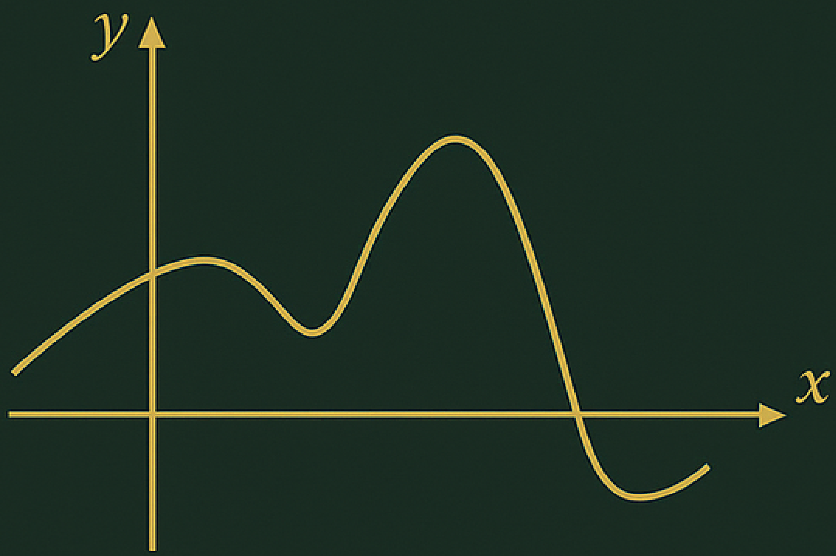
REAL ANALYSIS

A SELF-TAUGHT APPROACH



Luis Vasquez

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Contents

1 Introduction

1.1 Sources

The following notes are taken from the compilation of a few sources

- Zorich, V. A. (2004). Mathematical analysis I (R. Cooke, Trans.). Springer.
- MIT 18.100B Real Analysis, Spring 2025, available at the MIT OCW youtube channel

1.2 Purpose

This notes are taken in a way that is easy to understan math. No obscure proof or incomplete idea will be included, avoiding partial understanding of a certain topic. It also covers the need of having an easy-to-follow apporach to *Real Analysis*, making it possible to read this through to revisit known topics without the need of looking at other sources and rabbithole-ing into old books with russian lastnames on its cover. Since I will also be studying the course while taking this notes, the document as a whole will be written by hand without any AI slob nor blind copy/pasting, and since English is not my native language, typos may happen.

2 Base knowledge

2.1 What is a Field

"We can verify that a set is a field by checking that multiplication is a well-defined operation, i.e., it is independent of the representative."

For example, for arbitrary rational numbers Q:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1}$$

And evaluate an equivalent expression with different representatives of the same numbers:

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2}$$

Given that:

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \quad \text{and} \quad \frac{p_1}{q_1} = \frac{p_2}{q_2}$$

We want to verify that both multiplication results are the same. To review this, we start from the tautology (intuitive truth):

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \iff m_1 \times n_2 = m_2 \times n_1 \tag{A}$$

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \iff p_1 \times q_2 = p_2 \times q_1 \tag{B}$$

Then, operating the multiplication using both representatives:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_1 \cdot p_1}{n_1 \cdot q_1}$$

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2} = \frac{m_2 \cdot p_2}{n_2 \cdot q_2}$$

Conveniently, we want to form $m_1 \times n_2$ to use the first ground truth:

$$\begin{aligned} \frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 &= \frac{m_1 \cdot n_2 \cdot p_1}{n_1 \cdot q_1} \\ &= \frac{m_2 \cdot n_1 \cdot p_1}{n_1 \cdot q_1} \quad \text{(replacing using A)} \\ &= \frac{m_2 \cdot p_1}{q_1} \quad \text{(simplifying } n_1\text{)} \end{aligned}$$

Applying the same logic for $p_1 \times q_2$ to use the second ground truth:

$$\begin{aligned} \frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 &= \frac{m_2 \cdot p_1 \cdot q_2}{q_1} \\ &= \frac{m_2 \cdot p_2 \cdot q_1}{q_1} \quad \text{(replacing using B)} \\ &= m_2 \cdot p_2 \quad \text{(simplifying } q_1\text{)} \end{aligned}$$

Finally, rearranging:

$$\begin{aligned}\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 &= m_2 \cdot p_2 \\ \frac{m_1}{n_1} \times \frac{p_1}{q_1} &= \frac{m_2 \cdot p_2}{n_2 \cdot q_2} \\ \frac{m_1}{n_1} \times \frac{p_1}{q_1} &= \frac{m_2}{n_2} \times \frac{p_2}{q_2} \qquad \square\end{aligned}$$

This is not a rigorous demonstration, but gives us a first step to go from the intuition of a solution (particularly for $\mathbb{Q}$) to a more formal procedure based on the real definition of a field.

2.2 Formal definition

Definition
Field
2.1

A **field** $\mathbb{F}$ is a set with two operations: addition ($\oplus$) and multiplication ($\otimes$), with the following properties:

- $x, y \in \mathbb{F} \implies x \oplus y \in \mathbb{F}$
- $x, y \in \mathbb{F} \implies x \oplus y = y \oplus x$
- $x, y, z \in \mathbb{F} \implies (x \oplus y) \oplus z = x \oplus (y \oplus z)$
- $\exists\, 0 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}, x \oplus 0 = x$
- $\forall x \in \mathbb{F}, \exists\, (-x) \in \mathbb{F}$ such that $x \oplus (-x) = 0$
- $x, y \in \mathbb{F} \implies x \otimes y \in \mathbb{F}$
- $x, y \in \mathbb{F} \implies x \otimes y = y \otimes x$
- $x, y, z \in \mathbb{F} \implies (x \otimes y) \otimes z = x \otimes (y \otimes z)$
- $\exists\, 1 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}, x \otimes 1 = x$
- $\forall x \in \mathbb{F} \setminus \{0\}, \exists\, x^{-1} \in \mathbb{F}$ such that $x \otimes x^{-1} = 1$

The first five properties correspond to the addition operation, and the last five to the multiplication operation. In order to relate both sets of properties, the following axiom is stated:

Axiom
Distributive law
2.1

Let $x, y, z \in \mathbb{F}$. Then
$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$$

Theorem
Zero uniqueness
2.1

For any field $\mathbb{F}$, there exists only one zero element.

Proof Assume 0_1 and 0_2 are zeros for a field $\mathbb{F}$. Then

$$\forall x \in \mathbb{F} : \quad \begin{cases} 0_1 + x = x & \text{(A)} \\ 0_2 + x = x & \text{(B)} \end{cases}$$

For both cases, let x be 0_2 and 0_1 respectively:

$$x = 0_2 \implies \text{(A)}: \quad 0_1 + 0_2 = 0_2 \tag{C}$$
$$x = 0_1 \implies \text{(B)}: \quad 0_2 + 0_1 = 0_1 \tag{D}$$

From (C) and commutativity:

$$0_2 + 0_1 = 0_2$$

Comparing this result with (D):

$$\begin{aligned} 0_2 + 0_1 = 0_2 \quad \text{and} \quad 0_2 + 0_1 = 0_1 \\ \implies 0_1 = 0_2 \end{aligned}$$

$\square$

2.3 Order

Definition
Ordered sets
2.2

A set S is **ordered** when it has an ordering "<" such that for all $x, y \in S$, exactly one of the following properties holds:

1. $x = y$
2. $x < y$
3. $y < x$

Definition
Ordered fields
2.3

A field $\mathbb{F}$ is **ordered** if it is also an ordered set. As a consequence, the following properties apply:

- $x, y \in \mathbb{F}, x < y \implies \forall z \in \mathbb{F}, x \oplus z < y \oplus z$
- $x, y \in \mathbb{F}, 0 < x, y \implies 0 < x \otimes y$

Theorem
2.2

Given $\mathbb{F}$ an ordered field. If $x < y$ and $0 < z$, then $x \otimes z < y \otimes z$.

Proof

We prove by contradiction. Assume the opposite:

$$x \otimes z \geq y \otimes z$$

Then adding $(-x \otimes z)$ on both sides, we maintain the ordering of the expression:

$$\begin{aligned} x \otimes z \oplus (-x \otimes z) &\geq y \otimes z \oplus (-x \otimes z) \\ 0 &\geq y \otimes z \oplus (-x \otimes z) \end{aligned}$$

Now, using Axiom 2.1, we get:

$$0 \geq z \otimes (y \oplus (-x))$$

From the initial conditions, $x < y$ implies $y \oplus (-x) > 0$. Since we also have $z > 0$, we would expect the product of these two to be > 0 by Definition 2.3 (second property). Hence:

$$0 \geq z \otimes (y \oplus (-x)) \quad \text{and} \quad 0 < z \otimes (y \oplus (-x))$$

This is a contradiction. $\square$

From this point forward, for better readability, $+$ and $\cdot$ (or $\times$) will be used instead of $\oplus$ and $\otimes$. They will still represent the abstraction of a field's addition and multiplication operations, without necessarily being the familiar operations we might expect them to be.

2.4 Completeness

Definition Bounds 2.4 | A set $X \subset \mathbb{F}$ ($\mathbb{F}$ ordered field) is said to be *bounded above* (or respectively, *bounded below*) if $\exists\ c \in \mathbb{F}$ such that $\forall a \in \mathbb{F}, a \leq x$ (or respectively, $x \leq a$). c is called upper (or respectively lower) bound of X.

Definition 2.5 | A set that is both bounded above and below, is called *bounded.*

Definition 2.6 | An element $a \in X$ is called the *largest* element of X if $\forall x \in X, x \leq a$. Respectively, $a \in X$ is called the *smallest* element of X if $\forall x \in X, a \leq x$. Simplifying the notation:

$$(a = maxX) := (a \in X \land \forall x \in X, x \leq a)$$
$$(a = minX) := (a \in X \land \forall x \in X, a \leq x)$$

These read as *maximal* and *minimal* of X. Now, given this definition, is important to notice that not every set, not even every bounded set, has a maximal or minimal element. For example:

$$X = \{x \in \mathbb{F} \mid 0 \leq x < 1\}$$

Only has a minimal element (0), but no maximal element, since $1 \notin X$

Definition Least Upper Bound 2.7 | The smallest $s \in X \subset \mathbb{F}$ that bounds X from above is called the *least upper bound* of X, and denoted $\sup X$ (read "the supremum of X")

$$(s = \sup X) := \forall x \in X((x \leq s) \land (\forall s' < s\ \exists x' \in X(s' < x'))$$

Lets break this down by element

- $\forall x \in X$: The following definition applies to the whole set X.
- $(x \leq s)$: Given that s is an upper bound for X...
- $(\forall s' < s)\exists x' \in X(s' < x')$:
  - $(\forall s' < s)$: Considering any arbitrary s' smaller than our upper bound s
  - $\exists x' \in X$: There will be an element x' in X, so that...
  - $(s' < x')$: It is larger than the s', making s' to **fail** to be an upper bound.

So in summary, s is $\sup X$ if and only if, s is an upper bound, and no smaller number s' is an upper bound of X, because we can find an x' that is not bounded by it.

Definition Greatest Lower Bound 2.8

Similarly, the greatest $i \in X \subset \mathbb{F}$ that bounds X below is called the *greatest lower bound of* X, and denoted $\inf X$ (read "the infimum of X")

$$(i = \inf X) := \forall x \in X((i \leq x) \wedge (\forall i' < i \, \exists x' \in X(x' < i'))$$

Thus, we have now the following definitions:

$$\sup X := \min\{c \in \mathbb{F} | \forall x \in X(x \leq c)\}$$
$$\inf X := \max\{c \in \mathbb{F} | \forall x \in X(c \leq x)\}$$

It is important to note that the supremum and infimum of a set, as defined above, may not exist in an arbitrary ordered field $\mathbb{F}$. The definitions above specify what $\sup X$ and $\inf X$ mean *if they exist*, but they do not guarantee existence. We will address this issue shortly.

Theorem Uniqueness of Supremum 2.3

Let $X \subset \mathbb{F}$ be a nonempty set in an ordered field $\mathbb{F}$. If X has a supremum, then this supremum is unique.

Proof

Suppose s_1 and s_2 are both suprema of X. We will show that $s_1 = s_2$.
Since $s_1 = \sup X$, we know that s_1 is an upper bound of X. Since $s_2 = \sup X$, we know that s_2 is the *least* upper bound of X. Therefore:

$$s_2 \leq s_1$$

By the same reasoning (swapping the roles of s_1 and s_2), since s_2 is an upper bound and s_1 is the least upper bound:

$$s_1 \leq s_2$$

By the antisymmetry property of order in $\mathbb{F}$, we have $s_1 \leq s_2$ and $s_2 \leq s_1$, which implies:

$$s_1 = s_2$$

Therefore, the supremum is unique. □

The proof for the uniqueness of the infimum is analogous.

Example 2.1

Consider the ordered field $\mathbb{Q}$ of rational numbers, and let:

$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

The set S is nonempty (for instance, $1 \in S$) and bounded above (for instance, $2 \in \mathbb{F}$ is an upper bound). However, S does not have a supremum in $\mathbb{Q}$, since if it had, it would have to equal $\sqrt{2} \notin \mathbb{Q}$, so S has no least upper bound within the rational numbers.
This shows that not every ordered field has the property that bounded sets possess suprema.

Definition Complete Ordered Field 2.9

An ordered field $\mathbb{F}$ is called *complete* if every nonempty subset of $\mathbb{F}$ that is bounded above has a supremum in $\mathbb{F}$.

Proposition 2.1

If $\mathbb{F}$ is a complete ordered field, then every nonempty subset of $\mathbb{F}$ that is bounded below has an infimum in $\mathbb{F}$.

Proof Let $X \subset \mathbb{F}$ be nonempty and bounded below. Define:

$$X' = \{-x \mid x \in X\}$$

1. X' is nonempty and bounded above
Since X is nonempty, $\exists x_0 \in X \Rightarrow -x_0 \in X'$ by definition of X'. Thus $X' \neq \emptyset$.
Now, X is bounded below by the proposition, so there exists $c \in \mathbb{F}$ such that:

$$\begin{aligned} \forall x \in X, &\quad c \leq x \\ \forall x \in X, &\quad -x \leq -c \end{aligned}$$

Since $-x \in X^prime$ and $-c \in \mathbb{F}$, we can be sure that X^prime is bounded above in $\mathbb{F}$.
Considering $\mathbb{F}$ is complete, and we proved X' is nonempty and bounded above, X' has a supremum in $\mathbb{F}$.

2. $-s$ is a lower bound for X.
Let $s = \sup X'$ and $x \in X$ be arbitrary. Then $-x \in X'$ by definition of X'. Since s is an upper bound for X':

$$\begin{aligned} \forall - x \in X', &\quad -x \leq s \\ \Rightarrow &\quad - s \leq x \end{aligned}$$

Showing that $-s$ is a lower bound for X.

3: $-s$ is the greatest lower bound for X.
Let $\ell \in \mathbb{F}$ be any lower bound for X.
Since ℓ is a lower bound for X:

$$\begin{aligned} \forall x \in X, &\quad \ell \leq x \\ \Rightarrow &\quad - x \leq -\ell \end{aligned}$$

Meaning that $-\ell$ is an upper bound for any $-x \in X'$. Since $s = \sup X'$, we have:

$$\begin{aligned} & s \leq -\ell \\ \Rightarrow & \ell \leq -s \end{aligned}$$

Showing that $-s$ is greater than or equal to every lower bound of X. Therefore, $-s = \inf X$. □

Theorem
Least Upper Bound Principle
2.4

Let $\mathbb{F}$ be a complete ordered field. Every nonempty subset of $\mathbb{F}$ that is bounded above has a unique least upper bound in $\mathbb{F}$.

Proof

Let $X \subset \mathbb{F}$ be nonempty and bounded above.
Existence: Since $\mathbb{F}$ is complete, by definition of completeness, X has a supremum in $\mathbb{F}$.
Uniqueness: By the Uniqueness of Supremum theorem, this supremum is unique.
Therefore, X has a unique least upper bound. $\square$