:

- $x, y \in \mathbb{F}, x < y \rightarrow \forall z \in \mathbb{F}, x \oplus z < y \oplus z$
- $x, y \in \mathbb{F}, 0 < x, y \rightarrow 0 < x \otimes y$

Theorem 2.2. *Given \mathbb{F} : ordered field. $x < y \wedge 0 < z \Rightarrow x \otimes z < y \otimes z$*

Proof. Proving by contradiction, lets assume the opposite:

$$x \otimes z \geq y \otimes z$$

Then adding $(-x \otimes z)$ on both sides, we would maintain the ordering of the expression

$$\begin{aligned} x \otimes z \oplus (-x \otimes z) &\geq y \otimes z \oplus (-x \otimes z) \\ 0 &\geq y \otimes z \oplus (-x \otimes z) \end{aligned}$$

Now, using Axiom 2.1, we get:

$$0 \geq z \otimes (y \oplus (-x))$$

Considering the first initial consitions for $x < y \Rightarrow y \oplus (-x) > 0$ and having $z > 0$, we would expect that the product these two to be > 0 by Definition 2.3 (second property). Hence:

$$0 \geq z \otimes (y \oplus (-x)) \wedge 0 < z \otimes (y \oplus (-x)) \dashv \dashv \text{ (Contradiction found)}$$

□

From this point and for a better readability, $+$ and \cdot (or \times) will be used instead of \oplus or \otimes . They will still represent the abstraction of a field's addition and product operations, without necessarily being the known addition and multiplication we could expect them to be.

