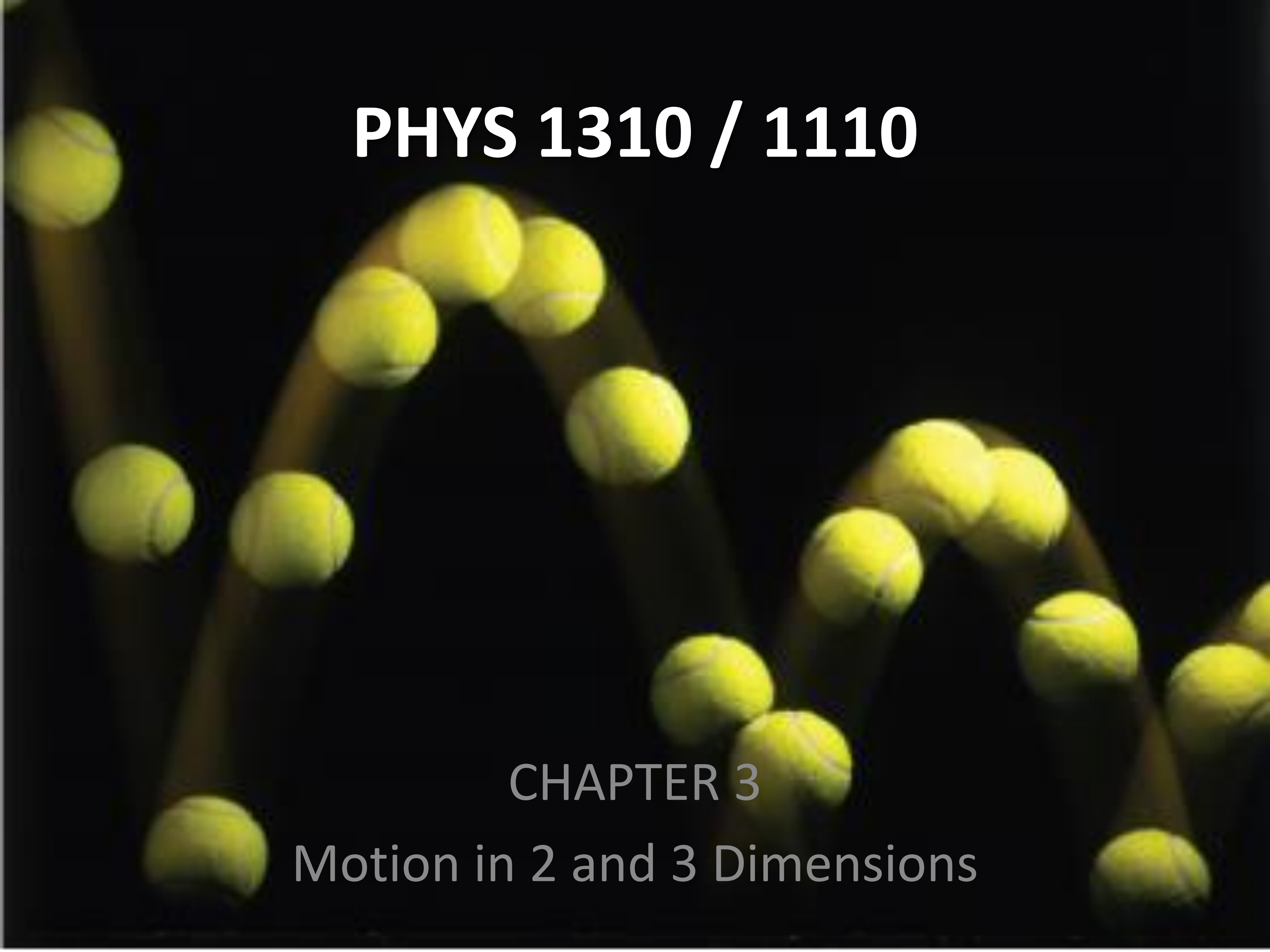


PHYS 1310 / 1110

CHAPTER 3

Motion in 2 and 3 Dimensions

CH3 in a Nutshell

- Position, Velocity and Acceleration Vectors:

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt}$$

$$\vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt}$$

- Projectile motion:

$$x = (v_o \cos\theta)\, t \Rightarrow v_x = \frac{dx}{dt} = v_o \cos\theta$$

$$y = (v_o \sin\theta)\, t - \frac{1}{2} g t^2 \Rightarrow v_y = \frac{dy}{dt} = v_o \sin\theta - gt$$

- Circular motion (radial velocity and acceleration):

$$v = \frac{2\pi R}{T}$$

$$a_{rad} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}$$

CH3 Overview

- **3.1 Position and Velocity Vectors**
- **3.2 The Acceleration Vector**
- **3.3 Projectile Motion**
- **3.4 Motion in a Circle**
- **3.5 Relative Velocity**

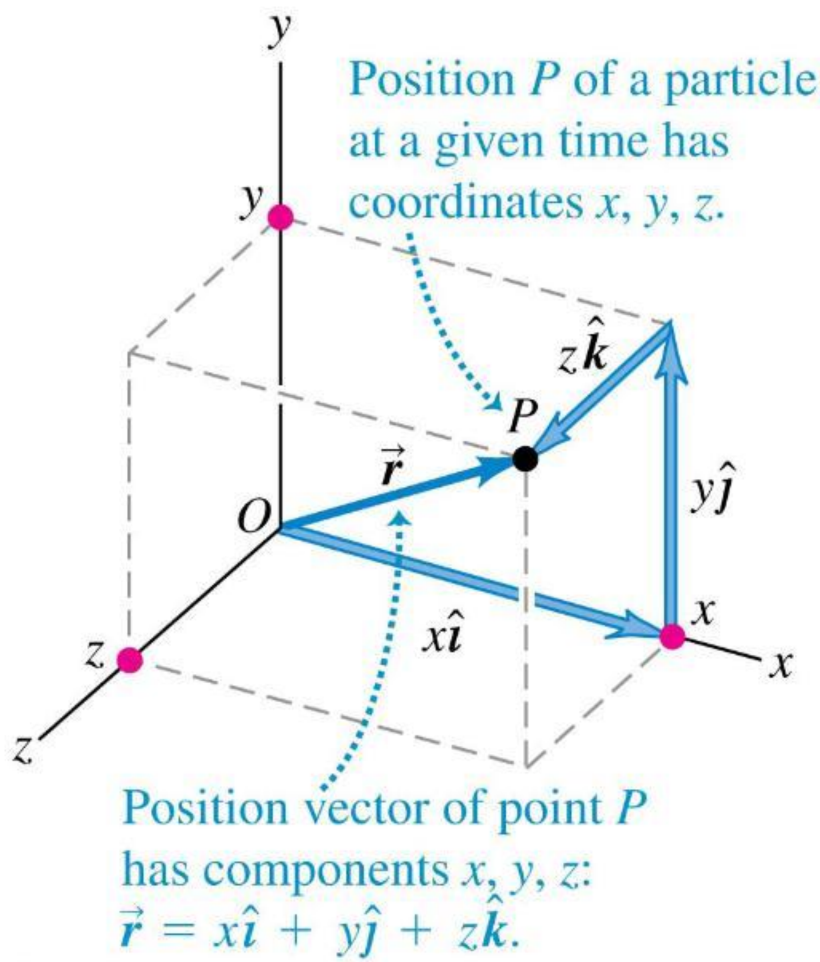
Introduction

- What determines where a batted baseball lands?
- How do you describe the motion of a roller coaster car along a curved track or the flight of a circling hawk?
- Which hits the ground first, a baseball that you simply drop or one that you throw horizontally?
- To answer these questions we need to extend our description of motion to two and three dimensions.

3.1 Position and Velocity Vectors

- The position vector from the origin to point *P* has components *x*, *y*, and *z*.

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\widehat{\boldsymbol{k}}$$



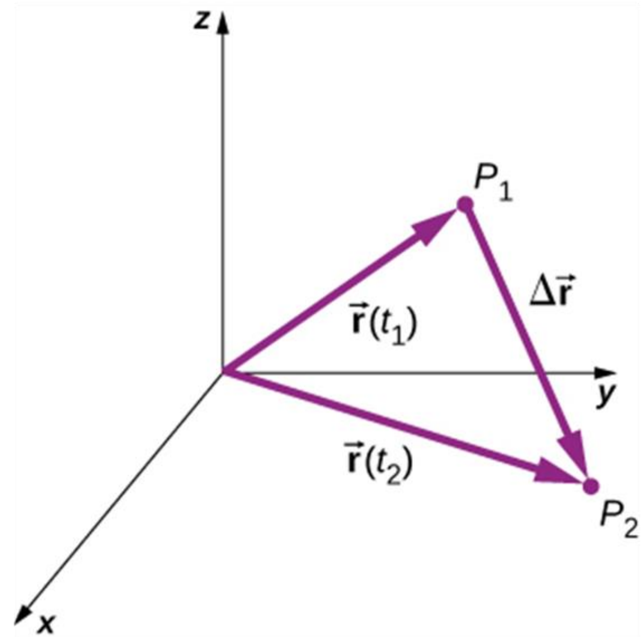
Vector Displacement

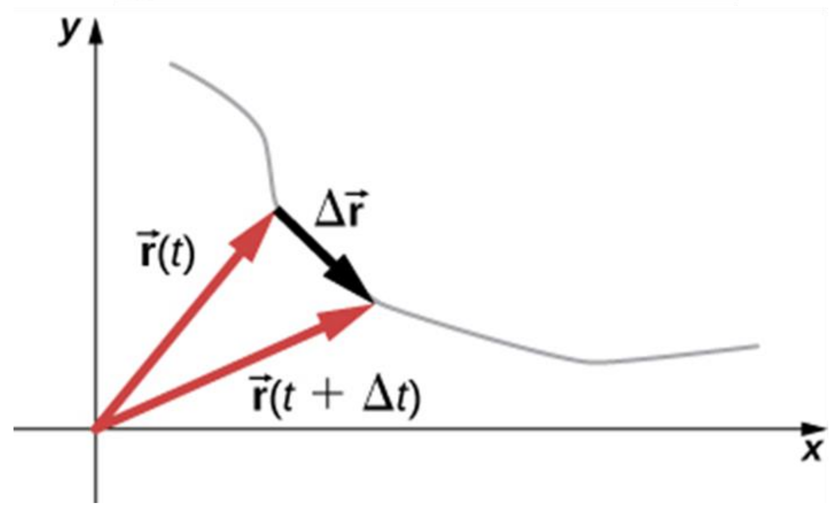
- The displacement vector:

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$$

is the vector from P_1 to P_2.

- The **average velocity** between two points is the displacement divided by the time interval between the two points, and it has the same direction as the displacement.





Velocity

- We define the **average velocity** as the displacement divided by the time interval:

Change in the particle's position vector

Average velocity vector of a particle during time interval from t_1 to t_2

$$\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

Final position minus initial position

Time interval

Final time minus initial time

- The **Instantaneous velocity** is the instantaneous rate of change of position with time:

The **instantaneous velocity vector** of a particle ...

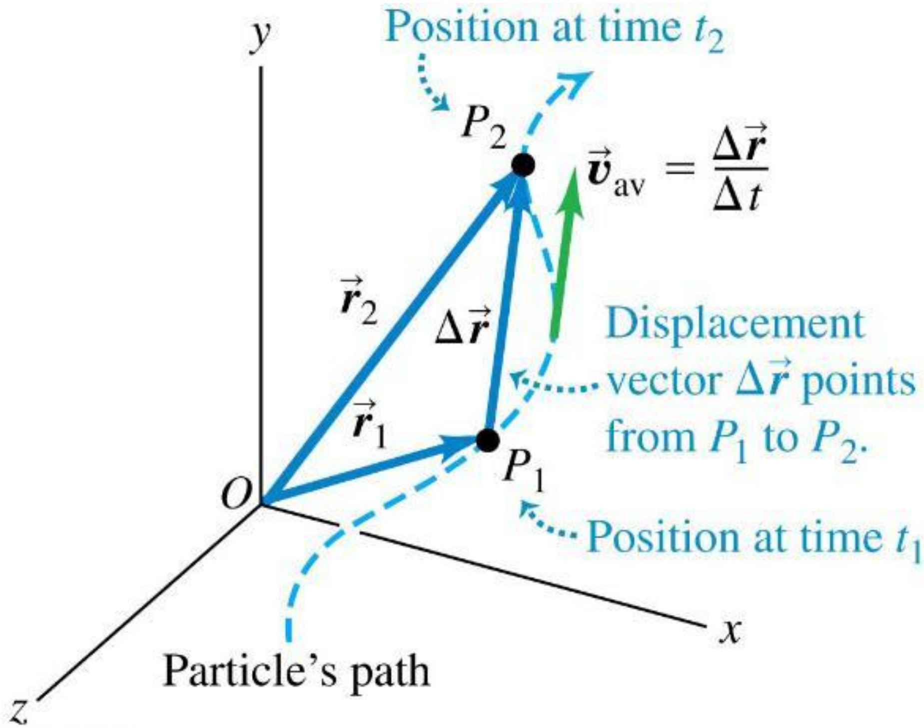
$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

... equals the limit of its average velocity vector as the time interval approaches zero ...

... and equals the instantaneous rate of change of its position vector.

Average Velocity

- The **average velocity** between two points is the displacement divided by the time interval between the two points, and it has the same direction as the displacement.

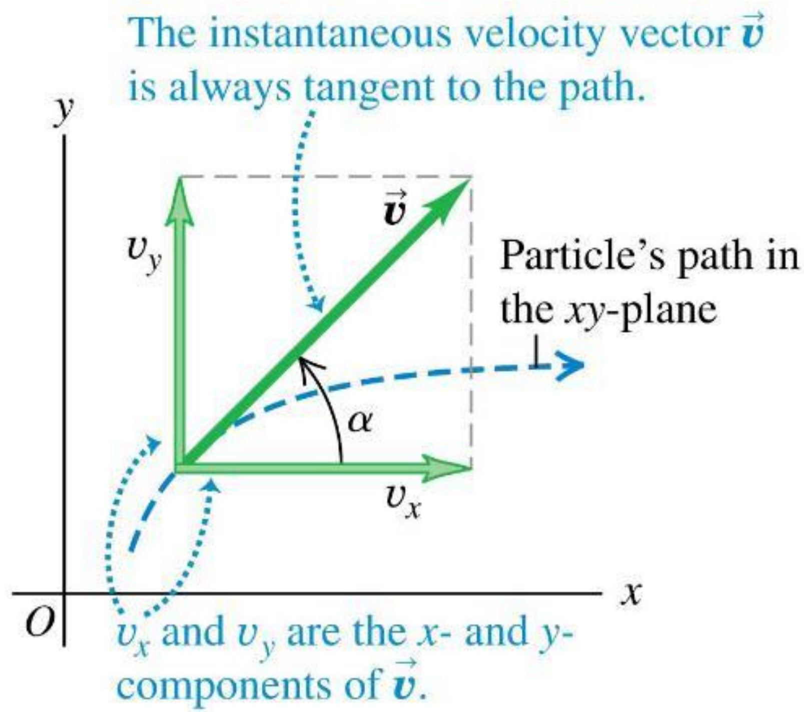


Instantaneous Velocity

- The **instantaneous velocity** is the instantaneous rate of change of position vector with respect to time.
- The components of the instantaneous velocity are

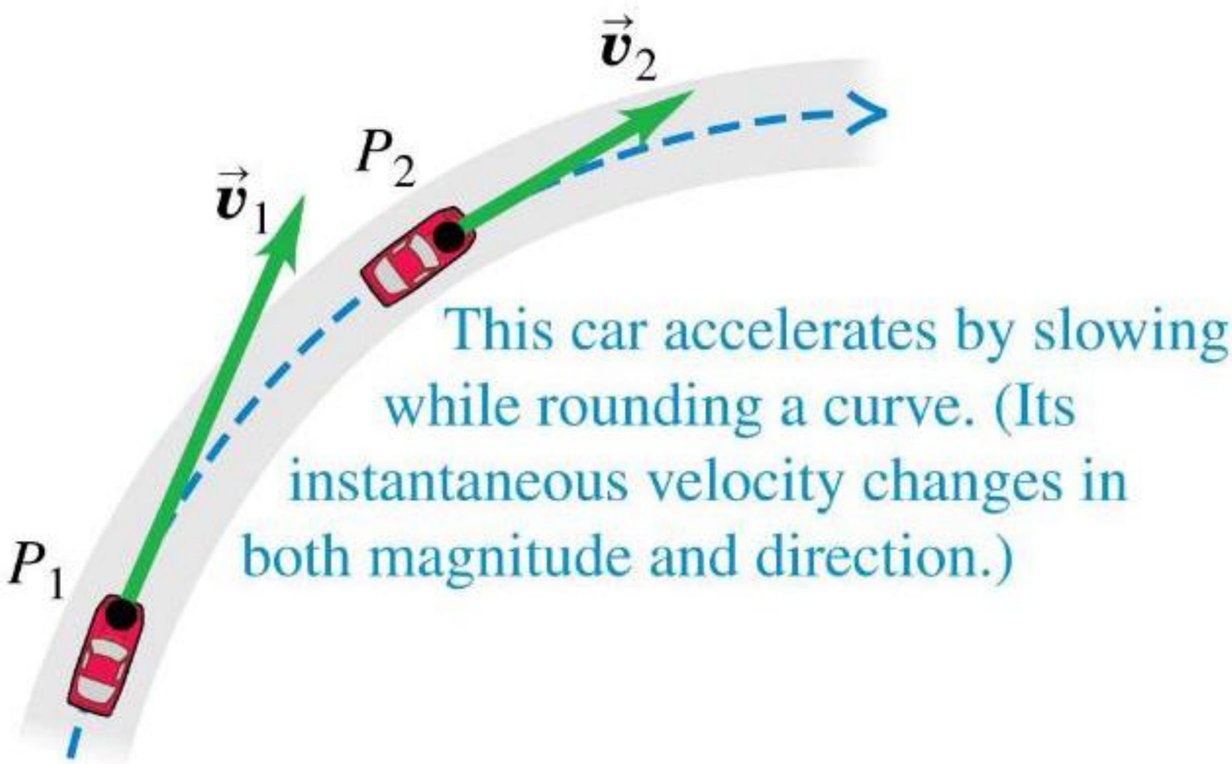
$$v_x = \frac{dx}{dt},\ v_y = \frac{dy}{dt},\ \text{and } v_z = \frac{dz}{dt}.$$

- The instantaneous velocity of a particle is always tangent to its path.



3.2 Acceleration Vectors

- Acceleration describes how the velocity changes.



Acceleration

- We define the **average acceleration** as the change in velocity divided by the time interval:

Average acceleration vector of a particle during time interval from t_1 to t_2

Change in the particle's velocity

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

Final velocity minus initial velocity

Time interval

Final time minus initial time

- **Instantaneous acceleration** is the instantaneous rate of change of velocity with time:

The **instantaneous acceleration vector** of a particle ...

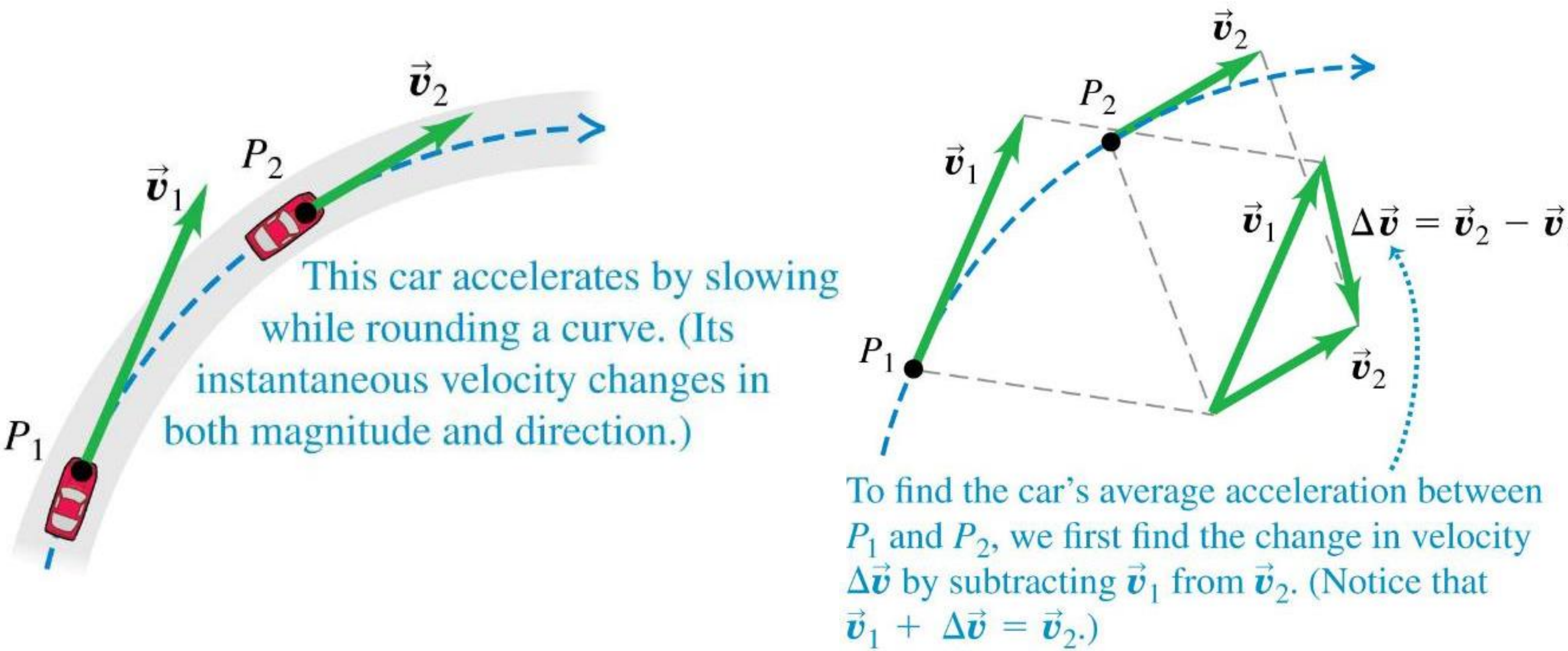
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

... equals the limit of its average acceleration vector as the time interval approaches zero ...

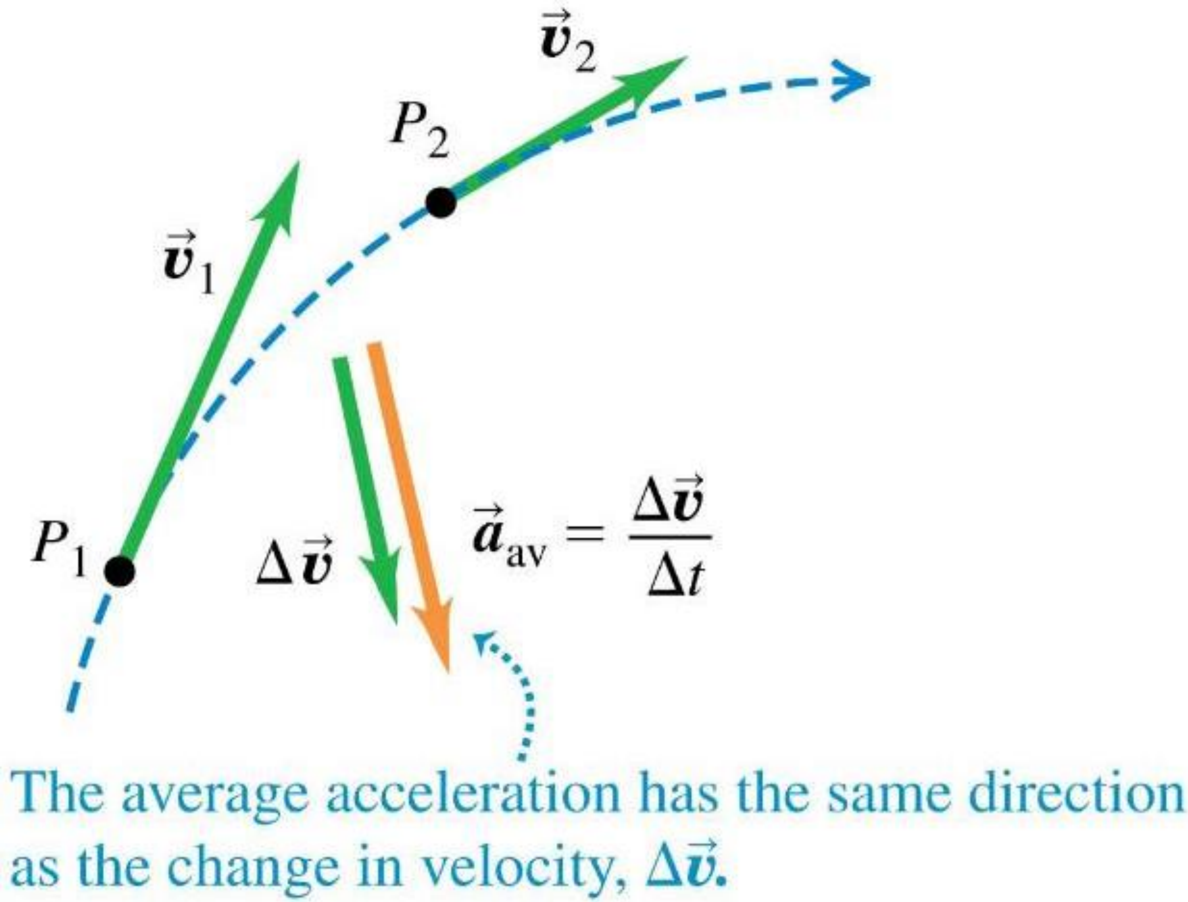
... and equals the instantaneous rate of change of its velocity vector.

Instantaneous vs Average Velocity

- The change in velocity between two points is determined by vector subtraction.



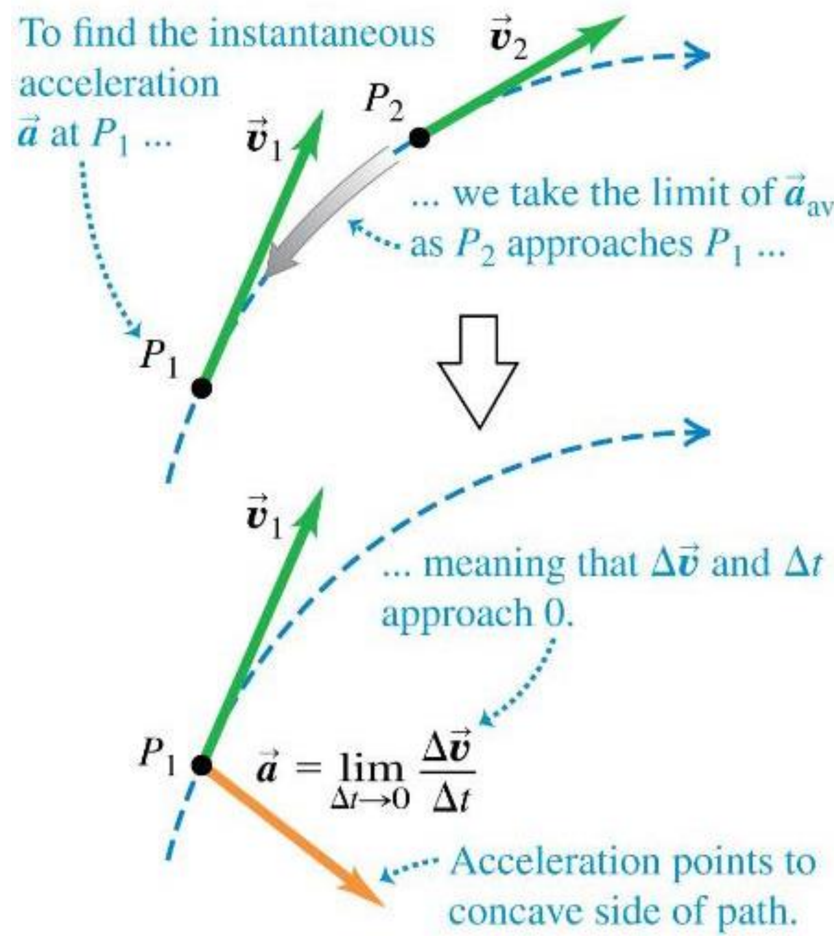
Average Acceleration



Instantaneous Acceleration

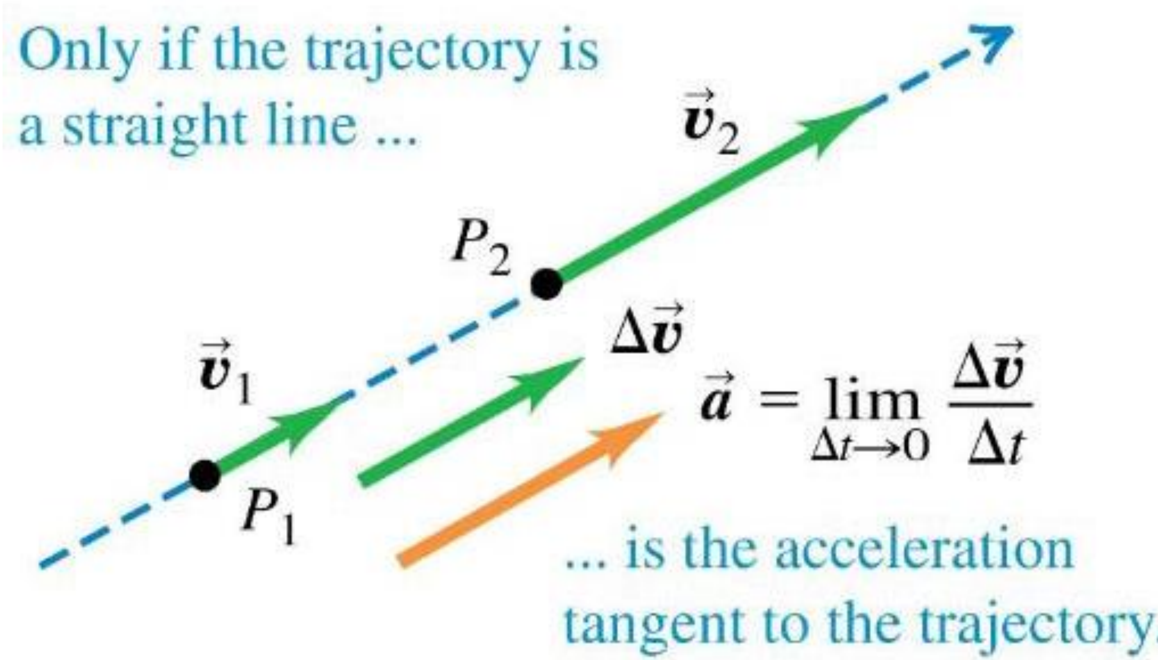
- The velocity vector is always tangent to the particle's path, but the instantaneous acceleration vector does **not** have to be tangent to the path.

- If the path is curved, the acceleration points toward the concave side of the path.



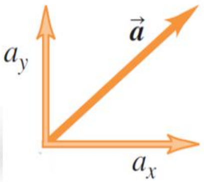
Instantaneous Acceleration

- When is the acceleration in the direction of motion?



Components of Acceleration

- Shooting an arrow is an example of an acceleration vector that has both *x*- and *y*-components.



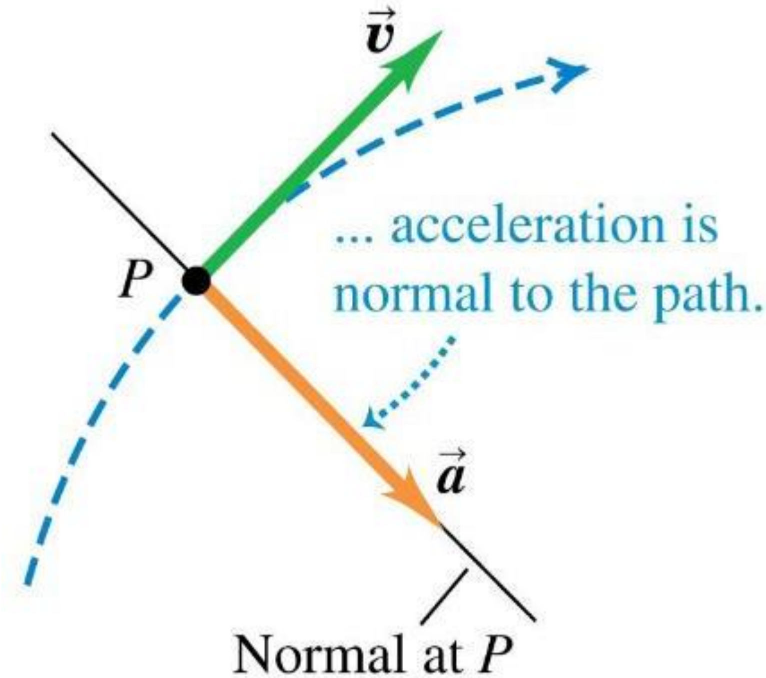
Each **component of** a particle's **instantaneous acceleration vector** ...

$$a_x = \frac{dv_x}{dt} \qquad a_y = \frac{dv_y}{dt} \qquad a_z = \frac{dv_z}{dt}$$

... equals the instantaneous rate of change of its corresponding velocity component.

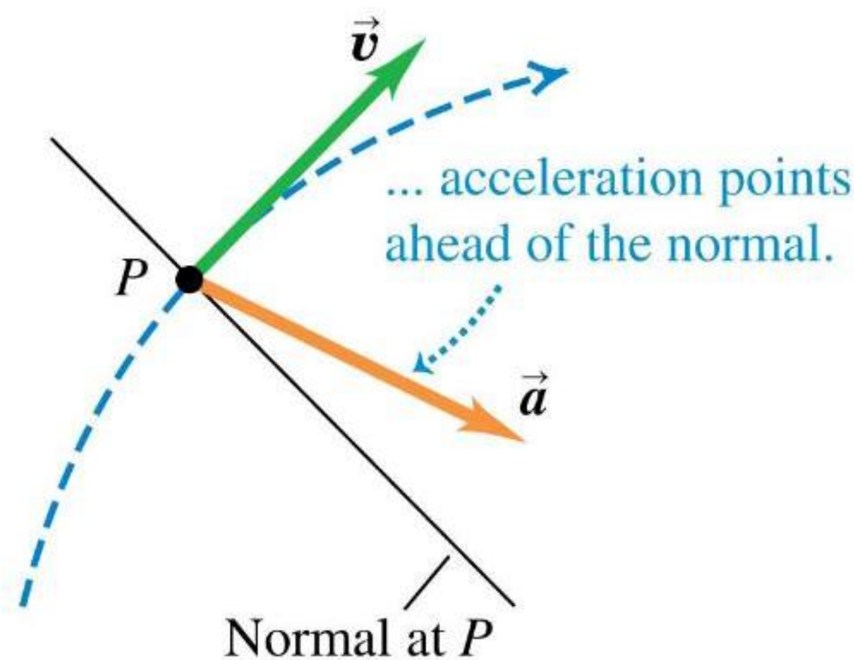
Parallel and Perpendicular Components of Acceleration

- Velocity and acceleration vectors for a particle moving through a point *P* on a curved path with **constant speed**

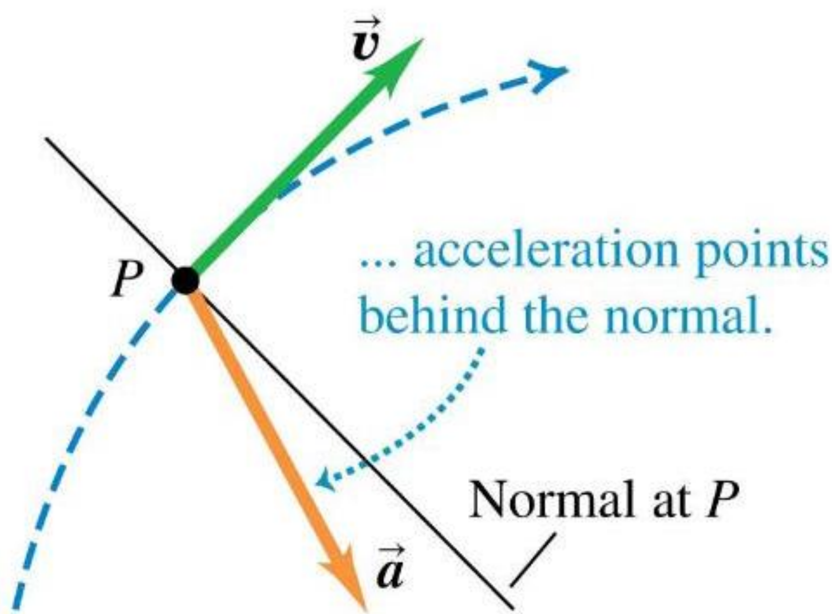


Parallel and Perpendicular Components of Acceleration

- Velocity and acceleration vectors for a particle moving through a point *P* on a curved path with **increasing speed**

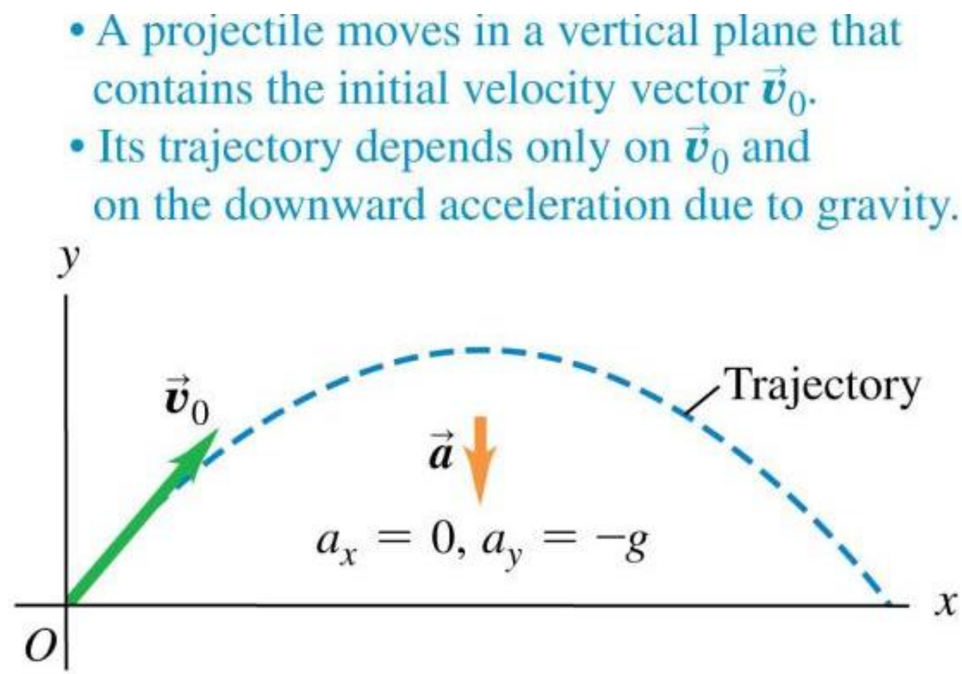


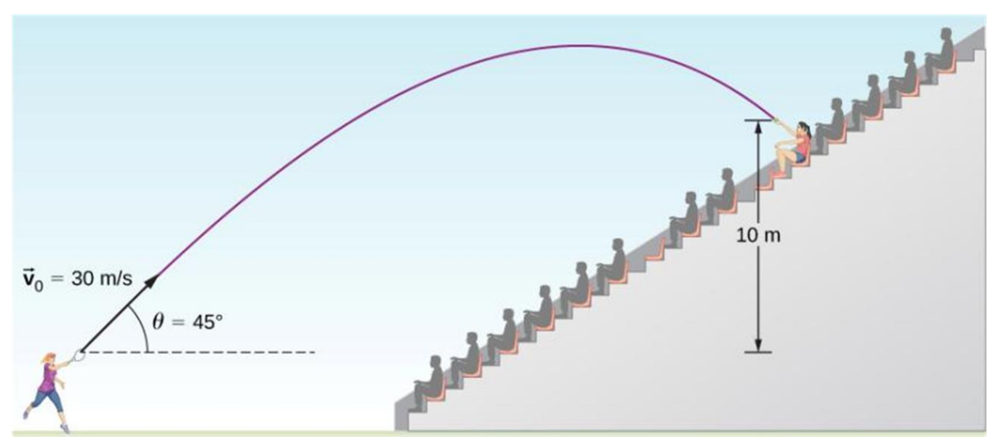
- Velocity and acceleration vectors for a particle moving through a point *P* on a curved path with **decreasing speed**



3.3 Projectile Motion

- A **projectile** is any object given an initial velocity that then follows a path determined by the effects of gravity and air resistance.
- Begin by neglecting resistance and the curvature and rotation of the earth.

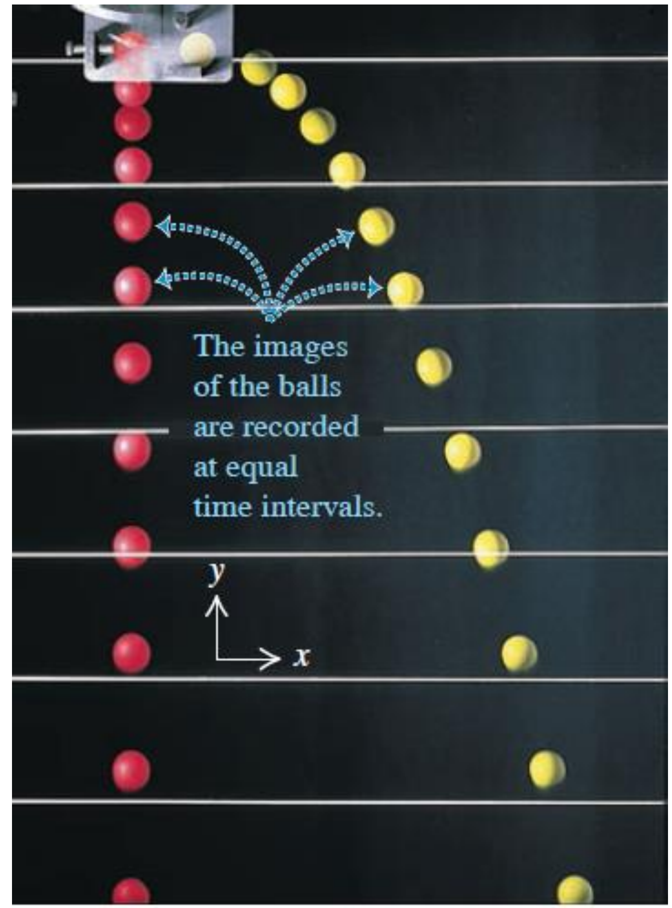




The *X*- and *Y*-Motion Are Separable

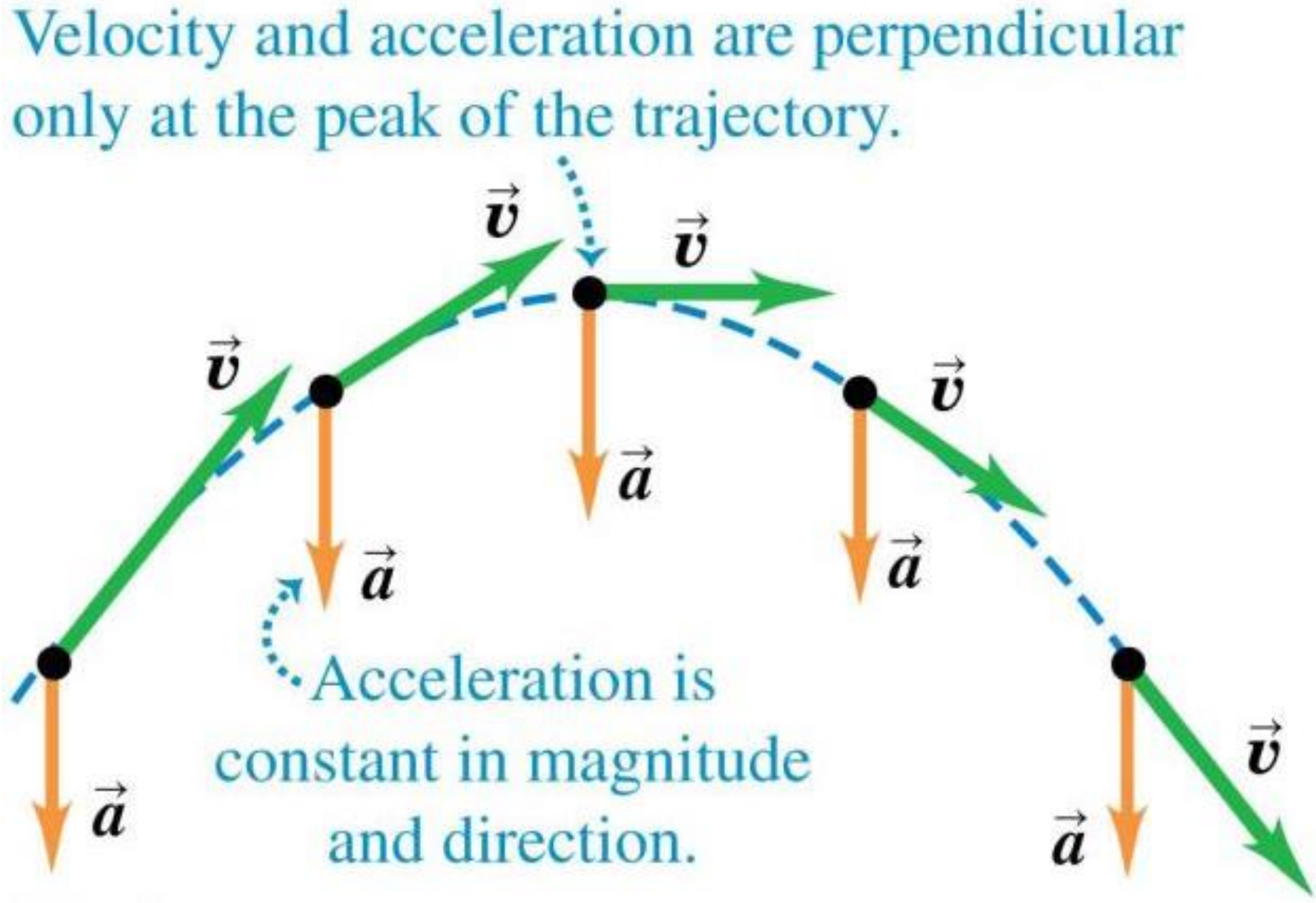
- The red ball is dropped at the same time that the yellow ball is fired horizontally.
- The strobe marks equal time intervals.
- We can analyze projectile motion as horizontal motion with constant velocity and vertical motion with constant acceleration:

$$a_x = 0 \text{ and } a_y = -g.$$



- At any time the two balls have different x-coordinates and x-velocities but the same y-coordinate, y-velocity, and y-acceleration.
- The horizontal motion of the yellow ball has no effect on its vertical motion.

Projectile Motion

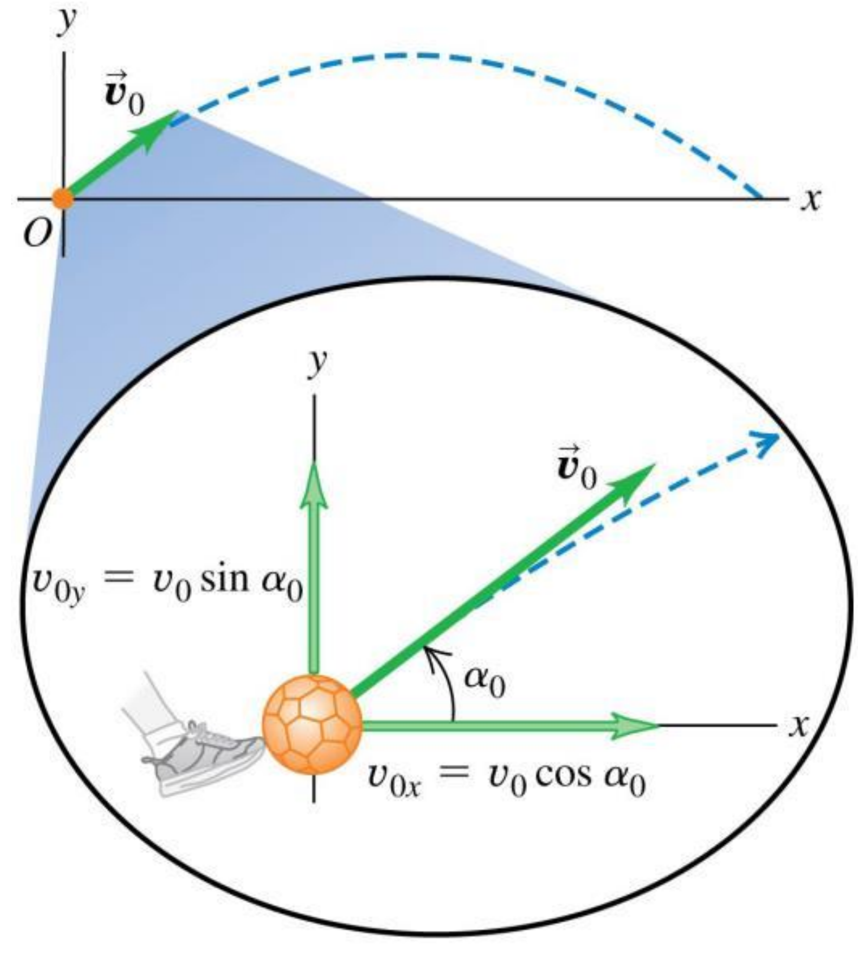


Projectile Motion – Initial Velocity

- The initial velocity components of a projectile (such as a kicked soccer ball) are related to the initial speed and initial angle.

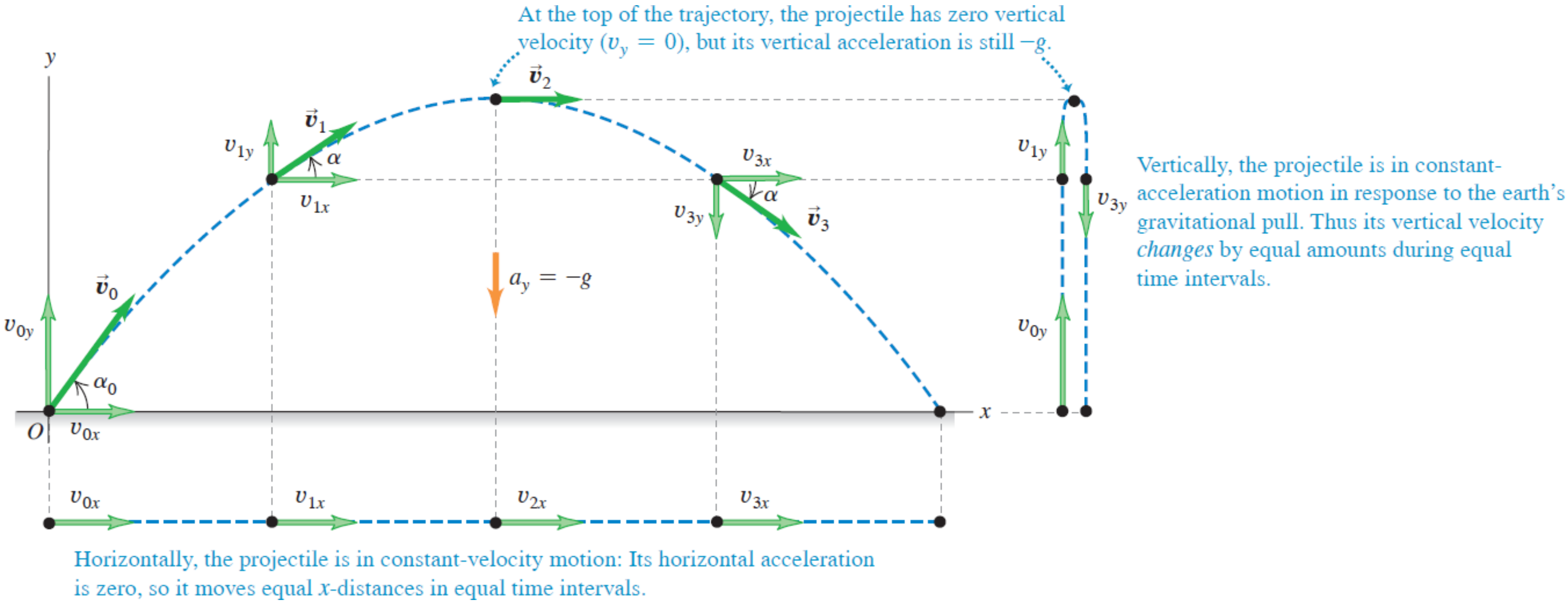
$$v_x = \frac{dx}{dt} = v_o \cos\theta$$

$$v_y = \frac{dy}{dt} = v_o \sin\theta - gt$$

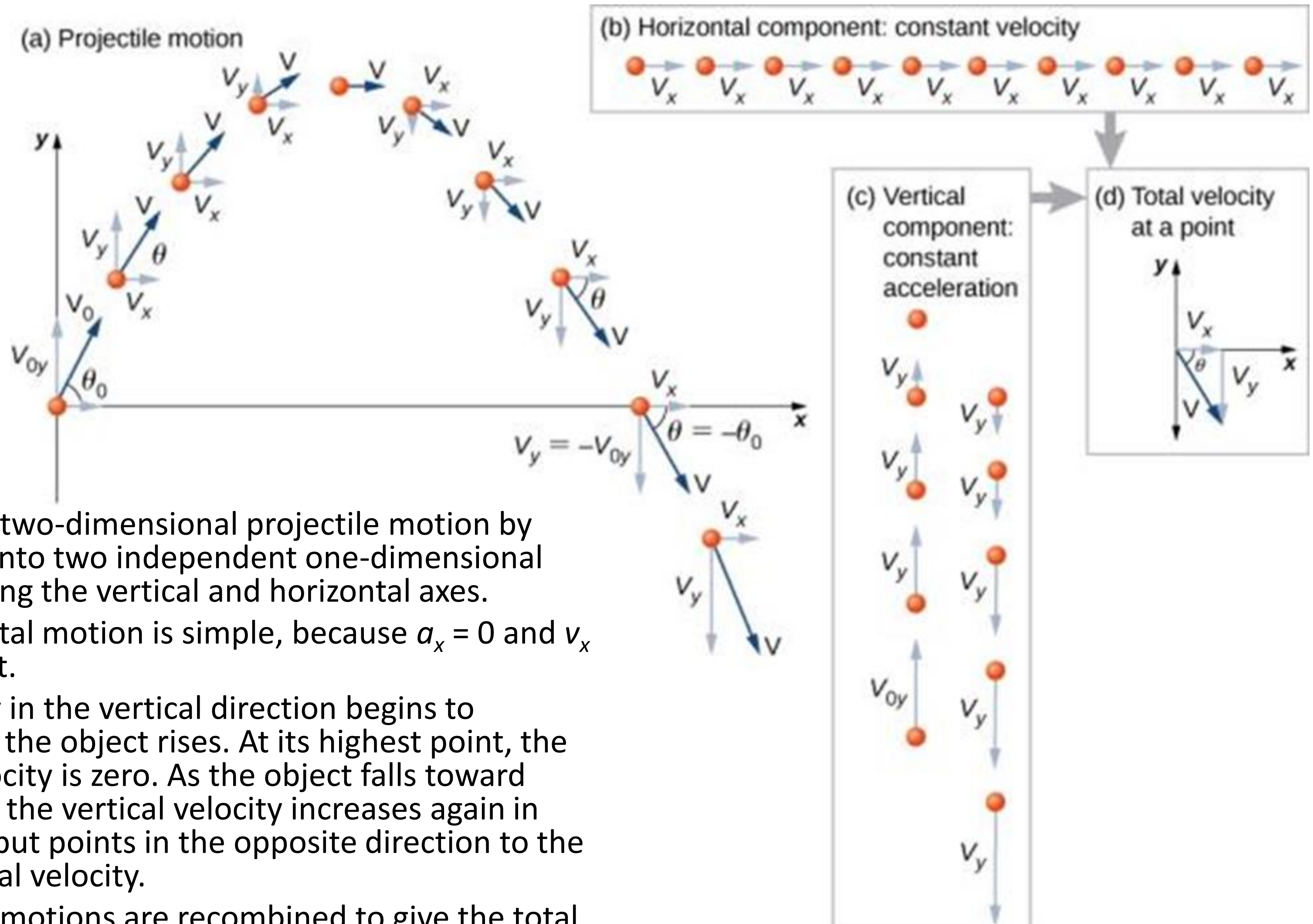


Projectile Motion

- If air resistance is negligible, the trajectory of a projectile is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration.



Components of Projectile Motion



(a) We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes.

(b) The horizontal motion is simple, because $a_x = 0$ and v_x is a constant.

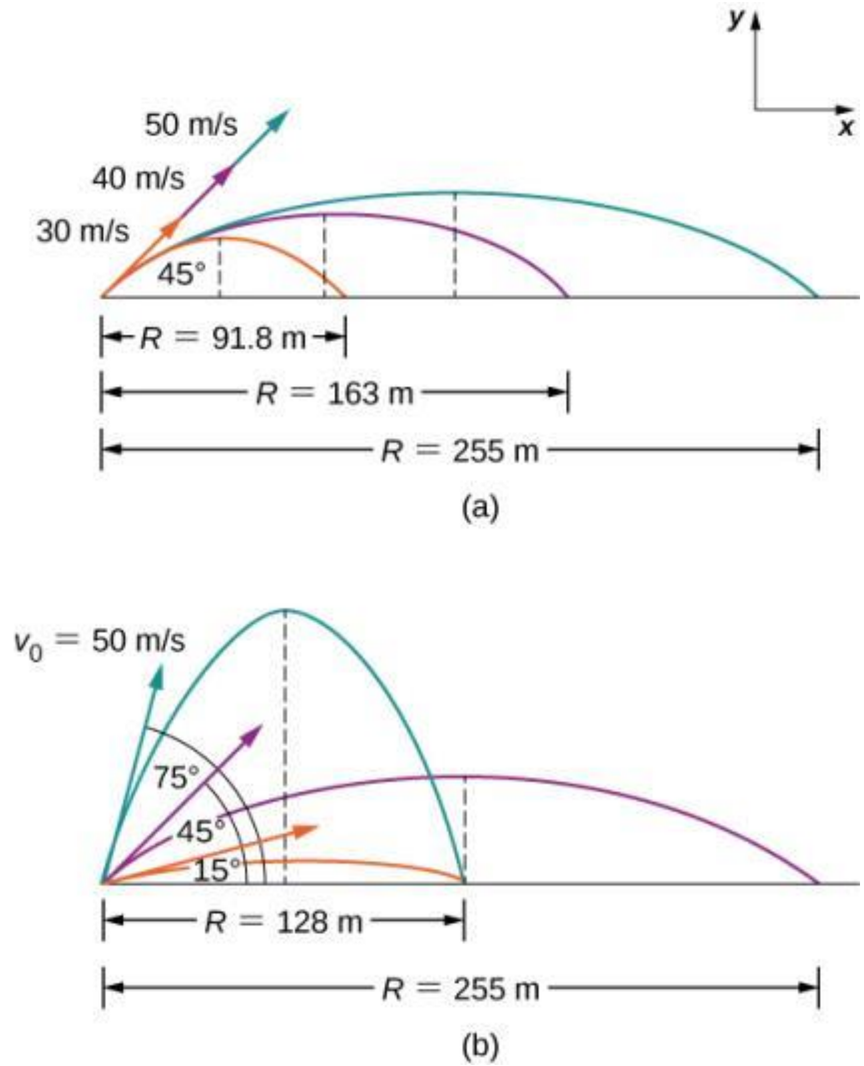
(c) The velocity in the vertical direction begins to decrease as the object rises. At its highest point, the vertical velocity is zero. As the object falls toward Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity.

(d) The x and y motions are recombined to give the total velocity at any given point on the trajectory.

Trajectories of Projectiles on Level Ground

(a)The greater the initial speed v_0, the greater the range for a given initial angle.

(b)The effect of initial angle θ_0 on the range of a projectile with a given initial speed. Note that the range is the same for initial angles of 15° and 75°, although the maximum heights of those paths are different.



The Equations for Projectile Motion

- If we set $x_0 = y_0 = 0$, the equations describing projectile motion are shown below:

Coordinates at time t of a **projectile** (positive y-direction is upward, and $x = y = 0$ at $t = 0$)

$$x = (v_0 \cos \alpha_0)t$$

Speed at $t = 0$ Direction at $t = 0$ Time

$$y = (v_0 \sin \alpha_0)t - \tfrac{1}{2}gt^2$$

Velocity components at time t of a **projectile** (positive y-direction is upward)

$$v_x = v_0 \cos \alpha_0$$

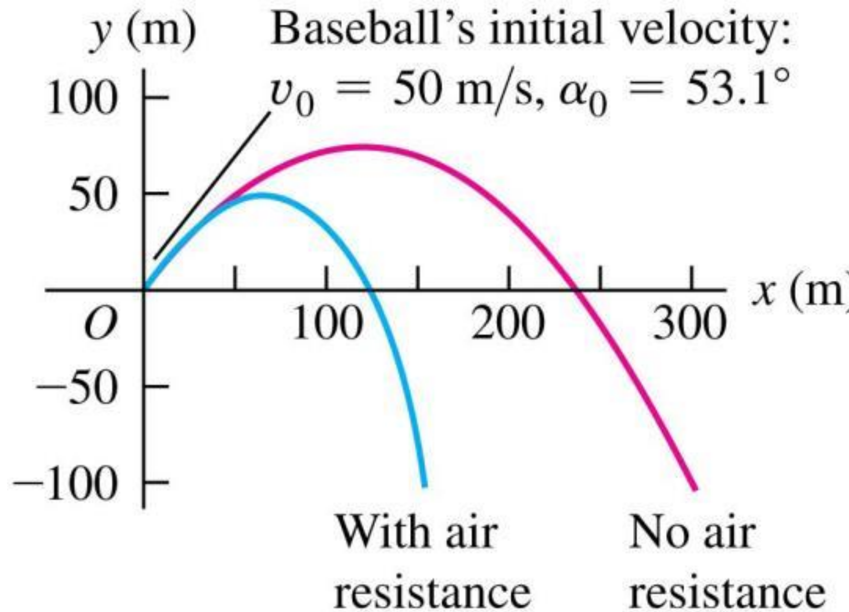
Speed at $t = 0$ Direction at $t = 0$

Acceleration due to gravity: Note $g > 0$.

$$v_y = v_0 \sin \alpha_0 - gt$$

Time

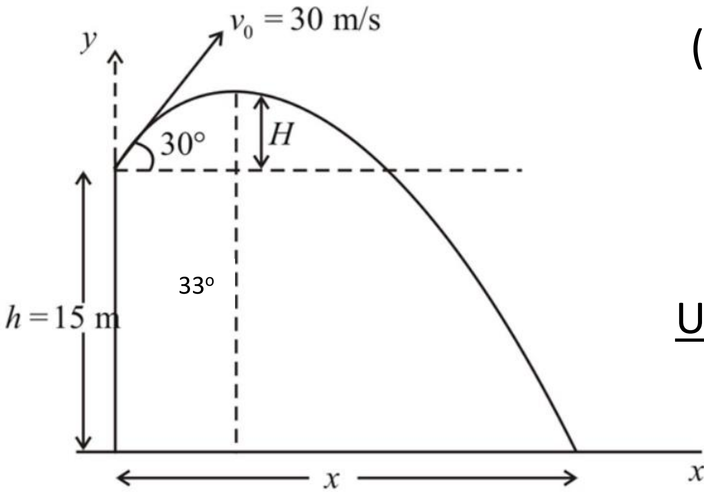
The Effects of Air Resistance



- Calculations become more complicated.
- Acceleration is not constant.
- Effects can be very large.
- Maximum height and range decrease.
- Trajectory is no longer a parabola.

Example: Projectile motion

- (3.21) A man stands on the roof of a 15 m tall building and throws a rock with a speed 30 m/s at an angle 33° above the horizon. Ignore air resistance. Calculate:



(a) the maximum height above the roof the rock reaches.

$$v_{ox} = v_o \cos(33^o) = 25.2\ m/s$$

$$v_{oy} = v_o \sin(33^o) = 16.3\ m/s$$

Use Eqn of Motion (iii): $v_y{}^2 = v_{oy}{}^2 + 2a_y(y - y_o)$

$$\Delta y = \frac{v_y{}^2 - v_{oy}{}^2}{2a_y} = \frac{-(16.3\ m/s)}{2\left(-9.8\frac{m}{s^2}\right)} = \mathbf{13.6\ m}$$

(b) The speed of the rock just before it reaches the ground.

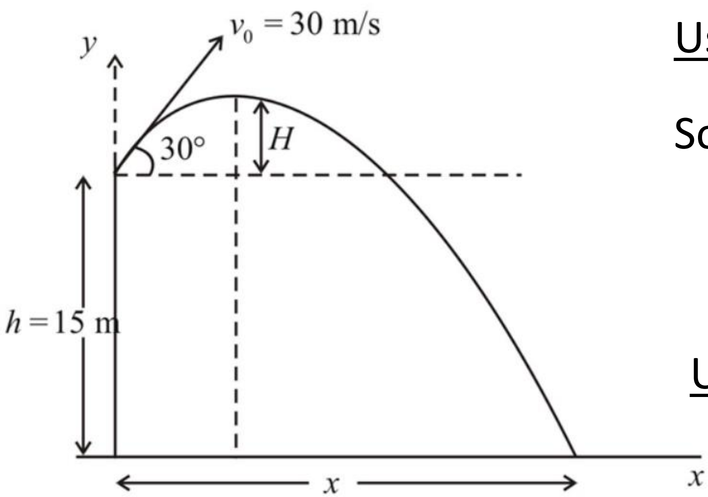
$$v = \sqrt{v_x{}^2 + v_y{}^2}$$

where $v_y{}^2 = v_o{}^2 + 2a_y(y - y_o) = (16.3\ m/s)^2 + 2\left(-9.8\frac{m}{s^2}\right)(-15m)$

$$v = \sqrt{(25.2\ m/s)^2 + (-23.7\ m/s)^2} = \mathbf{34.6\ m/s}$$

Example: Projectile motion

(c) The horizontal range from the base of the building to the point where the rock strikes the ground.



<u>Use Eqn of Motion (ii)</u>: $v_y = v_{oy} + a_y t$

Solve for t…

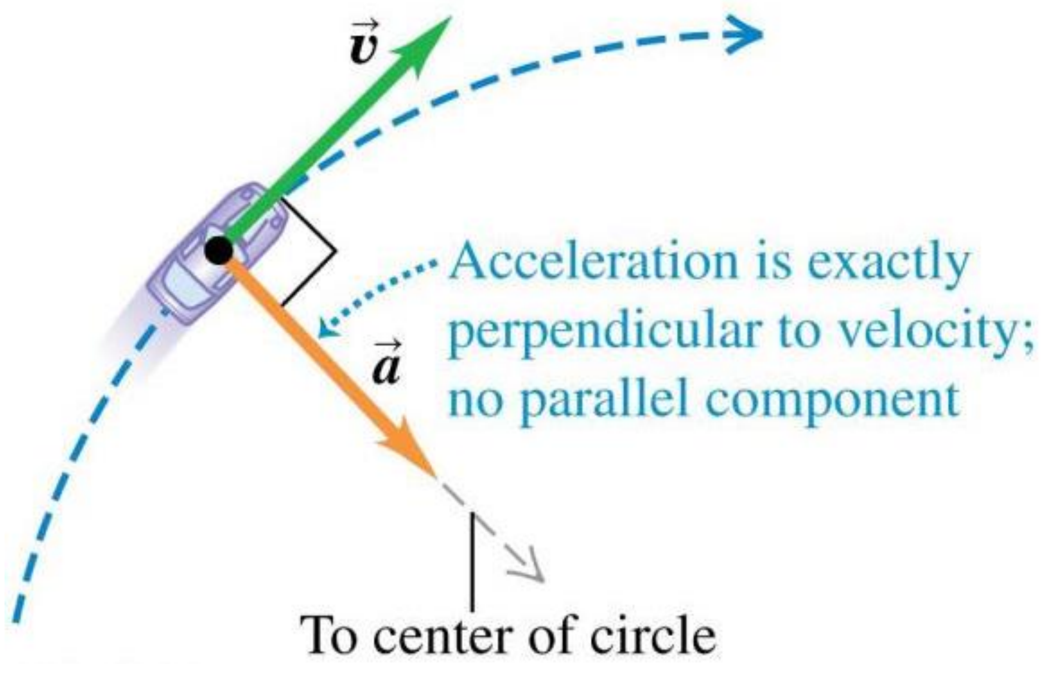
$$t = \frac{v_y - v_{oy}}{a_y} = \frac{(-23.7\, m/s) - (16.3\, m/s)}{-9.8\, m/s^2} = 4.1 \text{ s}$$

<u>Use Eqn of Motion (i)</u>: $x = x_o + v_o t + a_x t^2$

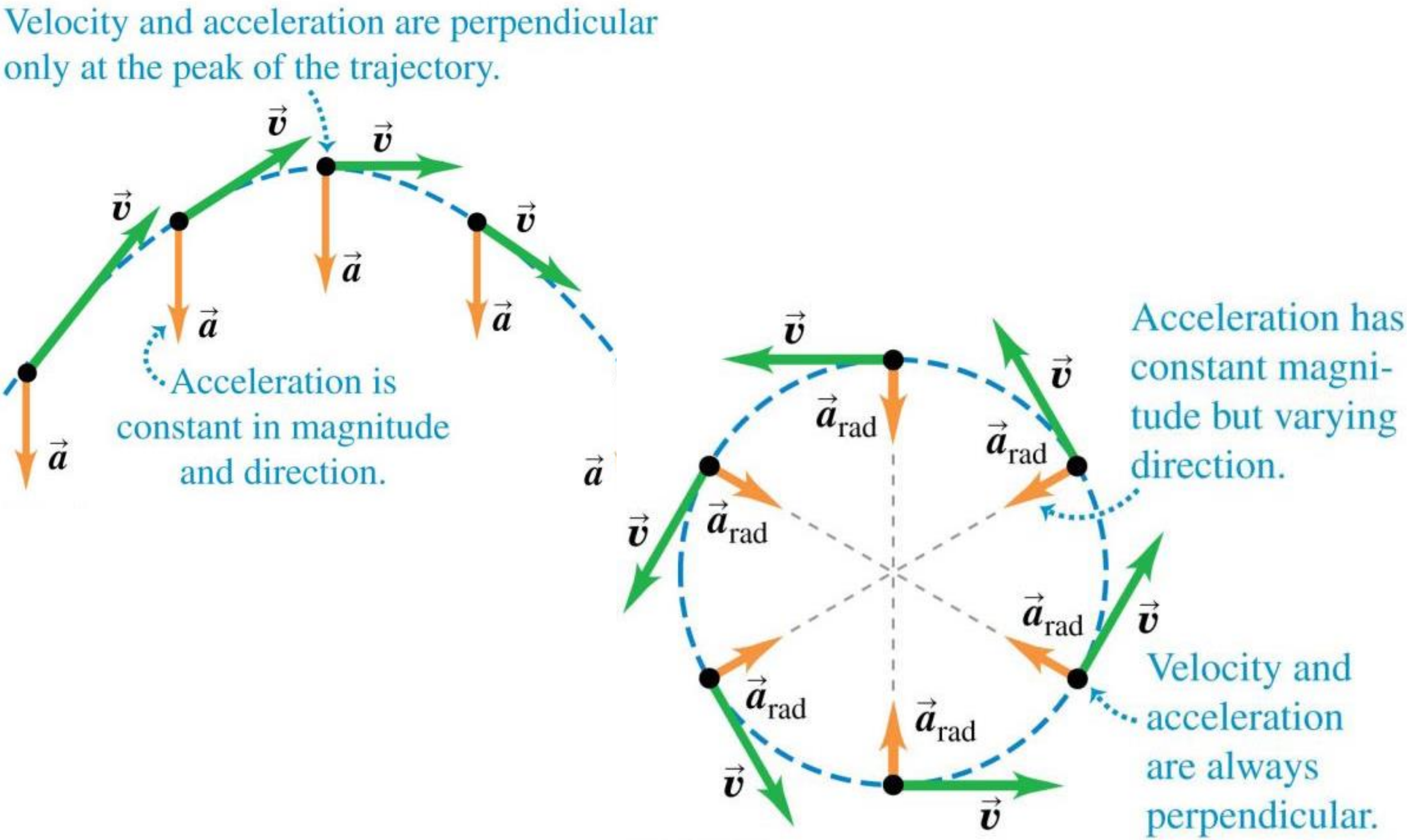
$$x(t) = x_o + v_{0x} t + a_x t^2 = 0 + \left(25.2 \frac{m}{s}\right)(4.1\ s) + 0 = \mathbf{102.8\ m}$$

3.4 Motion in a Circle

- **Uniform circular motion** is constant speed along a circular path.
- The **acceleration** is always directed towards the center of the circle.

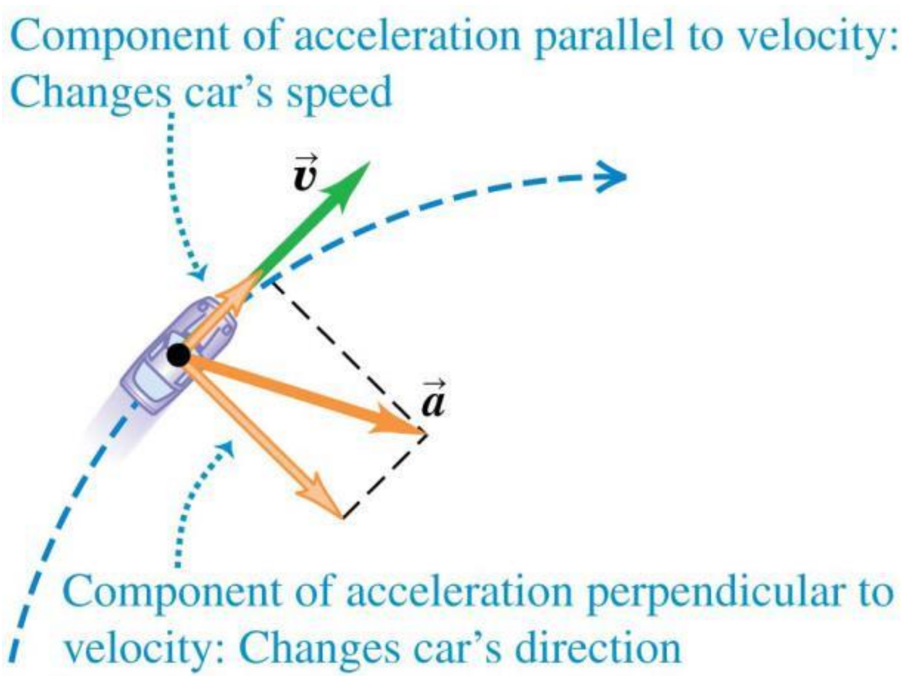


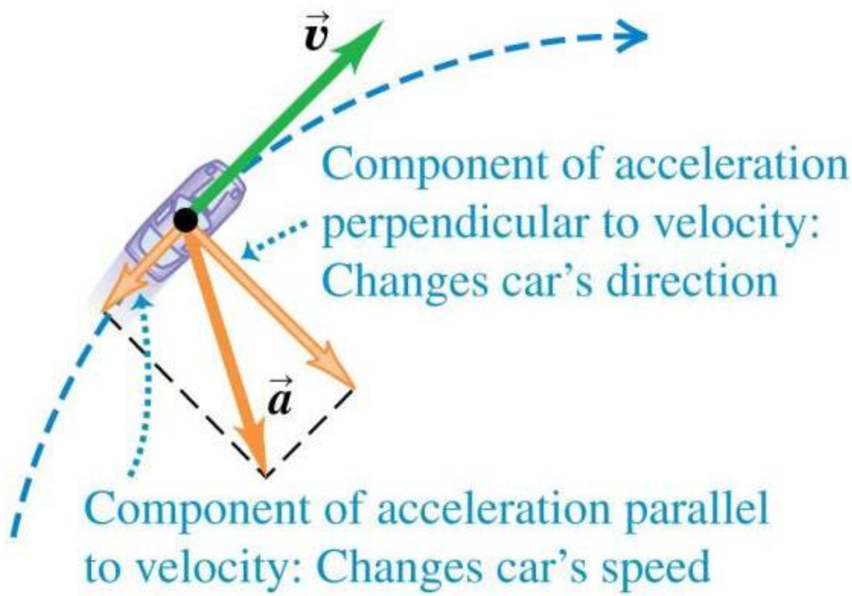
Projectile vs Uniform Circular Motion



Motion in a Circle / Changing Velocity

- Car speeding up along a circular path
- Car slowing down along a circular path



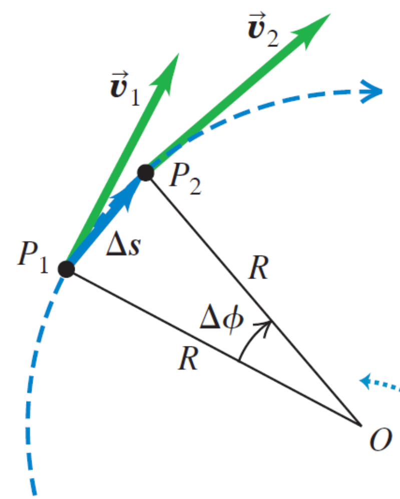


Velocity for Uniform Circular Motion

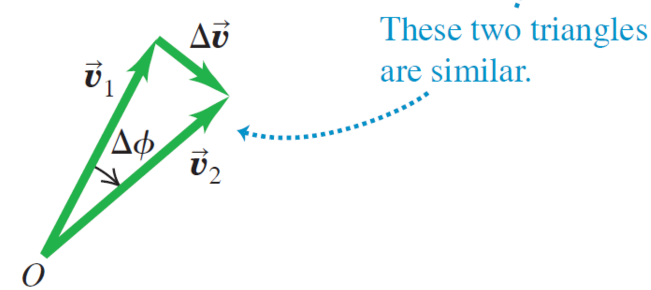
- The velocity is distance traveled (circumference = **2πR**) divided by the time it takes to complete one revolution (**period T**).

$$v = \frac{2\pi R}{T}$$

(a) A particle moves a distance Δs at constant speed along a circular path.



(b) The corresponding change in velocity $\Delta\vec{v}$. The average acceleration is in the same direction as $\Delta\vec{v}$.

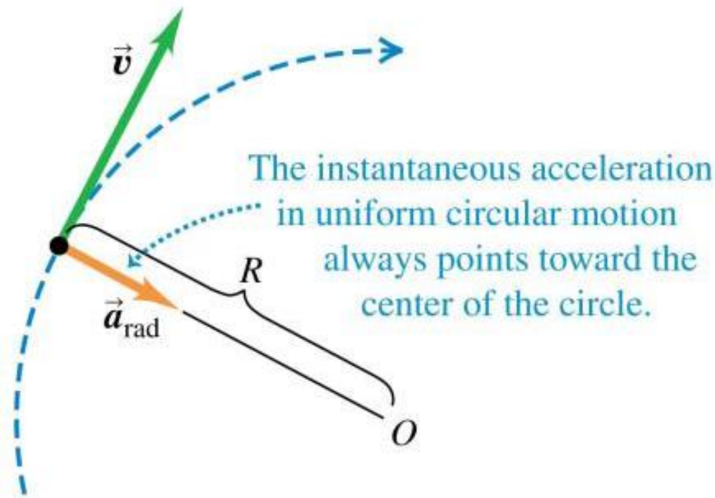


Acceleration for Uniform Circular Motion

- For uniform circular motion, the instantaneous acceleration always points toward the center of the circle and is called the **centripetal acceleration**.
- The magnitude of the acceleration is:

$$a_{rad} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}$$

where **period T** is the time to complete one revolution.



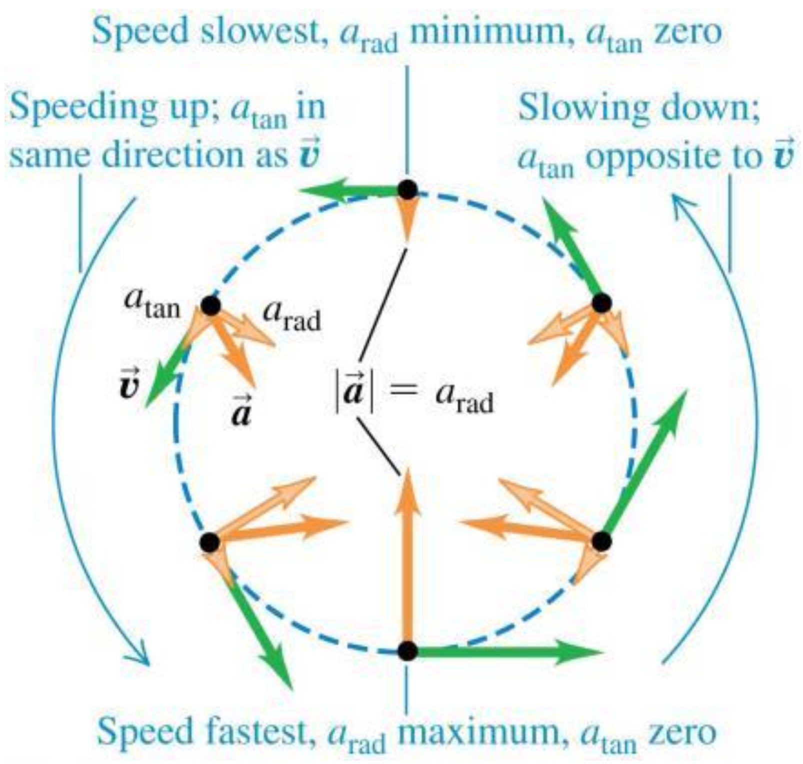
Magnitude of acceleration of an object in **uniform circular motion** $a_{rad} = \frac{v^2}{R}$ Speed of object (v); Radius of object's circular path (R)

Non-uniform Circular Motion

- If the speed varies, the motion is **non-uniform circular motion**.
- The radial acceleration component is still:

$$a_{rad} = \frac{v^2}{R} = \frac{4\pi^2 R}{T^2}$$

- but there is also a tangential acceleration component a_{tan} that is **parallel** to the instantaneous velocity.



Example problem

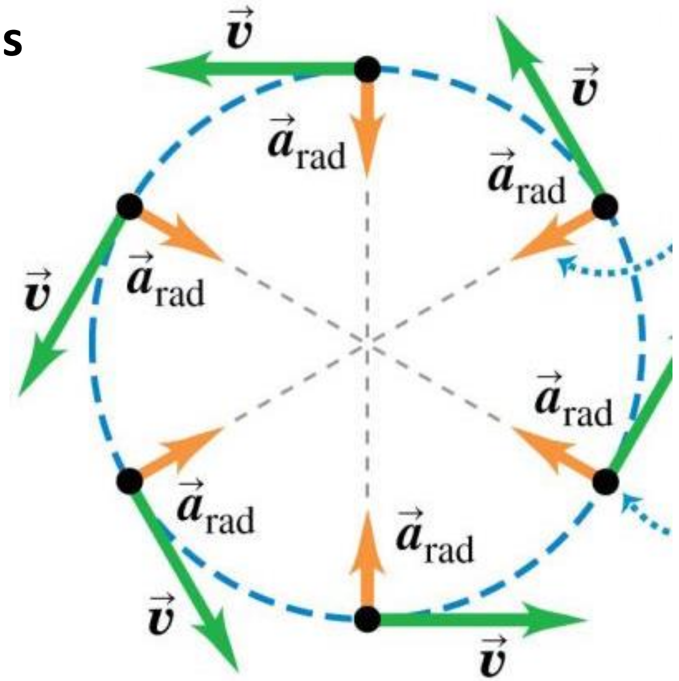
A Ferris wheel with a radius of 14 m is turning about a horizontal axis. The linear speed of a passenger on the rim is constant and equal to 6 m/s. What are the magnitude and direction of the passenger's acceleration as she passes through (a) the lowest point in her circular motion, and (b) the highest point in her circular motion? (c) How much time does it take to the Ferris wheel to make one revolution?

(a) $a_{rad} = \frac{v^2}{R} = \frac{(6\,m/s)^2}{14\,m} = \mathbf{2.57\ m/s2}$ **upwards**

(b) a_{rad} = $\mathbf{2.57\ m/s2}$ **downwards**

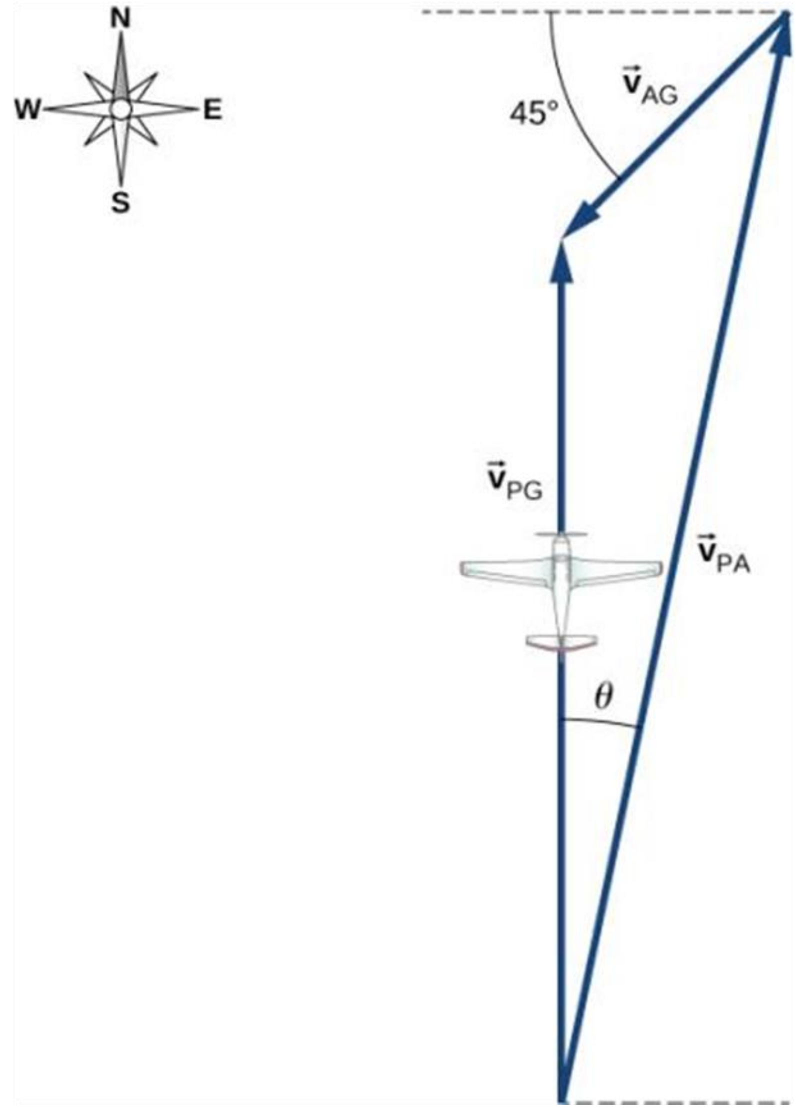
(c) $\boxed{v = \frac{2\pi R}{T}}$

$$T = \frac{2\pi R}{v} = \frac{2\pi(14\,m)}{6\,m/s} = \mathbf{14.7}\ \boldsymbol{s}$$

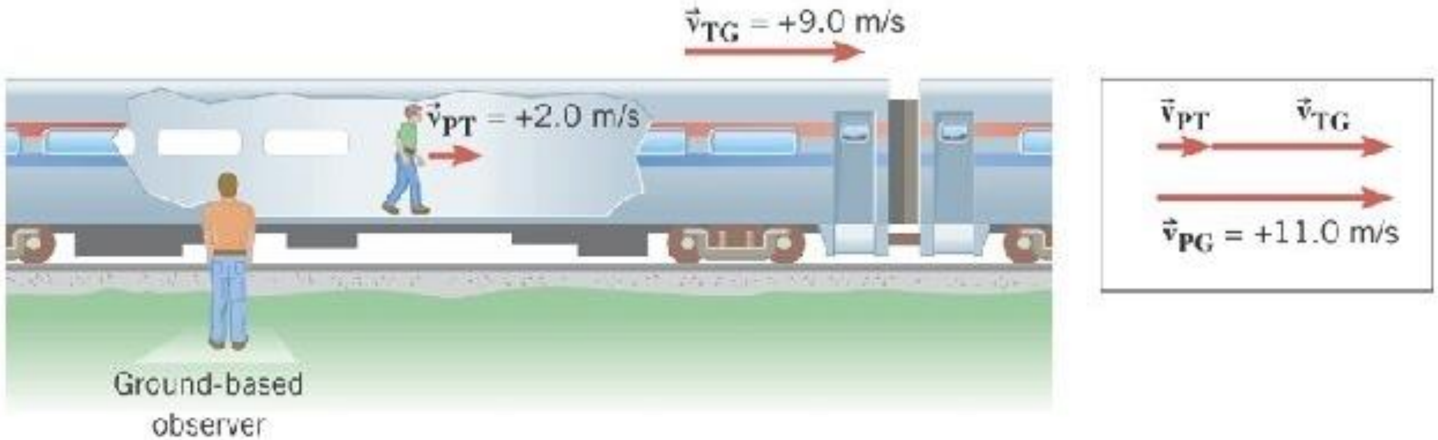


3.5 Relative Velocity

- The velocity of a moving object seen by a particular observer is called the velocity **relative** to that observer, or simply the **relative velocity**.
- A **frame of reference** is a coordinate system plus a time scale.
- In many situations relative velocity is extremely important.



Example of Relative Velocity



Consider the reference frame of the train (and its seated passengers).

Let V_{PT} = velocity of the (moving) Passenger relative to the Train = +2.0 m/s

Consider the reference frame of the ground-based observer.

Let V_{TG} = velocity of the Train relative to the Ground = +9.0 m/s

Then what is the velocity of the Passenger relative to the Ground (V_{PG})?

$$\vec{v}_{PG} = \vec{v}_{PT} + \vec{v}_{TG}$$