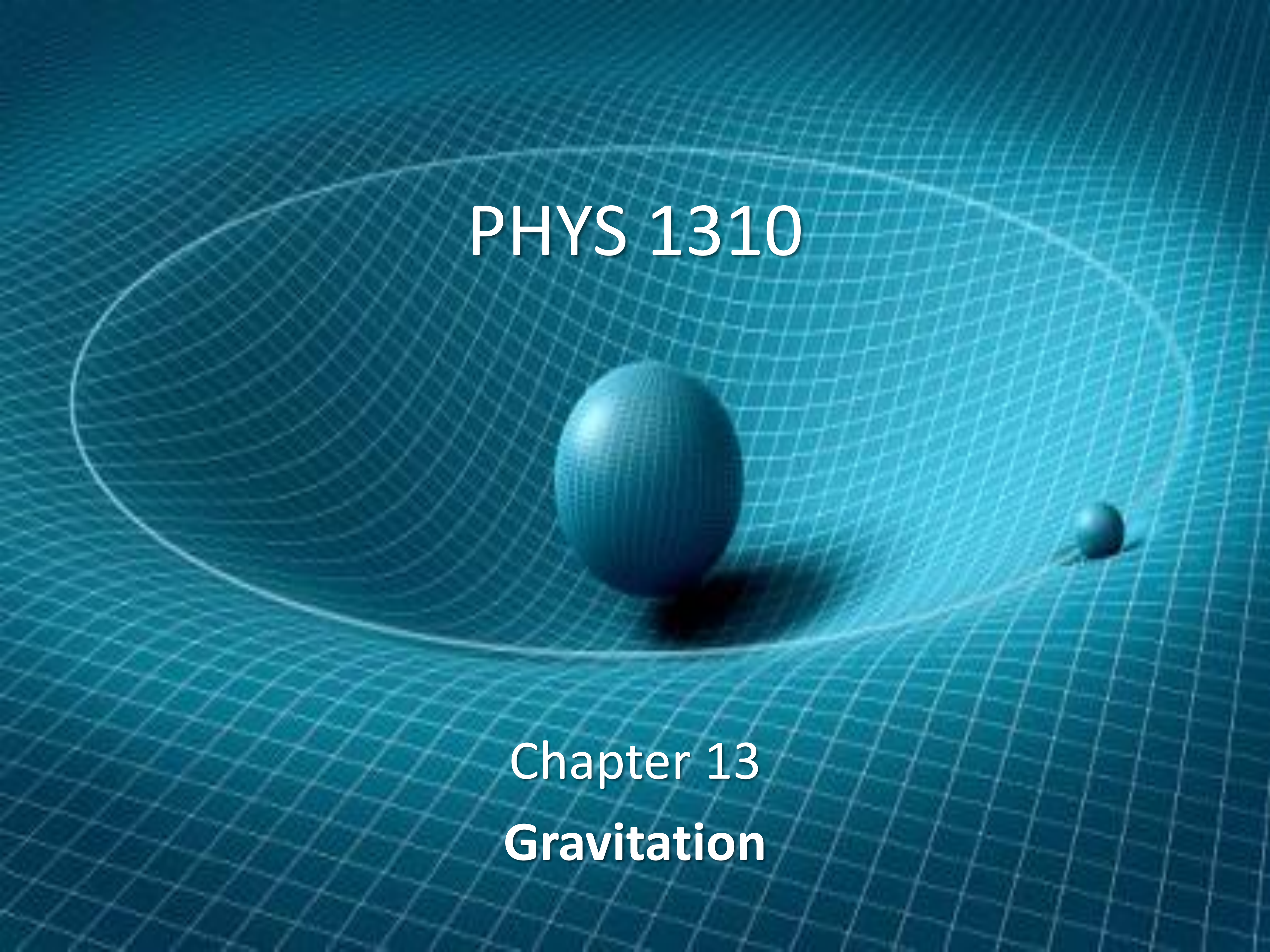


PHYS 1310

Chapter 13

Gravitation

CH13 Gravitation in brief

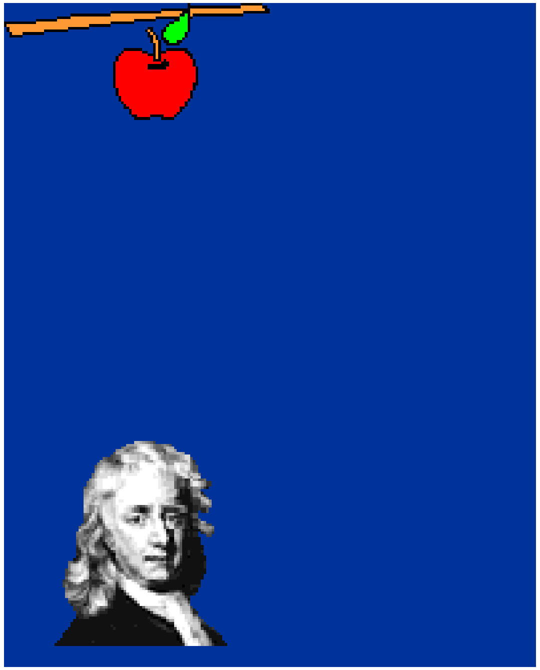
- Universal gravitation: $$F = \frac{GMm}{R^2}$$
- The gravitational constant: $$a_g = G\frac{M_E}{{R_E}^2} = 9.8\,\mathrm{m/s}^2$$
- Gravitational potential energy: $$U = \int F\,dr = \frac{-GMm}{R}$$
- Velocity of circular orbit: $$v = \frac{2\pi R}{T} = \sqrt{\frac{GM}{R}}$$
- Orbital period: $$T = \frac{2\pi R}{v} = \frac{2\pi R^{3/2}}{\sqrt{GM}}$$
- Schwarzschild radius: $$R = \frac{2GM}{c^2}$$

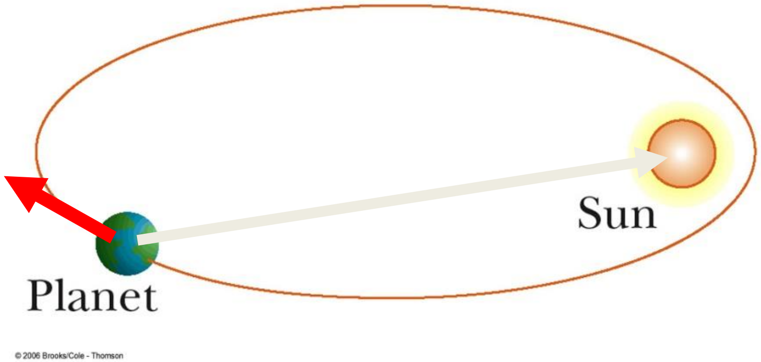
Overview

- 13.1 Newton's Law of Gravitation
- 13.2 Weight
- 13.3 Gravitational Potential Energy
- 13.4 Motion of Satellites
- 13.5 Kepler's Laws and Planetary Motion
- 13.6 Black Holes

13.1 Newton Universal Gravitation

- The apple was attracted to the Earth
- All objects in the Universe were attracted to each other in the same way the apple was attracted to the Earth

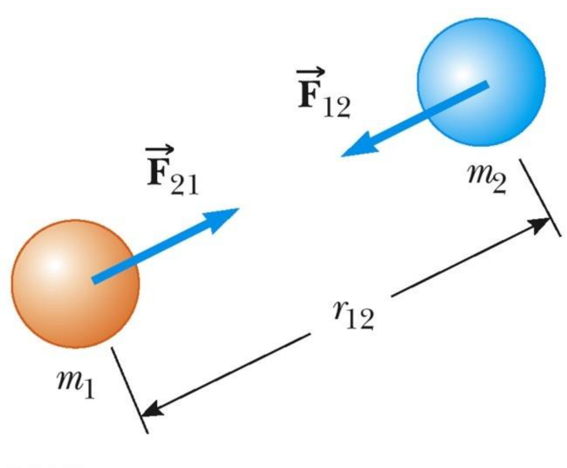




Newton's Law of Universal Gravitation

- Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of the masses and <u>inversely proportional to the square</u> of the distance between them.

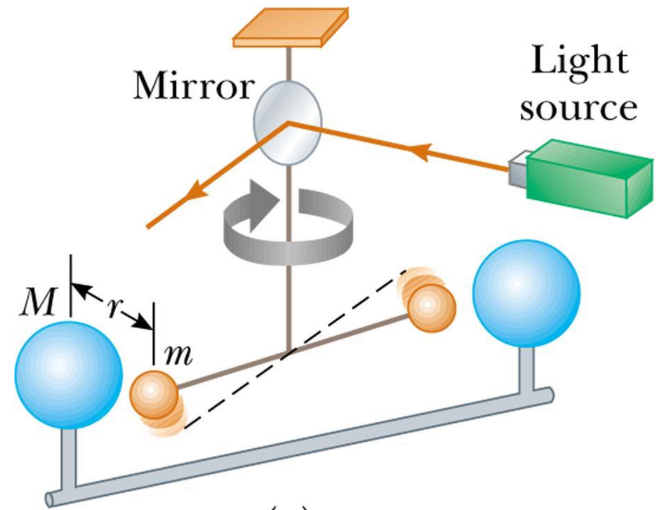
$$F = G\frac{m_1 m_2}{r^2}$$



Universal Gravitation

- G is the constant of universal gravitation
- $G = 6.673 \times 10^{-11}$ N m² /kg²
- This is an example of an *inverse square law*
- Determined experimentally
- Henry Cavendish in 1798

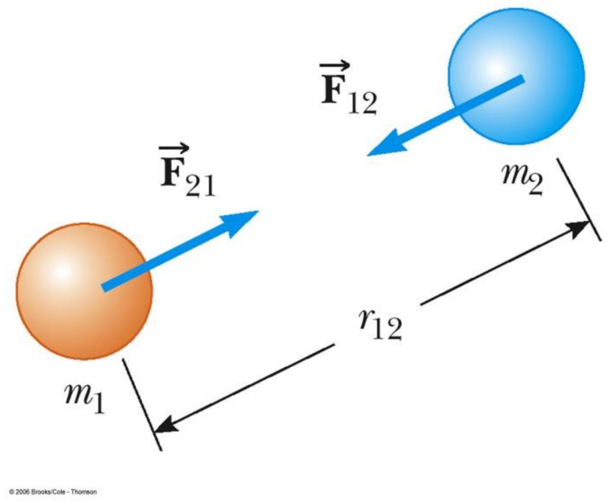
$$F = G\frac{m_1 m_2}{r^2}$$



(a)

Universal Gravitation

- The force that mass 1 exerts on mass 2 is equal and opposite to the force mass 2 exerts on mass 1
- The forces form a Newton's third law action-reaction



- The gravitational force exerted by a uniform sphere on a particle outside the sphere is the same as the force exerted if the entire mass of the sphere were concentrated on its center

Law of Universal Gravitation

$$F = \frac{mMG}{R^2}$$

where:

F = force between two bodies

m = body of lesser mass

M = body of greater mass

R = distance between the two mass centers

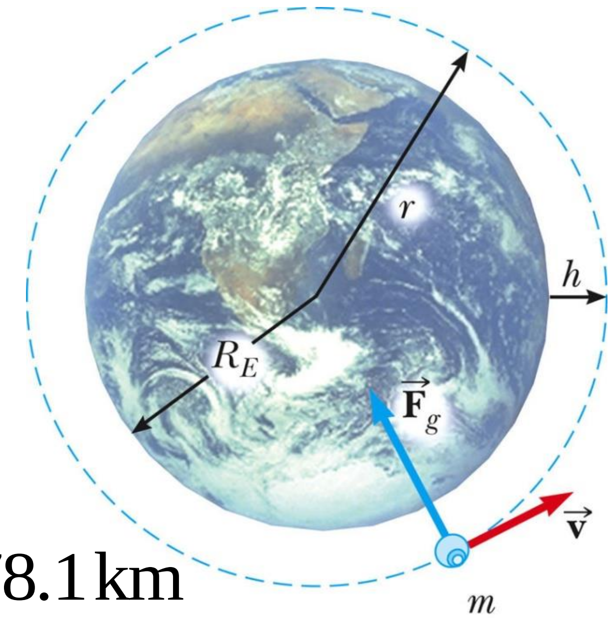
G (universal gravitational constant) = $6.673\text{x}10^{-11}$ N $(m/kg)^2$

13.2 **Weight**: Free-Fall Acceleration

- Have you heard this claim:
  - Astronauts are weightless in space, therefore there is no gravity in space?
- It is true that if an astronaut on the International Space Station (ISS) tries to step on a scale, he/she will weigh nothing.
- It may seem reasonable to think that if $\text{weight} = mg$, since $\text{weight} = 0$, $g = 0$, but this is **NOT true**.
- If you stand on a scale in an elevator and then the cables are cut, you will also weigh nothing ($ma = N - mg$, but in free-fall $a = g$, so the normal force $N = 0$). This does not mean $g = 0$!
- Astronauts in orbit are in free-fall around the Earth, just as you would be in the elevator. They do not fall to Earth, only because of their very high tangential speed.

Free-Fall Acceleration and the Gravitational Force

- Consider an object of mass m near the Earth's surface: $$F = G\frac{m_1 m_2}{r^2} = G\frac{mM_E}{{R_E}^2}$$



- Acceleration a_g due to gravity:

$$F = G\frac{mM_E}{{R_E}^2} = ma_g$$

- where $M_E = 5.9742 \times 10^{23}\ \text{kg}$ $\quad R_E = 6378.1\ \text{km}$

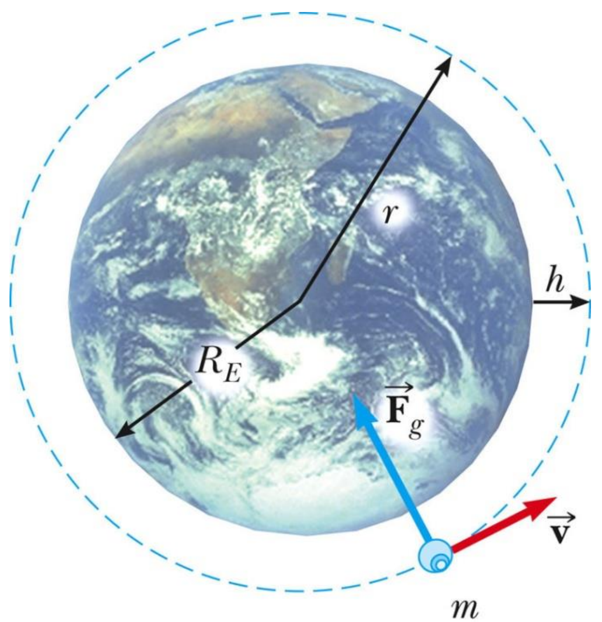
we find at the Earth's surface:

$$a_g = G\frac{M_E}{{R_E}^2} = 9.8\ \text{m/s}^2$$

Free-Fall Acceleration and the Gravitational Force

- Consider an object of mass m at a height h above the Earth's surface

$$F = G\frac{m_1 m_2}{r^2} = G\frac{mM_E}{(R_E + h)^2}$$



- Acceleration a_g due to gravity

$$F = G\frac{mM_E}{R_E^{\ 2}} = ma_g$$

- a_g will vary with altitude: $a_g = G\dfrac{M_E}{(R_E + h)^2}$

13.3 **Gravitational Potential Energy**

- $U = mgy$ is valid only near the earth's surface

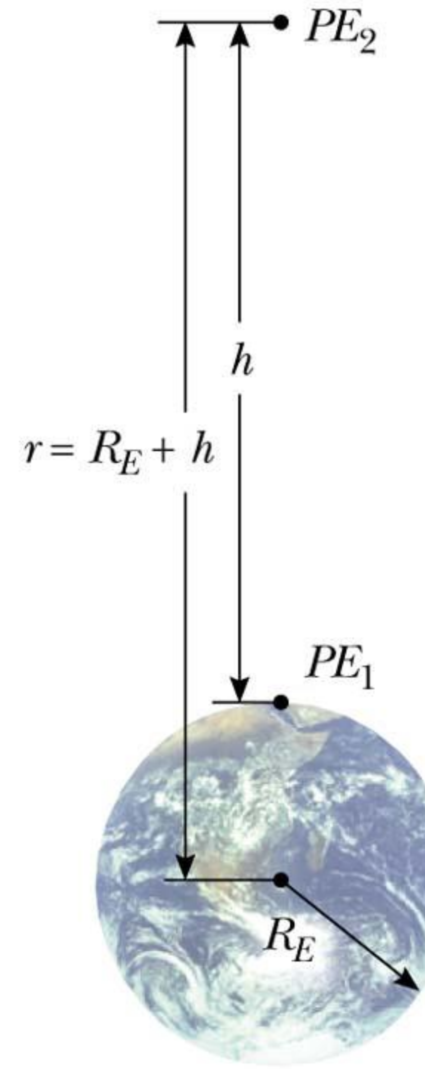
- For objects high above the earth's surface, an alternate expression is needed

$$U = -G\frac{M_E m}{r}$$

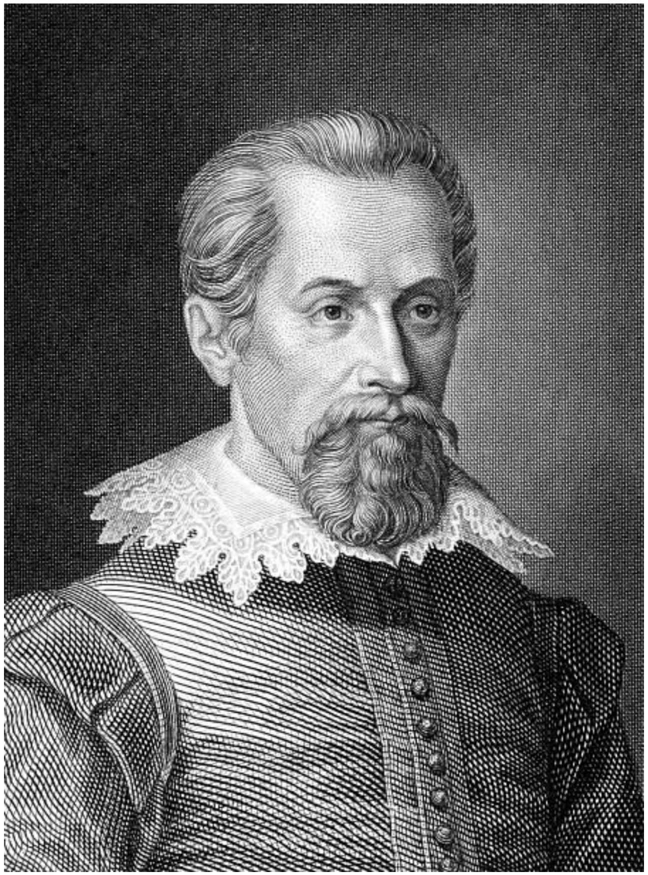
  - Zero reference level is infinitely far from the earth, so potential energy is everywhere negative!

- Energy conservation:

$$E = K + U = \frac{1}{2}mv^2 - G\frac{M_E m}{r}$$



13.5 **Kepler's Laws**

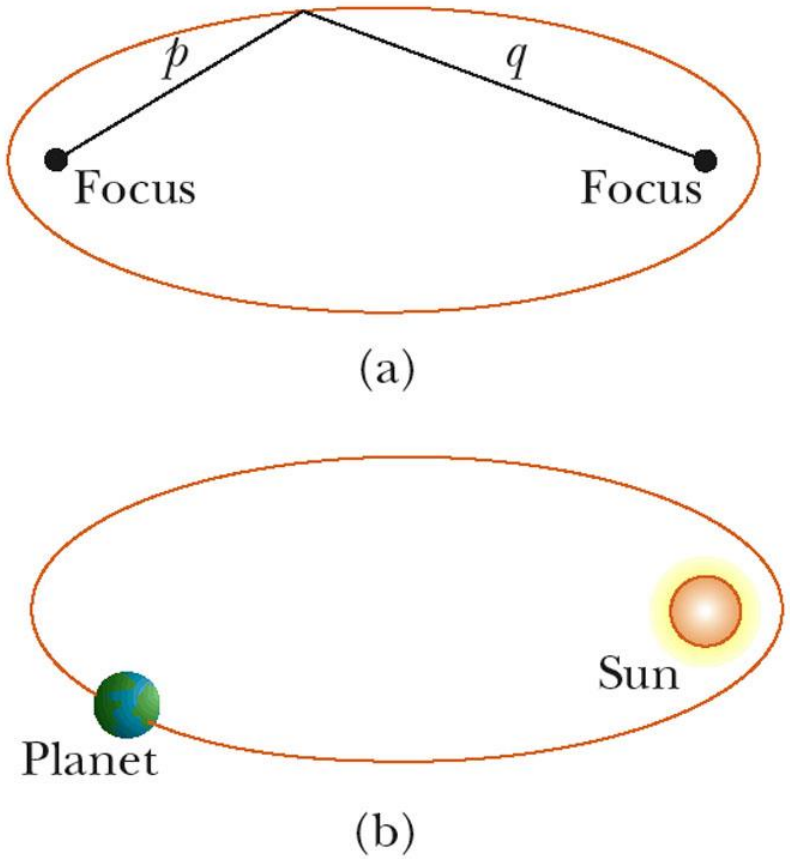


Johannes Kepler (1571-1630 –German Astronomer whose work provided the foundations for Newton’s theory of universal gravitation.

- All planets move in elliptical orbits with the Sun at one of the focal points.
- A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals.
- The square of the orbital period of any planet is proportional to cube of the average distance from the Sun to the planet.

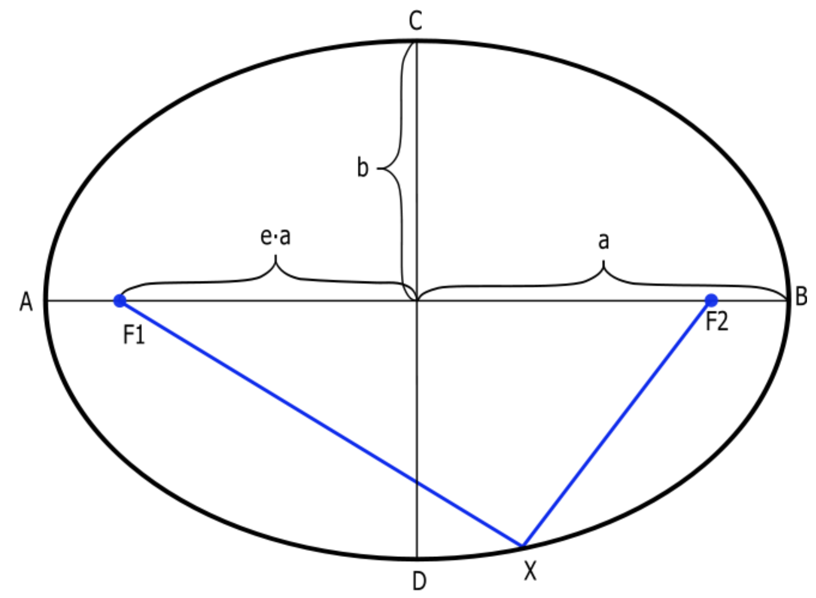
Kepler's First Law

- All planets move in elliptical orbits with the Sun at one **focus**.
  - Any object bound to another by an inverse square law will move in an elliptical path
  - Second focus is empty

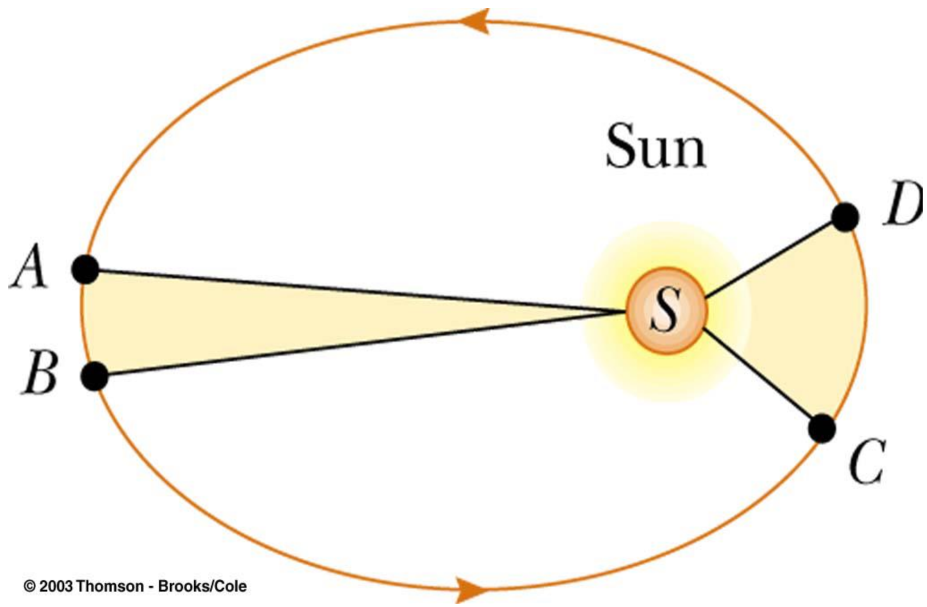


Ellipse Parameters

- Distance $a = AB/2$ is the **semi-major axis**
- Distance $b = CD/2$ is the **semi-minor axis**
- Distance from one focus to center of the ellipse is ea, where e is the **eccentricity**.
- Eccentricity is zero for a circular orbit, and gets larger as the ellipse gets more pronounced.

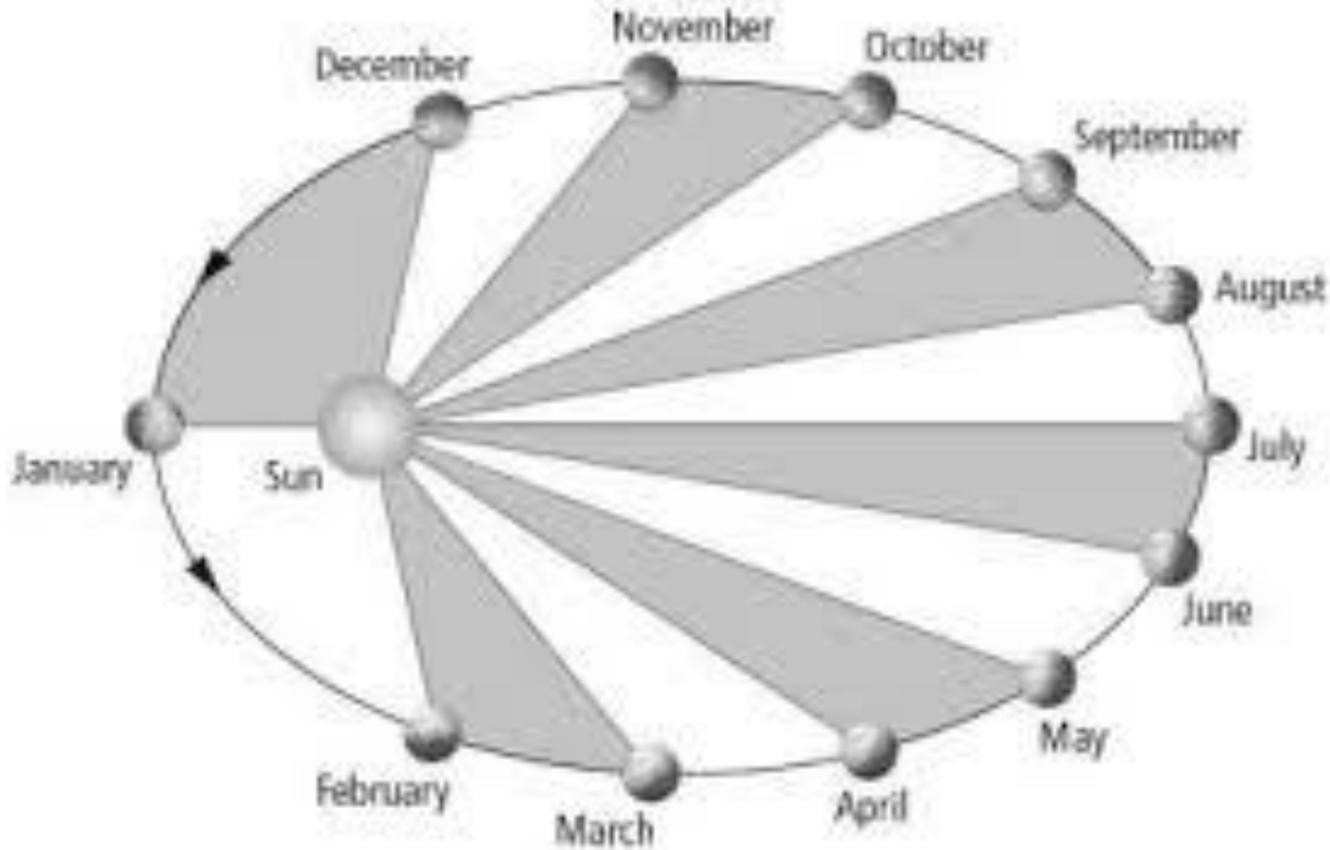


Kepler's Second Law



- A line drawn from the Sun to any planet will sweep out equal areas in equal times
  - Area from A to B and C to D are the same

Equal Time Intervals



- A line drawn from the Sun to any planet sweeps out equal areas in equal time intervals.

Kepler's Third Law

- The square of the orbital period of any planet is proportional to cube of the average distance from the Sun to the planet.

$$T^2 = Ka^3$$

  - T is the period of the planet
  - a is the average distance from the Sun. Or a is the length of the semi-major axis
  - For orbit around the Sun, $K = K_S = 2.97 \times 10^{-19}\ s^2/m^3$
  - K is independent of the mass of the planet

$$K_s = \frac{4\pi^2}{GM_s}$$

Orbital Velocity and Period

- From Newton's Second law and the Law of Universal Gravitation (recall: radial acceleration $a_{rad} = v^2/R$)…

$$F = ma = m\left(\frac{v^2}{R}\right) = \frac{GMm}{R^2}$$

- It follow that the orbital speed: $$v = \sqrt{\frac{GM}{R}}$$

- The orbital period, as defined the circumference traversed divided by the orbital speed of the object, is:

$$T = \frac{2\pi R}{v} = 2\pi R \cdot \sqrt{\frac{R}{GM}} = \frac{2\pi R^{3/2}}{\sqrt{GM}}$$ "or" $$T^2 = \frac{4\pi^2 R^3}{GM}$$

Example: Mass of the Sun

- Calculate the mass of the Sun noting that the period of the Earth’s orbit around the Sun is one year and its distance from the Sun is R = 1.496×10^{11} m.

$$T^2 = \frac{4\pi^2 R^3}{GM}$$

where

$$T = 365\ days\ x \left(\frac{24\ hrs}{day}\right) x \left(\frac{3600\ s}{hr}\right) = 3.156 \times 10^{7}\ \mathrm{s}$$

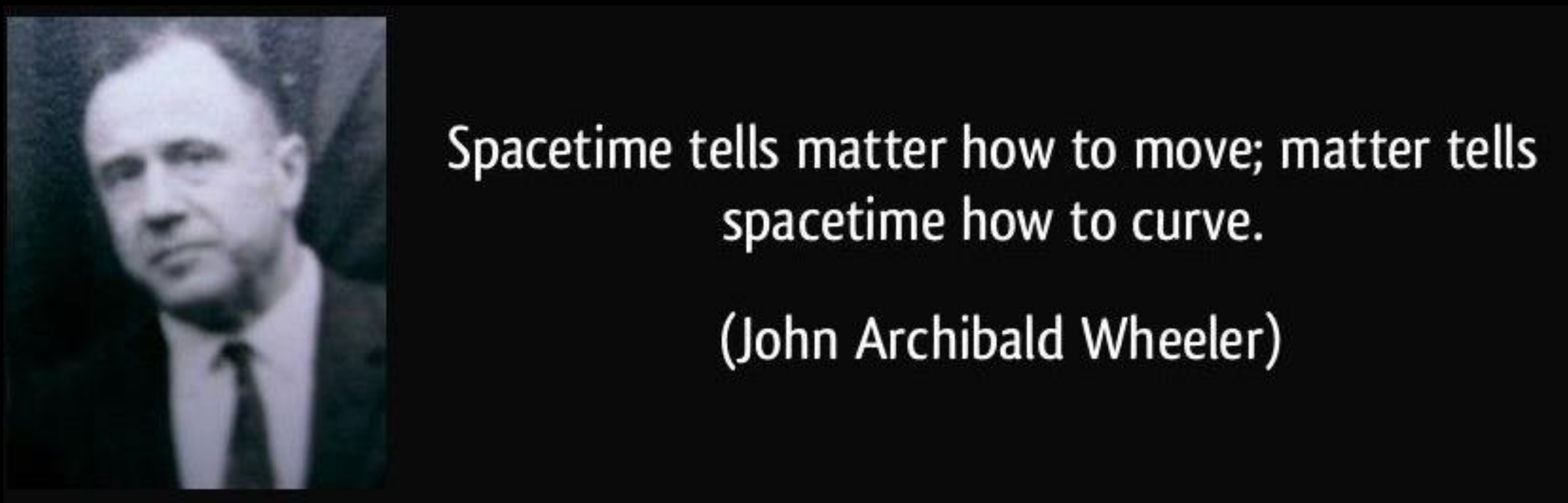
$$\mathrm{G} = 6.6743 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$$

- Thus,

$$M_s = \frac{4\pi^2 R^3}{GT^2} = \mathbf{1.99 \cdot 10^{30}}\ \boldsymbol{kg}$$

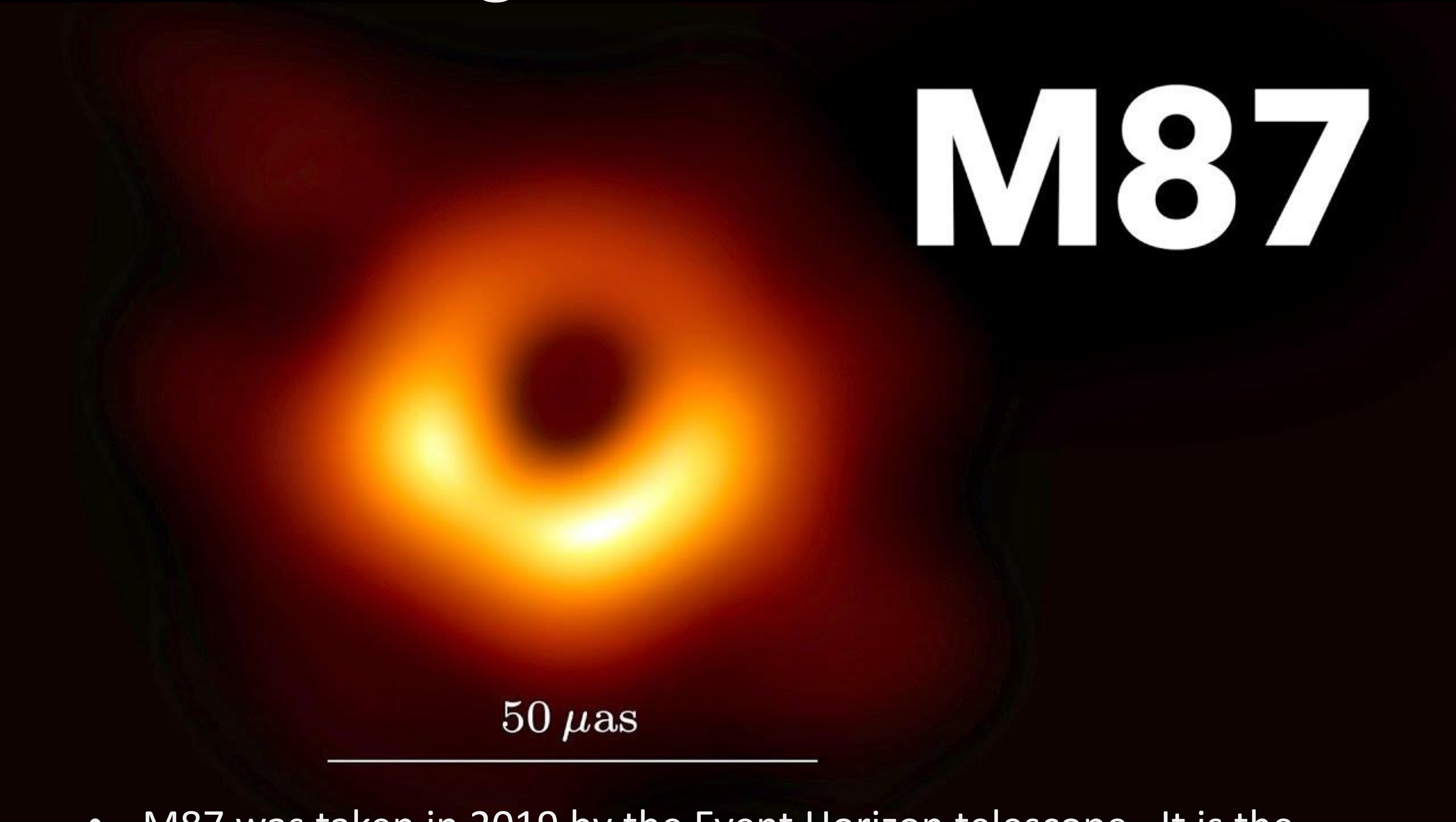
<u>NOTED</u>: M_s defines “one” solar mass

13.8 Black Holes



- **John A. Wheeler (1911-2008)**: American theoretical physicist. During WWII, he worked with the Manhattan Project in designing nuclear reactors for Hanford Site in Richland, Washington. In 1967, he was the first to coin the term “black hole” during a talk at the NASA Goodard Institute of Space Studies. Wheeler would go on play an instrumental role in creation and development of the LIGO project.

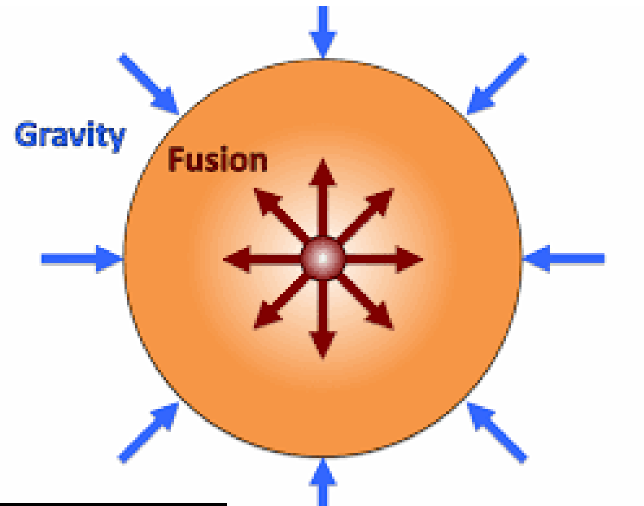
Image of a Black Hole

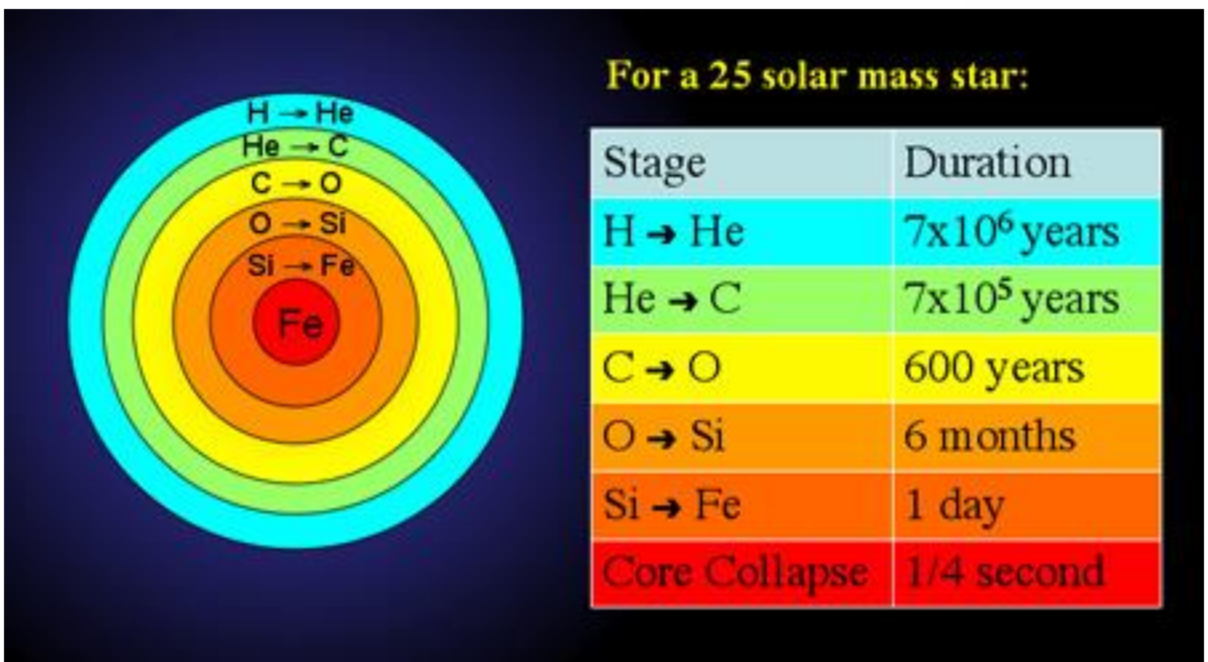


- M87 was taken in 2019 by the Event Horizon telescope. It is the image of the supermassive black hole at the center of our galaxy.

What makes a black hole?

- By process of stellar nucleosynthesis, a star will eventually succumb to it’s own gravitational collapse…



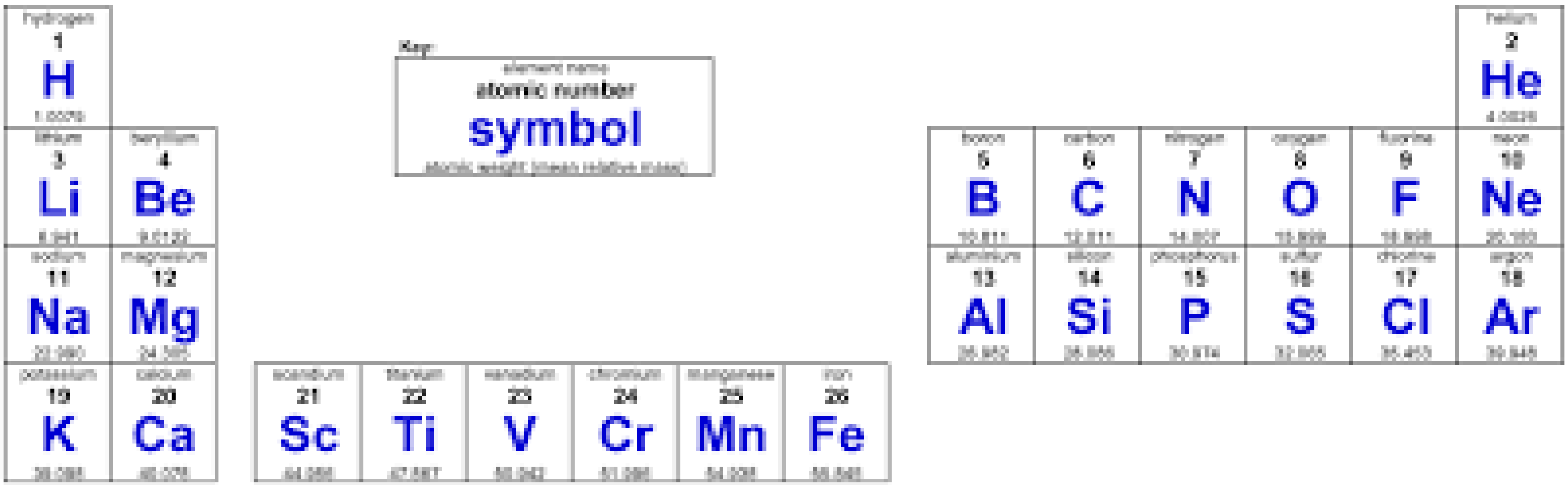


For a 25 solar mass star:

Stage	Duration
H → He	7×10^6 years
He → C	7×10^5 years
C → O	600 years
O → Si	6 months
Si → Fe	1 day
Core Collapse	1/4 second

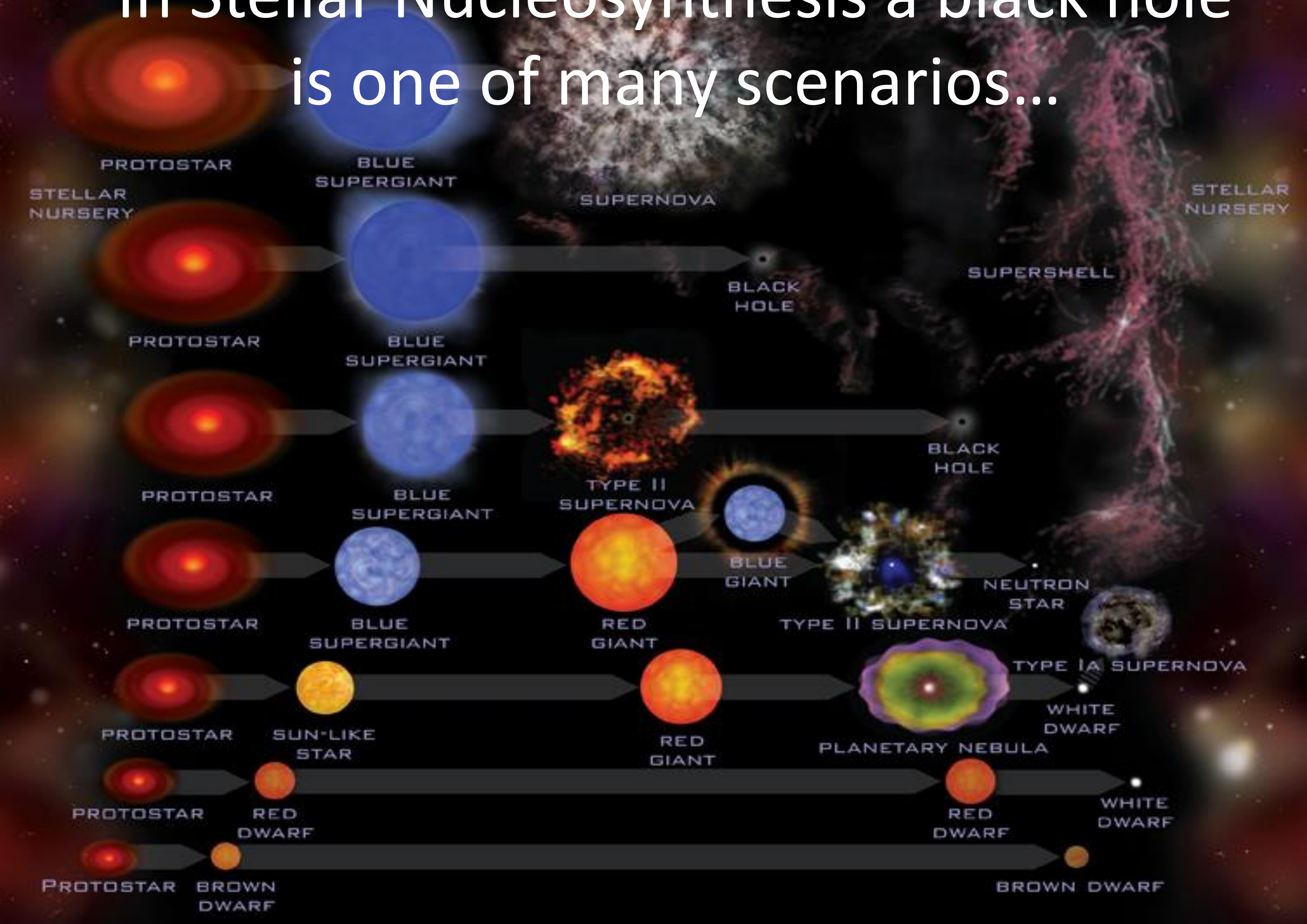
- As star creates heavier elements, the force of fusion can no longer counter-balance the force of gravity.

What elements can a star make?

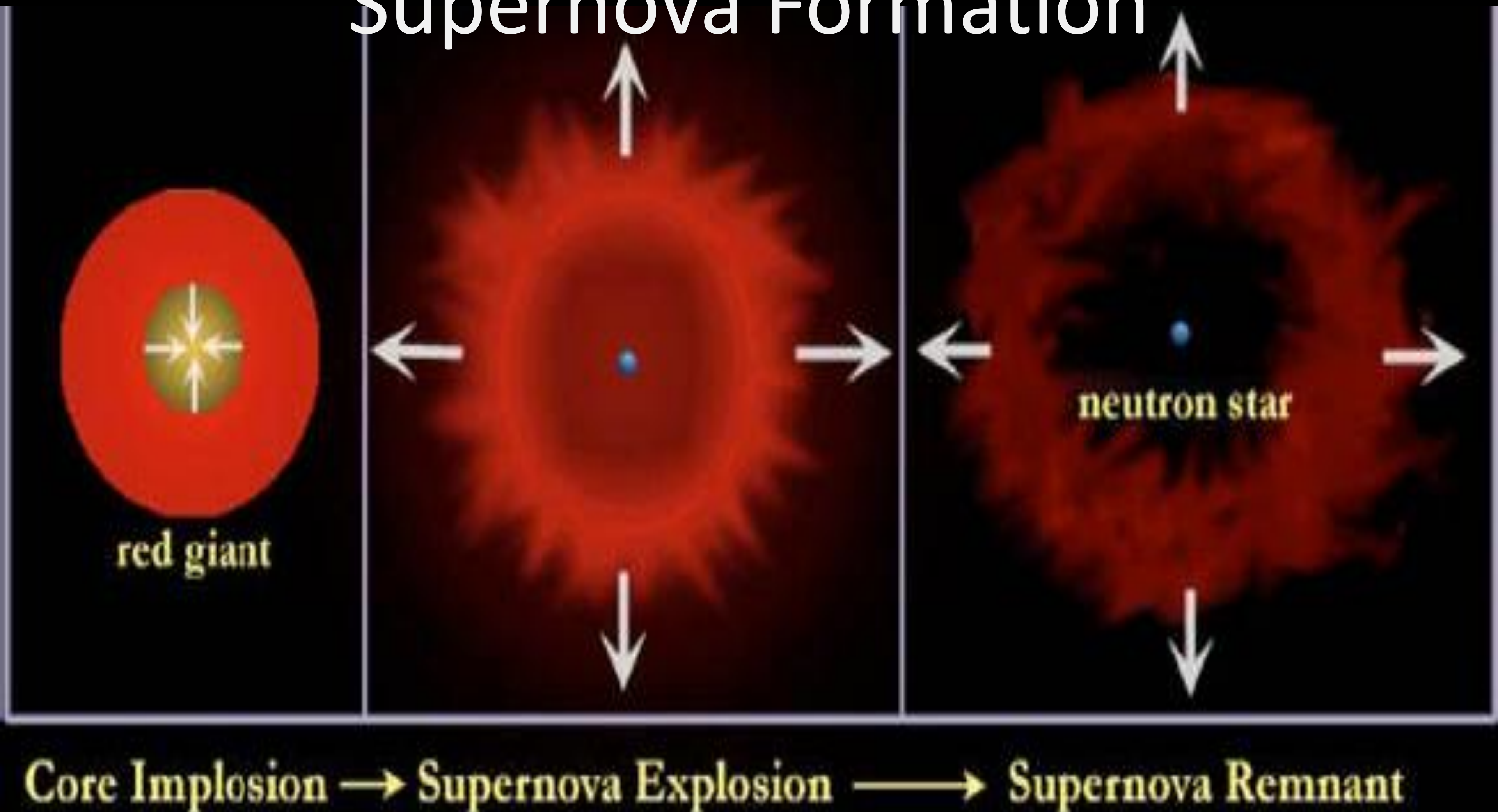


- The heavier elements we find on earth are thus the result of a supernova, over 5.2 billion years ago.

In Stellar Nucleosynthesis a black hole is one of many scenarios...



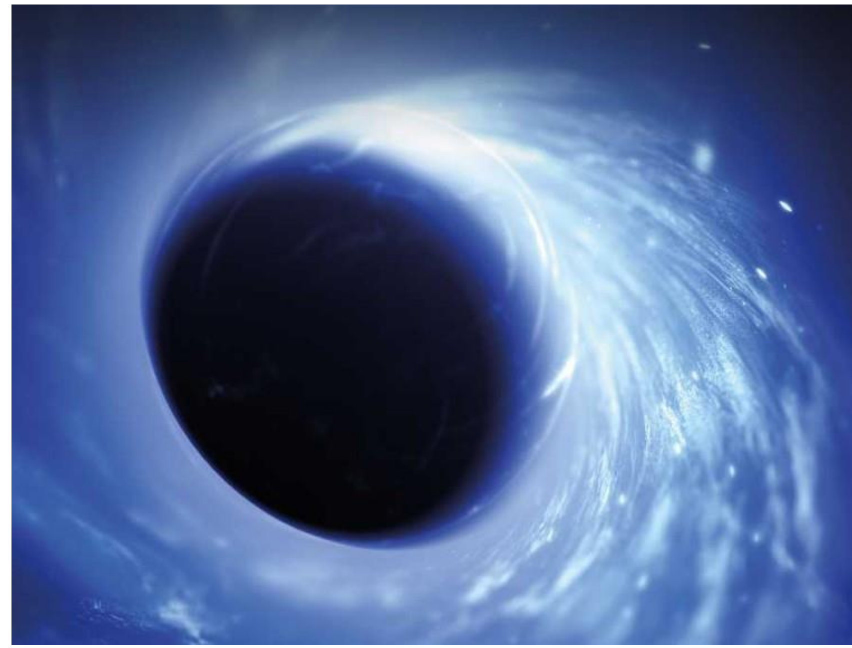
Supernova Formation



- The density of a neutron star can be considered extraordinary, considering the space between neutrons is essentially non-existent which gives rise to a mass of phenomenal proportions.

Black Holes are extremely massive!

- A typical stellar class black hole will have a mass 3 – 10X that of the sun ($2 \cdot 10^{30}$ kg = one solar mass).
- Supermassive black holes existing in the center of most galaxies range from millions to billions solar masses (SM).
- Although there is no theoretical upper limit, astronomers calculations have found that ultra massive black holes never seem to exceed roughly $10 \cdot 10^{9}$ SM.



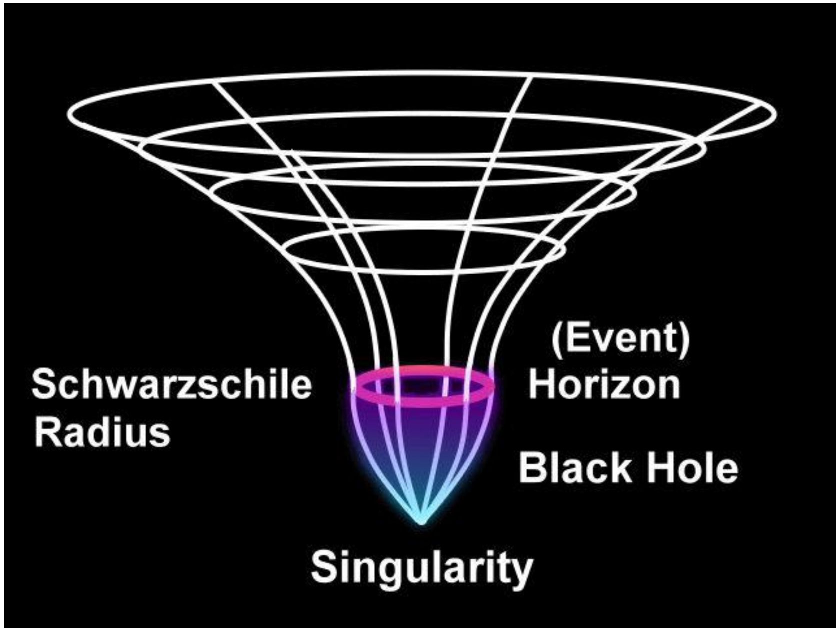
Schwarzschild Radius defined

- When an object attracted by the gravitational pull of a supermassive object exceeds the speed of light, it has crossed the boundary known as the “event horizon.”
- This boundary is defined via the conservation of energy…

$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$

- It follows that when velocity of a particle equals the speed of light (v = c)

$$R_s = \frac{2GM}{c^2}$$



Example Problem

- In 2005 astronomers announced the discovery of a large black hole in a nearby galaxy (Markarian 766). It was characterized by clumps of matter orbiting once every 27 hours at 30,000 km/s (a) How far are these clumps from the center of the black hole? (b) What is the mass of this black hole, assuming circular orbits? (c) What is the radius of its event horizon?

(a) Given: $v = 3{\cdot}10^7$ m/s and $T = 27$ hrs x (3600 s/hr) $= 9.72{\cdot}10^4$ s

$$v = \frac{2\pi R}{T} \qquad \mathrm{R} = \frac{vT}{2\pi} = \frac{(3{\cdot}10^7\ \mathrm{m/s})(9.72{\cdot}10^4\ \mathrm{s})}{2\pi} = \mathbf{4.64 \cdot 10^{11}}\ \boldsymbol{m}$$

(b) Mass:

$$M_{BH} = \frac{4\pi^2 R^3}{GT^2} = \mathbf{6.26 \cdot 10^{36}}\ \boldsymbol{kg}$$

(c) Schwarzschild radius:

$$R_S = \frac{2GM}{c^2} = \frac{2(6.6743 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}})(6.26{\cdot}10^{36}\ kg)}{(3{\cdot}10^8\ m/s)^2} = \mathbf{9.28 \cdot 10^9}\ \boldsymbol{m}$$

Energy of an Orbit

- Consider a circular orbit of a planet around the Sun. What keeps the planet moving in its circle?
- It is the centripetal force produced by the gravitational force, i.e. $$F = \frac{mv^2}{r} = G\frac{Mm}{r^2}$$
- That implies that $\frac{1}{2}mv^2 = \frac{GMm}{2r}$
- Making this substitution in the expression for total energy:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$E = -\frac{GMm}{2r} \quad \text{(circular orbits)}$$

- Note the total energy is negative, and is half the (negative) potential energy.
- For an elliptical orbit, r is replaced by a: $$E = -\frac{GMm}{2a} \quad \text{(elliptical orbits)}$$