

# Applications PSC-SNE

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## Cases

### Case 1

Sample on the  $(\mathbb{S}^1)^2$  where  $p = 1$  and  $r = 2$ .

Data:

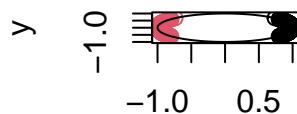
```
# Sample on the (S^1)^2
n <- 200
set.seed(100469000)
vmf11 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 10)
set.seed(100469000)
vmf12 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi)), kappa = 10)
x1 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf11),
                                DirStats::to_rad(vmf12)))

set.seed(100469000)
vmf21 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi / 2)), kappa = 5)
set.seed(100469000)
vmf22 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 5)
x2 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf21),DirStats::to_rad(vmf22)))
```

Let's take a look to the data we have generated for the first circumference:

```
r <- 1
plot(rbind(vmf11, vmf12),
      xlim = c(-r, r), ylim = c(-r, r),
      col = rep(1:2, each = n),
      xlab = "x", ylab="y")

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now, it is time to see the second circumference we have generated randomly before:

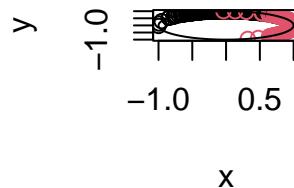
```
r <- 1
plot(rbind(vmf21, vmf22),
      xlim = c(-r, r), ylim = c(-r, r),
      col = rep(1:2, each = n),
```

```

xlab = "x", ylab="y"

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))

```

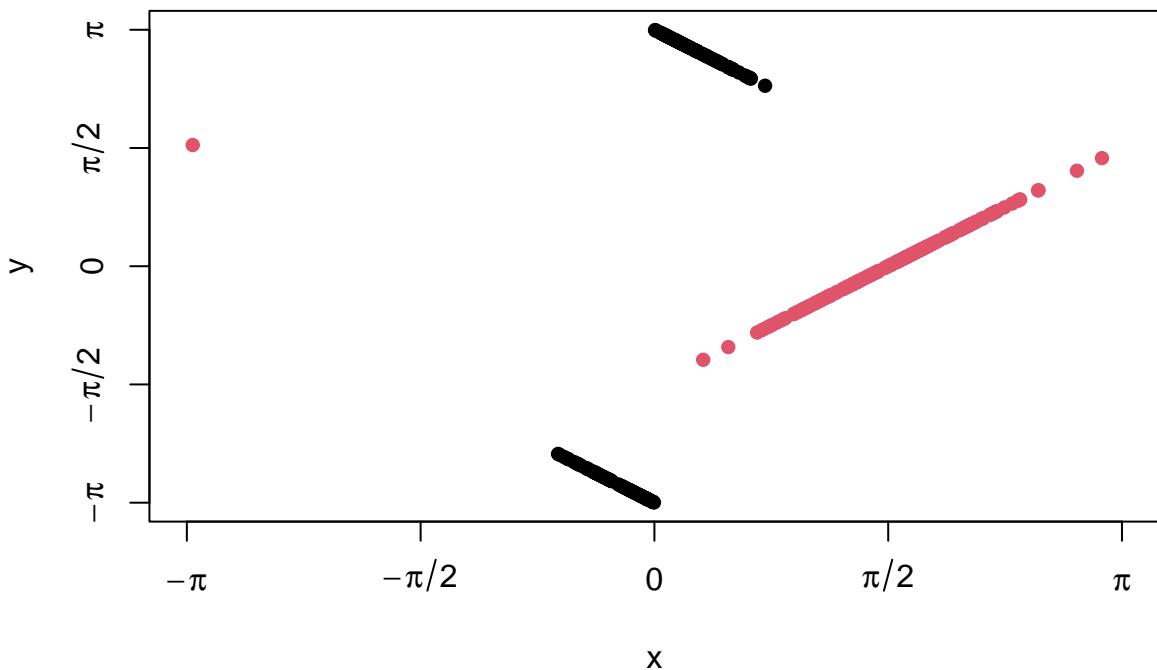


Now, let's do the same visualization with the data on the torus:

```

x <- rbind(x1, x2)
plot(x, xlim = c(-pi, pi), ylim = c(-pi, pi), axes = FALSE,
      col = rep(c(1, 2), each = n), pch = 16,
      xlab = "x", ylab = "y")
sdetorus::torusAxis()

```



```
# Cartesian coordinates
```

```

n <- 400
x_array <- array(dim = c(n, 2, 2))
x_array[, , 1] <- DirStats::to_cir(x[, 1])
x_array[, , 2] <- DirStats::to_cir(x[, 2])

```

Let's calculate the rho parameters based on a perplexity of 20:

```

# Time difference of 2.582784 mins
rho_first_20 <- rho_optim_parallel(x_array, 20)

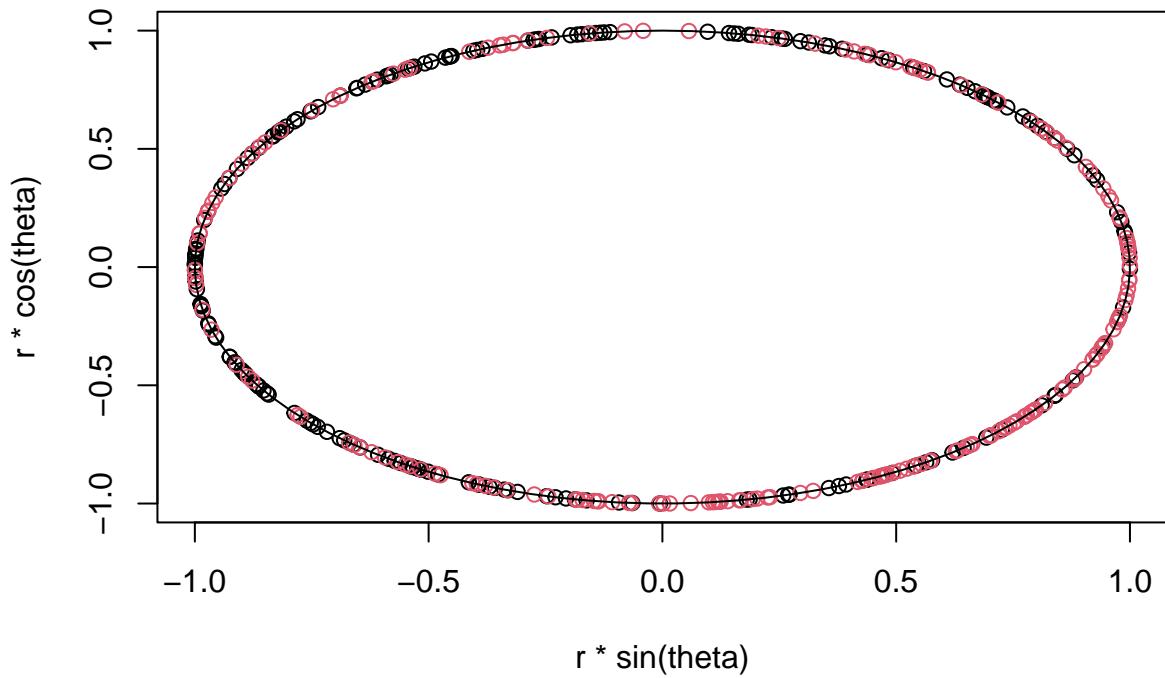
```

```
## Time difference of 2.512619 mins
```

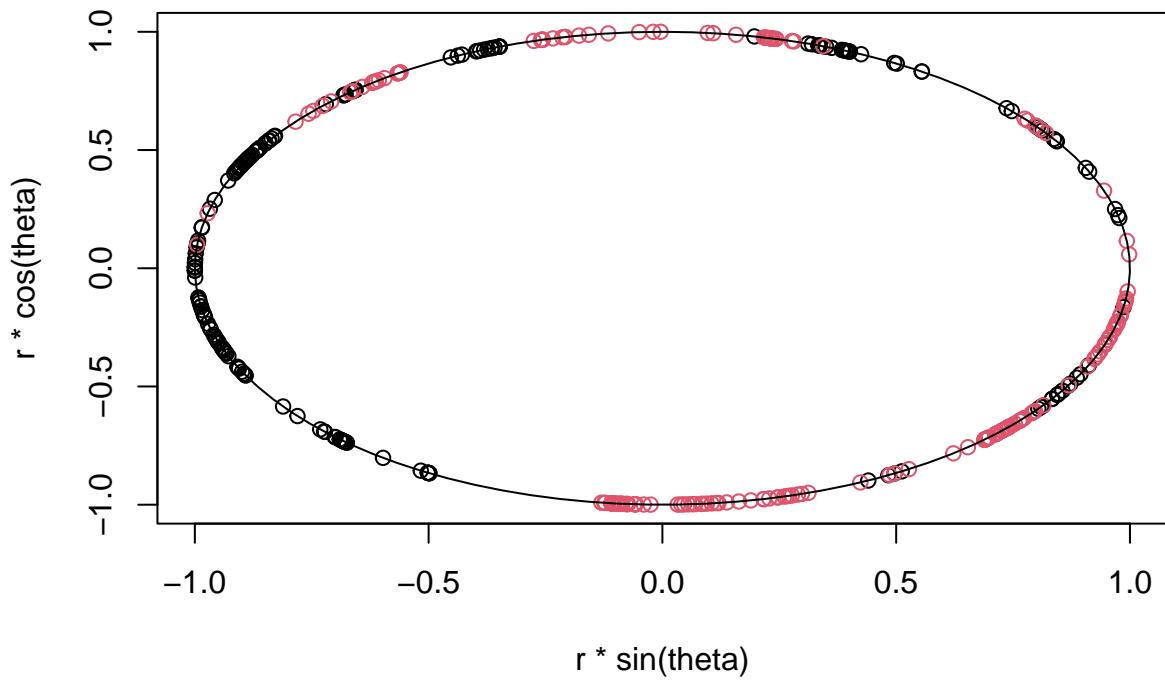
First, let's reduce to dimension  $\mathbb{S}^1$  then  $d = 1$  (circumference):

```
Y <- psc_sne(X=x_array, d=1, rho_psc_list = rho_first_20, num_iteration=125, colors=rep(c(1,2), each=nr)
```

```
## [1] "Iteration 1"  
## [1] "Iteration 1: objective function value is 16.712314"
```



```
## [1] "Iteration 2"  
## [1] "Iteration 3"  
## [1] "Iteration 4"  
## [1] "Iteration 5"  
## [1] "Iteration 5: objective function value is 14.604310"
```

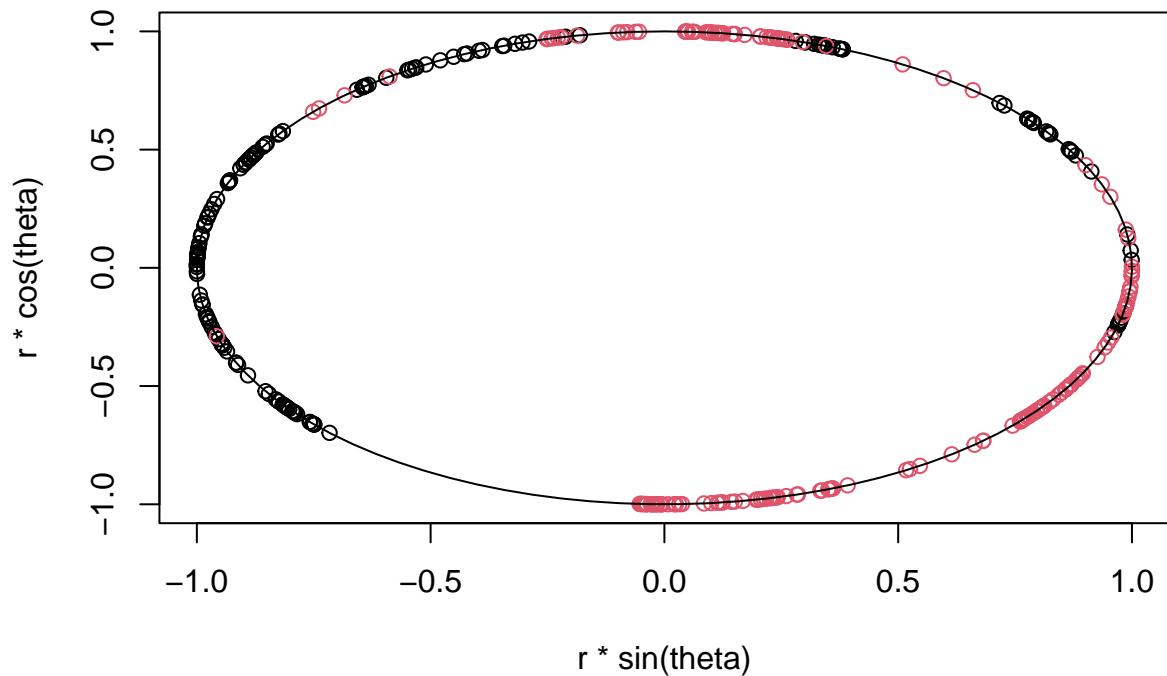


```
## [1] "Iteration 6"  
## [1] "Iteration 7"
```

```

## [1] "Iteration 8"
## [1] "Iteration 9"
## [1] "Iteration 10"
## [1] "Iteration 10: objective function value is 14.152355"

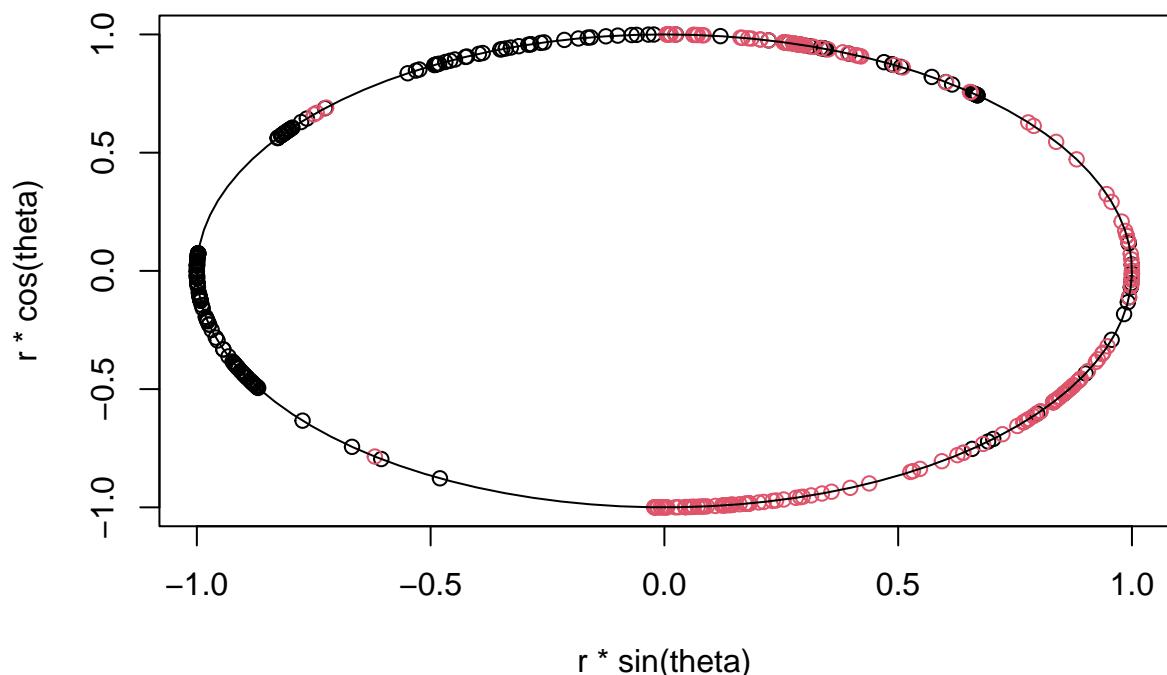
```



```

## [1] "Iteration 11"
## [1] "Iteration 12"
## [1] "Iteration 13"
## [1] "Iteration 14"
## [1] "Iteration 15"
## [1] "Iteration 15: objective function value is 13.883911"

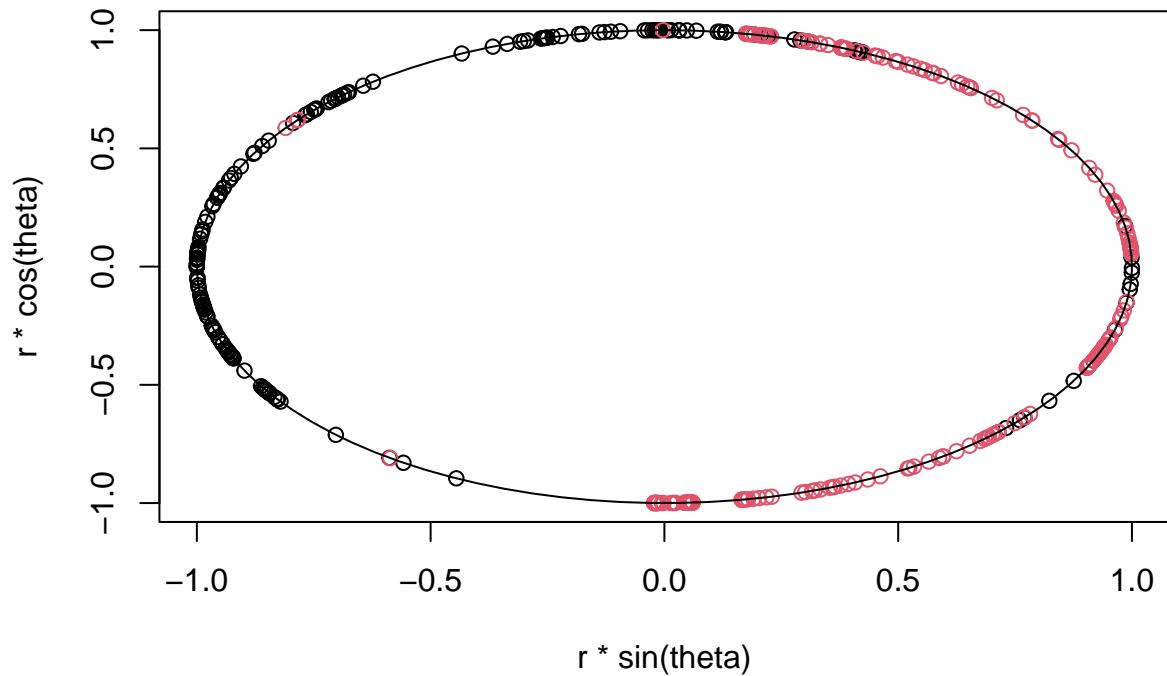
```



```

## [1] "Iteration 16"
## [1] "Iteration 17"
## [1] "Iteration 18"
## [1] "Iteration 19"
## [1] "Iteration 20"
## [1] "Iteration 20: objective function value is 13.687422"

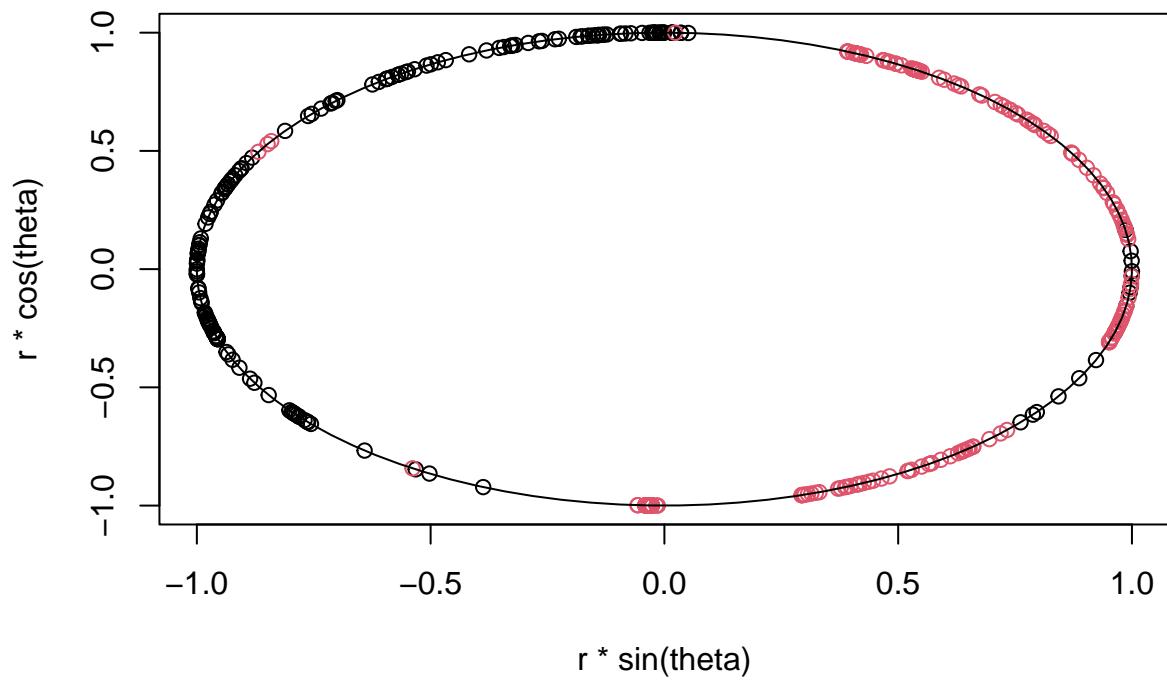
```



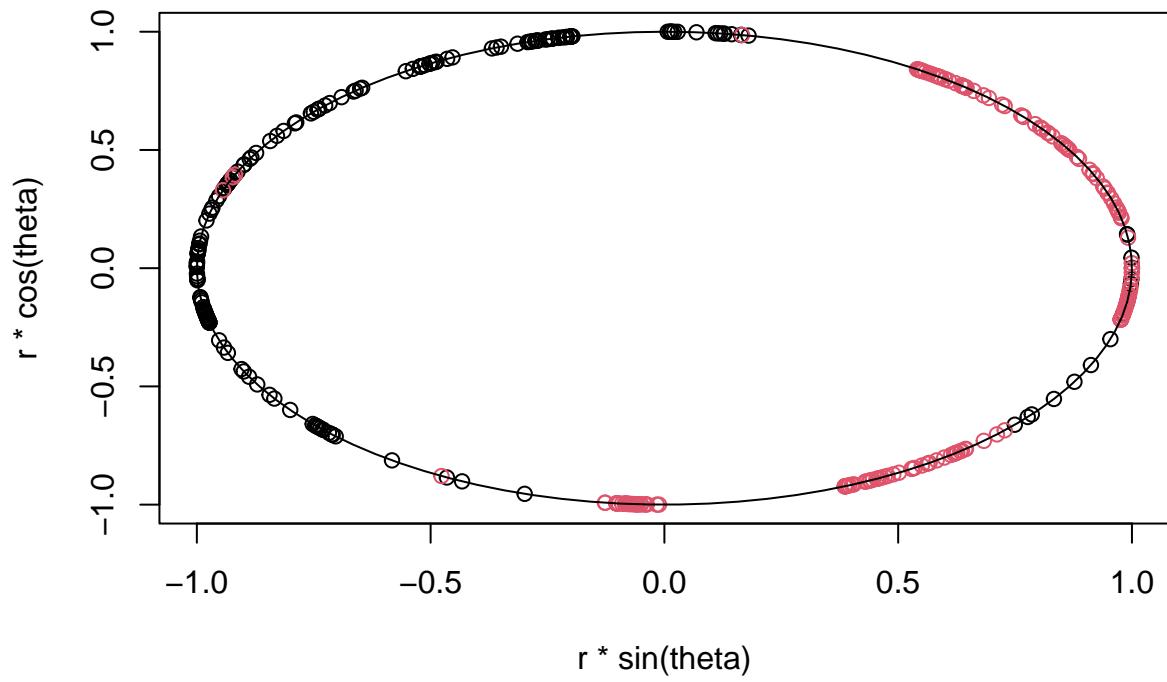
```

## [1] "Iteration 21"
## [1] "Iteration 22"
## [1] "Iteration 23"
## [1] "Iteration 24"
## [1] "Iteration 25"
## [1] "Iteration 25: objective function value is 13.534814"

```

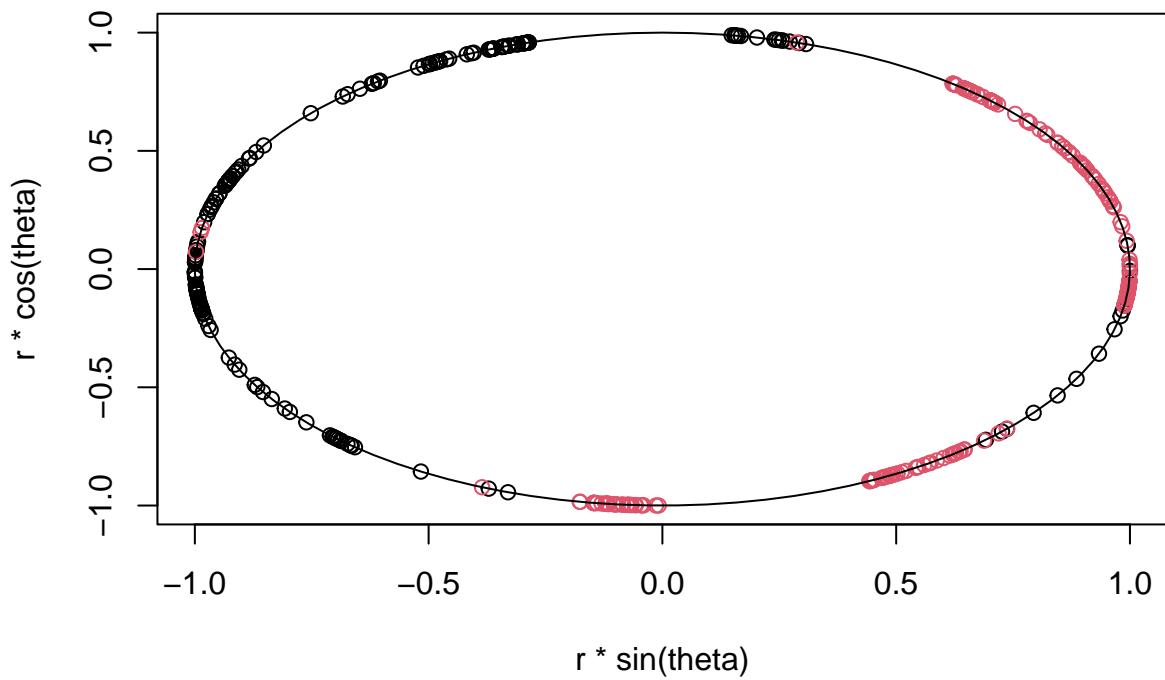


```
## [1] "Iteration 26"
## [1] "Iteration 27"
## [1] "Iteration 28"
## [1] "Iteration 29"
## [1] "Iteration 30"
## [1] "Iteration 30: objective function value is 13.462538"
```

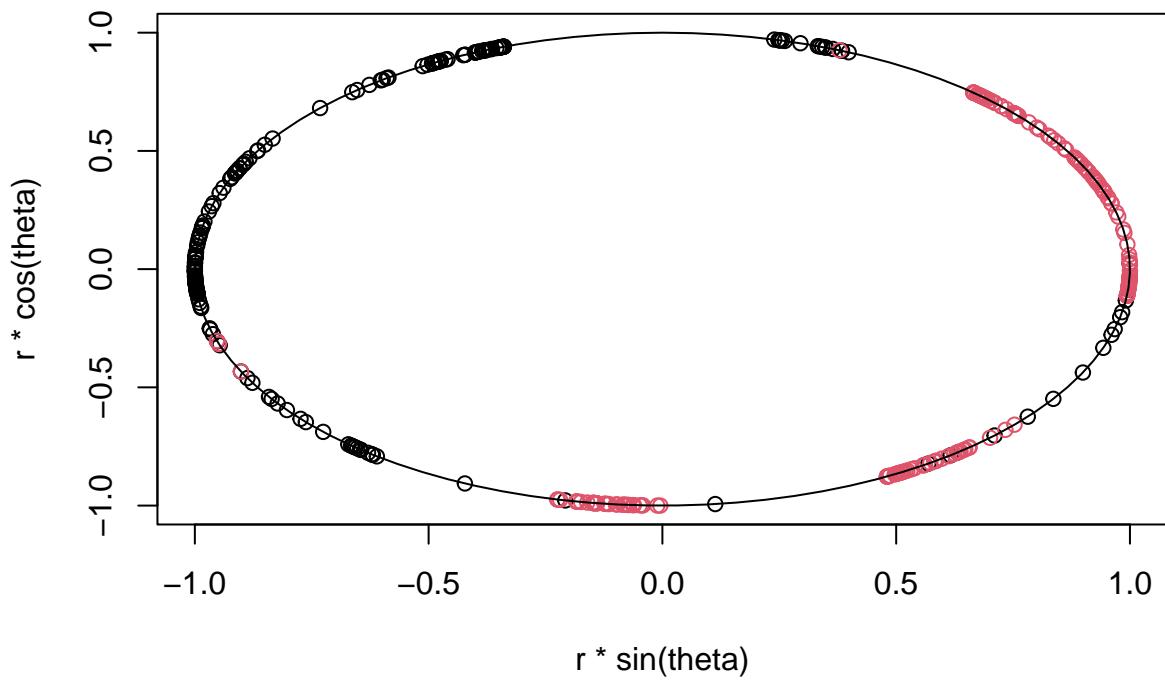


```
## [1] "Iteration 31"
## [1] "Iteration 32"
## [1] "Iteration 33"
## [1] "Iteration 34"
```

```
## [1] "Iteration 35"  
## [1] "Iteration 35: objective function value is 13.445353"
```

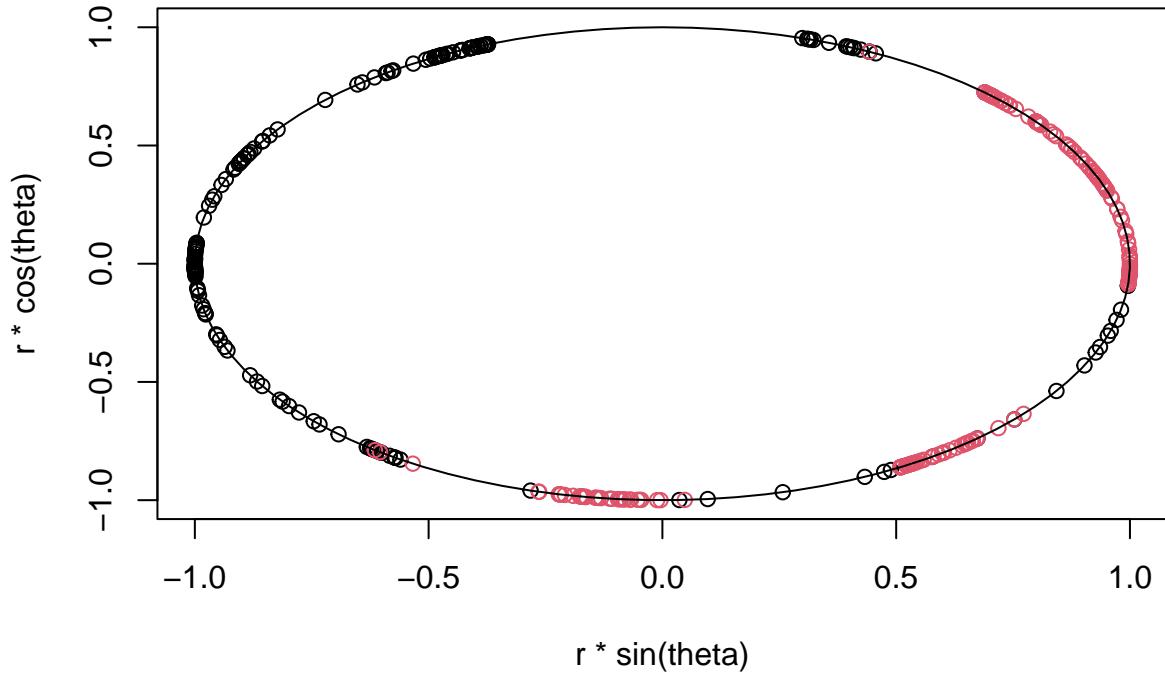


```
## [1] "Iteration 36"  
## [1] "Iteration 37"  
## [1] "Iteration 38"  
## [1] "Iteration 39"  
## [1] "Iteration 40"  
## [1] "Iteration 40: objective function value is 13.433003"
```

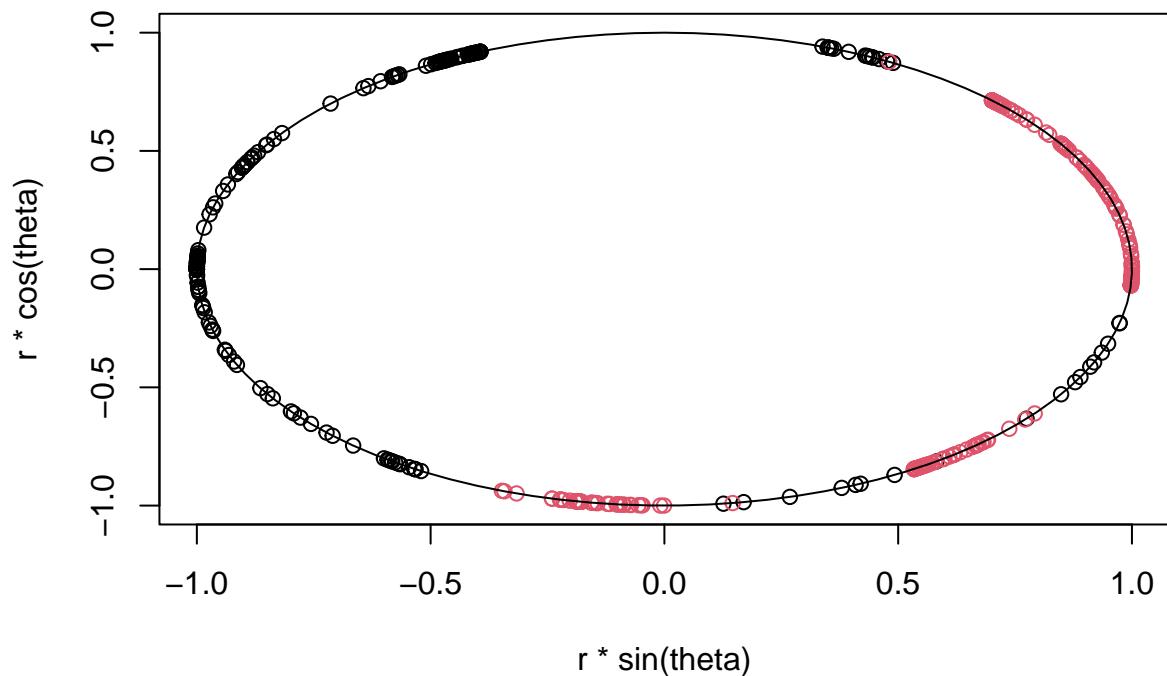


```
## [1] "Iteration 41"
```

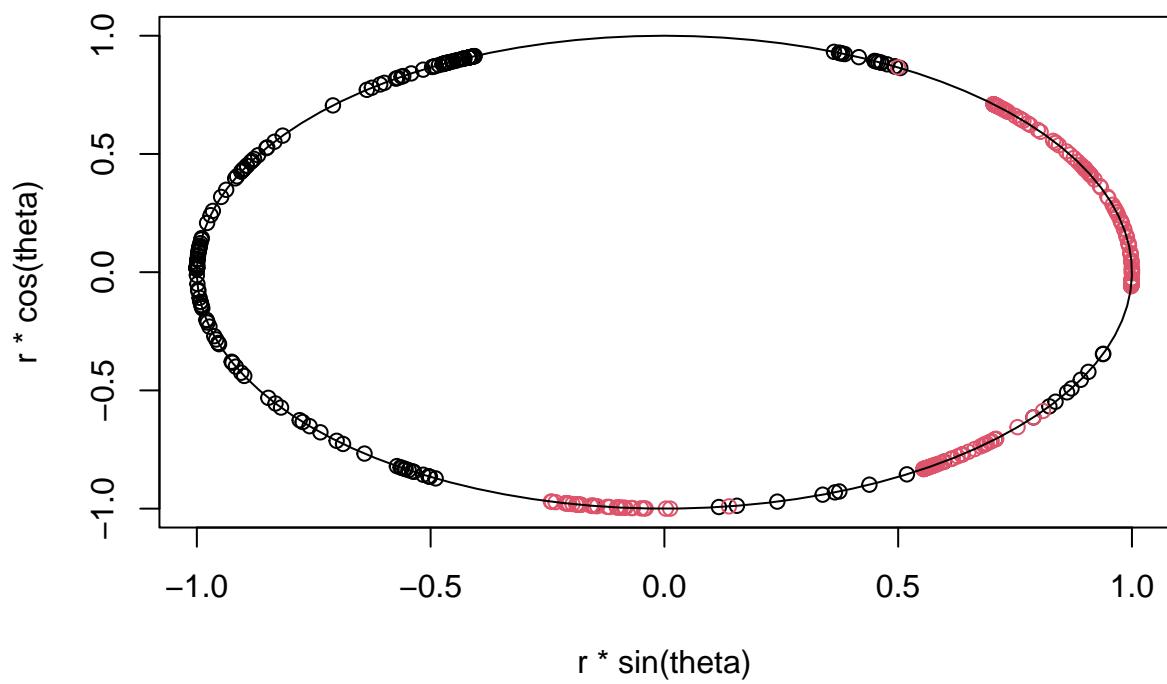
```
## [1] "Iteration 42"  
## [1] "Iteration 43"  
## [1] "Iteration 44"  
## [1] "Iteration 45"  
## [1] "Iteration 45: objective function value is 13.416203"
```



```
## [1] "Iteration 46"  
## [1] "Iteration 47"  
## [1] "Iteration 48"  
## [1] "Iteration 49"  
## [1] "Iteration 50"  
## [1] "Iteration 50: objective function value is 13.410235"
```

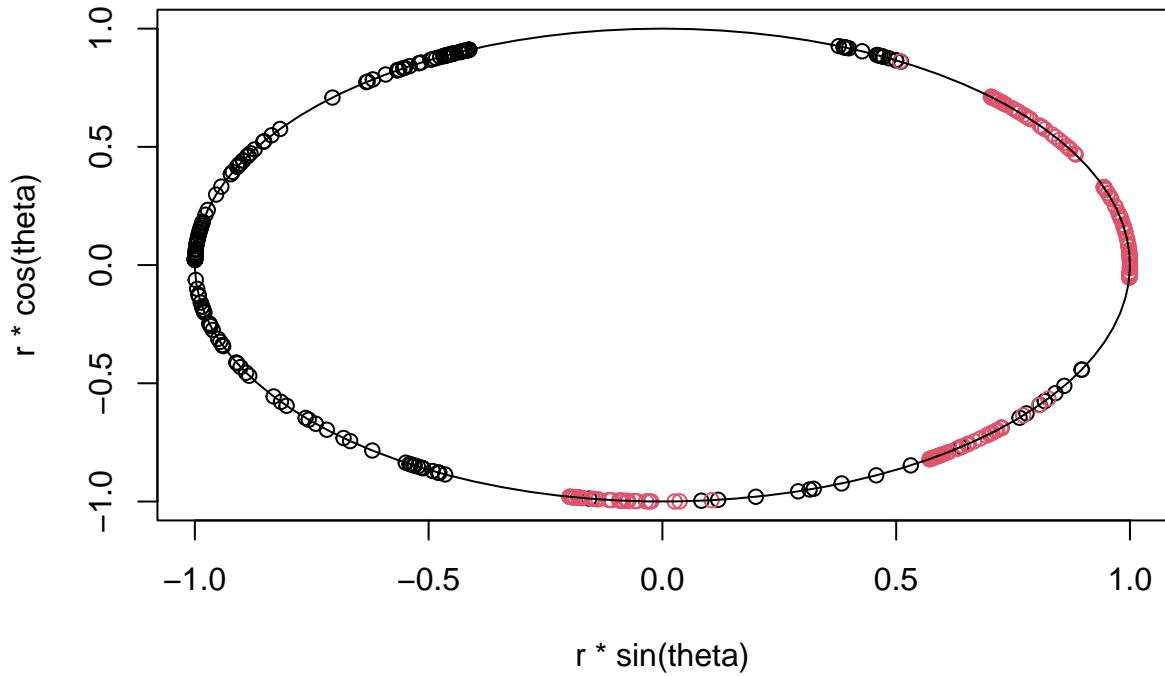


```
## [1] "Iteration 51"
## [1] "Iteration 52"
## [1] "Iteration 53"
## [1] "Iteration 54"
## [1] "Iteration 55"
## [1] "Iteration 55: objective function value is 13.403824"
```

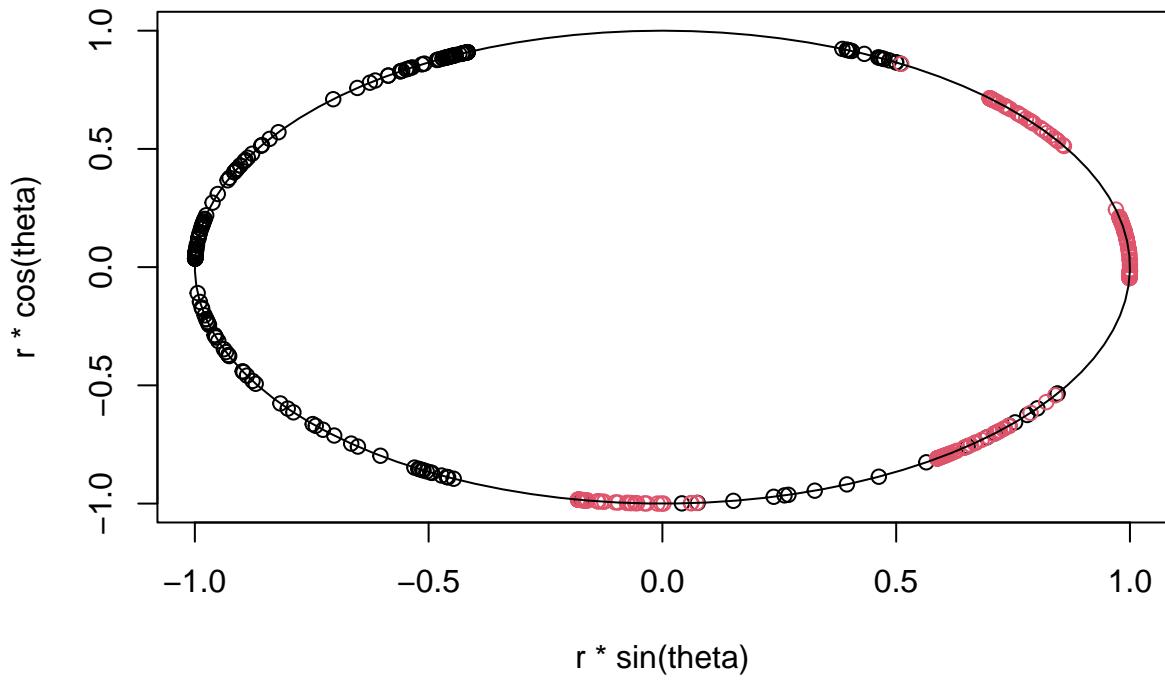


```
## [1] "Iteration 56"
## [1] "Iteration 57"
## [1] "Iteration 58"
## [1] "Iteration 59"
```

```
## [1] "Iteration 60"  
## [1] "Iteration 60: objective function value is 13.395454"
```



```
## [1] "Iteration 61"  
## [1] "Iteration 62"  
## [1] "Iteration 63"  
## [1] "Iteration 64"  
## [1] "Iteration 65"  
## [1] "Iteration 65: objective function value is 13.387166"
```

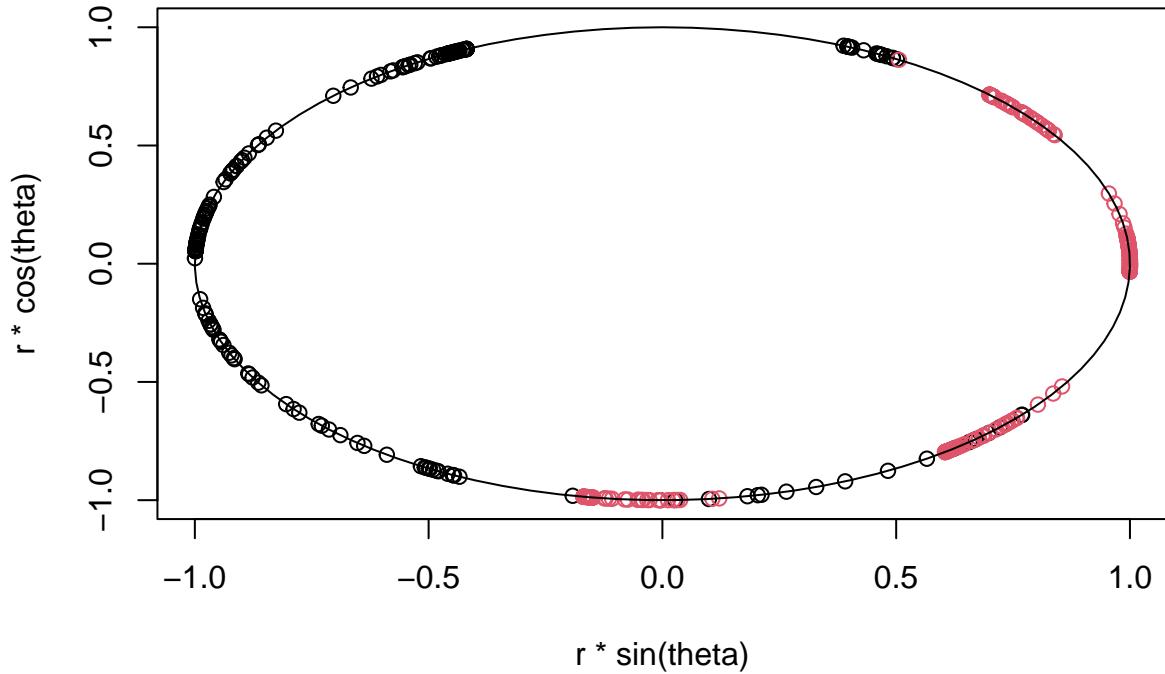


```
## [1] "Iteration 66"
```

```

## [1] "Iteration 67"
## [1] "Iteration 68"
## [1] "Iteration 69"
## [1] "Iteration 70"
## [1] "Iteration 70: objective function value is 13.377977"

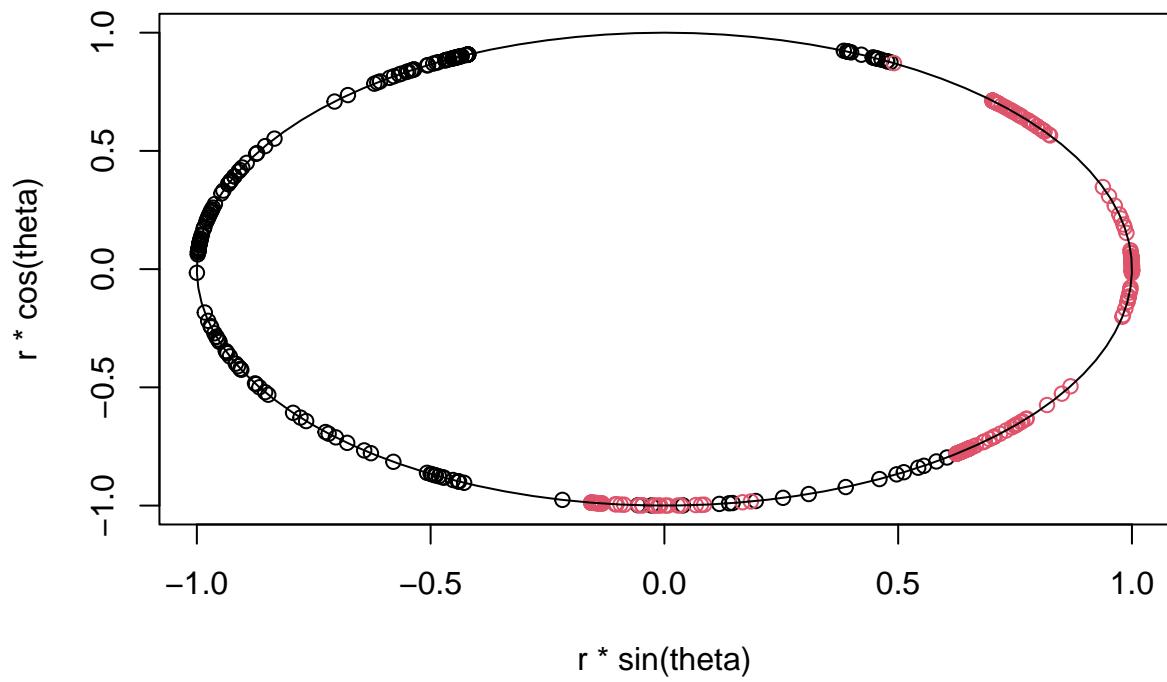
```



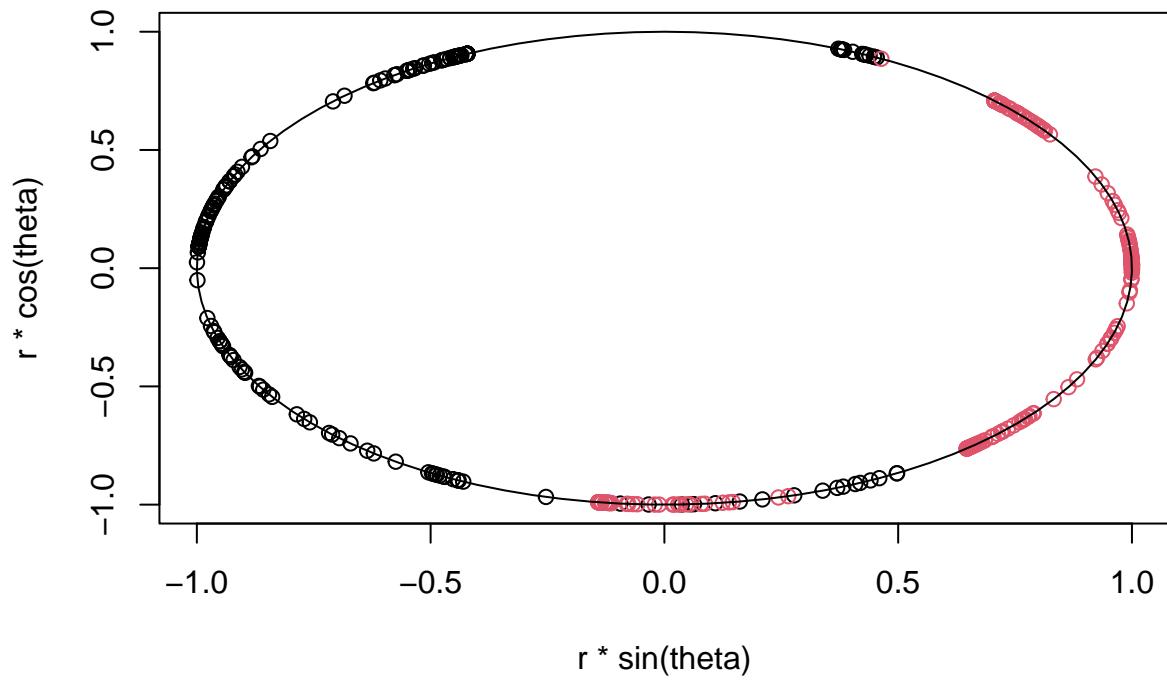
```

## [1] "Iteration 71"
## [1] "Iteration 72"
## [1] "Iteration 73"
## [1] "Iteration 74"
## [1] "Iteration 75"
## [1] "Iteration 75: objective function value is 13.362430"

```

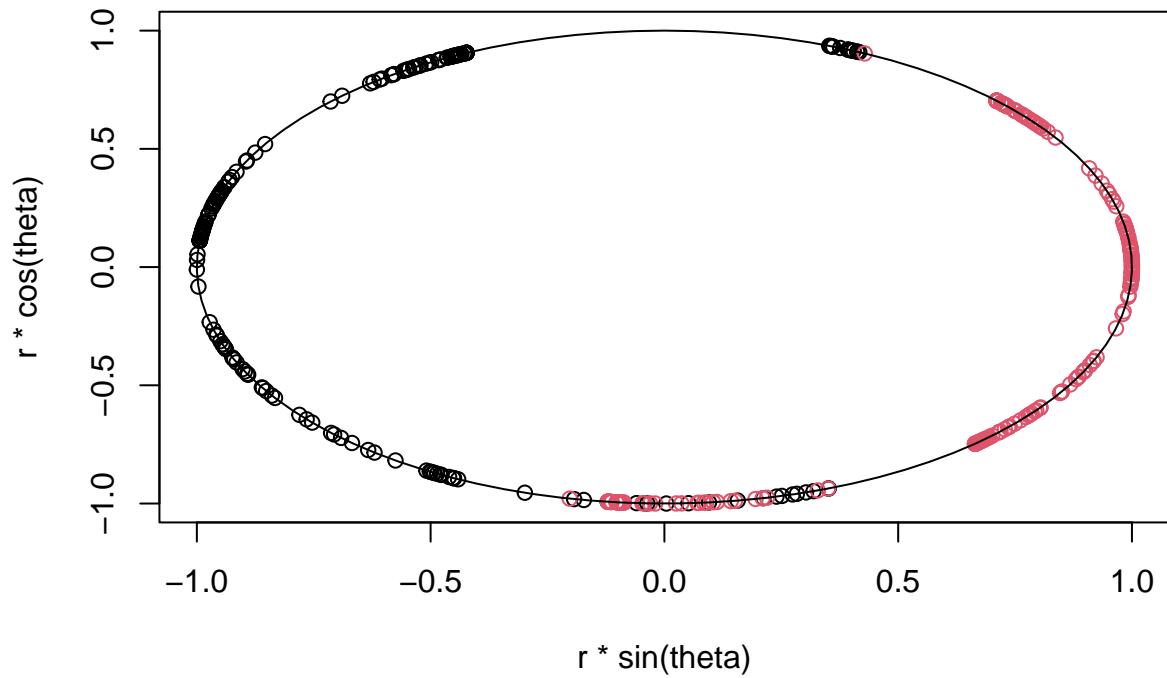


```
## [1] "Iteration 76"
## [1] "Iteration 77"
## [1] "Iteration 78"
## [1] "Iteration 79"
## [1] "Iteration 80"
## [1] "Iteration 80: objective function value is 13.342187"
```

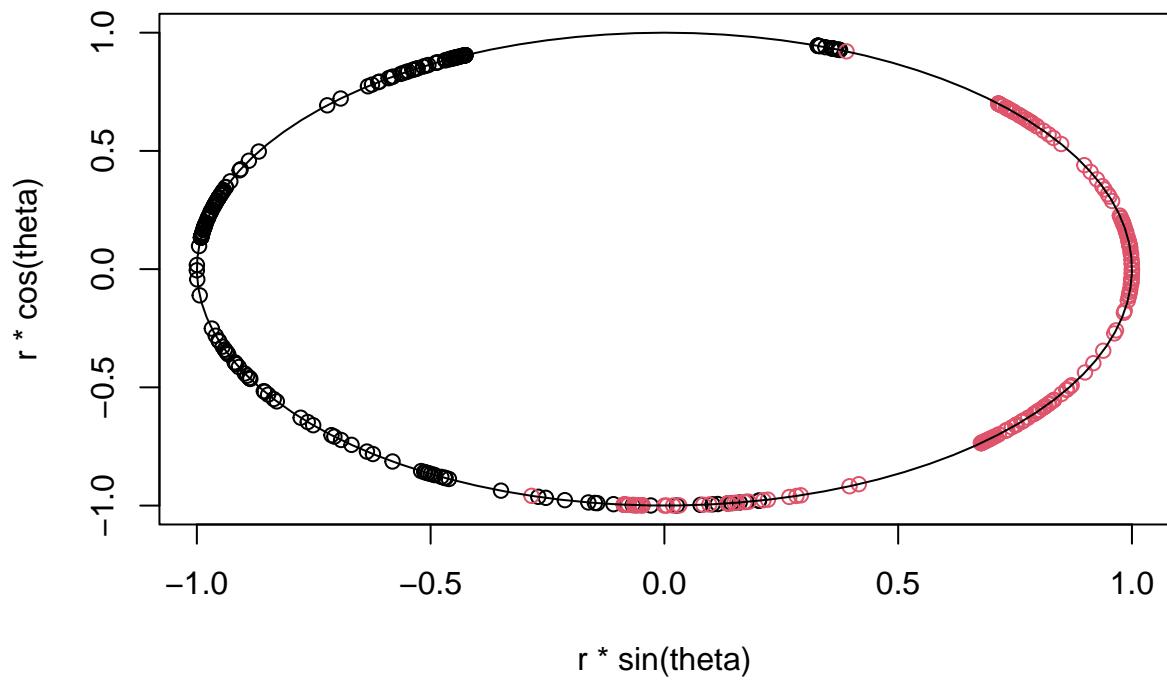


```
## [1] "Iteration 81"
## [1] "Iteration 82"
## [1] "Iteration 83"
## [1] "Iteration 84"
```

```
## [1] "Iteration 85"  
## [1] "Iteration 85: objective function value is 13.326353"
```

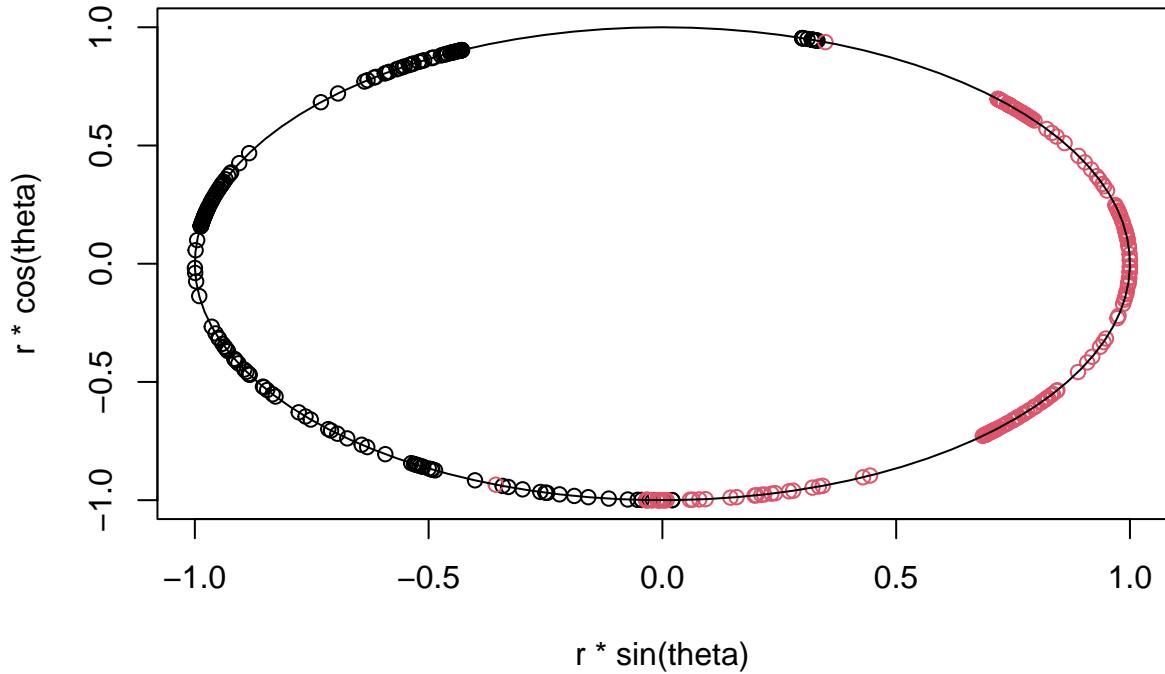


```
## [1] "Iteration 86"  
## [1] "Iteration 87"  
## [1] "Iteration 88"  
## [1] "Iteration 89"  
## [1] "Iteration 90"  
## [1] "Iteration 90: objective function value is 13.313369"
```

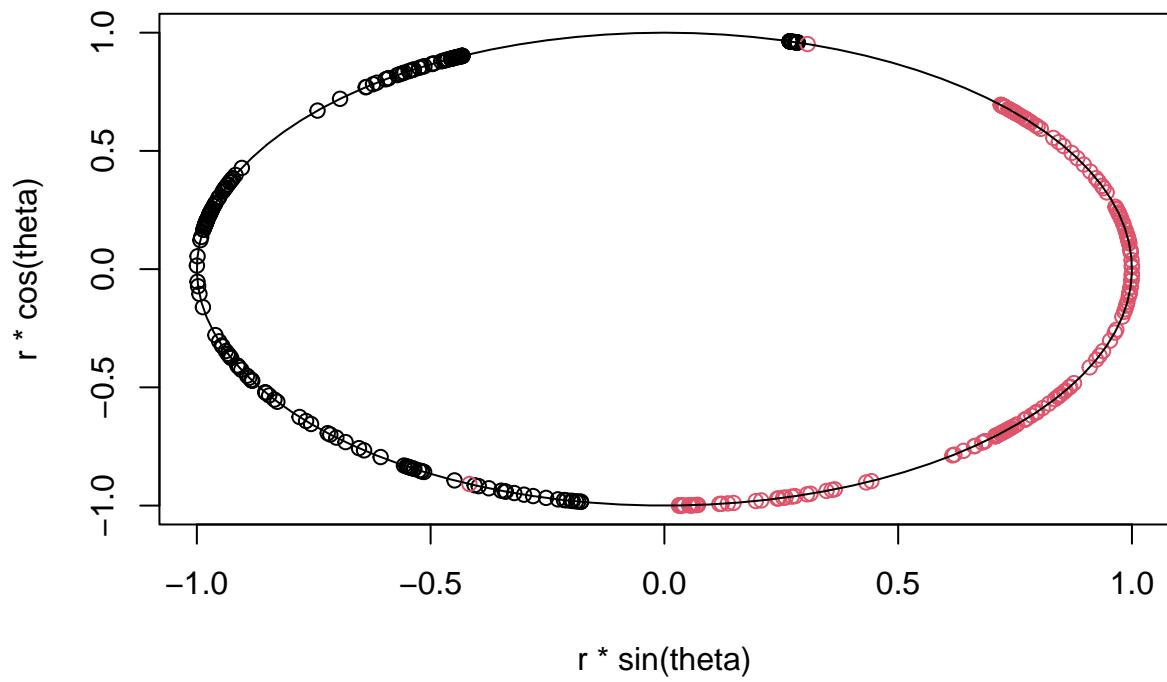


```
## [1] "Iteration 91"
```

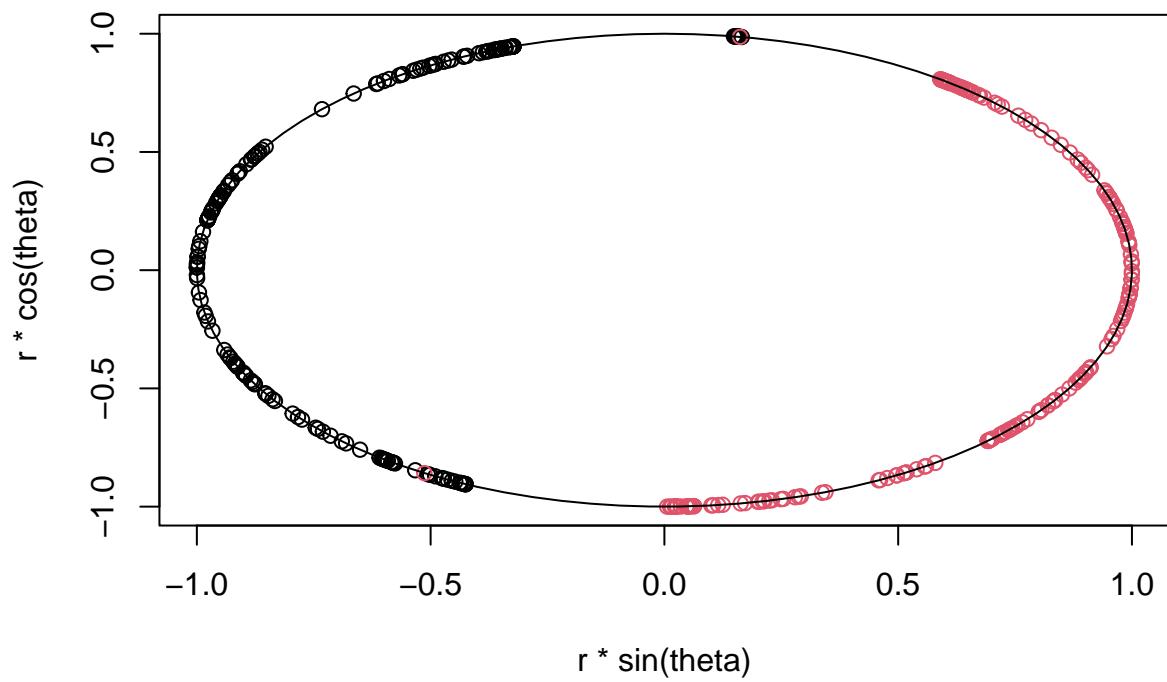
```
## [1] "Iteration 92"  
## [1] "Iteration 93"  
## [1] "Iteration 94"  
## [1] "Iteration 95"  
## [1] "Iteration 95: objective function value is 13.298804"
```



```
## [1] "Iteration 96"  
## [1] "Iteration 97"  
## [1] "Iteration 98"  
## [1] "Iteration 99"  
## [1] "Iteration 100"  
## [1] "Iteration 100: objective function value is 1.934331"
```

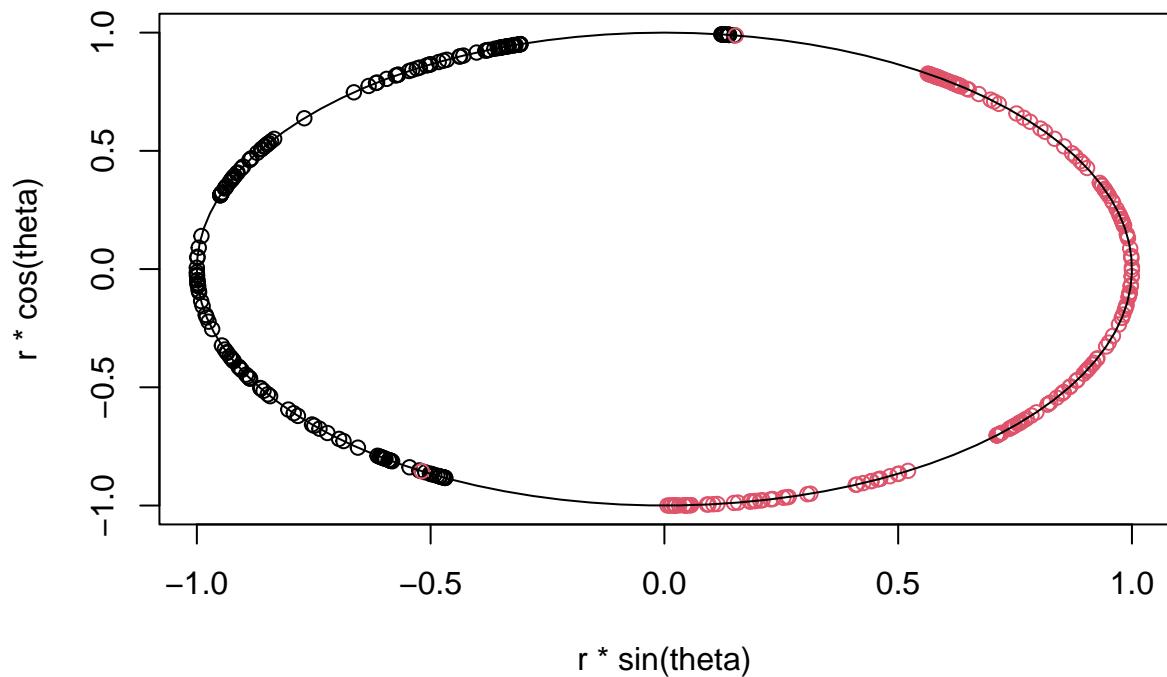


```
## [1] "Iteration 101"
## [1] "Iteration 102"
## [1] "Iteration 103"
## [1] "Iteration 104"
## [1] "Iteration 105"
## [1] "Iteration 105: objective function value is 1.922940"
```

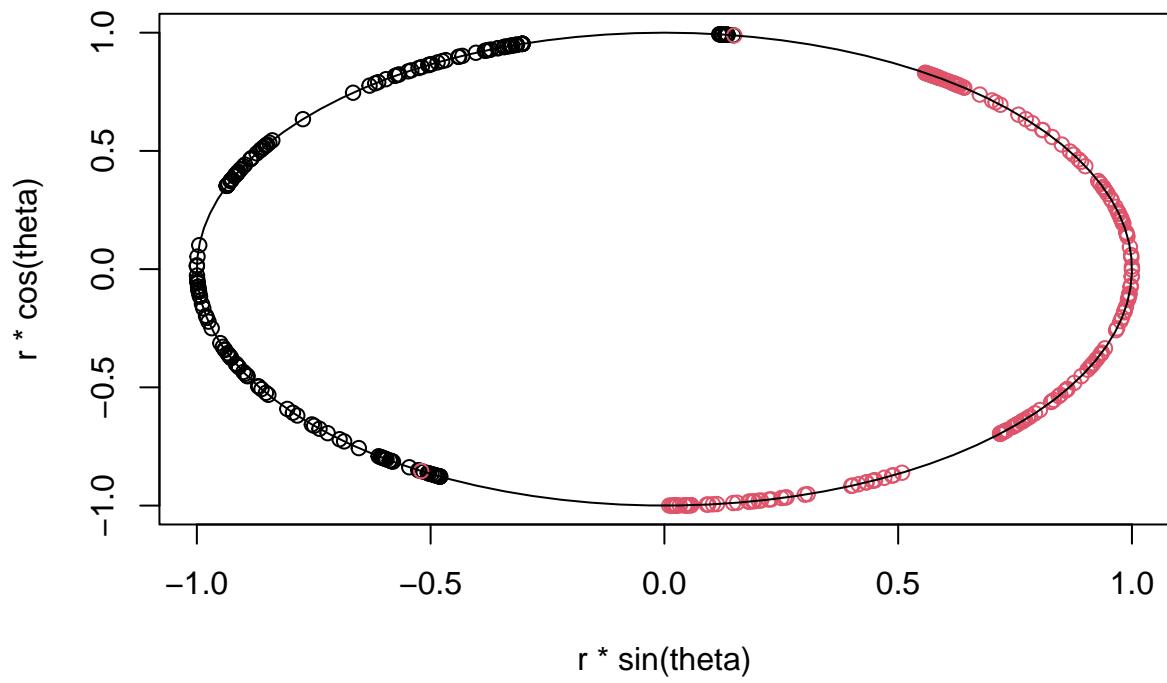


```
## [1] "Iteration 106"
## [1] "Iteration 107"
## [1] "Iteration 108"
## [1] "Iteration 109"
```

```
## [1] "Iteration 110"  
## [1] "Iteration 110: objective function value is 1.921026"
```

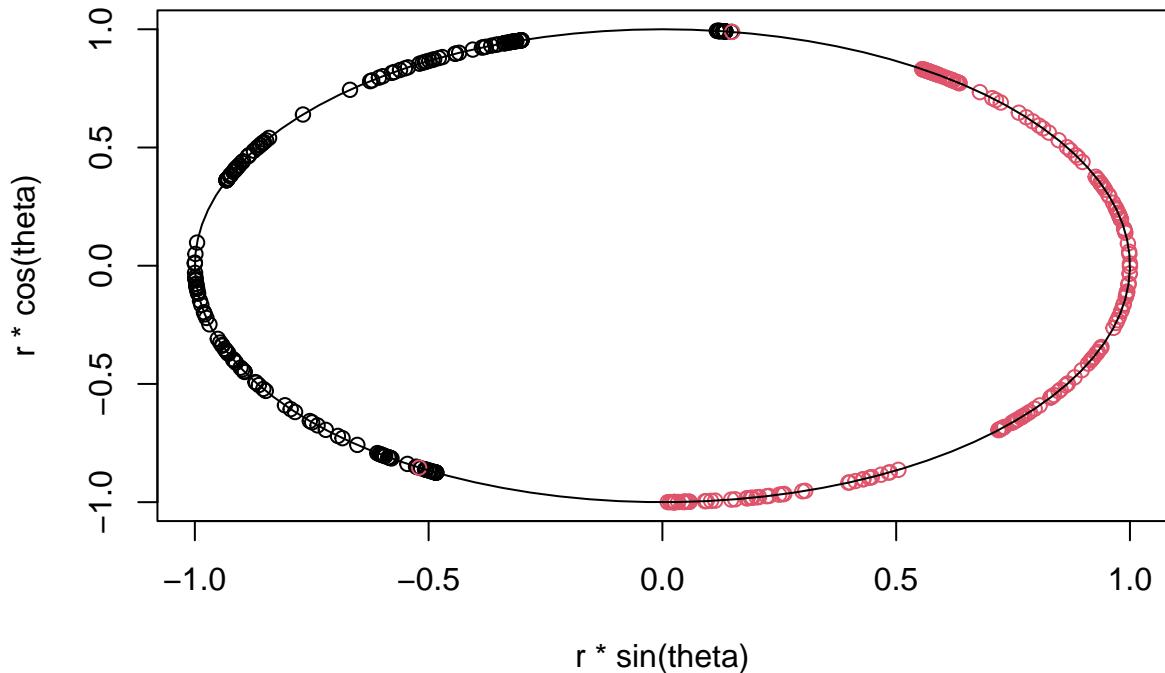


```
## [1] "Iteration 111"  
## [1] "Iteration 112"  
## [1] "Iteration 113"  
## [1] "Iteration 114"  
## [1] "Iteration 115"  
## [1] "Iteration 115: objective function value is 1.920853"
```

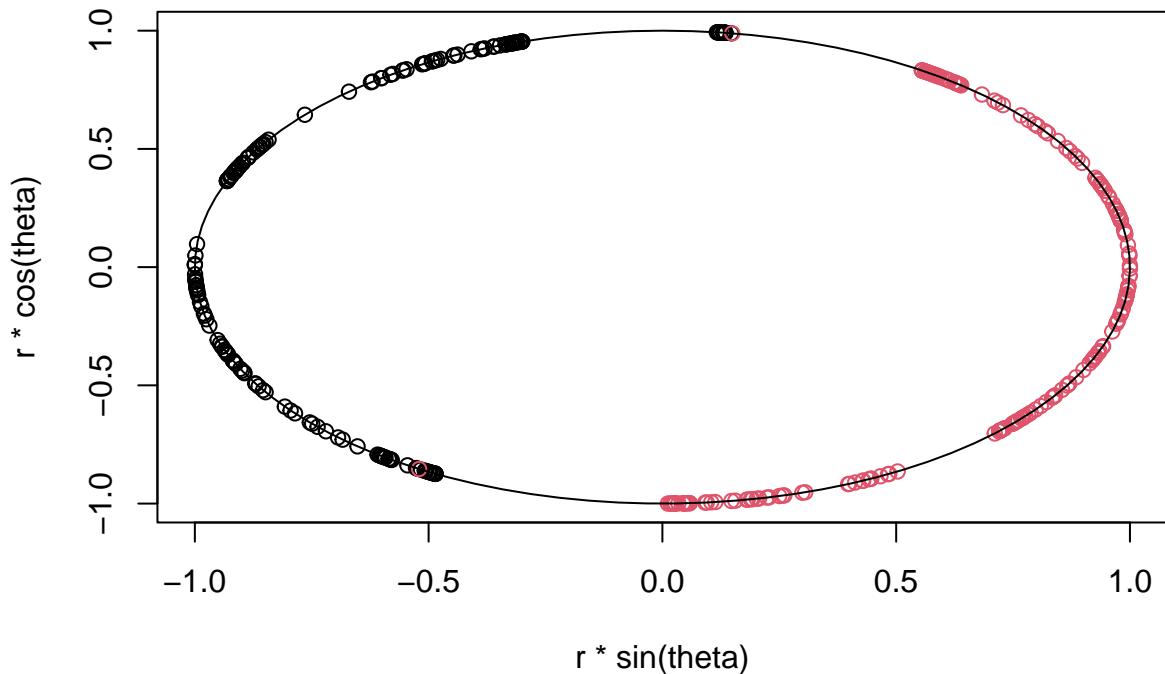


```
## [1] "Iteration 116"
```

```
## [1] "Iteration 117"  
## [1] "Iteration 118"  
## [1] "Iteration 119"  
## [1] "Iteration 120"  
## [1] "Iteration 120: objective function value is 1.920827"
```



```
## [1] "Iteration 121"  
## [1] "Iteration 122"  
## [1] "Iteration 123"  
## [1] "Iteration 124"  
## [1] "Iteration 125"  
## [1] "Iteration 125: objective function value is 1.920815"
```



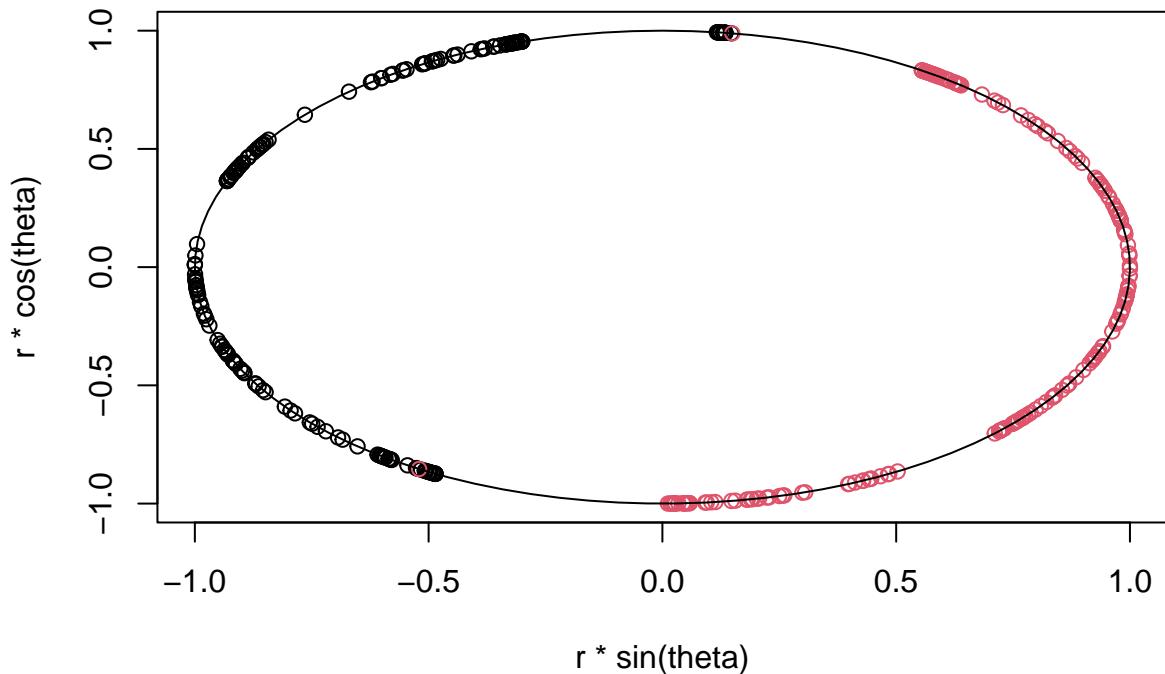
```

Y_rad <- DirStats::to_rad(Y[, , 127])
r <- 1
theta <- Y_rad

plot(r*sin(theta),
     r*cos(theta),
     col=rep(c(1,2), each=nrow(Y)/2),
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))

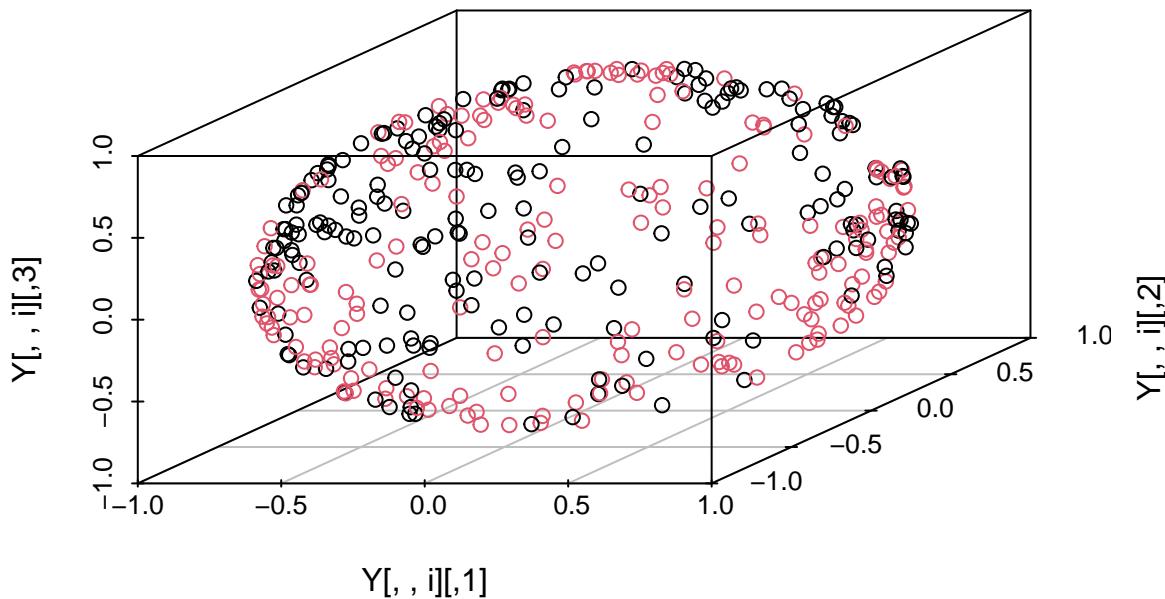
```



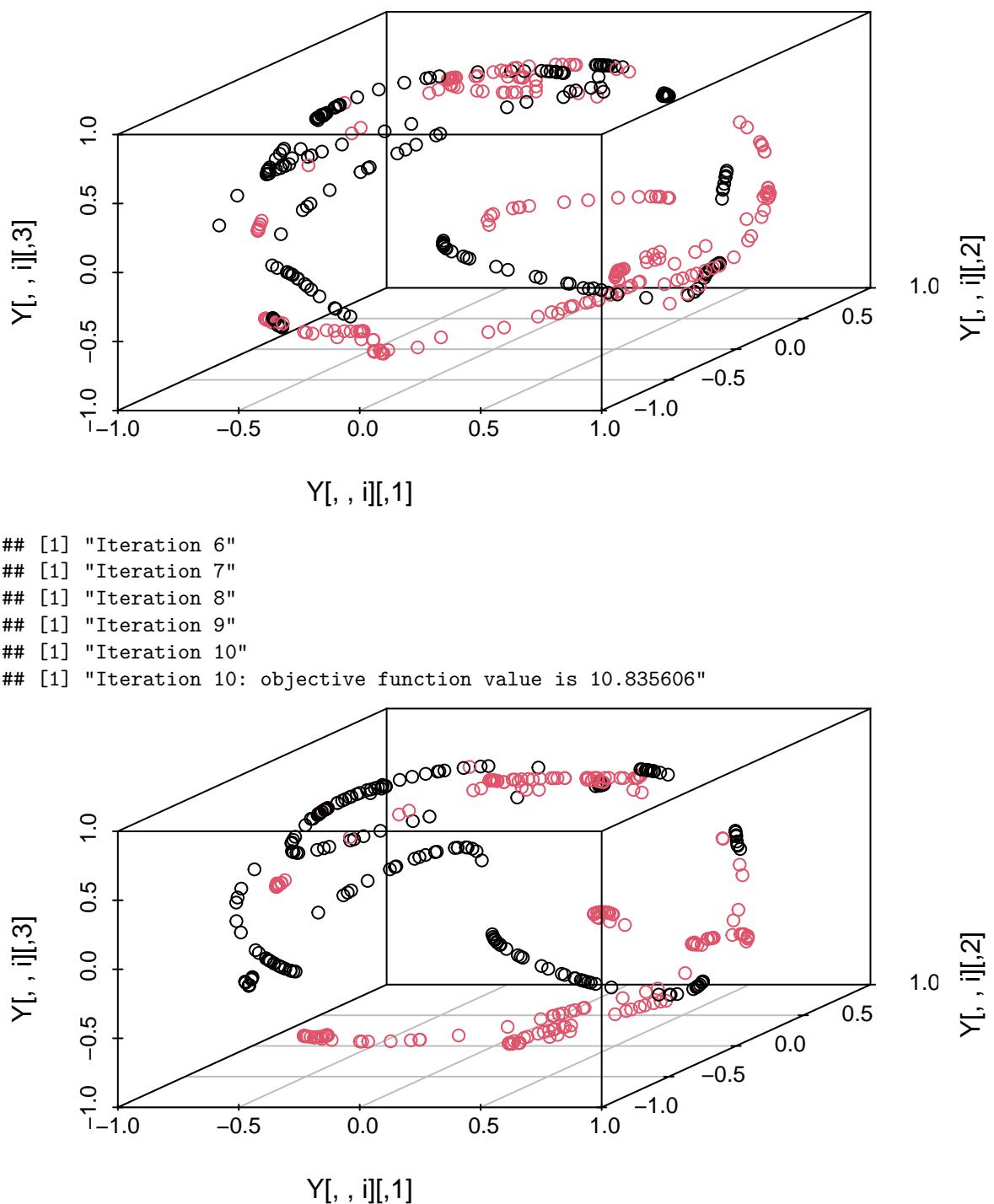
Now we are going to reduce to dimension  $\mathbb{S}^2$  then  $d = 2$  (sphere):

```
Y <- psc_sne(X=x_array, d=2, rho_psc_list = rho_first_20, num_iteration=125,
               colors=rep(c(1,2), each=nrow(x_array)/2), visualize_prog = TRUE, eta=100)
```

```
## [1] "Iteration 1"
## [1] "Iteration 1: objective function value is 15.975344"
```

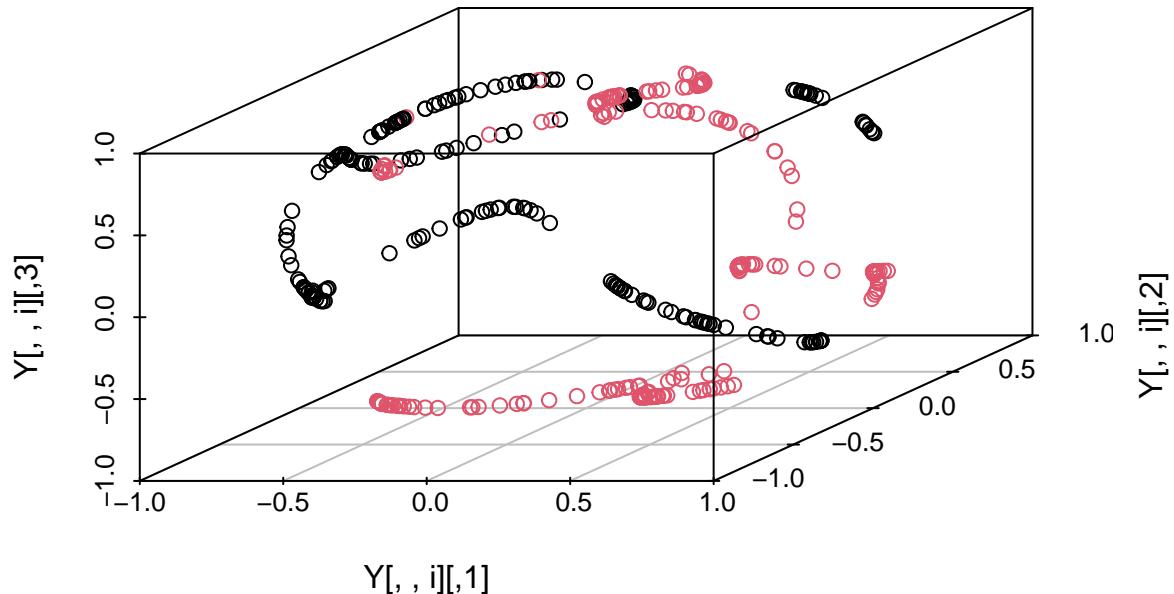


```
## [1] "Iteration 2"
## [1] "Iteration 3"
## [1] "Iteration 4"
## [1] "Iteration 5"
## [1] "Iteration 5: objective function value is 11.633564"
```

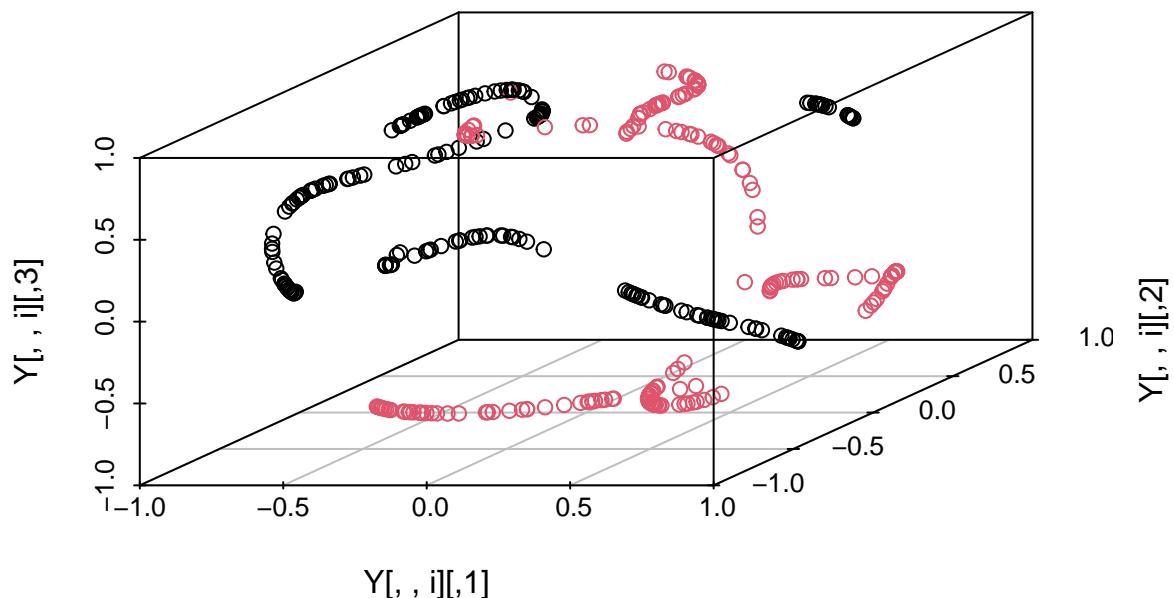


```
## [1] "Iteration 6"
## [1] "Iteration 7"
## [1] "Iteration 8"
## [1] "Iteration 9"
## [1] "Iteration 10"
## [1] "Iteration 10: objective function value is 10.835606"
```

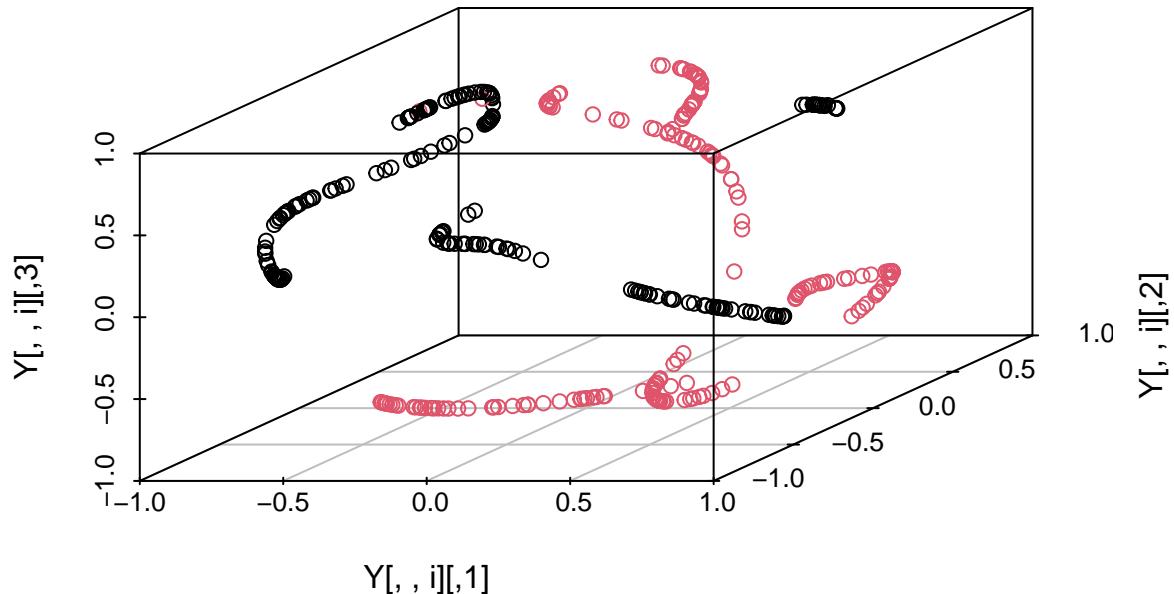
```
## [1] "Iteration 11"
## [1] "Iteration 12"
## [1] "Iteration 13"
## [1] "Iteration 14"
## [1] "Iteration 15"
## [1] "Iteration 15: objective function value is 10.544532"
```



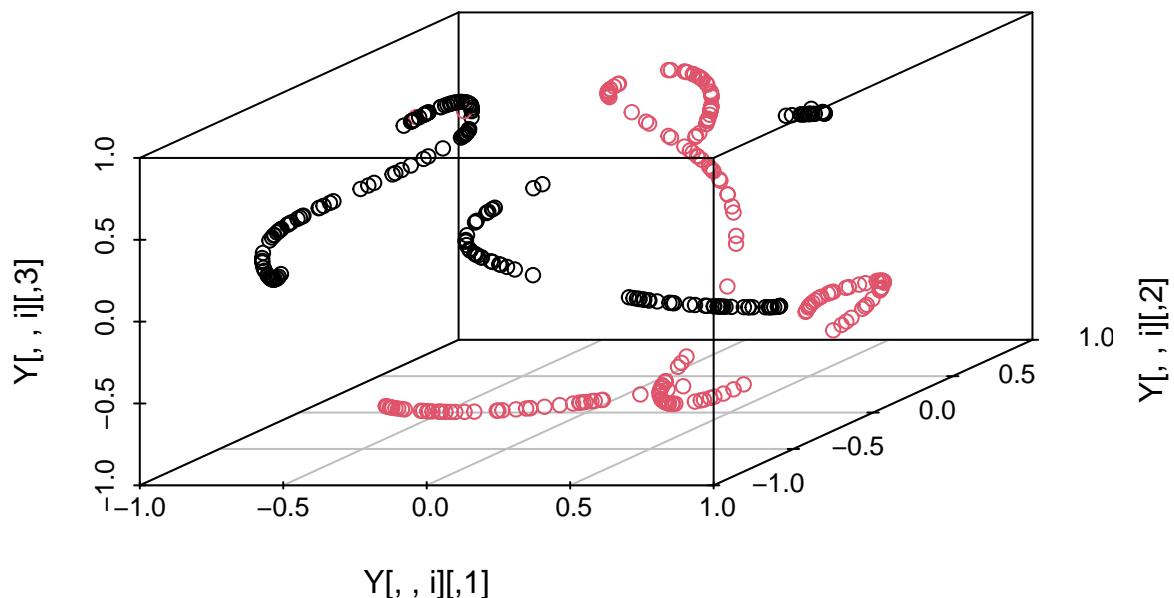
```
## [1] "Iteration 16"
## [1] "Iteration 17"
## [1] "Iteration 18"
## [1] "Iteration 19"
## [1] "Iteration 20"
## [1] "Iteration 20: objective function value is 10.182863"
```



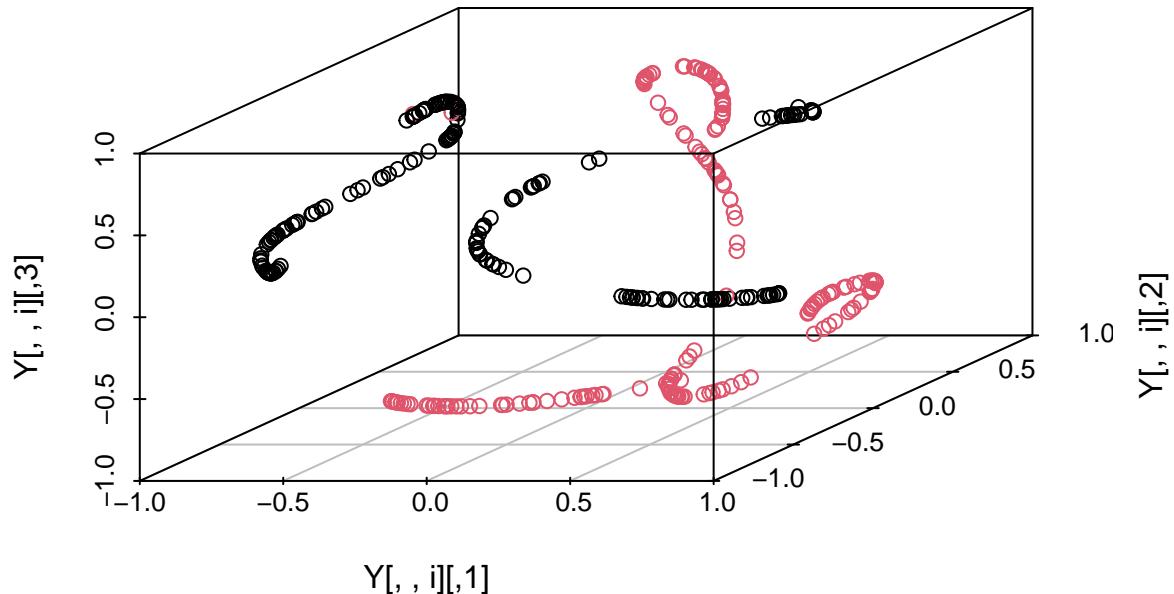
```
## [1] "Iteration 21"
## [1] "Iteration 22"
## [1] "Iteration 23"
## [1] "Iteration 24"
## [1] "Iteration 25"
## [1] "Iteration 25: objective function value is 10.025684"
```



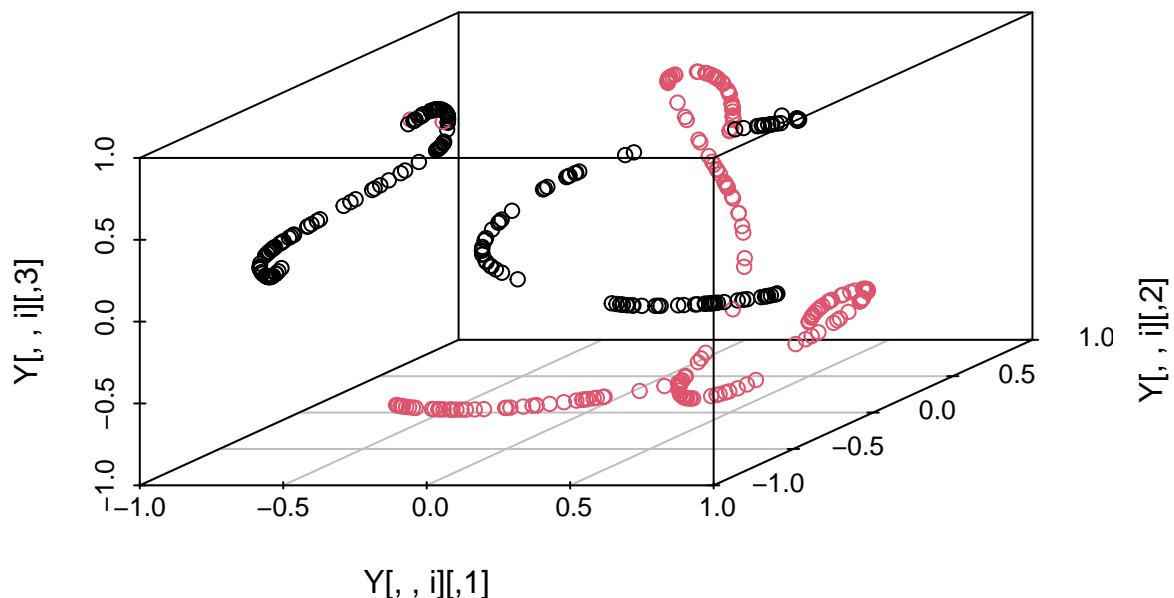
```
## [1] "Iteration 26"
## [1] "Iteration 27"
## [1] "Iteration 28"
## [1] "Iteration 29"
## [1] "Iteration 30"
## [1] "Iteration 30: objective function value is 9.983570"
```



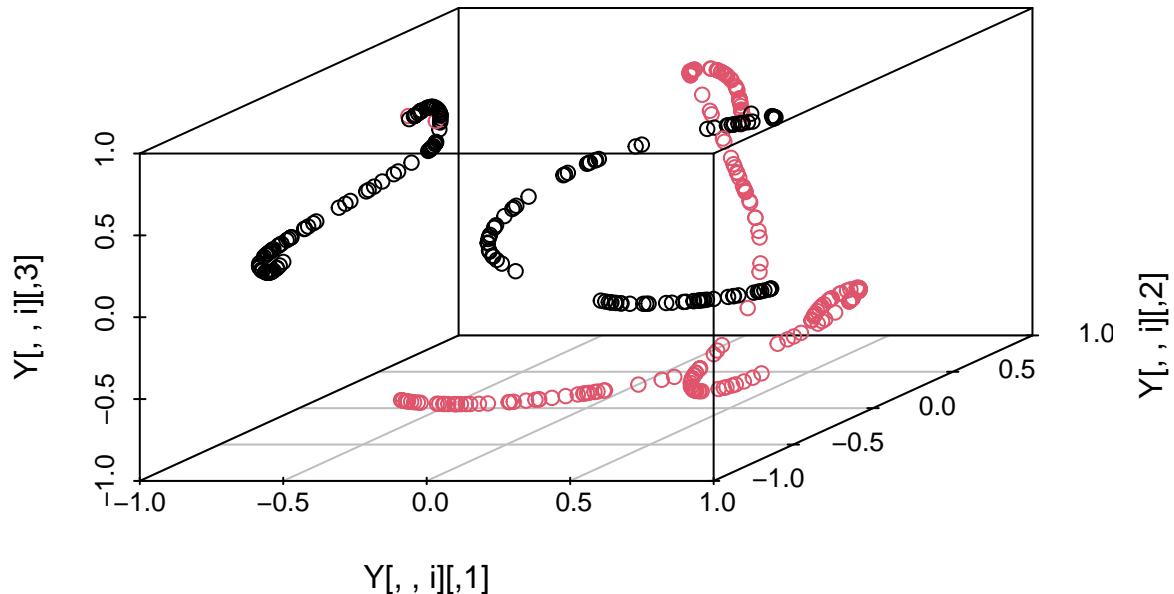
```
## [1] "Iteration 31"
## [1] "Iteration 32"
## [1] "Iteration 33"
## [1] "Iteration 34"
## [1] "Iteration 35"
## [1] "Iteration 35: objective function value is 9.978022"
```



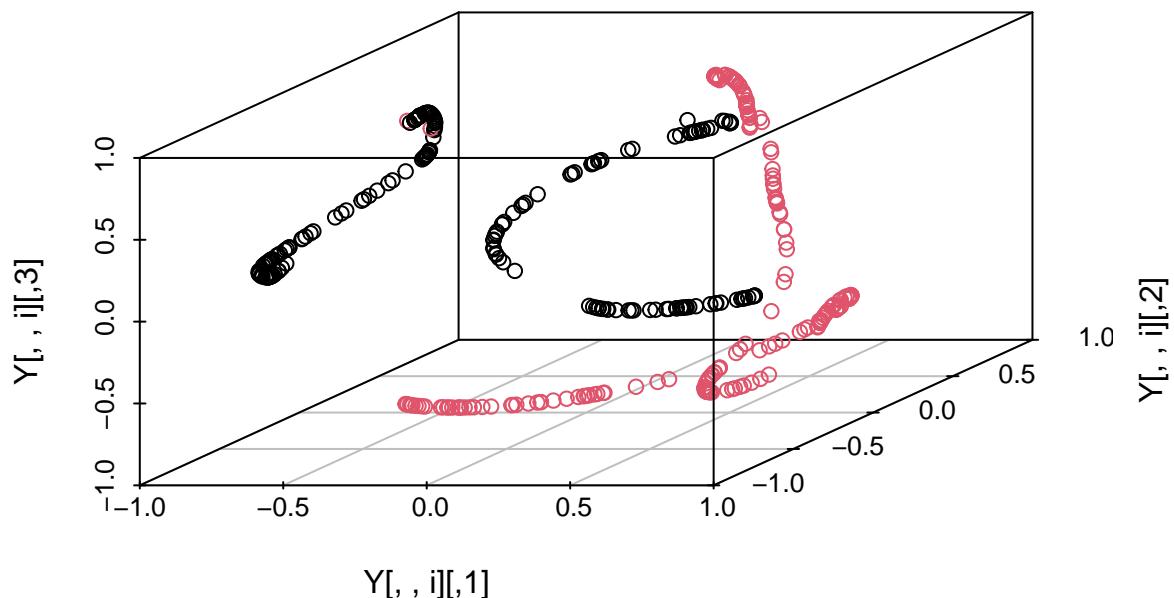
```
## [1] "Iteration 36"
## [1] "Iteration 37"
## [1] "Iteration 38"
## [1] "Iteration 39"
## [1] "Iteration 40"
## [1] "Iteration 40: objective function value is 9.987421"
```



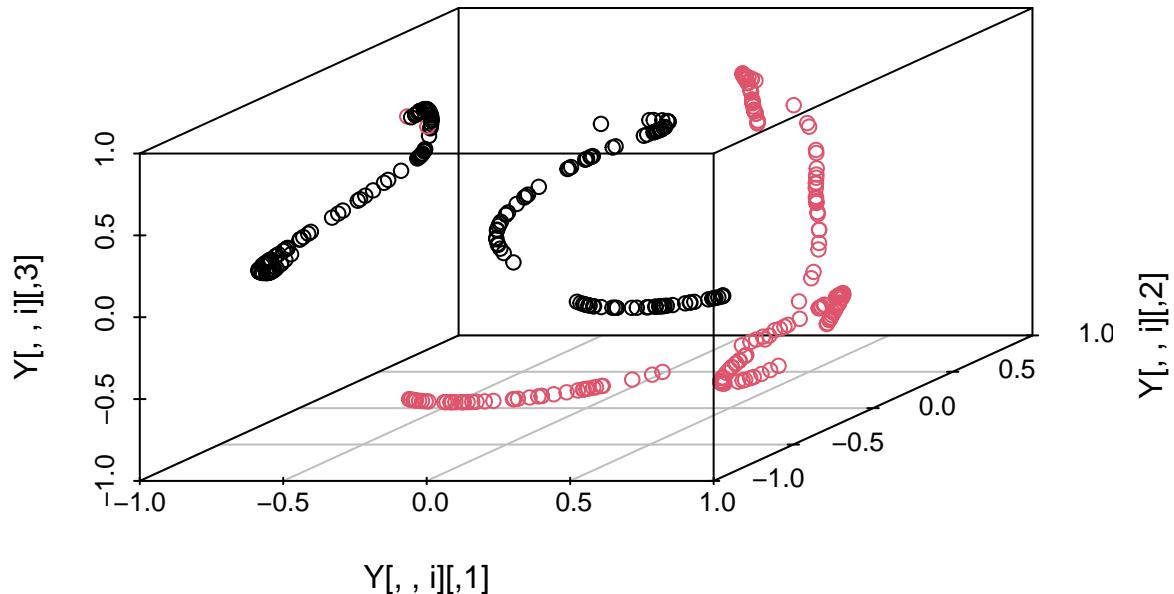
```
## [1] "Iteration 41"
## [1] "Iteration 42"
## [1] "Iteration 43"
## [1] "Iteration 44"
## [1] "Iteration 45"
## [1] "Iteration 45: objective function value is 9.993543"
```



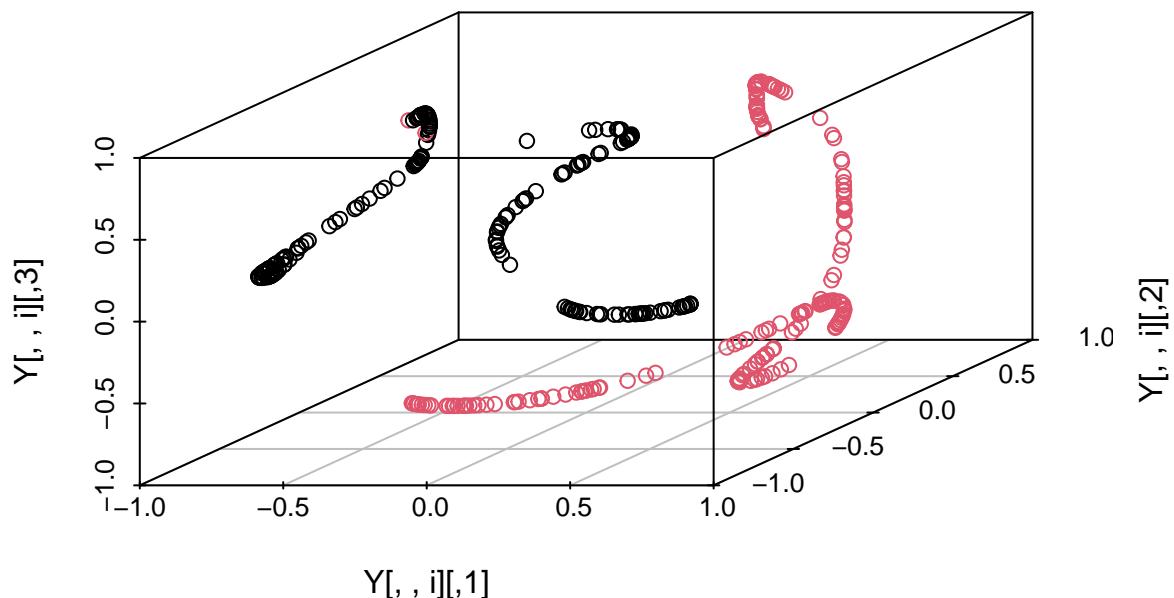
```
## [1] "Iteration 46"
## [1] "Iteration 47"
## [1] "Iteration 48"
## [1] "Iteration 49"
## [1] "Iteration 50"
## [1] "Iteration 50: objective function value is 9.955949"
```



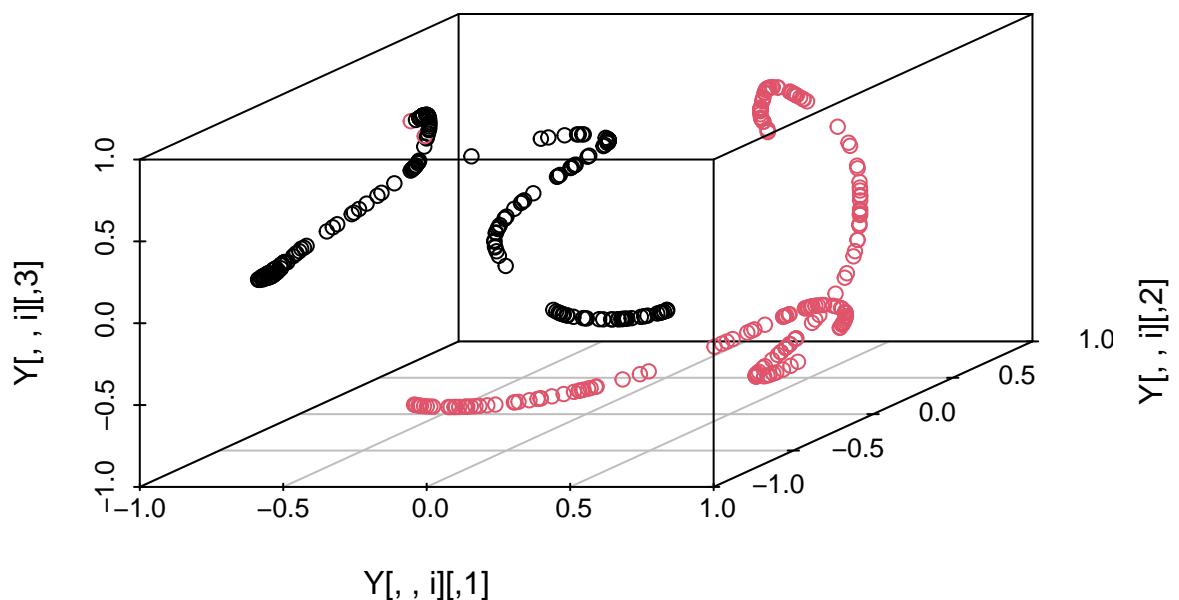
```
## [1] "Iteration 51"
## [1] "Iteration 52"
## [1] "Iteration 53"
## [1] "Iteration 54"
## [1] "Iteration 55"
## [1] "Iteration 55: objective function value is 9.865874"
```



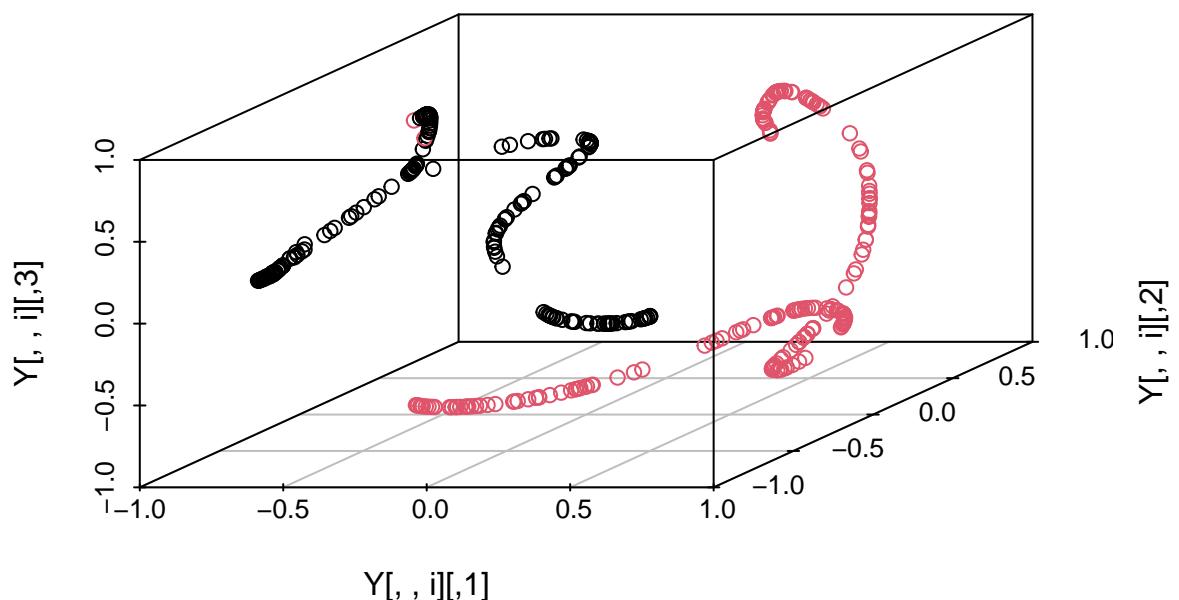
```
## [1] "Iteration 56"
## [1] "Iteration 57"
## [1] "Iteration 58"
## [1] "Iteration 59"
## [1] "Iteration 60"
## [1] "Iteration 60: objective function value is 9.793812"
```



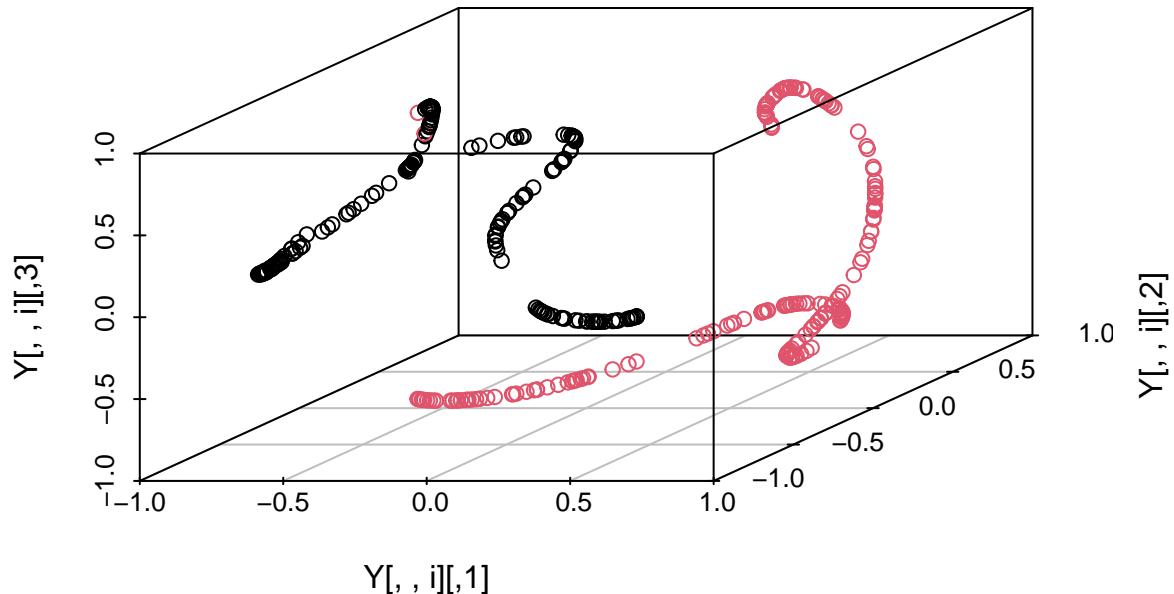
```
## [1] "Iteration 61"
## [1] "Iteration 62"
## [1] "Iteration 63"
## [1] "Iteration 64"
## [1] "Iteration 65"
## [1] "Iteration 65: objective function value is 9.754127"
```



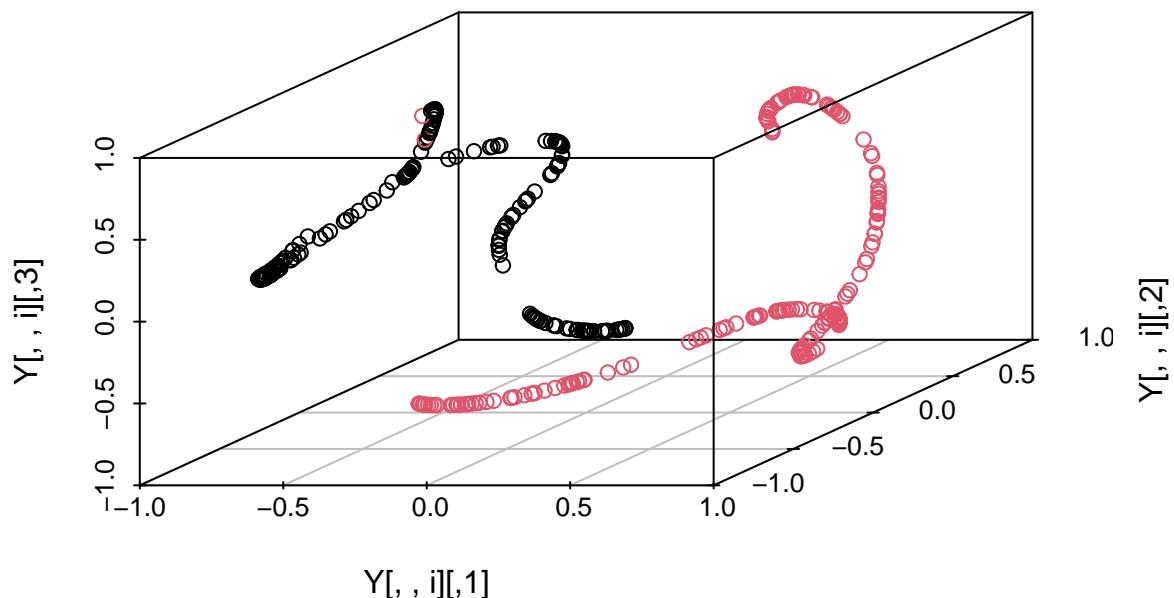
```
## [1] "Iteration 66"
## [1] "Iteration 67"
## [1] "Iteration 68"
## [1] "Iteration 69"
## [1] "Iteration 70"
## [1] "Iteration 70: objective function value is 9.733617"
```



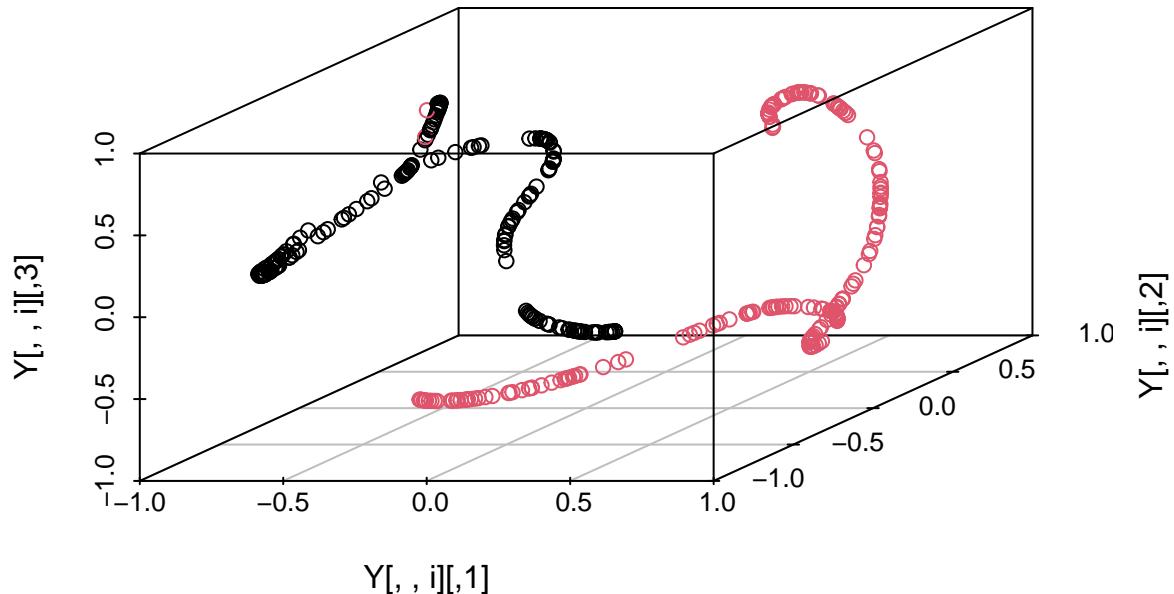
```
## [1] "Iteration 71"
## [1] "Iteration 72"
## [1] "Iteration 73"
## [1] "Iteration 74"
## [1] "Iteration 75"
## [1] "Iteration 75: objective function value is 9.722018"
```



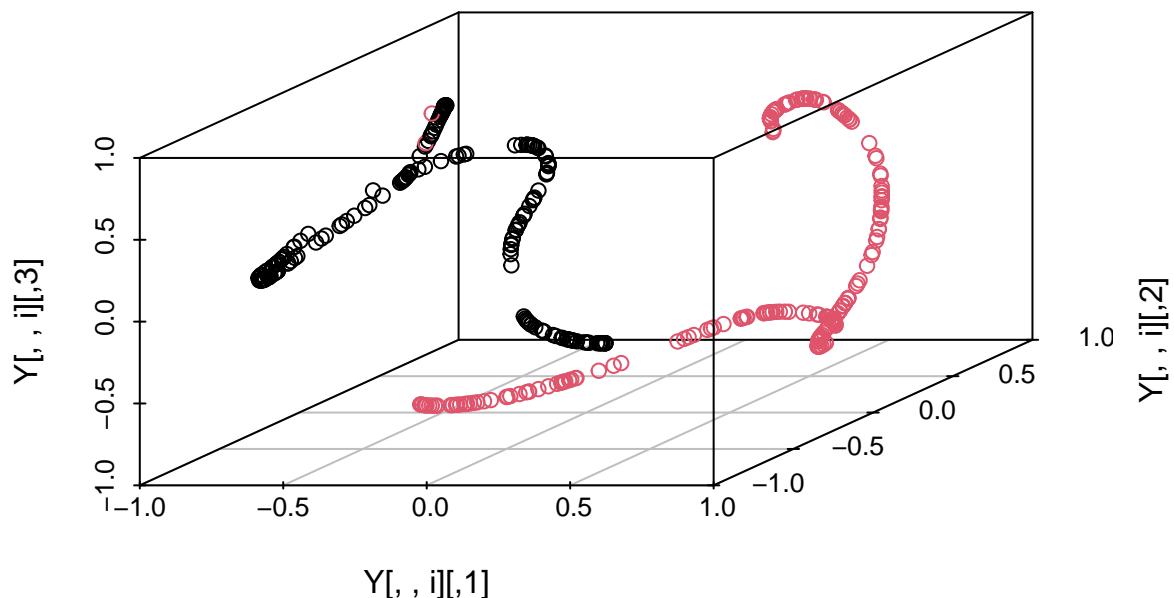
```
## [1] "Iteration 76"
## [1] "Iteration 77"
## [1] "Iteration 78"
## [1] "Iteration 79"
## [1] "Iteration 80"
## [1] "Iteration 80: objective function value is 9.714513"
```



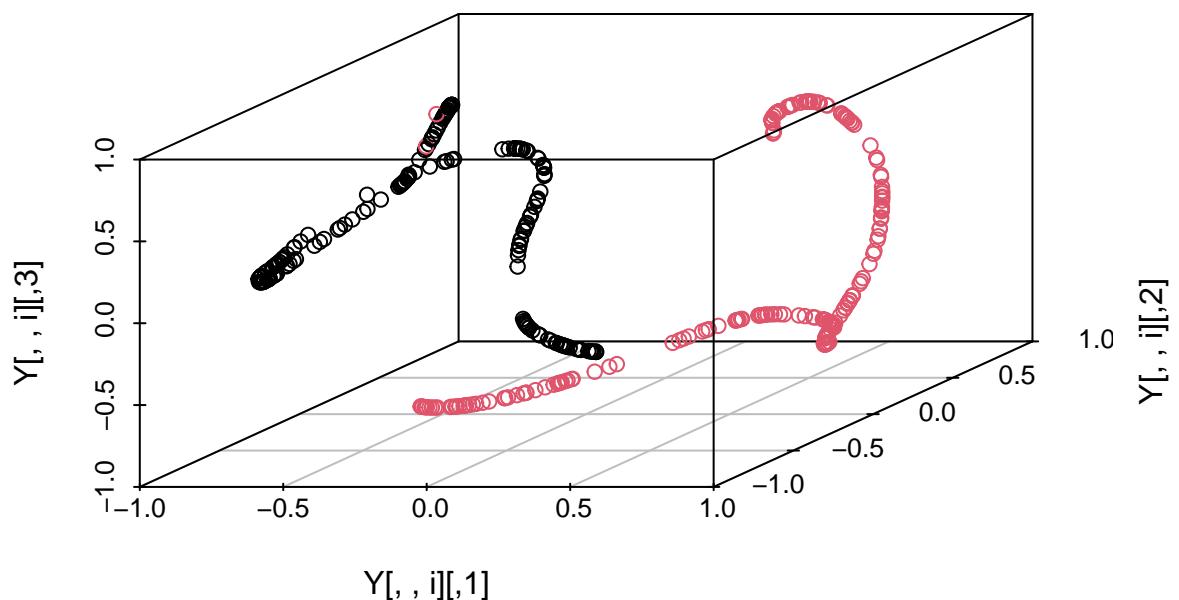
```
## [1] "Iteration 81"
## [1] "Iteration 82"
## [1] "Iteration 83"
## [1] "Iteration 84"
## [1] "Iteration 85"
## [1] "Iteration 85: objective function value is 9.708685"
```



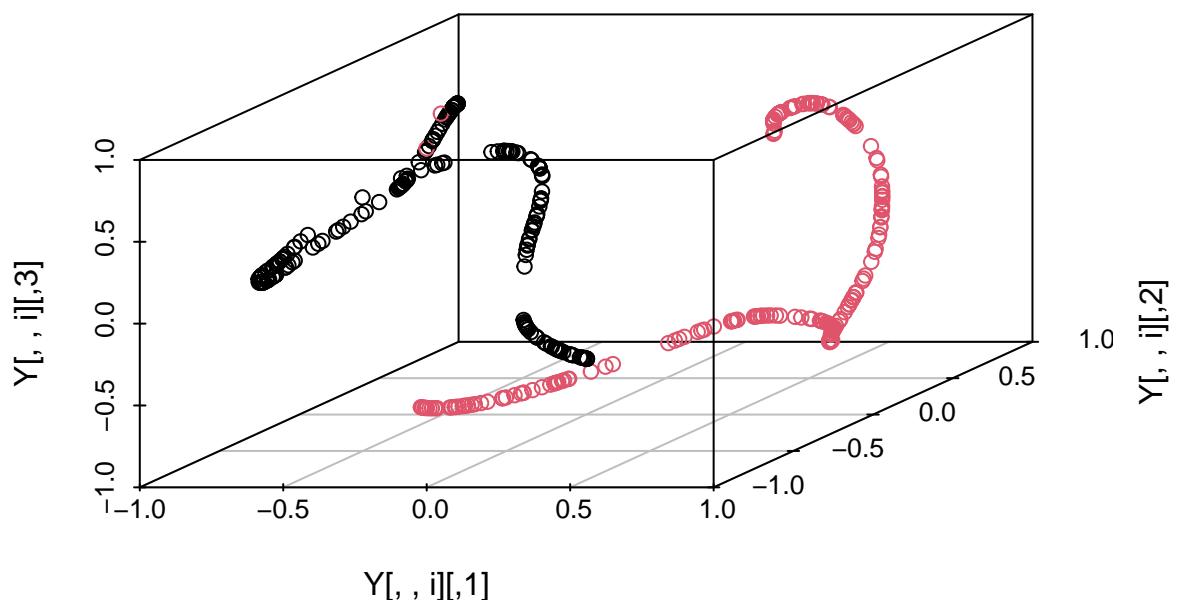
```
## [1] "Iteration 86"
## [1] "Iteration 87"
## [1] "Iteration 88"
## [1] "Iteration 89"
## [1] "Iteration 90"
## [1] "Iteration 90: objective function value is 9.703289"
```



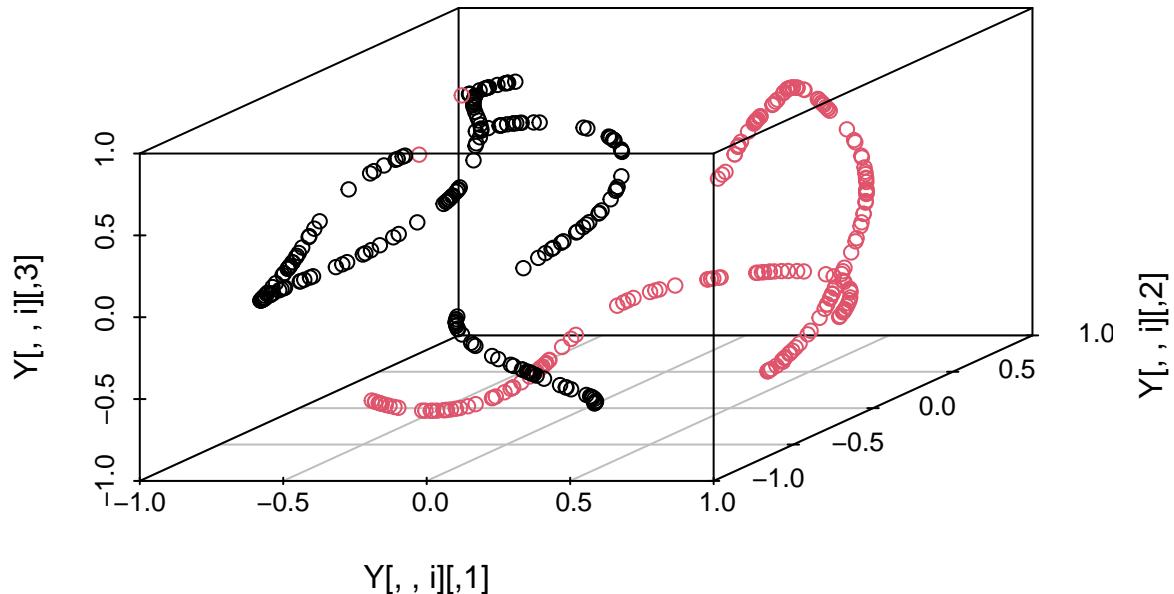
```
## [1] "Iteration 91"
## [1] "Iteration 92"
## [1] "Iteration 93"
## [1] "Iteration 94"
## [1] "Iteration 95"
## [1] "Iteration 95: objective function value is 9.697800"
```



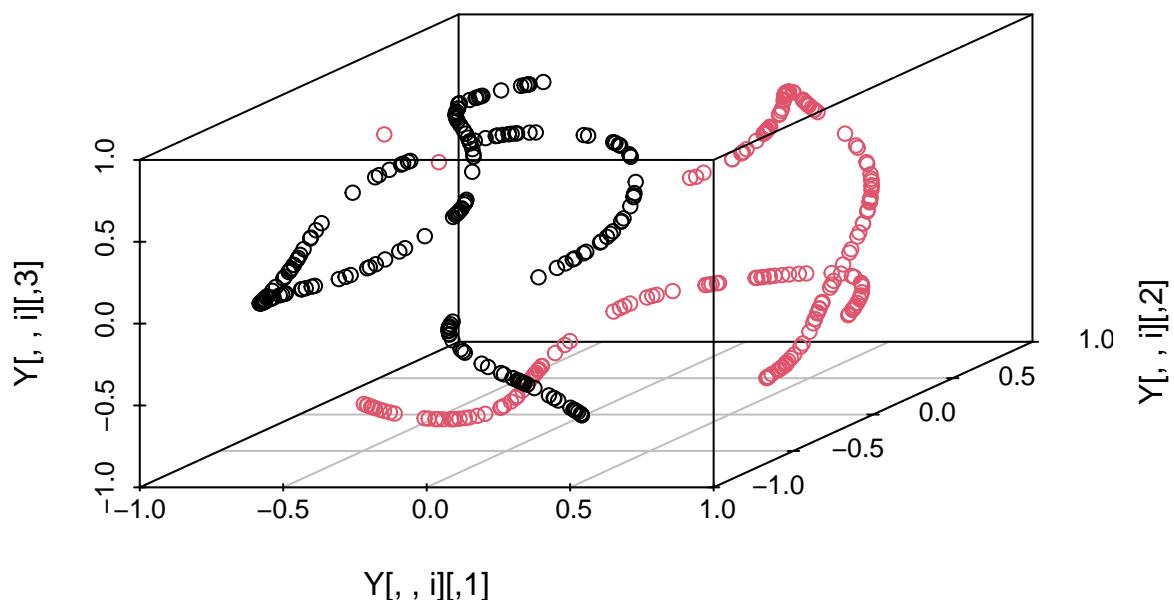
```
## [1] "Iteration 96"
## [1] "Iteration 97"
## [1] "Iteration 98"
## [1] "Iteration 99"
## [1] "Iteration 100"
## [1] "Iteration 100: objective function value is 1.036749"
```



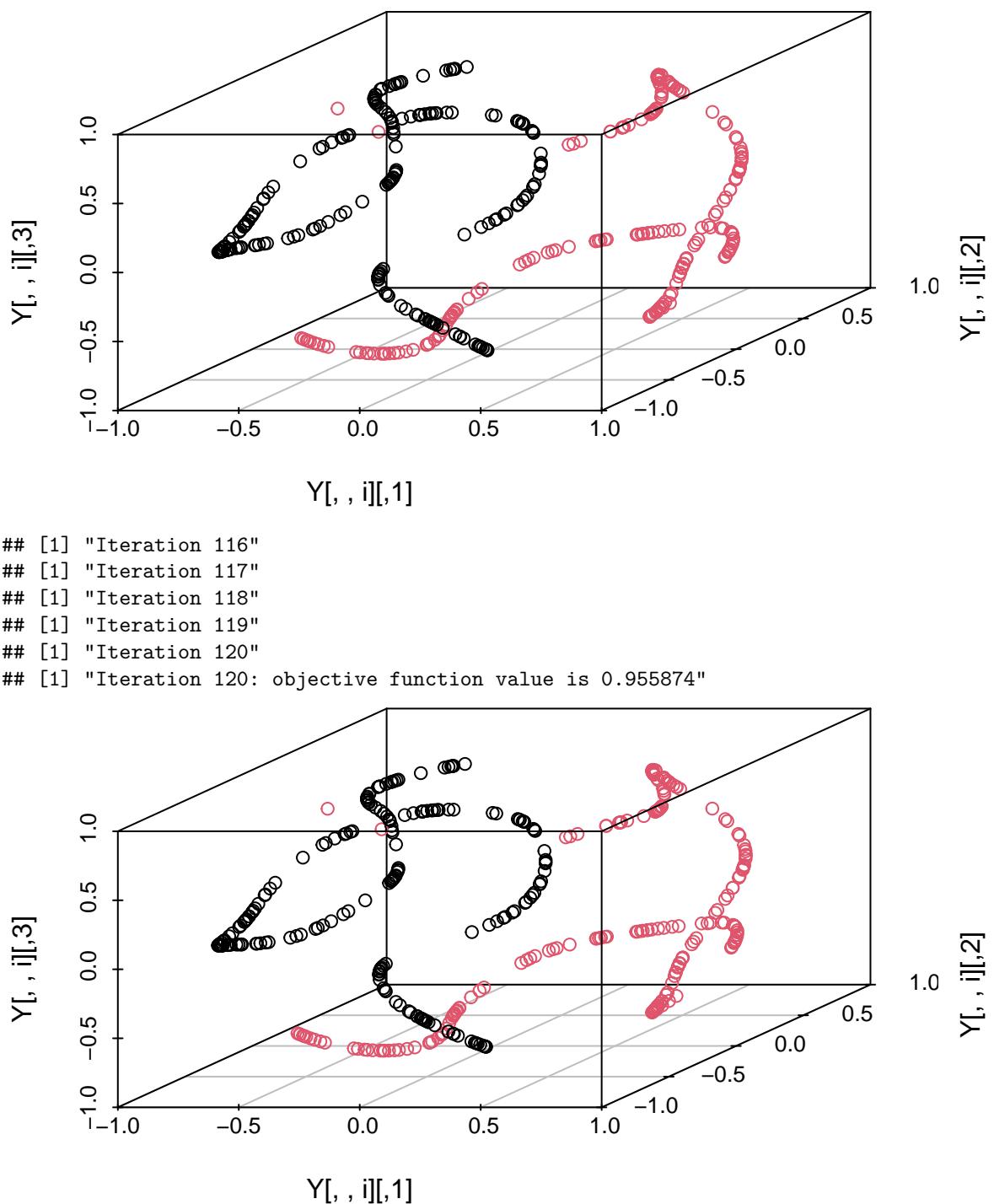
```
## [1] "Iteration 101"
## [1] "Iteration 102"
## [1] "Iteration 103"
## [1] "Iteration 104"
## [1] "Iteration 105"
## [1] "Iteration 105: objective function value is 0.961428"
```



```
## [1] "Iteration 106"
## [1] "Iteration 107"
## [1] "Iteration 108"
## [1] "Iteration 109"
## [1] "Iteration 110"
## [1] "Iteration 110: objective function value is 0.957950"
```



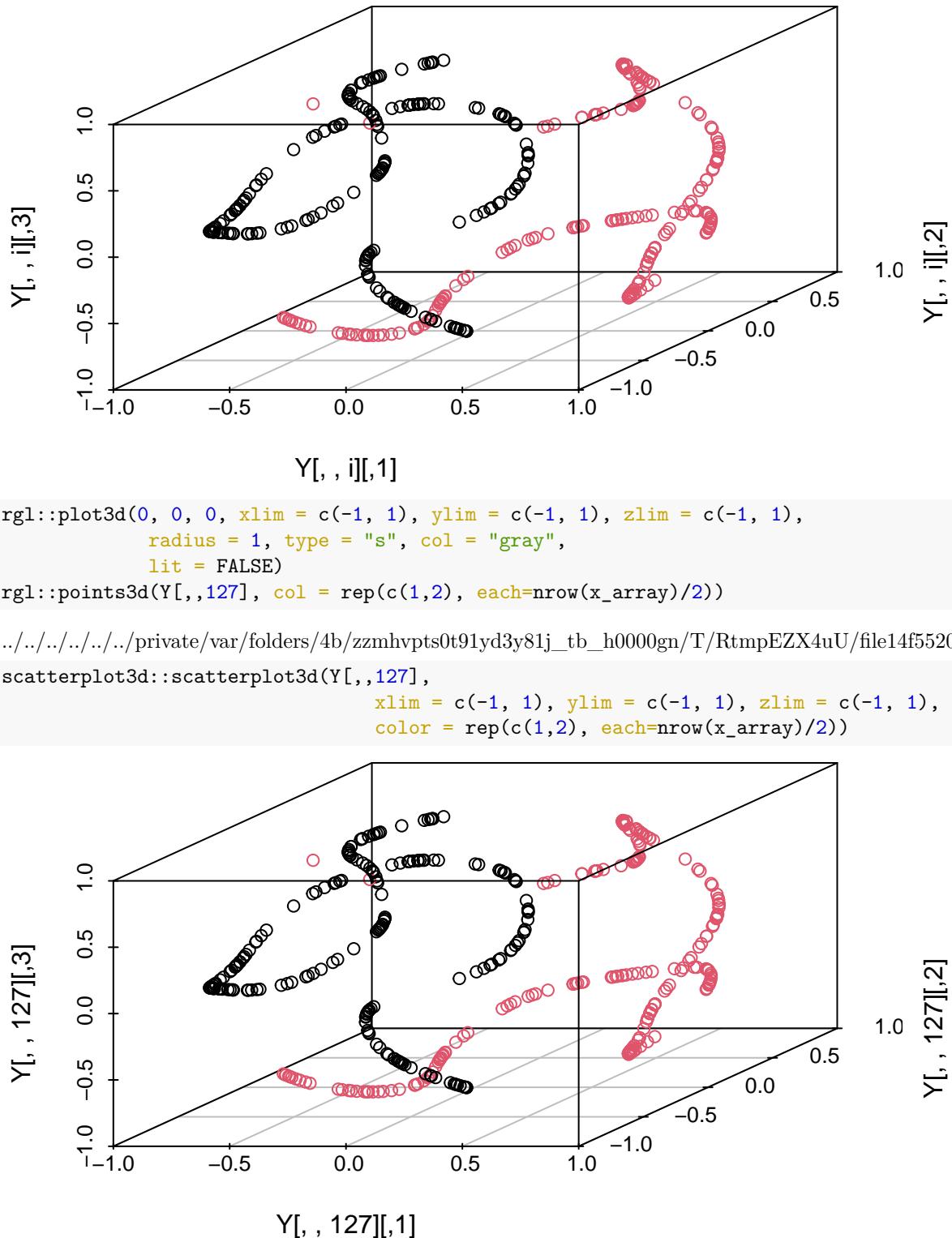
```
## [1] "Iteration 111"
## [1] "Iteration 112"
## [1] "Iteration 113"
## [1] "Iteration 114"
## [1] "Iteration 115"
## [1] "Iteration 115: objective function value is 0.956587"
```



```

## [1] "Iteration 121"
## [1] "Iteration 122"
## [1] "Iteration 123"
## [1] "Iteration 124"
## [1] "Iteration 125"
## [1] "Iteration 125: objective function value is 0.955467"

```

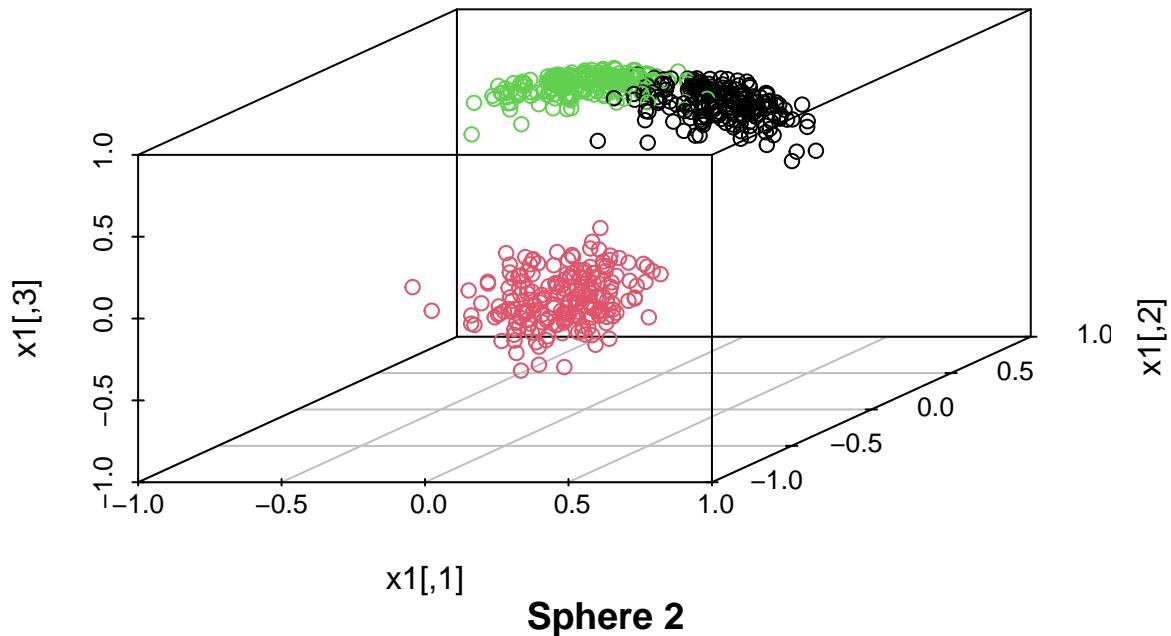


It's clearly clustered the two groups we had in the original generated data.

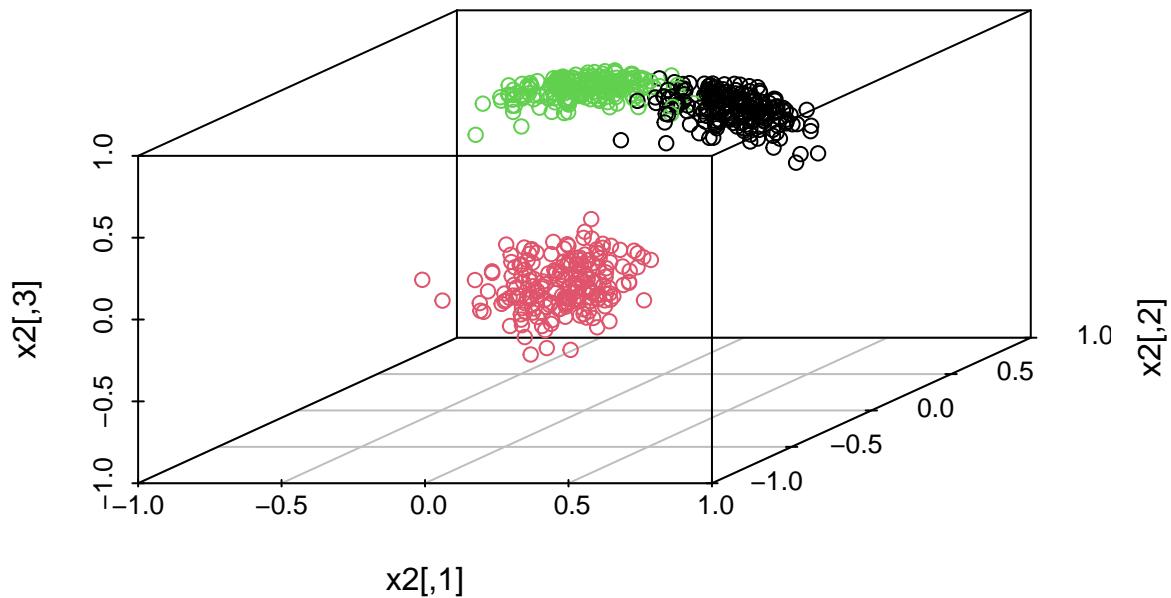
### Case 2

Sample on the  $(\mathbb{S}^2)^2$  where  $p = 2$  and  $r = 2$ .

## Sphere 1



## Sphere 2



Let's calculate the rho parameters based on a perplexity of 20:

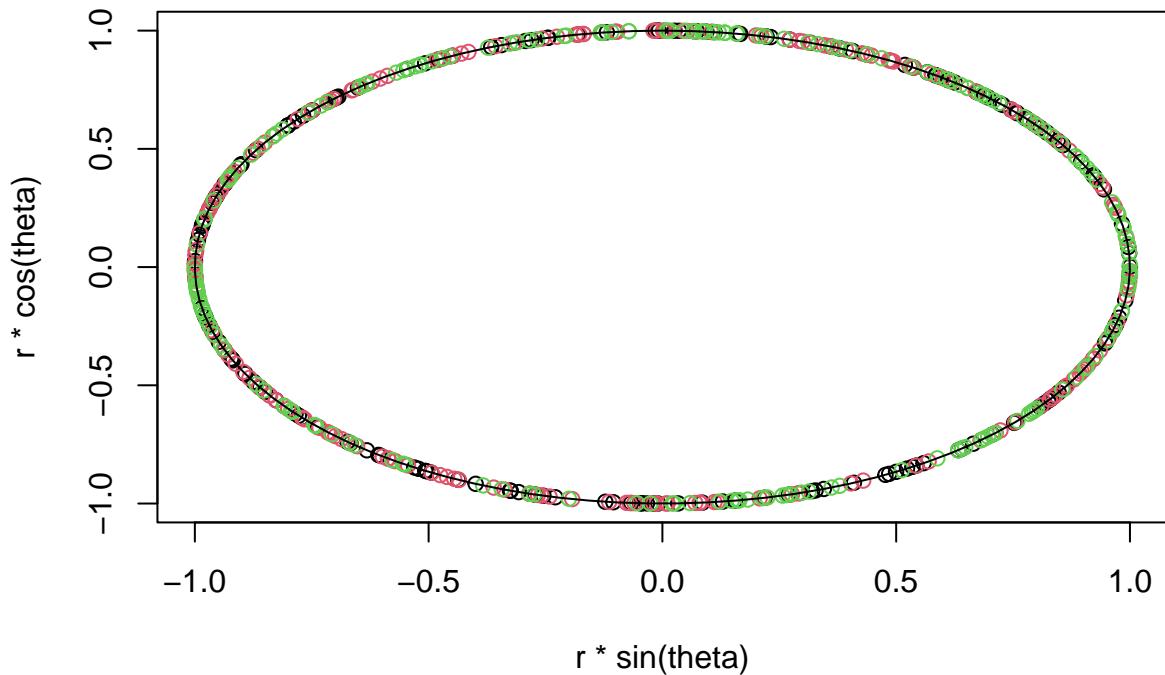
```
rho_second_perp20 <- rho_optim_parallel(x_2, 20)
```

```
## Time difference of 7.657965 mins
```

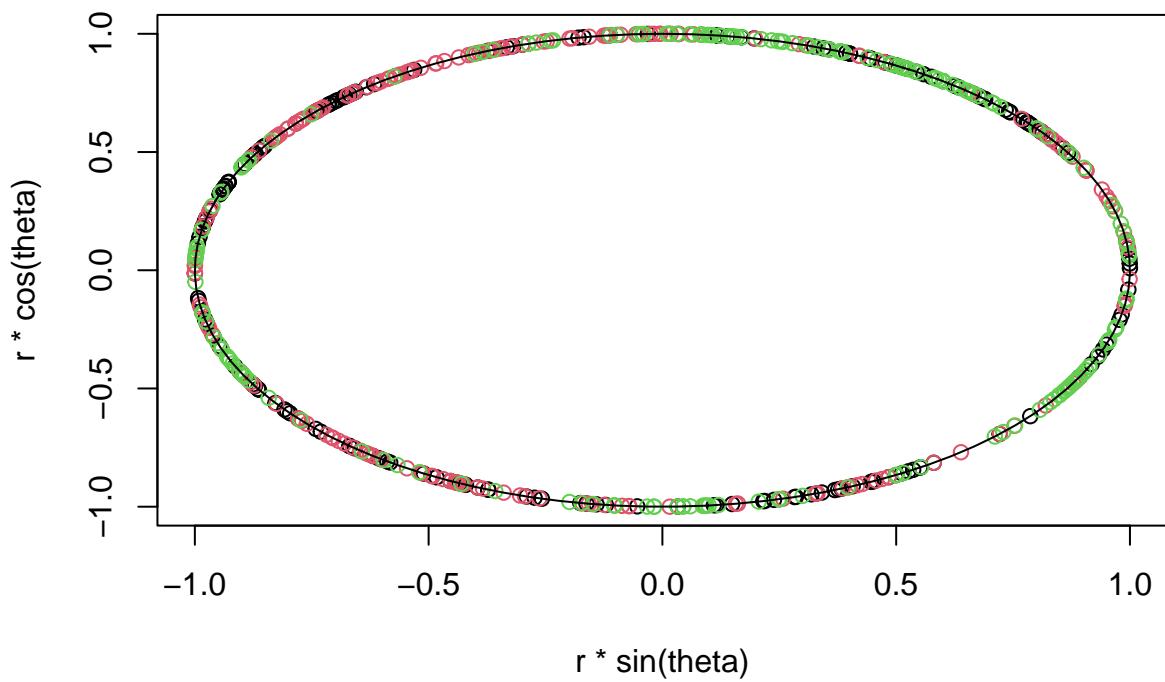
First, let's reduce to dimension  $\mathbb{S}^1$  then  $d = 1$  (circumference):

```
Y <- psc_sne(X=x_2, d=1, rho_psc_list = rho_second_perp20, num_iteration=250, eta=75,
               colors=rep(c(1,2,3), each=n/3), visualize_prog = TRUE)
```

```
## [1] "Iteration 1"  
## [1] "Iteration 1: objective function value is 19.767536"
```



```
## [1] "Iteration 2"  
## [1] "Iteration 3"  
## [1] "Iteration 4"  
## [1] "Iteration 5"  
## [1] "Iteration 5: objective function value is 17.734535"
```

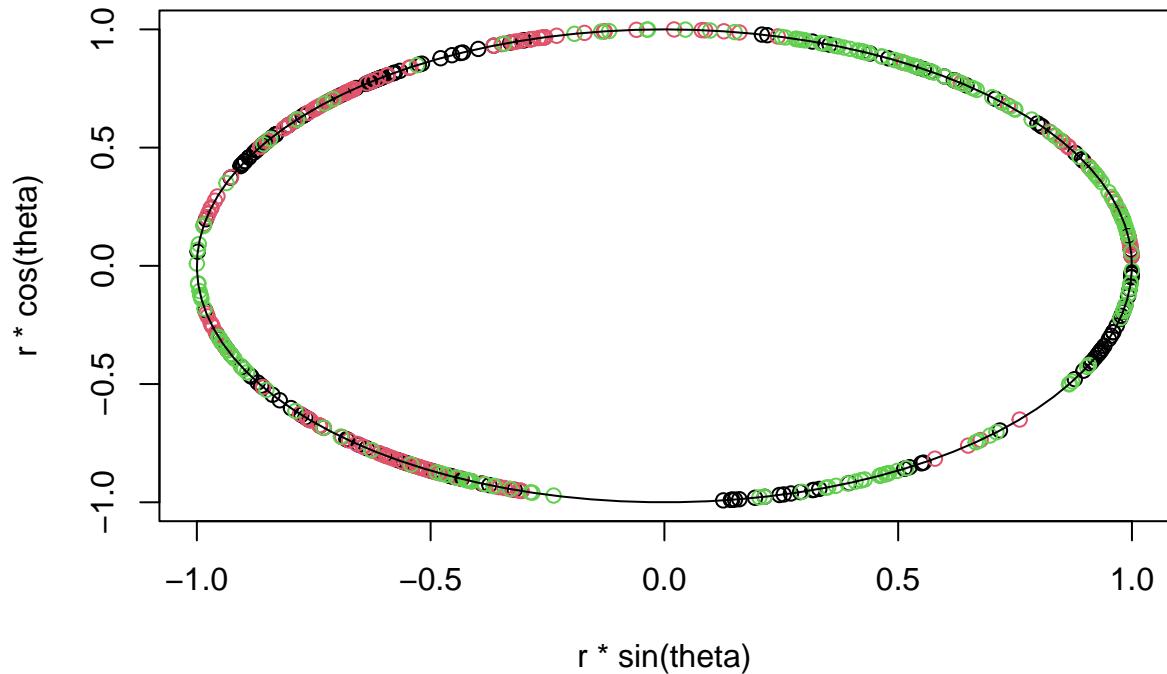


```
## [1] "Iteration 6"  
## [1] "Iteration 7"
```

```

## [1] "Iteration 8"
## [1] "Iteration 9"
## [1] "Iteration 10"
## [1] "Iteration 10: objective function value is 16.319043"

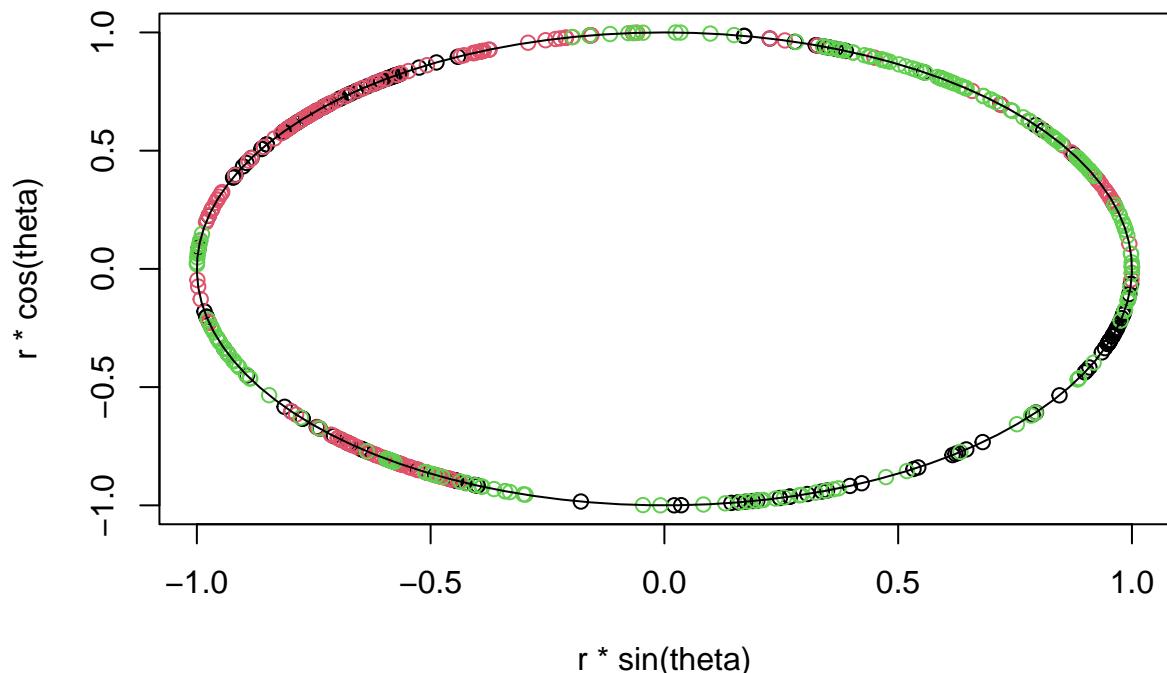
```



```

## [1] "Iteration 11"
## [1] "Iteration 12"
## [1] "Iteration 13"
## [1] "Iteration 14"
## [1] "Iteration 15"
## [1] "Iteration 15: objective function value is 15.896026"

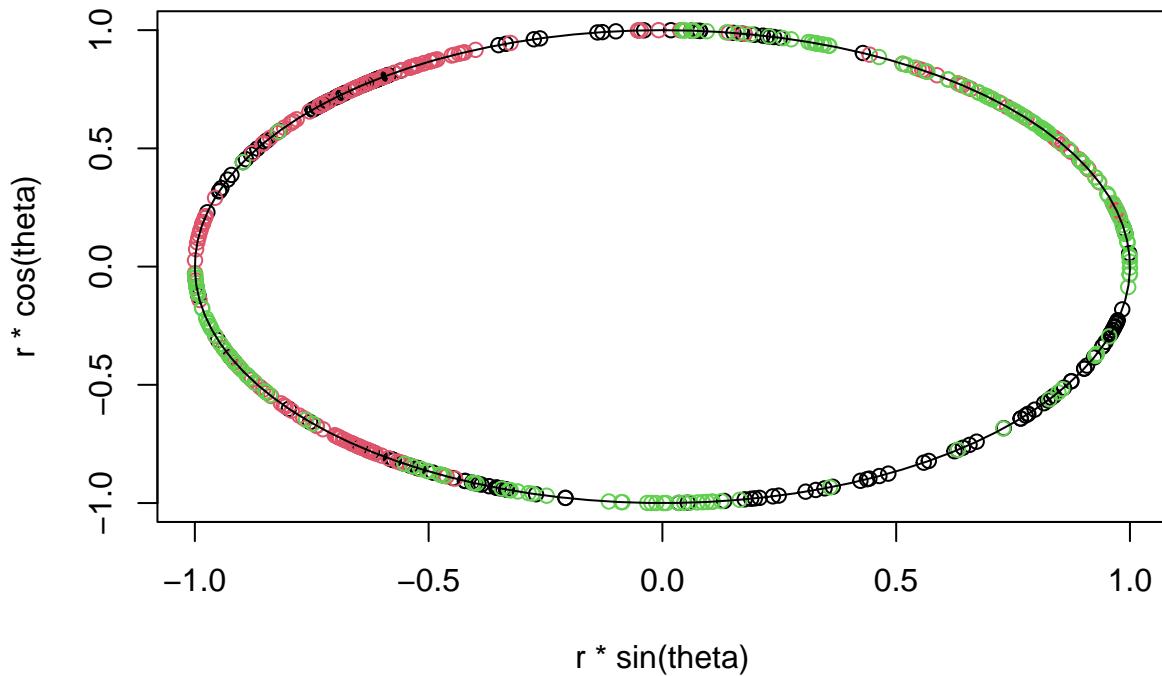
```



```

## [1] "Iteration 16"
## [1] "Iteration 17"
## [1] "Iteration 18"
## [1] "Iteration 19"
## [1] "Iteration 20"
## [1] "Iteration 20: objective function value is 15.723806"

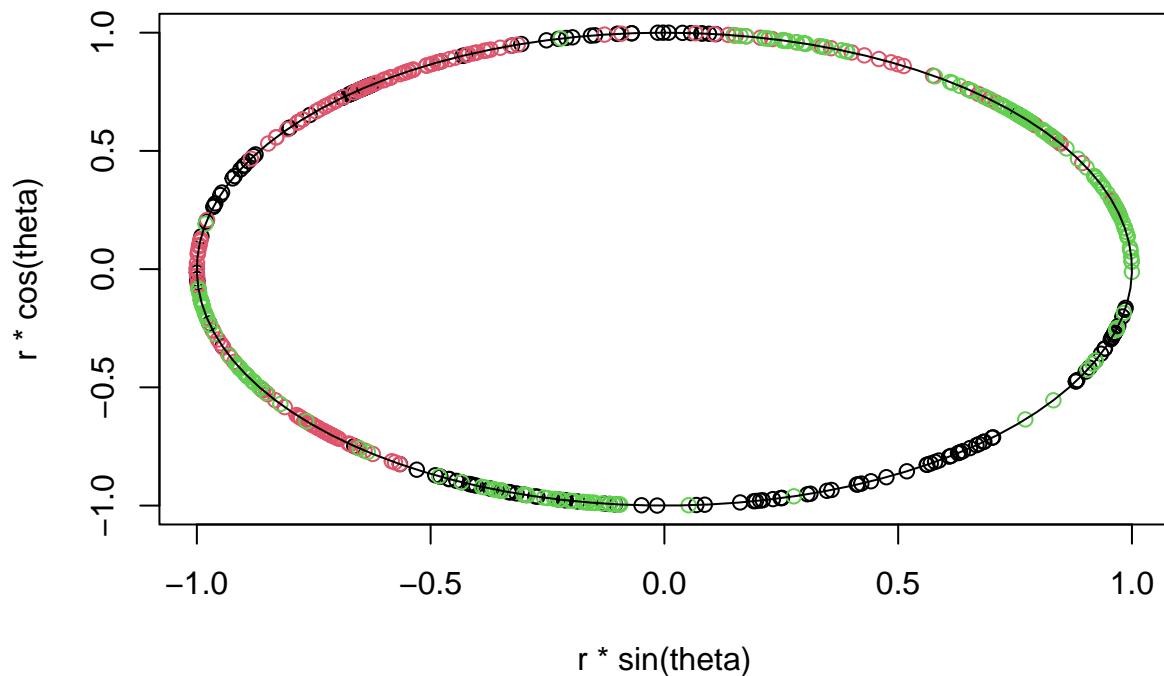
```



```

## [1] "Iteration 21"
## [1] "Iteration 22"
## [1] "Iteration 23"
## [1] "Iteration 24"
## [1] "Iteration 25"
## [1] "Iteration 25: objective function value is 15.614832"

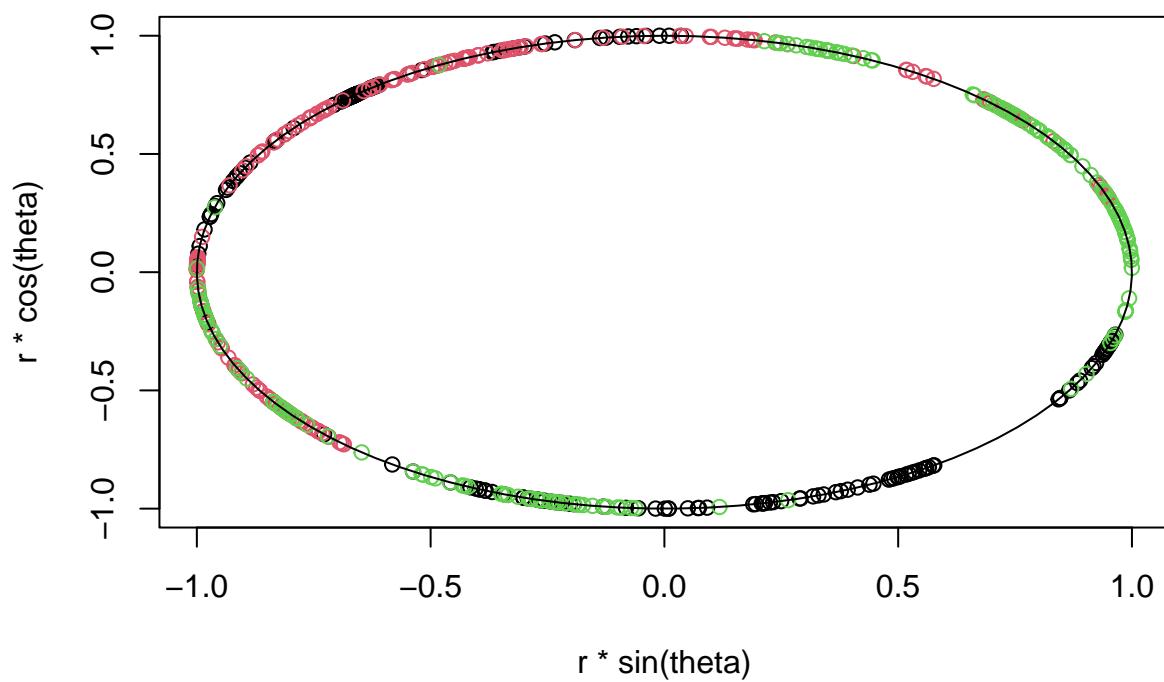
```



```

## [1] "Iteration 26"
## [1] "Iteration 27"
## [1] "Iteration 28"
## [1] "Iteration 29"
## [1] "Iteration 30"
## [1] "Iteration 30: objective function value is 15.554484"

```

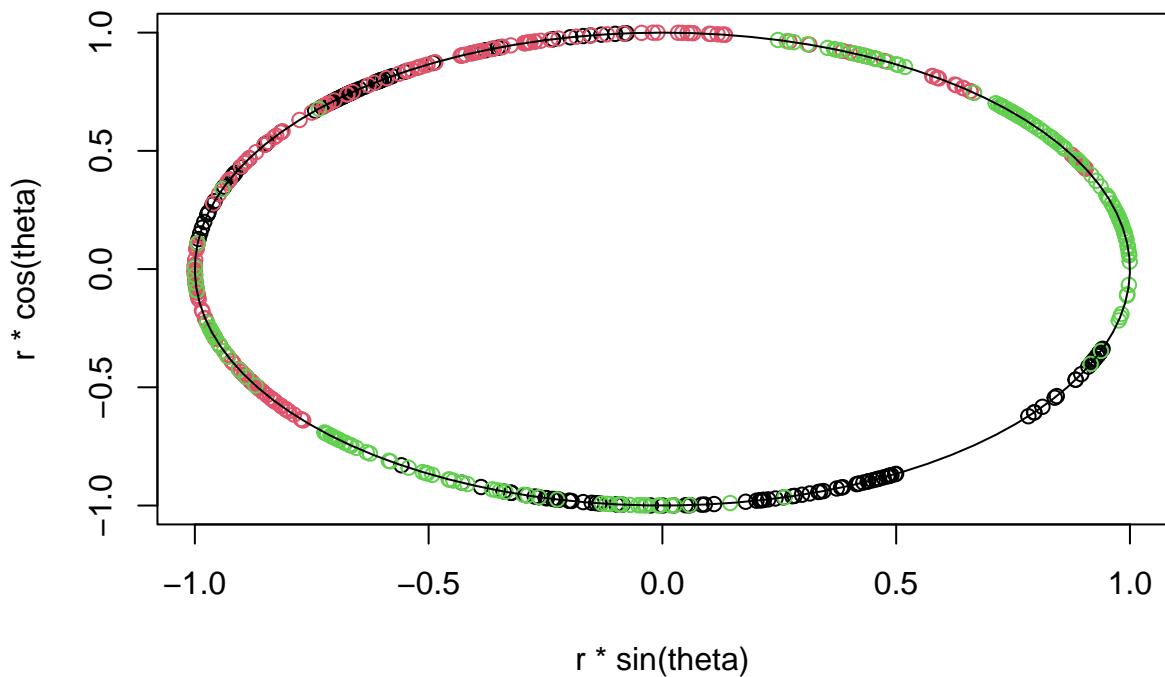


```

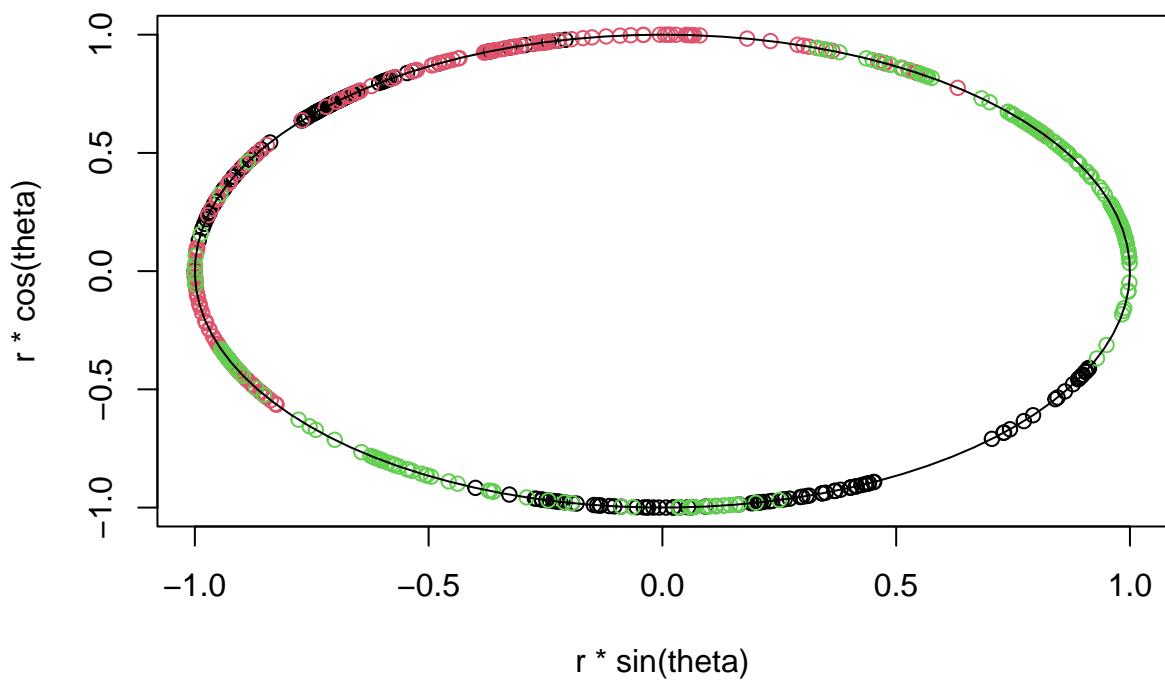
## [1] "Iteration 31"
## [1] "Iteration 32"
## [1] "Iteration 33"
## [1] "Iteration 34"

```

```
## [1] "Iteration 35"  
## [1] "Iteration 35: objective function value is 15.507556"
```



```
## [1] "Iteration 36"  
## [1] "Iteration 37"  
## [1] "Iteration 38"  
## [1] "Iteration 39"  
## [1] "Iteration 40"  
## [1] "Iteration 40: objective function value is 15.468256"
```

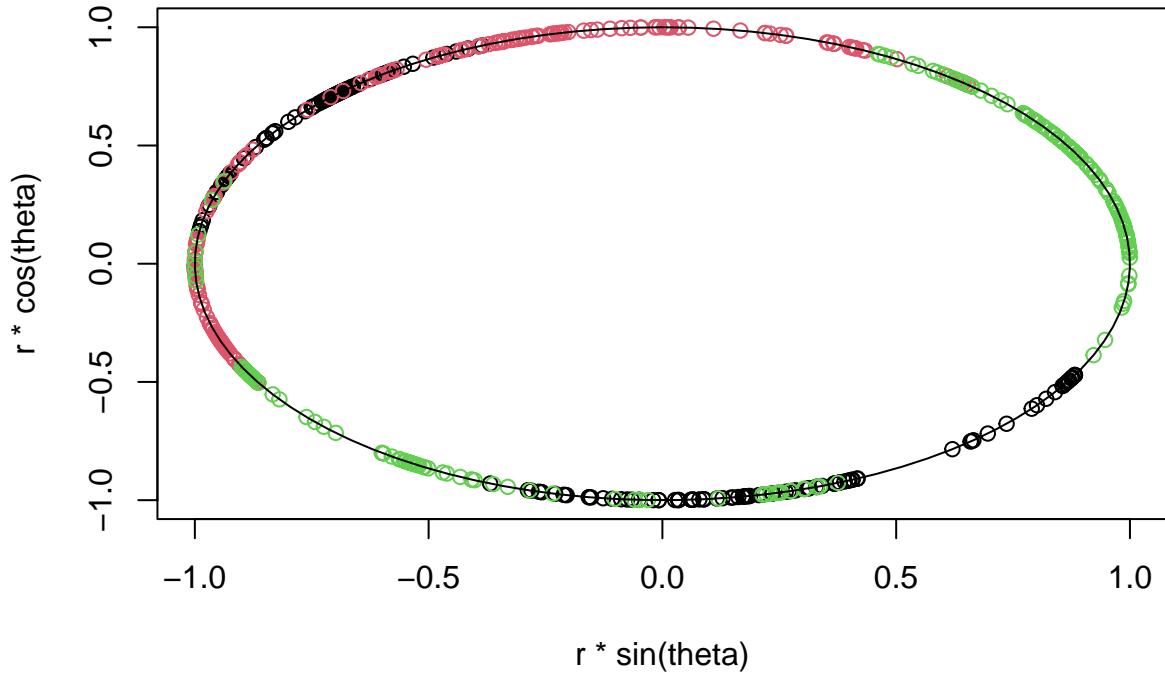


```
## [1] "Iteration 41"
```

```

## [1] "Iteration 42"
## [1] "Iteration 43"
## [1] "Iteration 44"
## [1] "Iteration 45"
## [1] "Iteration 45: objective function value is 15.429694"

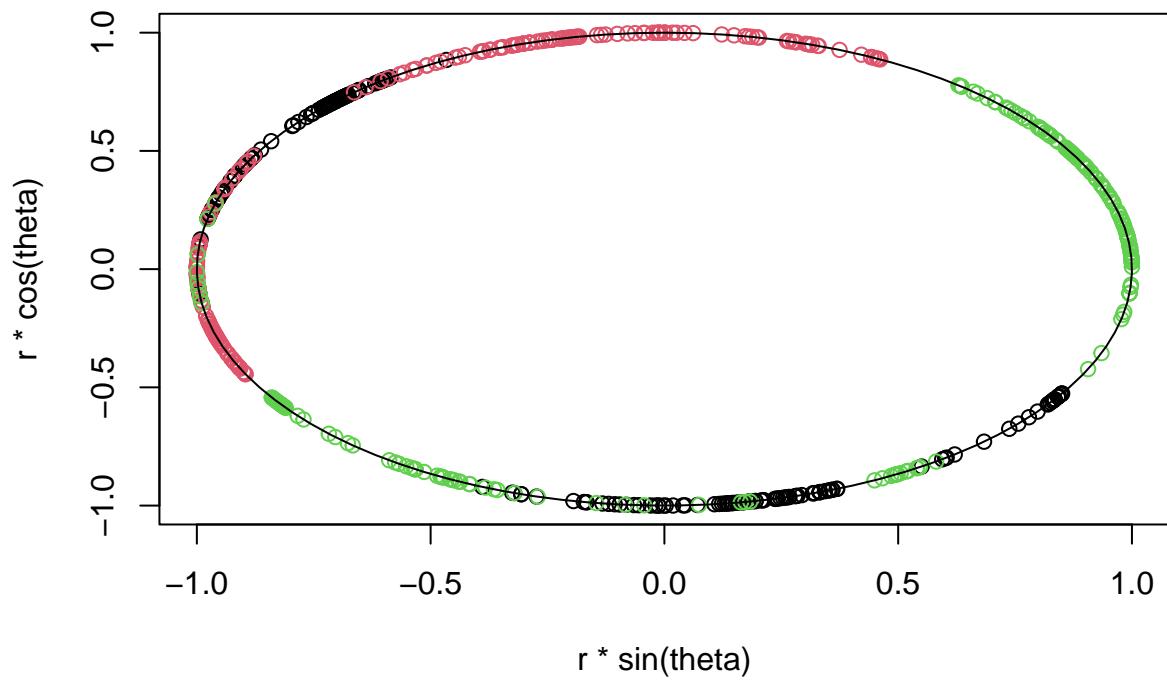
```



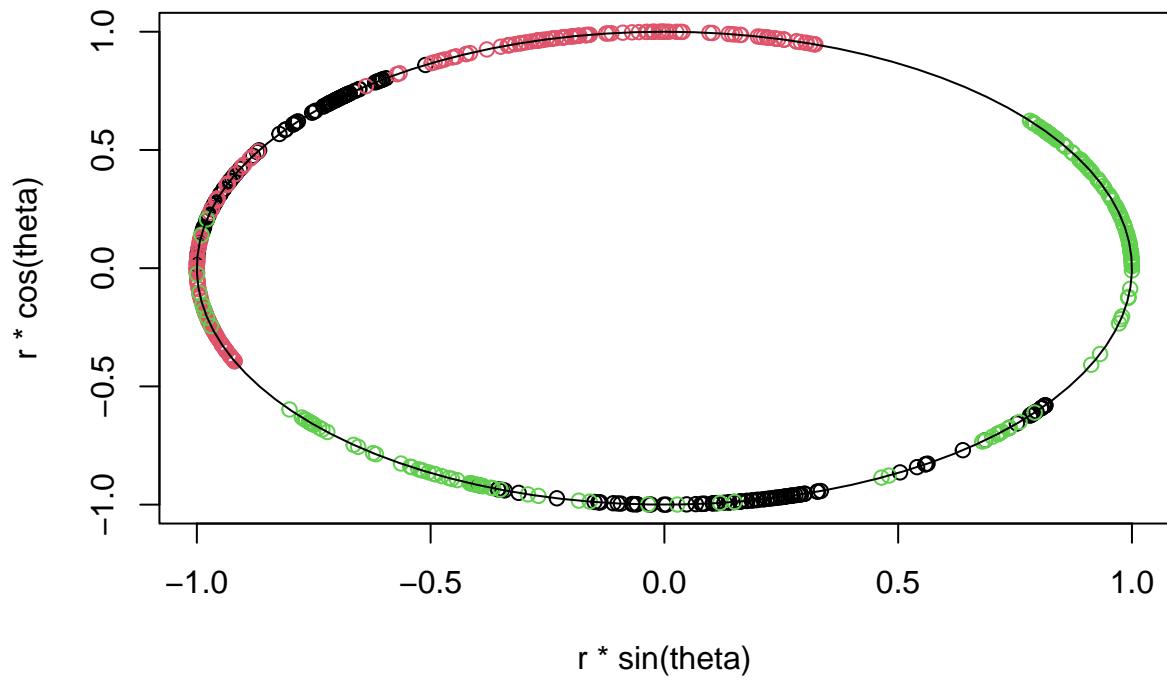
```

## [1] "Iteration 46"
## [1] "Iteration 47"
## [1] "Iteration 48"
## [1] "Iteration 49"
## [1] "Iteration 50"
## [1] "Iteration 50: objective function value is 15.381442"

```

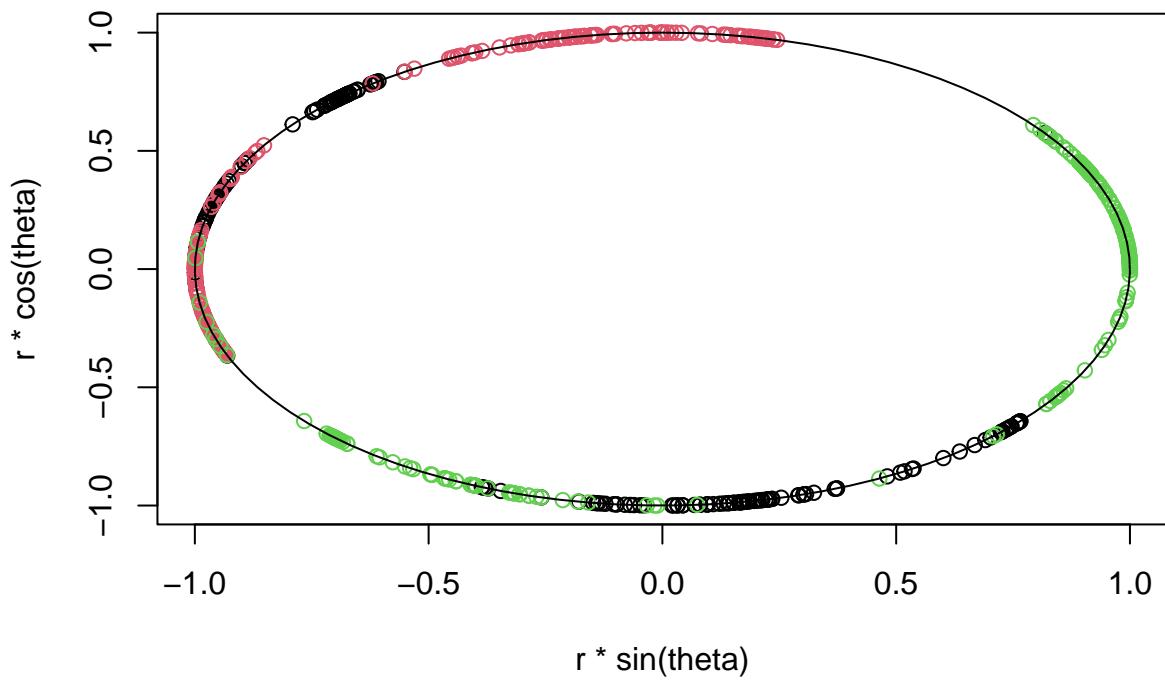


```
## [1] "Iteration 51"
## [1] "Iteration 52"
## [1] "Iteration 53"
## [1] "Iteration 54"
## [1] "Iteration 55"
## [1] "Iteration 55: objective function value is 15.338011"
```

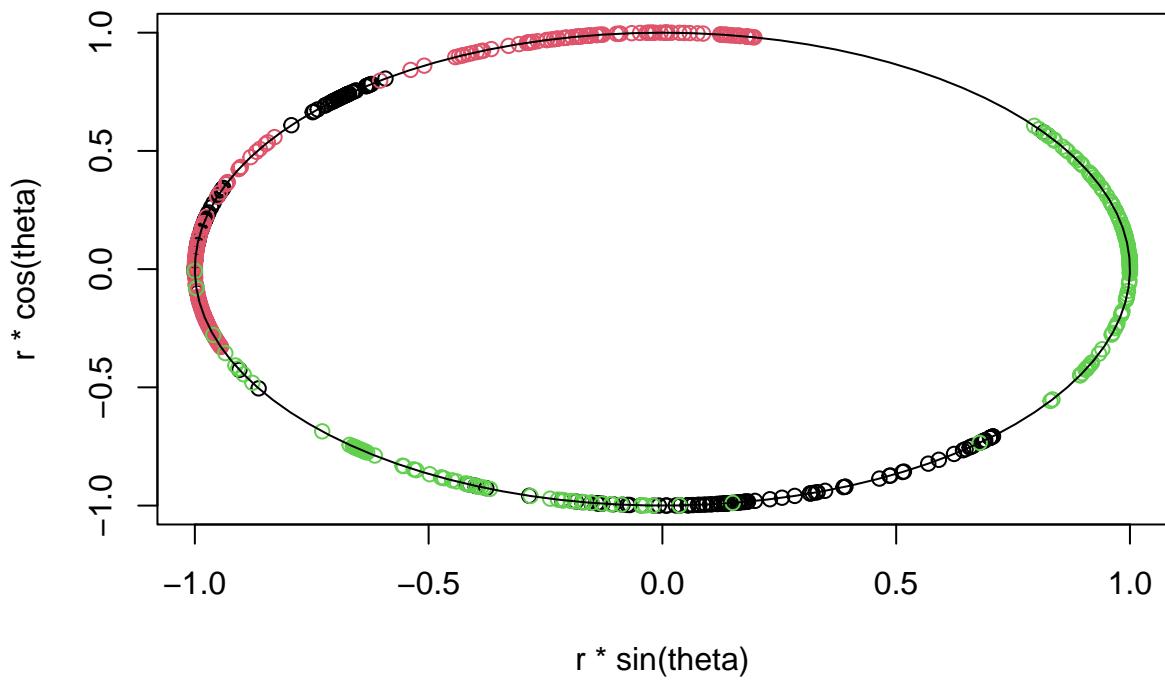


```
## [1] "Iteration 56"
## [1] "Iteration 57"
## [1] "Iteration 58"
## [1] "Iteration 59"
```

```
## [1] "Iteration 60"  
## [1] "Iteration 60: objective function value is 15.307749"
```

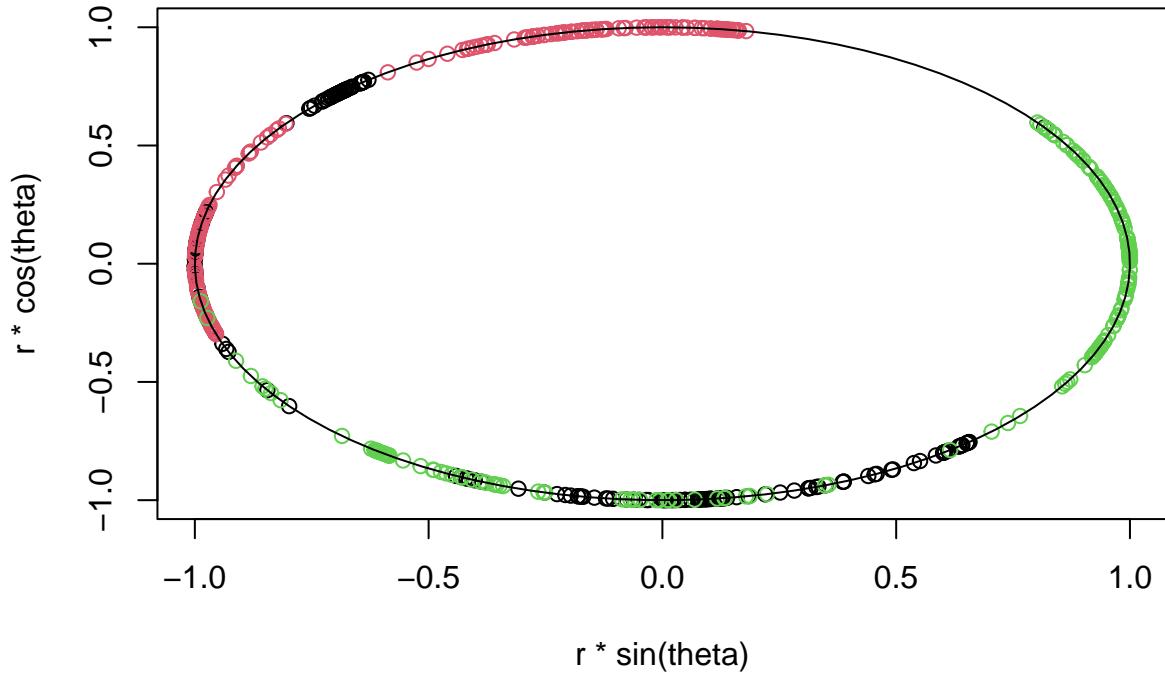


```
## [1] "Iteration 61"  
## [1] "Iteration 62"  
## [1] "Iteration 63"  
## [1] "Iteration 64"  
## [1] "Iteration 65"  
## [1] "Iteration 65: objective function value is 15.279330"
```

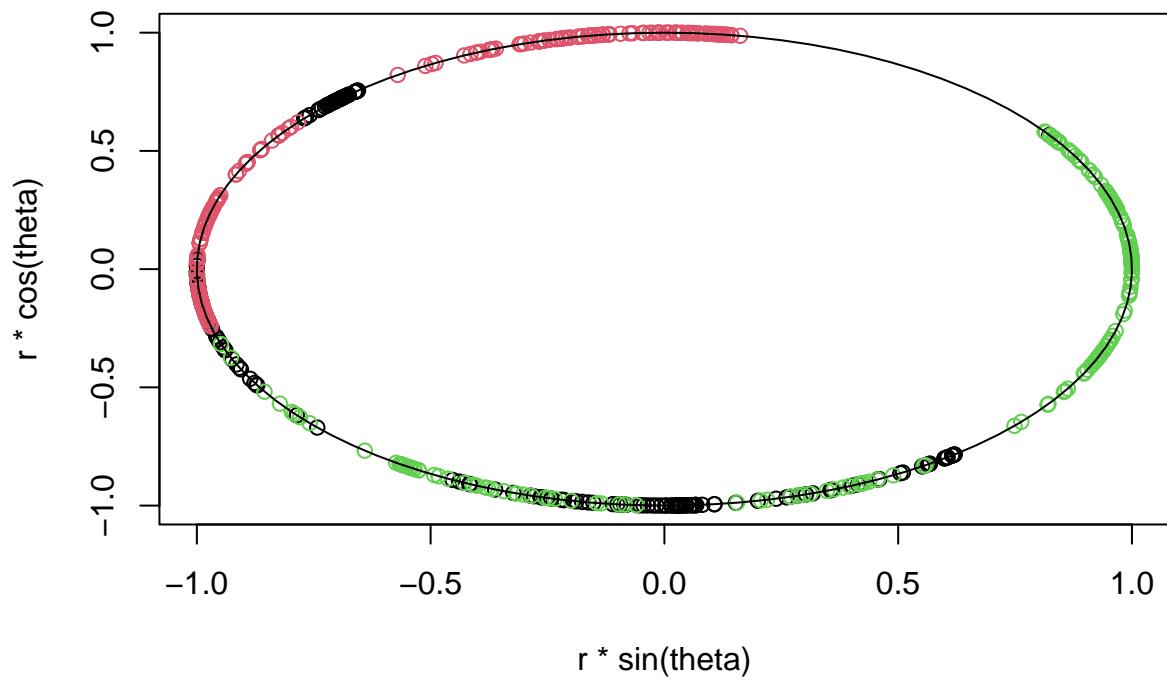


```
## [1] "Iteration 66"
```

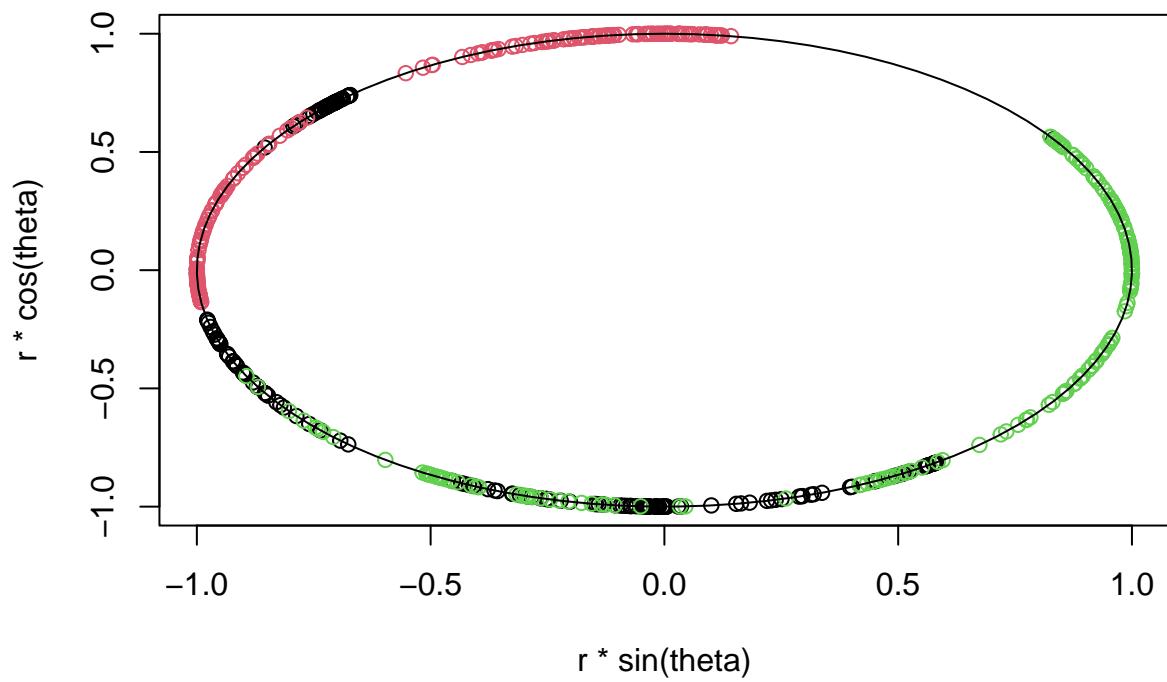
```
## [1] "Iteration 67"  
## [1] "Iteration 68"  
## [1] "Iteration 69"  
## [1] "Iteration 70"  
## [1] "Iteration 70: objective function value is 15.231950"
```



```
## [1] "Iteration 71"  
## [1] "Iteration 72"  
## [1] "Iteration 73"  
## [1] "Iteration 74"  
## [1] "Iteration 75"  
## [1] "Iteration 75: objective function value is 15.177195"
```

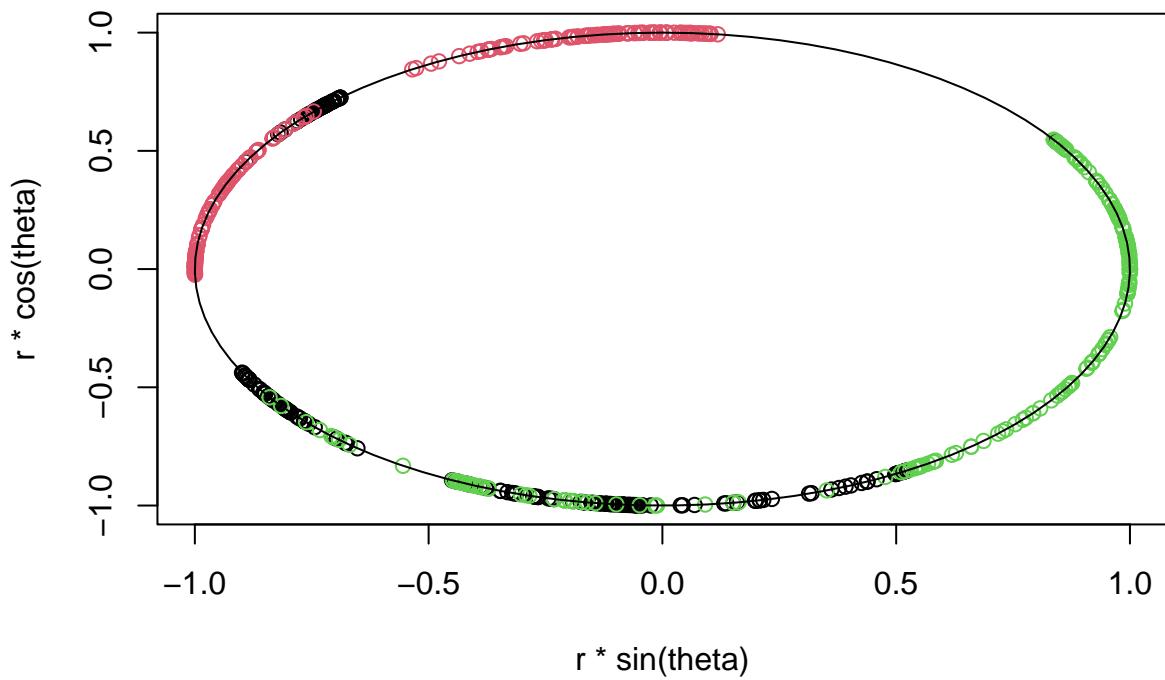


```
## [1] "Iteration 76"
## [1] "Iteration 77"
## [1] "Iteration 78"
## [1] "Iteration 79"
## [1] "Iteration 80"
## [1] "Iteration 80: objective function value is 15.127047"
```

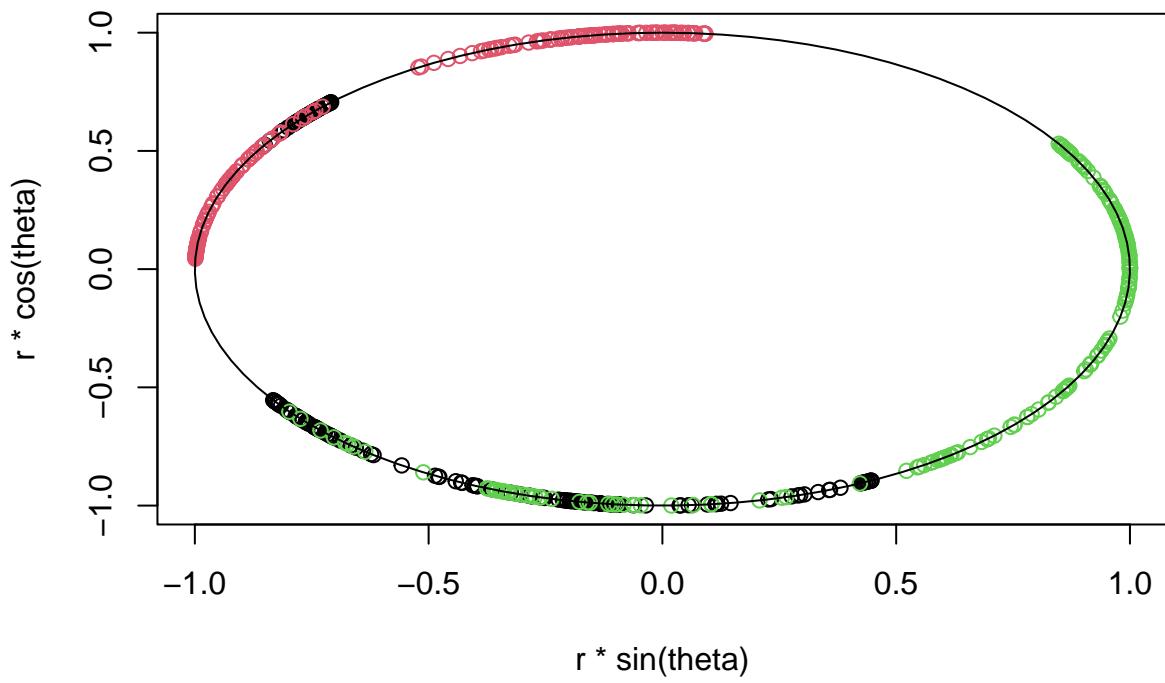


```
## [1] "Iteration 81"
## [1] "Iteration 82"
## [1] "Iteration 83"
## [1] "Iteration 84"
```

```
## [1] "Iteration 85"  
## [1] "Iteration 85: objective function value is 15.109272"
```

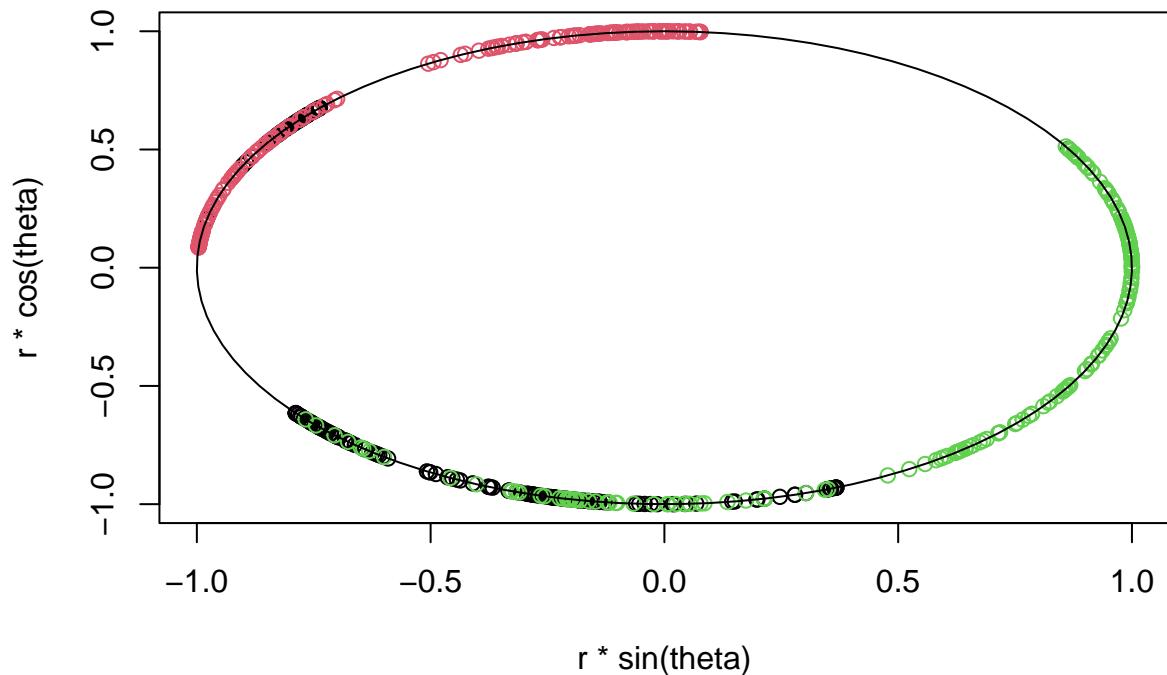


```
## [1] "Iteration 86"  
## [1] "Iteration 87"  
## [1] "Iteration 88"  
## [1] "Iteration 89"  
## [1] "Iteration 90"  
## [1] "Iteration 90: objective function value is 15.116447"
```

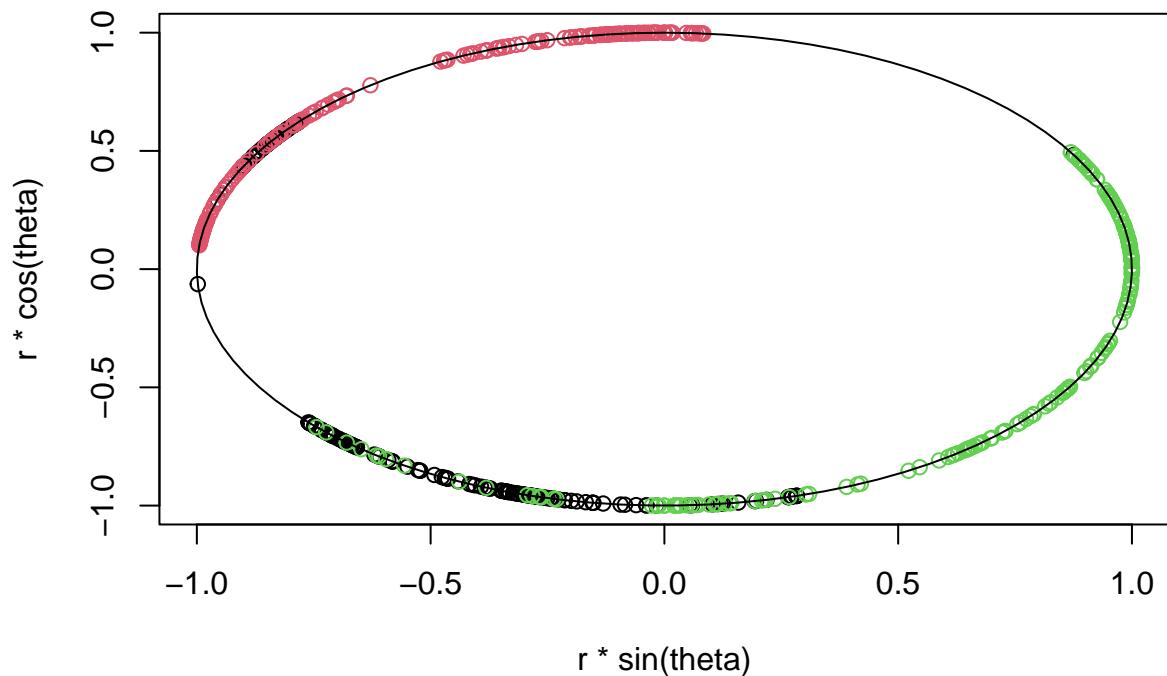


```
## [1] "Iteration 91"
```

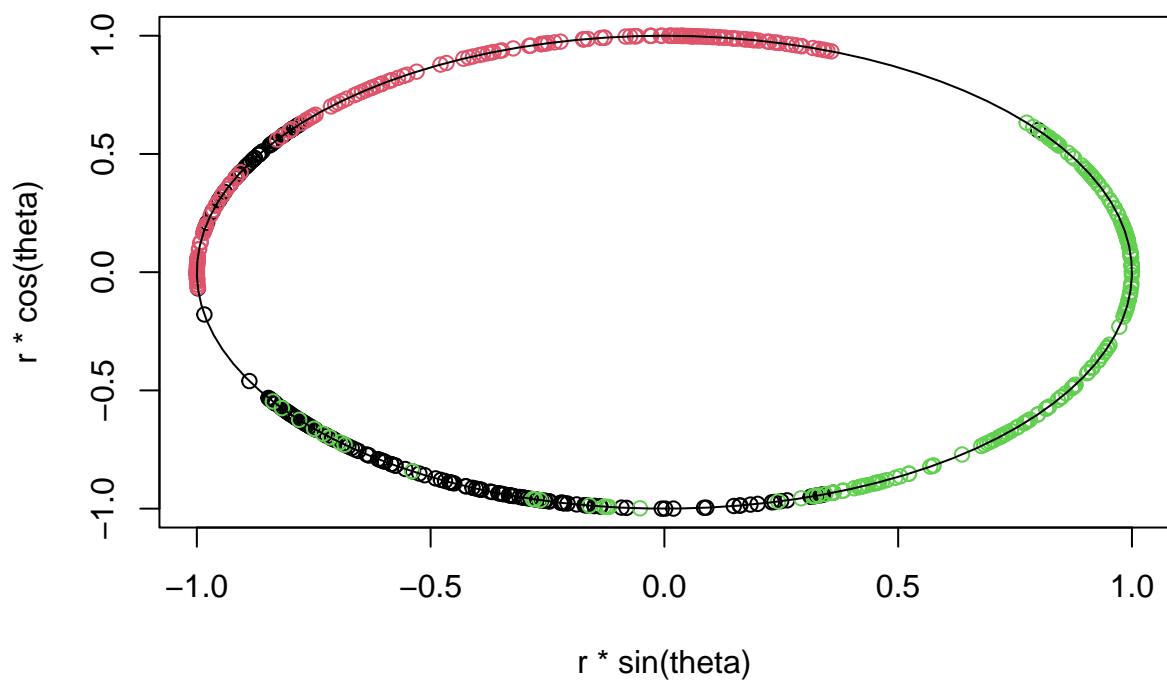
```
## [1] "Iteration 92"  
## [1] "Iteration 93"  
## [1] "Iteration 94"  
## [1] "Iteration 95"  
## [1] "Iteration 95: objective function value is 15.120641"
```



```
## [1] "Iteration 96"  
## [1] "Iteration 97"  
## [1] "Iteration 98"  
## [1] "Iteration 99"  
## [1] "Iteration 100"  
## [1] "Iteration 100: objective function value is 2.389553"
```

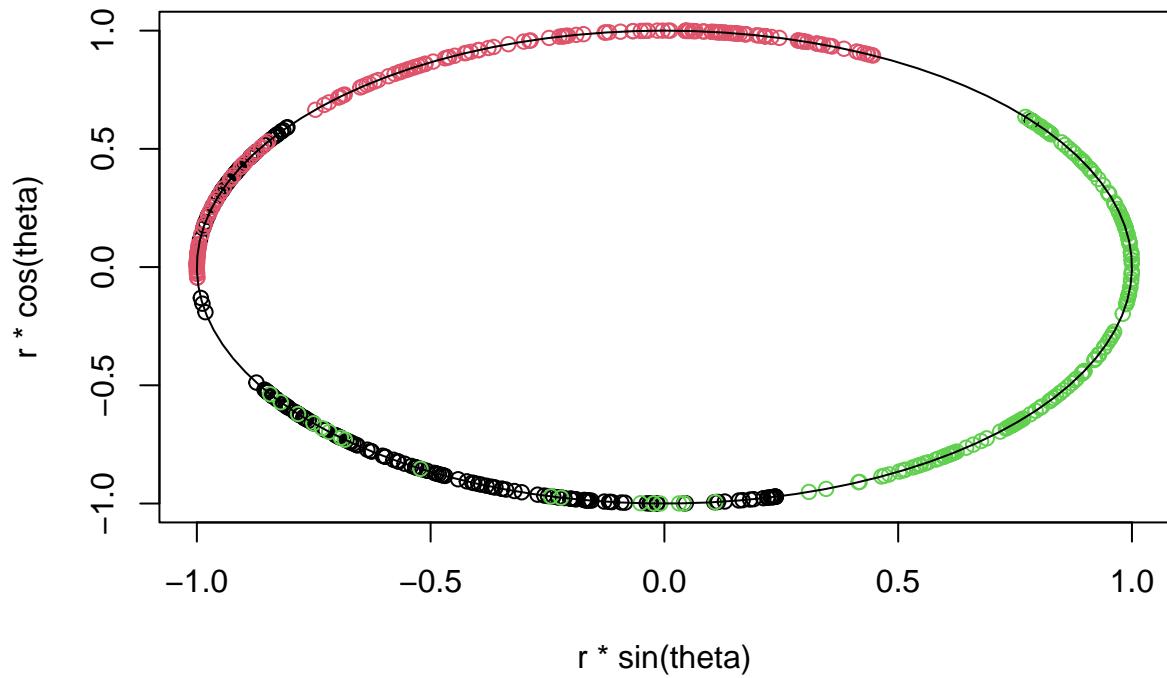


```
## [1] "Iteration 101"
## [1] "Iteration 102"
## [1] "Iteration 103"
## [1] "Iteration 104"
## [1] "Iteration 105"
## [1] "Iteration 105: objective function value is 2.357424"
```

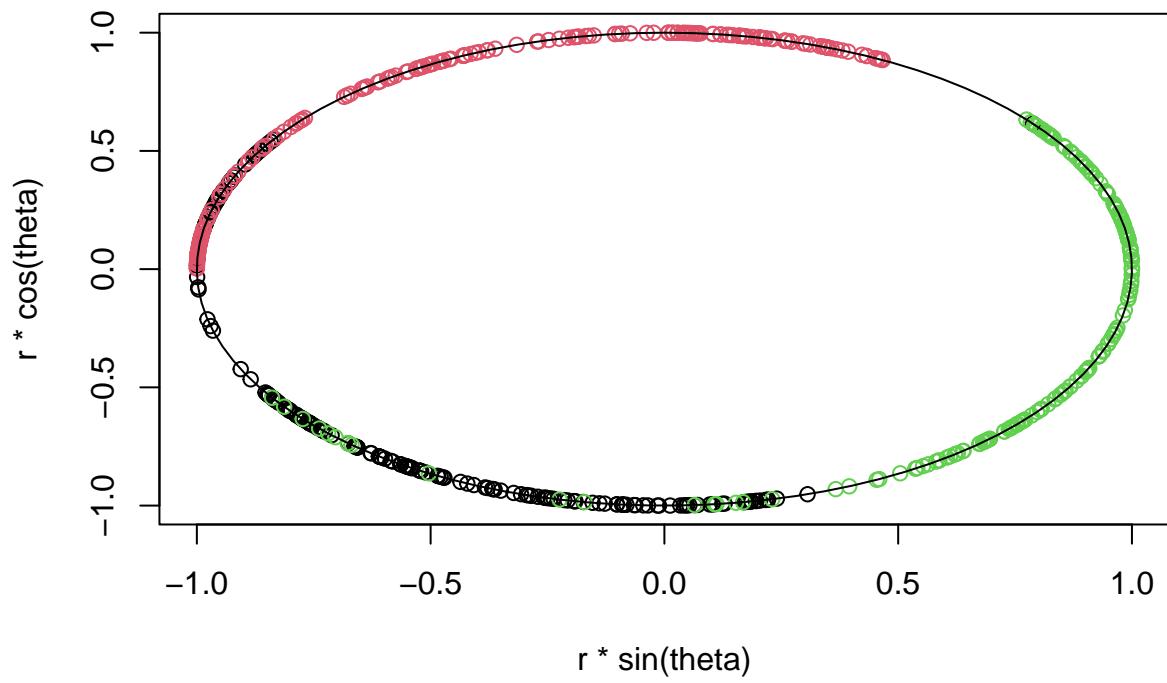


```
## [1] "Iteration 106"
## [1] "Iteration 107"
## [1] "Iteration 108"
## [1] "Iteration 109"
```

```
## [1] "Iteration 110"  
## [1] "Iteration 110: objective function value is 2.348125"
```

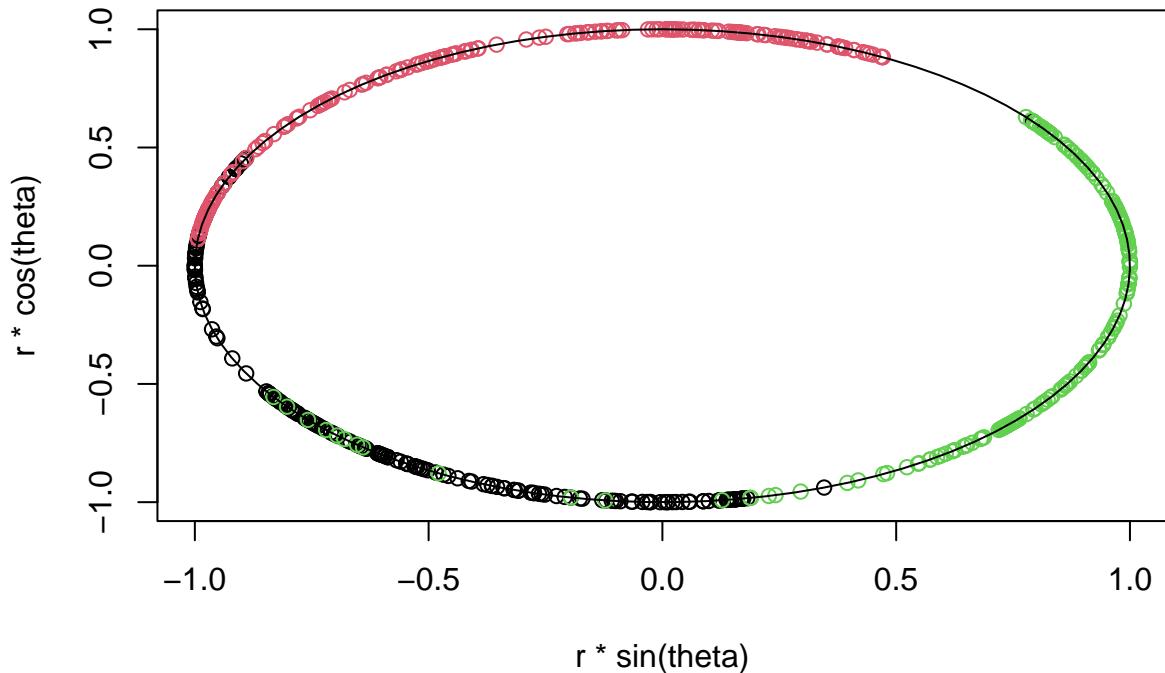


```
## [1] "Iteration 111"  
## [1] "Iteration 112"  
## [1] "Iteration 113"  
## [1] "Iteration 114"  
## [1] "Iteration 115"  
## [1] "Iteration 115: objective function value is 2.341223"
```

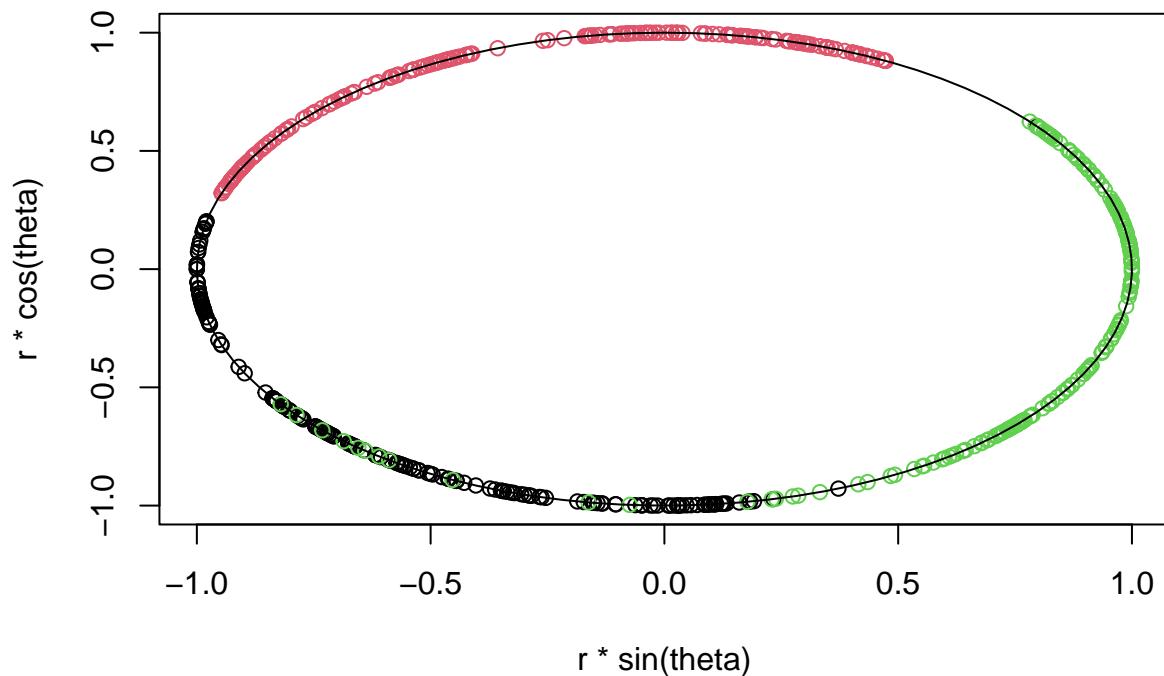


```
## [1] "Iteration 116"
```

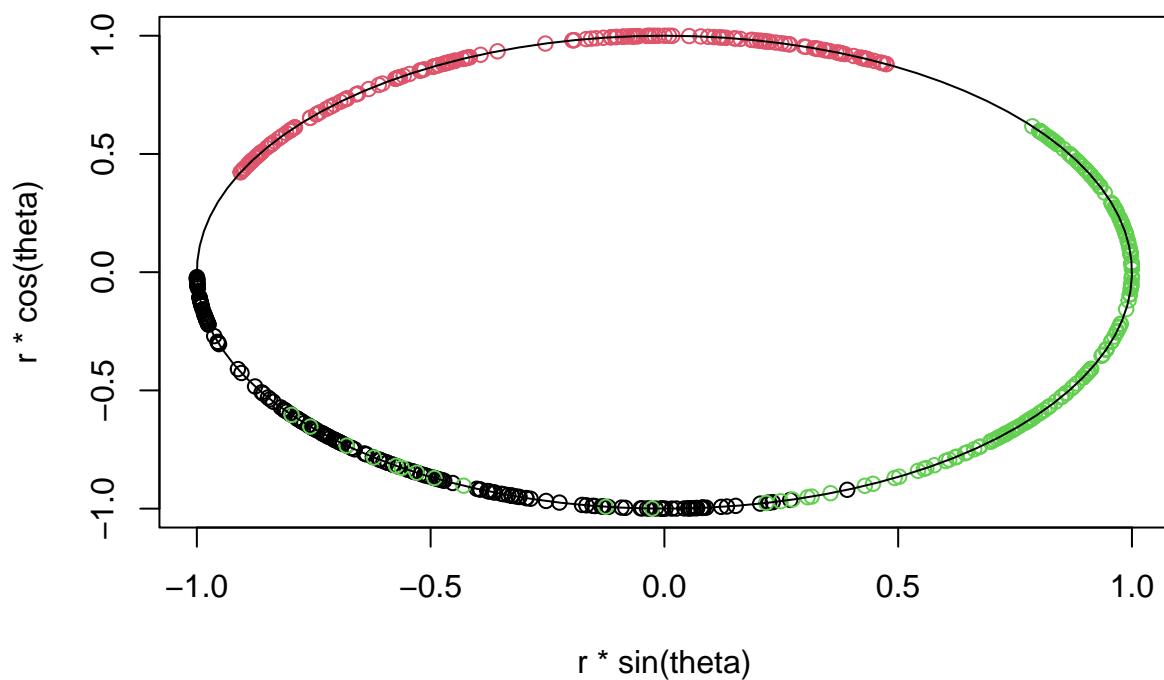
```
## [1] "Iteration 117"  
## [1] "Iteration 118"  
## [1] "Iteration 119"  
## [1] "Iteration 120"  
## [1] "Iteration 120: objective function value is 2.333425"
```



```
## [1] "Iteration 121"  
## [1] "Iteration 122"  
## [1] "Iteration 123"  
## [1] "Iteration 124"  
## [1] "Iteration 125"  
## [1] "Iteration 125: objective function value is 2.324420"
```

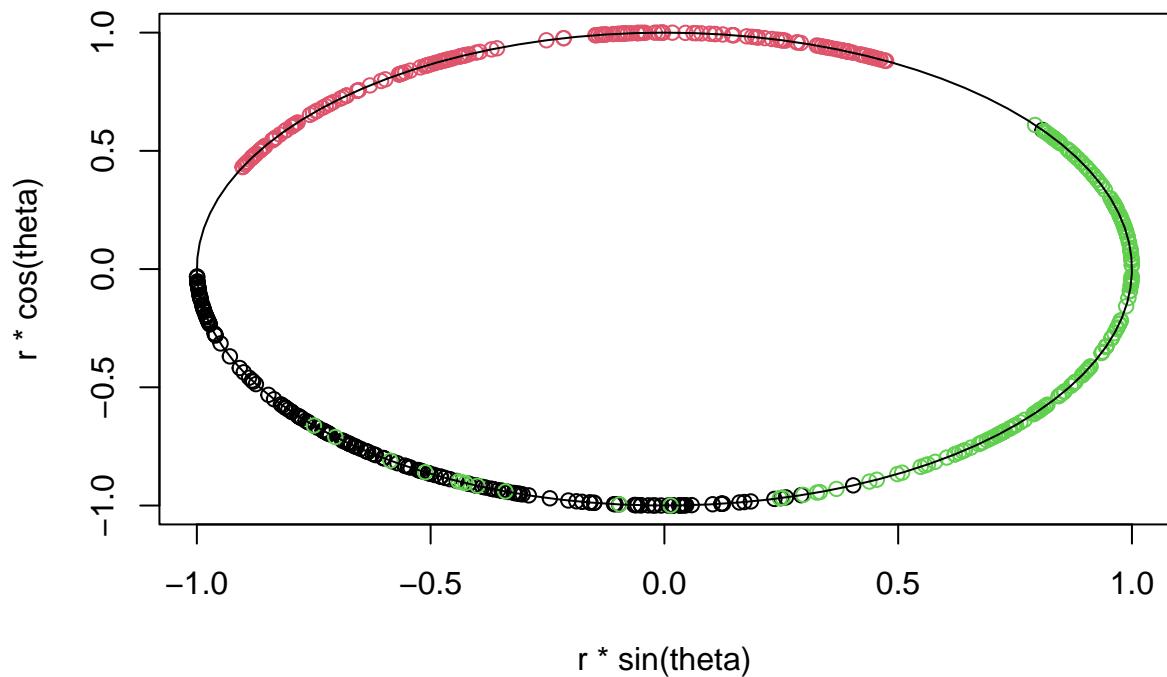


```
## [1] "Iteration 126"
## [1] "Iteration 127"
## [1] "Iteration 128"
## [1] "Iteration 129"
## [1] "Iteration 130"
## [1] "Iteration 130: objective function value is 2.321321"
```

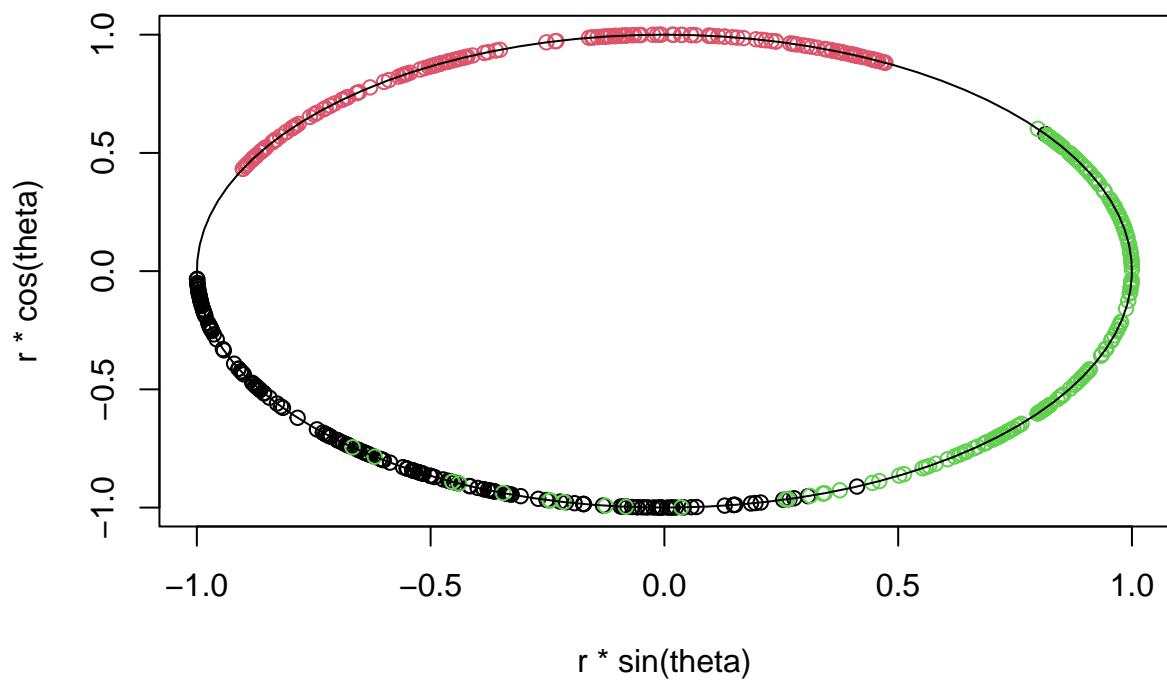


```
## [1] "Iteration 131"
## [1] "Iteration 132"
## [1] "Iteration 133"
## [1] "Iteration 134"
```

```
## [1] "Iteration 135"  
## [1] "Iteration 135: objective function value is 2.319558"
```

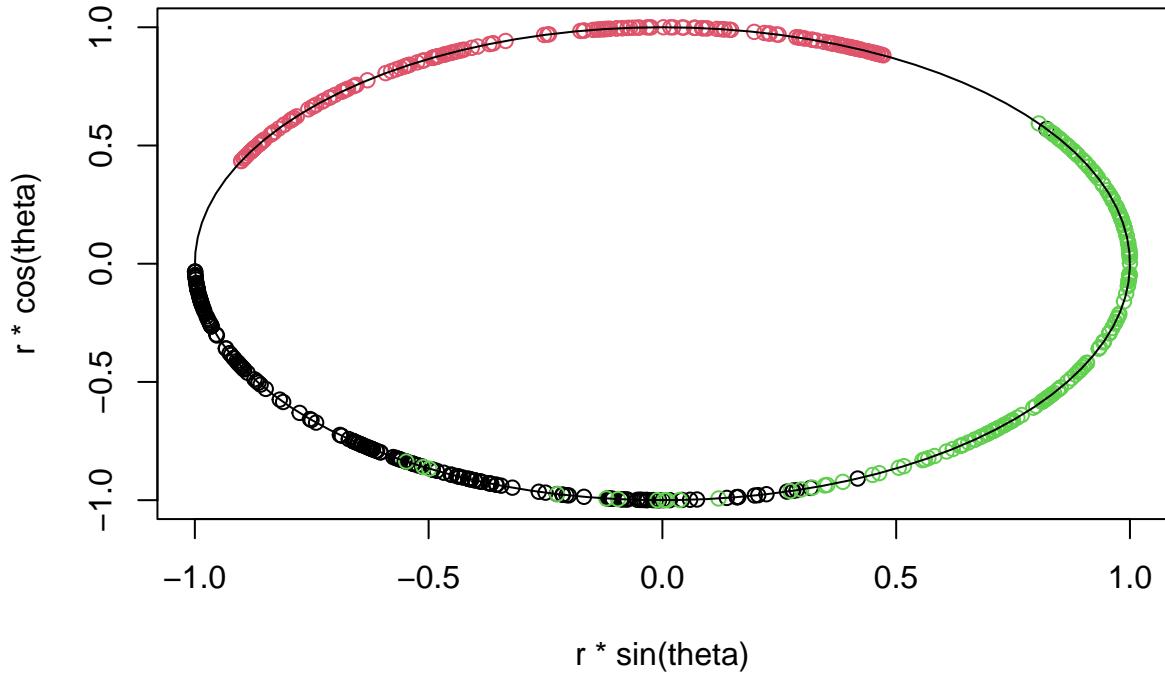


```
## [1] "Iteration 136"  
## [1] "Iteration 137"  
## [1] "Iteration 138"  
## [1] "Iteration 139"  
## [1] "Iteration 140"  
## [1] "Iteration 140: objective function value is 2.317507"
```

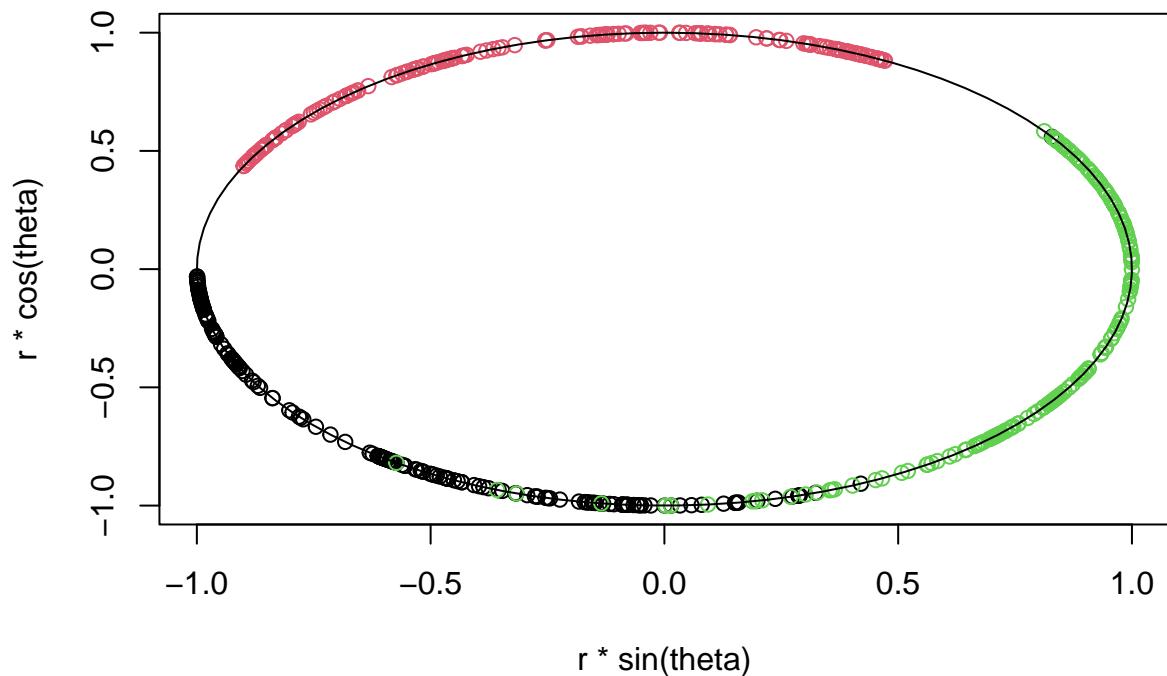


```
## [1] "Iteration 141"
```

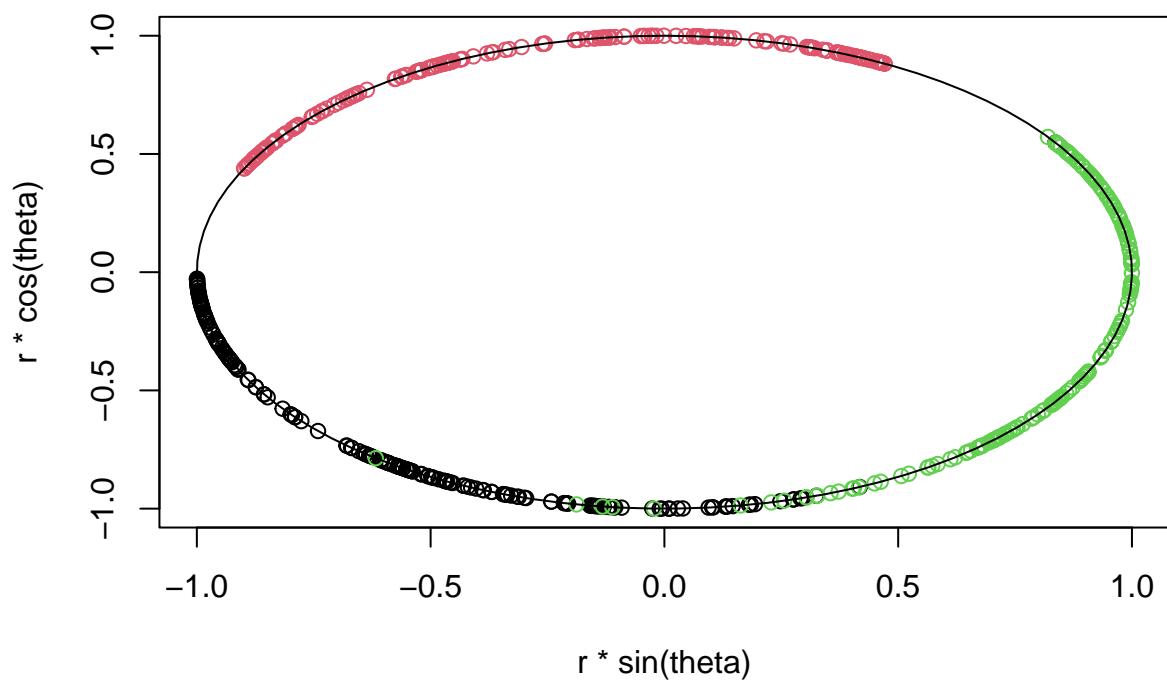
```
## [1] "Iteration 142"  
## [1] "Iteration 143"  
## [1] "Iteration 144"  
## [1] "Iteration 145"  
## [1] "Iteration 145: objective function value is 2.315169"
```



```
## [1] "Iteration 146"  
## [1] "Iteration 147"  
## [1] "Iteration 148"  
## [1] "Iteration 149"  
## [1] "Iteration 150"  
## [1] "Iteration 150: objective function value is 2.312527"
```

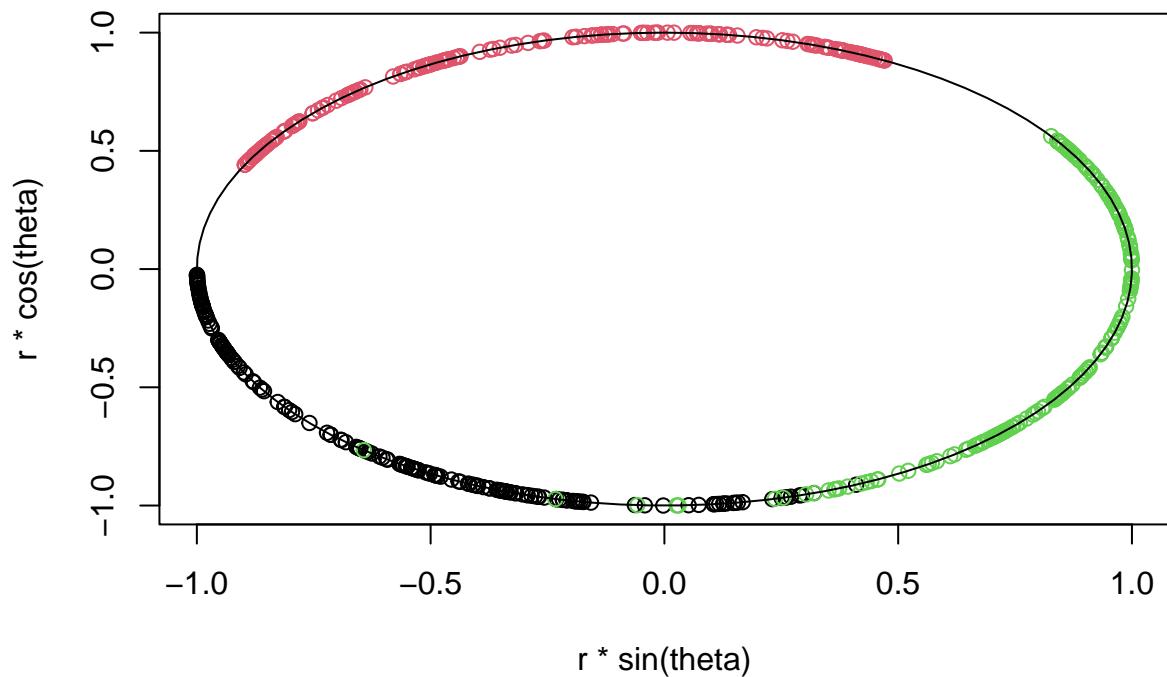


```
## [1] "Iteration 151"
## [1] "Iteration 152"
## [1] "Iteration 153"
## [1] "Iteration 154"
## [1] "Iteration 155"
## [1] "Iteration 155: objective function value is 2.310262"
```

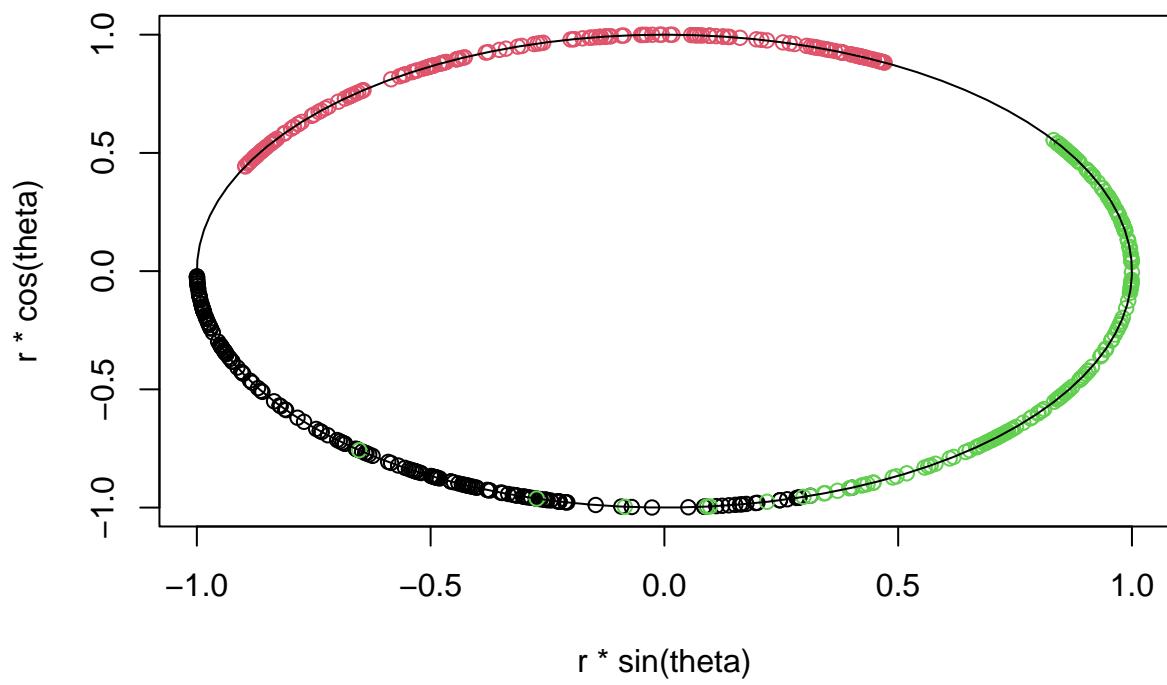


```
## [1] "Iteration 156"
## [1] "Iteration 157"
## [1] "Iteration 158"
## [1] "Iteration 159"
```

```
## [1] "Iteration 160"  
## [1] "Iteration 160: objective function value is 2.308720"
```

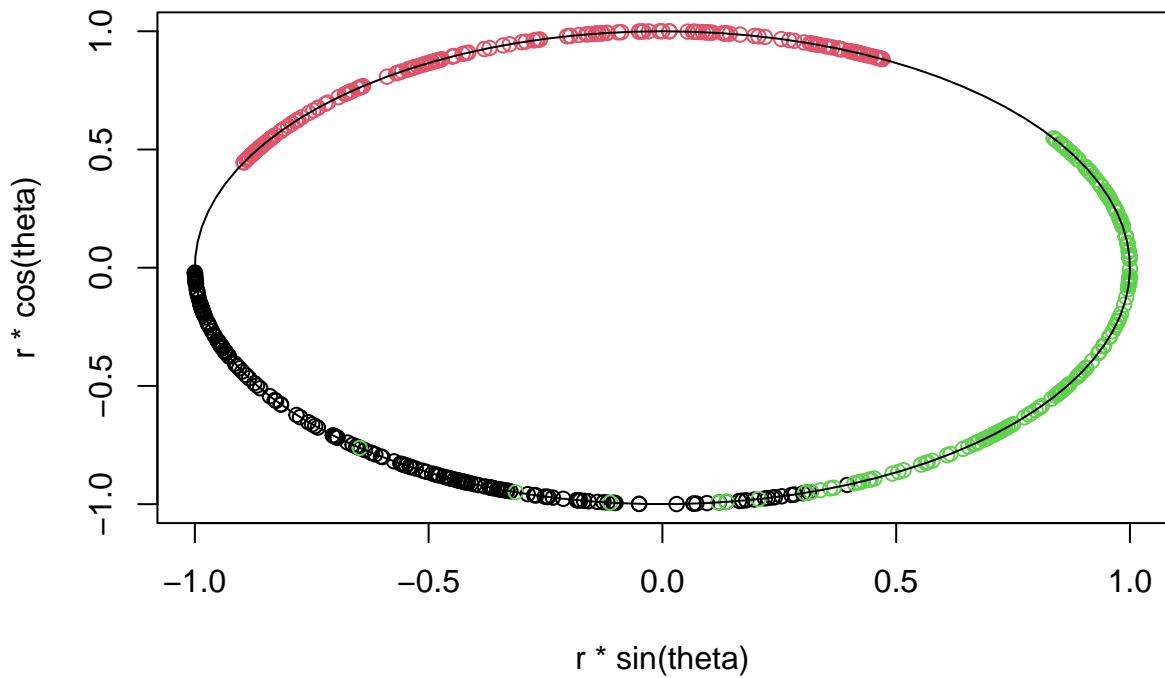


```
## [1] "Iteration 161"  
## [1] "Iteration 162"  
## [1] "Iteration 163"  
## [1] "Iteration 164"  
## [1] "Iteration 165"  
## [1] "Iteration 165: objective function value is 2.307447"
```

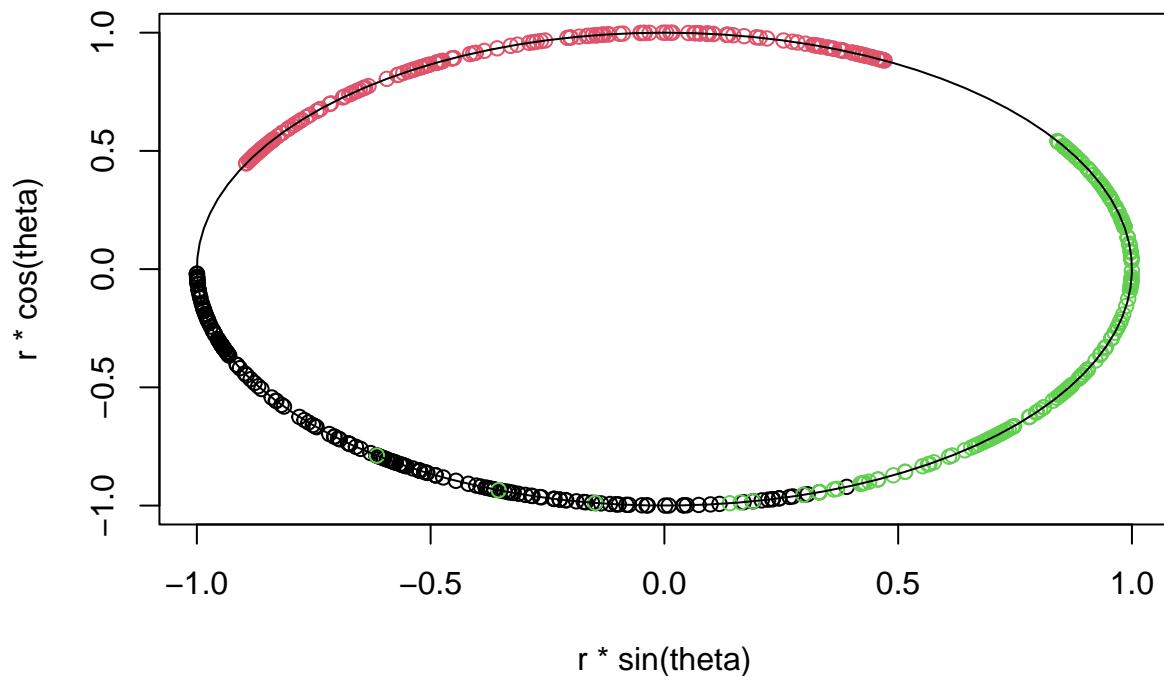


```
## [1] "Iteration 166"
```

```
## [1] "Iteration 167"  
## [1] "Iteration 168"  
## [1] "Iteration 169"  
## [1] "Iteration 170"  
## [1] "Iteration 170: objective function value is 2.305828"
```



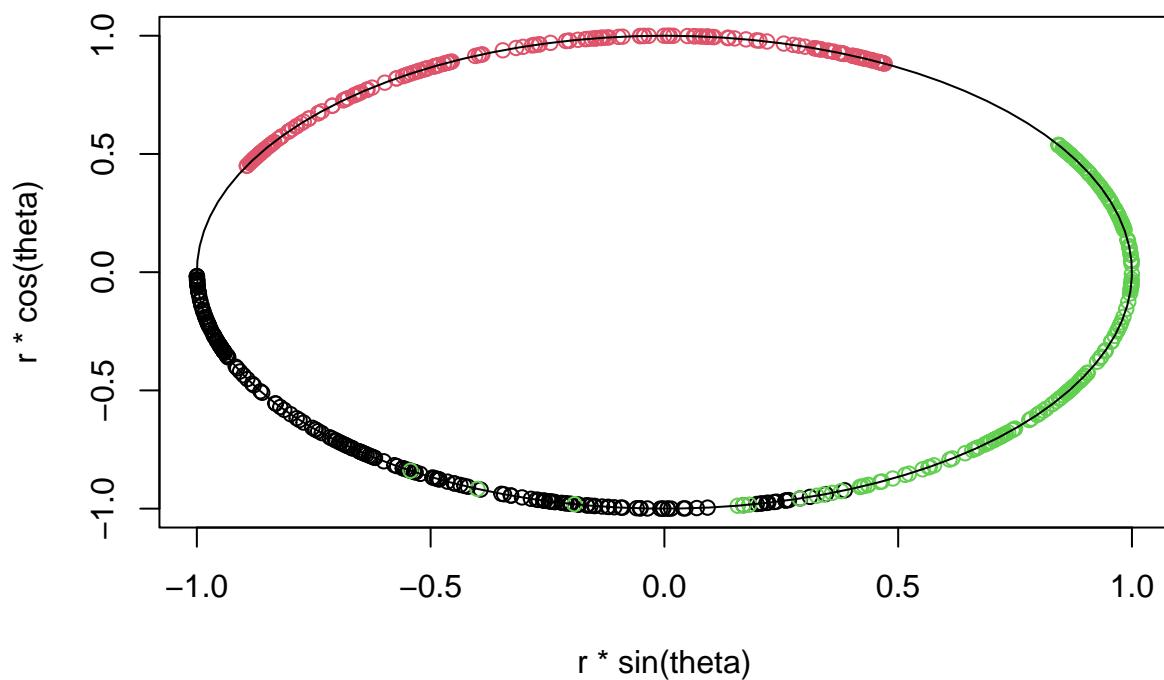
```
## [1] "Iteration 171"  
## [1] "Iteration 172"  
## [1] "Iteration 173"  
## [1] "Iteration 174"  
## [1] "Iteration 175"  
## [1] "Iteration 175: objective function value is 2.303758"
```



```

## [1] "Iteration 176"
## [1] "Iteration 177"
## [1] "Iteration 178"
## [1] "Iteration 179"
## [1] "Iteration 180"
## [1] "Iteration 180: objective function value is 2.302110"

```

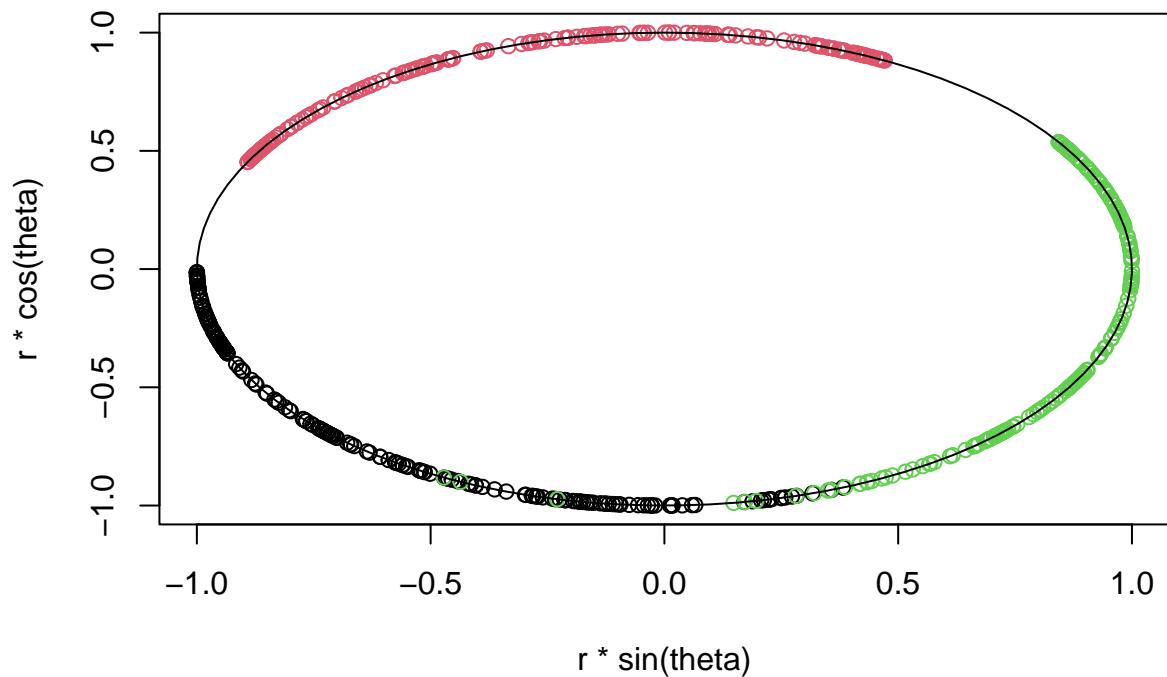


```

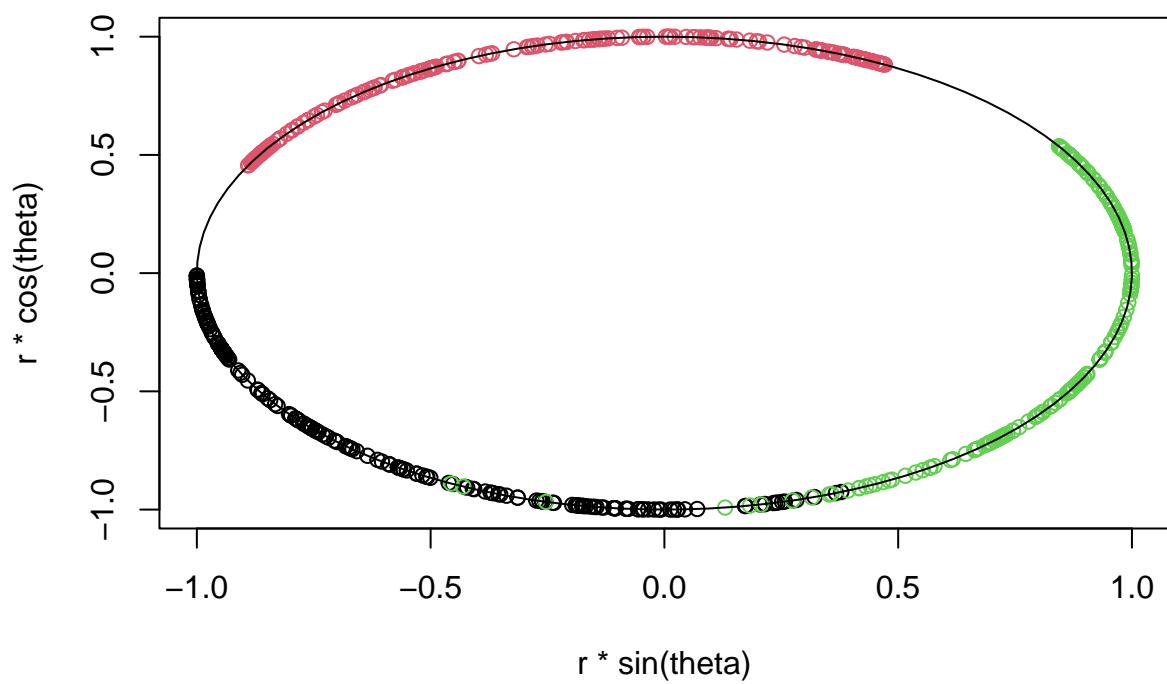
## [1] "Iteration 181"
## [1] "Iteration 182"
## [1] "Iteration 183"
## [1] "Iteration 184"

```

```
## [1] "Iteration 185"  
## [1] "Iteration 185: objective function value is 2.301293"
```

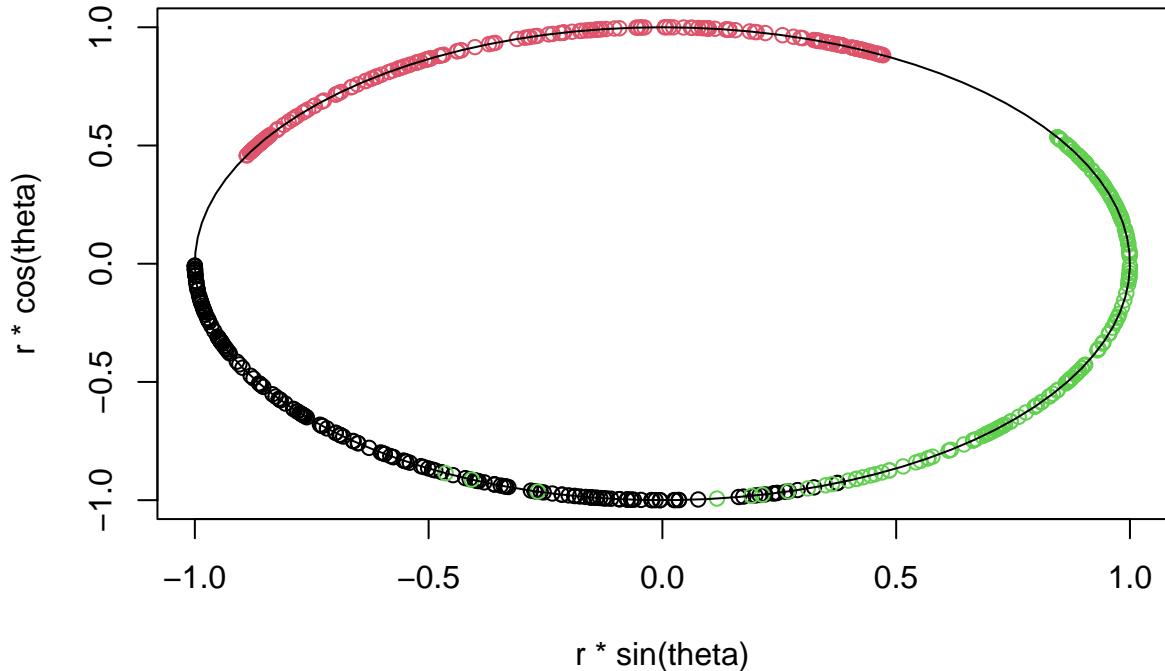


```
## [1] "Iteration 186"  
## [1] "Iteration 187"  
## [1] "Iteration 188"  
## [1] "Iteration 189"  
## [1] "Iteration 190"  
## [1] "Iteration 190: objective function value is 2.300968"
```

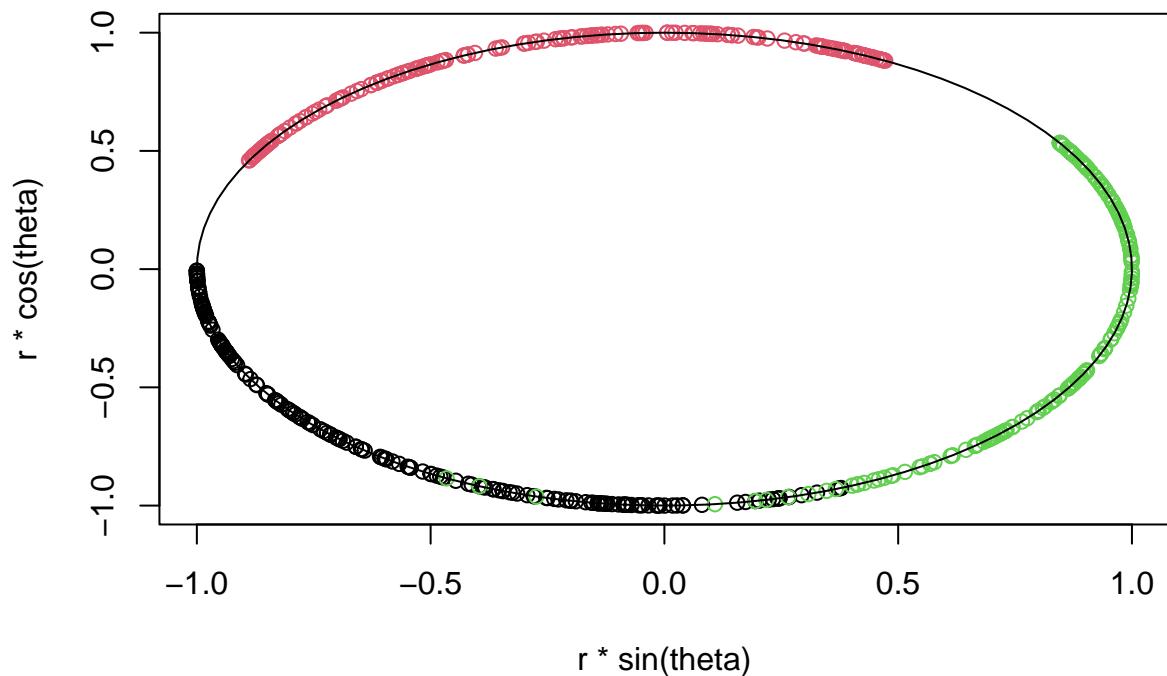


```
## [1] "Iteration 191"
```

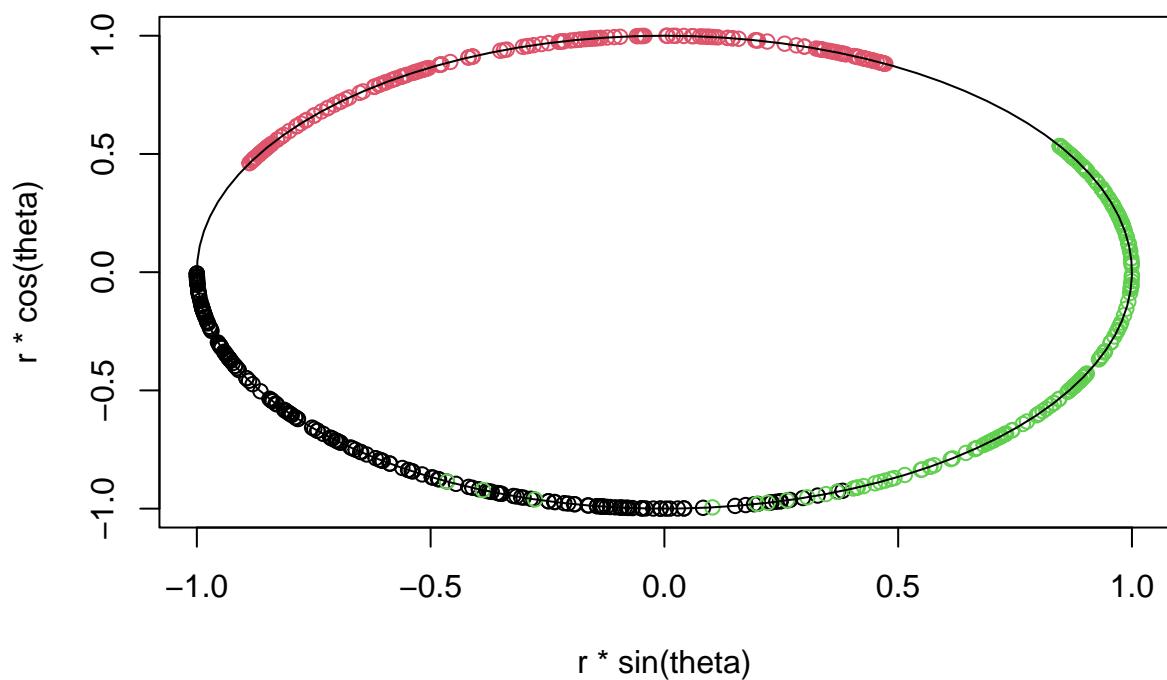
```
## [1] "Iteration 192"  
## [1] "Iteration 193"  
## [1] "Iteration 194"  
## [1] "Iteration 195"  
## [1] "Iteration 195: objective function value is 2.300825"
```



```
## [1] "Iteration 196"  
## [1] "Iteration 197"  
## [1] "Iteration 198"  
## [1] "Iteration 199"  
## [1] "Iteration 200"  
## [1] "Iteration 200: objective function value is 2.300744"
```

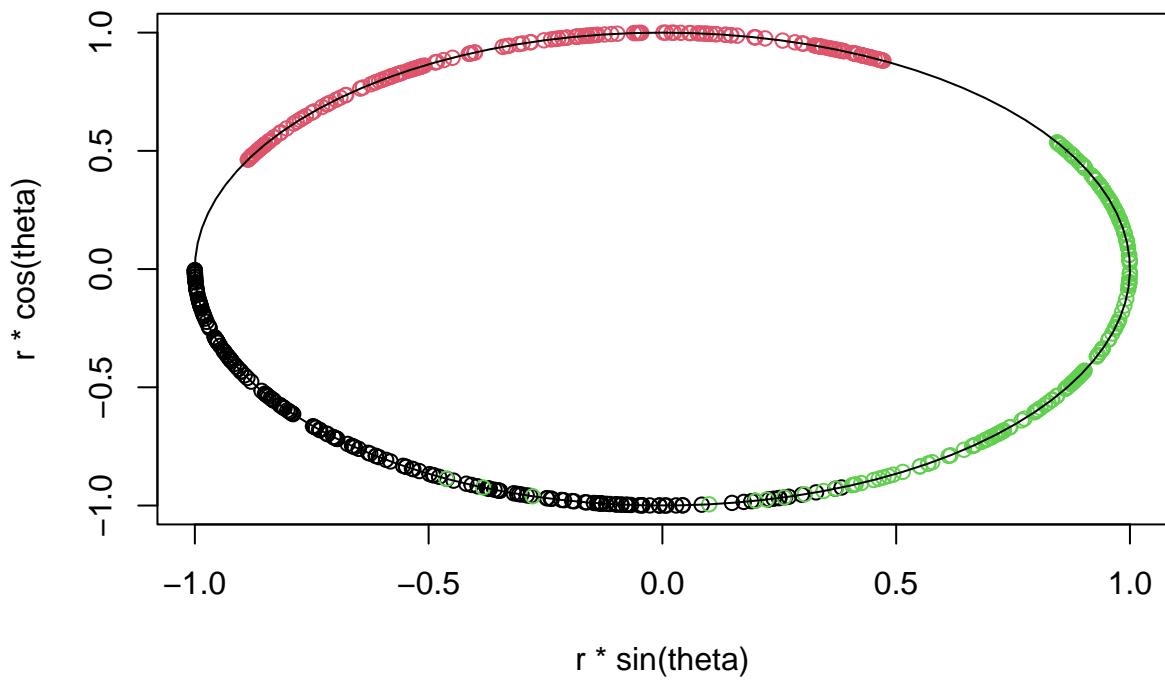


```
## [1] "Iteration 201"
## [1] "Iteration 202"
## [1] "Iteration 203"
## [1] "Iteration 204"
## [1] "Iteration 205"
## [1] "Iteration 205: objective function value is 2.300684"
```

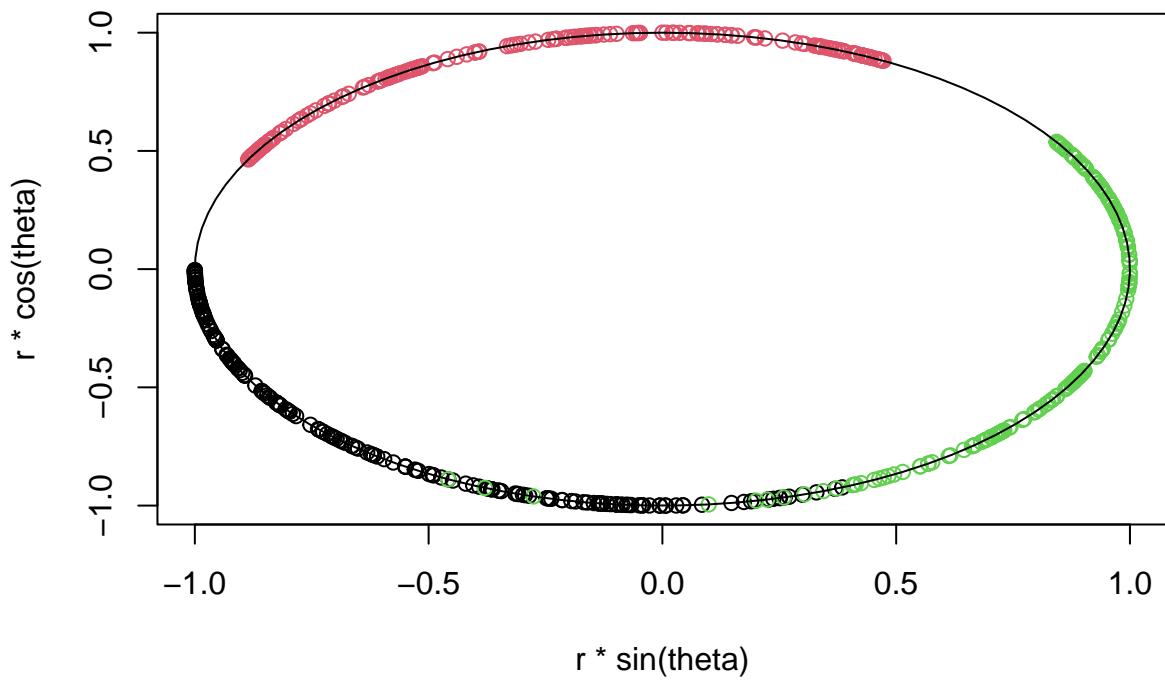


```
## [1] "Iteration 206"
## [1] "Iteration 207"
## [1] "Iteration 208"
## [1] "Iteration 209"
```

```
## [1] "Iteration 210"  
## [1] "Iteration 210: objective function value is 2.300634"
```

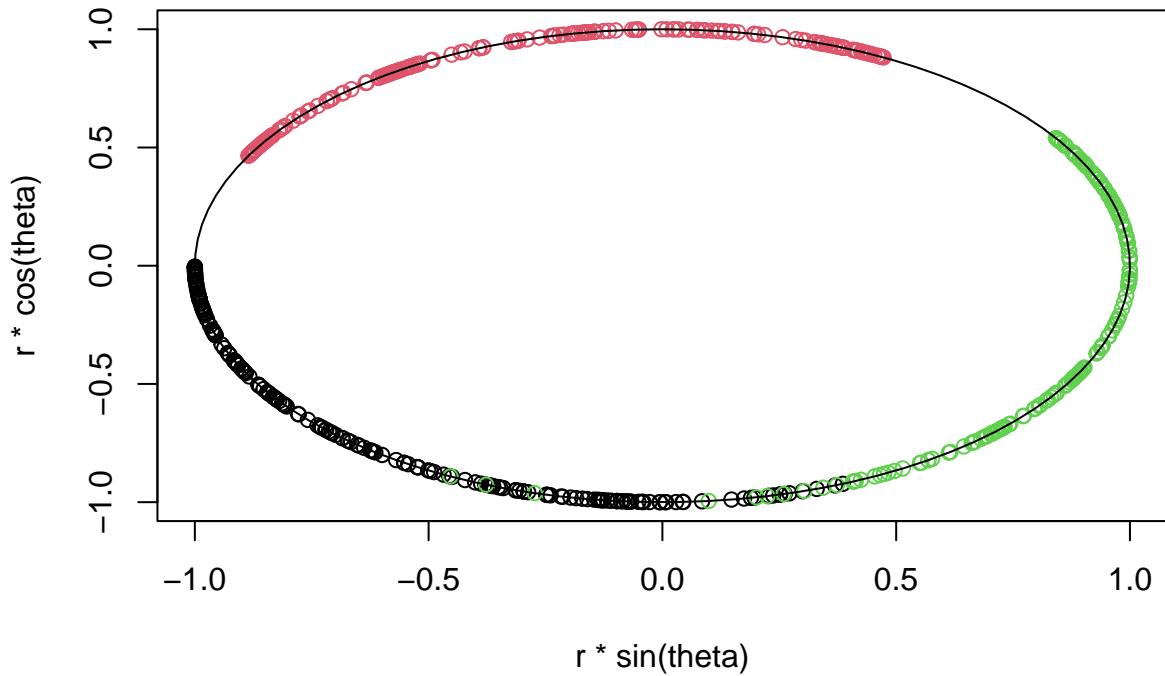


```
## [1] "Iteration 211"  
## [1] "Iteration 212"  
## [1] "Iteration 213"  
## [1] "Iteration 214"  
## [1] "Iteration 215"  
## [1] "Iteration 215: objective function value is 2.300587"
```

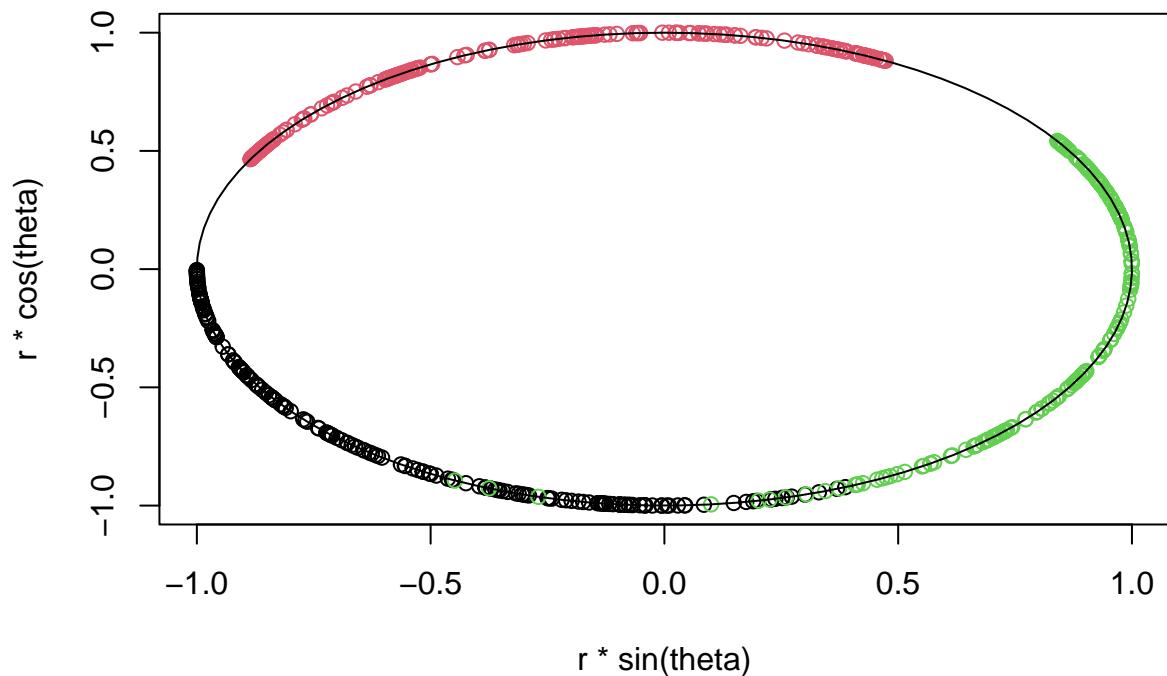


```
## [1] "Iteration 216"
```

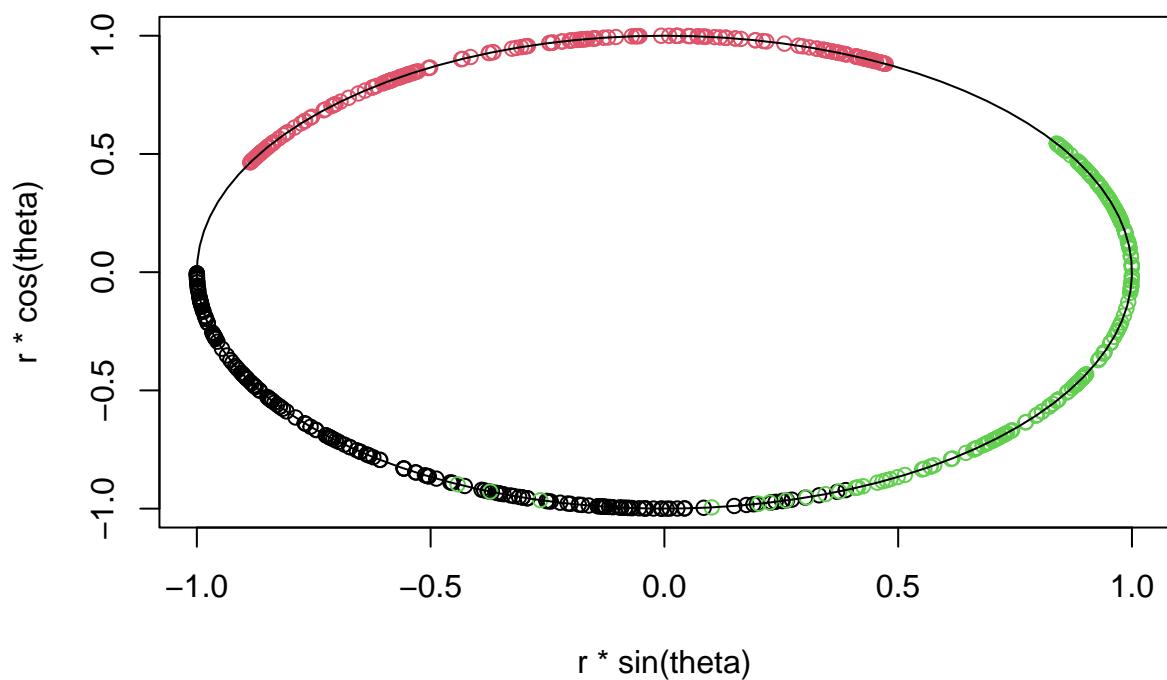
```
## [1] "Iteration 217"
## [1] "Iteration 218"
## [1] "Iteration 219"
## [1] "Iteration 220"
## [1] "Iteration 220: objective function value is 2.300544"
```



```
## [1] "Iteration 221"
## [1] "Iteration 222"
## [1] "Iteration 223"
## [1] "Iteration 224"
## [1] "Iteration 225"
## [1] "Iteration 225: objective function value is 2.300501"
```

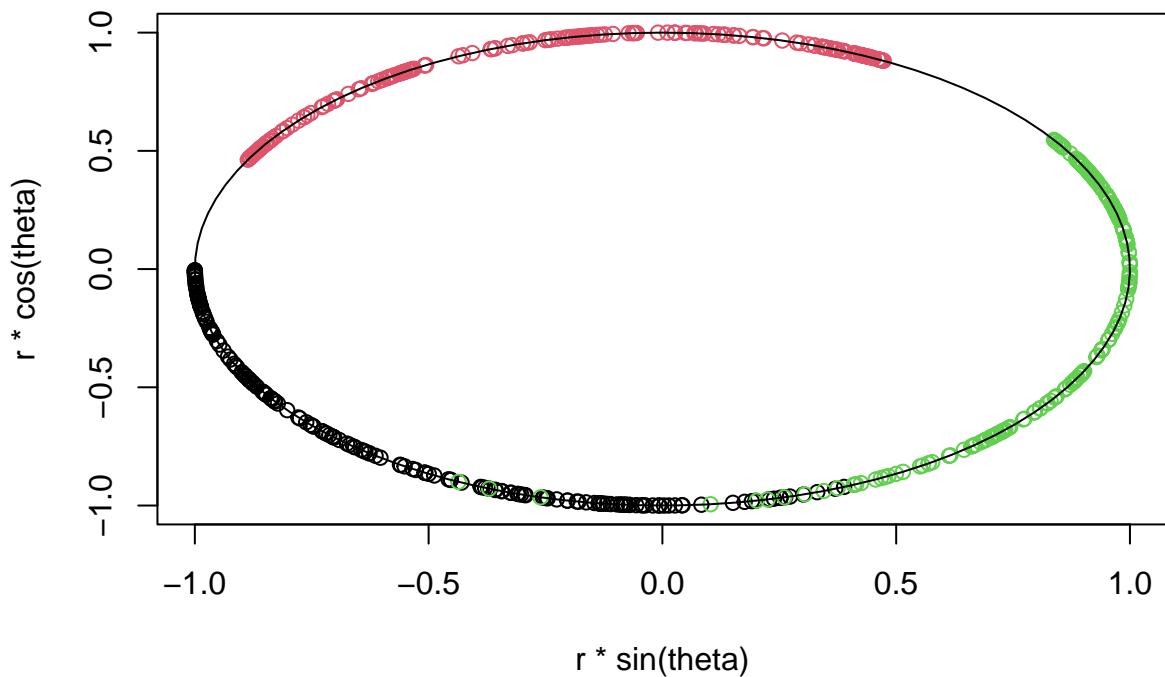


```
## [1] "Iteration 226"
## [1] "Iteration 227"
## [1] "Iteration 228"
## [1] "Iteration 229"
## [1] "Iteration 230"
## [1] "Iteration 230: objective function value is 2.300459"
```

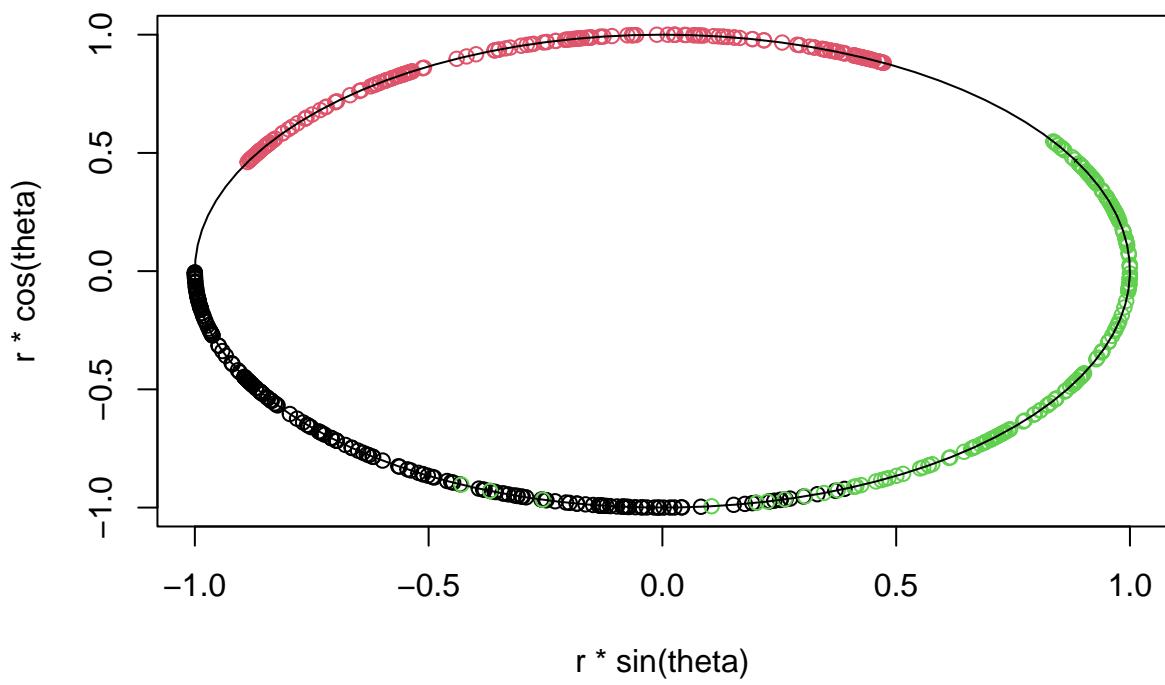


```
## [1] "Iteration 231"
## [1] "Iteration 232"
## [1] "Iteration 233"
## [1] "Iteration 234"
```

```
## [1] "Iteration 235"  
## [1] "Iteration 235: objective function value is 2.300417"
```

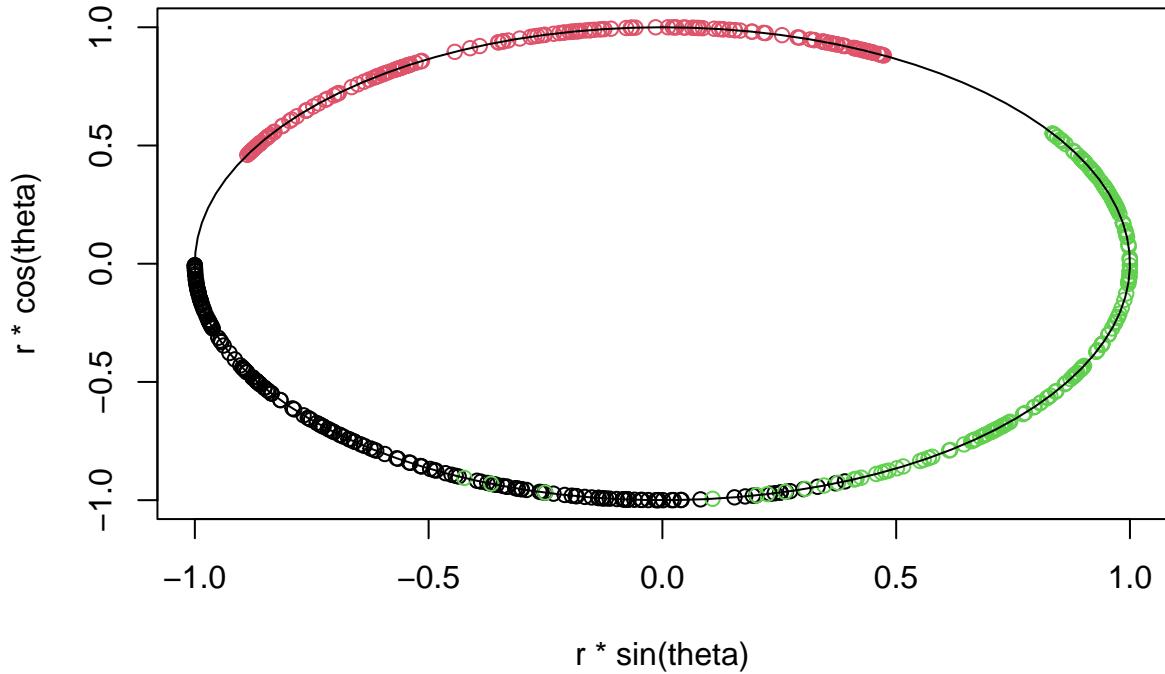


```
## [1] "Iteration 236"  
## [1] "Iteration 237"  
## [1] "Iteration 238"  
## [1] "Iteration 239"  
## [1] "Iteration 240"  
## [1] "Iteration 240: objective function value is 2.300373"
```

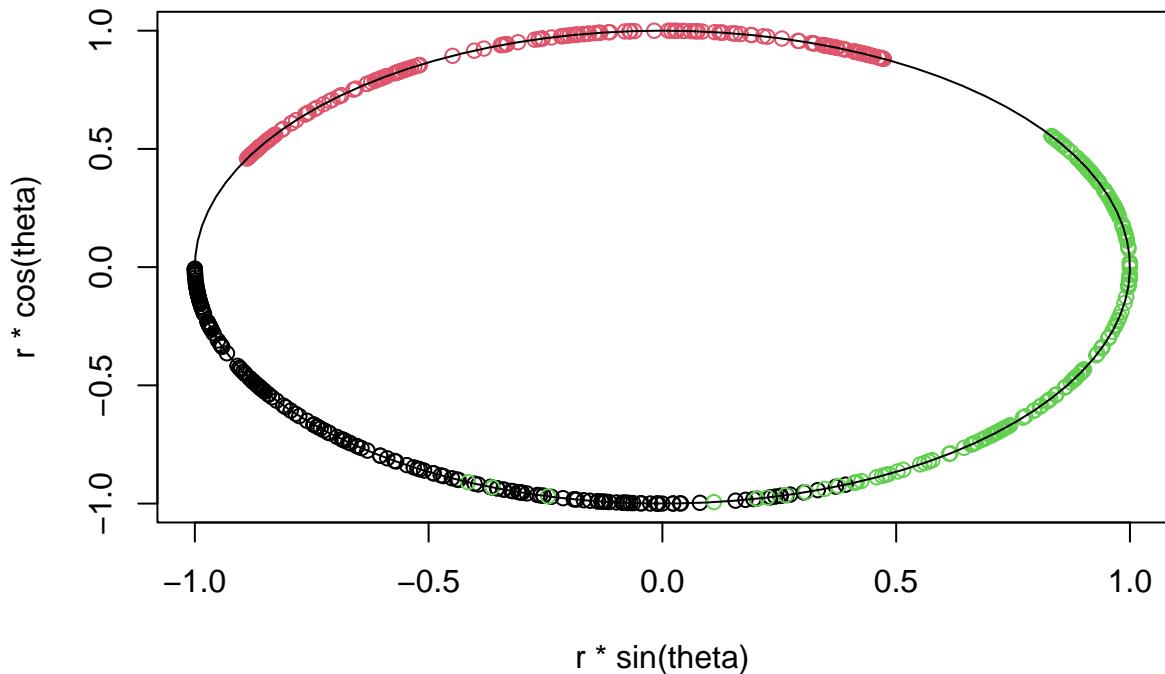


```
## [1] "Iteration 241"
```

```
## [1] "Iteration 242"  
## [1] "Iteration 243"  
## [1] "Iteration 244"  
## [1] "Iteration 245"  
## [1] "Iteration 245: objective function value is 2.300326"
```



```
## [1] "Iteration 246"  
## [1] "Iteration 247"  
## [1] "Iteration 248"  
## [1] "Iteration 249"  
## [1] "Iteration 250"  
## [1] "Iteration 250: objective function value is 2.300265"
```

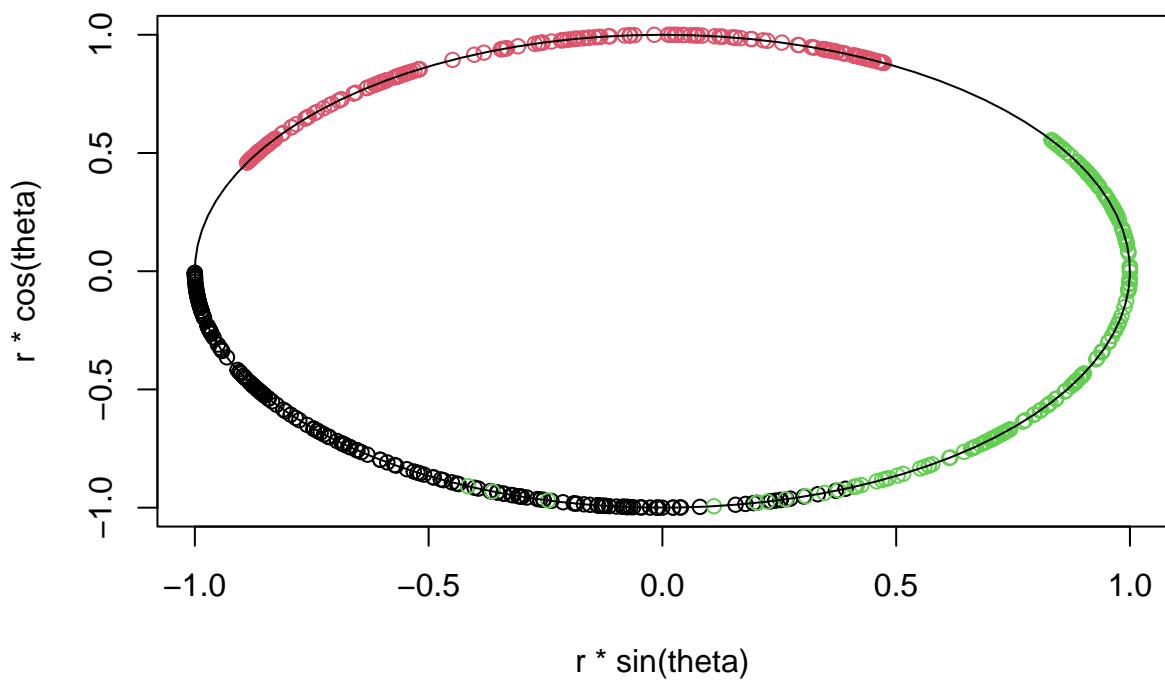


```

Y_rad <- DirStats::to_rad(Y[, , 252])
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=rep(c(1,2,3), each=n/3),
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))

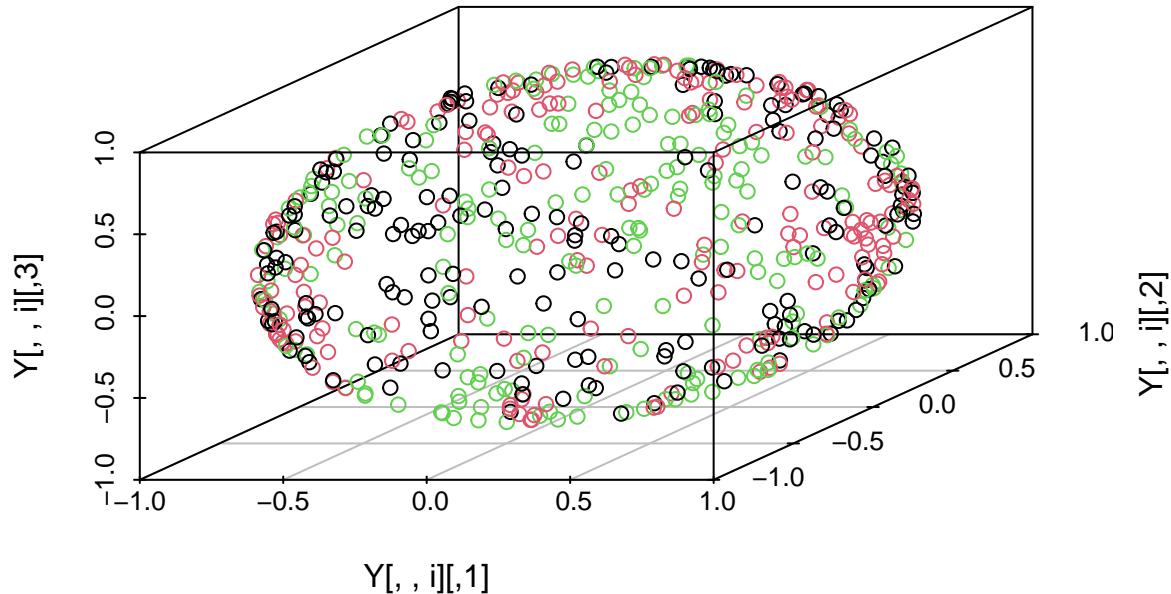
```



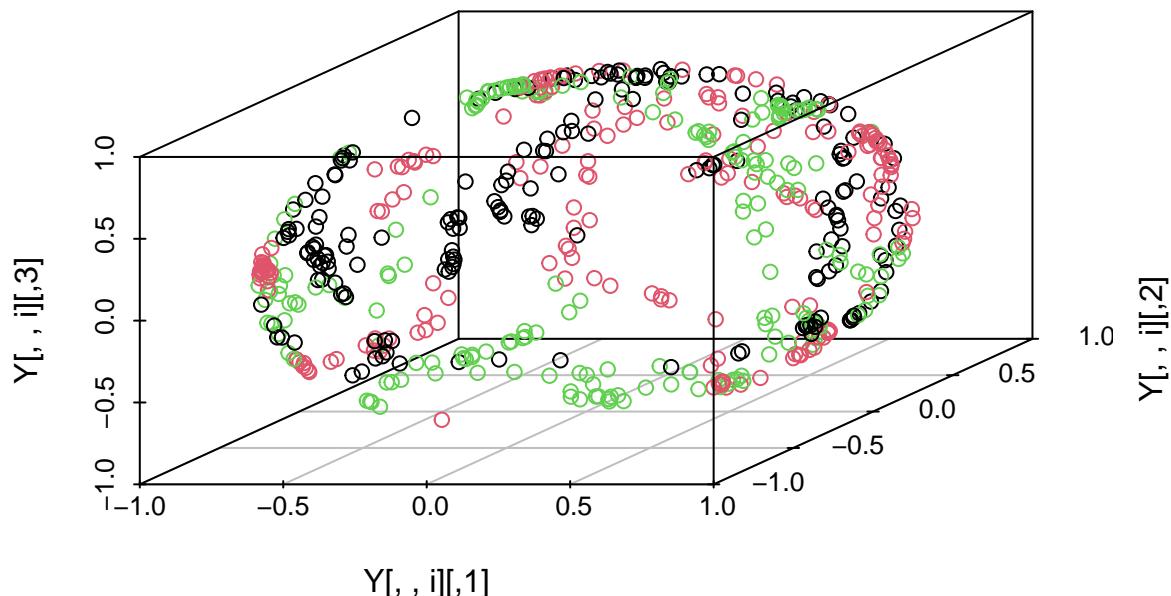
Now we are going to reduce to dimension  $\mathbb{S}^2$  then  $d = 2$  (sphere):

```
Y <- psc_sne(X=x_2, d=2, rho_psc_list = rho_second_perp20, num_iteration=250, eta=75,
               colors=rep(c(1,2,3), each=n/3), visualize_prog = TRUE)
```

```
## [1] "Iteration 1"
## [1] "Iteration 1: objective function value is 18.807525"
```



```
## [1] "Iteration 2"
## [1] "Iteration 3"
## [1] "Iteration 4"
## [1] "Iteration 5"
## [1] "Iteration 5: objective function value is 13.963458"
```

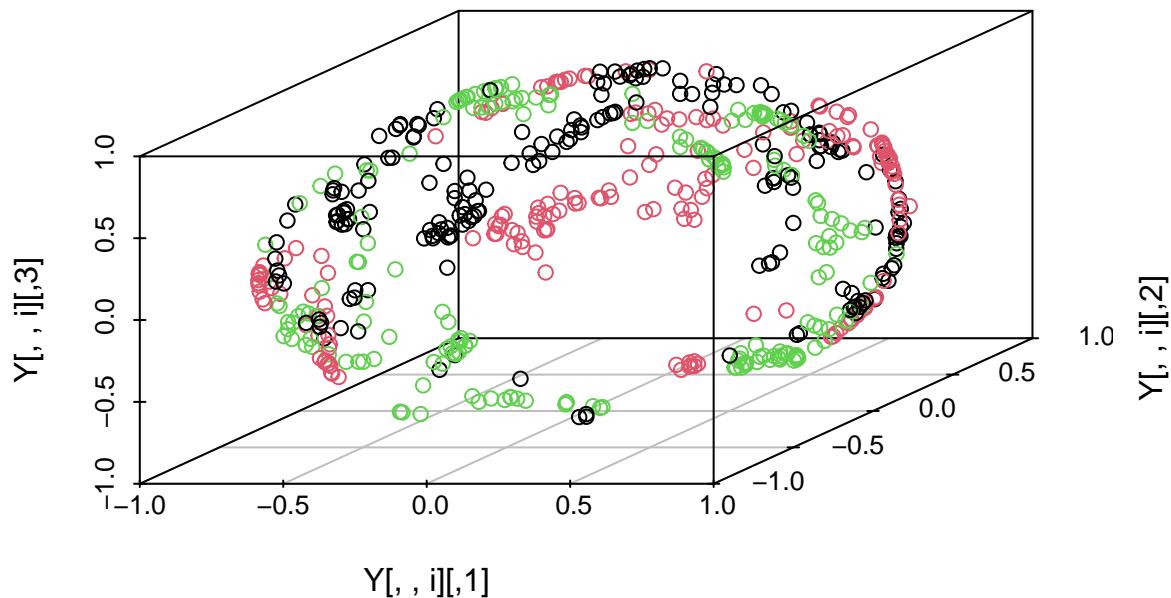


```
## [1] "Iteration 6"
## [1] "Iteration 7"
## [1] "Iteration 8"
```

```

## [1] "Iteration 9"
## [1] "Iteration 10"
## [1] "Iteration 10: objective function value is 13.022304"

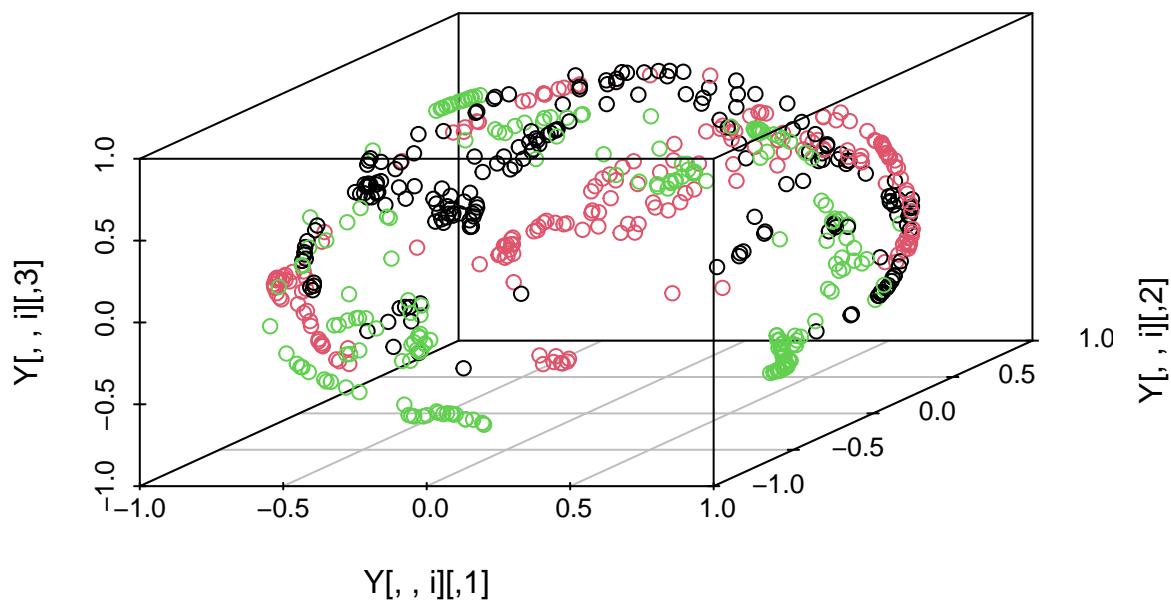
```



```

## [1] "Iteration 11"
## [1] "Iteration 12"
## [1] "Iteration 13"
## [1] "Iteration 14"
## [1] "Iteration 15"
## [1] "Iteration 15: objective function value is 12.661632"

```

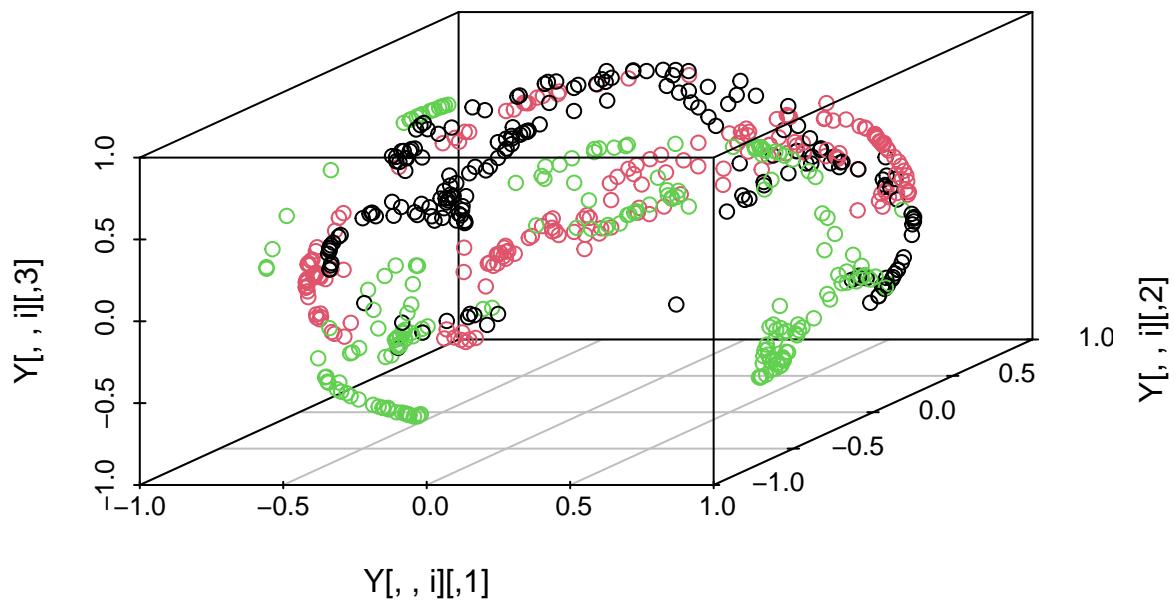


```

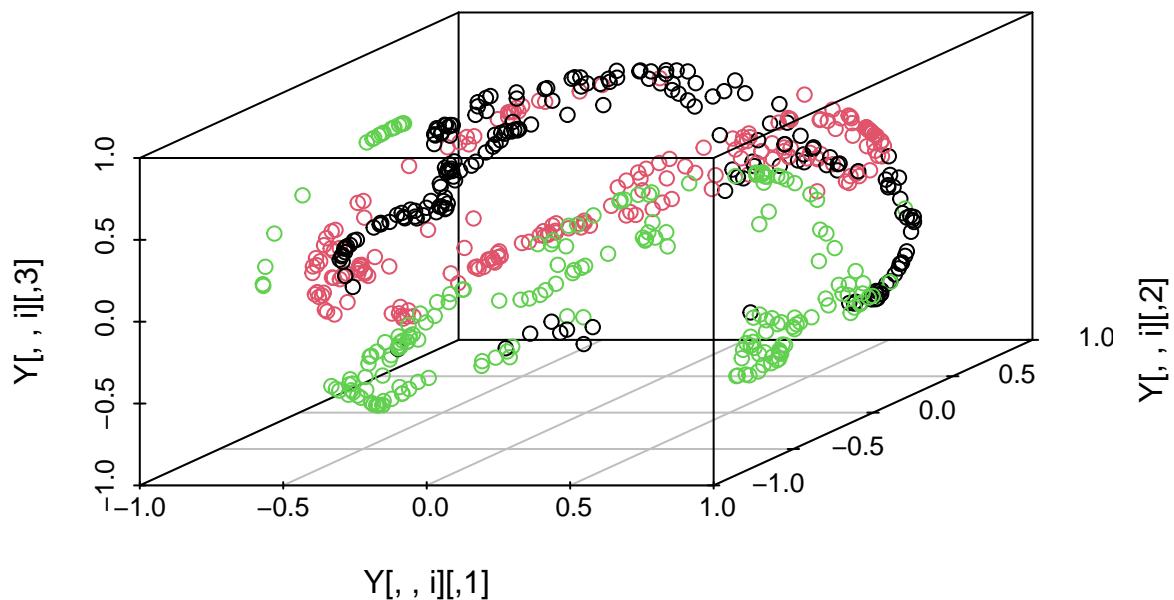
## [1] "Iteration 16"
## [1] "Iteration 17"
## [1] "Iteration 18"
## [1] "Iteration 19"
## [1] "Iteration 20"

```

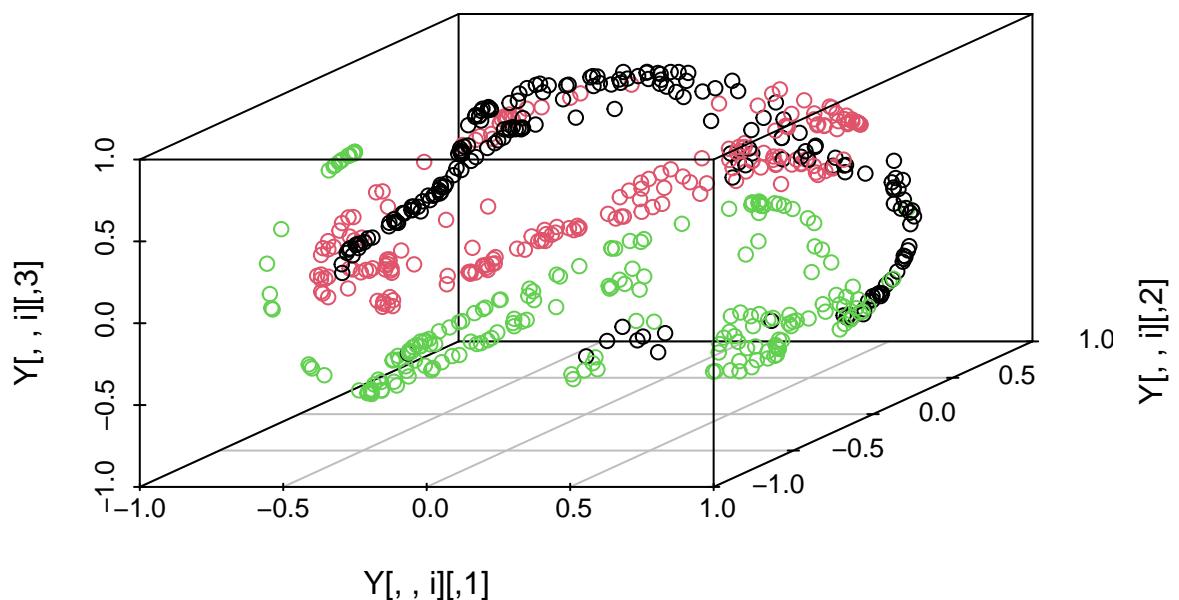
```
## [1] "Iteration 20: objective function value is 12.379957"
```



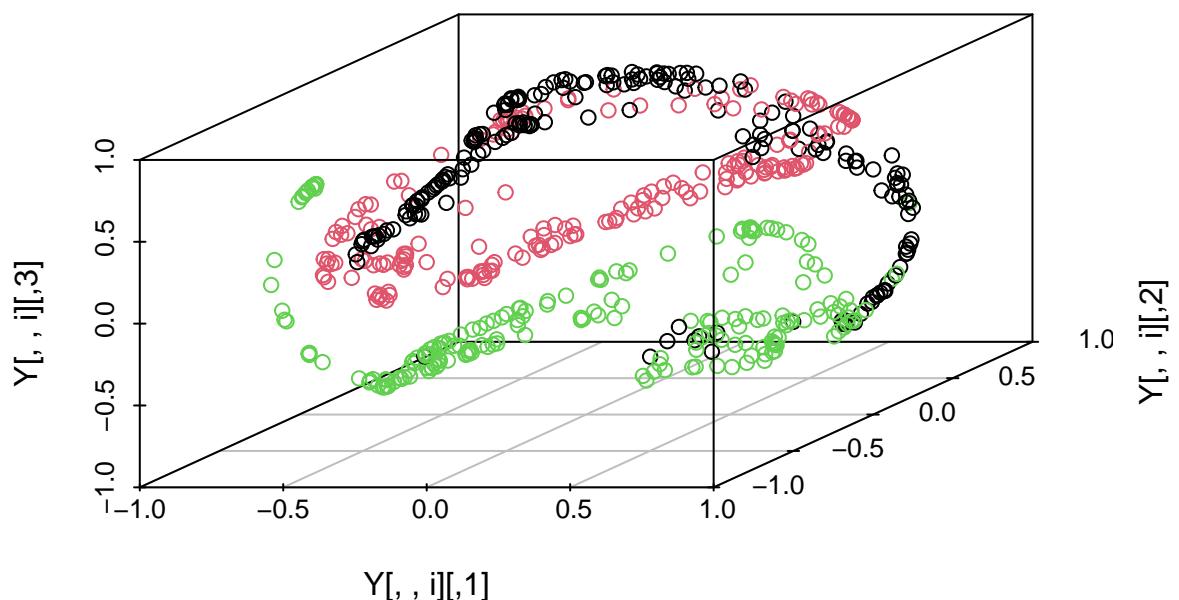
```
## [1] "Iteration 21"  
## [1] "Iteration 22"  
## [1] "Iteration 23"  
## [1] "Iteration 24"  
## [1] "Iteration 25"  
## [1] "Iteration 25: objective function value is 12.122764"
```



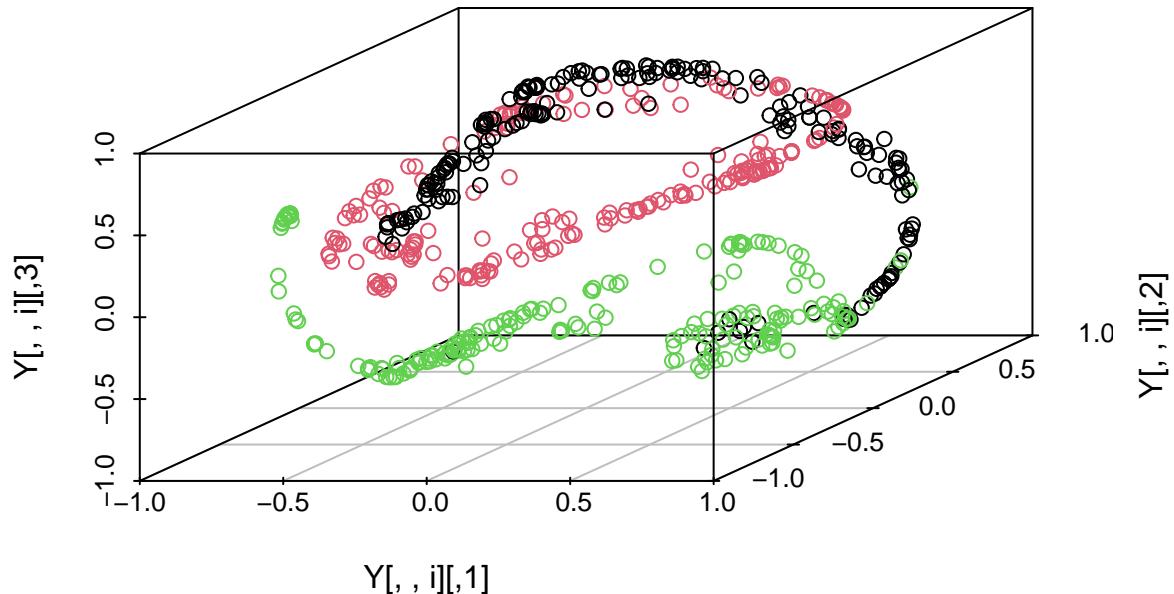
```
## [1] "Iteration 26"  
## [1] "Iteration 27"  
## [1] "Iteration 28"  
## [1] "Iteration 29"  
## [1] "Iteration 30"  
## [1] "Iteration 30: objective function value is 11.991483"
```



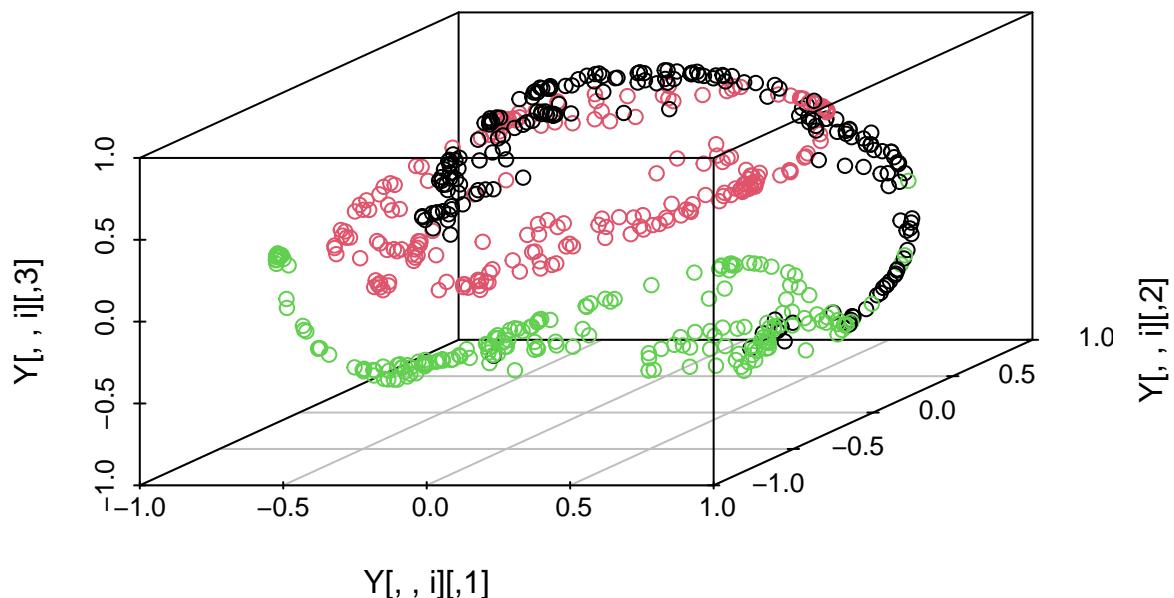
```
## [1] "Iteration 31"
## [1] "Iteration 32"
## [1] "Iteration 33"
## [1] "Iteration 34"
## [1] "Iteration 35"
## [1] "Iteration 35: objective function value is 11.937070"
```



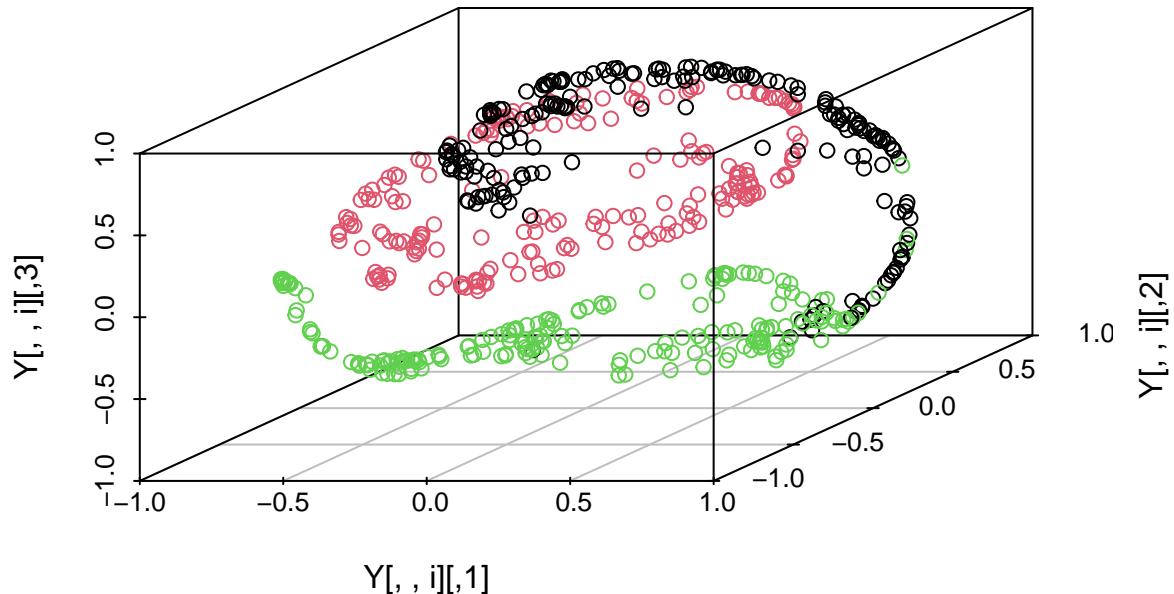
```
## [1] "Iteration 36"
## [1] "Iteration 37"
## [1] "Iteration 38"
## [1] "Iteration 39"
## [1] "Iteration 40"
## [1] "Iteration 40: objective function value is 11.880743"
```



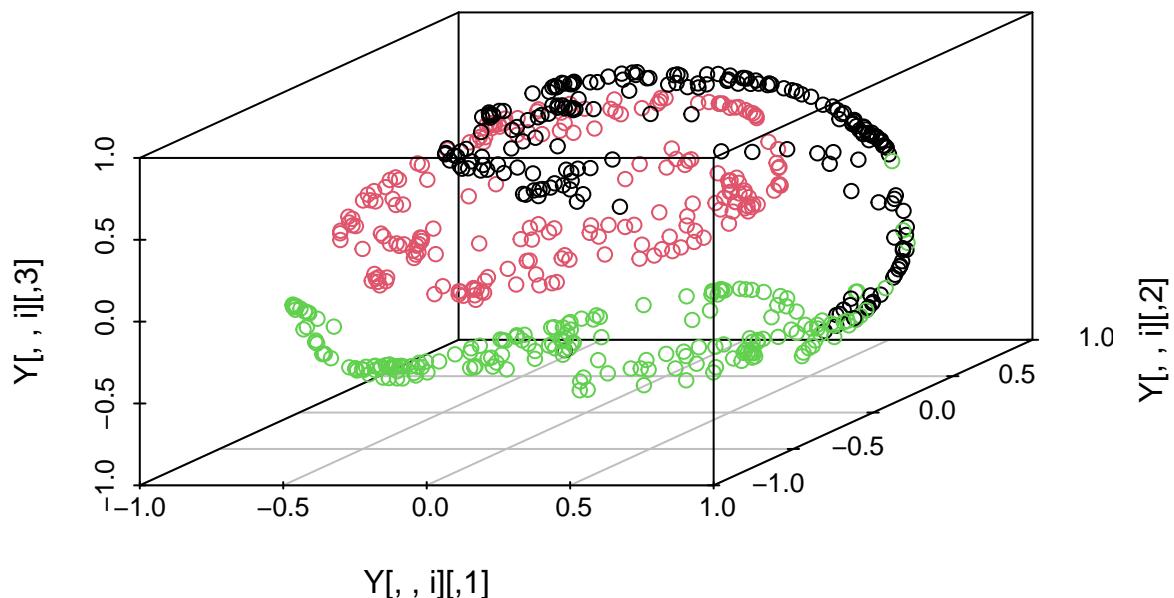
```
## [1] "Iteration 41"
## [1] "Iteration 42"
## [1] "Iteration 43"
## [1] "Iteration 44"
## [1] "Iteration 45"
## [1] "Iteration 45: objective function value is 11.810663"
```



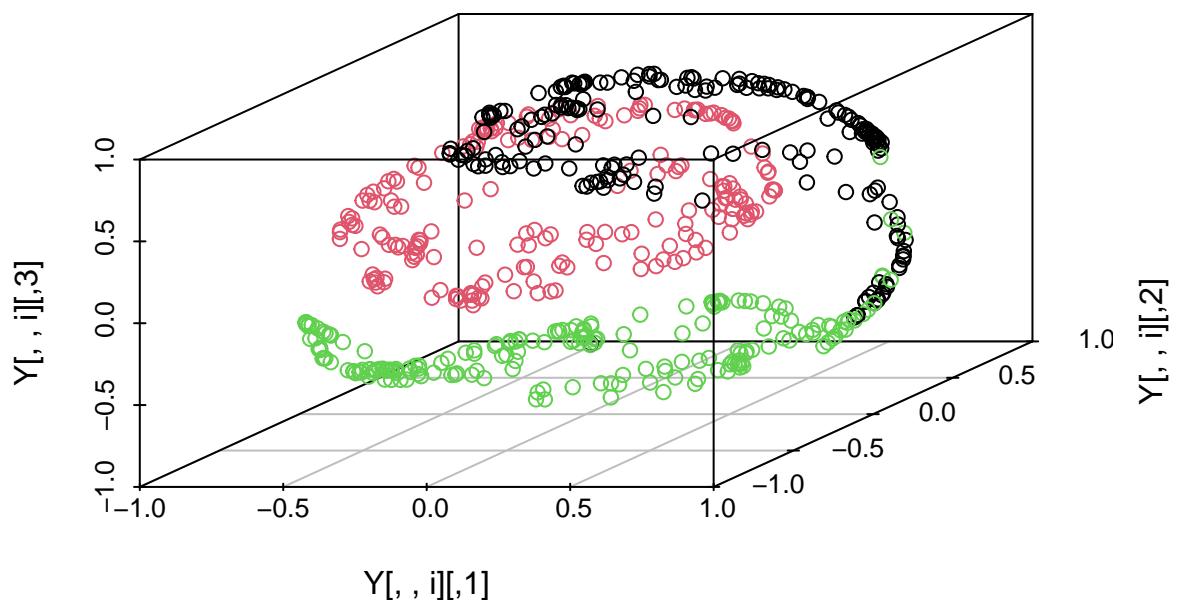
```
## [1] "Iteration 46"
## [1] "Iteration 47"
## [1] "Iteration 48"
## [1] "Iteration 49"
## [1] "Iteration 50"
## [1] "Iteration 50: objective function value is 11.724505"
```



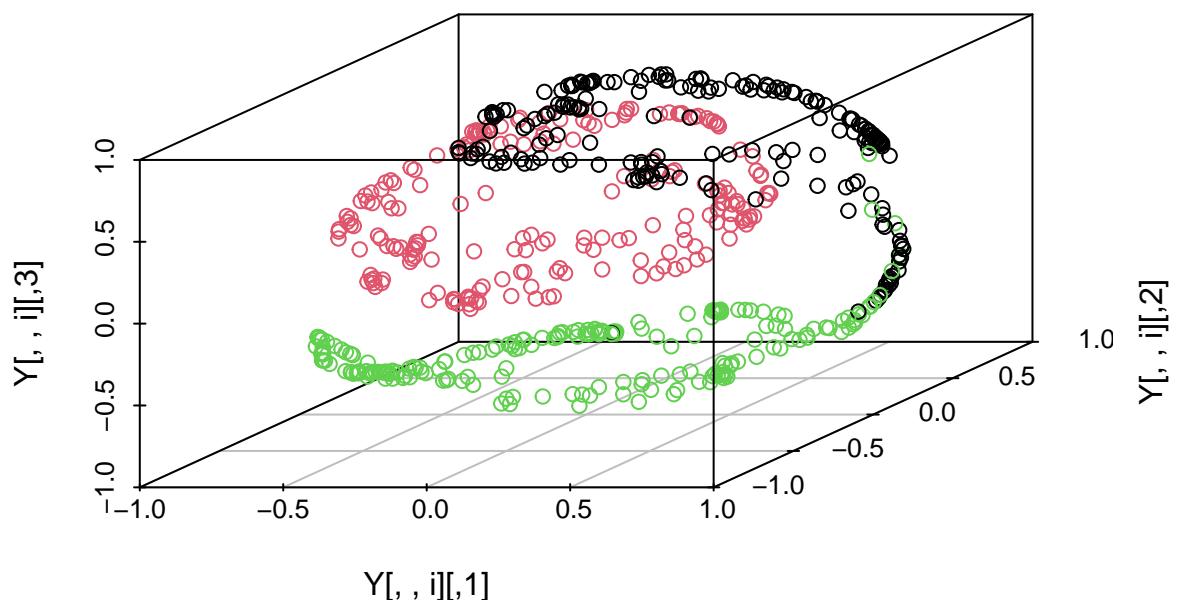
```
## [1] "Iteration 51"
## [1] "Iteration 52"
## [1] "Iteration 53"
## [1] "Iteration 54"
## [1] "Iteration 55"
## [1] "Iteration 55: objective function value is 11.660511"
```



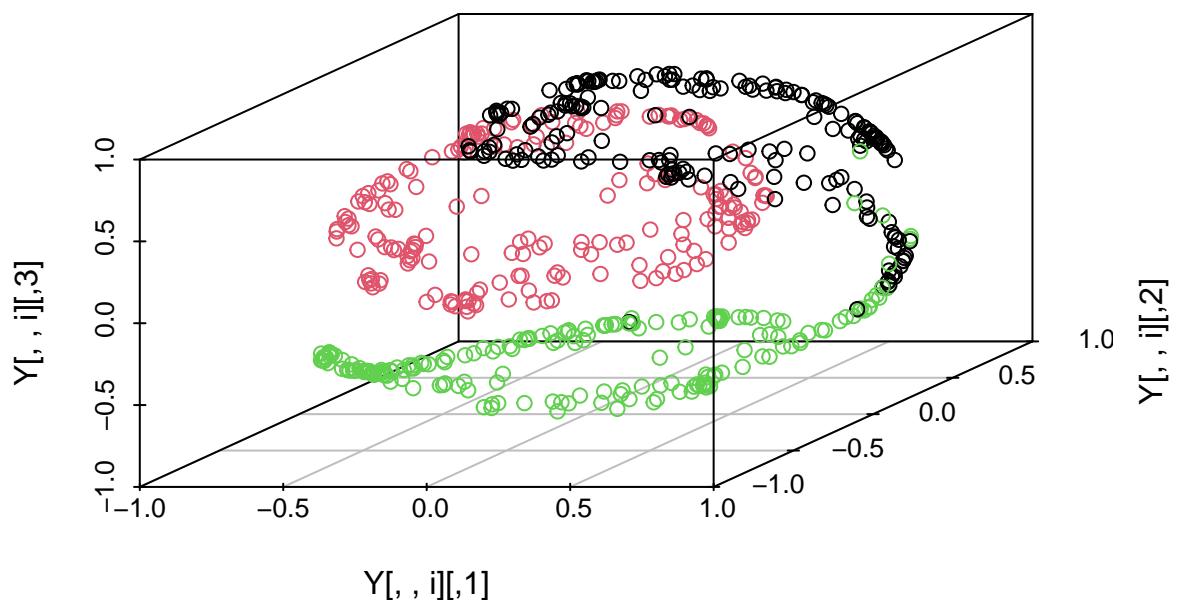
```
## [1] "Iteration 56"
## [1] "Iteration 57"
## [1] "Iteration 58"
## [1] "Iteration 59"
## [1] "Iteration 60"
## [1] "Iteration 60: objective function value is 11.625802"
```



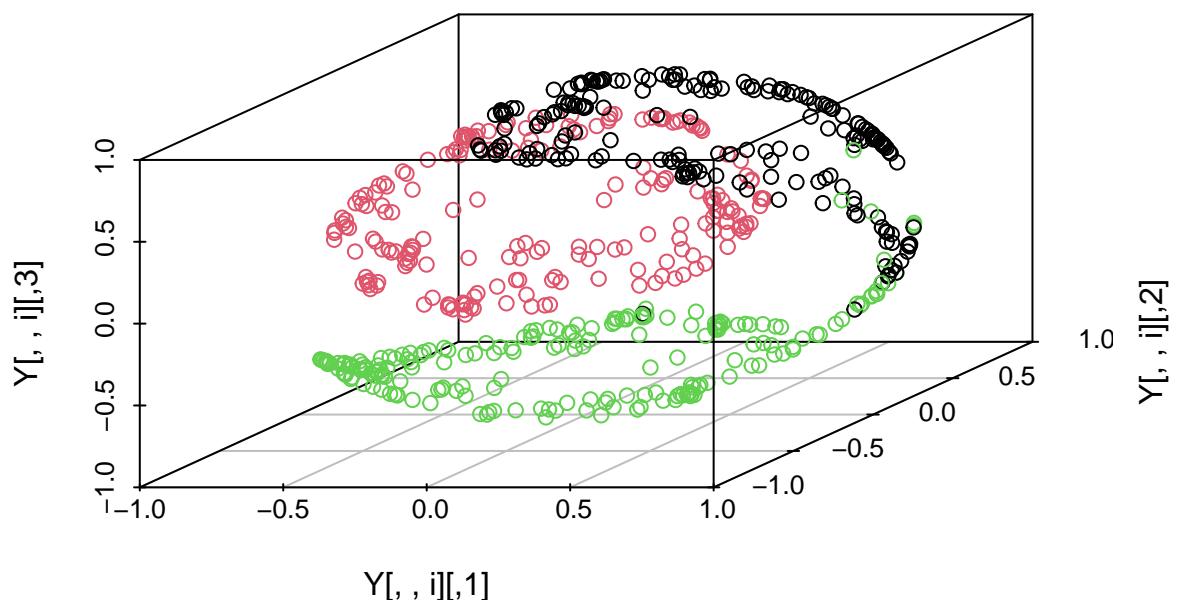
```
## [1] "Iteration 61"
## [1] "Iteration 62"
## [1] "Iteration 63"
## [1] "Iteration 64"
## [1] "Iteration 65"
## [1] "Iteration 65: objective function value is 11.600875"
```



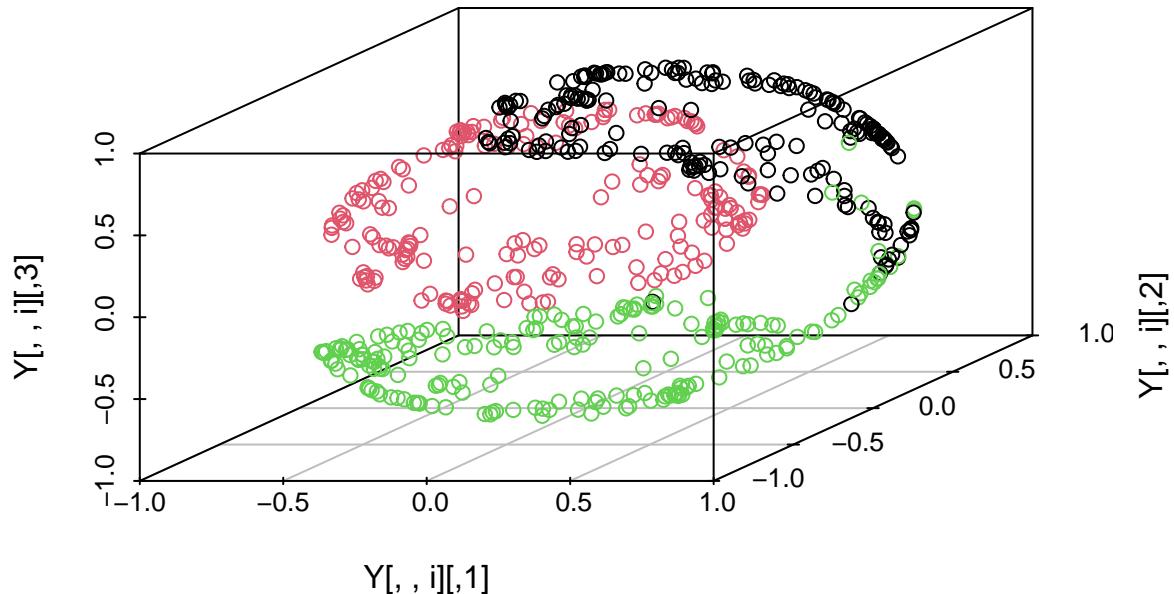
```
## [1] "Iteration 66"
## [1] "Iteration 67"
## [1] "Iteration 68"
## [1] "Iteration 69"
## [1] "Iteration 70"
## [1] "Iteration 70: objective function value is 11.575669"
```



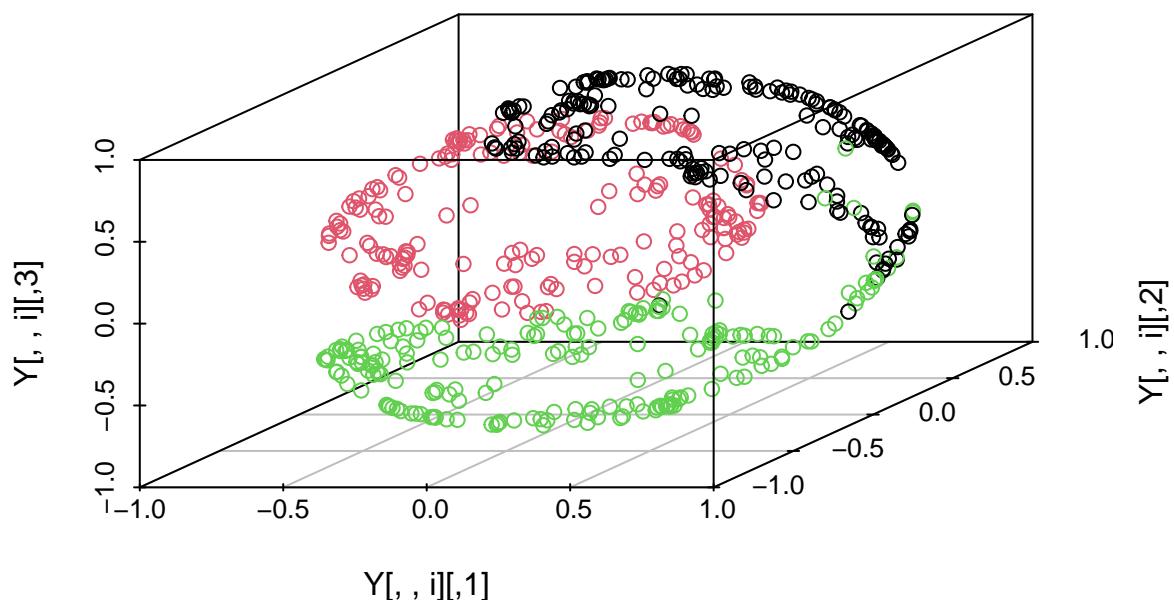
```
## [1] "Iteration 71"
## [1] "Iteration 72"
## [1] "Iteration 73"
## [1] "Iteration 74"
## [1] "Iteration 75"
## [1] "Iteration 75: objective function value is 11.533247"
```



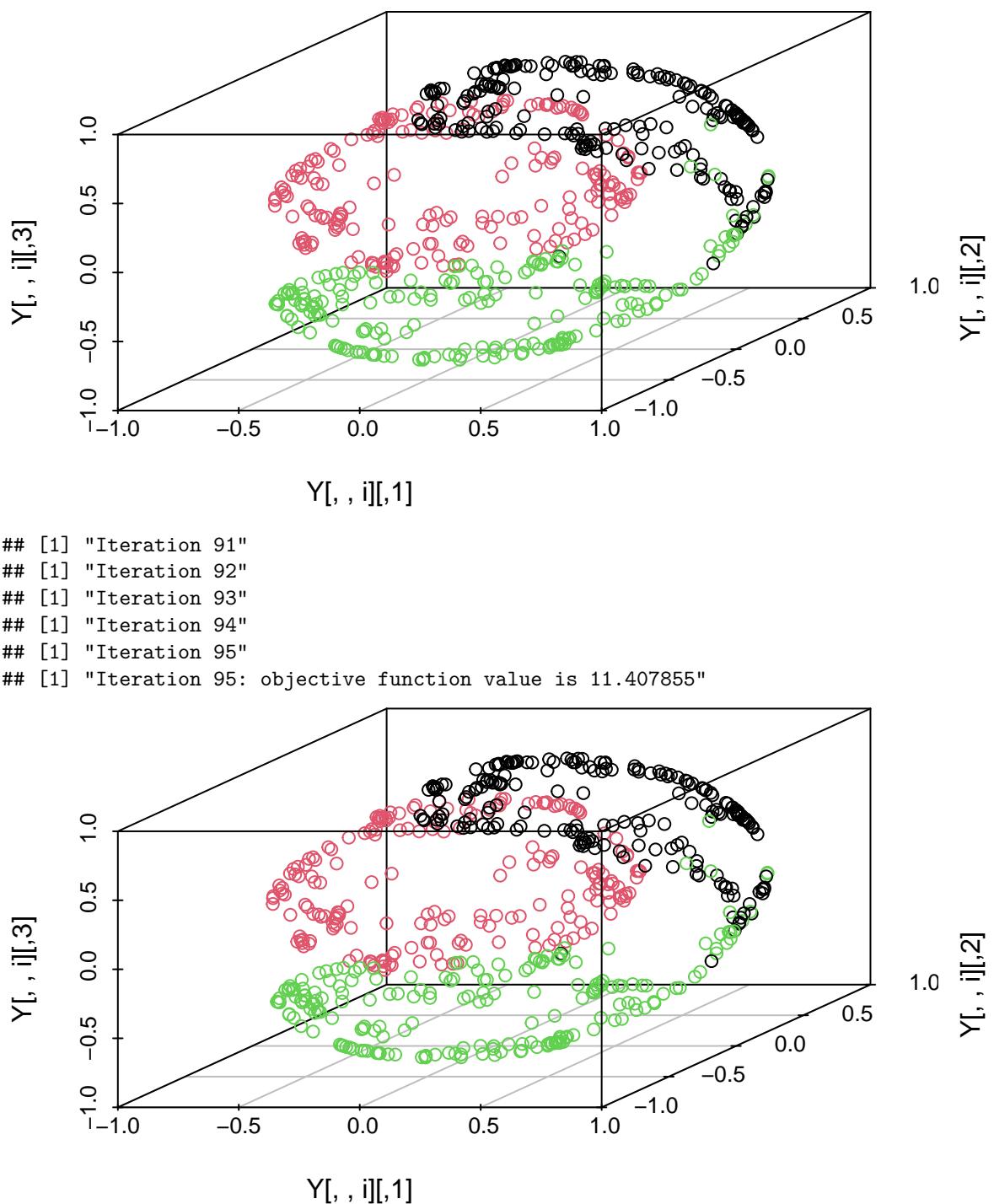
```
## [1] "Iteration 76"
## [1] "Iteration 77"
## [1] "Iteration 78"
## [1] "Iteration 79"
## [1] "Iteration 80"
## [1] "Iteration 80: objective function value is 11.479104"
```

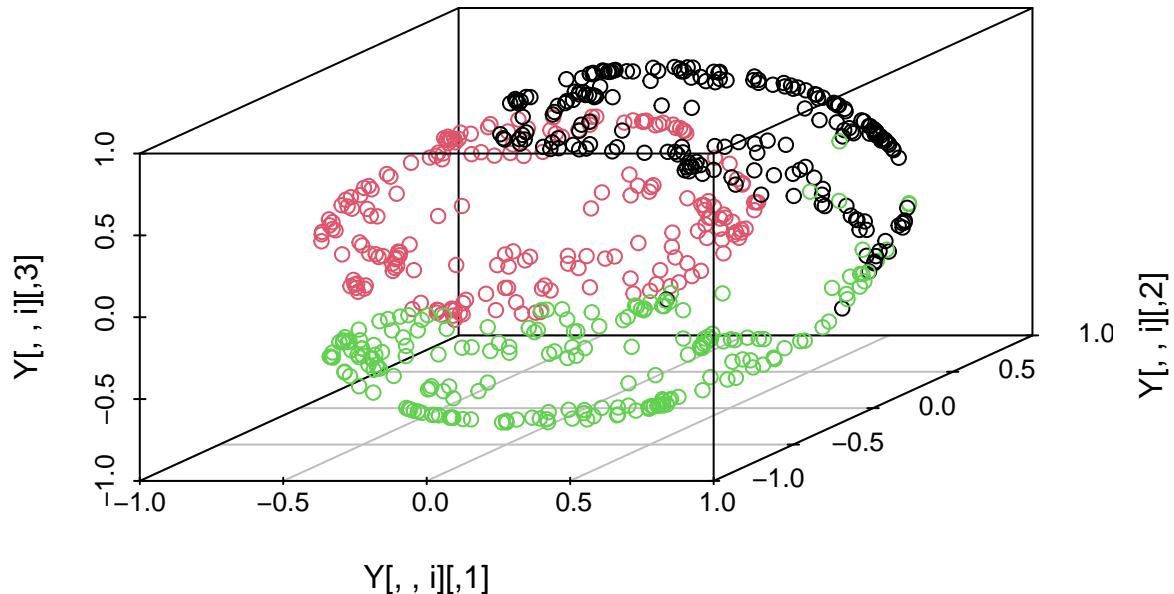


```
## [1] "Iteration 81"
## [1] "Iteration 82"
## [1] "Iteration 83"
## [1] "Iteration 84"
## [1] "Iteration 85"
## [1] "Iteration 85: objective function value is 11.442991"
```

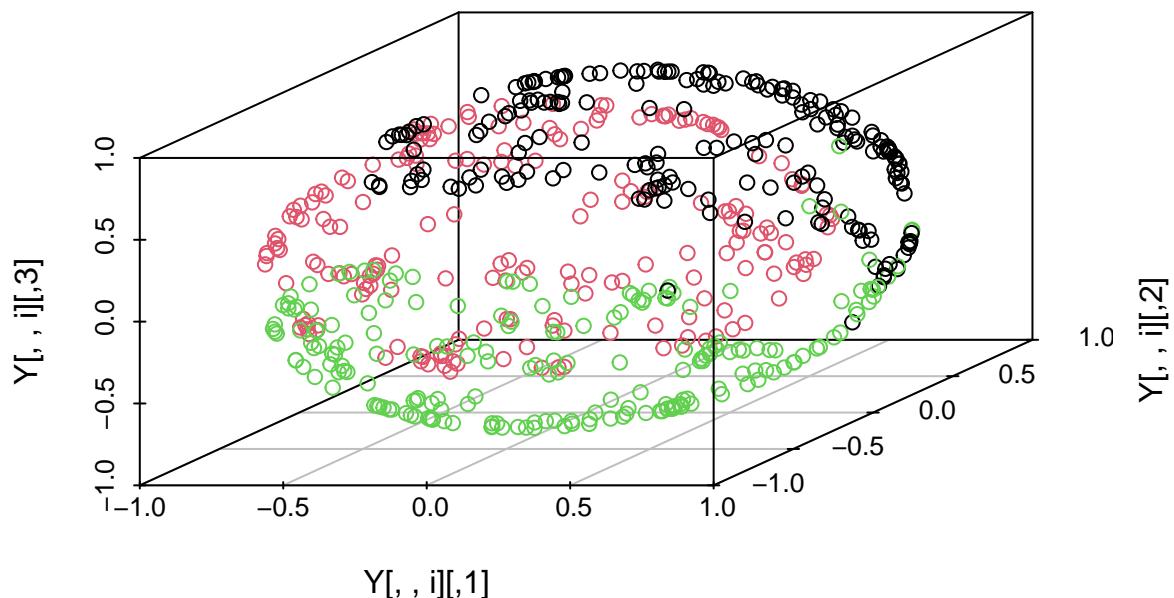


```
## [1] "Iteration 86"
## [1] "Iteration 87"
## [1] "Iteration 88"
## [1] "Iteration 89"
## [1] "Iteration 90"
## [1] "Iteration 90: objective function value is 11.421820"
```

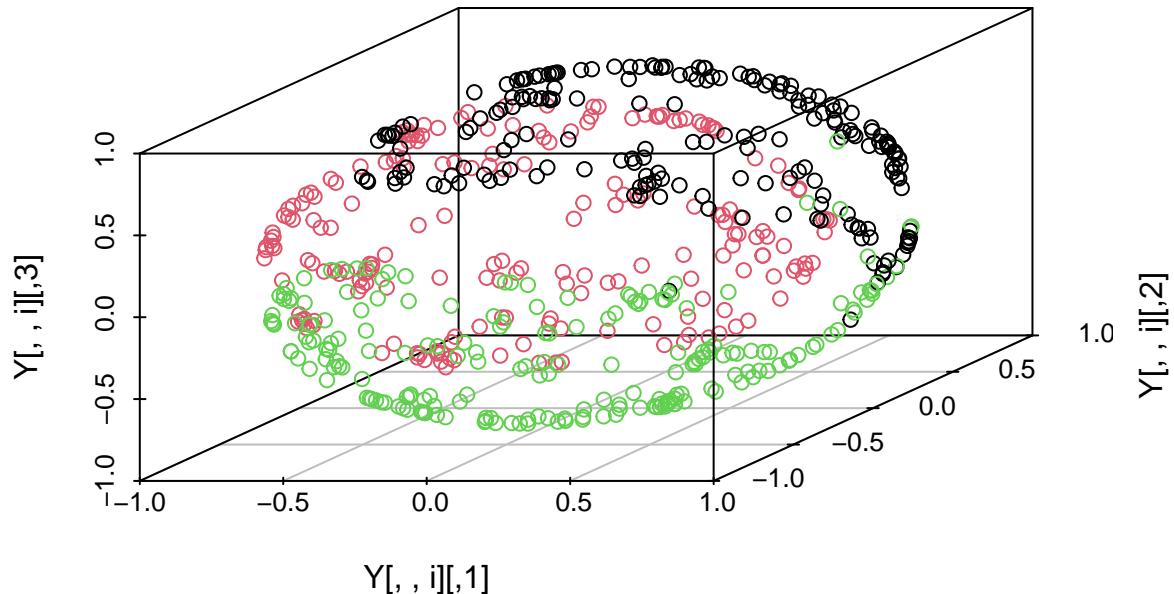




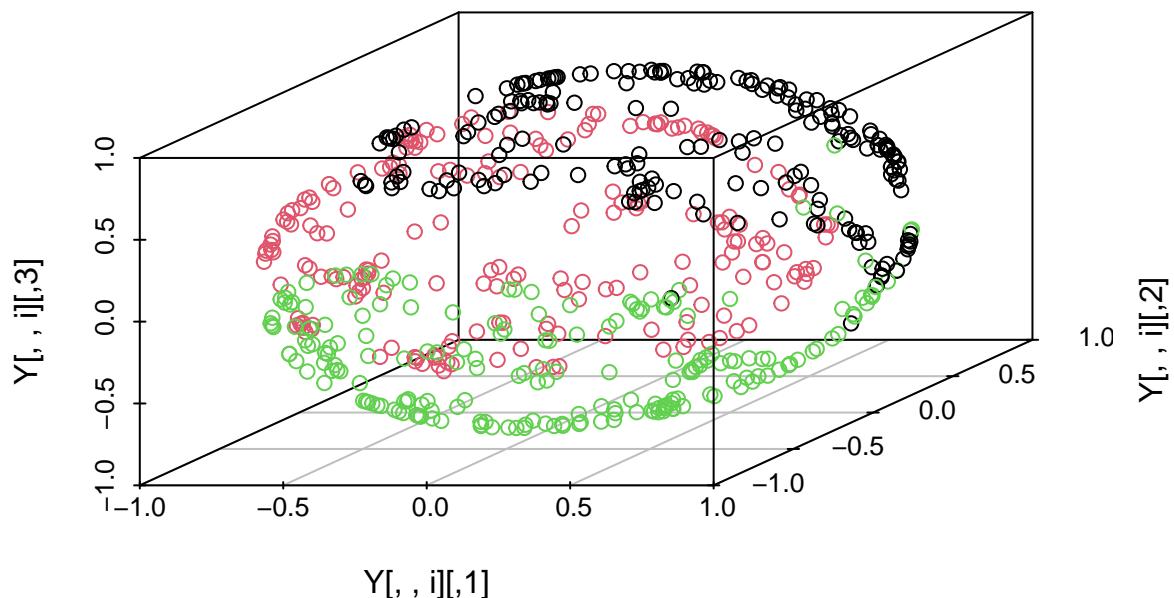
```
## [1] "Iteration 101"
## [1] "Iteration 102"
## [1] "Iteration 103"
## [1] "Iteration 104"
## [1] "Iteration 105"
## [1] "Iteration 105: objective function value is 1.354701"
```



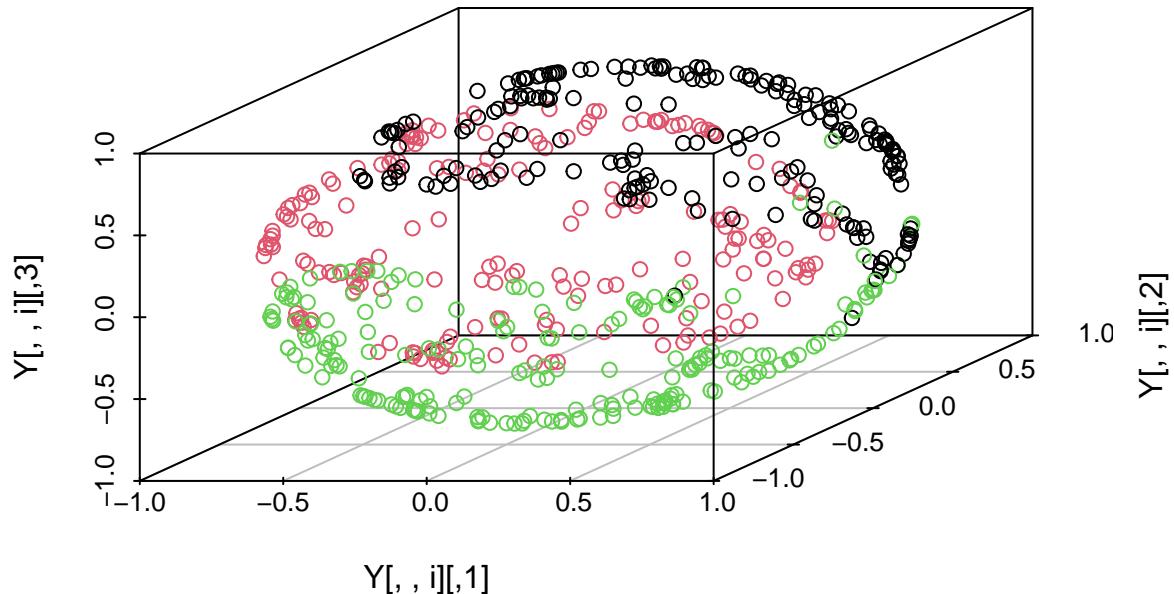
```
## [1] "Iteration 106"
## [1] "Iteration 107"
## [1] "Iteration 108"
## [1] "Iteration 109"
## [1] "Iteration 110"
## [1] "Iteration 110: objective function value is 1.351711"
```



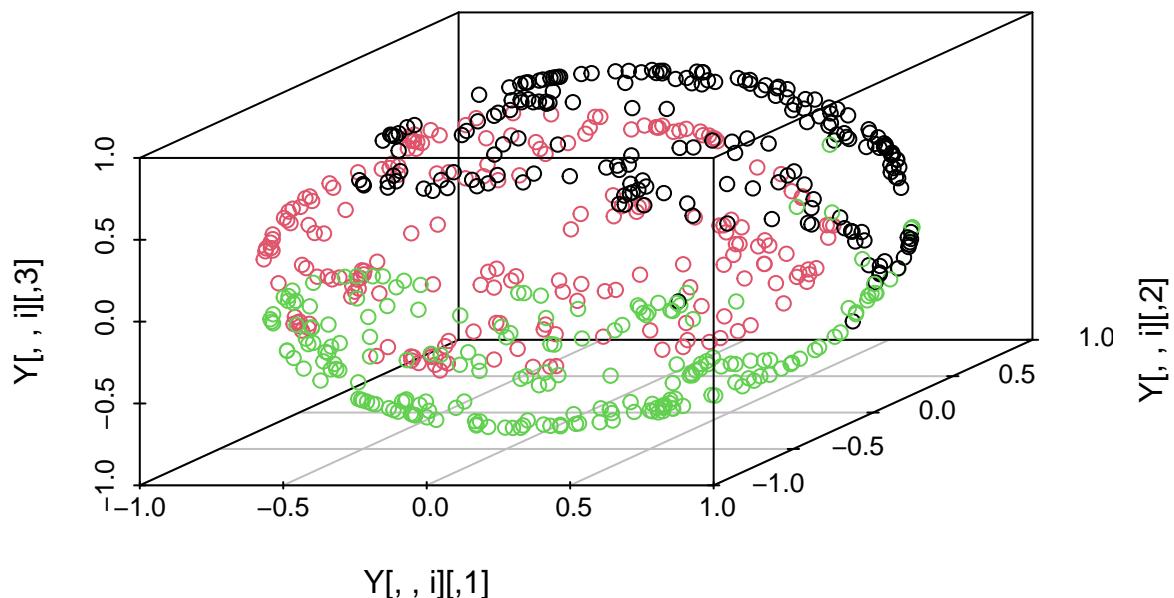
```
## [1] "Iteration 111"
## [1] "Iteration 112"
## [1] "Iteration 113"
## [1] "Iteration 114"
## [1] "Iteration 115"
## [1] "Iteration 115: objective function value is 1.351172"
```



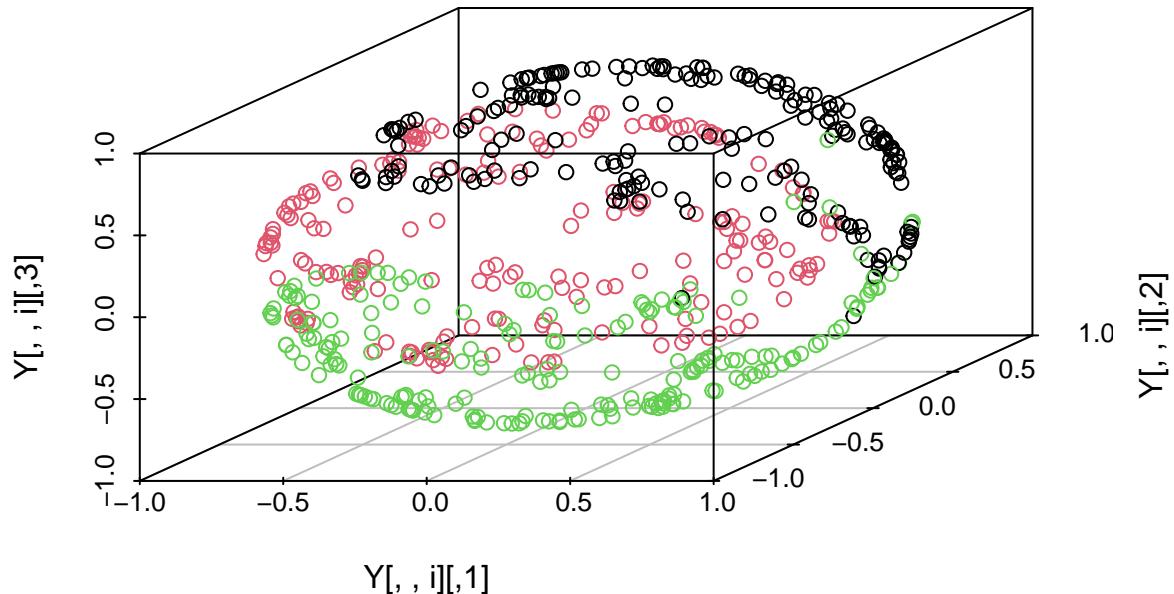
```
## [1] "Iteration 116"
## [1] "Iteration 117"
## [1] "Iteration 118"
## [1] "Iteration 119"
## [1] "Iteration 120"
## [1] "Iteration 120: objective function value is 1.350872"
```



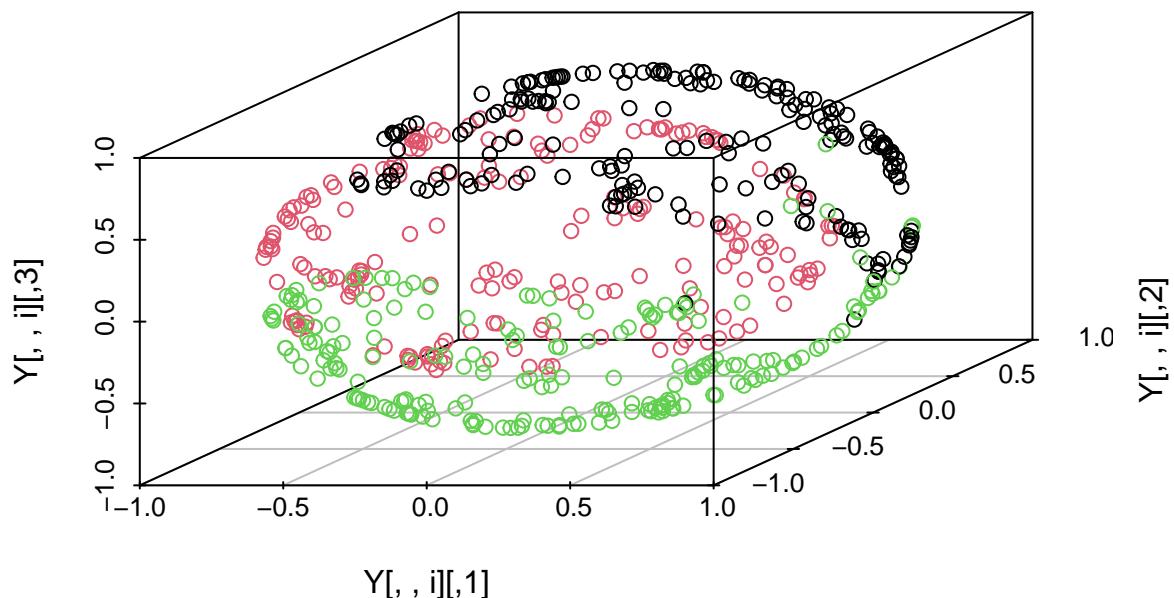
```
## [1] "Iteration 121"
## [1] "Iteration 122"
## [1] "Iteration 123"
## [1] "Iteration 124"
## [1] "Iteration 125"
## [1] "Iteration 125: objective function value is 1.350652"
```



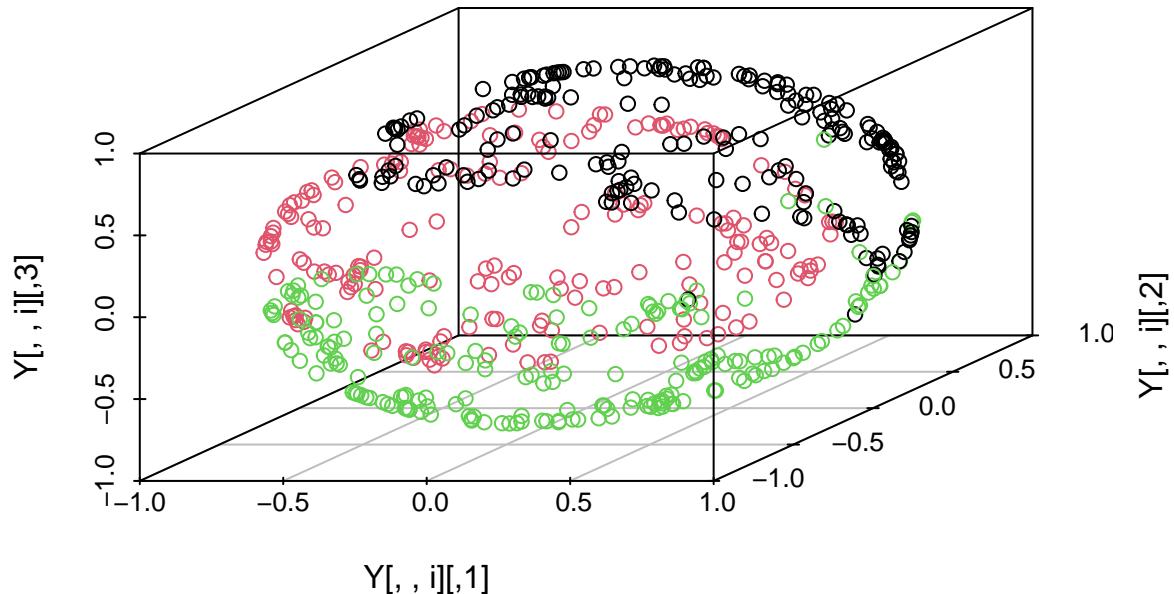
```
## [1] "Iteration 126"
## [1] "Iteration 127"
## [1] "Iteration 128"
## [1] "Iteration 129"
## [1] "Iteration 130"
## [1] "Iteration 130: objective function value is 1.350479"
```



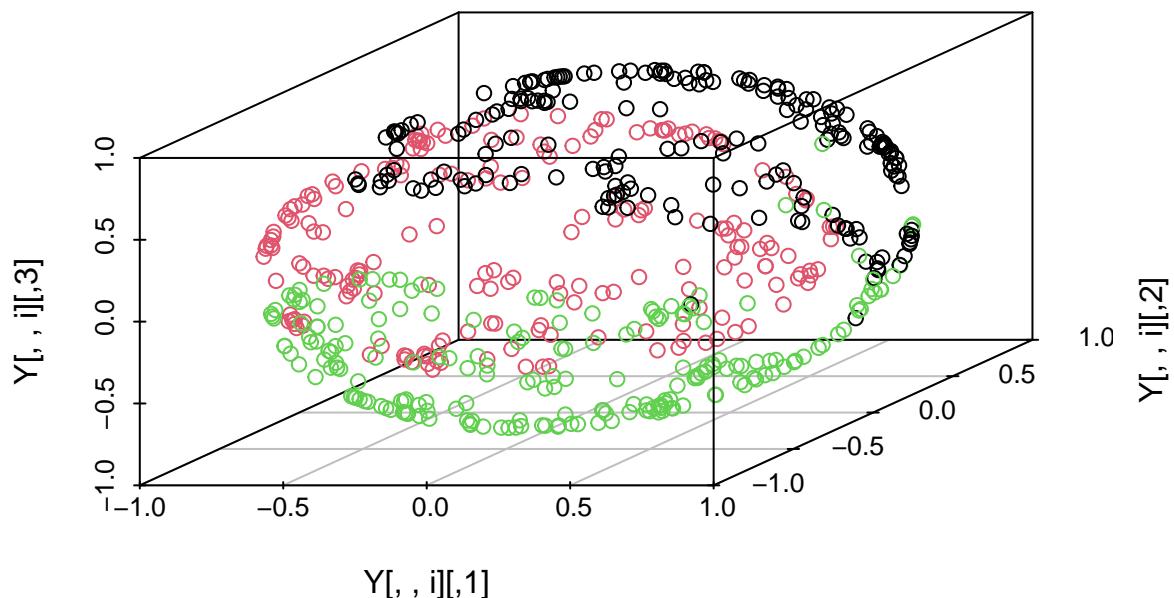
```
## [1] "Iteration 131"
## [1] "Iteration 132"
## [1] "Iteration 133"
## [1] "Iteration 134"
## [1] "Iteration 135"
## [1] "Iteration 135: objective function value is 1.350340"
```



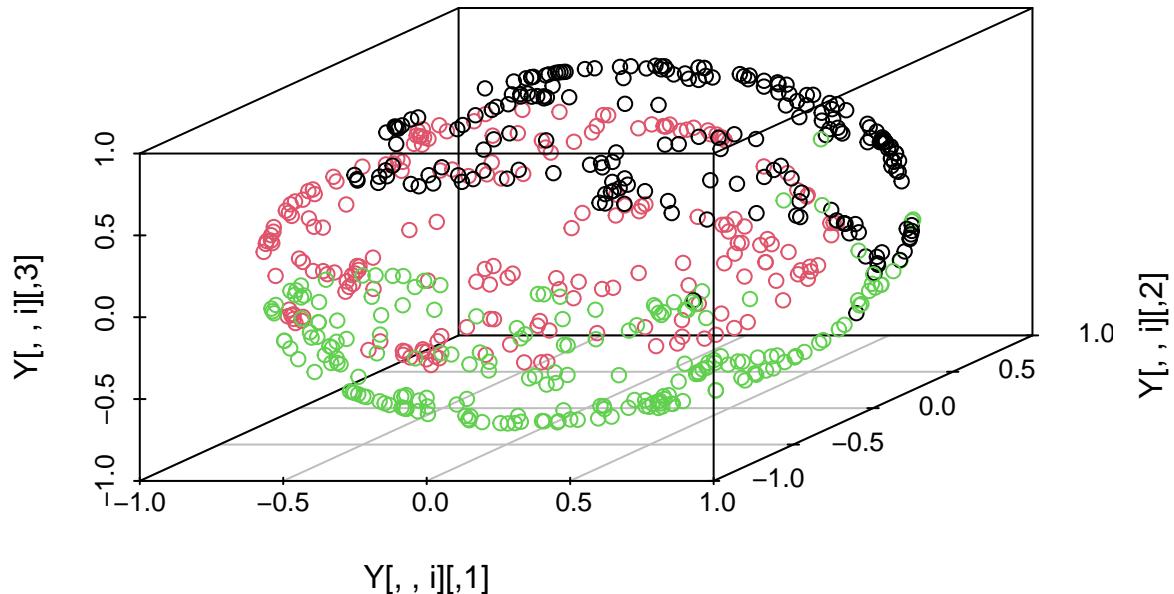
```
## [1] "Iteration 136"
## [1] "Iteration 137"
## [1] "Iteration 138"
## [1] "Iteration 139"
## [1] "Iteration 140"
## [1] "Iteration 140: objective function value is 1.350226"
```



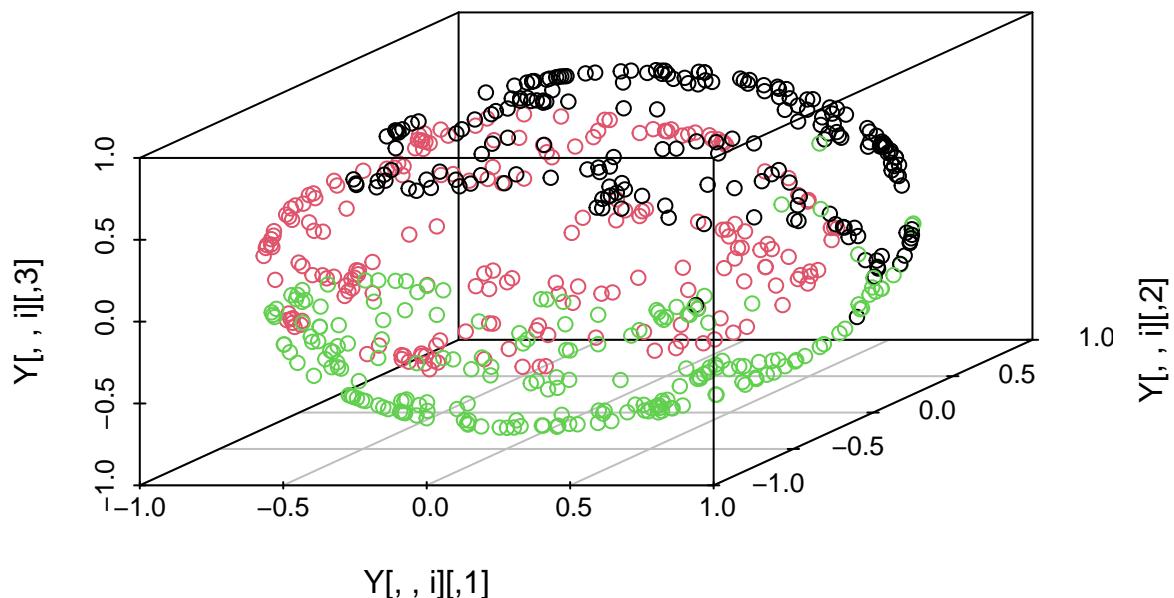
```
## [1] "Iteration 141"
## [1] "Iteration 142"
## [1] "Iteration 143"
## [1] "Iteration 144"
## [1] "Iteration 145"
## [1] "Iteration 145: objective function value is 1.350132"
```



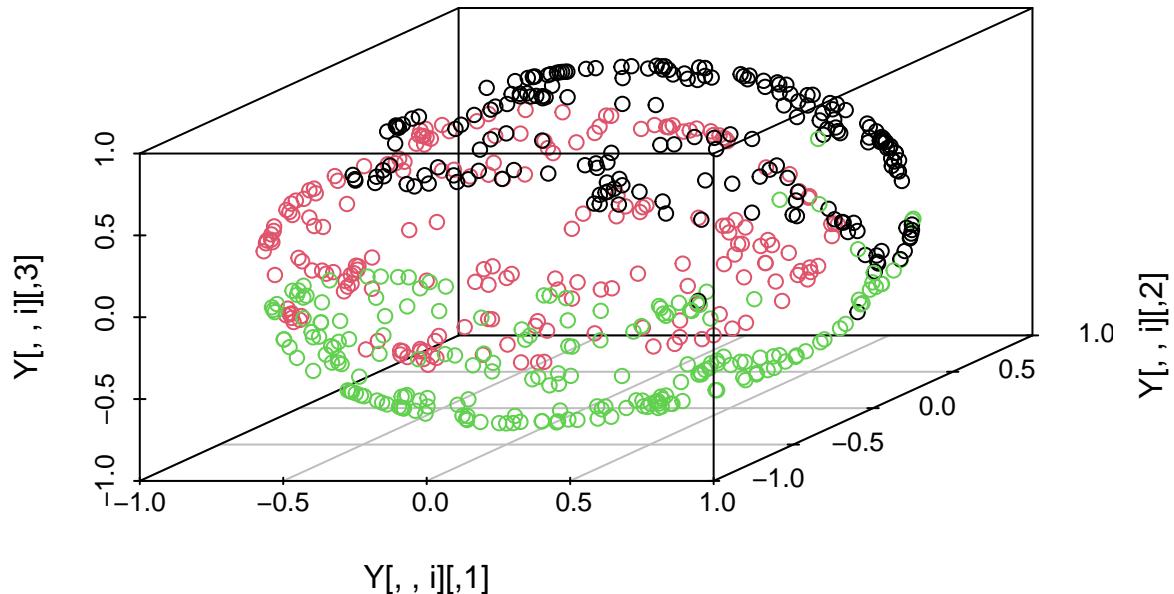
```
## [1] "Iteration 146"
## [1] "Iteration 147"
## [1] "Iteration 148"
## [1] "Iteration 149"
## [1] "Iteration 150"
## [1] "Iteration 150: objective function value is 1.350053"
```



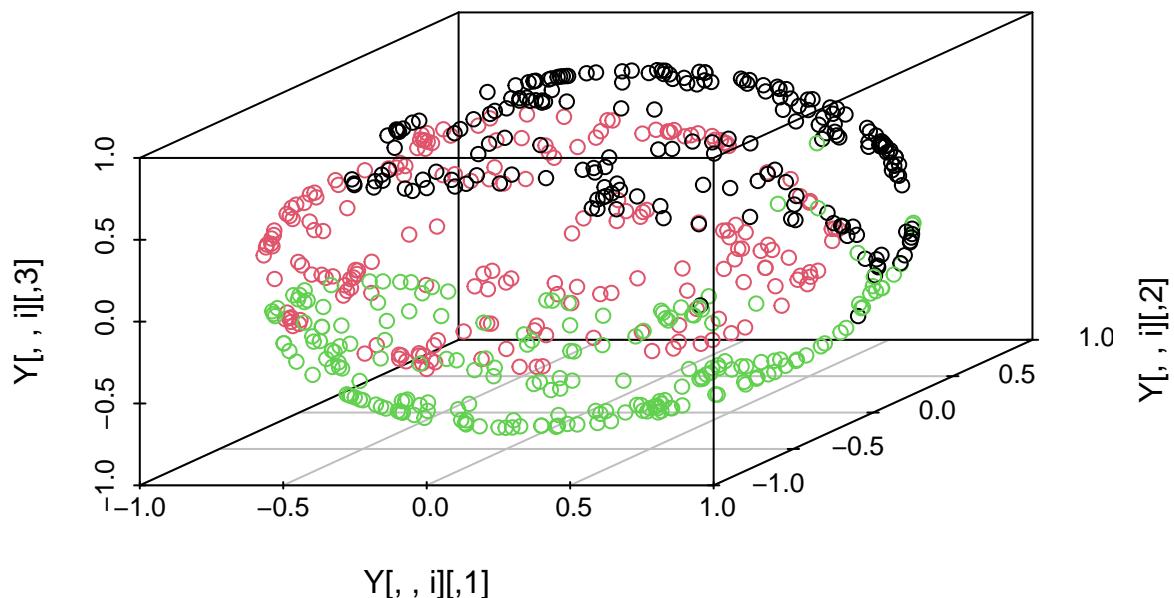
```
## [1] "Iteration 151"
## [1] "Iteration 152"
## [1] "Iteration 153"
## [1] "Iteration 154"
## [1] "Iteration 155"
## [1] "Iteration 155: objective function value is 1.349986"
```



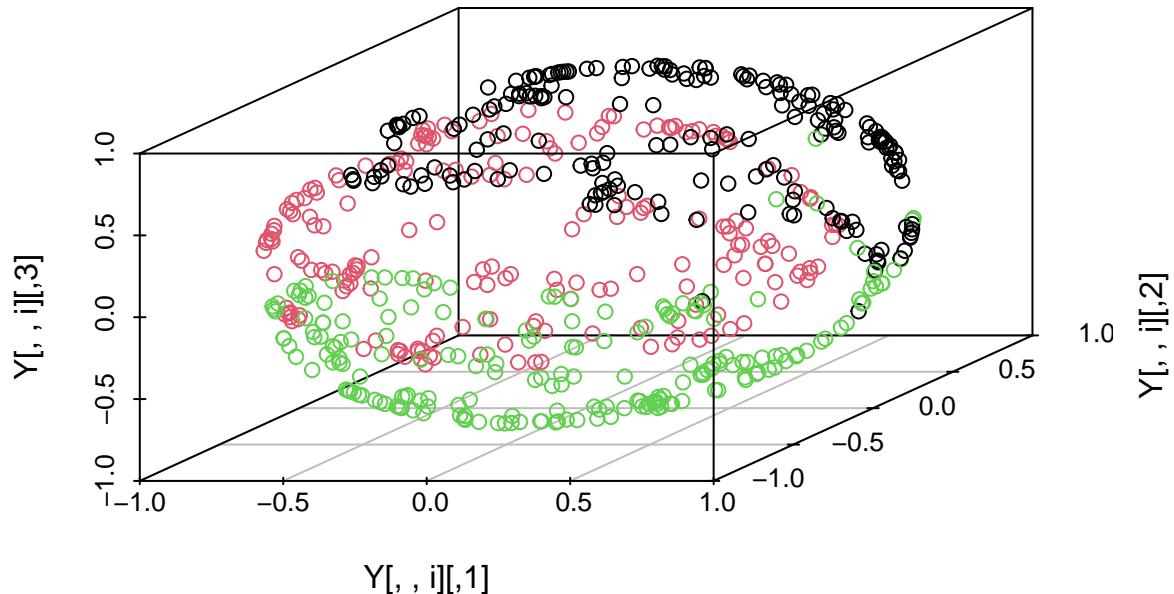
```
## [1] "Iteration 156"
## [1] "Iteration 157"
## [1] "Iteration 158"
## [1] "Iteration 159"
## [1] "Iteration 160"
## [1] "Iteration 160: objective function value is 1.349929"
```



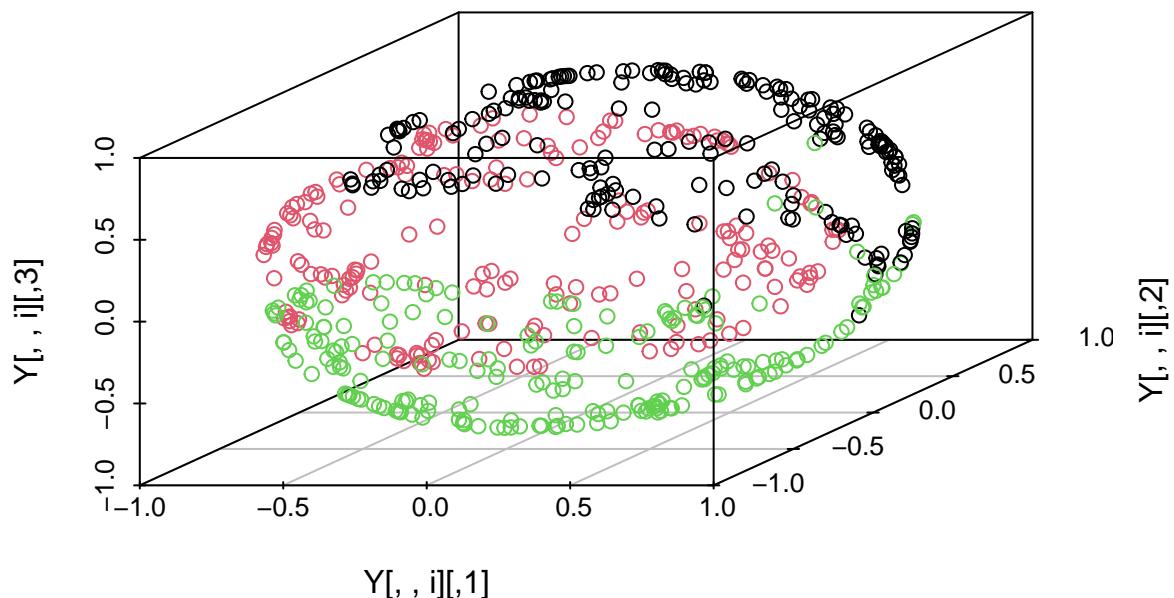
```
## [1] "Iteration 161"
## [1] "Iteration 162"
## [1] "Iteration 163"
## [1] "Iteration 164"
## [1] "Iteration 165"
## [1] "Iteration 165: objective function value is 1.349881"
```



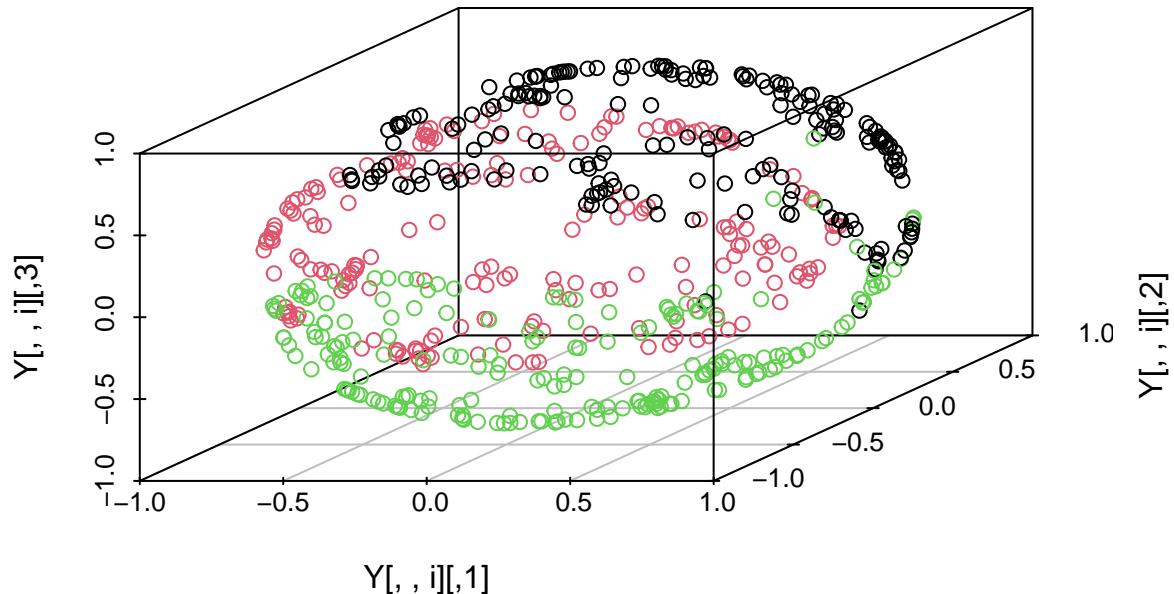
```
## [1] "Iteration 166"
## [1] "Iteration 167"
## [1] "Iteration 168"
## [1] "Iteration 169"
## [1] "Iteration 170"
## [1] "Iteration 170: objective function value is 1.349841"
```



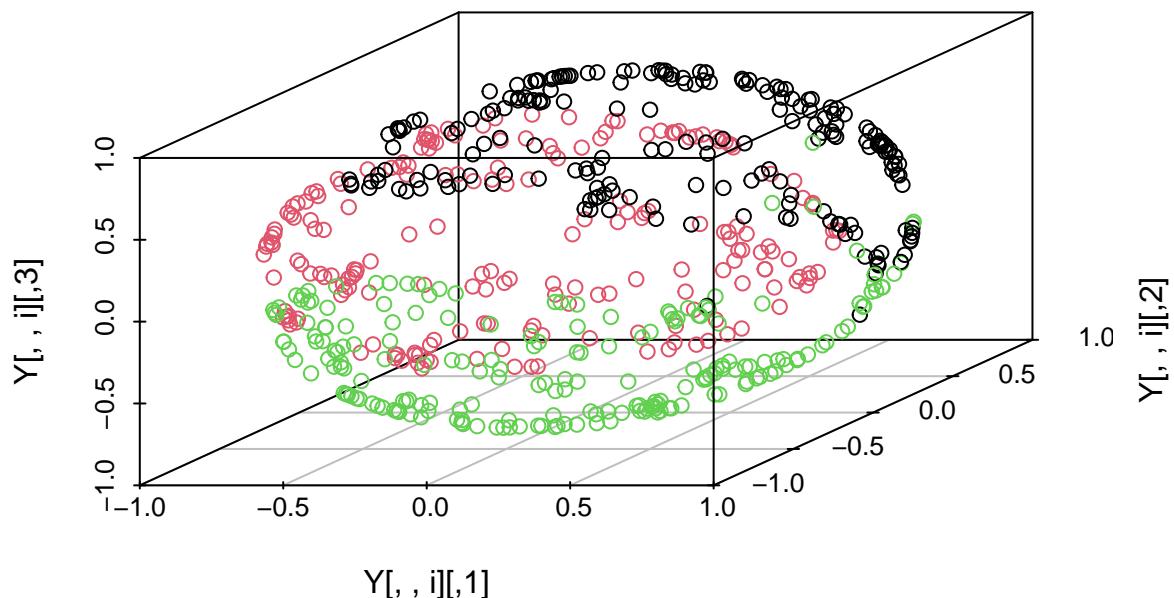
```
## [1] "Iteration 171"
## [1] "Iteration 172"
## [1] "Iteration 173"
## [1] "Iteration 174"
## [1] "Iteration 175"
## [1] "Iteration 175: objective function value is 1.349806"
```



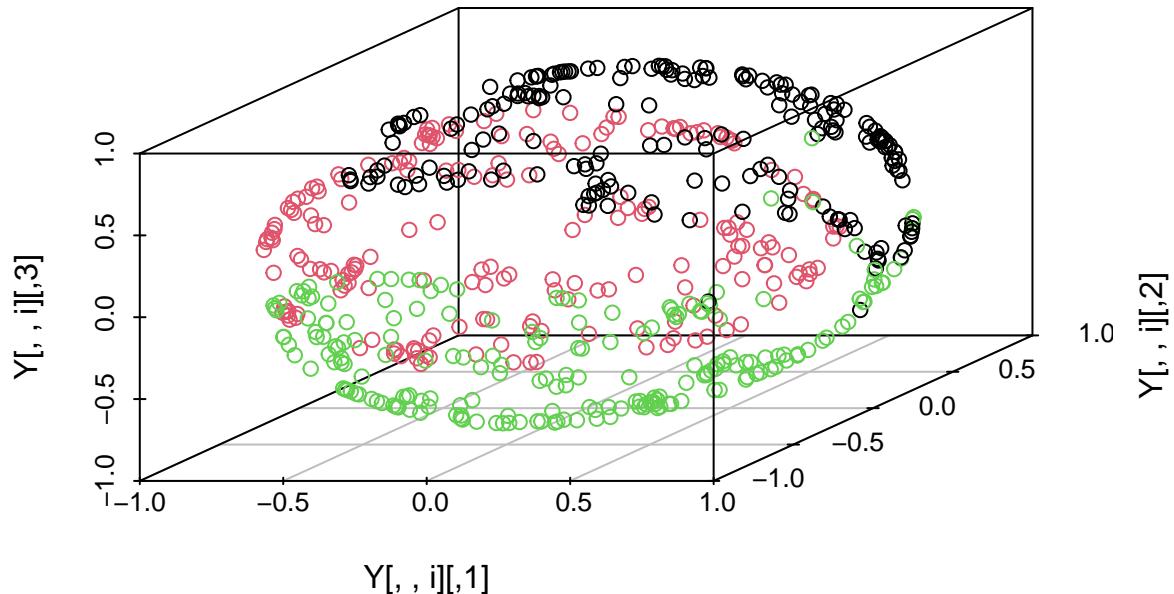
```
## [1] "Iteration 176"
## [1] "Iteration 177"
## [1] "Iteration 178"
## [1] "Iteration 179"
## [1] "Iteration 180"
## [1] "Iteration 180: objective function value is 1.349777"
```



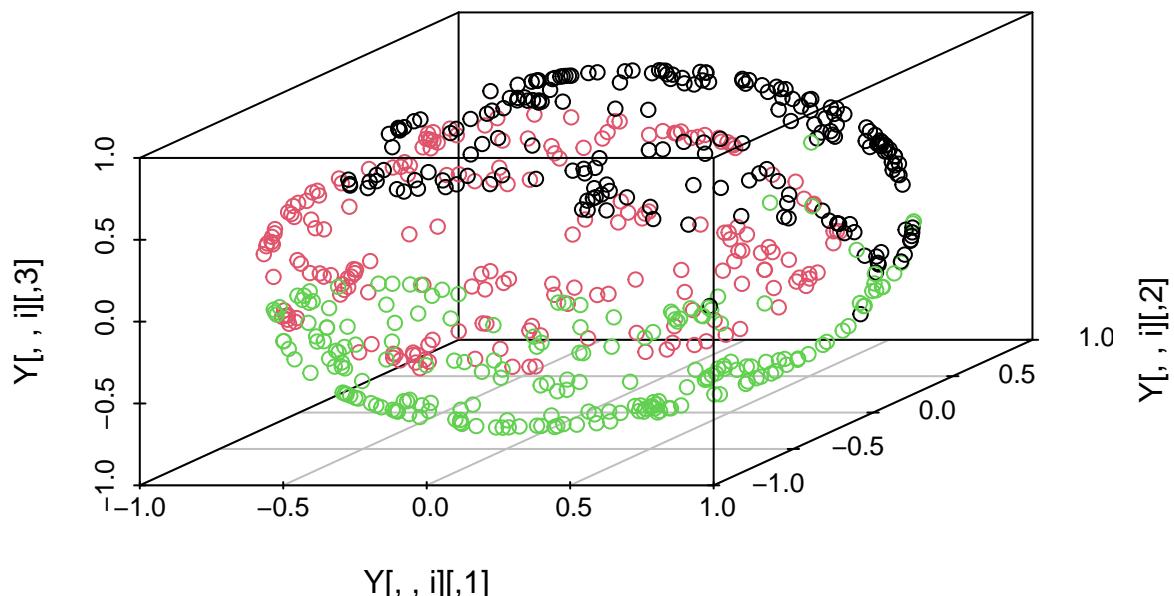
```
## [1] "Iteration 181"
## [1] "Iteration 182"
## [1] "Iteration 183"
## [1] "Iteration 184"
## [1] "Iteration 185"
## [1] "Iteration 185: objective function value is 1.349752"
```



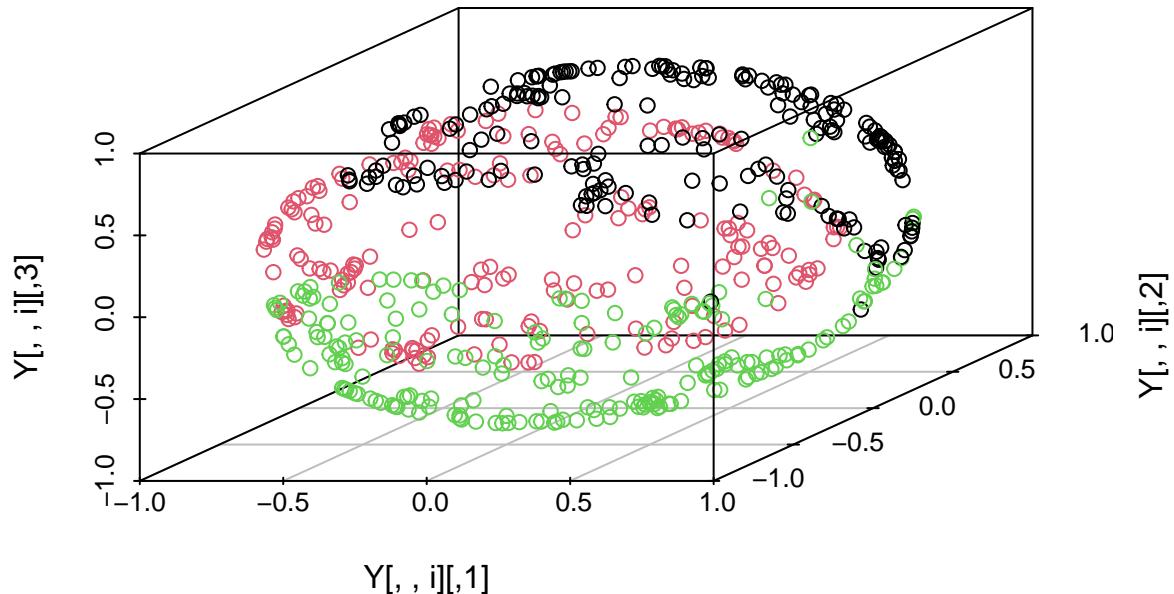
```
## [1] "Iteration 186"
## [1] "Iteration 187"
## [1] "Iteration 188"
## [1] "Iteration 189"
## [1] "Iteration 190"
## [1] "Iteration 190: objective function value is 1.349731"
```



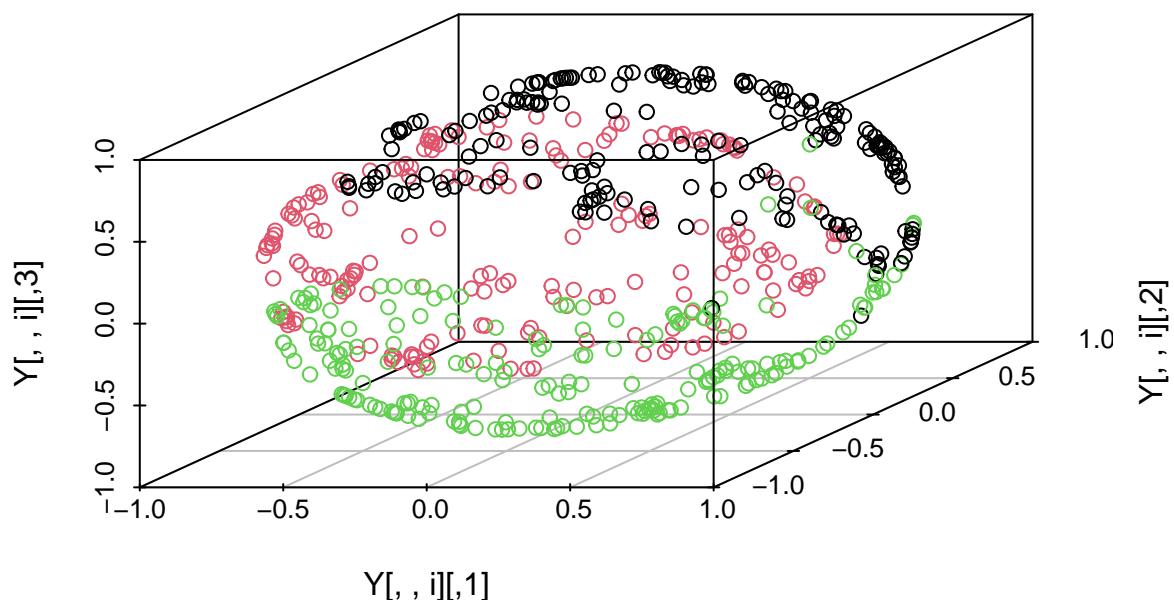
```
## [1] "Iteration 191"
## [1] "Iteration 192"
## [1] "Iteration 193"
## [1] "Iteration 194"
## [1] "Iteration 195"
## [1] "Iteration 195: objective function value is 1.349714"
```



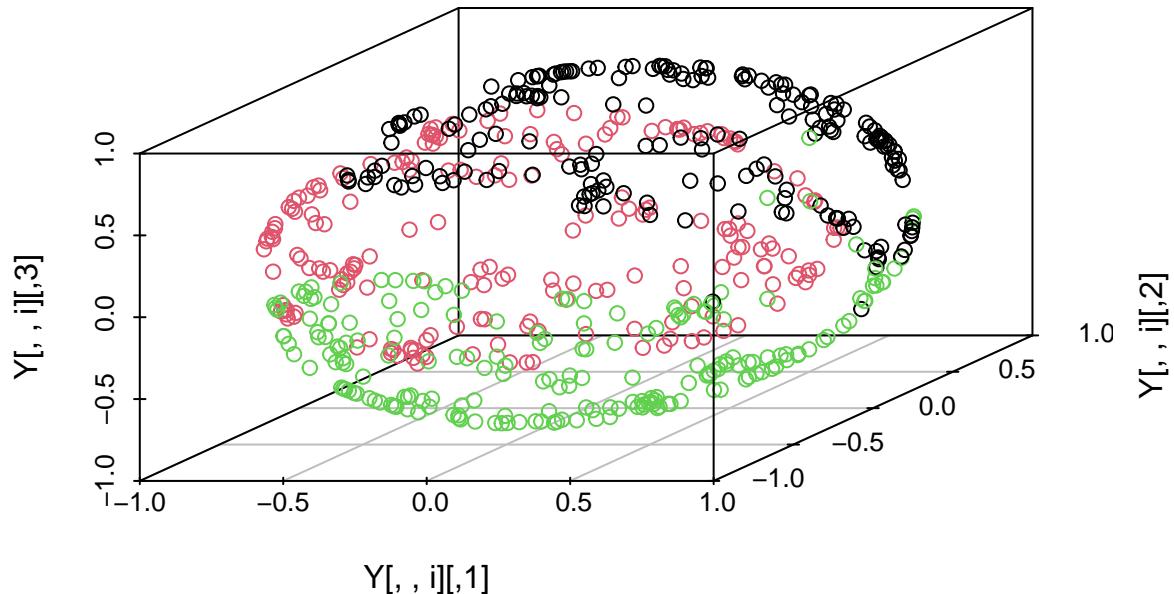
```
## [1] "Iteration 196"
## [1] "Iteration 197"
## [1] "Iteration 198"
## [1] "Iteration 199"
## [1] "Iteration 200"
## [1] "Iteration 200: objective function value is 1.349698"
```



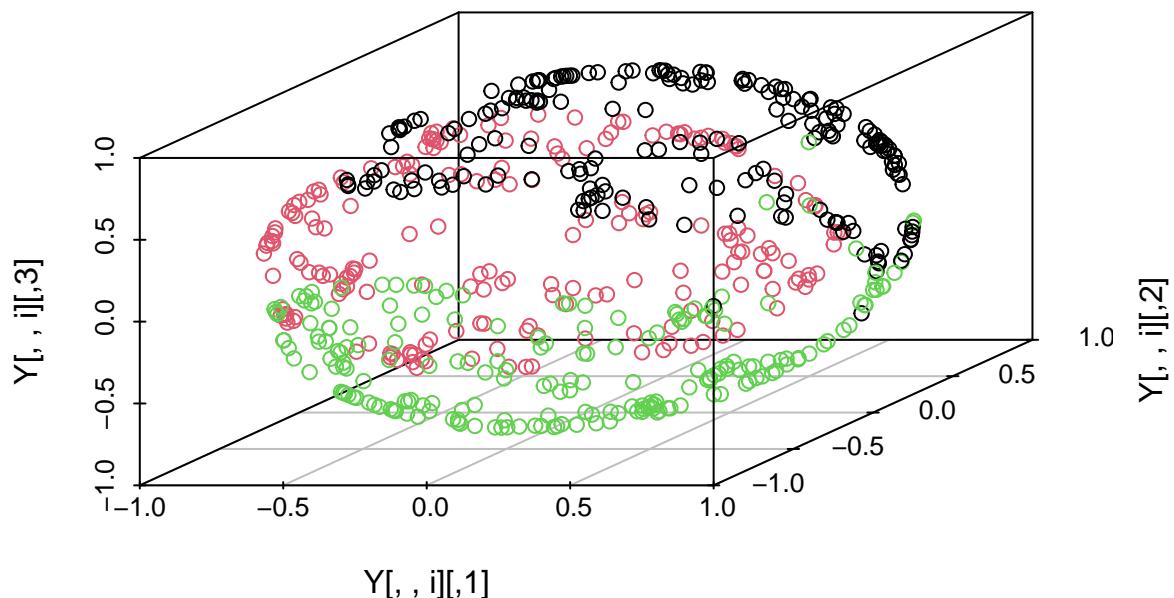
```
## [1] "Iteration 201"
## [1] "Iteration 202"
## [1] "Iteration 203"
## [1] "Iteration 204"
## [1] "Iteration 205"
## [1] "Iteration 205: objective function value is 1.349685"
```



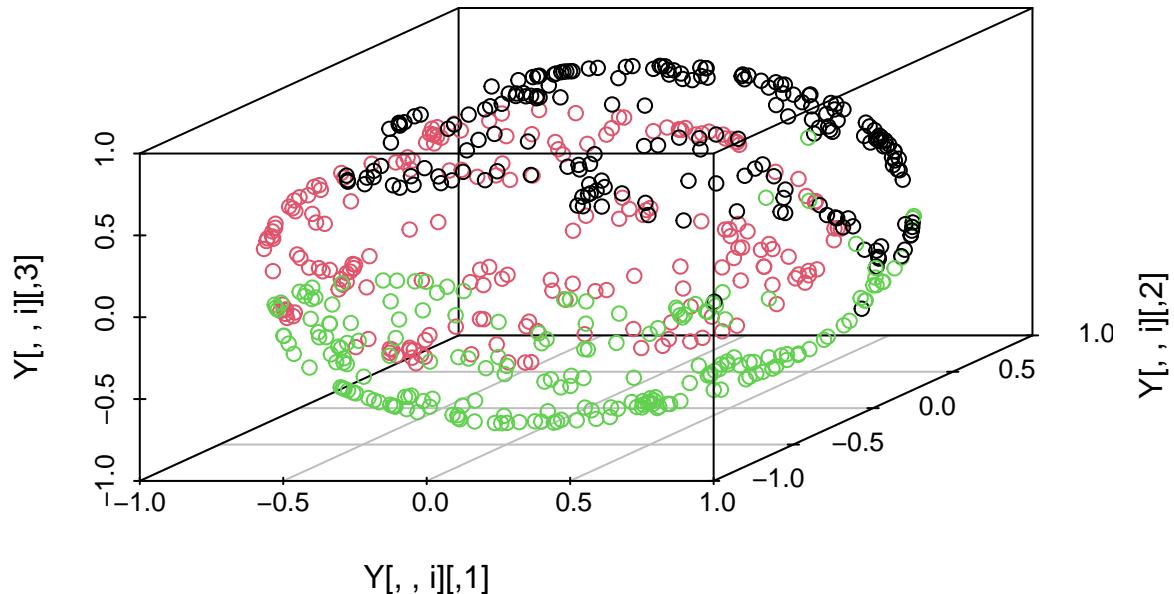
```
## [1] "Iteration 206"
## [1] "Iteration 207"
## [1] "Iteration 208"
## [1] "Iteration 209"
## [1] "Iteration 210"
## [1] "Iteration 210: objective function value is 1.349674"
```



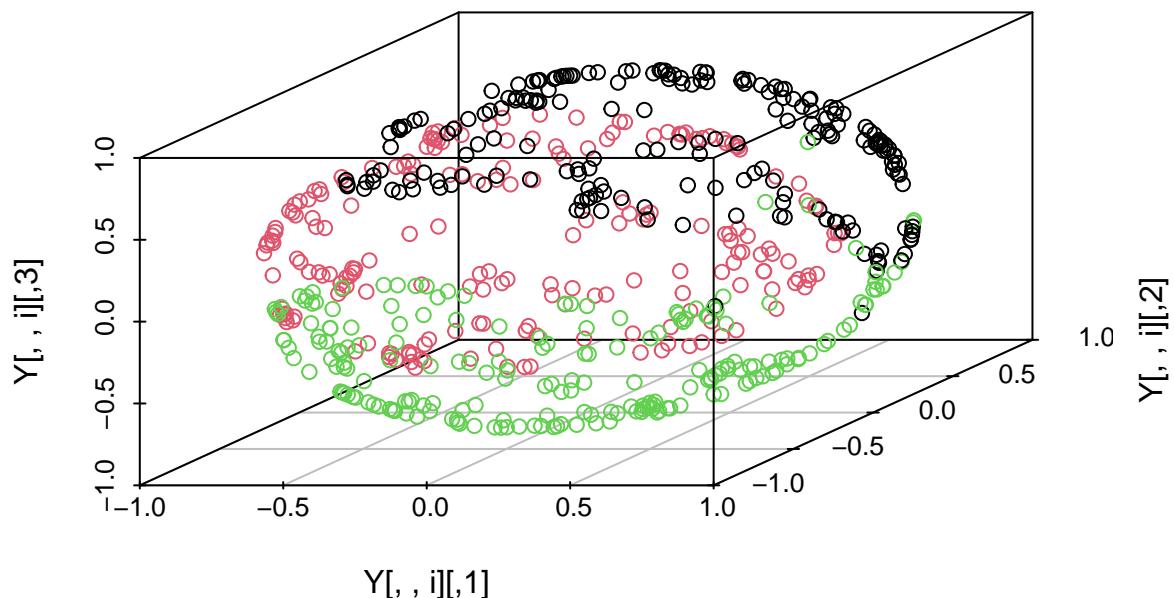
```
## [1] "Iteration 211"
## [1] "Iteration 212"
## [1] "Iteration 213"
## [1] "Iteration 214"
## [1] "Iteration 215"
## [1] "Iteration 215: objective function value is 1.349665"
```



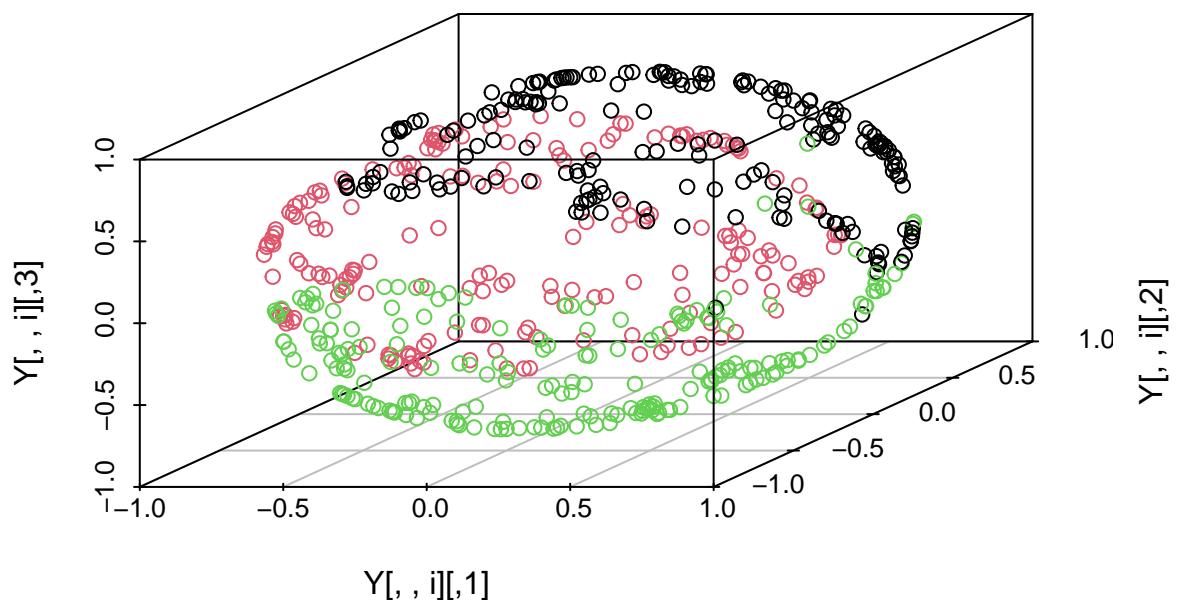
```
## [1] "Iteration 216"
## [1] "Iteration 217"
## [1] "Iteration 218"
## [1] "Iteration 219"
## [1] "Iteration 220"
## [1] "Iteration 220: objective function value is 1.349657"
```



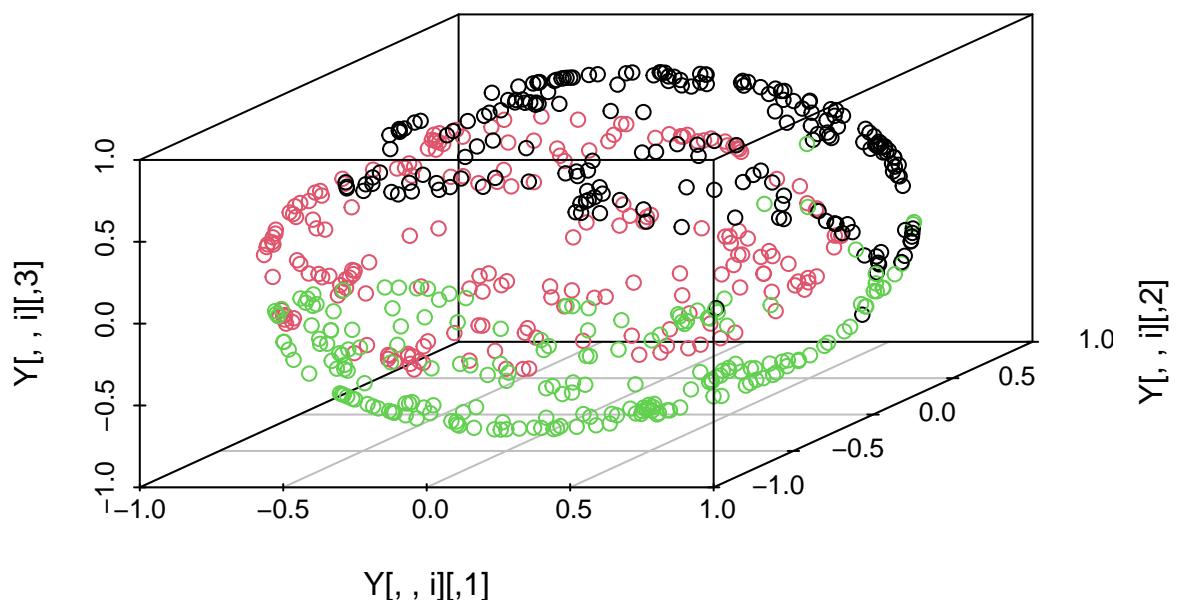
```
## [1] "Iteration 221"
## [1] "Iteration 222"
## [1] "Iteration 223"
## [1] "Iteration 224"
## [1] "Iteration 225"
## [1] "Iteration 225: objective function value is 1.349650"
```



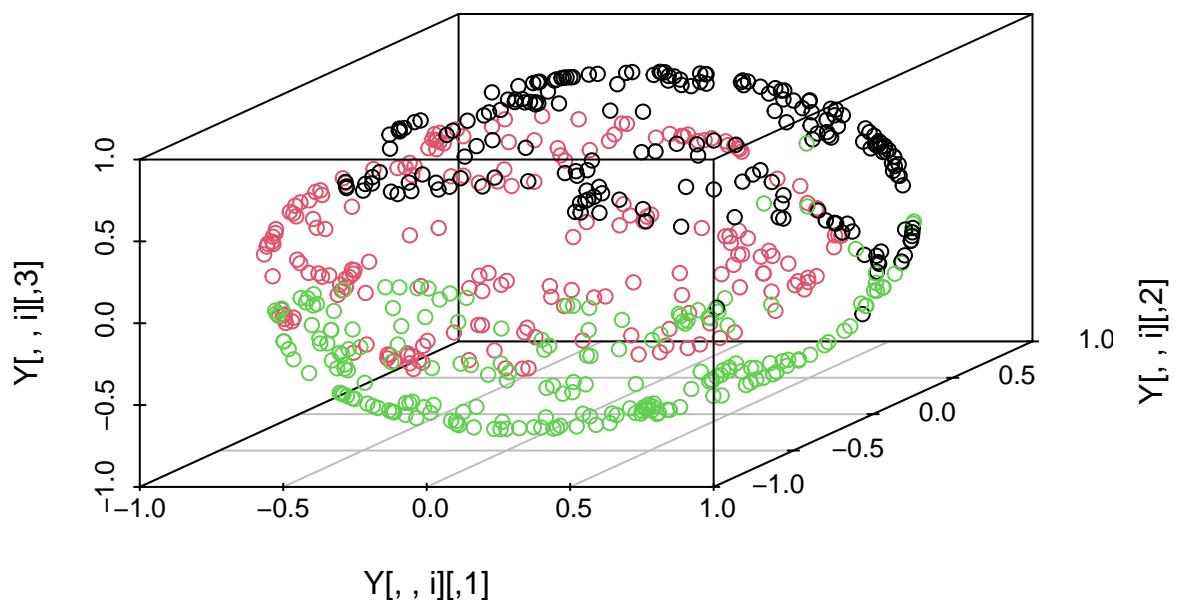
```
## [1] "Iteration 226"
## [1] "Iteration 227"
## [1] "Iteration 228"
## [1] "Iteration 229"
## [1] "Iteration 230"
## [1] "Iteration 230: objective function value is 1.349644"
```



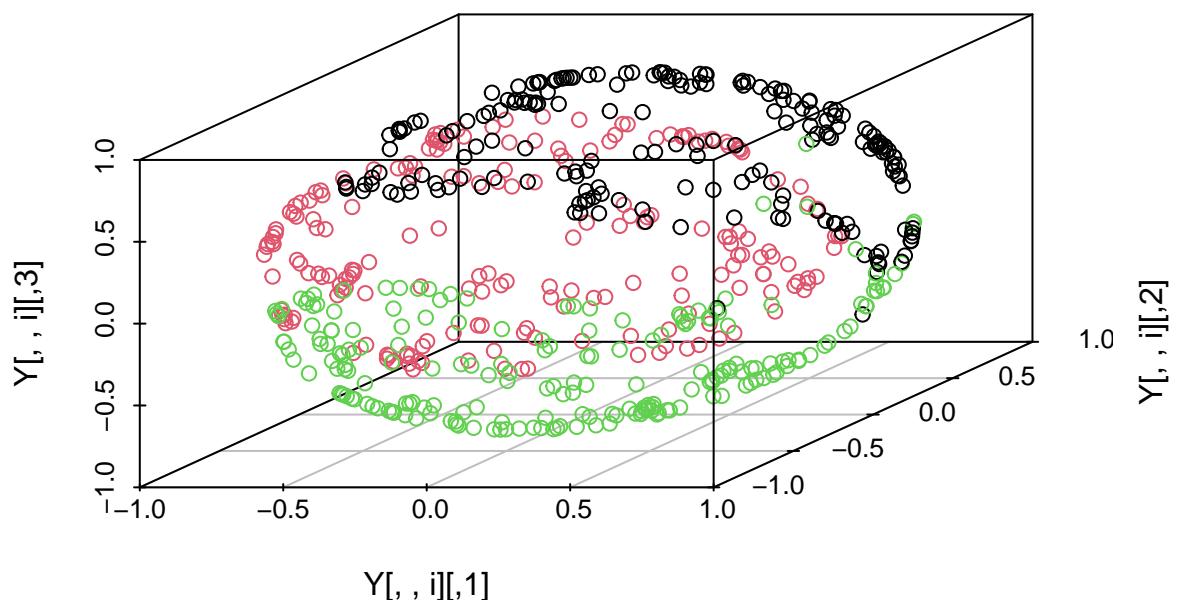
```
## [1] "Iteration 231"
## [1] "Iteration 232"
## [1] "Iteration 233"
## [1] "Iteration 234"
## [1] "Iteration 235"
## [1] "Iteration 235: objective function value is 1.349639"
```



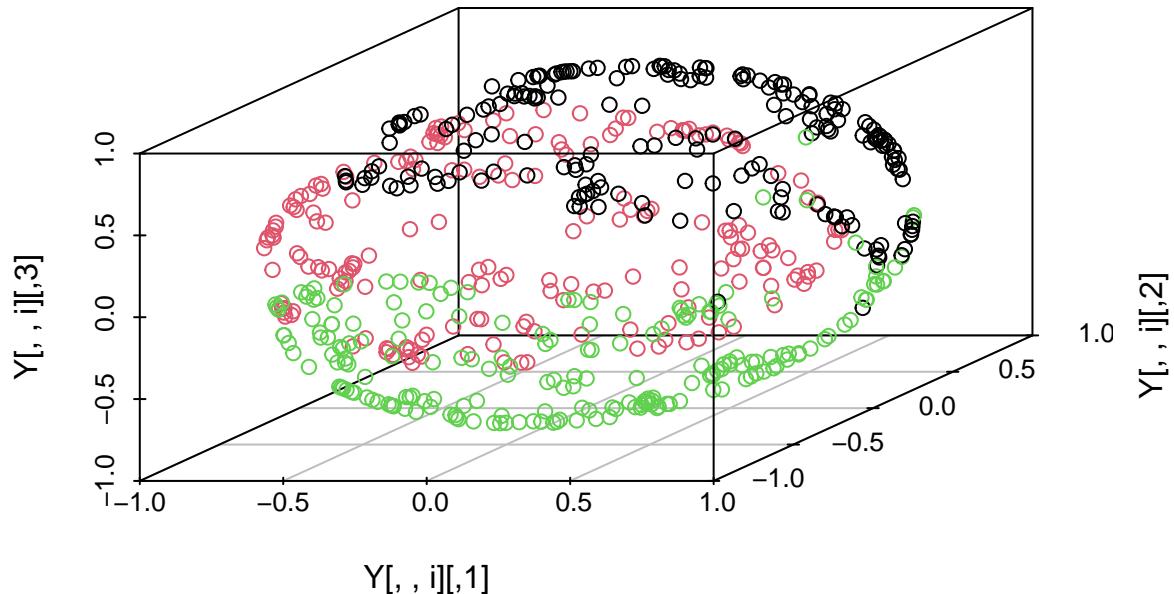
```
## [1] "Iteration 236"
## [1] "Iteration 237"
## [1] "Iteration 238"
## [1] "Iteration 239"
## [1] "Iteration 240"
## [1] "Iteration 240: objective function value is 1.349635"
```



```
## [1] "Iteration 241"
## [1] "Iteration 242"
## [1] "Iteration 243"
## [1] "Iteration 244"
## [1] "Iteration 245"
## [1] "Iteration 245: objective function value is 1.349631"
```

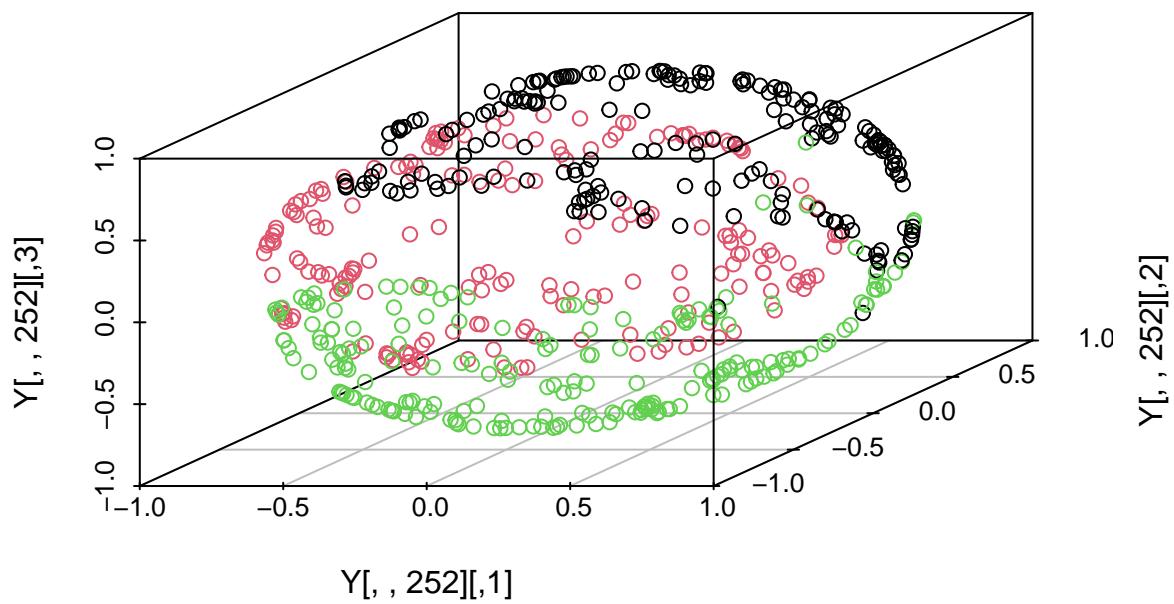


```
## [1] "Iteration 246"
## [1] "Iteration 247"
## [1] "Iteration 248"
## [1] "Iteration 249"
## [1] "Iteration 250"
## [1] "Iteration 250: objective function value is 1.349627"
```



```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "gray",
            lit = FALSE)
rgl::points3d(Y[, , 252], col = rep(c(1,2), each=nrow(Y)/2))
```

```
..../..../..../private/var/folders/4b/zzmhvpts0t91yd3y81j_tb_h0000gn/T/RtmpEZx4uU/file14f55645d8bbc.png
scatterplot3d::scatterplot3d(Y[, , 252], xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(c(1,2,3), each=n/3))
```



It's clearly clustered the three groups we had in the original generated data.