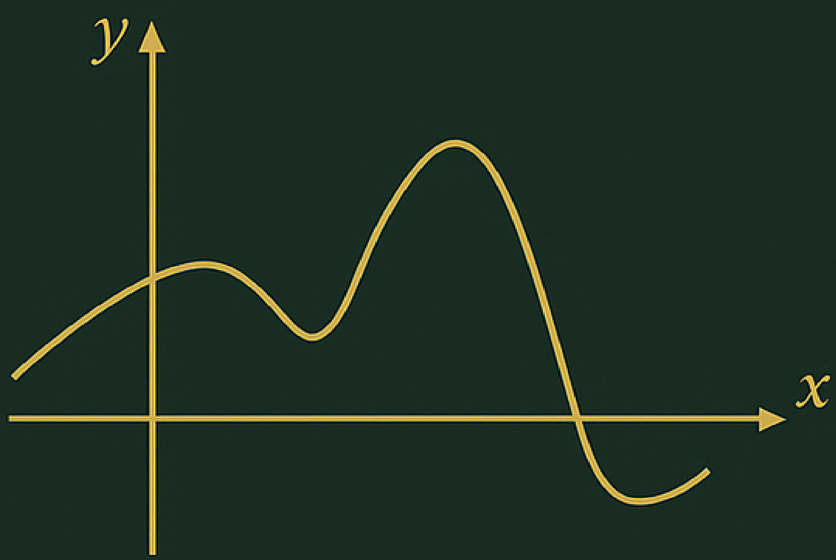
REAL ANALYSIS

A SELF-TAUGHT APPROACH



Luis Vasquez

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Contents

1 Introduction

1.1 Sources

The following notes are taken from the compilation of a few sources

- Zorich, V. A. (2004). Mathematical analysis I (R. Cooke, Trans.). Springer.
- MIT 18.100B Real Analysis, Spring 2025, available at the MIT OCW youtube channel

1.2 Purpose

This notes are taken in a way that is easy to understan math. No obscure proof or incomplete idea will be included, avoiding partial understanding of a certain topic. It also covers the need of having an easy-to-follow approach to *Real Analysis*, making it possible to read this through to revisit known topics without the need of looking at other sources and rabbithole-ing into old books with russian last names on its cover. Since I will also be studying the course while taking this notes, the document as a whole will be written by hand without any AI slob nor blind copy/pasting, and since English is not my native language, typos may happen.

When I say ***we****, it means you (the reader) and I. When I say* ***I****, responsability only lies on me.*

2 Base knowledge

2.1 What is a Field

"We can verify that a set is a field by checking that multiplication is a well-defined operation, i.e., it is independent of the representative."

For example, for arbitrary rational numbers Q:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1}$$

And evaluate an equivalent expression with different representatives of the same numbers:

$$\frac{m_2}{n_2} \times \frac{p_2}{q_2}$$

Given that:

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \quad \text{and} \quad \frac{p_1}{q_1} = \frac{p_2}{q_2}$$

We want to verify that both multiplication results are the same. To review this, we start from the tautology (intuitive truth):

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} \iff m_1 \times n_2 = m_2 \times n_1 \tag{A}$$

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} \iff p_1 \times q_2 = p_2 \times q_1 \tag{B}$$

Then, operating the multiplication using both representatives:

$$\frac{m_1}{n_1} \times \frac{p_1}{q_1} = \frac{m_1 \cdot p_1}{n_1 \cdot q_1}$$
$$\frac{m_2}{n_2} \times \frac{p_2}{q_2} = \frac{m_2 \cdot p_2}{n_2 \cdot q_2}$$

Conveniently, we want to form $m_1 \times n_2$ to use the first ground truth:

$$\begin{aligned}\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 &= \frac{m_1 \cdot n_2 \cdot p_1}{n_1 \cdot q_1} \\ &= \frac{m_2 \cdot n_1 \cdot p_1}{n_1 \cdot q_1} \quad \text{(replacing using A)} \\ &= \frac{m_2 \cdot p_1}{q_1} \quad \text{(simplifying } n_1\text{)}\end{aligned}$$

Applying the same logic for $p_1 \times q_2$ to use the second ground truth:

$$\begin{aligned}\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 &= \frac{m_2 \cdot p_1 \cdot q_2}{q_1} \\ &= \frac{m_2 \cdot p_2 \cdot q_1}{q_1} \quad \text{(replacing using B)} \\ &= m_2 \cdot p_2 \quad \text{(simplifying } q_1\text{)}\end{aligned}$$

Finally, rearranging:

$$\begin{aligned}\frac{m_1}{n_1} \times \frac{p_1}{q_1} \times n_2 \times q_2 &= m_2 \cdot p_2 \\ \frac{m_1}{n_1} \times \frac{p_1}{q_1} &= \frac{m_2 \cdot p_2}{n_2 \cdot q_2} \\ \frac{m_1}{n_1} \times \frac{p_1}{q_1} &= \frac{m_2}{n_2} \times \frac{p_2}{q_2} \qquad \square\end{aligned}$$

This is not a rigorous demonstration, but gives us a first step to go from the intuition of a solution (particularly for $\mathbb{Q}$) to a more formal procedure based on the real definition of a field.

2.2 Formal definition

Definition
Field
2.1

A **field** $\mathbb{F}$ is a set with two operations: addition ($\oplus$) and multiplication ($\otimes$), with the following properties:

- $x, y \in \mathbb{F} \implies x \oplus y \in \mathbb{F}$
- $x, y \in \mathbb{F} \implies x \oplus y = y \oplus x$
- $x, y, z \in \mathbb{F} \implies (x \oplus y) \oplus z = x \oplus (y \oplus z)$
- $\exists\, 0 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}, x \oplus 0 = x$
- $\forall x \in \mathbb{F}, \exists\, (-x) \in \mathbb{F}$ such that $x \oplus (-x) = 0$
- $x, y \in \mathbb{F} \implies x \otimes y \in \mathbb{F}$
- $x, y \in \mathbb{F} \implies x \otimes y = y \otimes x$
- $x, y, z \in \mathbb{F} \implies (x \otimes y) \otimes z = x \otimes (y \otimes z)$
- $\exists\, 1 \in \mathbb{F}$ such that $\forall x \in \mathbb{F}, x \otimes 1 = x$
- $\forall x \in \mathbb{F} \setminus \{0\}, \exists\, x^{-1} \in \mathbb{F}$ such that $x \otimes x^{-1} = 1$

The first five properties correspond to the addition operation, and the last five to the multiplication operation. In order to relate both sets of properties, the following axiom is stated:

Axiom
Distributive law
2.1

Let $x, y, z \in \mathbb{F}$. Then

$$x \otimes (y \oplus z) = x \otimes y \oplus x \otimes z$$

Theorem
Zero uniqueness
2.1

For any field $\mathbb{F}$, there exists only one zero element.

Proof Assume 0_1 and 0_2 are zeros for a field $\mathbb{F}$. Then

$$\forall x \in \mathbb{F} : \quad \begin{cases} 0_1 + x = x & \text{(A)} \\ 0_2 + x = x & \text{(B)} \end{cases}$$

For both cases, let x be 0_2 and 0_1 respectively:

$$x = 0_2 \implies \text{(A)} : \quad 0_1 + 0_2 = 0_2 \tag{C}$$
$$x = 0_1 \implies \text{(B)} : \quad 0_2 + 0_1 = 0_1 \tag{D}$$

From (C) and commutativity:

$$0_2 + 0_1 = 0_2$$

Comparing this result with (D):

$$0_2 + 0_1 = 0_2 \quad \text{and} \quad 0_2 + 0_1 = 0_1$$
$$\implies 0_1 = 0_2$$

$\square$

2.3 Order

Definition
Ordered sets
2.2

A set S is **ordered** when it has an ordering "$<$" such that for all $x, y \in S$, exactly one of the following properties holds:

1. $x = y$
2. $x < y$
3. $y < x$

Definition
Ordered fields
2.3

A field $\mathbb{F}$ is **ordered** if it is also an ordered set. As a consequence, the following properties apply:

- $x, y \in \mathbb{F}, x < y \implies \forall z \in \mathbb{F}, x \oplus z < y \oplus z$
- $x, y \in \mathbb{F}, 0 < x, y \implies 0 < x \otimes y$

Theorem
2.2

Given $\mathbb{F}$ an ordered field. If $x < y$ and $0 < z$, then $x \otimes z < y \otimes z$.

Proof

We prove by contradiction. Assume the opposite:

$$x \otimes z \geq y \otimes z$$

Then adding $(-x \otimes z)$ on both sides, we maintain the ordering of the expression:

$$\begin{aligned} x \otimes z \oplus (-x \otimes z) &\geq y \otimes z \oplus (-x \otimes z) \\ 0 &\geq y \otimes z \oplus (-x \otimes z) \end{aligned}$$

Now, using Axiom 2.1, we get:

$$0 \geq z \otimes (y \oplus (-x))$$

From the initial conditions, $x < y$ implies $y \oplus (-x) > 0$. Since we also have $z > 0$, we would expect the product of these two to be > 0 by Definition 2.3 (second property). Hence:

$$0 \geq z \otimes (y \oplus (-x)) \quad \text{and} \quad 0 < z \otimes (y \oplus (-x))$$

This is a contradiction. $\square$

From this point forward, for better readability, $+$ and $\cdot$ (or $\times$) will be used instead of $\oplus$ and $\otimes$. They will still represent the abstraction of a field's addition and multiplication operations, without necessarily being the familiar operations we might expect them to be.

2.4 Completeness

Definition Bounds 2.4 | A set $X \subset \mathbb{F}$ ($\mathbb{F}$ ordered field) is said to be *bounded above* (or respectively, *bounded below*) if $\exists\ c \in \mathbb{F}$ such that $\forall a \in \mathbb{F}, a \leq x$ (or respectively, $x \leq a$). c is called upper (or respectively lower) bound of X.

Definition 2.5 | A set that is both bounded above and below, is called *bounded.*

Definition 2.6 | An element $a \in X$ is called the *largest* element of X if $\forall x \in X, x \leq a$. Respectively, $a \in X$ is called the *smallest* element of X if $\forall x \in X, a \leq x$. Simplifying the notation:

$$(a = maxX) := (a \in X \wedge \forall x \in X, x \leq a)$$
$$(a = minX) := (a \in X \wedge \forall x \in X, a \leq x)$$

These read as *maximal* and *minimal* of X. Now, given this definition, is important to notice that not every set, not even every bounded set, has a maximal or minimal element. For example:

$$X = \{x \in \mathbb{F} \mid 0 \leq x < 1\}$$

Only has a minimal element (0), but no maximal element, since $1 \notin X$

Definition Least Upper Bound 2.7 | The smallest $s \in X \subset \mathbb{F}$ that bounds X from above is called the *least upper bound* of X, and denoted $\sup X$ (read "the supremum of X")

$$(s = \sup X) := \forall x \in X((x \leq s) \wedge (\forall s' < s\ \exists x' \in X(s' < x'))$$

Lets break this down by element

- $\forall x \in X$: The following definition applies to the whole set X.
- $(x \leq s)$: Given that s is an upper bound for X...
- $(\forall s' < s)\exists x' \in X(s' < x')$:
  - $(\forall s' < s)$: Considering any arbitrary s' smaller than our upper bound s
  - $\exists x' \in X$: There will be an element x' in X, so that...
  - $(s' < x')$: It is larger than the s', making s' to **fail** to be an upper bound.

So in summary, s is $\sup X$ if and only if, s is an upper bound, and no smaller number s' is an upper bound of X, because we can find an x' that is not bounded by it.

Definition
Greatest Lower Bound
2.8

Similarly, the greatest $i \in X \subset \mathbb{F}$ that bounds X below is called the *greatest lower bound of* X, and denoted $\inf X$ (read "the infimum of X")

$$(i = \inf X) := \forall x \in X((i \leq x) \land (\forall i' < i \; \exists x' \in X(x' < i'))$$

Thus, we have now the following definitions:

$$\sup X := \min\{c \in \mathbb{F} | \forall x \in X(x \leq c)\}$$
$$\inf X := \max\{c \in \mathbb{F} | \forall x \in X(c \leq x)\}$$

It is important to note that the supremum and infimum of a set, as defined above, may not exist in an arbitrary ordered field $\mathbb{F}$. The definitions above specify what $\sup X$ and $\inf X$ mean *if they exist*, but they do not guarantee existence. We will address this issue shortly.

Theorem
Uniqueness of Supremum
2.3

Let $X \subset \mathbb{F}$ be a nonempty set in an ordered field $\mathbb{F}$. If X has a supremum, then this supremum is unique.

Proof

Suppose s_1 and s_2 are both suprema of X. We will show that $s_1 = s_2$.
Since $s_1 = \sup X$, we know that s_1 is an upper bound of X. Since $s_2 = \sup X$, we know that s_2 is the *least* upper bound of X. Therefore:

$$s_2 \leq s_1$$

By the same reasoning (swapping the roles of s_1 and s_2), since s_2 is an upper bound and s_1 is the least upper bound:

$$s_1 \leq s_2$$

By the antisymmetry property of order in $\mathbb{F}$, we have $s_1 \leq s_2$ and $s_2 \leq s_1$, which implies:

$$s_1 = s_2$$

Therefore, the supremum is unique. □

The proof for the uniqueness of the infimum is analogous.

Example 2.1

Consider the ordered field $\mathbb{Q}$ of rational numbers, and let:

$$S = \{x \in \mathbb{Q} \mid x^2 < 2\}$$

The set S is nonempty (for instance, $1 \in S$) and bounded above (for instance, $2 \in \mathbb{F}$ is an upper bound). However, S does not have a supremum in $\mathbb{Q}$, since if it had, it would have to equal $\sqrt{2} \notin \mathbb{Q}$, so S has no least upper bound within the rational numbers.
This shows that not every ordered field has the property that bounded sets possess suprema.

Definition Complete Ordered Field 2.9

An ordered field $\mathbb{F}$ is called *complete* if every nonempty subset of $\mathbb{F}$ that is bounded above has a supremum in $\mathbb{F}$.

Proposition 2.1

If $\mathbb{F}$ is a complete ordered field, then every nonempty subset of $\mathbb{F}$ that is bounded below has an infimum in $\mathbb{F}$.

Proof Let $X \subset \mathbb{F}$ be nonempty and bounded below. Define:

$$X' = \{-x \mid x \in X\}$$

1. X' is nonempty and bounded above
Since X is nonempty, $\exists x_0 \in X \Rightarrow -x_0 \in X'$ by definition of X'. Thus $X' \neq \emptyset$.
Now, X is bounded below by the proposition, so there exists $c \in \mathbb{F}$ such that:

$$\begin{aligned} &\forall x \in X, \quad c \leq x \\ &\forall x \in X, \quad -x \leq -c \end{aligned}$$

Since $-x \in X'$ and $-c \in \mathbb{F}$, we can be sure that X' is bounded above in $\mathbb{F}$.
Considering $\mathbb{F}$ is complete, and we proved X' is nonempty and bounded above, X' has a supremum in $\mathbb{F}$.

2. $-s$ is a lower bound for X.
Let $s = \sup X'$ and $x \in X$ be arbitrary. Then $-x \in X'$ by definition of X'. Since s is an upper bound for X':

$$\begin{aligned} \forall -x \in X', &\quad -x \leq s \\ \Rightarrow &\quad -s \leq x \end{aligned}$$

Showing that $-s$ is a lower bound for X.

3: $-s$ is the greatest lower bound for X.
Let $\ell \in \mathbb{F}$ be any lower bound for X.
Since ℓ is a lower bound for X:

$$\begin{aligned} \forall x \in X, &\quad \ell \leq x \\ \Rightarrow &\quad -x \leq -\ell \end{aligned}$$

Meaning that $-\ell$ is an upper bound for any $-x \in X'$. Since $s = \sup X'$, we have:

$$\begin{aligned} &s \leq -\ell \\ \Rightarrow\ &\ell \leq -s \end{aligned}$$

Showing that $-s$ is greater than or equal to every lower bound of X. Therefore, $-s = \inf X$. $\square$

Theorem Least Upper Bound Principle 2.4

Let $\mathbb{F}$ be a complete ordered field. Every nonempty subset of $\mathbb{F}$ that is bounded above has a unique least upper bound in $\mathbb{F}$.

Proof

Let $X \subset \mathbb{F}$ be nonempty and bounded above.
Existence: Since $\mathbb{F}$ is complete, by definition of completeness, X has a supremum in $\mathbb{F}$.
Uniqueness: By the Uniqueness of Supremum theorem, this supremum is unique.
Therefore, X has a unique least upper bound. □

2.5 Archimedean property

Now that we achieved completeness on an ordered field, we can now extend the definitions and properties into a theorem/property that uses the result from the *Least Upper Bound principle* (Theorem 2.4).

Theorem 2.5

Let $\mathbb{F}$ be a complete ordered field, then

$$\forall x \in \mathbb{F}, \exists! n \in \mathbb{N} \subset \mathbb{F} : n > x$$

Proof We first define the subset $\mathbb{N} \subset \mathbb{F}$ **inductively**, to work with a generalized version of the set of "natural numbers"

1. $1_{\mathbb{F}}$ is the multiplicative identity in $\mathbb{F}$
2. $n + 1_{\mathbb{F}} \in \mathbb{N}, \forall n \in \mathbb{N}$ defines the induction step over $\mathbb{N}$, using the addition operation from the field $\mathbb{F}$

We now proceed to prove by contradiction, assuming $\exists x \in \mathbb{F}, \forall n \in \mathbb{N}$:

$$n \leq x$$

Meaning that $\mathbb{N}$ is bounded above. Now, since $\mathbb{N}$ is not empty, and is bounded by an element of the (complete ordered) field that contains it ($x \in \mathbb{F}$), then by *L.U.B. principle* (Theorem 2.4), $\mathbb{N}$ must have a supremum.

Let $s := \sup \mathbb{N} \in \mathbb{F}$ be that supremum of $\mathbb{N}$. Then, since s is the least upper bound, any element that is less than s will no longer be an upper bound for $\mathbb{N}$. Conveniently we take $s - 1_{\mathbb{F}}$:

$$\begin{aligned} \exists n_0 \in \mathbb{N} : & s - 1_{\mathbb{F}} < n_0 \\ & \Rightarrow s < n_0 + 1_{\mathbb{F}} \end{aligned}$$

But by our inductive definition of $\mathbb{N} : (n_0 + 1_{\mathbb{F}}) \in \mathbb{N}$. So we found out that s ($\sup \mathbb{N}$) fails to be an upper bound for an element in $\mathbb{N}$.

Therefore, our original assumption of x must be wrong. $\square$

Now, is important to notice that $\mathbb{F}$ implies an archimedien field, but the opposite won't necessarily be true.

Example 2.2 Using an Archimedean field, that is not complete, lets take $\mathbb{F} = \mathbb{Q}$

$$x \in \mathbb{Q} \wedge n \in \mathbb{N} \subset \mathbb{Q}$$

It is true that, for any x there is a natural number n so that

$$n > x$$

But invoking the previous example (Example 2.1) we can take a subset of $\mathbb{Q}$ that is not bounded in $\mathbb{Q}$, failing to properly form a complete set.

Closing this chapter, lets emphazise the fact that the construction of $\mathbb{N} \subset \mathbb{F}$ by induction was crucial to generalize the property for any complete ordered field. Next we will explain why this wasn't necessary at all.

3 $\mathbb{R}$ as the complete ordered field

3.1 $\mathbb{F}$ is $\mathbb{R}$

Turns out that there is a reason why it is called ***Real*** Analysis. So far we have worked over a generic *Complete Ordered Field* $\mathbb{F}$, that was also an *Archimedean* following the expected properties. We have finally reached the point were we can state that **any** set that manages to follow all these properties is actually identical to the set of real numbers $\mathbb{R}$. In fact, $\mathbb{R}$ is the only set that accomplishes this.

Most Real Analysis courses don't go into proving this, but as I already mentioned in the first chapter, I want to cover all doubts and missing point existent during a course like this.

We will go through the initial proofs needed to understand the main one. At the beginning its possible that some of them look unecessary or trivial, but I don't want to assume that much. Better to be safe than sorry.

3.1.1 Isomorphism

Our starting point will be defining what it means to have "equal fields".

Definition Field homomorphism 3.1

Let F, F' fields. A function $\phi : F \to F'$ is a homomorphism between F and F' iif. it preserves the defined operations in both fields:

$$\phi(x + y) = \phi(x) + \phi(y), \forall x, y \in F$$
$$\phi(x.y) = \phi(x).\phi(y), \forall x, y \in F$$

Definition Field isomorphism 3.2

Let F, F' fields. A function $\phi : F \to F'$ is a isomorphism iif. it is a homomorphism and also a bijection

Lemma
Injectivity by order preservation
3.1

Let F, F' fields, $\phi : F \to F'$. If ϕ preserves order from $\mathbb{F}$:

$$x < y \Rightarrow \phi(x) < \phi(y), x, y \in F$$

Then ϕ is injective.

Proof

To prove injectivity, we need to show that for any two identical images of the function ϕ, we get that their arguments were the same, guaranteeing the uniqueness of the image:
Let $x, y \in \mathbb{F}$ so that

$$\phi(x) = \phi(y)$$

Using the tricotomy over F:
Case 1: $x < y$:

$$\to \phi(x) < \phi(y)(\to\leftarrow)$$

Case 2: $y < x$:

$$\to \phi(y) < \phi(x)(\to\leftarrow)$$

$$\Rightarrow x = y$$

Then, ϕ is injective. $\square$

Lemma
Surjectivity by construction
3.2

Let F, F' fields, $\phi : F \to F'$. Define:

$$\text{Im}_F := \{\phi(x), x \in F\}$$

Then, the bounded function $\phi' : F \to \text{Im}_F$ is surjective

Proof To prove surjectivity, we need to show that every element in $y \in \mathrm{Im}_F$ has a $x \in F$ such that $\phi'(x) = y$
Using the definition of our constructed set:

$$\forall y \in \mathrm{Im}_F \Leftrightarrow y \in \{\phi(x), x \in F\}$$
$$\Rightarrow \exists x \in F / \forall y \in \mathrm{Im}_F : y = \phi'(x)$$

Finally proving that for any image of $\phi' \in \mathrm{Im}_F$ we will have a preimage $x \in F$.
Then ϕ' is surjective. □

On this first part, this surjectiveness by construction might be the bit that doesn't quite seem correct. Isn't it trivial to have surjectiveness if we grab the method ϕ and limit it to its image ϕ'?

The key here is that we defined the set of images Im_F first, and *then* proved the surjectiveness in it. The lemma confirms that the set we defined was *just right*. Not so big to leave some elements unreached, and not too small to make undefined preimages at some spots.

3.1.2 Sets construction