

Applications PSC-SNE

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Cases

Case 1

Sample on the $(\mathbb{S}^1)^2$ where $p = 1$ and $r = 2$.

Data:

```
# Sample on the  $(\mathbb{S}^1)^2$ 
n <- 200
vmf11 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 10)
vmf12 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi)), kappa = 10)
x1 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf11),
                             DirStats::to_rad(vmf12)))

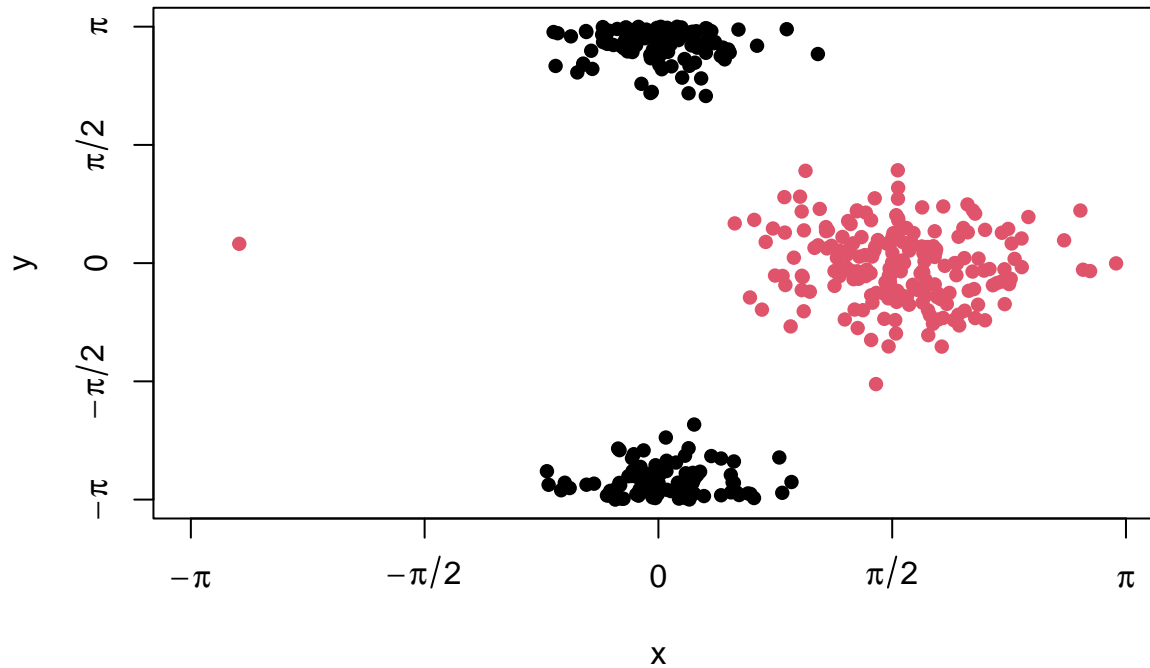
vmf21 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi / 2)), kappa = 5)
vmf22 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 5)
x2 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf21), DirStats::to_rad(vmf22)))

x <- rbind(x1, x2)

# Cartesian coordinates
n <- 400
x_array <- array(dim = c(n, 2, 2))
x_array[, , 1] <- DirStats::to_cir(x[, 1])
x_array[, , 2] <- DirStats::to_cir(x[, 2])

indexes <- sample(1:n)
x_array <- x_array[indexes,,]
colors <- rep(c(1, 2), each = n/2)[indexes]

plot(x, xlim = c(-pi, pi), ylim = c(-pi, pi), axes = FALSE,
     col = rep(c(1, 2), each = n/2), pch = 16,
     xlab = "x", ylab = "y")
sdetorus::torusAxis()
```



Let's calculate the rho parameters based on a perplexity of 20:

```
# Time difference of 2.582784 mins
rho_first_20 <- rho_optim_bst(x_array, 20)
```

```
## Time difference of 27.66424 secs
```

First, let's reduce to dimension \mathbb{S}^1 then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_array, d=1, rho_psc_list = rho_first_20, num_iteration=125,
             colors=colors, visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 16.666327, abs 0.000000, rel 0.000000, norm 0.168640"
## [1] "Iter 2, obj 15.880444, abs 0.785883, rel 0.047154, norm 0.416610"
## [1] "Iter 3, obj 15.459048, abs 0.421395, rel 0.026535, norm 0.547835"
## [1] "Iter 4, obj 15.203613, abs 0.255436, rel 0.016523, norm 0.623848"
## [1] "Iter 5, obj 15.010616, abs 0.192996, rel 0.012694, norm 0.658429"
## [1] "Iter 6, obj 14.856618, abs 0.153998, rel 0.010259, norm 0.683585"
## [1] "Iter 7, obj 14.749971, abs 0.106647, rel 0.007178, norm 0.711267"
## [1] "Iter 8, obj 14.659648, abs 0.090323, rel 0.006124, norm 0.731386"
## [1] "Iter 9, obj 14.581870, abs 0.077779, rel 0.005306, norm 0.747610"
## [1] "Iter 10, obj 14.514048, abs 0.067822, rel 0.004651, norm 0.762143"
## [1] "Iter 11, obj 14.453935, abs 0.060113, rel 0.004142, norm 0.775963"
## [1] "Iter 12, obj 14.407242, abs 0.046692, rel 0.003230, norm 0.789473"
## [1] "Iter 13, obj 14.361328, abs 0.045914, rel 0.003187, norm 0.800504"
## [1] "Iter 14, obj 14.315771, abs 0.045557, rel 0.003172, norm 0.809056"
## [1] "Iter 15, obj 14.278257, abs 0.037514, rel 0.002621, norm 0.816354"
## [1] "Iter 16, obj 14.248492, abs 0.029764, rel 0.002085, norm 0.822383"
## [1] "Iter 17, obj 14.219419, abs 0.029073, rel 0.002040, norm 0.827598"
## [1] "Iter 18, obj 14.192835, abs 0.026584, rel 0.001870, norm 0.832745"
## [1] "Iter 19, obj 14.166619, abs 0.026216, rel 0.001847, norm 0.838100"
## [1] "Iter 20, obj 14.139393, abs 0.027227, rel 0.001922, norm 0.843604"
## [1] "Iter 21, obj 14.112009, abs 0.027384, rel 0.001937, norm 0.849796"
## [1] "Iter 22, obj 14.085316, abs 0.026693, rel 0.001892, norm 0.856958"
```

```

## [1] "Iter 23, obj 14.060543, abs 0.024773, rel 0.001759, norm 0.864842"
## [1] "Iter 24, obj 14.039077, abs 0.021466, rel 0.001527, norm 0.872794"
## [1] "Iter 25, obj 14.022547, abs 0.016531, rel 0.001177, norm 0.880355"

## [1] "Iter 26, obj 14.011092, abs 0.011455, rel 0.000817, norm 0.887140"
## [1] "Iter 27, obj 14.003730, abs 0.007362, rel 0.000525, norm 0.893096"
## [1] "Iter 28, obj 13.999423, abs 0.004307, rel 0.000308, norm 0.898396"
## [1] "Iter 29, obj 13.997370, abs 0.002053, rel 0.000147, norm 0.903174"
## [1] "Iter 30, obj 13.996940, abs 0.000429, rel 0.000031, norm 0.907493"
## [1] "Iter 31, obj 13.997650, abs 0.000710, rel 0.000051, norm 0.911418"
## [1] "Iter 32, obj 13.999169, abs 0.001519, rel 0.000108, norm 0.915034"
## [1] "Iter 33, obj 14.001296, abs 0.002127, rel 0.000152, norm 0.918425"
## [1] "Iter 34, obj 14.003915, abs 0.002620, rel 0.000187, norm 0.921656"
## [1] "Iter 35, obj 14.006953, abs 0.003037, rel 0.000217, norm 0.924772"
## [1] "Iter 36, obj 14.010334, abs 0.003382, rel 0.000241, norm 0.927802"
## [1] "Iter 37, obj 14.013957, abs 0.003623, rel 0.000259, norm 0.930765"
## [1] "Iter 38, obj 14.017662, abs 0.003705, rel 0.000264, norm 0.933677"
## [1] "Iter 39, obj 14.021217, abs 0.003554, rel 0.000254, norm 0.936550"
## [1] "Iter 40, obj 14.024300, abs 0.003083, rel 0.000220, norm 0.939402"
## [1] "Iter 41, obj 14.026496, abs 0.002197, rel 0.000157, norm 0.942255"
## [1] "Iter 42, obj 14.027308, abs 0.000812, rel 0.000058, norm 0.945142"
## [1] "Iter 43, obj 14.026213, abs 0.001095, rel 0.000078, norm 0.948111"
## [1] "Iter 44, obj 14.022814, abs 0.003399, rel 0.000242, norm 0.951218"
## [1] "Iter 45, obj 14.017066, abs 0.005748, rel 0.000410, norm 0.954522"
## [1] "Iter 46, obj 14.009466, abs 0.007600, rel 0.000542, norm 0.958036"
## [1] "Iter 47, obj 14.000949, abs 0.008518, rel 0.000608, norm 0.961700"
## [1] "Iter 48, obj 13.992462, abs 0.008486, rel 0.000606, norm 0.965401"
## [1] "Iter 49, obj 13.984577, abs 0.007885, rel 0.000564, norm 0.969027"
## [1] "Iter 50, obj 13.977315, abs 0.007262, rel 0.000519, norm 0.972488"

## [1] "Iter 51, obj 13.970189, abs 0.007126, rel 0.000510, norm 0.975707"
## [1] "Iter 52, obj 13.962233, abs 0.007956, rel 0.000570, norm 0.978645"
## [1] "Iter 53, obj 13.951866, abs 0.010367, rel 0.000742, norm 0.981325"
## [1] "Iter 54, obj 13.936780, abs 0.015086, rel 0.001081, norm 0.983864"
## [1] "Iter 55, obj 13.914172, abs 0.022608, rel 0.001622, norm 0.986480"
## [1] "Iter 56, obj 13.881915, abs 0.032258, rel 0.002318, norm 0.989483"
## [1] "Iter 57, obj 13.841379, abs 0.040536, rel 0.002920, norm 0.993231"
## [1] "Iter 58, obj 13.798943, abs 0.042435, rel 0.003066, norm 0.998006"
## [1] "Iter 59, obj 13.762466, abs 0.036477, rel 0.002643, norm 1.003887"
## [1] "Iter 60, obj 13.736154, abs 0.026312, rel 0.001912, norm 1.010783"
## [1] "Iter 61, obj 13.719745, abs 0.016409, rel 0.001195, norm 1.018551"
## [1] "Iter 62, obj 13.711004, abs 0.008741, rel 0.000637, norm 1.027018"
## [1] "Iter 63, obj 13.707874, abs 0.003130, rel 0.000228, norm 1.035977"
## [1] "Iter 64, obj 13.709055, abs 0.001182, rel 0.000086, norm 1.045217"
## [1] "Iter 65, obj 13.713575, abs 0.004519, rel 0.000330, norm 1.054516"
## [1] "Iter 66, obj 13.720526, abs 0.006951, rel 0.000507, norm 1.063650"
## [1] "Iter 67, obj 13.729109, abs 0.008583, rel 0.000626, norm 1.072457"
## [1] "Iter 68, obj 13.738737, abs 0.009628, rel 0.000701, norm 1.080851"
## [1] "Iter 69, obj 13.748997, abs 0.010260, rel 0.000747, norm 1.088789"
## [1] "Iter 70, obj 13.759585, abs 0.010588, rel 0.000770, norm 1.096251"
## [1] "Iter 71, obj 13.770269, abs 0.010684, rel 0.000776, norm 1.103227"
## [1] "Iter 72, obj 13.780873, abs 0.010604, rel 0.000770, norm 1.109717"
## [1] "Iter 73, obj 13.791264, abs 0.010391, rel 0.000754, norm 1.115733"
## [1] "Iter 74, obj 13.801345, abs 0.010081, rel 0.000731, norm 1.121292"
## [1] "Iter 75, obj 13.811048, abs 0.009703, rel 0.000703, norm 1.126416"

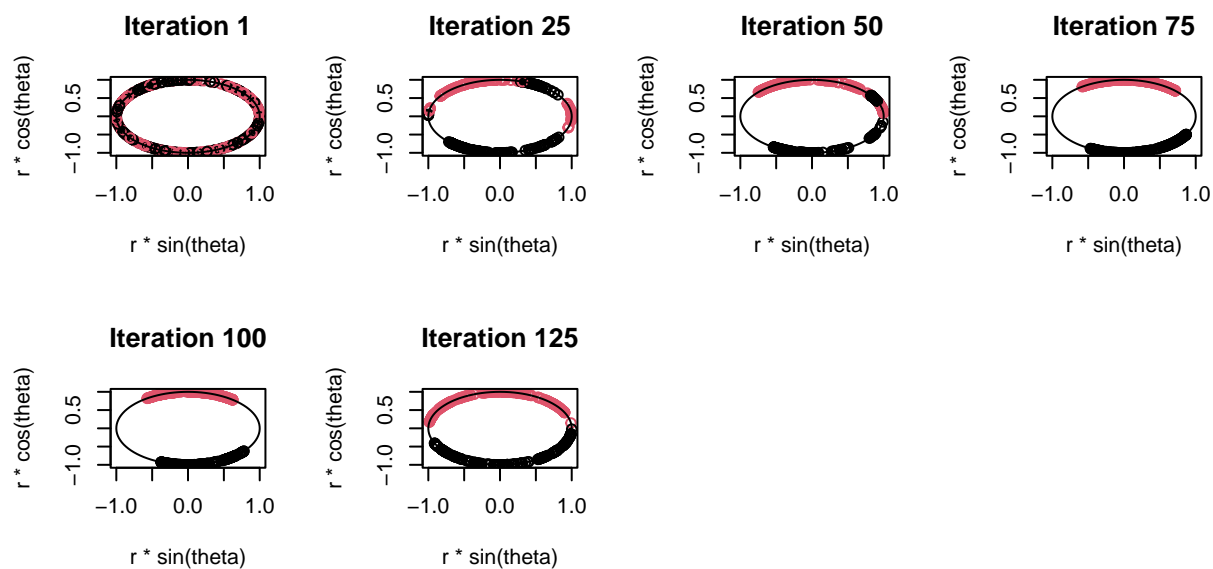
```

```

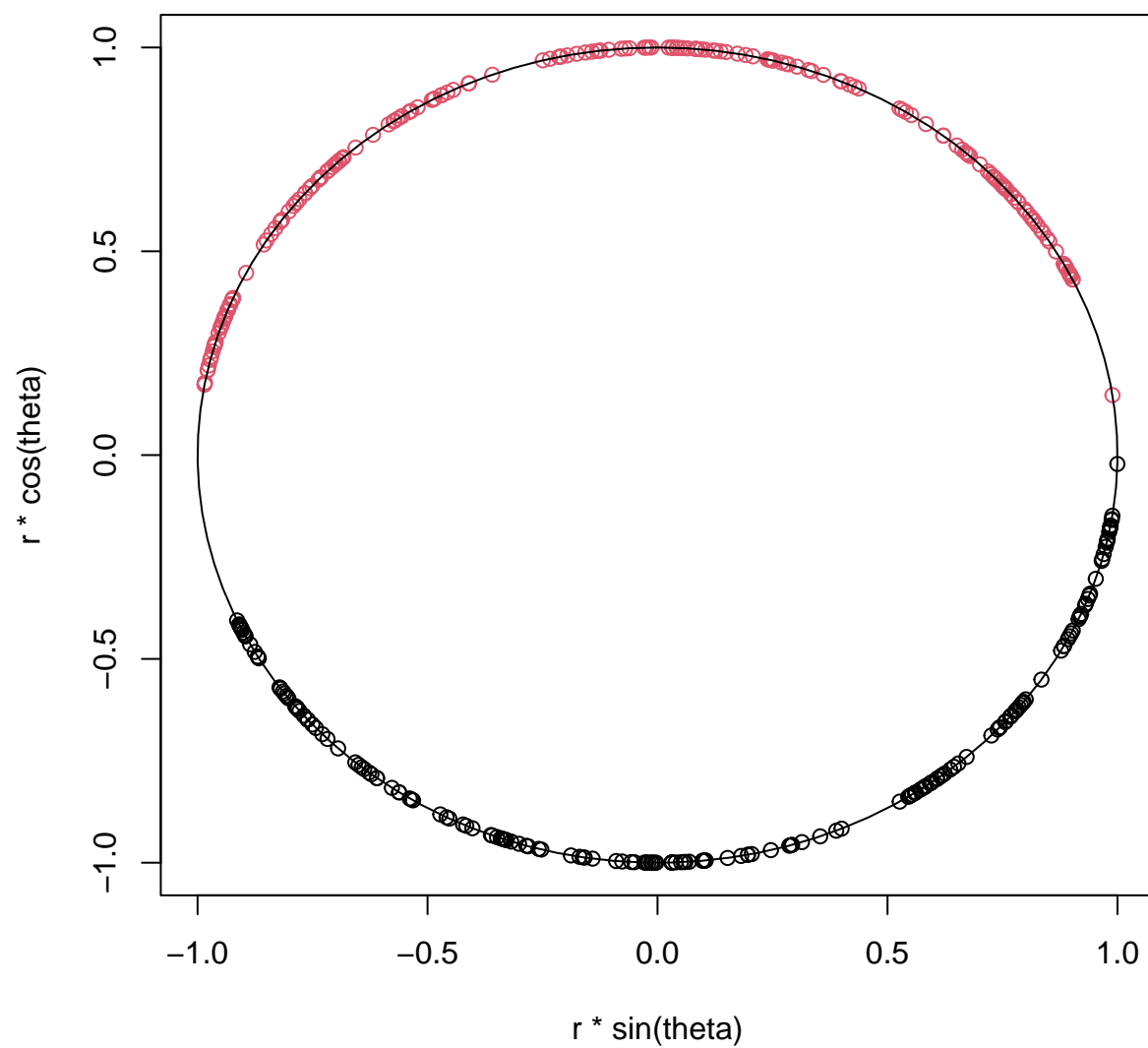
## [1] "Iter 76, obj 13.820328, abs 0.009280, rel 0.000672, norm 1.131133"
## [1] "Iter 77, obj 13.829159, abs 0.008831, rel 0.000639, norm 1.135470"
## [1] "Iter 78, obj 13.837527, abs 0.008369, rel 0.000605, norm 1.139456"
## [1] "Iter 79, obj 13.845431, abs 0.007904, rel 0.000571, norm 1.143117"
## [1] "Iter 80, obj 13.852875, abs 0.007444, rel 0.000538, norm 1.146482"
## [1] "Iter 81, obj 13.859870, abs 0.006995, rel 0.000505, norm 1.149574"
## [1] "Iter 82, obj 13.866429, abs 0.006559, rel 0.000473, norm 1.152418"
## [1] "Iter 83, obj 13.872569, abs 0.006140, rel 0.000443, norm 1.155035"
## [1] "Iter 84, obj 13.878307, abs 0.005738, rel 0.000414, norm 1.157445"
## [1] "Iter 85, obj 13.883663, abs 0.005356, rel 0.000386, norm 1.159667"
## [1] "Iter 86, obj 13.888655, abs 0.004992, rel 0.000360, norm 1.161716"
## [1] "Iter 87, obj 13.893303, abs 0.004648, rel 0.000335, norm 1.163608"
## [1] "Iter 88, obj 13.897625, abs 0.004322, rel 0.000311, norm 1.165357"
## [1] "Iter 89, obj 13.901640, abs 0.004015, rel 0.000289, norm 1.166976"
## [1] "Iter 90, obj 13.905365, abs 0.003726, rel 0.000268, norm 1.168475"
## [1] "Iter 91, obj 13.908819, abs 0.003454, rel 0.000248, norm 1.169865"
## [1] "Iter 92, obj 13.912017, abs 0.003198, rel 0.000230, norm 1.171155"
## [1] "Iter 93, obj 13.914975, abs 0.002958, rel 0.000213, norm 1.172353"
## [1] "Iter 94, obj 13.917709, abs 0.002734, rel 0.000196, norm 1.173468"
## [1] "Iter 95, obj 13.920233, abs 0.002524, rel 0.000181, norm 1.174506"
## [1] "Iter 96, obj 13.922560, abs 0.002327, rel 0.000167, norm 1.175474"
## [1] "Iter 97, obj 13.924704, abs 0.002144, rel 0.000154, norm 1.176377"
## [1] "Iter 98, obj 13.926676, abs 0.001972, rel 0.000142, norm 1.177220"
## [1] "Iter 99, obj 13.928489, abs 0.001812, rel 0.000130, norm 1.178008"
## [1] "Iter 100, obj 13.930152, abs 0.001663, rel 0.000119, norm 1.178745"

## [1] "Iter 101, obj 2.042664, abs 11.887488, rel 0.853364, norm 0.124433"
## [1] "Iter 102, obj 2.000371, abs 0.042293, rel 0.020705, norm 0.128438"
## [1] "Iter 103, obj 1.982226, abs 0.018144, rel 0.009071, norm 0.131267"
## [1] "Iter 104, obj 1.975884, abs 0.006343, rel 0.003200, norm 0.132990"
## [1] "Iter 105, obj 1.972867, abs 0.003016, rel 0.001527, norm 0.133652"
## [1] "Iter 106, obj 1.969725, abs 0.003142, rel 0.001593, norm 0.133695"
## [1] "Iter 107, obj 1.969341, abs 0.000384, rel 0.000195, norm 0.133402"
## [1] "Iter 108, obj 1.971102, abs 0.001761, rel 0.000894, norm 0.133343"
## [1] "Iter 109, obj 1.970166, abs 0.000936, rel 0.000475, norm 0.133450"
## [1] "Iter 110, obj 1.966857, abs 0.003310, rel 0.001680, norm 0.133404"
## [1] "Iter 111, obj 1.967613, abs 0.000756, rel 0.000384, norm 0.133335"
## [1] "Iter 112, obj 1.969272, abs 0.001660, rel 0.000843, norm 0.133339"
## [1] "Iter 113, obj 1.966756, abs 0.002516, rel 0.001278, norm 0.133213"
## [1] "Iter 114, obj 1.966375, abs 0.000380, rel 0.000193, norm 0.133309"
## [1] "Iter 115, obj 1.967557, abs 0.001182, rel 0.000601, norm 0.133286"
## [1] "Iter 116, obj 1.969129, abs 0.001573, rel 0.000799, norm 0.133332"
## [1] "Iter 117, obj 1.966683, abs 0.002447, rel 0.001242, norm 0.133200"
## [1] "Iter 118, obj 1.966665, abs 0.000018, rel 0.000009, norm 0.133305"
## [1] "Iter 119, obj 1.971452, abs 0.004788, rel 0.002434, norm 0.133179"
## [1] "Iter 120, obj 1.969321, abs 0.002132, rel 0.001081, norm 0.133232"
## [1] "Iter 121, obj 1.967090, abs 0.002231, rel 0.001133, norm 0.133224"
## [1] "Iter 122, obj 1.969355, abs 0.002265, rel 0.001151, norm 0.133293"
## [1] "Iter 123, obj 1.968498, abs 0.000856, rel 0.000435, norm 0.133440"
## [1] "Iter 124, obj 1.969130, abs 0.000632, rel 0.000321, norm 0.133332"
## [1] "Iter 125, obj 1.967761, abs 0.001370, rel 0.000696, norm 0.133340"

```



Iteration 125



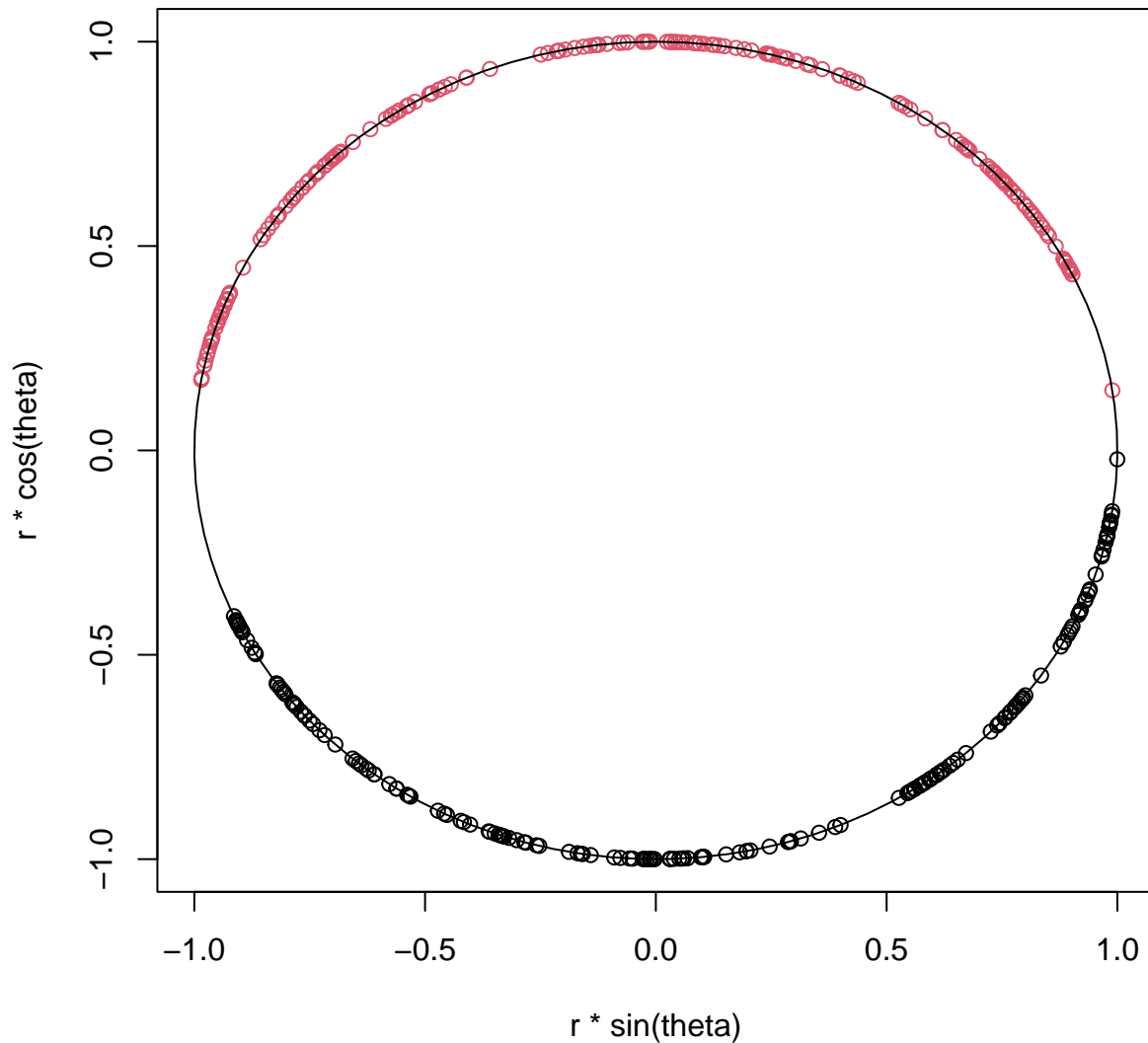
```

Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad

plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))

```



Now we are going to reduce to dimension \mathbb{S}^2 then $d = 2$ (sphere):

```

Y <- psc_sne(X=x_array, d=2, rho_psc_list = rho_first_20, num_iteration=125,
             colors=colors, visualize_prog = TRUE, eta=100, check=F)

```

```

## [1] "Iter 1, obj 17.211976, abs 0.000000, rel 0.000000, norm 0.198372"
## [1] "Iter 2, obj 14.189697, abs 3.022279, rel 0.175592, norm 0.529104"
## [1] "Iter 3, obj 13.152058, abs 1.037639, rel 0.073126, norm 0.905916"

```

```

## [1] "Iter 4, obj 12.598493, abs 0.553565, rel 0.042090, norm 1.077377"
## [1] "Iter 5, obj 12.278474, abs 0.320019, rel 0.025401, norm 1.169797"
## [1] "Iter 6, obj 12.061235, abs 0.217240, rel 0.017693, norm 1.224623"
## [1] "Iter 7, obj 11.903710, abs 0.157525, rel 0.013060, norm 1.265725"
## [1] "Iter 8, obj 11.777265, abs 0.126445, rel 0.010622, norm 1.301194"
## [1] "Iter 9, obj 11.664213, abs 0.113052, rel 0.009599, norm 1.334444"
## [1] "Iter 10, obj 11.562290, abs 0.101923, rel 0.008738, norm 1.366371"
## [1] "Iter 11, obj 11.480038, abs 0.082252, rel 0.007114, norm 1.396695"
## [1] "Iter 12, obj 11.416135, abs 0.063902, rel 0.005566, norm 1.425153"
## [1] "Iter 13, obj 11.358809, abs 0.057327, rel 0.005022, norm 1.451888"
## [1] "Iter 14, obj 11.297591, abs 0.061218, rel 0.005389, norm 1.477197"
## [1] "Iter 15, obj 11.232974, abs 0.064617, rel 0.005720, norm 1.501774"
## [1] "Iter 16, obj 11.171743, abs 0.061232, rel 0.005451, norm 1.526657"
## [1] "Iter 17, obj 11.120305, abs 0.051438, rel 0.004604, norm 1.552598"
## [1] "Iter 18, obj 11.082195, abs 0.038110, rel 0.003427, norm 1.579719"
## [1] "Iter 19, obj 11.056592, abs 0.025603, rel 0.002310, norm 1.607627"
## [1] "Iter 20, obj 11.041182, abs 0.015410, rel 0.001394, norm 1.635793"
## [1] "Iter 21, obj 11.034251, abs 0.006931, rel 0.000628, norm 1.664008"
## [1] "Iter 22, obj 11.034757, abs 0.000506, rel 0.000046, norm 1.692317"
## [1] "Iter 23, obj 11.042073, abs 0.007315, rel 0.000663, norm 1.720779"
## [1] "Iter 24, obj 11.055833, abs 0.013760, rel 0.001246, norm 1.749377"
## [1] "Iter 25, obj 11.075808, abs 0.019976, rel 0.001807, norm 1.778029"

## [1] "Iter 26, obj 11.101781, abs 0.025972, rel 0.002345, norm 1.806595"
## [1] "Iter 27, obj 11.133447, abs 0.031666, rel 0.002852, norm 1.834900"
## [1] "Iter 28, obj 11.170372, abs 0.036925, rel 0.003317, norm 1.862752"
## [1] "Iter 29, obj 11.211988, abs 0.041616, rel 0.003726, norm 1.889959"
## [1] "Iter 30, obj 11.257624, abs 0.045636, rel 0.004070, norm 1.916346"
## [1] "Iter 31, obj 11.306552, abs 0.048928, rel 0.004346, norm 1.941763"
## [1] "Iter 32, obj 11.358028, abs 0.051476, rel 0.004553, norm 1.966089"
## [1] "Iter 33, obj 11.411325, abs 0.053297, rel 0.004692, norm 1.989234"
## [1] "Iter 34, obj 11.465750, abs 0.054424, rel 0.004769, norm 2.011136"
## [1] "Iter 35, obj 11.520650, abs 0.054900, rel 0.004788, norm 2.031757"
## [1] "Iter 36, obj 11.575426, abs 0.054777, rel 0.004755, norm 2.051086"
## [1] "Iter 37, obj 11.629541, abs 0.054115, rel 0.004675, norm 2.069131"
## [1] "Iter 38, obj 11.682524, abs 0.052983, rel 0.004556, norm 2.085922"
## [1] "Iter 39, obj 11.733980, abs 0.051456, rel 0.004405, norm 2.101502"
## [1] "Iter 40, obj 11.783590, abs 0.049609, rel 0.004228, norm 2.115926"
## [1] "Iter 41, obj 11.831106, abs 0.047517, rel 0.004032, norm 2.129255"
## [1] "Iter 42, obj 11.876351, abs 0.045245, rel 0.003824, norm 2.141554"
## [1] "Iter 43, obj 11.919207, abs 0.042855, rel 0.003608, norm 2.152888"
## [1] "Iter 44, obj 11.959605, abs 0.040398, rel 0.003389, norm 2.163320"
## [1] "Iter 45, obj 11.997522, abs 0.037917, rel 0.003170, norm 2.172915"
## [1] "Iter 46, obj 12.032970, abs 0.035447, rel 0.002955, norm 2.181732"
## [1] "Iter 47, obj 12.065986, abs 0.033017, rel 0.002744, norm 2.189828"
## [1] "Iter 48, obj 12.096635, abs 0.030648, rel 0.002540, norm 2.197257"
## [1] "Iter 49, obj 12.124994, abs 0.028360, rel 0.002344, norm 2.204070"
## [1] "Iter 50, obj 12.151162, abs 0.026168, rel 0.002158, norm 2.210314"

## [1] "Iter 51, obj 12.175247, abs 0.024084, rel 0.001982, norm 2.216030"
## [1] "Iter 52, obj 12.197366, abs 0.022119, rel 0.001817, norm 2.221257"
## [1] "Iter 53, obj 12.217644, abs 0.020278, rel 0.001663, norm 2.226031"
## [1] "Iter 54, obj 12.236208, abs 0.018564, rel 0.001519, norm 2.230386"
## [1] "Iter 55, obj 12.253184, abs 0.016976, rel 0.001387, norm 2.234353"
## [1] "Iter 56, obj 12.268693, abs 0.015508, rel 0.001266, norm 2.237964"

```

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## [1] "Iter 57, obj 12.282845, abs 0.014153, rel 0.001154, norm 2.241247"
## [1] "Iter 58, obj 12.295744, abs 0.012898, rel 0.001050, norm 2.244231"
## [1] "Iter 59, obj 12.307476, abs 0.011732, rel 0.000954, norm 2.246942"
## [1] "Iter 60, obj 12.318115, abs 0.010639, rel 0.000864, norm 2.249408"
## [1] "Iter 61, obj 12.327717, abs 0.009602, rel 0.000780, norm 2.251650"
## [1] "Iter 62, obj 12.336319, abs 0.008602, rel 0.000698, norm 2.253693"
## [1] "Iter 63, obj 12.343937, abs 0.007617, rel 0.000617, norm 2.255555"
## [1] "Iter 64, obj 12.350563, abs 0.006627, rel 0.000537, norm 2.257254"
## [1] "Iter 65, obj 12.356175, abs 0.005611, rel 0.000454, norm 2.258807"
## [1] "Iter 66, obj 12.360735, abs 0.004560, rel 0.000369, norm 2.260228"
## [1] "Iter 67, obj 12.364204, abs 0.003469, rel 0.000281, norm 2.261529"
## [1] "Iter 68, obj 12.366538, abs 0.002334, rel 0.000189, norm 2.262723"
## [1] "Iter 69, obj 12.367685, abs 0.001147, rel 0.000093, norm 2.263821"
## [1] "Iter 70, obj 12.367587, abs 0.000098, rel 0.000008, norm 2.264831"
## [1] "Iter 71, obj 12.366191, abs 0.001396, rel 0.000113, norm 2.265760"
## [1] "Iter 72, obj 12.363473, abs 0.002718, rel 0.000220, norm 2.266609"
## [1] "Iter 73, obj 12.359451, abs 0.004021, rel 0.000325, norm 2.267374"
## [1] "Iter 74, obj 12.354197, abs 0.005254, rel 0.000425, norm 2.268048"
## [1] "Iter 75, obj 12.347823, abs 0.006374, rel 0.000516, norm 2.268621"

## [1] "Iter 76, obj 12.340471, abs 0.007352, rel 0.000595, norm 2.269084"
## [1] "Iter 77, obj 12.332290, abs 0.008181, rel 0.000663, norm 2.269430"
## [1] "Iter 78, obj 12.323423, abs 0.008867, rel 0.000719, norm 2.269656"
## [1] "Iter 79, obj 12.313996, abs 0.009426, rel 0.000765, norm 2.269764"
## [1] "Iter 80, obj 12.304118, abs 0.009879, rel 0.000802, norm 2.269760"
## [1] "Iter 81, obj 12.293873, abs 0.010244, rel 0.000833, norm 2.269652"
## [1] "Iter 82, obj 12.283333, abs 0.010540, rel 0.000857, norm 2.269450"
## [1] "Iter 83, obj 12.272554, abs 0.010779, rel 0.000878, norm 2.269163"
## [1] "Iter 84, obj 12.261584, abs 0.010970, rel 0.000894, norm 2.268800"
## [1] "Iter 85, obj 12.250466, abs 0.011118, rel 0.000907, norm 2.268370"
## [1] "Iter 86, obj 12.239241, abs 0.011225, rel 0.000916, norm 2.267881"
## [1] "Iter 87, obj 12.227952, abs 0.011289, rel 0.000922, norm 2.267339"
## [1] "Iter 88, obj 12.216646, abs 0.011306, rel 0.000925, norm 2.266749"
## [1] "Iter 89, obj 12.205372, abs 0.011274, rel 0.000923, norm 2.266115"
## [1] "Iter 90, obj 12.194185, abs 0.011187, rel 0.000917, norm 2.265443"
## [1] "Iter 91, obj 12.183143, abs 0.011043, rel 0.000906, norm 2.264736"
## [1] "Iter 92, obj 12.172302, abs 0.010841, rel 0.000890, norm 2.263999"
## [1] "Iter 93, obj 12.161719, abs 0.010583, rel 0.000869, norm 2.263236"
## [1] "Iter 94, obj 12.151446, abs 0.010273, rel 0.000845, norm 2.262453"
## [1] "Iter 95, obj 12.141527, abs 0.009918, rel 0.000816, norm 2.261655"
## [1] "Iter 96, obj 12.132002, abs 0.009526, rel 0.000785, norm 2.260849"
## [1] "Iter 97, obj 12.122898, abs 0.009104, rel 0.000750, norm 2.260040"
## [1] "Iter 98, obj 12.114235, abs 0.008662, rel 0.000715, norm 2.259234"
## [1] "Iter 99, obj 12.106026, abs 0.008209, rel 0.000678, norm 2.258437"
## [1] "Iter 100, obj 12.098275, abs 0.007751, rel 0.000640, norm 2.257655"

## [1] "Iter 101, obj 1.423207, abs 10.675068, rel 0.882363, norm 0.184220"
## [1] "Iter 102, obj 1.252678, abs 0.170530, rel 0.119821, norm 0.177528"
## [1] "Iter 103, obj 1.192807, abs 0.059871, rel 0.047794, norm 0.186448"
## [1] "Iter 104, obj 1.186864, abs 0.005943, rel 0.004982, norm 0.193813"
## [1] "Iter 105, obj 1.180716, abs 0.006148, rel 0.005180, norm 0.195413"
## [1] "Iter 106, obj 1.183594, abs 0.002878, rel 0.002437, norm 0.195975"
## [1] "Iter 107, obj 1.181904, abs 0.001690, rel 0.001428, norm 0.196389"
## [1] "Iter 108, obj 1.173957, abs 0.007946, rel 0.006723, norm 0.196021"
## [1] "Iter 109, obj 1.172031, abs 0.001926, rel 0.001641, norm 0.195597"

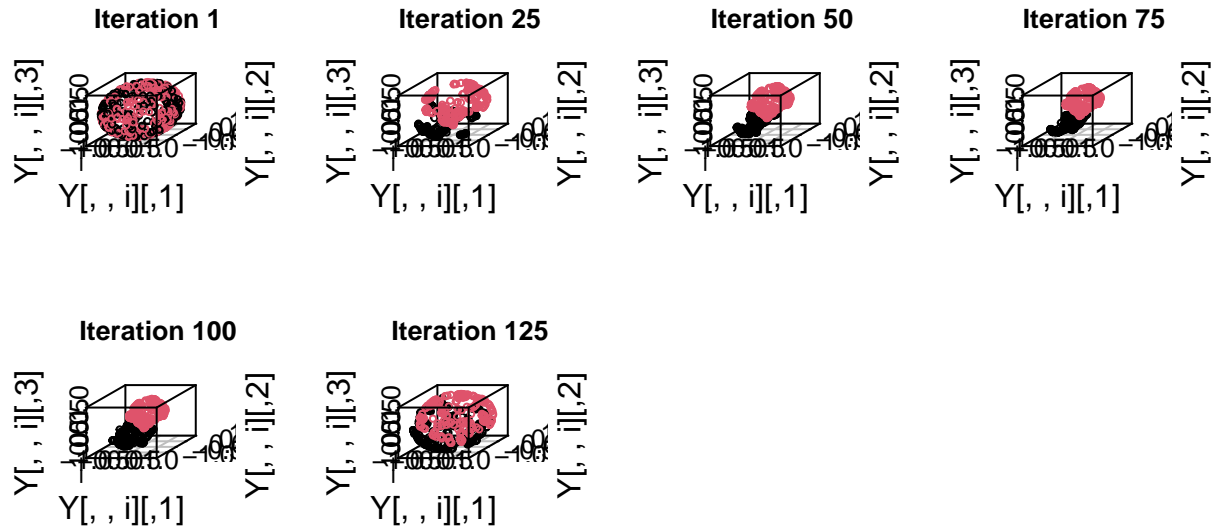
```



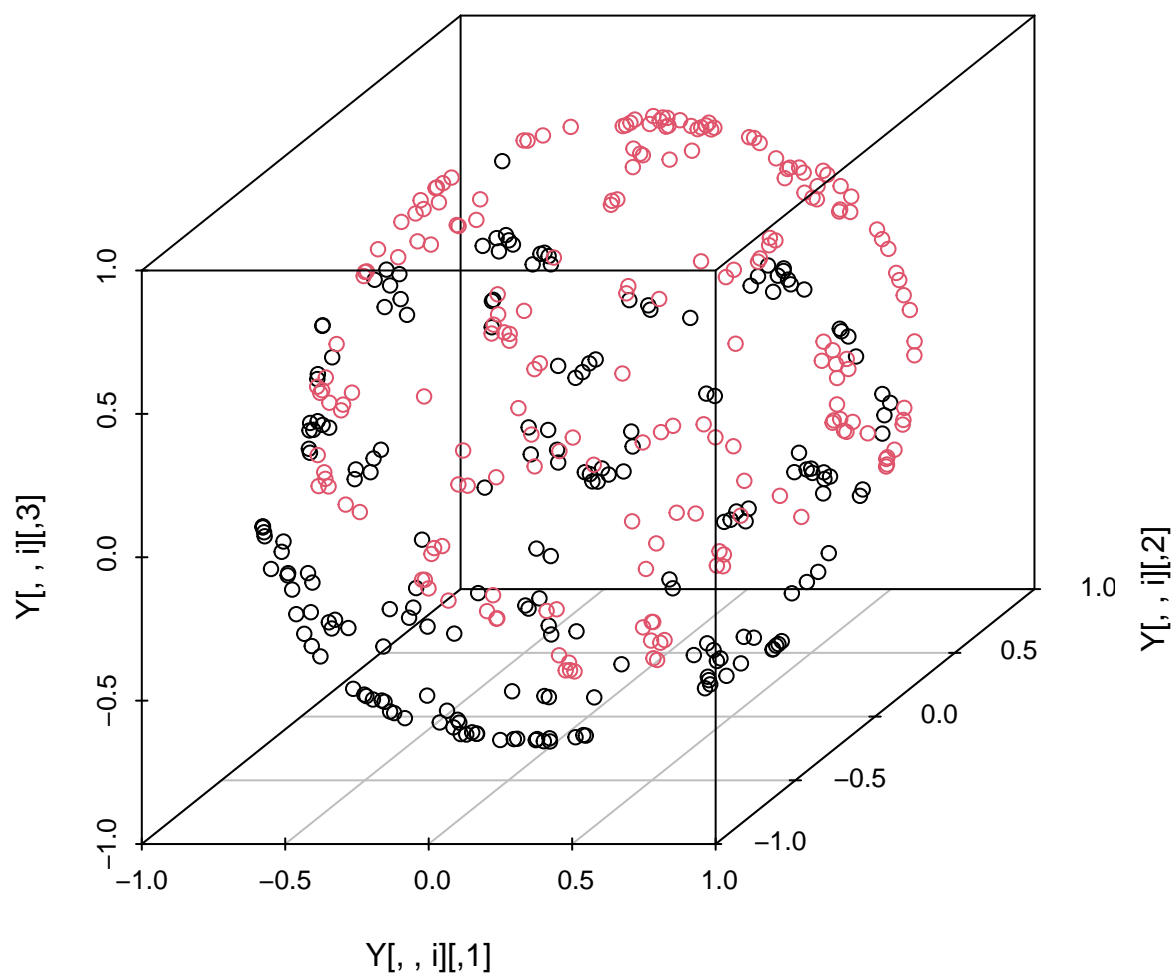
```

## [1] "Iter 110, obj 1.177936, abs 0.005905, rel 0.005038, norm 0.195580"
## [1] "Iter 111, obj 1.183240, abs 0.005304, rel 0.004503, norm 0.195559"
## [1] "Iter 112, obj 1.177091, abs 0.006149, rel 0.005197, norm 0.195780"
## [1] "Iter 113, obj 1.178091, abs 0.001001, rel 0.000850, norm 0.195404"
## [1] "Iter 114, obj 1.180293, abs 0.002202, rel 0.001869, norm 0.195325"
## [1] "Iter 115, obj 1.179933, abs 0.000360, rel 0.000305, norm 0.195449"
## [1] "Iter 116, obj 1.177315, abs 0.002619, rel 0.002219, norm 0.195610"
## [1] "Iter 117, obj 1.181242, abs 0.003928, rel 0.003336, norm 0.195093"
## [1] "Iter 118, obj 1.182509, abs 0.001267, rel 0.001073, norm 0.195386"
## [1] "Iter 119, obj 1.174059, abs 0.008451, rel 0.007146, norm 0.195714"
## [1] "Iter 120, obj 1.176154, abs 0.002096, rel 0.001785, norm 0.195193"
## [1] "Iter 121, obj 1.179937, abs 0.003783, rel 0.003216, norm 0.195305"
## [1] "Iter 122, obj 1.176055, abs 0.003881, rel 0.003290, norm 0.195216"
## [1] "Iter 123, obj 1.176762, abs 0.000707, rel 0.000601, norm 0.195671"
## [1] "Iter 124, obj 1.181732, abs 0.004970, rel 0.004224, norm 0.195009"
## [1] "Iter 125, obj 1.173373, abs 0.008359, rel 0.007074, norm 0.195336"

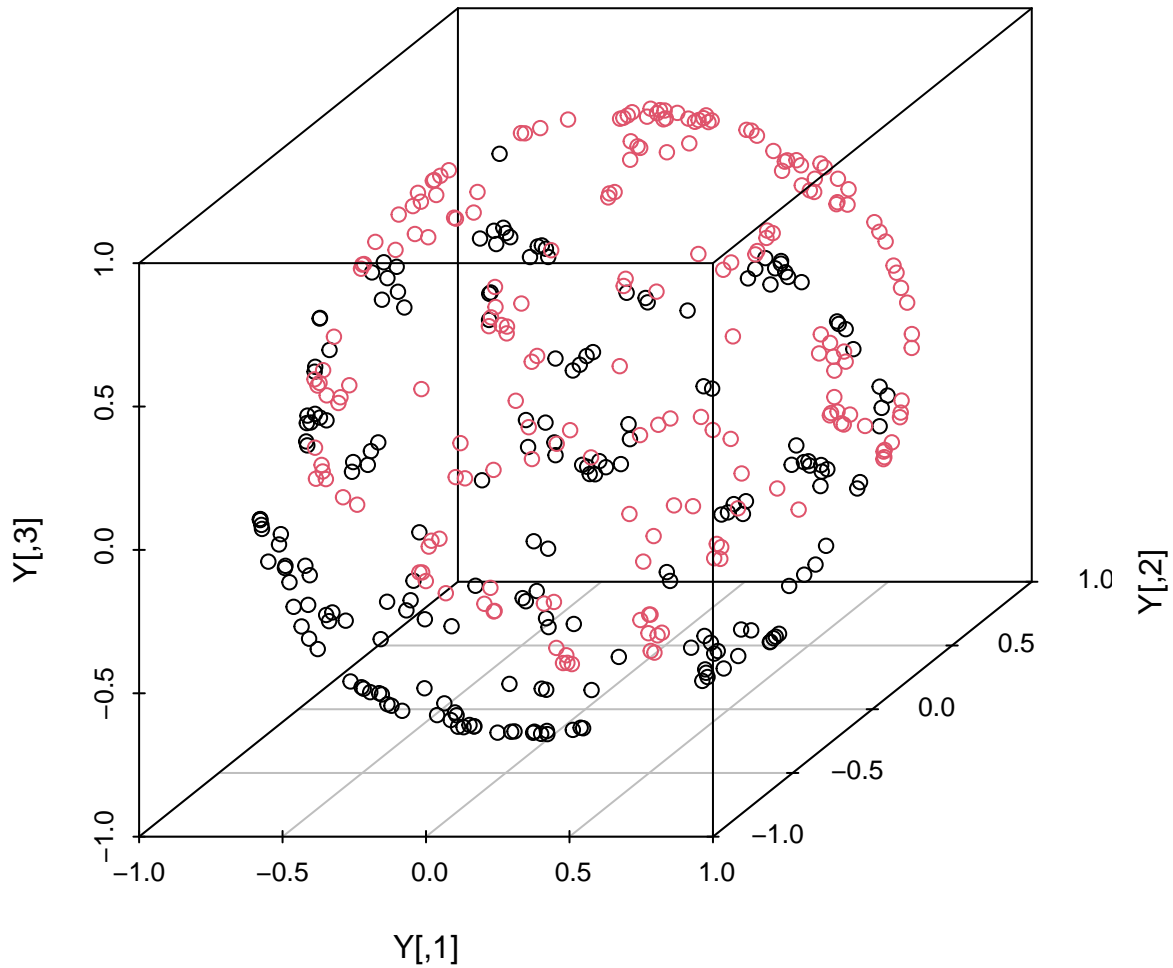
```



Iteration 125



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1),  
                             zlim = c(-1, 1), color = colors)
```



It's clearly clustered the two groups we had in the original generated data.

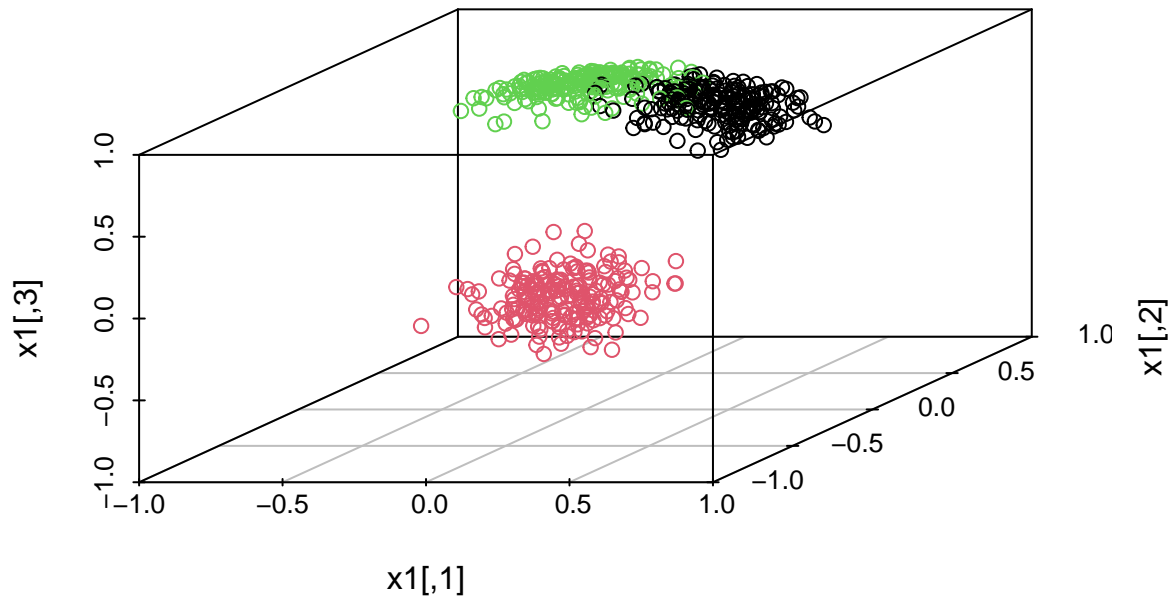
Case 2

Sample on the $(\mathbb{S}^2)^2$ where $p = 2$ and $r = 2$.

```
n=200
d=2
r=2
samp1 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 0, ph = 0.5)),
                        kappa = 50)
samp2 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 2, ph = -1.5)),
                        kappa = 50)
samp3 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = -1, ph = 0)),
                        kappa = 50)
x1 <- rbind(samp1, samp2, samp3)

scatterplot3d::scatterplot3d(x1,
                             xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(1:3, each = n), main="Sphere 1")
```

Sphere 1

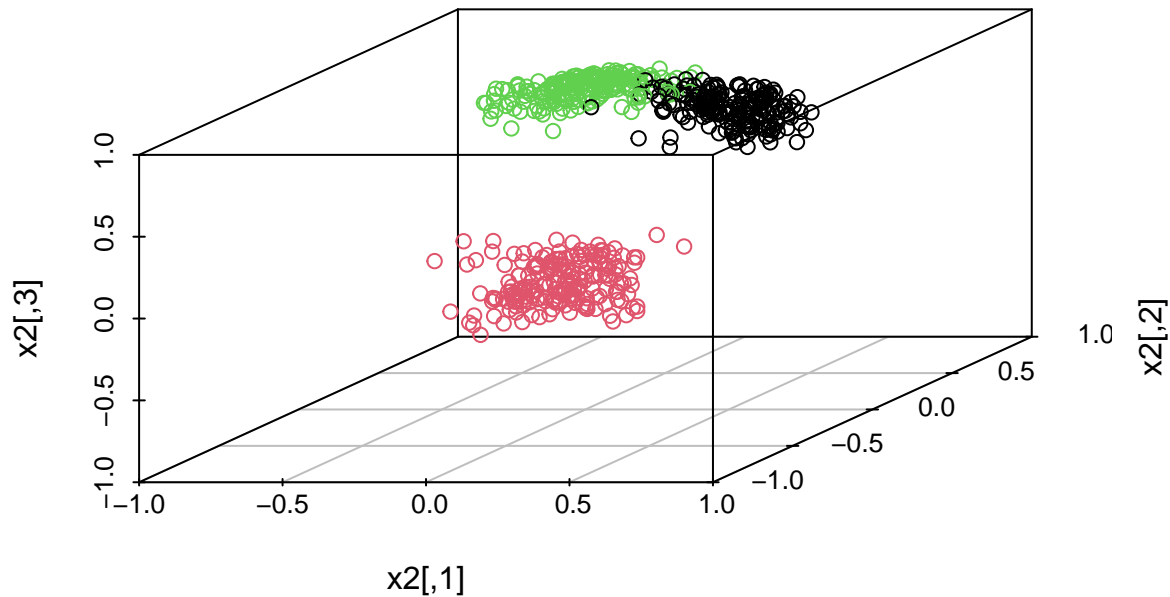


```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue", alpha = 0.25,
            lit = FALSE)
rgl::points3d(x1, col = rep(c(1,2,3), each=n))

samp4 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 0, ph = 0.55)),
                        kappa = 60)
samp5 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 2, ph = -1.41)),
                        kappa = 60)
samp6 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = -1.1, ph = 0.05)),
                        kappa = 60)
x2 <- rbind(samp4, samp5, samp6)

scatterplot3d::scatterplot3d(x2,
                              xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                              color = rep(1:3, each = n), main="Sphere 2")
```

Sphere 2



```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue", alpha = 0.25,
            lit = FALSE)
rgl::points3d(x2, col = rep(c(1,2,3), each=n))

x_2 <- array(dim = c(n*3, 3, 2))
x_2[, , 1] <- x1
x_2[, , 2] <- x2

n <- nrow(x_2)
indexes <- sample(1:n)
x_2 <- x_2[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]
```

Let's calculate the rho parameters based on a perplexity of 20:

```
rho_second_perp20 <- rho_optim_bst(x_2, 20)
```

```
## Time difference of 1.033727 mins
```

First, let's reduce to dimension \mathbb{S}^1 then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_2, d=1, rho_psc_list = rho_second_perp20, num_iteration=200,
            colors=colors, visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 18.706907, abs 0.000000, rel 0.000000, norm 0.135621"
## [1] "Iter 2, obj 18.002281, abs 0.704626, rel 0.037667, norm 0.254412"
## [1] "Iter 3, obj 17.584508, abs 0.417773, rel 0.023207, norm 0.352465"
## [1] "Iter 4, obj 17.310169, abs 0.274338, rel 0.015601, norm 0.412903"
## [1] "Iter 5, obj 17.119358, abs 0.190812, rel 0.011023, norm 0.462528"
## [1] "Iter 6, obj 16.957311, abs 0.162046, rel 0.009466, norm 0.496918"
## [1] "Iter 7, obj 16.822534, abs 0.134777, rel 0.007948, norm 0.527691"
```

```

## [1] "Iter 8, obj 16.694737, abs 0.127797, rel 0.007597, norm 0.550548"
## [1] "Iter 9, obj 16.593227, abs 0.101510, rel 0.006080, norm 0.573904"
## [1] "Iter 10, obj 16.516961, abs 0.076266, rel 0.004596, norm 0.596494"
## [1] "Iter 11, obj 16.447546, abs 0.069416, rel 0.004203, norm 0.617838"
## [1] "Iter 12, obj 16.375951, abs 0.071595, rel 0.004353, norm 0.638751"
## [1] "Iter 13, obj 16.305419, abs 0.070532, rel 0.004307, norm 0.661001"
## [1] "Iter 14, obj 16.228248, abs 0.077171, rel 0.004733, norm 0.683706"
## [1] "Iter 15, obj 16.147617, abs 0.080631, rel 0.004969, norm 0.705246"
## [1] "Iter 16, obj 16.074806, abs 0.072811, rel 0.004509, norm 0.726584"
## [1] "Iter 17, obj 16.004753, abs 0.070053, rel 0.004358, norm 0.746543"
## [1] "Iter 18, obj 15.933913, abs 0.070840, rel 0.004426, norm 0.764477"
## [1] "Iter 19, obj 15.869125, abs 0.064788, rel 0.004066, norm 0.780711"
## [1] "Iter 20, obj 15.810628, abs 0.058497, rel 0.003686, norm 0.794731"
## [1] "Iter 21, obj 15.755996, abs 0.054632, rel 0.003455, norm 0.806644"
## [1] "Iter 22, obj 15.704883, abs 0.051113, rel 0.003244, norm 0.817242"
## [1] "Iter 23, obj 15.656595, abs 0.048288, rel 0.003075, norm 0.827144"
## [1] "Iter 24, obj 15.607614, abs 0.048981, rel 0.003128, norm 0.836803"
## [1] "Iter 25, obj 15.557086, abs 0.050528, rel 0.003237, norm 0.846597"

## [1] "Iter 26, obj 15.510054, abs 0.047032, rel 0.003023, norm 0.856971"
## [1] "Iter 27, obj 15.475884, abs 0.034169, rel 0.002203, norm 0.867178"
## [1] "Iter 28, obj 15.451566, abs 0.024318, rel 0.001571, norm 0.875299"
## [1] "Iter 29, obj 15.431229, abs 0.020337, rel 0.001316, norm 0.881079"
## [1] "Iter 30, obj 15.414265, abs 0.016965, rel 0.001099, norm 0.885353"
## [1] "Iter 31, obj 15.404991, abs 0.009274, rel 0.000602, norm 0.888434"
## [1] "Iter 32, obj 15.404263, abs 0.000728, rel 0.000047, norm 0.889992"
## [1] "Iter 33, obj 15.408329, abs 0.004067, rel 0.000264, norm 0.890144"
## [1] "Iter 34, obj 15.414466, abs 0.006137, rel 0.000398, norm 0.889526"
## [1] "Iter 35, obj 15.421270, abs 0.006804, rel 0.000441, norm 0.888550"
## [1] "Iter 36, obj 15.428049, abs 0.006779, rel 0.000440, norm 0.887418"
## [1] "Iter 37, obj 15.434477, abs 0.006428, rel 0.000417, norm 0.886233"
## [1] "Iter 38, obj 15.440419, abs 0.005942, rel 0.000385, norm 0.885047"
## [1] "Iter 39, obj 15.445839, abs 0.005420, rel 0.000351, norm 0.883892"
## [1] "Iter 40, obj 15.450750, abs 0.004911, rel 0.000318, norm 0.882782"
## [1] "Iter 41, obj 15.455188, abs 0.004438, rel 0.000287, norm 0.881727"
## [1] "Iter 42, obj 15.459199, abs 0.004011, rel 0.000260, norm 0.880731"
## [1] "Iter 43, obj 15.462832, abs 0.003633, rel 0.000235, norm 0.879794"
## [1] "Iter 44, obj 15.466137, abs 0.003305, rel 0.000214, norm 0.878917"
## [1] "Iter 45, obj 15.469162, abs 0.003025, rel 0.000196, norm 0.878099"
## [1] "Iter 46, obj 15.471954, abs 0.002792, rel 0.000181, norm 0.877340"
## [1] "Iter 47, obj 15.474547, abs 0.002592, rel 0.000168, norm 0.876640"
## [1] "Iter 48, obj 15.476920, abs 0.002374, rel 0.000153, norm 0.875998"
## [1] "Iter 49, obj 15.478849, abs 0.001929, rel 0.000125, norm 0.875415"
## [1] "Iter 50, obj 15.479296, abs 0.000448, rel 0.000029, norm 0.874894"

## [1] "Iter 51, obj 15.474741, abs 0.004555, rel 0.000294, norm 0.874441"
## [1] "Iter 52, obj 15.461133, abs 0.013608, rel 0.000879, norm 0.874141"
## [1] "Iter 53, obj 15.442810, abs 0.018323, rel 0.001185, norm 0.874346"
## [1] "Iter 54, obj 15.426888, abs 0.015922, rel 0.001031, norm 0.875321"
## [1] "Iter 55, obj 15.418799, abs 0.008088, rel 0.000524, norm 0.876733"
## [1] "Iter 56, obj 15.416051, abs 0.002748, rel 0.000178, norm 0.877655"
## [1] "Iter 57, obj 15.415451, abs 0.000600, rel 0.000039, norm 0.877735"
## [1] "Iter 58, obj 15.415349, abs 0.000102, rel 0.000007, norm 0.877136"
## [1] "Iter 59, obj 15.415192, abs 0.000157, rel 0.000010, norm 0.876147"
## [1] "Iter 60, obj 15.414875, abs 0.000318, rel 0.000021, norm 0.874977"

```

```

## [1] "Iter 61, obj 15.414413, abs 0.000462, rel 0.000030, norm 0.873735"
## [1] "Iter 62, obj 15.413840, abs 0.000572, rel 0.000037, norm 0.872469"
## [1] "Iter 63, obj 15.413186, abs 0.000654, rel 0.000042, norm 0.871201"
## [1] "Iter 64, obj 15.412471, abs 0.000715, rel 0.000046, norm 0.869942"
## [1] "Iter 65, obj 15.411710, abs 0.000761, rel 0.000049, norm 0.868695"
## [1] "Iter 66, obj 15.410913, abs 0.000797, rel 0.000052, norm 0.867465"
## [1] "Iter 67, obj 15.410089, abs 0.000824, rel 0.000053, norm 0.866250"
## [1] "Iter 68, obj 15.409243, abs 0.000846, rel 0.000055, norm 0.865052"
## [1] "Iter 69, obj 15.408380, abs 0.000863, rel 0.000056, norm 0.863869"
## [1] "Iter 70, obj 15.407504, abs 0.000875, rel 0.000057, norm 0.862702"
## [1] "Iter 71, obj 15.406620, abs 0.000885, rel 0.000057, norm 0.861550"
## [1] "Iter 72, obj 15.405728, abs 0.000892, rel 0.000058, norm 0.860411"
## [1] "Iter 73, obj 15.404832, abs 0.000896, rel 0.000058, norm 0.859287"
## [1] "Iter 74, obj 15.403934, abs 0.000898, rel 0.000058, norm 0.858175"
## [1] "Iter 75, obj 15.403036, abs 0.000898, rel 0.000058, norm 0.857075"

## [1] "Iter 76, obj 15.402138, abs 0.000897, rel 0.000058, norm 0.855988"
## [1] "Iter 77, obj 15.401244, abs 0.000895, rel 0.000058, norm 0.854911"
## [1] "Iter 78, obj 15.400352, abs 0.000891, rel 0.000058, norm 0.853845"
## [1] "Iter 79, obj 15.399465, abs 0.000887, rel 0.000058, norm 0.852789"
## [1] "Iter 80, obj 15.398583, abs 0.000882, rel 0.000057, norm 0.851743"
## [1] "Iter 81, obj 15.397706, abs 0.000877, rel 0.000057, norm 0.850706"
## [1] "Iter 82, obj 15.396833, abs 0.000872, rel 0.000057, norm 0.849678"
## [1] "Iter 83, obj 15.395965, abs 0.000868, rel 0.000056, norm 0.848659"
## [1] "Iter 84, obj 15.395102, abs 0.000864, rel 0.000056, norm 0.847648"
## [1] "Iter 85, obj 15.394241, abs 0.000861, rel 0.000056, norm 0.846645"
## [1] "Iter 86, obj 15.393381, abs 0.000859, rel 0.000056, norm 0.845650"
## [1] "Iter 87, obj 15.392522, abs 0.000859, rel 0.000056, norm 0.844662"
## [1] "Iter 88, obj 15.391660, abs 0.000862, rel 0.000056, norm 0.843681"
## [1] "Iter 89, obj 15.390794, abs 0.000866, rel 0.000056, norm 0.842707"
## [1] "Iter 90, obj 15.389920, abs 0.000874, rel 0.000057, norm 0.841740"
## [1] "Iter 91, obj 15.389036, abs 0.000884, rel 0.000057, norm 0.840779"
## [1] "Iter 92, obj 15.388138, abs 0.000898, rel 0.000058, norm 0.839824"
## [1] "Iter 93, obj 15.387223, abs 0.000915, rel 0.000059, norm 0.838876"
## [1] "Iter 94, obj 15.386287, abs 0.000936, rel 0.000061, norm 0.837933"
## [1] "Iter 95, obj 15.385327, abs 0.000960, rel 0.000062, norm 0.836996"
## [1] "Iter 96, obj 15.384340, abs 0.000986, rel 0.000064, norm 0.836064"
## [1] "Iter 97, obj 15.383325, abs 0.001015, rel 0.000066, norm 0.835137"
## [1] "Iter 98, obj 15.382279, abs 0.001046, rel 0.000068, norm 0.834215"
## [1] "Iter 99, obj 15.381202, abs 0.001077, rel 0.000070, norm 0.833297"
## [1] "Iter 100, obj 15.380094, abs 0.001108, rel 0.000072, norm 0.832384"

## [1] "Iter 101, obj 2.426495, abs 12.953599, rel 0.842231, norm 0.083538"
## [1] "Iter 102, obj 2.390907, abs 0.035587, rel 0.014666, norm 0.090137"
## [1] "Iter 103, obj 2.369106, abs 0.021801, rel 0.009118, norm 0.099935"
## [1] "Iter 104, obj 2.360122, abs 0.008984, rel 0.003792, norm 0.105777"
## [1] "Iter 105, obj 2.356735, abs 0.003387, rel 0.001435, norm 0.108239"
## [1] "Iter 106, obj 2.354965, abs 0.001770, rel 0.000751, norm 0.109188"
## [1] "Iter 107, obj 2.353792, abs 0.001173, rel 0.000498, norm 0.109594"
## [1] "Iter 108, obj 2.352939, abs 0.000853, rel 0.000362, norm 0.109815"
## [1] "Iter 109, obj 2.352298, abs 0.000640, rel 0.000272, norm 0.109968"
## [1] "Iter 110, obj 2.351787, abs 0.000511, rel 0.000217, norm 0.110082"
## [1] "Iter 111, obj 2.351412, abs 0.000375, rel 0.000160, norm 0.110158"
## [1] "Iter 112, obj 2.351157, abs 0.000255, rel 0.000108, norm 0.110203"
## [1] "Iter 113, obj 2.350960, abs 0.000197, rel 0.000084, norm 0.110236"

```

```

## [1] "Iter 114, obj 2.350797, abs 0.000163, rel 0.000070, norm 0.110263"
## [1] "Iter 115, obj 2.350737, abs 0.000060, rel 0.000025, norm 0.110296"
## [1] "Iter 116, obj 2.350631, abs 0.000106, rel 0.000045, norm 0.110328"
## [1] "Iter 117, obj 2.350541, abs 0.000090, rel 0.000038, norm 0.110328"
## [1] "Iter 118, obj 2.350546, abs 0.000005, rel 0.000002, norm 0.110305"
## [1] "Iter 119, obj 2.350583, abs 0.000036, rel 0.000016, norm 0.110300"
## [1] "Iter 120, obj 2.350571, abs 0.000012, rel 0.000005, norm 0.110328"
## [1] "Iter 121, obj 2.350563, abs 0.000008, rel 0.000003, norm 0.110420"
## [1] "Iter 122, obj 2.350103, abs 0.000459, rel 0.000195, norm 0.110412"
## [1] "Iter 123, obj 2.350429, abs 0.000325, rel 0.000138, norm 0.110319"
## [1] "Iter 124, obj 2.350414, abs 0.000015, rel 0.000006, norm 0.110377"
## [1] "Iter 125, obj 2.350172, abs 0.000242, rel 0.000103, norm 0.110408"

## [1] "Iter 126, obj 2.350404, abs 0.000232, rel 0.000099, norm 0.110394"
## [1] "Iter 127, obj 2.350460, abs 0.000056, rel 0.000024, norm 0.110454"
## [1] "Iter 128, obj 2.350245, abs 0.000215, rel 0.000092, norm 0.110445"
## [1] "Iter 129, obj 2.350642, abs 0.000397, rel 0.000169, norm 0.110367"
## [1] "Iter 130, obj 2.350915, abs 0.000273, rel 0.000116, norm 0.110492"
## [1] "Iter 131, obj 2.350344, abs 0.000571, rel 0.000243, norm 0.110560"
## [1] "Iter 132, obj 2.350119, abs 0.000225, rel 0.000096, norm 0.110508"
## [1] "Iter 133, obj 2.350243, abs 0.000124, rel 0.000053, norm 0.110548"
## [1] "Iter 134, obj 2.350058, abs 0.000185, rel 0.000079, norm 0.110510"
## [1] "Iter 135, obj 2.350428, abs 0.000370, rel 0.000157, norm 0.110419"
## [1] "Iter 136, obj 2.350807, abs 0.000380, rel 0.000162, norm 0.110547"
## [1] "Iter 137, obj 2.350196, abs 0.000612, rel 0.000260, norm 0.110633"
## [1] "Iter 138, obj 2.349981, abs 0.000215, rel 0.000091, norm 0.110598"
## [1] "Iter 139, obj 2.350087, abs 0.000107, rel 0.000045, norm 0.110604"
## [1] "Iter 140, obj 2.349990, abs 0.000097, rel 0.000041, norm 0.110569"
## [1] "Iter 141, obj 2.350298, abs 0.000308, rel 0.000131, norm 0.110469"
## [1] "Iter 142, obj 2.350702, abs 0.000404, rel 0.000172, norm 0.110588"
## [1] "Iter 143, obj 2.350105, abs 0.000597, rel 0.000254, norm 0.110672"
## [1] "Iter 144, obj 2.349900, abs 0.000205, rel 0.000087, norm 0.110631"
## [1] "Iter 145, obj 2.350013, abs 0.000113, rel 0.000048, norm 0.110633"
## [1] "Iter 146, obj 2.349905, abs 0.000108, rel 0.000046, norm 0.110591"
## [1] "Iter 147, obj 2.350238, abs 0.000333, rel 0.000142, norm 0.110487"
## [1] "Iter 148, obj 2.350640, abs 0.000402, rel 0.000171, norm 0.110604"
## [1] "Iter 149, obj 2.350051, abs 0.000589, rel 0.000250, norm 0.110685"
## [1] "Iter 150, obj 2.349847, abs 0.000205, rel 0.000087, norm 0.110639"

## [1] "Iter 151, obj 2.349969, abs 0.000122, rel 0.000052, norm 0.110642"
## [1] "Iter 152, obj 2.349849, abs 0.000120, rel 0.000051, norm 0.110597"
## [1] "Iter 153, obj 2.350201, abs 0.000352, rel 0.000150, norm 0.110491"
## [1] "Iter 154, obj 2.350601, abs 0.000400, rel 0.000170, norm 0.110609"
## [1] "Iter 155, obj 2.350019, abs 0.000582, rel 0.000248, norm 0.110688"
## [1] "Iter 156, obj 2.349815, abs 0.000204, rel 0.000087, norm 0.110640"
## [1] "Iter 157, obj 2.349943, abs 0.000128, rel 0.000054, norm 0.110643"
## [1] "Iter 158, obj 2.349815, abs 0.000127, rel 0.000054, norm 0.110597"
## [1] "Iter 159, obj 2.350179, abs 0.000364, rel 0.000155, norm 0.110491"
## [1] "Iter 160, obj 2.350577, abs 0.000398, rel 0.000169, norm 0.110609"
## [1] "Iter 161, obj 2.350000, abs 0.000577, rel 0.000246, norm 0.110687"
## [1] "Iter 162, obj 2.349798, abs 0.000203, rel 0.000086, norm 0.110639"
## [1] "Iter 163, obj 2.349928, abs 0.000131, rel 0.000056, norm 0.110643"
## [1] "Iter 164, obj 2.349797, abs 0.000132, rel 0.000056, norm 0.110595"
## [1] "Iter 165, obj 2.350167, abs 0.000371, rel 0.000158, norm 0.110490"
## [1] "Iter 166, obj 2.350564, abs 0.000397, rel 0.000169, norm 0.110608"

```

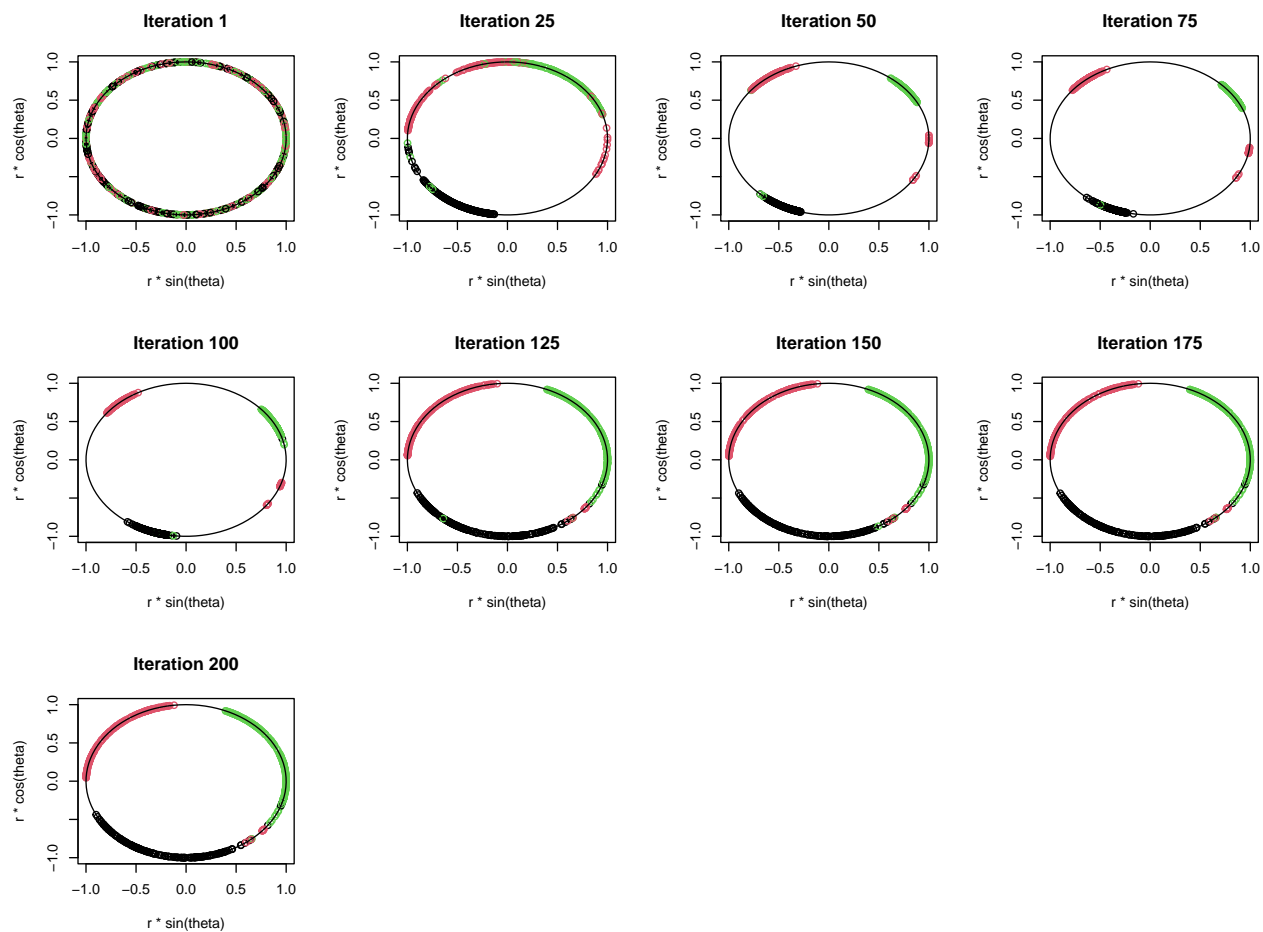


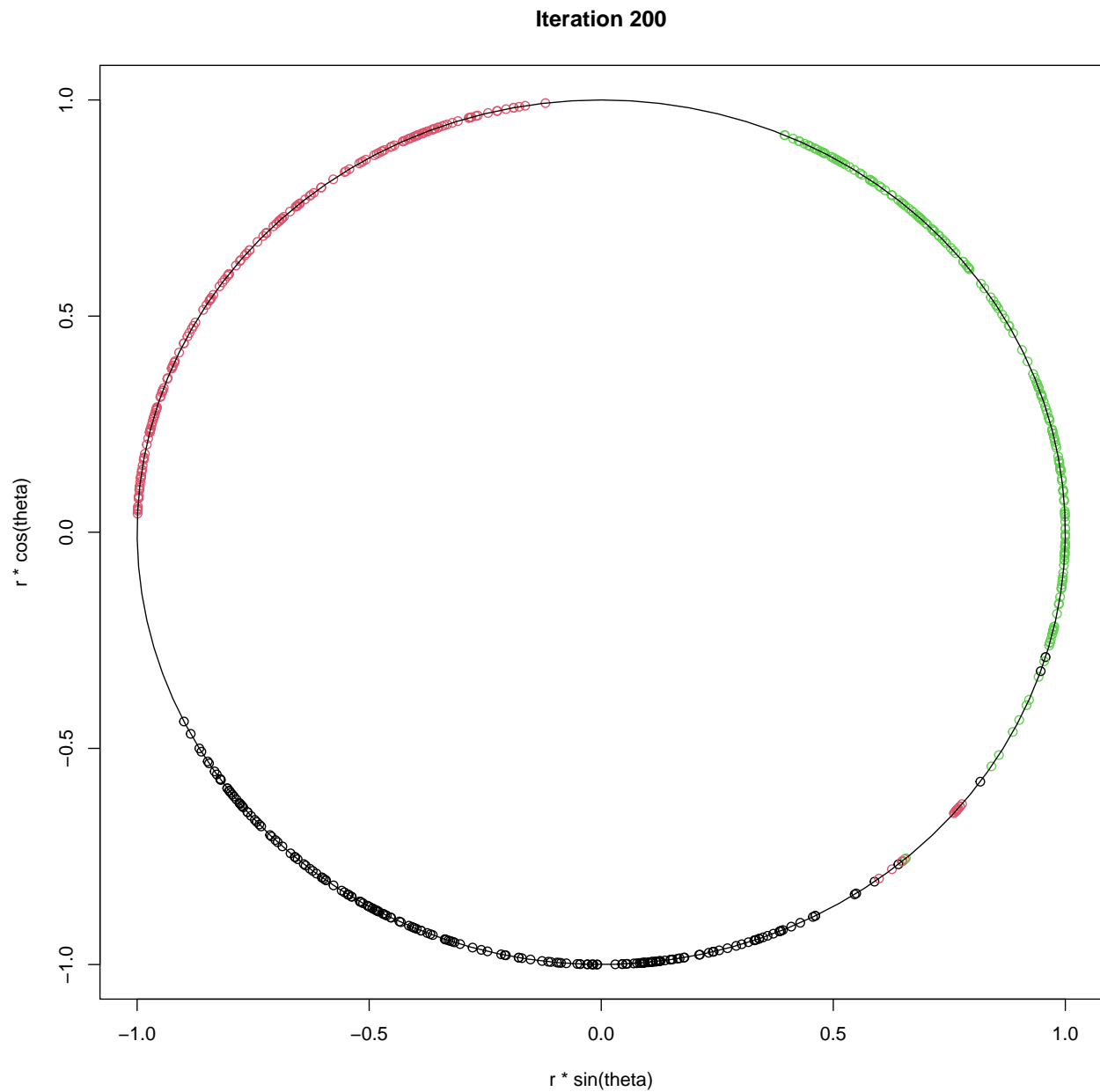
```

## [1] "Iter 167, obj 2.349990, abs 0.000574, rel 0.000244, norm 0.110686"
## [1] "Iter 168, obj 2.349789, abs 0.000202, rel 0.000086, norm 0.110638"
## [1] "Iter 169, obj 2.349921, abs 0.000132, rel 0.000056, norm 0.110642"
## [1] "Iter 170, obj 2.349786, abs 0.000134, rel 0.000057, norm 0.110594"
## [1] "Iter 171, obj 2.350161, abs 0.000374, rel 0.000159, norm 0.110489"
## [1] "Iter 172, obj 2.350557, abs 0.000396, rel 0.000169, norm 0.110608"
## [1] "Iter 173, obj 2.349985, abs 0.000572, rel 0.000243, norm 0.110685"
## [1] "Iter 174, obj 2.349784, abs 0.000201, rel 0.000085, norm 0.110638"
## [1] "Iter 175, obj 2.349917, abs 0.000133, rel 0.000057, norm 0.110642"

## [1] "Iter 176, obj 2.349781, abs 0.000136, rel 0.000058, norm 0.110593"
## [1] "Iter 177, obj 2.350158, abs 0.000377, rel 0.000160, norm 0.110489"
## [1] "Iter 178, obj 2.350553, abs 0.000396, rel 0.000168, norm 0.110608"
## [1] "Iter 179, obj 2.349982, abs 0.000571, rel 0.000243, norm 0.110685"
## [1] "Iter 180, obj 2.349782, abs 0.000200, rel 0.000085, norm 0.110638"
## [1] "Iter 181, obj 2.349915, abs 0.000133, rel 0.000057, norm 0.110642"
## [1] "Iter 182, obj 2.349778, abs 0.000137, rel 0.000058, norm 0.110593"
## [1] "Iter 183, obj 2.350156, abs 0.000378, rel 0.000161, norm 0.110489"
## [1] "Iter 184, obj 2.350552, abs 0.000396, rel 0.000168, norm 0.110608"
## [1] "Iter 185, obj 2.349981, abs 0.000570, rel 0.000243, norm 0.110685"
## [1] "Iter 186, obj 2.349781, abs 0.000200, rel 0.000085, norm 0.110638"
## [1] "Iter 187, obj 2.349914, abs 0.000133, rel 0.000057, norm 0.110642"
## [1] "Iter 188, obj 2.349777, abs 0.000138, rel 0.000059, norm 0.110593"
## [1] "Iter 189, obj 2.350155, abs 0.000379, rel 0.000161, norm 0.110489"
## [1] "Iter 190, obj 2.350550, abs 0.000395, rel 0.000168, norm 0.110608"
## [1] "Iter 191, obj 2.349980, abs 0.000570, rel 0.000243, norm 0.110685"
## [1] "Iter 192, obj 2.349781, abs 0.000200, rel 0.000085, norm 0.110638"
## [1] "Iter 193, obj 2.349914, abs 0.000133, rel 0.000057, norm 0.110643"
## [1] "Iter 194, obj 2.349776, abs 0.000138, rel 0.000059, norm 0.110593"
## [1] "Iter 195, obj 2.350155, abs 0.000379, rel 0.000161, norm 0.110489"
## [1] "Iter 196, obj 2.350550, abs 0.000395, rel 0.000168, norm 0.110609"
## [1] "Iter 197, obj 2.349980, abs 0.000570, rel 0.000242, norm 0.110686"
## [1] "Iter 198, obj 2.349780, abs 0.000200, rel 0.000085, norm 0.110638"
## [1] "Iter 199, obj 2.349914, abs 0.000133, rel 0.000057, norm 0.110643"
## [1] "Iter 200, obj 2.349775, abs 0.000139, rel 0.000059, norm 0.110593"

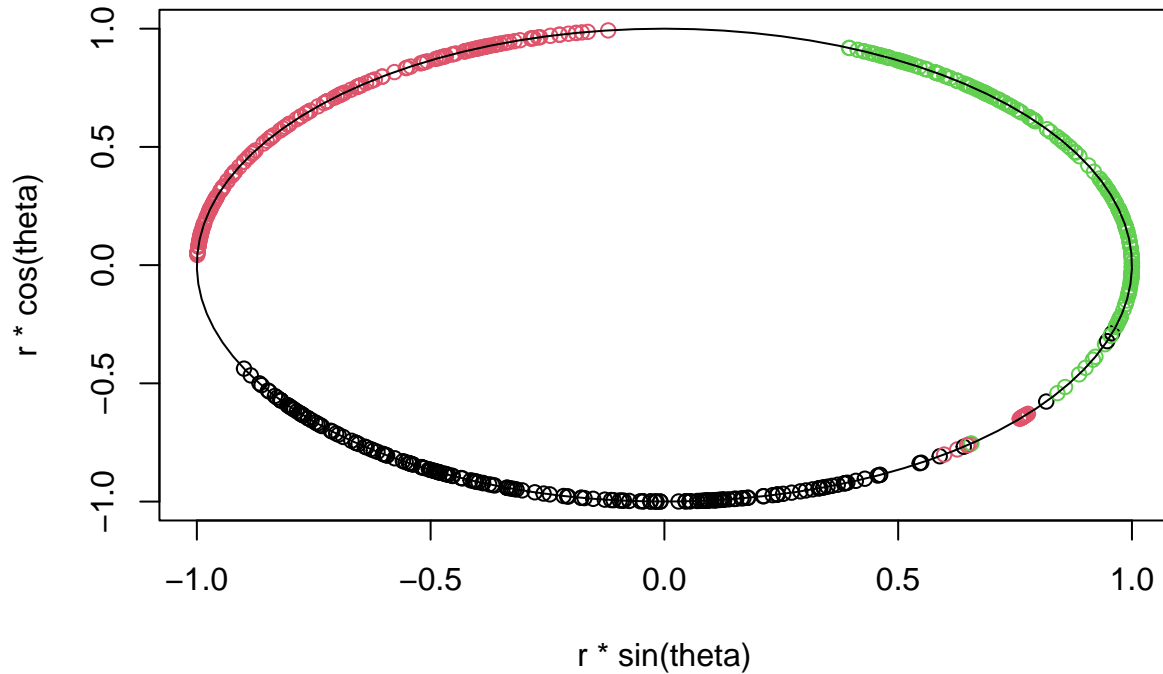
```





```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension \mathbb{S}^2 then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_2, d=2, rho_psc_list = rho_second_perp20, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 19.394365, abs 0.000000, rel 0.000000, norm 0.175546"
## [1] "Iter 2, obj 16.465063, abs 2.929302, rel 0.151039, norm 0.353526"
## [1] "Iter 3, obj 15.189535, abs 1.275528, rel 0.077469, norm 0.626357"
## [1] "Iter 4, obj 14.561233, abs 0.628303, rel 0.041364, norm 0.788881"
## [1] "Iter 5, obj 14.101342, abs 0.459891, rel 0.031583, norm 0.903470"
## [1] "Iter 6, obj 13.744759, abs 0.356583, rel 0.025287, norm 0.996351"
## [1] "Iter 7, obj 13.486043, abs 0.258716, rel 0.018823, norm 1.082091"
## [1] "Iter 8, obj 13.314368, abs 0.171675, rel 0.012730, norm 1.159303"
## [1] "Iter 9, obj 13.211315, abs 0.103053, rel 0.007740, norm 1.224989"
## [1] "Iter 10, obj 13.148397, abs 0.062918, rel 0.004762, norm 1.280454"
## [1] "Iter 11, obj 13.106941, abs 0.041456, rel 0.003153, norm 1.329852"
## [1] "Iter 12, obj 13.079171, abs 0.027770, rel 0.002119, norm 1.376476"
## [1] "Iter 13, obj 13.061870, abs 0.017302, rel 0.001323, norm 1.420807"
## [1] "Iter 14, obj 13.052431, abs 0.009439, rel 0.000723, norm 1.461797"
## [1] "Iter 15, obj 13.053095, abs 0.000664, rel 0.000051, norm 1.498202"
## [1] "Iter 16, obj 13.069225, abs 0.016129, rel 0.001236, norm 1.529423"
## [1] "Iter 17, obj 13.103422, abs 0.034197, rel 0.002617, norm 1.554878"
## [1] "Iter 18, obj 13.154122, abs 0.050701, rel 0.003869, norm 1.574690"
## [1] "Iter 19, obj 13.214248, abs 0.060126, rel 0.004571, norm 1.589562"
## [1] "Iter 20, obj 13.277192, abs 0.062944, rel 0.004763, norm 1.600608"
## [1] "Iter 21, obj 13.338975, abs 0.061784, rel 0.004653, norm 1.608802"
## [1] "Iter 22, obj 13.397335, abs 0.058360, rel 0.004375, norm 1.614813"
## [1] "Iter 23, obj 13.451016, abs 0.053681, rel 0.004007, norm 1.619091"
## [1] "Iter 24, obj 13.499387, abs 0.048370, rel 0.003596, norm 1.621961"
## [1] "Iter 25, obj 13.542271, abs 0.042884, rel 0.003177, norm 1.623681"
## [1] "Iter 26, obj 13.579796, abs 0.037526, rel 0.002771, norm 1.624465"
## [1] "Iter 27, obj 13.612293, abs 0.032497, rel 0.002393, norm 1.624494"
```

```

## [1] "Iter 28, obj 13.640188, abs 0.027895, rel 0.002049, norm 1.623916"
## [1] "Iter 29, obj 13.663948, abs 0.023760, rel 0.001742, norm 1.622854"
## [1] "Iter 30, obj 13.684036, abs 0.020088, rel 0.001470, norm 1.621406"
## [1] "Iter 31, obj 13.700889, abs 0.016853, rel 0.001232, norm 1.619653"
## [1] "Iter 32, obj 13.714906, abs 0.014017, rel 0.001023, norm 1.617659"
## [1] "Iter 33, obj 13.726444, abs 0.011538, rel 0.000841, norm 1.615474"
## [1] "Iter 34, obj 13.735820, abs 0.009376, rel 0.000683, norm 1.613139"
## [1] "Iter 35, obj 13.743312, abs 0.007492, rel 0.000545, norm 1.610690"
## [1] "Iter 36, obj 13.749161, abs 0.005849, rel 0.000426, norm 1.608152"
## [1] "Iter 37, obj 13.753580, abs 0.004419, rel 0.000321, norm 1.605548"
## [1] "Iter 38, obj 13.756751, abs 0.003171, rel 0.000231, norm 1.602897"
## [1] "Iter 39, obj 13.758834, abs 0.002083, rel 0.000151, norm 1.600215"
## [1] "Iter 40, obj 13.759969, abs 0.001135, rel 0.000082, norm 1.597516"
## [1] "Iter 41, obj 13.760276, abs 0.000307, rel 0.000022, norm 1.594809"
## [1] "Iter 42, obj 13.759860, abs 0.000416, rel 0.000030, norm 1.592107"
## [1] "Iter 43, obj 13.758814, abs 0.001046, rel 0.000076, norm 1.589416"
## [1] "Iter 44, obj 13.757217, abs 0.001597, rel 0.000116, norm 1.586746"
## [1] "Iter 45, obj 13.755140, abs 0.002077, rel 0.000151, norm 1.584104"
## [1] "Iter 46, obj 13.752645, abs 0.002495, rel 0.000181, norm 1.581496"
## [1] "Iter 47, obj 13.749785, abs 0.002860, rel 0.000208, norm 1.578929"
## [1] "Iter 48, obj 13.746608, abs 0.003177, rel 0.000231, norm 1.576407"
## [1] "Iter 49, obj 13.743155, abs 0.003453, rel 0.000251, norm 1.573938"
## [1] "Iter 50, obj 13.739463, abs 0.003692, rel 0.000269, norm 1.571525"

## [1] "Iter 51, obj 13.735564, abs 0.003899, rel 0.000284, norm 1.569175"
## [1] "Iter 52, obj 13.731487, abs 0.004077, rel 0.000297, norm 1.566892"
## [1] "Iter 53, obj 13.727257, abs 0.004230, rel 0.000308, norm 1.564680"
## [1] "Iter 54, obj 13.722897, abs 0.004359, rel 0.000318, norm 1.562544"
## [1] "Iter 55, obj 13.718429, abs 0.004469, rel 0.000326, norm 1.560490"
## [1] "Iter 56, obj 13.713870, abs 0.004559, rel 0.000332, norm 1.558520"
## [1] "Iter 57, obj 13.709238, abs 0.004632, rel 0.000338, norm 1.556639"
## [1] "Iter 58, obj 13.704550, abs 0.004688, rel 0.000342, norm 1.554851"
## [1] "Iter 59, obj 13.699822, abs 0.004729, rel 0.000345, norm 1.553159"
## [1] "Iter 60, obj 13.695068, abs 0.004754, rel 0.000347, norm 1.551568"
## [1] "Iter 61, obj 13.690303, abs 0.004765, rel 0.000348, norm 1.550079"
## [1] "Iter 62, obj 13.685543, abs 0.004760, rel 0.000348, norm 1.548697"
## [1] "Iter 63, obj 13.680801, abs 0.004742, rel 0.000346, norm 1.547422"
## [1] "Iter 64, obj 13.676092, abs 0.004709, rel 0.000344, norm 1.546259"
## [1] "Iter 65, obj 13.671431, abs 0.004661, rel 0.000341, norm 1.545208"
## [1] "Iter 66, obj 13.666830, abs 0.004600, rel 0.000336, norm 1.544270"
## [1] "Iter 67, obj 13.662305, abs 0.004526, rel 0.000331, norm 1.543447"
## [1] "Iter 68, obj 13.657866, abs 0.004439, rel 0.000325, norm 1.542740"
## [1] "Iter 69, obj 13.653527, abs 0.004339, rel 0.000318, norm 1.542149"
## [1] "Iter 70, obj 13.649297, abs 0.004229, rel 0.000310, norm 1.541672"
## [1] "Iter 71, obj 13.645188, abs 0.004110, rel 0.000301, norm 1.541311"
## [1] "Iter 72, obj 13.641207, abs 0.003981, rel 0.000292, norm 1.541064"
## [1] "Iter 73, obj 13.637361, abs 0.003845, rel 0.000282, norm 1.540929"
## [1] "Iter 74, obj 13.633657, abs 0.003704, rel 0.000272, norm 1.540905"
## [1] "Iter 75, obj 13.630100, abs 0.003557, rel 0.000261, norm 1.540990"

## [1] "Iter 76, obj 13.626693, abs 0.003407, rel 0.000250, norm 1.541180"
## [1] "Iter 77, obj 13.623439, abs 0.003254, rel 0.000239, norm 1.541475"
## [1] "Iter 78, obj 13.620339, abs 0.003100, rel 0.000228, norm 1.541869"
## [1] "Iter 79, obj 13.617393, abs 0.002945, rel 0.000216, norm 1.542360"
## [1] "Iter 80, obj 13.614602, abs 0.002791, rel 0.000205, norm 1.542945"

```

```

## [1] "Iter 81, obj 13.611965, abs 0.002638, rel 0.000194, norm 1.543618"
## [1] "Iter 82, obj 13.609479, abs 0.002486, rel 0.000183, norm 1.544377"
## [1] "Iter 83, obj 13.607143, abs 0.002336, rel 0.000172, norm 1.545218"
## [1] "Iter 84, obj 13.604954, abs 0.002189, rel 0.000161, norm 1.546135"
## [1] "Iter 85, obj 13.602909, abs 0.002045, rel 0.000150, norm 1.547124"
## [1] "Iter 86, obj 13.601005, abs 0.001904, rel 0.000140, norm 1.548182"
## [1] "Iter 87, obj 13.599239, abs 0.001766, rel 0.000130, norm 1.549303"
## [1] "Iter 88, obj 13.597606, abs 0.001633, rel 0.000120, norm 1.550484"
## [1] "Iter 89, obj 13.596103, abs 0.001503, rel 0.000111, norm 1.551719"
## [1] "Iter 90, obj 13.594726, abs 0.001377, rel 0.000101, norm 1.553006"
## [1] "Iter 91, obj 13.593472, abs 0.001255, rel 0.000092, norm 1.554339"
## [1] "Iter 92, obj 13.592335, abs 0.001137, rel 0.000084, norm 1.555715"
## [1] "Iter 93, obj 13.591311, abs 0.001023, rel 0.000075, norm 1.557130"
## [1] "Iter 94, obj 13.590398, abs 0.000913, rel 0.000067, norm 1.558580"
## [1] "Iter 95, obj 13.589590, abs 0.000808, rel 0.000059, norm 1.560061"
## [1] "Iter 96, obj 13.588883, abs 0.000707, rel 0.000052, norm 1.561570"
## [1] "Iter 97, obj 13.588274, abs 0.000609, rel 0.000045, norm 1.563104"
## [1] "Iter 98, obj 13.587758, abs 0.000516, rel 0.000038, norm 1.564659"
## [1] "Iter 99, obj 13.587332, abs 0.000427, rel 0.000031, norm 1.566233"
## [1] "Iter 100, obj 13.586991, abs 0.000341, rel 0.000025, norm 1.567823"

## [1] "Iter 101, obj 1.911800, abs 11.675190, rel 0.859292, norm 0.127960"
## [1] "Iter 102, obj 1.769101, abs 0.142699, rel 0.074641, norm 0.129621"
## [1] "Iter 103, obj 1.689574, abs 0.079527, rel 0.044954, norm 0.144599"
## [1] "Iter 104, obj 1.679590, abs 0.009984, rel 0.005909, norm 0.148791"
## [1] "Iter 105, obj 1.693853, abs 0.014264, rel 0.008492, norm 0.148953"
## [1] "Iter 106, obj 1.696874, abs 0.003021, rel 0.001783, norm 0.147623"
## [1] "Iter 107, obj 1.676885, abs 0.019990, rel 0.011780, norm 0.147033"
## [1] "Iter 108, obj 1.684194, abs 0.007310, rel 0.004359, norm 0.147639"
## [1] "Iter 109, obj 1.684246, abs 0.000052, rel 0.000031, norm 0.147441"
## [1] "Iter 110, obj 1.689106, abs 0.004860, rel 0.002886, norm 0.148102"
## [1] "Iter 111, obj 1.670394, abs 0.018712, rel 0.011078, norm 0.147632"
## [1] "Iter 112, obj 1.665895, abs 0.004499, rel 0.002693, norm 0.148832"
## [1] "Iter 113, obj 1.684919, abs 0.019023, rel 0.011419, norm 0.146822"
## [1] "Iter 114, obj 1.671694, abs 0.013225, rel 0.007849, norm 0.147407"
## [1] "Iter 115, obj 1.676750, abs 0.005056, rel 0.003024, norm 0.147404"
## [1] "Iter 116, obj 1.686239, abs 0.009489, rel 0.005659, norm 0.146985"
## [1] "Iter 117, obj 1.680780, abs 0.005459, rel 0.003237, norm 0.147771"
## [1] "Iter 118, obj 1.683876, abs 0.003096, rel 0.001842, norm 0.147917"
## [1] "Iter 119, obj 1.689499, abs 0.005623, rel 0.003339, norm 0.146726"
## [1] "Iter 120, obj 1.686137, abs 0.003362, rel 0.001990, norm 0.147438"
## [1] "Iter 121, obj 1.685390, abs 0.000747, rel 0.000443, norm 0.147091"
## [1] "Iter 122, obj 1.697406, abs 0.012016, rel 0.007130, norm 0.148153"
## [1] "Iter 123, obj 1.686058, abs 0.011347, rel 0.006685, norm 0.147325"
## [1] "Iter 124, obj 1.679559, abs 0.006500, rel 0.003855, norm 0.147402"
## [1] "Iter 125, obj 1.683945, abs 0.004387, rel 0.002612, norm 0.147372"

## [1] "Iter 126, obj 1.678316, abs 0.005629, rel 0.003343, norm 0.147849"
## [1] "Iter 127, obj 1.679871, abs 0.001555, rel 0.000927, norm 0.147902"
## [1] "Iter 128, obj 1.694070, abs 0.014199, rel 0.008452, norm 0.148242"
## [1] "Iter 129, obj 1.695048, abs 0.000978, rel 0.000577, norm 0.147416"
## [1] "Iter 130, obj 1.690579, abs 0.004469, rel 0.002637, norm 0.147537"
## [1] "Iter 131, obj 1.699986, abs 0.009407, rel 0.005564, norm 0.147250"
## [1] "Iter 132, obj 1.707047, abs 0.007061, rel 0.004154, norm 0.148345"
## [1] "Iter 133, obj 1.718609, abs 0.011561, rel 0.006773, norm 0.147694"

```

```

## [1] "Iter 134, obj 1.729984, abs 0.011375, rel 0.006619, norm 0.148598"
## [1] "Iter 135, obj 1.703276, abs 0.026707, rel 0.015438, norm 0.147571"
## [1] "Iter 136, obj 1.704300, abs 0.001024, rel 0.000601, norm 0.147924"
## [1] "Iter 137, obj 1.699582, abs 0.004718, rel 0.002769, norm 0.146664"
## [1] "Iter 138, obj 1.683625, abs 0.015956, rel 0.009388, norm 0.146915"
## [1] "Iter 139, obj 1.691389, abs 0.007764, rel 0.004611, norm 0.146504"
## [1] "Iter 140, obj 1.693446, abs 0.002057, rel 0.001216, norm 0.147531"
## [1] "Iter 141, obj 1.711327, abs 0.017881, rel 0.010559, norm 0.147446"
## [1] "Iter 142, obj 1.715414, abs 0.004087, rel 0.002388, norm 0.148078"
## [1] "Iter 143, obj 1.727681, abs 0.012268, rel 0.007151, norm 0.147494"
## [1] "Iter 144, obj 1.710263, abs 0.017418, rel 0.010082, norm 0.147707"
## [1] "Iter 145, obj 1.686322, abs 0.023941, rel 0.013998, norm 0.147611"
## [1] "Iter 146, obj 1.695063, abs 0.008741, rel 0.005184, norm 0.147507"
## [1] "Iter 147, obj 1.710283, abs 0.015219, rel 0.008978, norm 0.147580"
## [1] "Iter 148, obj 1.701874, abs 0.008408, rel 0.004916, norm 0.148304"
## [1] "Iter 149, obj 1.711480, abs 0.009606, rel 0.005644, norm 0.147898"
## [1] "Iter 150, obj 1.688179, abs 0.023301, rel 0.013615, norm 0.148466"

## [1] "Iter 151, obj 1.695293, abs 0.007114, rel 0.004214, norm 0.147581"
## [1] "Iter 152, obj 1.698501, abs 0.003208, rel 0.001892, norm 0.147282"
## [1] "Iter 153, obj 1.683705, abs 0.014796, rel 0.008711, norm 0.147212"
## [1] "Iter 154, obj 1.696539, abs 0.012834, rel 0.007623, norm 0.147556"
## [1] "Iter 155, obj 1.712665, abs 0.016127, rel 0.009506, norm 0.147570"
## [1] "Iter 156, obj 1.705873, abs 0.006792, rel 0.003966, norm 0.147856"
## [1] "Iter 157, obj 1.702831, abs 0.003042, rel 0.001783, norm 0.148628"
## [1] "Iter 158, obj 1.703225, abs 0.000394, rel 0.000231, norm 0.148891"
## [1] "Iter 159, obj 1.689361, abs 0.013864, rel 0.008140, norm 0.147048"
## [1] "Iter 160, obj 1.694612, abs 0.005250, rel 0.003108, norm 0.147549"
## [1] "Iter 161, obj 1.704161, abs 0.009549, rel 0.005635, norm 0.147710"
## [1] "Iter 162, obj 1.686209, abs 0.017952, rel 0.010534, norm 0.146954"
## [1] "Iter 163, obj 1.680452, abs 0.005756, rel 0.003414, norm 0.146931"
## [1] "Iter 164, obj 1.687812, abs 0.007360, rel 0.004380, norm 0.147055"
## [1] "Iter 165, obj 1.679156, abs 0.008656, rel 0.005129, norm 0.146554"
## [1] "Iter 166, obj 1.699325, abs 0.020169, rel 0.012012, norm 0.146778"
## [1] "Iter 167, obj 1.708987, abs 0.009662, rel 0.005686, norm 0.148428"
## [1] "Iter 168, obj 1.676199, abs 0.032788, rel 0.019186, norm 0.146526"
## [1] "Iter 169, obj 1.685320, abs 0.009121, rel 0.005441, norm 0.145724"
## [1] "Iter 170, obj 1.729392, abs 0.044072, rel 0.026151, norm 0.147291"
## [1] "Iter 171, obj 1.686360, abs 0.043032, rel 0.024883, norm 0.146799"
## [1] "Iter 172, obj 1.674207, abs 0.012153, rel 0.007207, norm 0.146331"
## [1] "Iter 173, obj 1.680289, abs 0.006082, rel 0.003633, norm 0.147649"
## [1] "Iter 174, obj 1.689811, abs 0.009522, rel 0.005667, norm 0.146947"
## [1] "Iter 175, obj 1.696029, abs 0.006218, rel 0.003680, norm 0.147489"

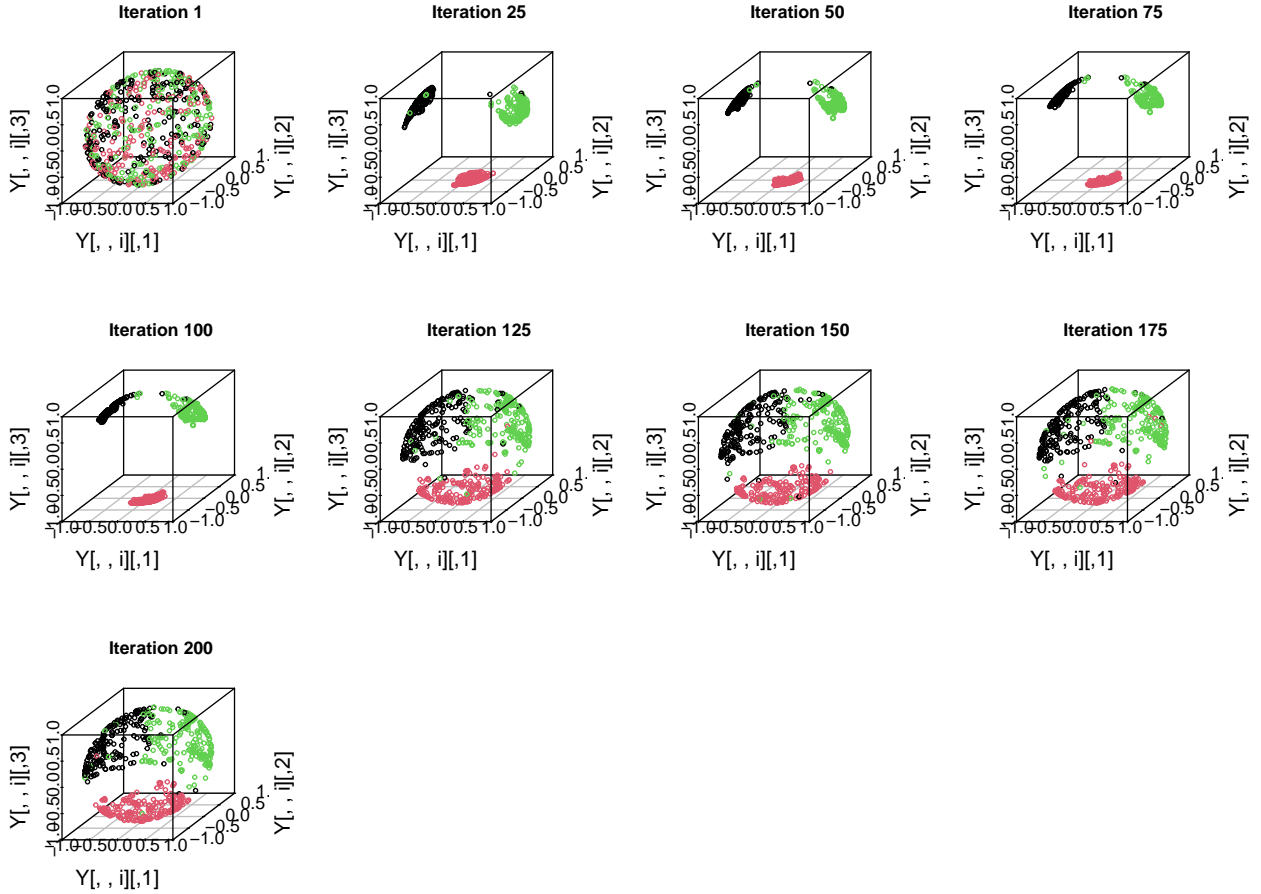
## [1] "Iter 176, obj 1.692277, abs 0.003752, rel 0.002212, norm 0.147633"
## [1] "Iter 177, obj 1.678474, abs 0.013803, rel 0.008157, norm 0.146817"
## [1] "Iter 178, obj 1.678227, abs 0.000247, rel 0.000147, norm 0.145981"
## [1] "Iter 179, obj 1.702913, abs 0.024686, rel 0.014710, norm 0.147302"
## [1] "Iter 180, obj 1.696460, abs 0.006452, rel 0.003789, norm 0.147619"
## [1] "Iter 181, obj 1.689067, abs 0.007393, rel 0.004358, norm 0.147247"
## [1] "Iter 182, obj 1.714813, abs 0.025746, rel 0.015243, norm 0.148117"
## [1] "Iter 183, obj 1.692644, abs 0.022169, rel 0.012928, norm 0.147467"
## [1] "Iter 184, obj 1.691392, abs 0.001253, rel 0.000740, norm 0.146944"
## [1] "Iter 185, obj 1.712791, abs 0.021399, rel 0.012652, norm 0.147843"
## [1] "Iter 186, obj 1.706258, abs 0.006534, rel 0.003815, norm 0.147396"

```

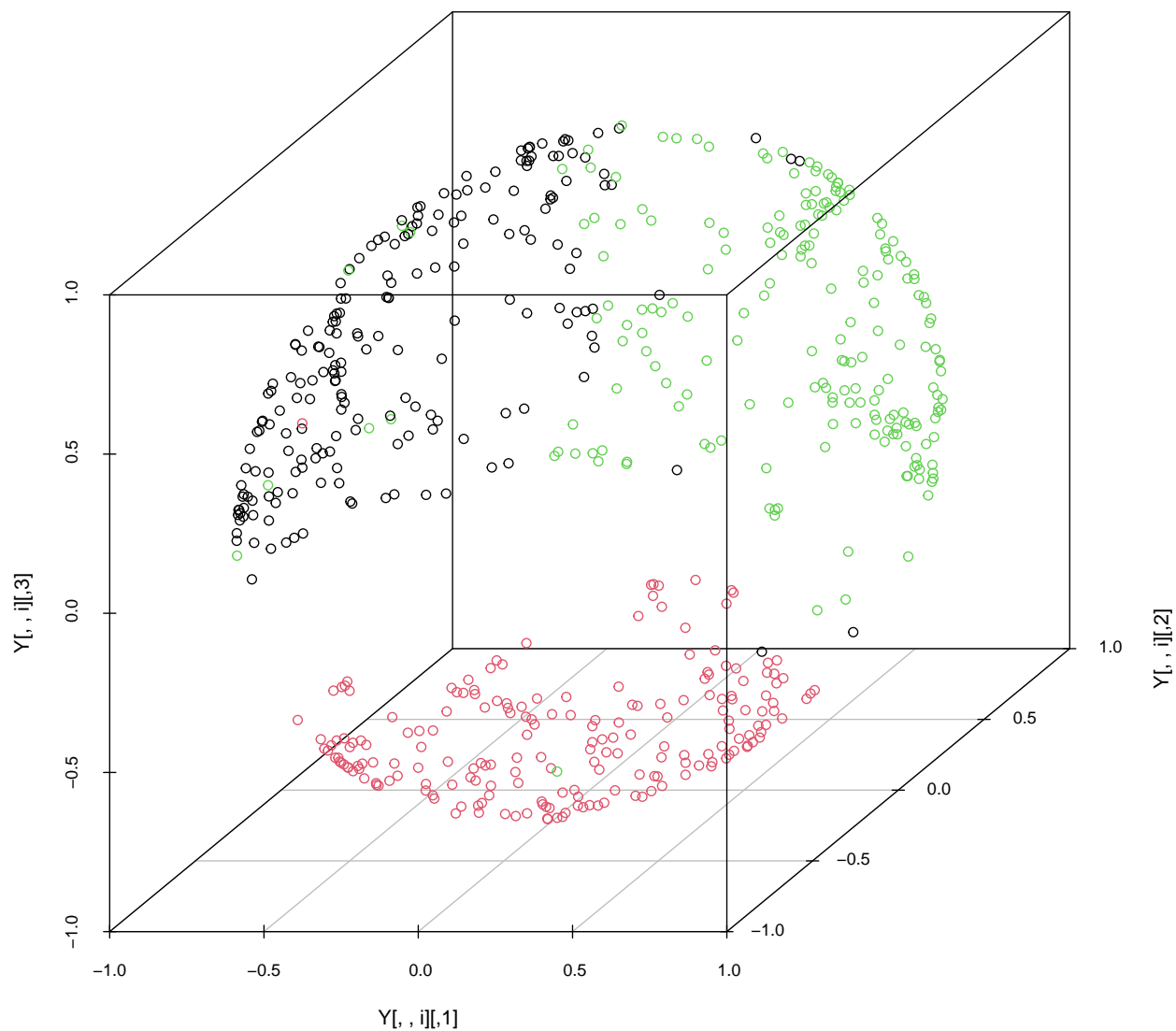
```

## [1] "Iter 187, obj 1.688773, abs 0.017485, rel 0.010248, norm 0.146813"
## [1] "Iter 188, obj 1.698376, abs 0.009604, rel 0.005687, norm 0.147865"
## [1] "Iter 189, obj 1.676901, abs 0.021475, rel 0.012645, norm 0.146732"
## [1] "Iter 190, obj 1.668000, abs 0.008901, rel 0.005308, norm 0.146581"
## [1] "Iter 191, obj 1.701051, abs 0.033052, rel 0.019815, norm 0.148130"
## [1] "Iter 192, obj 1.689071, abs 0.011980, rel 0.007043, norm 0.146859"
## [1] "Iter 193, obj 1.688740, abs 0.000331, rel 0.000196, norm 0.146833"
## [1] "Iter 194, obj 1.710834, abs 0.022094, rel 0.013083, norm 0.147970"
## [1] "Iter 195, obj 1.689904, abs 0.020930, rel 0.012234, norm 0.147555"
## [1] "Iter 196, obj 1.673514, abs 0.016390, rel 0.009699, norm 0.147102"
## [1] "Iter 197, obj 1.695583, abs 0.022069, rel 0.013187, norm 0.146957"
## [1] "Iter 198, obj 1.685391, abs 0.010192, rel 0.006011, norm 0.147224"
## [1] "Iter 199, obj 1.682462, abs 0.002929, rel 0.001738, norm 0.147072"
## [1] "Iter 200, obj 1.699426, abs 0.016964, rel 0.010083, norm 0.147615"

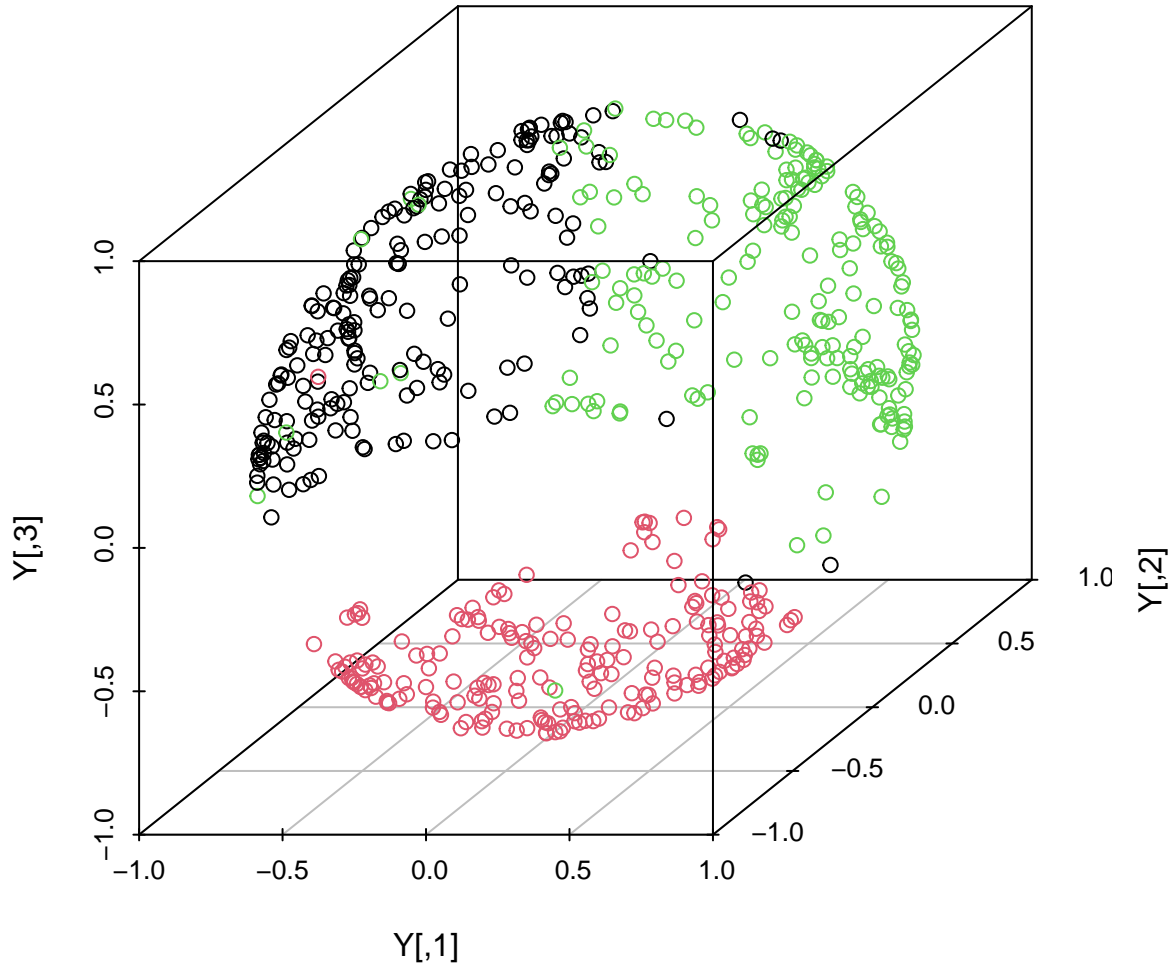
```



Iteration 200



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),  
                             color = colors)
```



It's clearly clustered the three groups we had in the original generated data.

Case 3

Sample on the $(S^1)^3$ where $p = 1$ and $r = 3$.

```
# Sample on the  $(S^1)^2$ 
n_ori <- 200
vmf11 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 10)
vmf12 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 10)
vmf13 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi/2)), kappa = 10)
x1 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf11),
                             DirStats::to_rad(vmf12),
                             DirStats::to_rad(vmf13)))

vmf21 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi / 2)), kappa = 5)
vmf22 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 5)
vmf23 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 5)
x2 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf21), DirStats::to_rad(vmf22),
                             DirStats::to_rad(vmf23)))

vmf31 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 30)
vmf32 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 30)
```

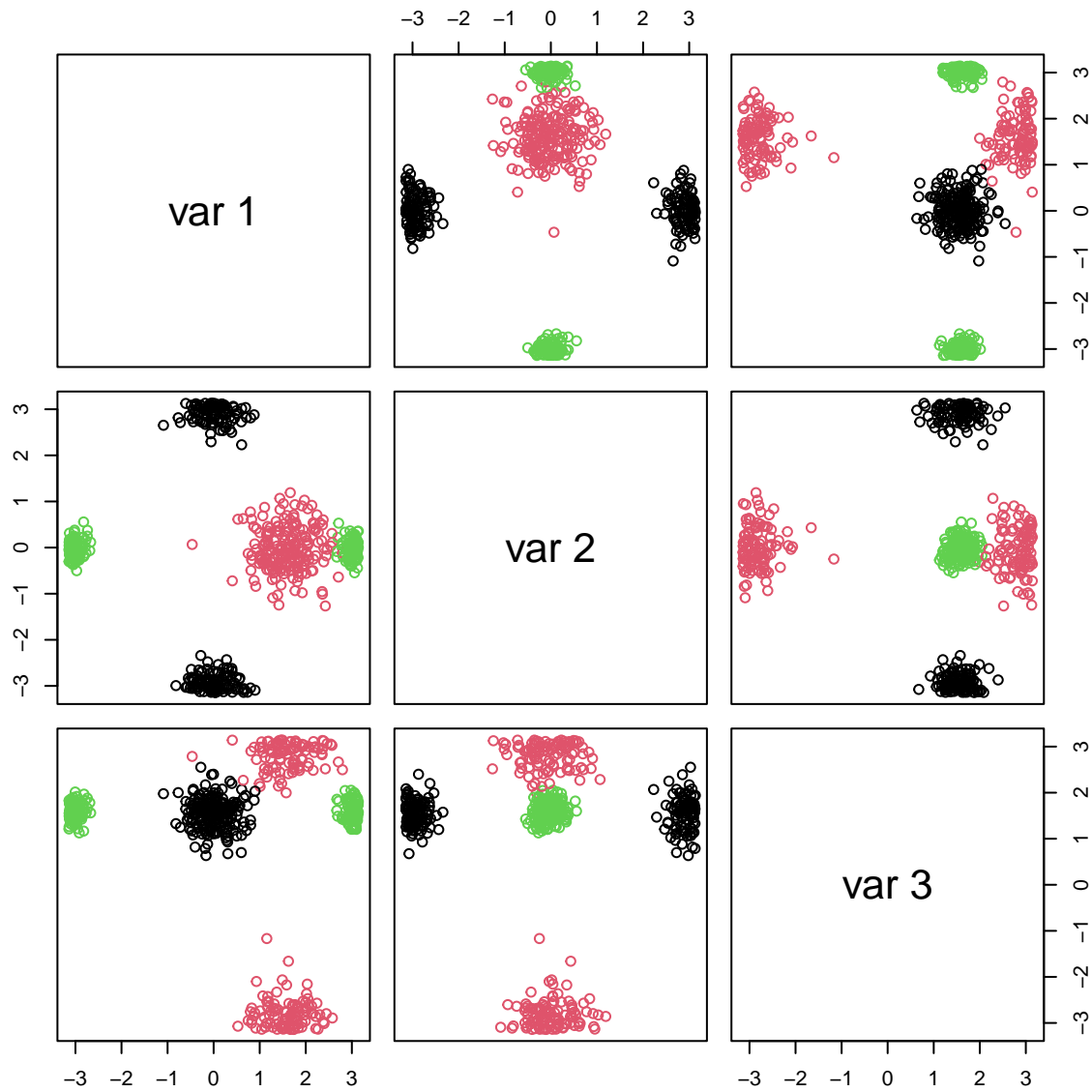
```
vmf33 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi/2)), kappa = 30)
x3 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf31), DirStats::to_rad(vmf32),
                             DirStats::to_rad(vmf33)))
```

Now, let's do the same visualization with the data on the torus:

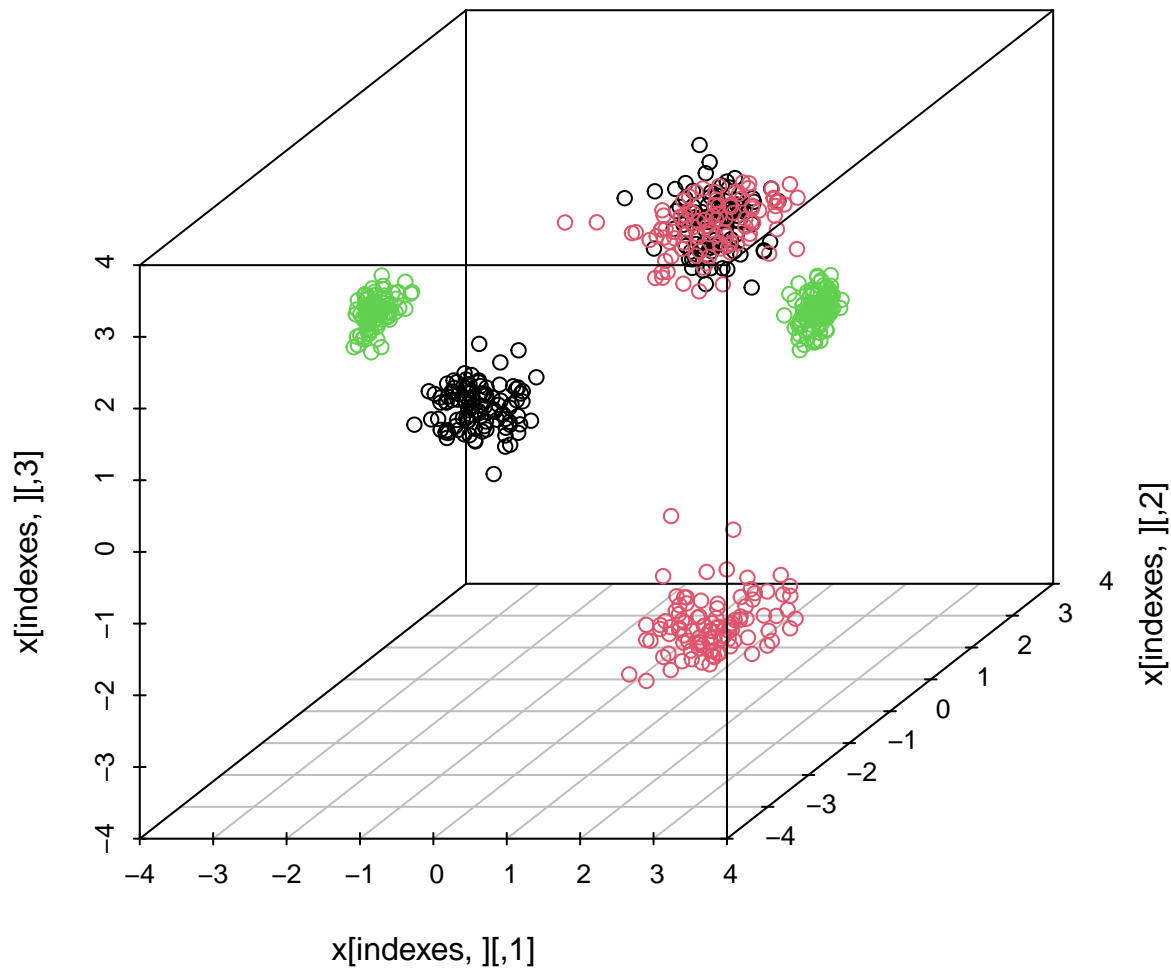
```
x <- rbind(x1, x2, x3)
n <- n_ori * 3
x_3 <- array(dim = c(n, 2, 3))
x_3[, , 1] <- DirStats::to_cir(x[, 1])
x_3[, , 2] <- DirStats::to_cir(x[, 2])
x_3[, , 3] <- DirStats::to_cir(x[, 3])

n <- nrow(x_3)
indexes <- sample(1:n)
x_3 <- x_3[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]

pairs(x[indexes,], col = colors)
```



```
scatterplot3d::scatterplot3d(x[indexes,], xlim = c(-pi, pi), ylim = c(-pi, pi), zlim = c(-pi, pi),
                             color = colors)
```



Let's calculate the rho parameters based on a perplexity of 25:

```
rho_third_perp25 <- rho_optim_bst(x_3, 25)
```

```
## Time difference of 1.284627 mins
```

First, let's reduce to dimension \mathbb{S}^1 then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_3, d=1, rho_psc_list = rho_third_perp25, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 17.900844, abs 0.000000, rel 0.000000, norm 0.126072"
## [1] "Iter 2, obj 17.253370, abs 0.647474, rel 0.036170, norm 0.252853"
## [1] "Iter 3, obj 16.863880, abs 0.389490, rel 0.022575, norm 0.337146"
## [1] "Iter 4, obj 16.591956, abs 0.271924, rel 0.016125, norm 0.392923"
## [1] "Iter 5, obj 16.363333, abs 0.228622, rel 0.013779, norm 0.433542"
## [1] "Iter 6, obj 16.133692, abs 0.229641, rel 0.014034, norm 0.464060"
## [1] "Iter 7, obj 15.912810, abs 0.220882, rel 0.013691, norm 0.495877"
## [1] "Iter 8, obj 15.734087, abs 0.178723, rel 0.011231, norm 0.533433"
## [1] "Iter 9, obj 15.590271, abs 0.143816, rel 0.009140, norm 0.568293"
## [1] "Iter 10, obj 15.465374, abs 0.124897, rel 0.008011, norm 0.598349"
## [1] "Iter 11, obj 15.360239, abs 0.105135, rel 0.006798, norm 0.624403"
## [1] "Iter 12, obj 15.273849, abs 0.086390, rel 0.005624, norm 0.648868"
## [1] "Iter 13, obj 15.203426, abs 0.070424, rel 0.004611, norm 0.672396"
```

```

## [1] "Iter 14, obj 15.144754, abs 0.058671, rel 0.003859, norm 0.692742"
## [1] "Iter 15, obj 15.096771, abs 0.047983, rel 0.003168, norm 0.711071"
## [1] "Iter 16, obj 15.058575, abs 0.038196, rel 0.002530, norm 0.729196"
## [1] "Iter 17, obj 15.027545, abs 0.031030, rel 0.002061, norm 0.747299"
## [1] "Iter 18, obj 15.000862, abs 0.026683, rel 0.001776, norm 0.765114"
## [1] "Iter 19, obj 14.978637, abs 0.022225, rel 0.001482, norm 0.782239"
## [1] "Iter 20, obj 14.961403, abs 0.017234, rel 0.001151, norm 0.798009"
## [1] "Iter 21, obj 14.948954, abs 0.012449, rel 0.000832, norm 0.812053"
## [1] "Iter 22, obj 14.942646, abs 0.006308, rel 0.000422, norm 0.824891"
## [1] "Iter 23, obj 14.942322, abs 0.000324, rel 0.000022, norm 0.836933"
## [1] "Iter 24, obj 14.946968, abs 0.004646, rel 0.000311, norm 0.848658"
## [1] "Iter 25, obj 14.956806, abs 0.009838, rel 0.000658, norm 0.860444"

## [1] "Iter 26, obj 14.971131, abs 0.014325, rel 0.000958, norm 0.871782"
## [1] "Iter 27, obj 14.987778, abs 0.016647, rel 0.001112, norm 0.882536"
## [1] "Iter 28, obj 15.004490, abs 0.016713, rel 0.001115, norm 0.893159"
## [1] "Iter 29, obj 15.018264, abs 0.013773, rel 0.000918, norm 0.903627"
## [1] "Iter 30, obj 15.025706, abs 0.007442, rel 0.000496, norm 0.913829"
## [1] "Iter 31, obj 15.023813, abs 0.001893, rel 0.000126, norm 0.923837"
## [1] "Iter 32, obj 15.010726, abs 0.013087, rel 0.000871, norm 0.933749"
## [1] "Iter 33, obj 14.986778, abs 0.023948, rel 0.001595, norm 0.943459"
## [1] "Iter 34, obj 14.954440, abs 0.032339, rel 0.002158, norm 0.952472"
## [1] "Iter 35, obj 14.917210, abs 0.037230, rel 0.002490, norm 0.960052"
## [1] "Iter 36, obj 14.879934, abs 0.037275, rel 0.002499, norm 0.965669"
## [1] "Iter 37, obj 14.848276, abs 0.031659, rel 0.002128, norm 0.969008"
## [1] "Iter 38, obj 14.824910, abs 0.023366, rel 0.001574, norm 0.969873"
## [1] "Iter 39, obj 14.808642, abs 0.016268, rel 0.001097, norm 0.968624"
## [1] "Iter 40, obj 14.797370, abs 0.011272, rel 0.000761, norm 0.966096"
## [1] "Iter 41, obj 14.789558, abs 0.007812, rel 0.000528, norm 0.962983"
## [1] "Iter 42, obj 14.784235, abs 0.005323, rel 0.000360, norm 0.959680"
## [1] "Iter 43, obj 14.780778, abs 0.003457, rel 0.000234, norm 0.956382"
## [1] "Iter 44, obj 14.778763, abs 0.002014, rel 0.000136, norm 0.953184"
## [1] "Iter 45, obj 14.777866, abs 0.000897, rel 0.000061, norm 0.950127"
## [1] "Iter 46, obj 14.777738, abs 0.000128, rel 0.000009, norm 0.947225"
## [1] "Iter 47, obj 14.777716, abs 0.000022, rel 0.000001, norm 0.944483"
## [1] "Iter 48, obj 14.775987, abs 0.001729, rel 0.000117, norm 0.941911"
## [1] "Iter 49, obj 14.767775, abs 0.008212, rel 0.000556, norm 0.939578"
## [1] "Iter 50, obj 14.744844, abs 0.022931, rel 0.001553, norm 0.937782"

## [1] "Iter 51, obj 14.702829, abs 0.042015, rel 0.002849, norm 0.937262"
## [1] "Iter 52, obj 14.657725, abs 0.045104, rel 0.003068, norm 0.938749"
## [1] "Iter 53, obj 14.627780, abs 0.029945, rel 0.002043, norm 0.940792"
## [1] "Iter 54, obj 14.611115, abs 0.016665, rel 0.001139, norm 0.940940"
## [1] "Iter 55, obj 14.600996, abs 0.010119, rel 0.000693, norm 0.938920"
## [1] "Iter 56, obj 14.593587, abs 0.007410, rel 0.000507, norm 0.935507"
## [1] "Iter 57, obj 14.587353, abs 0.006234, rel 0.000427, norm 0.931432"
## [1] "Iter 58, obj 14.581755, abs 0.005598, rel 0.000384, norm 0.927125"
## [1] "Iter 59, obj 14.576603, abs 0.005152, rel 0.000353, norm 0.922792"
## [1] "Iter 60, obj 14.571819, abs 0.004784, rel 0.000328, norm 0.918522"
## [1] "Iter 61, obj 14.567361, abs 0.004458, rel 0.000306, norm 0.914354"
## [1] "Iter 62, obj 14.563201, abs 0.004160, rel 0.000286, norm 0.910302"
## [1] "Iter 63, obj 14.559316, abs 0.003885, rel 0.000267, norm 0.906372"
## [1] "Iter 64, obj 14.555684, abs 0.003632, rel 0.000249, norm 0.902563"
## [1] "Iter 65, obj 14.552287, abs 0.003397, rel 0.000233, norm 0.898873"
## [1] "Iter 66, obj 14.549107, abs 0.003180, rel 0.000219, norm 0.895298"

```

```

## [1] "Iter 67, obj 14.546127, abs 0.002980, rel 0.000205, norm 0.891834"
## [1] "Iter 68, obj 14.543333, abs 0.002794, rel 0.000192, norm 0.888477"
## [1] "Iter 69, obj 14.540711, abs 0.002623, rel 0.000180, norm 0.885223"
## [1] "Iter 70, obj 14.538247, abs 0.002464, rel 0.000169, norm 0.882067"
## [1] "Iter 71, obj 14.535931, abs 0.002316, rel 0.000159, norm 0.879006"
## [1] "Iter 72, obj 14.533752, abs 0.002179, rel 0.000150, norm 0.876035"
## [1] "Iter 73, obj 14.531699, abs 0.002052, rel 0.000141, norm 0.873151"
## [1] "Iter 74, obj 14.529765, abs 0.001934, rel 0.000133, norm 0.870351"
## [1] "Iter 75, obj 14.527940, abs 0.001825, rel 0.000126, norm 0.867631"

## [1] "Iter 76, obj 14.526218, abs 0.001722, rel 0.000119, norm 0.864989"
## [1] "Iter 77, obj 14.524591, abs 0.001627, rel 0.000112, norm 0.862420"
## [1] "Iter 78, obj 14.523053, abs 0.001538, rel 0.000106, norm 0.859924"
## [1] "Iter 79, obj 14.521599, abs 0.001455, rel 0.000100, norm 0.857496"
## [1] "Iter 80, obj 14.520222, abs 0.001377, rel 0.000095, norm 0.855134"
## [1] "Iter 81, obj 14.518918, abs 0.001304, rel 0.000090, norm 0.852837"
## [1] "Iter 82, obj 14.517682, abs 0.001236, rel 0.000085, norm 0.850601"
## [1] "Iter 83, obj 14.516510, abs 0.001172, rel 0.000081, norm 0.848425"
## [1] "Iter 84, obj 14.515399, abs 0.001112, rel 0.000077, norm 0.846306"
## [1] "Iter 85, obj 14.514343, abs 0.001055, rel 0.000073, norm 0.844243"
## [1] "Iter 86, obj 14.513341, abs 0.001003, rel 0.000069, norm 0.842233"
## [1] "Iter 87, obj 14.512388, abs 0.000953, rel 0.000066, norm 0.840276"
## [1] "Iter 88, obj 14.511482, abs 0.000906, rel 0.000062, norm 0.838369"
## [1] "Iter 89, obj 14.510620, abs 0.000862, rel 0.000059, norm 0.836510"
## [1] "Iter 90, obj 14.509800, abs 0.000820, rel 0.000057, norm 0.834699"
## [1] "Iter 91, obj 14.509019, abs 0.000781, rel 0.000054, norm 0.832933"
## [1] "Iter 92, obj 14.508275, abs 0.000744, rel 0.000051, norm 0.831212"
## [1] "Iter 93, obj 14.507566, abs 0.000709, rel 0.000049, norm 0.829533"
## [1] "Iter 94, obj 14.506889, abs 0.000676, rel 0.000047, norm 0.827897"
## [1] "Iter 95, obj 14.506244, abs 0.000645, rel 0.000044, norm 0.826300"
## [1] "Iter 96, obj 14.505628, abs 0.000616, rel 0.000042, norm 0.824743"
## [1] "Iter 97, obj 14.505040, abs 0.000588, rel 0.000041, norm 0.823225"
## [1] "Iter 98, obj 14.504478, abs 0.000562, rel 0.000039, norm 0.821743"
## [1] "Iter 99, obj 14.503940, abs 0.000537, rel 0.000037, norm 0.820297"
## [1] "Iter 100, obj 14.503427, abs 0.000514, rel 0.000035, norm 0.818886"

## [1] "Iter 101, obj 2.228460, abs 12.274967, rel 0.846349, norm 0.076416"
## [1] "Iter 102, obj 2.205259, abs 0.023201, rel 0.010411, norm 0.079154"
## [1] "Iter 103, obj 2.175501, abs 0.029758, rel 0.013494, norm 0.086089"
## [1] "Iter 104, obj 2.149179, abs 0.026322, rel 0.012099, norm 0.093498"
## [1] "Iter 105, obj 2.133650, abs 0.015530, rel 0.007226, norm 0.098240"
## [1] "Iter 106, obj 2.126808, abs 0.006842, rel 0.003207, norm 0.100882"
## [1] "Iter 107, obj 2.123334, abs 0.003474, rel 0.001633, norm 0.102583"
## [1] "Iter 108, obj 2.122213, abs 0.001121, rel 0.000528, norm 0.103777"
## [1] "Iter 109, obj 2.121179, abs 0.001034, rel 0.000487, norm 0.104599"
## [1] "Iter 110, obj 2.120535, abs 0.000644, rel 0.000304, norm 0.105032"
## [1] "Iter 111, obj 2.120132, abs 0.000402, rel 0.000190, norm 0.105255"
## [1] "Iter 112, obj 2.119730, abs 0.000402, rel 0.000190, norm 0.105394"
## [1] "Iter 113, obj 2.119361, abs 0.000369, rel 0.000174, norm 0.105532"
## [1] "Iter 114, obj 2.119032, abs 0.000329, rel 0.000155, norm 0.105673"
## [1] "Iter 115, obj 2.118703, abs 0.000329, rel 0.000155, norm 0.105787"
## [1] "Iter 116, obj 2.118389, abs 0.000314, rel 0.000148, norm 0.105866"
## [1] "Iter 117, obj 2.118114, abs 0.000275, rel 0.000130, norm 0.105921"
## [1] "Iter 118, obj 2.117870, abs 0.000244, rel 0.000115, norm 0.105958"
## [1] "Iter 119, obj 2.117654, abs 0.000216, rel 0.000102, norm 0.105989"

```

```

## [1] "Iter 120, obj 2.117467, abs 0.000187, rel 0.000088, norm 0.106017"
## [1] "Iter 121, obj 2.117303, abs 0.000164, rel 0.000077, norm 0.106041"
## [1] "Iter 122, obj 2.117162, abs 0.000142, rel 0.000067, norm 0.106059"
## [1] "Iter 123, obj 2.117039, abs 0.000123, rel 0.000058, norm 0.106073"
## [1] "Iter 124, obj 2.116931, abs 0.000108, rel 0.000051, norm 0.106083"
## [1] "Iter 125, obj 2.116838, abs 0.000094, rel 0.000044, norm 0.106091"

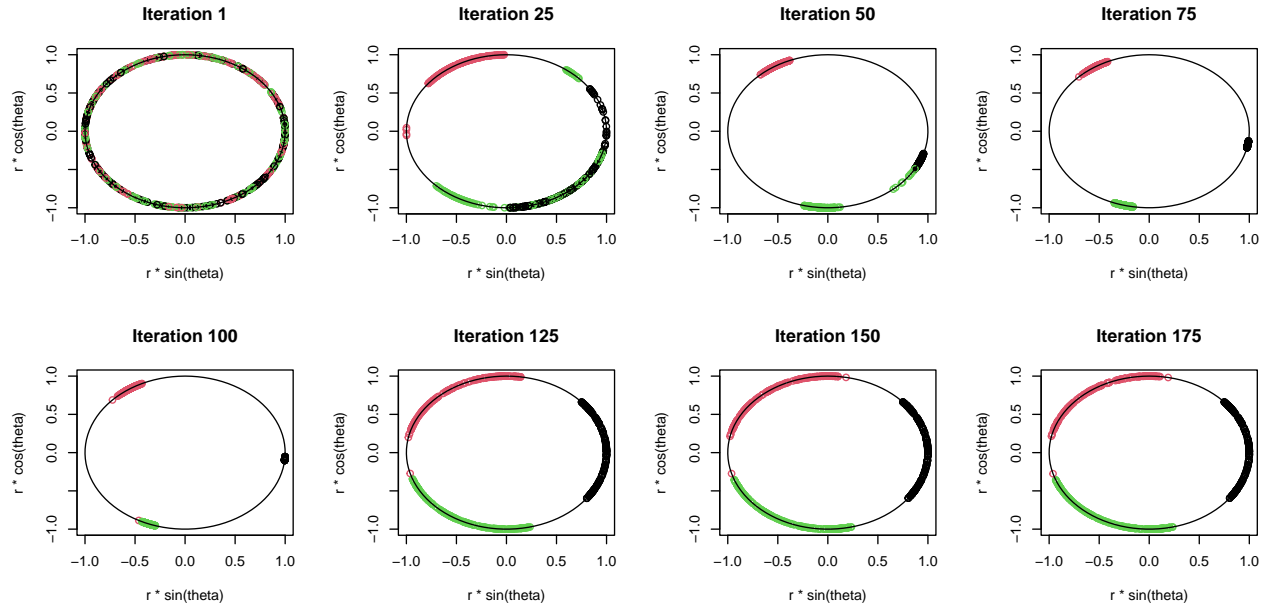
## [1] "Iter 126, obj 2.116756, abs 0.000081, rel 0.000038, norm 0.106095"
## [1] "Iter 127, obj 2.116686, abs 0.000071, rel 0.000033, norm 0.106096"
## [1] "Iter 128, obj 2.116624, abs 0.000061, rel 0.000029, norm 0.106095"
## [1] "Iter 129, obj 2.116572, abs 0.000053, rel 0.000025, norm 0.106093"
## [1] "Iter 130, obj 2.116526, abs 0.000046, rel 0.000022, norm 0.106090"
## [1] "Iter 131, obj 2.116486, abs 0.000040, rel 0.000019, norm 0.106088"
## [1] "Iter 132, obj 2.116451, abs 0.000035, rel 0.000017, norm 0.106084"
## [1] "Iter 133, obj 2.116420, abs 0.000031, rel 0.000014, norm 0.106080"
## [1] "Iter 134, obj 2.116393, abs 0.000027, rel 0.000013, norm 0.106076"
## [1] "Iter 135, obj 2.116370, abs 0.000024, rel 0.000011, norm 0.106072"
## [1] "Iter 136, obj 2.116349, abs 0.000021, rel 0.000010, norm 0.106068"
## [1] "Iter 137, obj 2.116331, abs 0.000018, rel 0.000009, norm 0.106064"
## [1] "Iter 138, obj 2.116315, abs 0.000016, rel 0.000008, norm 0.106060"
## [1] "Iter 139, obj 2.116301, abs 0.000014, rel 0.000007, norm 0.106057"
## [1] "Iter 140, obj 2.116289, abs 0.000012, rel 0.000006, norm 0.106053"
## [1] "Iter 141, obj 2.116279, abs 0.000010, rel 0.000005, norm 0.106050"
## [1] "Iter 142, obj 2.116270, abs 0.000009, rel 0.000004, norm 0.106047"
## [1] "Iter 143, obj 2.116262, abs 0.000008, rel 0.000004, norm 0.106045"
## [1] "Iter 144, obj 2.116256, abs 0.000006, rel 0.000003, norm 0.106042"
## [1] "Iter 145, obj 2.116250, abs 0.000005, rel 0.000003, norm 0.106040"
## [1] "Iter 146, obj 2.116246, abs 0.000005, rel 0.000002, norm 0.106038"
## [1] "Iter 147, obj 2.116242, abs 0.000004, rel 0.000002, norm 0.106036"
## [1] "Iter 148, obj 2.116238, abs 0.000003, rel 0.000002, norm 0.106035"
## [1] "Iter 149, obj 2.116236, abs 0.000003, rel 0.000001, norm 0.106034"
## [1] "Iter 150, obj 2.116233, abs 0.000002, rel 0.000001, norm 0.106033"

## [1] "Iter 151, obj 2.116232, abs 0.000002, rel 0.000001, norm 0.106032"
## [1] "Iter 152, obj 2.116230, abs 0.000002, rel 0.000001, norm 0.106031"
## [1] "Iter 153, obj 2.116229, abs 0.000001, rel 0.000001, norm 0.106030"
## [1] "Iter 154, obj 2.116228, abs 0.000001, rel 0.000001, norm 0.106030"
## [1] "Iter 155, obj 2.116227, abs 0.000001, rel 0.000000, norm 0.106029"
## [1] "Iter 156, obj 2.116226, abs 0.000001, rel 0.000000, norm 0.106029"
## [1] "Iter 157, obj 2.116225, abs 0.000001, rel 0.000000, norm 0.106028"
## [1] "Iter 158, obj 2.116225, abs 0.000001, rel 0.000000, norm 0.106028"
## [1] "Iter 159, obj 2.116224, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 160, obj 2.116224, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 161, obj 2.116224, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 162, obj 2.116223, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 163, obj 2.116223, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 164, obj 2.116223, abs 0.000000, rel 0.000000, norm 0.106027"
## [1] "Iter 165, obj 2.116223, abs 0.000000, rel 0.000000, norm 0.106027"
## [1] "Iter 166, obj 2.116223, abs 0.000000, rel 0.000000, norm 0.106027"
## [1] "Iter 167, obj 2.116223, abs 0.000000, rel 0.000000, norm 0.106027"
## [1] "Iter 168, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106027"
## [1] "Iter 169, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106027"
## [1] "Iter 170, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106027"
## [1] "Iter 171, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 172, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"

```

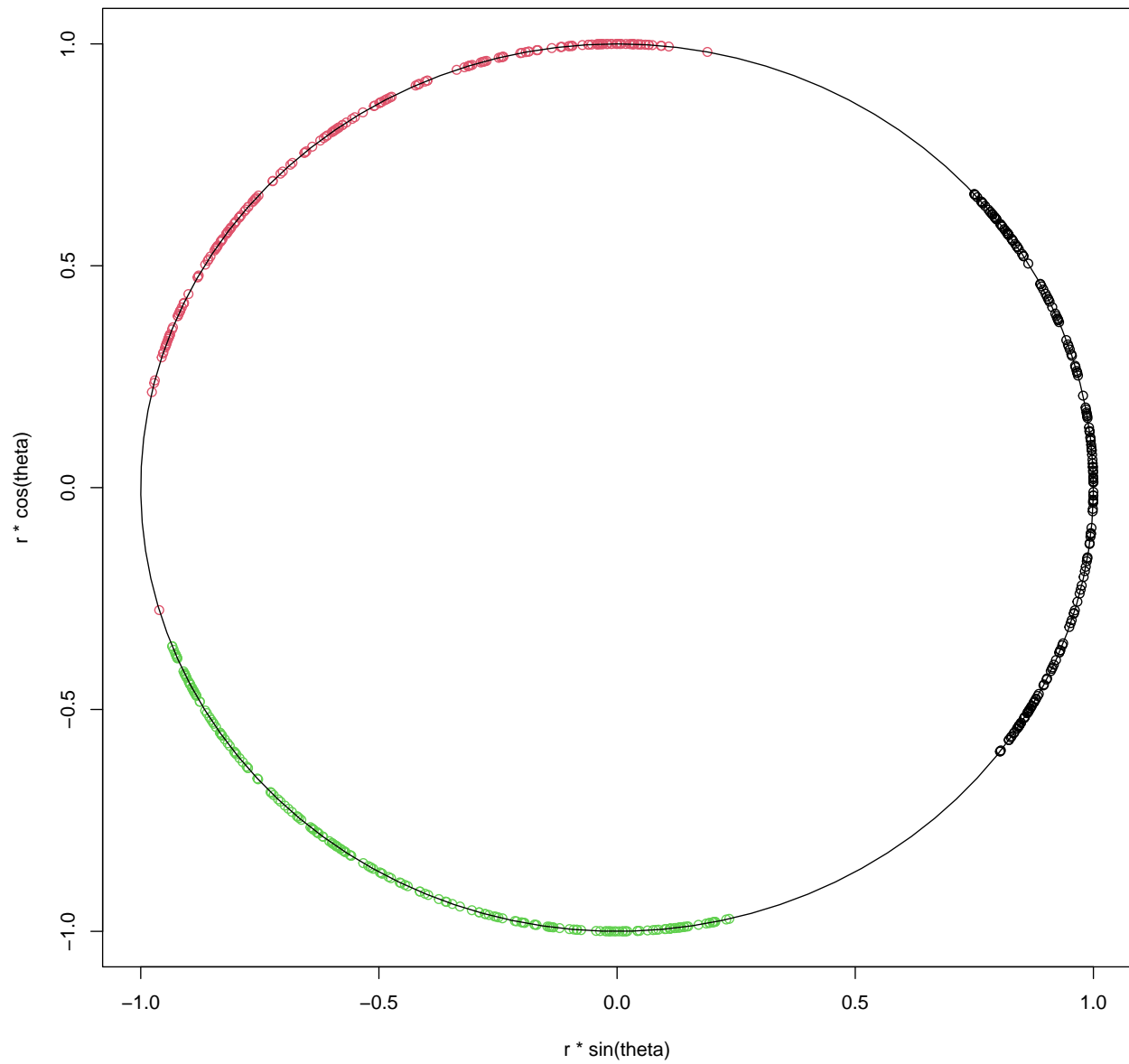


```
## [1] "Iter 173, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 174, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 175, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
```



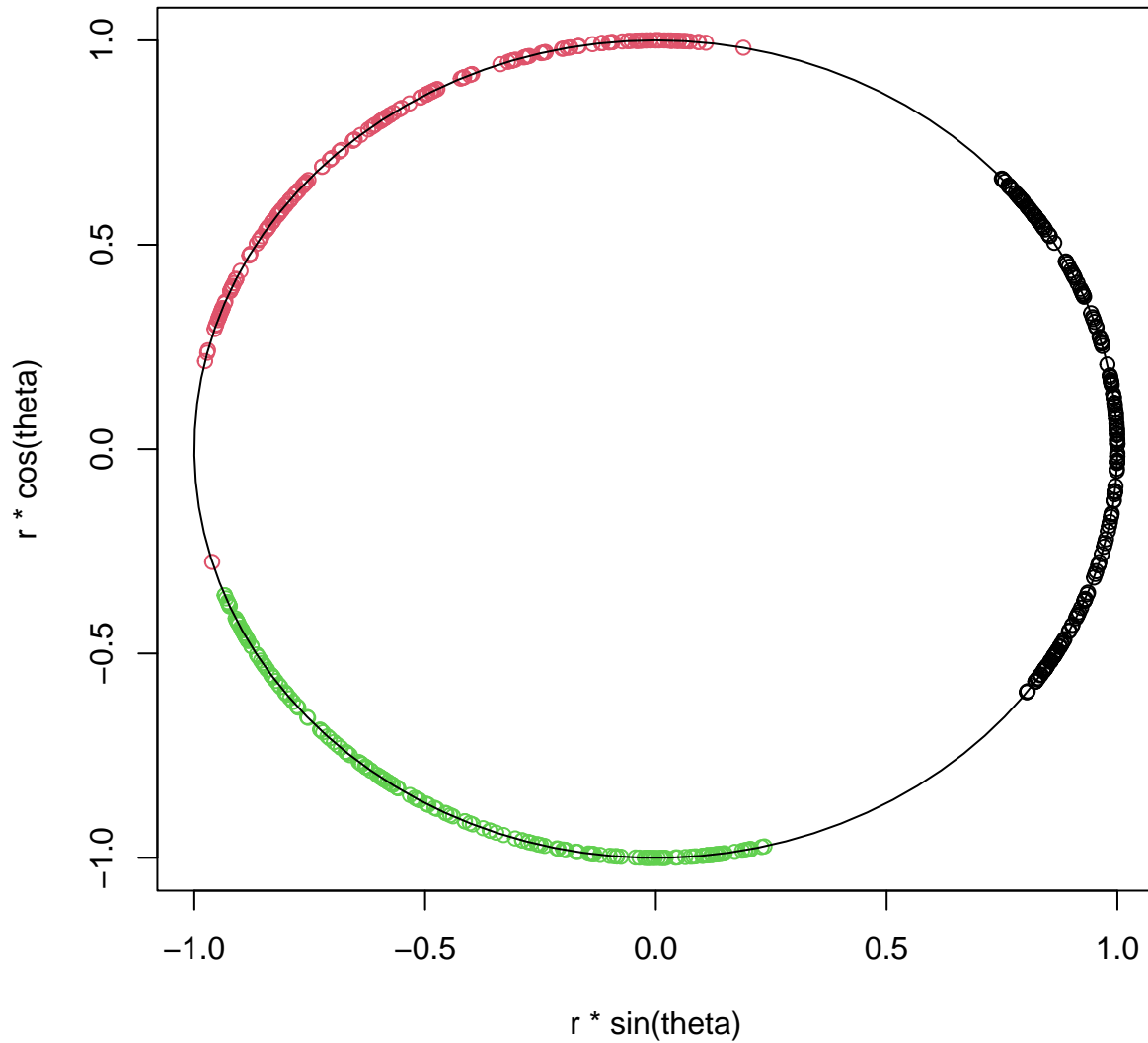
```
## [1] "Iter 176, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 177, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 178, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 179, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 180, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 181, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 182, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 183, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 184, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 185, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 186, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 187, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 188, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 189, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 190, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
## [1] "Iter 191, obj 2.116222, abs 0.000000, rel 0.000000, norm 0.106028"
```

Iteration 191



```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension \mathbb{S}^2 then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_3, d=2, rho_psc_list = rho_third_perp25, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 18.397529, abs 0.000000, rel 0.000000, norm 0.146664"
## [1] "Iter 2, obj 15.231125, abs 3.166404, rel 0.172110, norm 0.387666"
## [1] "Iter 3, obj 13.972530, abs 1.258595, rel 0.082633, norm 0.748061"
## [1] "Iter 4, obj 13.322082, abs 0.650448, rel 0.046552, norm 0.920246"
## [1] "Iter 5, obj 12.897777, abs 0.424304, rel 0.031850, norm 1.033736"
## [1] "Iter 6, obj 12.558150, abs 0.339628, rel 0.026332, norm 1.132709"
## [1] "Iter 7, obj 12.331803, abs 0.226347, rel 0.018024, norm 1.221607"
## [1] "Iter 8, obj 12.209180, abs 0.122624, rel 0.009944, norm 1.297654"
## [1] "Iter 9, obj 12.168264, abs 0.040916, rel 0.003351, norm 1.358185"
## [1] "Iter 10, obj 12.180314, abs 0.012051, rel 0.000990, norm 1.402491"
## [1] "Iter 11, obj 12.222653, abs 0.042338, rel 0.003476, norm 1.433271"
## [1] "Iter 12, obj 12.281393, abs 0.058740, rel 0.004806, norm 1.454878"
## [1] "Iter 13, obj 12.346681, abs 0.065289, rel 0.005316, norm 1.470586"
## [1] "Iter 14, obj 12.411137, abs 0.064456, rel 0.005220, norm 1.482516"
## [1] "Iter 15, obj 12.470520, abs 0.059383, rel 0.004785, norm 1.491862"
```

```

## [1] "Iter 16, obj 12.523804, abs 0.053284, rel 0.004273, norm 1.499356"
## [1] "Iter 17, obj 12.573456, abs 0.049652, rel 0.003965, norm 1.505548"
## [1] "Iter 18, obj 12.623340, abs 0.049884, rel 0.003967, norm 1.510926"
## [1] "Iter 19, obj 12.673505, abs 0.050165, rel 0.003974, norm 1.515810"
## [1] "Iter 20, obj 12.720649, abs 0.047144, rel 0.003720, norm 1.520150"
## [1] "Iter 21, obj 12.762811, abs 0.042162, rel 0.003314, norm 1.523775"
## [1] "Iter 22, obj 12.799813, abs 0.037001, rel 0.002899, norm 1.526705"
## [1] "Iter 23, obj 12.832132, abs 0.032320, rel 0.002525, norm 1.529059"
## [1] "Iter 24, obj 12.860374, abs 0.028242, rel 0.002201, norm 1.530959"
## [1] "Iter 25, obj 12.885105, abs 0.024731, rel 0.001923, norm 1.532500"

## [1] "Iter 26, obj 12.906819, abs 0.021714, rel 0.001685, norm 1.533757"
## [1] "Iter 27, obj 12.925934, abs 0.019115, rel 0.001481, norm 1.534788"
## [1] "Iter 28, obj 12.942804, abs 0.016870, rel 0.001305, norm 1.535635"
## [1] "Iter 29, obj 12.957726, abs 0.014922, rel 0.001153, norm 1.536332"
## [1] "Iter 30, obj 12.970952, abs 0.013226, rel 0.001021, norm 1.536907"
## [1] "Iter 31, obj 12.982697, abs 0.011744, rel 0.000905, norm 1.537381"
## [1] "Iter 32, obj 12.993142, abs 0.010445, rel 0.000805, norm 1.537770"
## [1] "Iter 33, obj 13.002444, abs 0.009302, rel 0.000716, norm 1.538089"
## [1] "Iter 34, obj 13.010737, abs 0.008293, rel 0.000638, norm 1.538348"
## [1] "Iter 35, obj 13.018138, abs 0.007401, rel 0.000569, norm 1.538557"
## [1] "Iter 36, obj 13.024748, abs 0.006610, rel 0.000508, norm 1.538724"
## [1] "Iter 37, obj 13.030654, abs 0.005906, rel 0.000453, norm 1.538854"
## [1] "Iter 38, obj 13.035934, abs 0.005280, rel 0.000405, norm 1.538953"
## [1] "Iter 39, obj 13.040655, abs 0.004721, rel 0.000362, norm 1.539026"
## [1] "Iter 40, obj 13.044876, abs 0.004221, rel 0.000324, norm 1.539077"
## [1] "Iter 41, obj 13.048650, abs 0.003774, rel 0.000289, norm 1.539108"
## [1] "Iter 42, obj 13.052022, abs 0.003373, rel 0.000258, norm 1.539122"
## [1] "Iter 43, obj 13.055035, abs 0.003013, rel 0.000231, norm 1.539122"
## [1] "Iter 44, obj 13.057725, abs 0.002690, rel 0.000206, norm 1.539110"
## [1] "Iter 45, obj 13.060124, abs 0.002399, rel 0.000184, norm 1.539087"
## [1] "Iter 46, obj 13.062261, abs 0.002138, rel 0.000164, norm 1.539055"
## [1] "Iter 47, obj 13.064163, abs 0.001902, rel 0.000146, norm 1.539016"
## [1] "Iter 48, obj 13.065853, abs 0.001690, rel 0.000129, norm 1.538970"
## [1] "Iter 49, obj 13.067353, abs 0.001499, rel 0.000115, norm 1.538919"
## [1] "Iter 50, obj 13.068680, abs 0.001327, rel 0.000102, norm 1.538862"

## [1] "Iter 51, obj 13.069851, abs 0.001172, rel 0.000090, norm 1.538802"
## [1] "Iter 52, obj 13.070883, abs 0.001032, rel 0.000079, norm 1.538739"
## [1] "Iter 53, obj 13.071789, abs 0.000906, rel 0.000069, norm 1.538673"
## [1] "Iter 54, obj 13.072582, abs 0.000792, rel 0.000061, norm 1.538605"
## [1] "Iter 55, obj 13.073272, abs 0.000690, rel 0.000053, norm 1.538535"
## [1] "Iter 56, obj 13.073869, abs 0.000598, rel 0.000046, norm 1.538464"
## [1] "Iter 57, obj 13.074384, abs 0.000515, rel 0.000039, norm 1.538392"
## [1] "Iter 58, obj 13.074825, abs 0.000440, rel 0.000034, norm 1.538320"
## [1] "Iter 59, obj 13.075198, abs 0.000373, rel 0.000029, norm 1.538247"
## [1] "Iter 60, obj 13.075510, abs 0.000313, rel 0.000024, norm 1.538174"
## [1] "Iter 61, obj 13.075769, abs 0.000259, rel 0.000020, norm 1.538102"
## [1] "Iter 62, obj 13.075979, abs 0.000210, rel 0.000016, norm 1.538029"
## [1] "Iter 63, obj 13.076146, abs 0.000167, rel 0.000013, norm 1.537957"
## [1] "Iter 64, obj 13.076273, abs 0.000128, rel 0.000010, norm 1.537886"
## [1] "Iter 65, obj 13.076366, abs 0.000093, rel 0.000007, norm 1.537816"
## [1] "Iter 66, obj 13.076427, abs 0.000062, rel 0.000005, norm 1.537746"
## [1] "Iter 67, obj 13.076461, abs 0.000034, rel 0.000003, norm 1.537678"
## [1] "Iter 68, obj 13.076471, abs 0.000009, rel 0.000001, norm 1.537611"

```

```

## [1] "Iter 69, obj 13.076458, abs 0.000013, rel 0.000001, norm 1.537545"
## [1] "Iter 70, obj 13.076426, abs 0.000032, rel 0.000002, norm 1.537480"
## [1] "Iter 71, obj 13.076377, abs 0.000049, rel 0.000004, norm 1.537416"
## [1] "Iter 72, obj 13.076312, abs 0.000064, rel 0.000005, norm 1.537354"
## [1] "Iter 73, obj 13.076235, abs 0.000078, rel 0.000006, norm 1.537294"
## [1] "Iter 74, obj 13.076145, abs 0.000089, rel 0.000007, norm 1.537234"
## [1] "Iter 75, obj 13.076046, abs 0.000100, rel 0.000008, norm 1.537176"

## [1] "Iter 76, obj 13.075938, abs 0.000108, rel 0.000008, norm 1.537120"
## [1] "Iter 77, obj 13.075821, abs 0.000116, rel 0.000009, norm 1.537065"
## [1] "Iter 78, obj 13.075699, abs 0.000123, rel 0.000009, norm 1.537012"
## [1] "Iter 79, obj 13.075571, abs 0.000128, rel 0.000010, norm 1.536961"
## [1] "Iter 80, obj 13.075437, abs 0.000133, rel 0.000010, norm 1.536910"
## [1] "Iter 81, obj 13.075300, abs 0.000137, rel 0.000010, norm 1.536862"
## [1] "Iter 82, obj 13.075160, abs 0.000140, rel 0.000011, norm 1.536815"
## [1] "Iter 83, obj 13.075017, abs 0.000143, rel 0.000011, norm 1.536769"
## [1] "Iter 84, obj 13.074872, abs 0.000145, rel 0.000011, norm 1.536725"
## [1] "Iter 85, obj 13.074725, abs 0.000147, rel 0.000011, norm 1.536683"
## [1] "Iter 86, obj 13.074576, abs 0.000148, rel 0.000011, norm 1.536642"
## [1] "Iter 87, obj 13.074427, abs 0.000149, rel 0.000011, norm 1.536603"
## [1] "Iter 88, obj 13.074277, abs 0.000150, rel 0.000011, norm 1.536565"
## [1] "Iter 89, obj 13.074127, abs 0.000150, rel 0.000011, norm 1.536529"
## [1] "Iter 90, obj 13.073977, abs 0.000150, rel 0.000011, norm 1.536494"
## [1] "Iter 91, obj 13.073828, abs 0.000150, rel 0.000011, norm 1.536461"
## [1] "Iter 92, obj 13.073678, abs 0.000149, rel 0.000011, norm 1.536429"
## [1] "Iter 93, obj 13.073529, abs 0.000149, rel 0.000011, norm 1.536398"
## [1] "Iter 94, obj 13.073381, abs 0.000148, rel 0.000011, norm 1.536369"
## [1] "Iter 95, obj 13.073234, abs 0.000147, rel 0.000011, norm 1.536342"
## [1] "Iter 96, obj 13.073088, abs 0.000146, rel 0.000011, norm 1.536315"
## [1] "Iter 97, obj 13.072943, abs 0.000145, rel 0.000011, norm 1.536291"
## [1] "Iter 98, obj 13.072799, abs 0.000144, rel 0.000011, norm 1.536267"
## [1] "Iter 99, obj 13.072656, abs 0.000143, rel 0.000011, norm 1.536245"
## [1] "Iter 100, obj 13.072514, abs 0.000142, rel 0.000011, norm 1.536224"

## [1] "Iter 101, obj 1.801132, abs 11.271383, rel 0.862220, norm 0.105926"
## [1] "Iter 102, obj 1.602240, abs 0.198891, rel 0.110426, norm 0.117554"
## [1] "Iter 103, obj 1.511840, abs 0.090400, rel 0.056421, norm 0.139118"
## [1] "Iter 104, obj 1.507266, abs 0.004575, rel 0.003026, norm 0.141898"
## [1] "Iter 105, obj 1.524248, abs 0.016982, rel 0.011267, norm 0.142854"
## [1] "Iter 106, obj 1.489285, abs 0.034963, rel 0.022938, norm 0.140910"
## [1] "Iter 107, obj 1.488886, abs 0.000398, rel 0.000267, norm 0.140944"
## [1] "Iter 108, obj 1.494257, abs 0.005371, rel 0.003607, norm 0.142803"
## [1] "Iter 109, obj 1.480020, abs 0.014237, rel 0.009528, norm 0.141621"
## [1] "Iter 110, obj 1.483610, abs 0.003590, rel 0.002426, norm 0.141413"
## [1] "Iter 111, obj 1.489652, abs 0.006042, rel 0.004072, norm 0.142377"
## [1] "Iter 112, obj 1.480532, abs 0.009120, rel 0.006122, norm 0.141594"
## [1] "Iter 113, obj 1.501092, abs 0.020560, rel 0.013887, norm 0.140759"
## [1] "Iter 114, obj 1.497730, abs 0.003362, rel 0.002240, norm 0.142138"
## [1] "Iter 115, obj 1.494882, abs 0.002848, rel 0.001901, norm 0.142409"
## [1] "Iter 116, obj 1.503123, abs 0.008241, rel 0.005513, norm 0.142472"
## [1] "Iter 117, obj 1.494809, abs 0.008314, rel 0.005531, norm 0.142006"
## [1] "Iter 118, obj 1.507181, abs 0.012372, rel 0.008277, norm 0.141231"
## [1] "Iter 119, obj 1.509887, abs 0.002706, rel 0.001795, norm 0.141660"
## [1] "Iter 120, obj 1.499947, abs 0.009940, rel 0.006583, norm 0.141298"
## [1] "Iter 121, obj 1.511375, abs 0.011428, rel 0.007619, norm 0.141137"

```

```

## [1] "Iter 122, obj 1.512195, abs 0.000820, rel 0.000542, norm 0.141303"
## [1] "Iter 123, obj 1.503964, abs 0.008230, rel 0.005443, norm 0.141549"
## [1] "Iter 124, obj 1.512061, abs 0.008097, rel 0.005384, norm 0.141426"
## [1] "Iter 125, obj 1.518368, abs 0.006307, rel 0.004171, norm 0.140958"

## [1] "Iter 126, obj 1.510816, abs 0.007552, rel 0.004974, norm 0.141452"
## [1] "Iter 127, obj 1.507091, abs 0.003725, rel 0.002465, norm 0.140677"
## [1] "Iter 128, obj 1.513779, abs 0.006687, rel 0.004437, norm 0.140991"
## [1] "Iter 129, obj 1.509670, abs 0.004108, rel 0.002714, norm 0.141576"
## [1] "Iter 130, obj 1.508866, abs 0.000805, rel 0.000533, norm 0.140922"
## [1] "Iter 131, obj 1.527823, abs 0.018957, rel 0.012564, norm 0.140843"
## [1] "Iter 132, obj 1.511885, abs 0.015938, rel 0.010432, norm 0.142019"
## [1] "Iter 133, obj 1.512699, abs 0.000814, rel 0.000539, norm 0.140853"
## [1] "Iter 134, obj 1.530811, abs 0.018112, rel 0.011973, norm 0.141570"
## [1] "Iter 135, obj 1.514361, abs 0.016450, rel 0.010746, norm 0.141959"
## [1] "Iter 136, obj 1.508928, abs 0.005432, rel 0.003587, norm 0.140561"
## [1] "Iter 137, obj 1.531029, abs 0.022101, rel 0.014647, norm 0.141863"
## [1] "Iter 138, obj 1.513791, abs 0.017238, rel 0.011259, norm 0.141699"
## [1] "Iter 139, obj 1.507240, abs 0.006551, rel 0.004327, norm 0.140207"
## [1] "Iter 140, obj 1.525054, abs 0.017814, rel 0.011819, norm 0.141177"
## [1] "Iter 141, obj 1.518520, abs 0.006533, rel 0.004284, norm 0.141443"
## [1] "Iter 142, obj 1.509301, abs 0.009219, rel 0.006071, norm 0.141012"
## [1] "Iter 143, obj 1.530370, abs 0.021068, rel 0.013959, norm 0.141190"
## [1] "Iter 144, obj 1.504159, abs 0.026210, rel 0.017127, norm 0.141380"
## [1] "Iter 145, obj 1.493536, abs 0.010624, rel 0.007063, norm 0.140371"
## [1] "Iter 146, obj 1.516839, abs 0.023303, rel 0.015603, norm 0.141598"
## [1] "Iter 147, obj 1.500784, abs 0.016055, rel 0.010585, norm 0.142108"
## [1] "Iter 148, obj 1.496670, abs 0.004113, rel 0.002741, norm 0.140950"
## [1] "Iter 149, obj 1.508541, abs 0.011871, rel 0.007932, norm 0.141362"
## [1] "Iter 150, obj 1.497427, abs 0.011114, rel 0.007367, norm 0.141104"

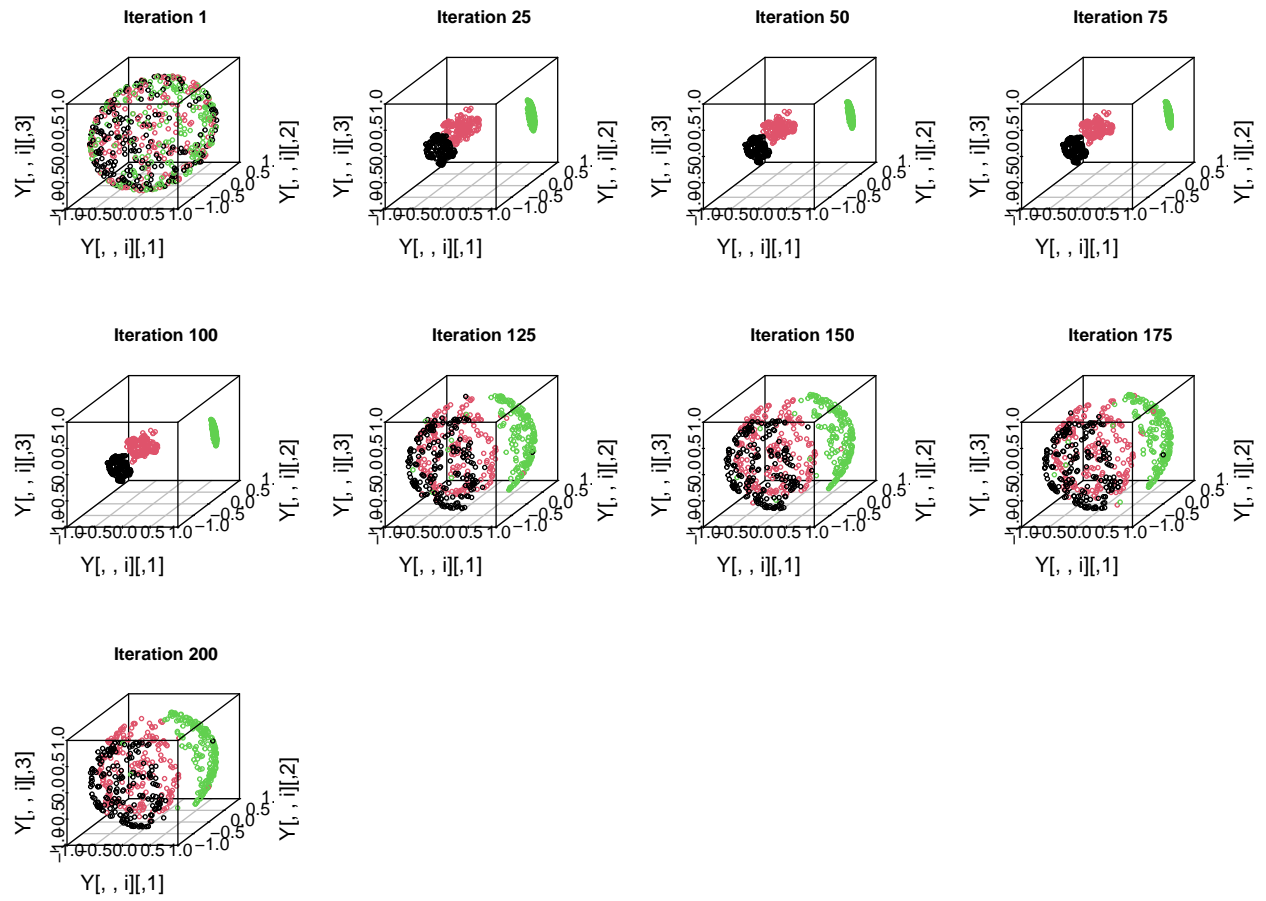
## [1] "Iter 151, obj 1.503439, abs 0.006011, rel 0.004015, norm 0.140442"
## [1] "Iter 152, obj 1.519362, abs 0.015923, rel 0.010591, norm 0.142094"
## [1] "Iter 153, obj 1.502652, abs 0.016711, rel 0.010998, norm 0.141259"
## [1] "Iter 154, obj 1.503571, abs 0.000920, rel 0.000612, norm 0.140898"
## [1] "Iter 155, obj 1.525106, abs 0.021534, rel 0.014322, norm 0.141163"
## [1] "Iter 156, obj 1.519303, abs 0.005803, rel 0.003805, norm 0.141254"
## [1] "Iter 157, obj 1.502880, abs 0.016423, rel 0.010809, norm 0.141105"
## [1] "Iter 158, obj 1.509467, abs 0.006587, rel 0.004383, norm 0.140172"
## [1] "Iter 159, obj 1.508842, abs 0.000625, rel 0.000414, norm 0.141047"
## [1] "Iter 160, obj 1.507675, abs 0.001167, rel 0.000773, norm 0.140962"
## [1] "Iter 161, obj 1.503928, abs 0.003747, rel 0.002485, norm 0.140995"
## [1] "Iter 162, obj 1.513021, abs 0.009092, rel 0.006046, norm 0.140907"
## [1] "Iter 163, obj 1.519940, abs 0.006919, rel 0.004573, norm 0.141569"
## [1] "Iter 164, obj 1.512304, abs 0.007636, rel 0.005024, norm 0.140555"
## [1] "Iter 165, obj 1.506405, abs 0.005899, rel 0.003901, norm 0.141005"
## [1] "Iter 166, obj 1.505055, abs 0.001350, rel 0.000896, norm 0.140795"
## [1] "Iter 167, obj 1.505837, abs 0.000782, rel 0.000520, norm 0.140826"
## [1] "Iter 168, obj 1.504154, abs 0.001683, rel 0.001118, norm 0.141533"
## [1] "Iter 169, obj 1.497876, abs 0.006278, rel 0.004174, norm 0.140622"
## [1] "Iter 170, obj 1.514557, abs 0.016682, rel 0.011137, norm 0.141033"
## [1] "Iter 171, obj 1.511005, abs 0.003552, rel 0.002345, norm 0.141163"
## [1] "Iter 172, obj 1.507634, abs 0.003371, rel 0.002231, norm 0.141507"
## [1] "Iter 173, obj 1.507001, abs 0.000633, rel 0.000420, norm 0.141470"
## [1] "Iter 174, obj 1.507186, abs 0.000185, rel 0.000123, norm 0.141473"

```

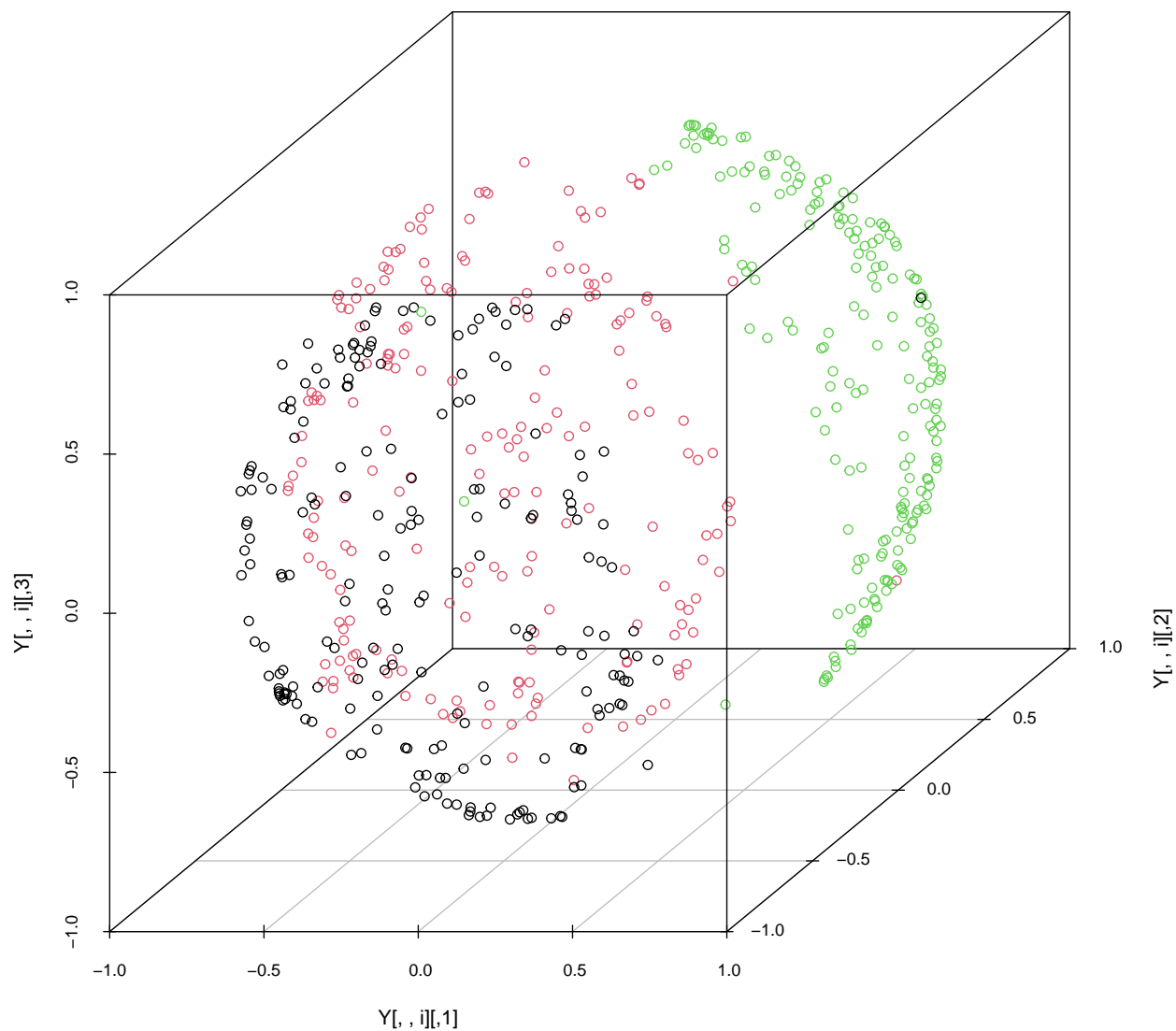
```

## [1] "Iter 175, obj 1.502448, abs 0.004738, rel 0.003143, norm 0.141234"
## [1] "Iter 176, obj 1.509543, abs 0.007095, rel 0.004722, norm 0.141118"
## [1] "Iter 177, obj 1.512587, abs 0.003044, rel 0.002017, norm 0.141625"
## [1] "Iter 178, obj 1.512352, abs 0.000235, rel 0.000156, norm 0.141922"
## [1] "Iter 179, obj 1.513429, abs 0.001078, rel 0.000712, norm 0.140895"
## [1] "Iter 180, obj 1.524007, abs 0.010577, rel 0.006989, norm 0.141142"
## [1] "Iter 181, obj 1.511071, abs 0.012936, rel 0.008488, norm 0.141119"
## [1] "Iter 182, obj 1.512862, abs 0.001791, rel 0.001185, norm 0.140466"
## [1] "Iter 183, obj 1.530005, abs 0.017143, rel 0.011332, norm 0.140977"
## [1] "Iter 184, obj 1.515994, abs 0.014011, rel 0.009158, norm 0.141836"
## [1] "Iter 185, obj 1.499284, abs 0.016710, rel 0.011022, norm 0.140394"
## [1] "Iter 186, obj 1.516177, abs 0.016893, rel 0.011268, norm 0.141620"
## [1] "Iter 187, obj 1.510181, abs 0.005996, rel 0.003955, norm 0.141442"
## [1] "Iter 188, obj 1.497860, abs 0.012321, rel 0.008159, norm 0.141180"
## [1] "Iter 189, obj 1.518229, abs 0.020369, rel 0.013599, norm 0.141286"
## [1] "Iter 190, obj 1.509421, abs 0.008808, rel 0.005801, norm 0.142006"
## [1] "Iter 191, obj 1.508717, abs 0.000704, rel 0.000467, norm 0.141732"
## [1] "Iter 192, obj 1.527600, abs 0.018883, rel 0.012516, norm 0.141610"
## [1] "Iter 193, obj 1.506474, abs 0.021127, rel 0.013830, norm 0.141563"
## [1] "Iter 194, obj 1.497606, abs 0.008867, rel 0.005886, norm 0.140219"
## [1] "Iter 195, obj 1.530352, abs 0.032746, rel 0.021865, norm 0.141323"
## [1] "Iter 196, obj 1.505266, abs 0.025086, rel 0.016393, norm 0.141241"
## [1] "Iter 197, obj 1.499782, abs 0.005484, rel 0.003643, norm 0.140214"
## [1] "Iter 198, obj 1.539628, abs 0.039847, rel 0.026568, norm 0.141767"
## [1] "Iter 199, obj 1.518774, abs 0.020855, rel 0.013545, norm 0.141879"
## [1] "Iter 200, obj 1.484580, abs 0.034193, rel 0.022514, norm 0.140034"

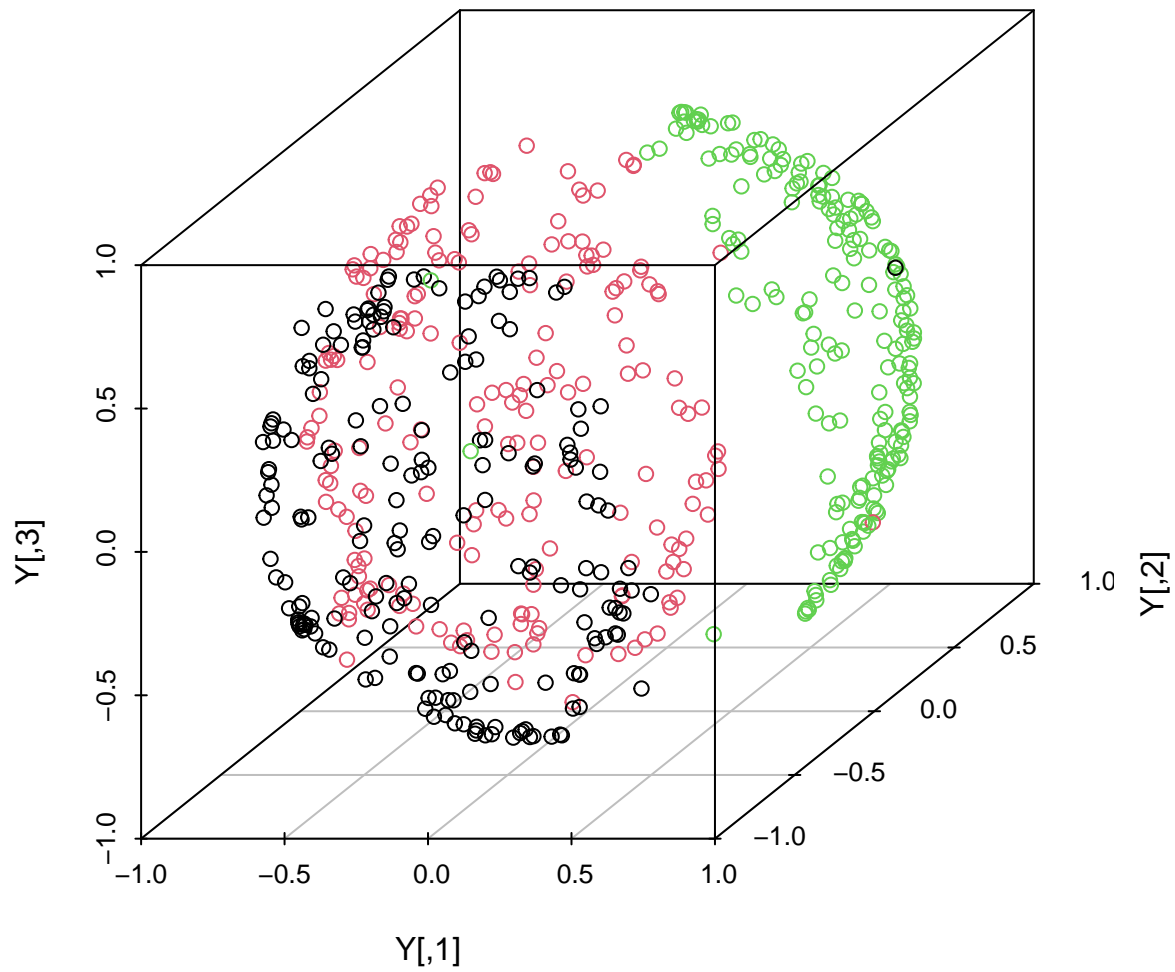
```



Iteration 200

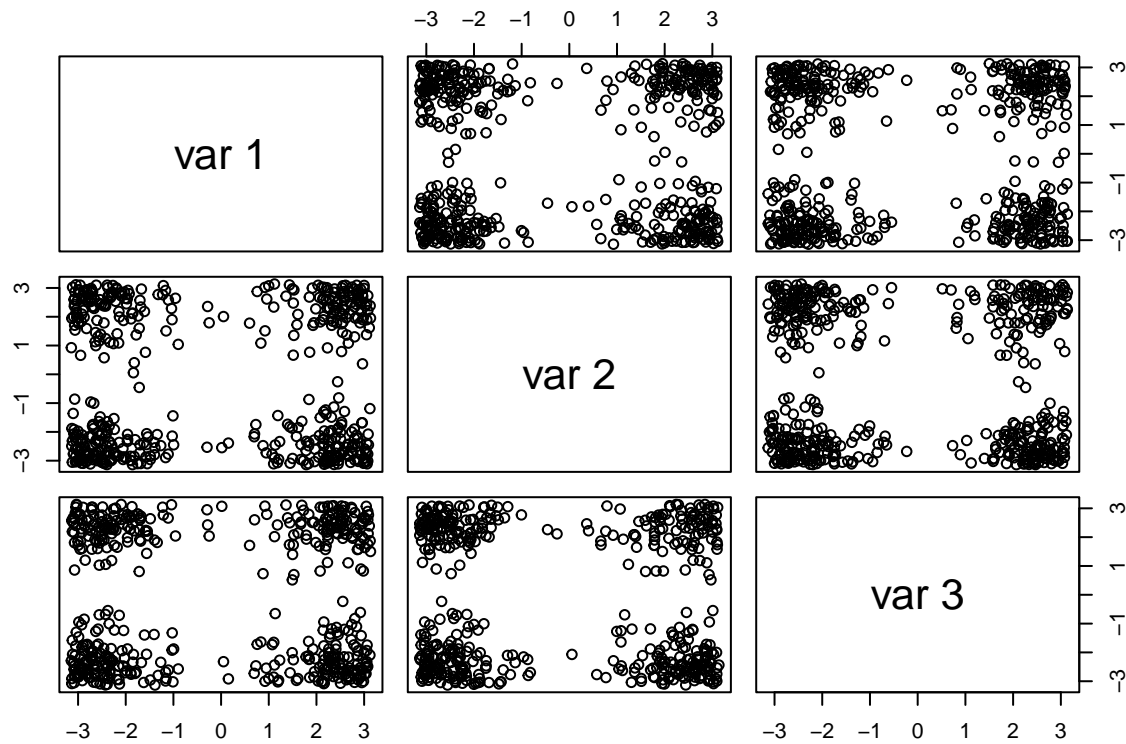


```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),  
                             color = colors)
```

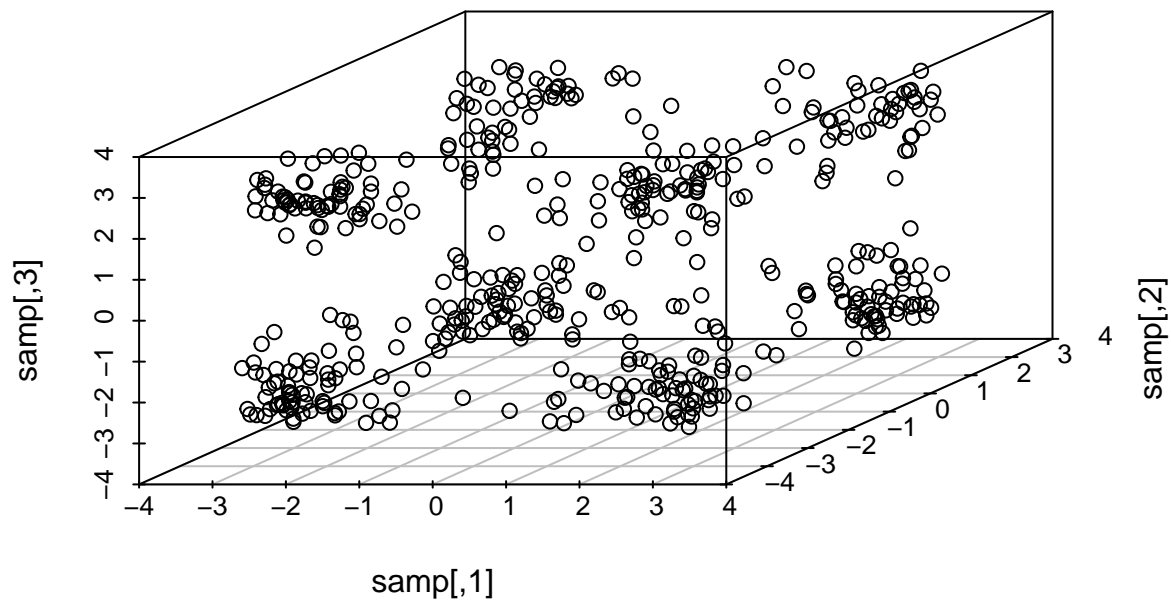


Case 4

```
d <- 3
n <- 5e2
samp <- sdetorus::toPiInt(mvtnorm::rmvnorm(n = n, mean = rep(pi, d)))
pairs(samp)
```



```
scatterplot3d::scatterplot3d(samp, xlim = c(-pi, pi), ylim = c(-pi, pi), zlim = c(-pi, pi))
```



```
x_4 <- array(dim = c(n, 2, 3))
x_4[, , 1] <- DirStats::to_cir(samp[, 1])
x_4[, , 2] <- DirStats::to_cir(samp[, 2])
x_4[, , 3] <- DirStats::to_cir(samp[, 3])
```

Let's calculate the rho parameters based on a perplexity of 25:

```
rho_four_perp20 <- rho_optim_bst(x_4, 20)
```

```
## Time difference of 51.75982 secs
```

First, let's reduce to dimension \mathbb{S}^1 then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_4, d=1, rho_psc_list = rho_four_perp20, num_iteration=200,  
             visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 17.832530, abs 0.000000, rel 0.000000, norm 0.149851"  
## [1] "Iter 2, obj 17.132231, abs 0.700299, rel 0.039271, norm 0.314131"  
## [1] "Iter 3, obj 16.749832, abs 0.382399, rel 0.022320, norm 0.417791"  
## [1] "Iter 4, obj 16.488882, abs 0.260950, rel 0.015579, norm 0.480186"  
## [1] "Iter 5, obj 16.293884, abs 0.194998, rel 0.011826, norm 0.528585"  
## [1] "Iter 6, obj 16.145337, abs 0.148547, rel 0.009117, norm 0.570134"  
## [1] "Iter 7, obj 16.014375, abs 0.130962, rel 0.008111, norm 0.603219"  
## [1] "Iter 8, obj 15.915075, abs 0.099299, rel 0.006201, norm 0.629362"  
## [1] "Iter 9, obj 15.836971, abs 0.078105, rel 0.004908, norm 0.649268"  
## [1] "Iter 10, obj 15.764933, abs 0.072038, rel 0.004549, norm 0.663807"  
## [1] "Iter 11, obj 15.698385, abs 0.066548, rel 0.004221, norm 0.677135"  
## [1] "Iter 12, obj 15.650151, abs 0.048233, rel 0.003073, norm 0.690621"  
## [1] "Iter 13, obj 15.612826, abs 0.037326, rel 0.002385, norm 0.701445"  
## [1] "Iter 14, obj 15.574167, abs 0.038659, rel 0.002476, norm 0.709705"  
## [1] "Iter 15, obj 15.534982, abs 0.039185, rel 0.002516, norm 0.716724"  
## [1] "Iter 16, obj 15.508732, abs 0.026250, rel 0.001690, norm 0.723596"  
## [1] "Iter 17, obj 15.483371, abs 0.025361, rel 0.001635, norm 0.730432"  
## [1] "Iter 18, obj 15.465264, abs 0.018107, rel 0.001169, norm 0.737400"  
## [1] "Iter 19, obj 15.455548, abs 0.009715, rel 0.000628, norm 0.743576"  
## [1] "Iter 20, obj 15.450715, abs 0.004834, rel 0.000313, norm 0.749020"  
## [1] "Iter 21, obj 15.448617, abs 0.002097, rel 0.000136, norm 0.753844"  
## [1] "Iter 22, obj 15.447110, abs 0.001508, rel 0.000098, norm 0.757991"  
## [1] "Iter 23, obj 15.439345, abs 0.007765, rel 0.000503, norm 0.761532"  
## [1] "Iter 24, obj 15.436860, abs 0.002485, rel 0.000161, norm 0.764897"  
## [1] "Iter 25, obj 15.434262, abs 0.002598, rel 0.000168, norm 0.767267"  
  
## [1] "Iter 26, obj 15.431198, abs 0.003064, rel 0.000199, norm 0.769285"  
## [1] "Iter 27, obj 15.427731, abs 0.003466, rel 0.000225, norm 0.771003"  
## [1] "Iter 28, obj 15.424151, abs 0.003581, rel 0.000232, norm 0.772482"  
## [1] "Iter 29, obj 15.420734, abs 0.003416, rel 0.000222, norm 0.773734"  
## [1] "Iter 30, obj 15.417678, abs 0.003056, rel 0.000198, norm 0.774759"  
## [1] "Iter 31, obj 15.415097, abs 0.002581, rel 0.000167, norm 0.775572"  
## [1] "Iter 32, obj 15.413033, abs 0.002064, rel 0.000134, norm 0.776200"  
## [1] "Iter 33, obj 15.411459, abs 0.001574, rel 0.000102, norm 0.776681"  
## [1] "Iter 34, obj 15.410270, abs 0.001189, rel 0.000077, norm 0.777054"  
## [1] "Iter 35, obj 15.409196, abs 0.001074, rel 0.000070, norm 0.777358"  
## [1] "Iter 36, obj 15.407934, abs 0.001261, rel 0.000082, norm 0.777639"  
## [1] "Iter 37, obj 15.406547, abs 0.001388, rel 0.000090, norm 0.777949"  
## [1] "Iter 38, obj 15.405471, abs 0.001076, rel 0.000070, norm 0.778333"  
## [1] "Iter 39, obj 15.404877, abs 0.000594, rel 0.000039, norm 0.778636"  
## [1] "Iter 40, obj 15.404493, abs 0.000384, rel 0.000025, norm 0.778657"  
## [1] "Iter 41, obj 15.404187, abs 0.000306, rel 0.000020, norm 0.778566"  
## [1] "Iter 42, obj 15.403854, abs 0.000333, rel 0.000022, norm 0.778451"  
## [1] "Iter 43, obj 15.403442, abs 0.000412, rel 0.000027, norm 0.778329"  
## [1] "Iter 44, obj 15.402927, abs 0.000515, rel 0.000033, norm 0.778209"  
## [1] "Iter 45, obj 15.402299, abs 0.000628, rel 0.000041, norm 0.778099"  
## [1] "Iter 46, obj 15.401559, abs 0.000741, rel 0.000048, norm 0.778006"  
## [1] "Iter 47, obj 15.400714, abs 0.000844, rel 0.000055, norm 0.777934"  
## [1] "Iter 48, obj 15.399776, abs 0.000938, rel 0.000061, norm 0.777882"  
## [1] "Iter 49, obj 15.398748, abs 0.001028, rel 0.000067, norm 0.777853"
```

```

## [1] "Iter 50, obj 15.397622, abs 0.001126, rel 0.000073, norm 0.777843"
## [1] "Iter 51, obj 15.396391, abs 0.001231, rel 0.000080, norm 0.777854"
## [1] "Iter 52, obj 15.395051, abs 0.001340, rel 0.000087, norm 0.777886"
## [1] "Iter 53, obj 15.393590, abs 0.001461, rel 0.000095, norm 0.777935"
## [1] "Iter 54, obj 15.391962, abs 0.001627, rel 0.000106, norm 0.777998"
## [1] "Iter 55, obj 15.390105, abs 0.001858, rel 0.000121, norm 0.778073"
## [1] "Iter 56, obj 15.387967, abs 0.002138, rel 0.000139, norm 0.778159"
## [1] "Iter 57, obj 15.385536, abs 0.002430, rel 0.000158, norm 0.778248"
## [1] "Iter 58, obj 15.382844, abs 0.002692, rel 0.000175, norm 0.778326"
## [1] "Iter 59, obj 15.379957, abs 0.002887, rel 0.000188, norm 0.778372"
## [1] "Iter 60, obj 15.376961, abs 0.002996, rel 0.000195, norm 0.778367"
## [1] "Iter 61, obj 15.373953, abs 0.003008, rel 0.000196, norm 0.778290"
## [1] "Iter 62, obj 15.371029, abs 0.002924, rel 0.000190, norm 0.778129"
## [1] "Iter 63, obj 15.368303, abs 0.002726, rel 0.000177, norm 0.777876"
## [1] "Iter 64, obj 15.365942, abs 0.002361, rel 0.000154, norm 0.777530"
## [1] "Iter 65, obj 15.363747, abs 0.002195, rel 0.000143, norm 0.777085"
## [1] "Iter 66, obj 15.357113, abs 0.006634, rel 0.000432, norm 0.776593"
## [1] "Iter 67, obj 15.351982, abs 0.005130, rel 0.000334, norm 0.776082"
## [1] "Iter 68, obj 15.348965, abs 0.003017, rel 0.000197, norm 0.775280"
## [1] "Iter 69, obj 15.346284, abs 0.002680, rel 0.000175, norm 0.774423"
## [1] "Iter 70, obj 15.343714, abs 0.002571, rel 0.000168, norm 0.773519"
## [1] "Iter 71, obj 15.341113, abs 0.002601, rel 0.000170, norm 0.772585"
## [1] "Iter 72, obj 15.338344, abs 0.002769, rel 0.000180, norm 0.771635"
## [1] "Iter 73, obj 15.335285, abs 0.003058, rel 0.000199, norm 0.770681"
## [1] "Iter 74, obj 15.331851, abs 0.003435, rel 0.000224, norm 0.769735"
## [1] "Iter 75, obj 15.328023, abs 0.003828, rel 0.000250, norm 0.768804"

## [1] "Iter 76, obj 15.323890, abs 0.004133, rel 0.000270, norm 0.767889"
## [1] "Iter 77, obj 15.319660, abs 0.004230, rel 0.000276, norm 0.766976"
## [1] "Iter 78, obj 15.315628, abs 0.004032, rel 0.000263, norm 0.766030"
## [1] "Iter 79, obj 15.312104, abs 0.003524, rel 0.000230, norm 0.765010"
## [1] "Iter 80, obj 15.309321, abs 0.002784, rel 0.000182, norm 0.763881"
## [1] "Iter 81, obj 15.307375, abs 0.001946, rel 0.000127, norm 0.762632"
## [1] "Iter 82, obj 15.306237, abs 0.001138, rel 0.000074, norm 0.761279"
## [1] "Iter 83, obj 15.305804, abs 0.000433, rel 0.000028, norm 0.759860"
## [1] "Iter 84, obj 15.305950, abs 0.000146, rel 0.000010, norm 0.758415"
## [1] "Iter 85, obj 15.306558, abs 0.000608, rel 0.000040, norm 0.756981"
## [1] "Iter 86, obj 15.307530, abs 0.000971, rel 0.000063, norm 0.755586"
## [1] "Iter 87, obj 15.308782, abs 0.001252, rel 0.000082, norm 0.754249"
## [1] "Iter 88, obj 15.310244, abs 0.001462, rel 0.000095, norm 0.752979"
## [1] "Iter 89, obj 15.311853, abs 0.001609, rel 0.000105, norm 0.751782"
## [1] "Iter 90, obj 15.313550, abs 0.001698, rel 0.000111, norm 0.750657"
## [1] "Iter 91, obj 15.315284, abs 0.001733, rel 0.000113, norm 0.749600"
## [1] "Iter 92, obj 15.317006, abs 0.001723, rel 0.000112, norm 0.748605"
## [1] "Iter 93, obj 15.318684, abs 0.001678, rel 0.000110, norm 0.747663"
## [1] "Iter 94, obj 15.320297, abs 0.001613, rel 0.000105, norm 0.746760"
## [1] "Iter 95, obj 15.321840, abs 0.001543, rel 0.000101, norm 0.745884"
## [1] "Iter 96, obj 15.323316, abs 0.001476, rel 0.000096, norm 0.745018"
## [1] "Iter 97, obj 15.324726, abs 0.001410, rel 0.000092, norm 0.744150"
## [1] "Iter 98, obj 15.326071, abs 0.001345, rel 0.000088, norm 0.743274"
## [1] "Iter 99, obj 15.327359, abs 0.001288, rel 0.000084, norm 0.742389"
## [1] "Iter 100, obj 15.328626, abs 0.001267, rel 0.000083, norm 0.741511"

## [1] "Iter 101, obj 2.421696, abs 12.906931, rel 0.842015, norm 0.106687"

```

```

## [1] "Iter 102, obj 2.399466, abs 0.022230, rel 0.009179, norm 0.102895"
## [1] "Iter 103, obj 2.381080, abs 0.018385, rel 0.007662, norm 0.100992"
## [1] "Iter 104, obj 2.367509, abs 0.013572, rel 0.005700, norm 0.100689"
## [1] "Iter 105, obj 2.357305, abs 0.010203, rel 0.004310, norm 0.100837"
## [1] "Iter 106, obj 2.349682, abs 0.007623, rel 0.003234, norm 0.101161"
## [1] "Iter 107, obj 2.344804, abs 0.004879, rel 0.002076, norm 0.101241"
## [1] "Iter 108, obj 2.342046, abs 0.002758, rel 0.001176, norm 0.101211"
## [1] "Iter 109, obj 2.342567, abs 0.000521, rel 0.000222, norm 0.101201"
## [1] "Iter 110, obj 2.343417, abs 0.000851, rel 0.000363, norm 0.101403"
## [1] "Iter 111, obj 2.340769, abs 0.002648, rel 0.001130, norm 0.101376"
## [1] "Iter 112, obj 2.338235, abs 0.002534, rel 0.001083, norm 0.101690"
## [1] "Iter 113, obj 2.337779, abs 0.000456, rel 0.000195, norm 0.101696"
## [1] "Iter 114, obj 2.339735, abs 0.001956, rel 0.000837, norm 0.101491"
## [1] "Iter 115, obj 2.338937, abs 0.000799, rel 0.000341, norm 0.101745"
## [1] "Iter 116, obj 2.338315, abs 0.000622, rel 0.000266, norm 0.101679"
## [1] "Iter 117, obj 2.337934, abs 0.000380, rel 0.000163, norm 0.101409"
## [1] "Iter 118, obj 2.337205, abs 0.000729, rel 0.000312, norm 0.101247"
## [1] "Iter 119, obj 2.337276, abs 0.000071, rel 0.000030, norm 0.101688"
## [1] "Iter 120, obj 2.338882, abs 0.001606, rel 0.000687, norm 0.101848"
## [1] "Iter 121, obj 2.337654, abs 0.001228, rel 0.000525, norm 0.101642"
## [1] "Iter 122, obj 2.338353, abs 0.000699, rel 0.000299, norm 0.101769"
## [1] "Iter 123, obj 2.336514, abs 0.001838, rel 0.000786, norm 0.101559"
## [1] "Iter 124, obj 2.337270, abs 0.000756, rel 0.000324, norm 0.101642"
## [1] "Iter 125, obj 2.337177, abs 0.000093, rel 0.000040, norm 0.101314"

## [1] "Iter 126, obj 2.336438, abs 0.000740, rel 0.000316, norm 0.101488"
## [1] "Iter 127, obj 2.337486, abs 0.001049, rel 0.000449, norm 0.102134"
## [1] "Iter 128, obj 2.336393, abs 0.001094, rel 0.000468, norm 0.101924"
## [1] "Iter 129, obj 2.335934, abs 0.000459, rel 0.000197, norm 0.101946"
## [1] "Iter 130, obj 2.336462, abs 0.000528, rel 0.000226, norm 0.101801"
## [1] "Iter 131, obj 2.337309, abs 0.000847, rel 0.000363, norm 0.101553"
## [1] "Iter 132, obj 2.336204, abs 0.001105, rel 0.000473, norm 0.101798"
## [1] "Iter 133, obj 2.335788, abs 0.000417, rel 0.000178, norm 0.101839"
## [1] "Iter 134, obj 2.335804, abs 0.000017, rel 0.000007, norm 0.101838"
## [1] "Iter 135, obj 2.335926, abs 0.000122, rel 0.000052, norm 0.101934"
## [1] "Iter 136, obj 2.335285, abs 0.000641, rel 0.000274, norm 0.102160"
## [1] "Iter 137, obj 2.335894, abs 0.000609, rel 0.000261, norm 0.102211"
## [1] "Iter 138, obj 2.335702, abs 0.000192, rel 0.000082, norm 0.102193"
## [1] "Iter 139, obj 2.336214, abs 0.000512, rel 0.000219, norm 0.101913"
## [1] "Iter 140, obj 2.335675, abs 0.000539, rel 0.000231, norm 0.101857"
## [1] "Iter 141, obj 2.335361, abs 0.000314, rel 0.000134, norm 0.101938"
## [1] "Iter 142, obj 2.336013, abs 0.000652, rel 0.000279, norm 0.102060"
## [1] "Iter 143, obj 2.334695, abs 0.001318, rel 0.000564, norm 0.102309"
## [1] "Iter 144, obj 2.335766, abs 0.001071, rel 0.000459, norm 0.102323"
## [1] "Iter 145, obj 2.335036, abs 0.000730, rel 0.000313, norm 0.102007"
## [1] "Iter 146, obj 2.336741, abs 0.001705, rel 0.000730, norm 0.101864"
## [1] "Iter 147, obj 2.335281, abs 0.001460, rel 0.000625, norm 0.101730"
## [1] "Iter 148, obj 2.333269, abs 0.002012, rel 0.000862, norm 0.101923"
## [1] "Iter 149, obj 2.334420, abs 0.001151, rel 0.000493, norm 0.101988"
## [1] "Iter 150, obj 2.336548, abs 0.002128, rel 0.000912, norm 0.102245"

## [1] "Iter 151, obj 2.336938, abs 0.000390, rel 0.000167, norm 0.102479"
## [1] "Iter 152, obj 2.334983, abs 0.001956, rel 0.000837, norm 0.102183"
## [1] "Iter 153, obj 2.334491, abs 0.000491, rel 0.000210, norm 0.102122"
## [1] "Iter 154, obj 2.335519, abs 0.001027, rel 0.000440, norm 0.101749"

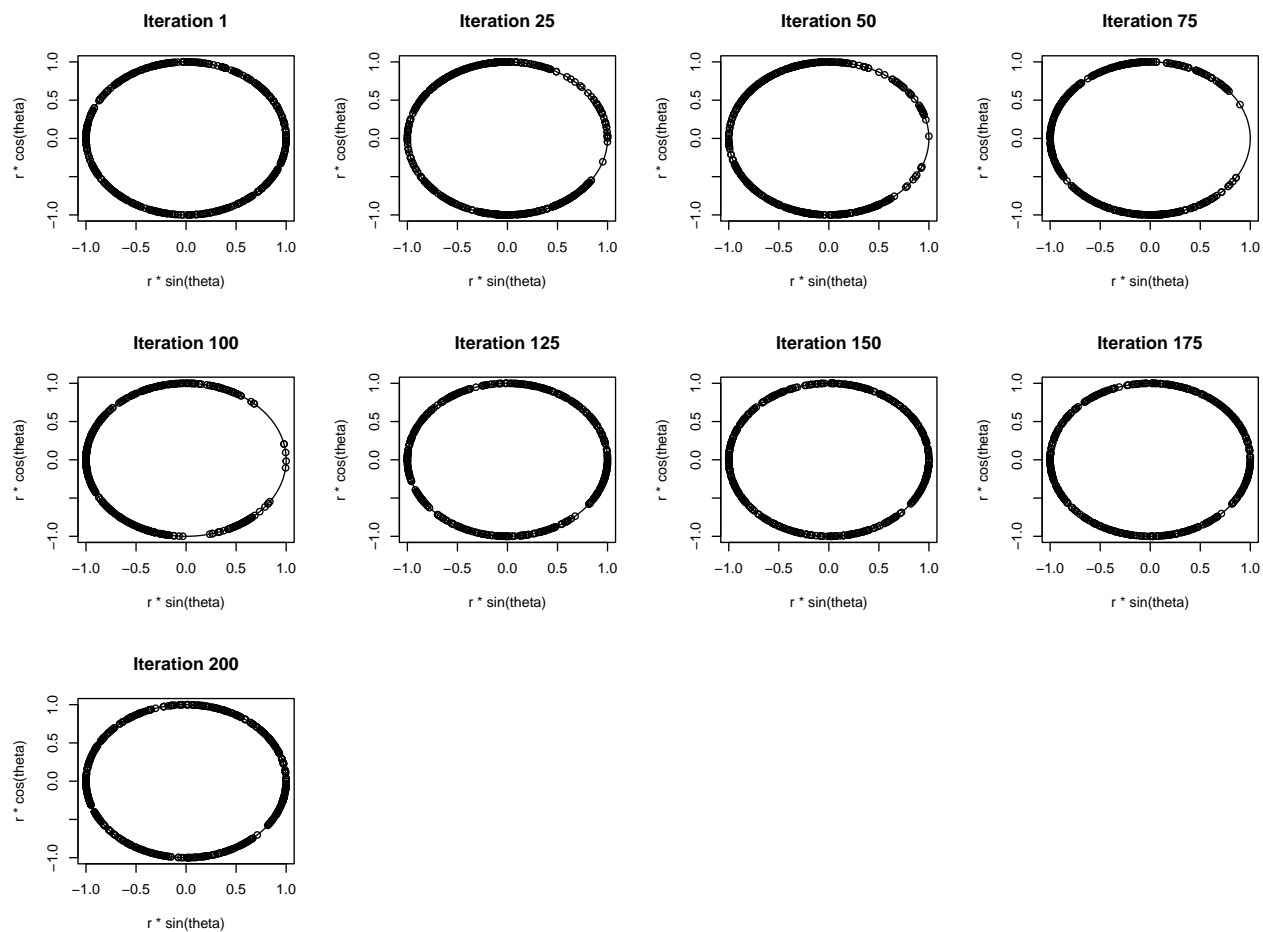
```

```

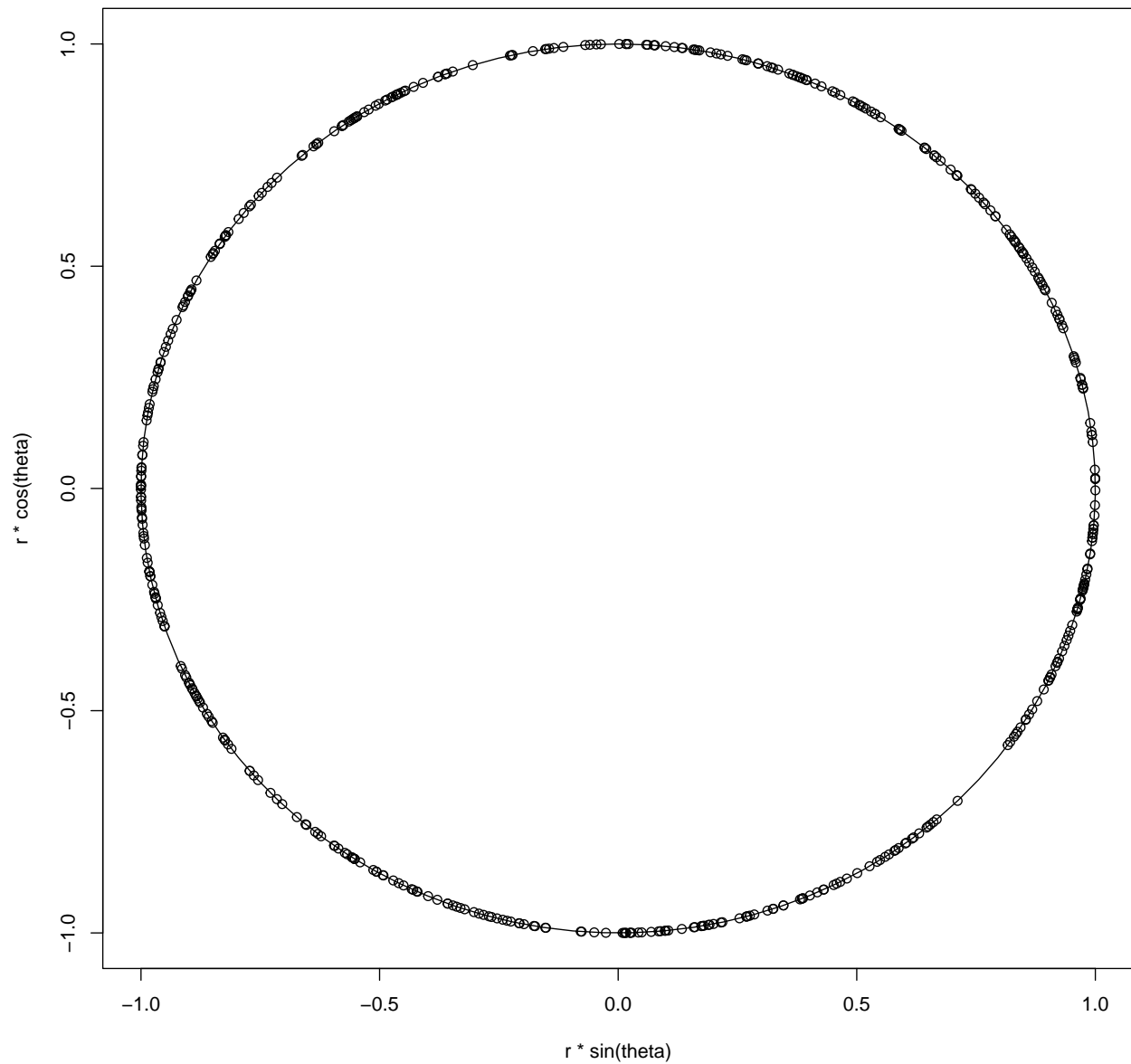
## [1] "Iter 155, obj 2.335810, abs 0.000292, rel 0.000125, norm 0.101938"
## [1] "Iter 156, obj 2.335940, abs 0.000130, rel 0.000056, norm 0.101985"
## [1] "Iter 157, obj 2.337461, abs 0.001521, rel 0.000651, norm 0.101948"
## [1] "Iter 158, obj 2.335018, abs 0.002443, rel 0.001045, norm 0.102132"
## [1] "Iter 159, obj 2.334634, abs 0.000384, rel 0.000165, norm 0.102261"
## [1] "Iter 160, obj 2.335610, abs 0.000976, rel 0.000418, norm 0.102207"
## [1] "Iter 161, obj 2.336467, abs 0.000857, rel 0.000367, norm 0.102113"
## [1] "Iter 162, obj 2.336150, abs 0.000317, rel 0.000136, norm 0.102200"
## [1] "Iter 163, obj 2.334720, abs 0.001430, rel 0.000612, norm 0.101993"
## [1] "Iter 164, obj 2.333887, abs 0.000833, rel 0.000357, norm 0.102091"
## [1] "Iter 165, obj 2.334851, abs 0.000963, rel 0.000413, norm 0.102216"
## [1] "Iter 166, obj 2.334953, abs 0.000102, rel 0.000044, norm 0.102258"
## [1] "Iter 167, obj 2.335287, abs 0.000334, rel 0.000143, norm 0.102127"
## [1] "Iter 168, obj 2.334294, abs 0.000993, rel 0.000425, norm 0.101915"
## [1] "Iter 169, obj 2.333834, abs 0.000461, rel 0.000197, norm 0.101929"
## [1] "Iter 170, obj 2.334223, abs 0.000389, rel 0.000167, norm 0.101742"
## [1] "Iter 171, obj 2.336152, abs 0.001929, rel 0.000827, norm 0.101874"
## [1] "Iter 172, obj 2.337344, abs 0.001192, rel 0.000510, norm 0.102254"
## [1] "Iter 173, obj 2.335771, abs 0.001573, rel 0.000673, norm 0.102127"
## [1] "Iter 174, obj 2.335528, abs 0.000243, rel 0.000104, norm 0.102306"
## [1] "Iter 175, obj 2.334642, abs 0.000886, rel 0.000379, norm 0.102242"

## [1] "Iter 176, obj 2.334498, abs 0.000145, rel 0.000062, norm 0.101899"
## [1] "Iter 177, obj 2.337666, abs 0.003169, rel 0.001357, norm 0.102171"
## [1] "Iter 178, obj 2.337364, abs 0.000302, rel 0.000129, norm 0.101956"
## [1] "Iter 179, obj 2.335008, abs 0.002356, rel 0.001008, norm 0.101976"
## [1] "Iter 180, obj 2.334904, abs 0.000104, rel 0.000045, norm 0.101869"
## [1] "Iter 181, obj 2.334597, abs 0.000307, rel 0.000132, norm 0.101902"
## [1] "Iter 182, obj 2.336579, abs 0.001983, rel 0.000849, norm 0.101988"
## [1] "Iter 183, obj 2.338124, abs 0.001544, rel 0.000661, norm 0.101999"
## [1] "Iter 184, obj 2.334078, abs 0.004045, rel 0.001730, norm 0.101847"
## [1] "Iter 185, obj 2.334293, abs 0.000215, rel 0.000092, norm 0.102122"
## [1] "Iter 186, obj 2.336062, abs 0.001769, rel 0.000758, norm 0.101967"
## [1] "Iter 187, obj 2.334780, abs 0.001283, rel 0.000549, norm 0.102034"
## [1] "Iter 188, obj 2.335962, abs 0.001182, rel 0.000506, norm 0.101999"
## [1] "Iter 189, obj 2.335531, abs 0.000431, rel 0.000185, norm 0.102042"
## [1] "Iter 190, obj 2.335152, abs 0.000379, rel 0.000162, norm 0.102269"
## [1] "Iter 191, obj 2.333991, abs 0.001161, rel 0.000497, norm 0.102147"
## [1] "Iter 192, obj 2.334976, abs 0.000985, rel 0.000422, norm 0.101743"
## [1] "Iter 193, obj 2.336159, abs 0.001183, rel 0.000507, norm 0.101973"
## [1] "Iter 194, obj 2.336761, abs 0.000601, rel 0.000257, norm 0.101899"
## [1] "Iter 195, obj 2.334699, abs 0.002062, rel 0.000882, norm 0.101813"
## [1] "Iter 196, obj 2.334279, abs 0.000419, rel 0.000180, norm 0.102235"
## [1] "Iter 197, obj 2.334538, abs 0.000259, rel 0.000111, norm 0.101807"
## [1] "Iter 198, obj 2.336642, abs 0.002104, rel 0.000901, norm 0.102009"
## [1] "Iter 199, obj 2.336332, abs 0.000310, rel 0.000133, norm 0.102310"
## [1] "Iter 200, obj 2.335422, abs 0.000910, rel 0.000390, norm 0.101967"

```

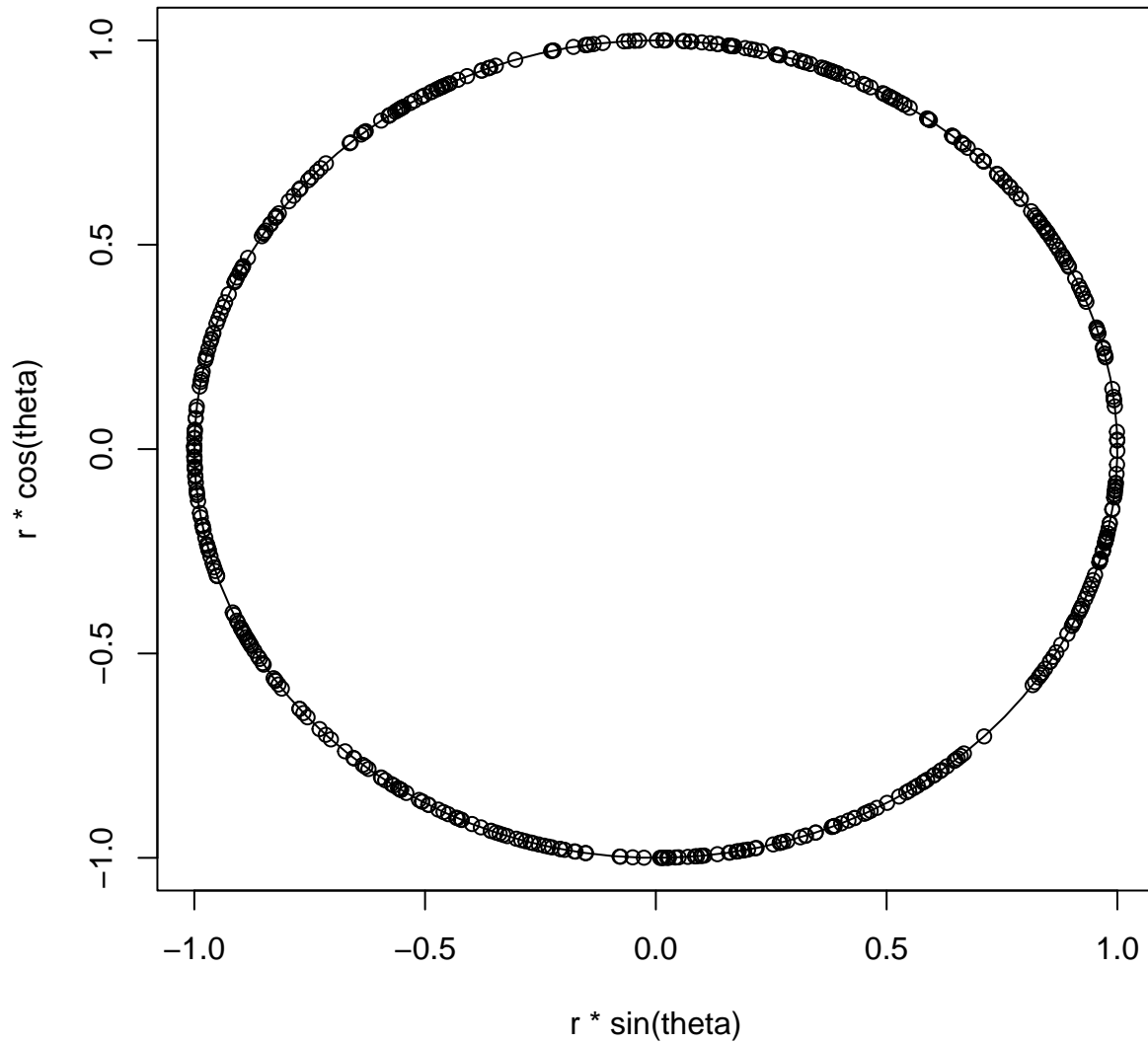


Iteration 200



```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension \mathbb{S}^2 then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_4, d=2, rho_psc_list = rho_four_perp20, num_iteration=200,
             visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 19.650201, abs 0.000000, rel 0.000000, norm 0.195000"
## [1] "Iter 2, obj 17.016095, abs 2.634106, rel 0.134050, norm 0.328324"
## [1] "Iter 3, obj 15.473101, abs 1.542994, rel 0.090678, norm 0.588128"
## [1] "Iter 4, obj 14.839706, abs 0.633395, rel 0.040935, norm 0.775793"
## [1] "Iter 5, obj 14.573023, abs 0.266683, rel 0.017971, norm 0.866767"
## [1] "Iter 6, obj 14.413832, abs 0.159191, rel 0.010924, norm 0.915788"
## [1] "Iter 7, obj 14.271768, abs 0.142064, rel 0.009856, norm 0.945896"
## [1] "Iter 8, obj 14.135612, abs 0.136156, rel 0.009540, norm 0.972447"
## [1] "Iter 9, obj 14.033909, abs 0.101703, rel 0.007195, norm 0.993914"
## [1] "Iter 10, obj 13.927297, abs 0.106613, rel 0.007597, norm 1.012557"
## [1] "Iter 11, obj 13.849873, abs 0.077423, rel 0.005559, norm 1.029098"
## [1] "Iter 12, obj 13.800079, abs 0.049794, rel 0.003595, norm 1.044554"
## [1] "Iter 13, obj 13.769878, abs 0.030201, rel 0.002188, norm 1.058118"
## [1] "Iter 14, obj 13.741709, abs 0.028169, rel 0.002046, norm 1.070779"
## [1] "Iter 15, obj 13.714671, abs 0.027038, rel 0.001968, norm 1.084450"
```

```

## [1] "Iter 16, obj 13.684498, abs 0.030174, rel 0.002200, norm 1.098861"
## [1] "Iter 17, obj 13.675329, abs 0.009169, rel 0.000670, norm 1.115065"
## [1] "Iter 18, obj 13.676035, abs 0.000707, rel 0.000052, norm 1.131044"
## [1] "Iter 19, obj 13.684034, abs 0.007999, rel 0.000585, norm 1.147164"
## [1] "Iter 20, obj 13.693639, abs 0.009604, rel 0.000702, norm 1.163542"
## [1] "Iter 21, obj 13.706715, abs 0.013076, rel 0.000955, norm 1.180201"
## [1] "Iter 22, obj 13.722389, abs 0.015674, rel 0.001144, norm 1.196965"
## [1] "Iter 23, obj 13.741357, abs 0.018967, rel 0.001382, norm 1.213577"
## [1] "Iter 24, obj 13.763982, abs 0.022625, rel 0.001646, norm 1.229585"
## [1] "Iter 25, obj 13.789605, abs 0.025623, rel 0.001862, norm 1.244362"

## [1] "Iter 26, obj 13.817363, abs 0.027759, rel 0.002013, norm 1.257668"
## [1] "Iter 27, obj 13.846471, abs 0.029107, rel 0.002107, norm 1.269463"
## [1] "Iter 28, obj 13.876471, abs 0.030000, rel 0.002167, norm 1.279804"
## [1] "Iter 29, obj 13.906899, abs 0.030428, rel 0.002193, norm 1.288620"
## [1] "Iter 30, obj 13.937842, abs 0.030943, rel 0.002225, norm 1.295951"
## [1] "Iter 31, obj 13.969289, abs 0.031447, rel 0.002256, norm 1.301807"
## [1] "Iter 32, obj 14.001624, abs 0.032335, rel 0.002315, norm 1.306420"
## [1] "Iter 33, obj 14.035034, abs 0.033410, rel 0.002386, norm 1.310163"
## [1] "Iter 34, obj 14.070199, abs 0.035165, rel 0.002505, norm 1.313741"
## [1] "Iter 35, obj 14.107627, abs 0.037428, rel 0.002660, norm 1.318248"
## [1] "Iter 36, obj 14.147875, abs 0.040248, rel 0.002853, norm 1.325274"
## [1] "Iter 37, obj 14.191050, abs 0.043175, rel 0.003052, norm 1.336667"
## [1] "Iter 38, obj 14.237214, abs 0.046164, rel 0.003253, norm 1.353832"
## [1] "Iter 39, obj 14.286319, abs 0.049105, rel 0.003449, norm 1.377108"
## [1] "Iter 40, obj 14.338678, abs 0.052358, rel 0.003665, norm 1.405803"
## [1] "Iter 41, obj 14.394989, abs 0.056311, rel 0.003927, norm 1.438791"
## [1] "Iter 42, obj 14.456172, abs 0.061183, rel 0.004250, norm 1.474834"
## [1] "Iter 43, obj 14.522635, abs 0.066463, rel 0.004598, norm 1.512789"
## [1] "Iter 44, obj 14.594152, abs 0.071517, rel 0.004924, norm 1.551735"
## [1] "Iter 45, obj 14.669994, abs 0.075842, rel 0.005197, norm 1.591042"
## [1] "Iter 46, obj 14.749290, abs 0.079296, rel 0.005405, norm 1.630206"
## [1] "Iter 47, obj 14.831069, abs 0.081779, rel 0.005545, norm 1.668731"
## [1] "Iter 48, obj 14.914357, abs 0.083287, rel 0.005616, norm 1.706089"
## [1] "Iter 49, obj 14.998062, abs 0.083706, rel 0.005612, norm 1.741822"
## [1] "Iter 50, obj 15.081088, abs 0.083026, rel 0.005536, norm 1.775572"

## [1] "Iter 51, obj 15.162391, abs 0.081303, rel 0.005391, norm 1.807099"
## [1] "Iter 52, obj 15.241199, abs 0.078808, rel 0.005198, norm 1.836303"
## [1] "Iter 53, obj 15.316985, abs 0.075785, rel 0.004972, norm 1.863206"
## [1] "Iter 54, obj 15.389439, abs 0.072454, rel 0.004730, norm 1.887887"
## [1] "Iter 55, obj 15.458356, abs 0.068917, rel 0.004478, norm 1.910449"
## [1] "Iter 56, obj 15.523618, abs 0.065262, rel 0.004222, norm 1.931002"
## [1] "Iter 57, obj 15.585145, abs 0.061527, rel 0.003963, norm 1.949661"
## [1] "Iter 58, obj 15.642909, abs 0.057764, rel 0.003706, norm 1.966545"
## [1] "Iter 59, obj 15.696915, abs 0.054005, rel 0.003452, norm 1.981779"
## [1] "Iter 60, obj 15.747207, abs 0.050292, rel 0.003204, norm 1.995489"
## [1] "Iter 61, obj 15.793860, abs 0.046653, rel 0.002963, norm 2.007799"
## [1] "Iter 62, obj 15.836984, abs 0.043124, rel 0.002730, norm 2.018831"
## [1] "Iter 63, obj 15.876710, abs 0.039726, rel 0.002508, norm 2.028703"
## [1] "Iter 64, obj 15.913193, abs 0.036483, rel 0.002298, norm 2.037526"
## [1] "Iter 65, obj 15.946600, abs 0.033408, rel 0.002099, norm 2.045404"
## [1] "Iter 66, obj 15.977112, abs 0.030512, rel 0.001913, norm 2.052435"
## [1] "Iter 67, obj 16.004913, abs 0.027800, rel 0.001740, norm 2.058707"
## [1] "Iter 68, obj 16.030187, abs 0.025275, rel 0.001579, norm 2.064300"

```

```

## [1] "Iter 69, obj 16.053120, abs 0.022932, rel 0.001431, norm 2.069288"
## [1] "Iter 70, obj 16.073889, abs 0.020770, rel 0.001294, norm 2.073736"
## [1] "Iter 71, obj 16.092668, abs 0.018779, rel 0.001168, norm 2.077702"
## [1] "Iter 72, obj 16.109621, abs 0.016953, rel 0.001053, norm 2.081241"
## [1] "Iter 73, obj 16.124903, abs 0.015282, rel 0.000949, norm 2.084397"
## [1] "Iter 74, obj 16.138661, abs 0.013757, rel 0.000853, norm 2.087215"
## [1] "Iter 75, obj 16.151030, abs 0.012369, rel 0.000766, norm 2.089729"

## [1] "Iter 76, obj 16.162136, abs 0.011107, rel 0.000688, norm 2.091974"
## [1] "Iter 77, obj 16.172097, abs 0.009961, rel 0.000616, norm 2.093978"
## [1] "Iter 78, obj 16.181021, abs 0.008923, rel 0.000552, norm 2.095769"
## [1] "Iter 79, obj 16.189005, abs 0.007984, rel 0.000493, norm 2.097368"
## [1] "Iter 80, obj 16.196141, abs 0.007136, rel 0.000441, norm 2.098797"
## [1] "Iter 81, obj 16.202511, abs 0.006370, rel 0.000393, norm 2.100073"
## [1] "Iter 82, obj 16.208191, abs 0.005680, rel 0.000351, norm 2.101214"
## [1] "Iter 83, obj 16.213250, abs 0.005058, rel 0.000312, norm 2.102233"
## [1] "Iter 84, obj 16.217749, abs 0.004499, rel 0.000277, norm 2.103143"
## [1] "Iter 85, obj 16.221745, abs 0.003996, rel 0.000246, norm 2.103957"
## [1] "Iter 86, obj 16.225290, abs 0.003545, rel 0.000219, norm 2.104684"
## [1] "Iter 87, obj 16.228430, abs 0.003140, rel 0.000194, norm 2.105333"
## [1] "Iter 88, obj 16.231208, abs 0.002778, rel 0.000171, norm 2.105913"
## [1] "Iter 89, obj 16.233661, abs 0.002453, rel 0.000151, norm 2.106430"
## [1] "Iter 90, obj 16.235823, abs 0.002162, rel 0.000133, norm 2.106892"
## [1] "Iter 91, obj 16.237726, abs 0.001903, rel 0.000117, norm 2.107305"
## [1] "Iter 92, obj 16.239397, abs 0.001671, rel 0.000103, norm 2.107673"
## [1] "Iter 93, obj 16.240862, abs 0.001465, rel 0.000090, norm 2.108001"
## [1] "Iter 94, obj 16.242143, abs 0.001281, rel 0.000079, norm 2.108293"
## [1] "Iter 95, obj 16.243260, abs 0.001117, rel 0.000069, norm 2.108554"
## [1] "Iter 96, obj 16.244232, abs 0.000972, rel 0.000060, norm 2.108786"
## [1] "Iter 97, obj 16.245076, abs 0.000843, rel 0.000052, norm 2.108993"
## [1] "Iter 98, obj 16.245805, abs 0.000729, rel 0.000045, norm 2.109177"
## [1] "Iter 99, obj 16.246434, abs 0.000629, rel 0.000039, norm 2.109341"
## [1] "Iter 100, obj 16.246973, abs 0.000540, rel 0.000033, norm 2.109487"

## [1] "Iter 101, obj 2.450621, abs 13.796352, rel 0.849164, norm 0.172704"
## [1] "Iter 102, obj 2.122606, abs 0.328015, rel 0.133850, norm 0.083758"
## [1] "Iter 103, obj 2.132541, abs 0.009934, rel 0.004680, norm 0.102006"
## [1] "Iter 104, obj 2.141349, abs 0.008808, rel 0.004130, norm 0.110057"
## [1] "Iter 105, obj 2.146608, abs 0.005259, rel 0.002456, norm 0.130382"
## [1] "Iter 106, obj 2.177216, abs 0.030608, rel 0.014259, norm 0.135796"
## [1] "Iter 107, obj 2.319942, abs 0.142726, rel 0.065554, norm 0.144041"
## [1] "Iter 108, obj 2.408593, abs 0.088652, rel 0.038213, norm 0.146524"
## [1] "Iter 109, obj 2.549304, abs 0.140711, rel 0.058420, norm 0.155394"
## [1] "Iter 110, obj 2.711410, abs 0.162106, rel 0.063588, norm 0.173490"
## [1] "Iter 111, obj 2.837623, abs 0.126212, rel 0.046549, norm 0.204542"
## [1] "Iter 112, obj 2.968229, abs 0.130606, rel 0.046027, norm 0.252781"
## [1] "Iter 113, obj 3.037687, abs 0.069458, rel 0.023401, norm 0.286069"
## [1] "Iter 114, obj 3.067415, abs 0.029728, rel 0.009786, norm 0.336029"
## [1] "Iter 115, obj 3.093384, abs 0.025969, rel 0.008466, norm 0.365589"
## [1] "Iter 116, obj 3.068787, abs 0.024597, rel 0.007951, norm 0.391926"
## [1] "Iter 117, obj 3.022701, abs 0.046087, rel 0.015018, norm 0.408805"
## [1] "Iter 118, obj 3.028353, abs 0.005652, rel 0.001870, norm 0.423925"
## [1] "Iter 119, obj 3.001185, abs 0.027168, rel 0.008971, norm 0.429844"
## [1] "Iter 120, obj 2.965708, abs 0.035476, rel 0.011821, norm 0.440621"
## [1] "Iter 121, obj 2.951538, abs 0.014171, rel 0.004778, norm 0.449172"

```

```

## [1] "Iter 122, obj 2.935592, abs 0.015946, rel 0.005403, norm 0.456325"
## [1] "Iter 123, obj 2.938793, abs 0.003202, rel 0.001091, norm 0.467224"
## [1] "Iter 124, obj 2.885769, abs 0.053025, rel 0.018043, norm 0.467946"
## [1] "Iter 125, obj 2.872745, abs 0.013024, rel 0.004513, norm 0.473313"

## [1] "Iter 126, obj 2.866529, abs 0.006216, rel 0.002164, norm 0.474692"
## [1] "Iter 127, obj 2.859953, abs 0.006576, rel 0.002294, norm 0.471162"
## [1] "Iter 128, obj 2.837988, abs 0.021965, rel 0.007680, norm 0.471814"
## [1] "Iter 129, obj 2.833123, abs 0.004866, rel 0.001714, norm 0.471739"
## [1] "Iter 130, obj 2.786954, abs 0.046169, rel 0.016296, norm 0.478522"
## [1] "Iter 131, obj 2.763069, abs 0.023885, rel 0.008570, norm 0.473304"
## [1] "Iter 132, obj 2.746530, abs 0.016539, rel 0.005986, norm 0.469127"
## [1] "Iter 133, obj 2.678347, abs 0.068182, rel 0.024825, norm 0.462370"
## [1] "Iter 134, obj 2.644443, abs 0.033904, rel 0.012659, norm 0.465197"
## [1] "Iter 135, obj 2.675692, abs 0.031249, rel 0.011817, norm 0.453245"
## [1] "Iter 136, obj 2.672177, abs 0.003515, rel 0.001314, norm 0.461644"
## [1] "Iter 137, obj 2.672277, abs 0.000100, rel 0.000037, norm 0.465751"
## [1] "Iter 138, obj 2.654232, abs 0.018046, rel 0.006753, norm 0.476616"
## [1] "Iter 139, obj 2.590513, abs 0.063718, rel 0.024006, norm 0.477955"
## [1] "Iter 140, obj 2.619336, abs 0.028823, rel 0.011126, norm 0.474930"
## [1] "Iter 141, obj 2.609442, abs 0.009895, rel 0.003778, norm 0.471199"
## [1] "Iter 142, obj 2.616011, abs 0.006569, rel 0.002518, norm 0.478788"
## [1] "Iter 143, obj 2.601779, abs 0.014232, rel 0.005440, norm 0.482000"
## [1] "Iter 144, obj 2.615670, abs 0.013890, rel 0.005339, norm 0.483247"
## [1] "Iter 145, obj 2.600779, abs 0.014891, rel 0.005693, norm 0.483508"
## [1] "Iter 146, obj 2.588863, abs 0.011916, rel 0.004582, norm 0.487568"
## [1] "Iter 147, obj 2.587837, abs 0.001026, rel 0.000396, norm 0.482116"
## [1] "Iter 148, obj 2.599057, abs 0.011220, rel 0.004335, norm 0.485671"
## [1] "Iter 149, obj 2.568953, abs 0.030104, rel 0.011583, norm 0.487821"
## [1] "Iter 150, obj 2.596964, abs 0.028012, rel 0.010904, norm 0.489947"

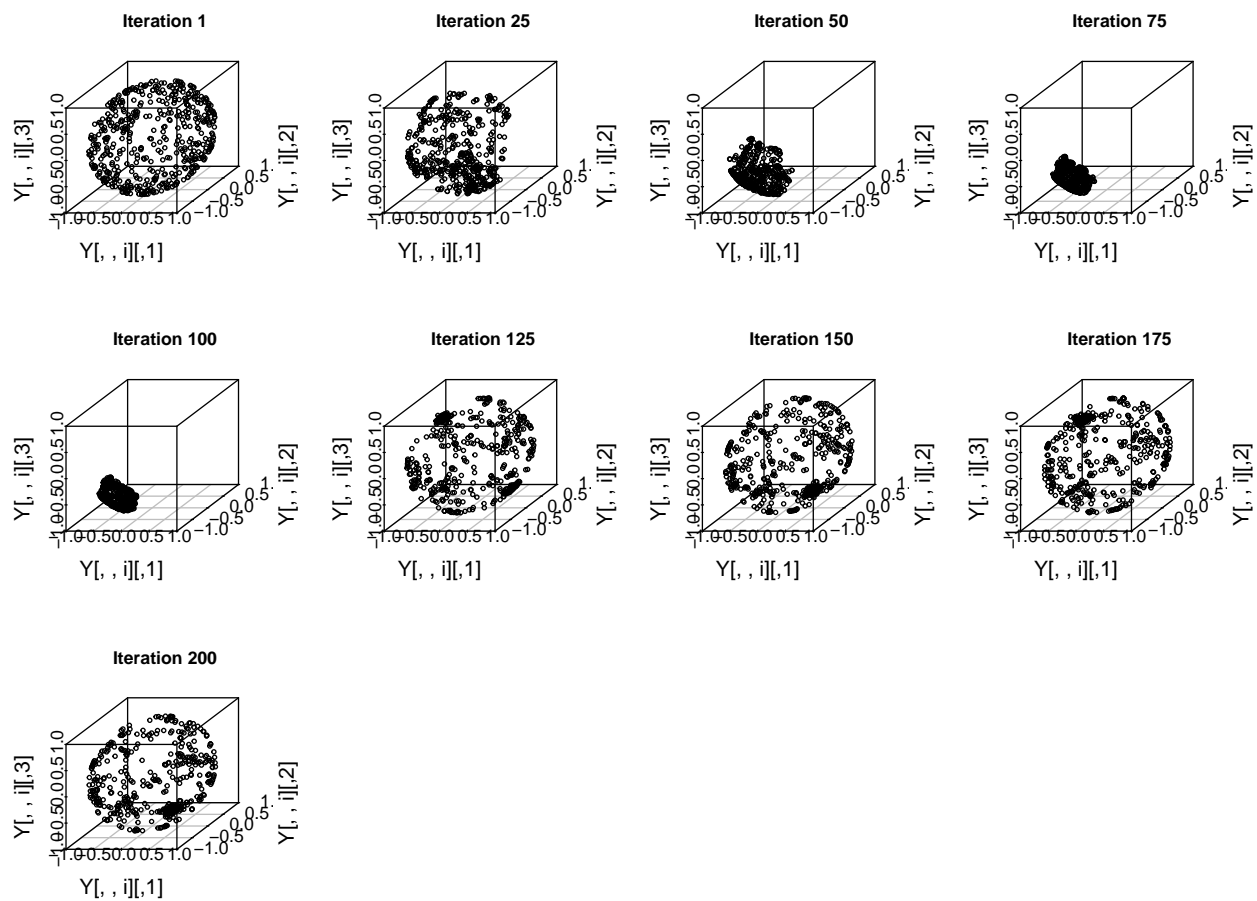
## [1] "Iter 151, obj 2.557077, abs 0.039887, rel 0.015359, norm 0.492496"
## [1] "Iter 152, obj 2.576018, abs 0.018941, rel 0.007407, norm 0.491482"
## [1] "Iter 153, obj 2.555025, abs 0.020993, rel 0.008149, norm 0.484830"
## [1] "Iter 154, obj 2.562273, abs 0.007249, rel 0.002837, norm 0.491733"
## [1] "Iter 155, obj 2.546813, abs 0.015461, rel 0.006034, norm 0.487141"
## [1] "Iter 156, obj 2.557880, abs 0.011067, rel 0.004346, norm 0.487085"
## [1] "Iter 157, obj 2.536774, abs 0.021106, rel 0.008252, norm 0.485349"
## [1] "Iter 158, obj 2.537298, abs 0.000524, rel 0.000206, norm 0.488933"
## [1] "Iter 159, obj 2.545881, abs 0.008583, rel 0.003383, norm 0.486263"
## [1] "Iter 160, obj 2.549314, abs 0.003433, rel 0.001349, norm 0.486914"
## [1] "Iter 161, obj 2.545114, abs 0.004200, rel 0.001648, norm 0.484603"
## [1] "Iter 162, obj 2.564578, abs 0.019464, rel 0.007648, norm 0.488283"
## [1] "Iter 163, obj 2.549175, abs 0.015404, rel 0.006006, norm 0.490575"
## [1] "Iter 164, obj 2.578021, abs 0.028847, rel 0.011316, norm 0.491588"
## [1] "Iter 165, obj 2.557545, abs 0.020476, rel 0.007943, norm 0.491865"
## [1] "Iter 166, obj 2.558199, abs 0.000654, rel 0.000256, norm 0.491857"
## [1] "Iter 167, obj 2.561788, abs 0.003589, rel 0.001403, norm 0.486956"
## [1] "Iter 168, obj 2.555404, abs 0.006383, rel 0.002492, norm 0.493590"
## [1] "Iter 169, obj 2.543371, abs 0.012033, rel 0.004709, norm 0.487990"
## [1] "Iter 170, obj 2.532662, abs 0.010709, rel 0.004211, norm 0.490085"
## [1] "Iter 171, obj 2.536138, abs 0.003476, rel 0.001373, norm 0.484635"
## [1] "Iter 172, obj 2.523181, abs 0.012957, rel 0.005109, norm 0.484644"
## [1] "Iter 173, obj 2.527156, abs 0.003975, rel 0.001575, norm 0.474405"
## [1] "Iter 174, obj 2.536050, abs 0.008893, rel 0.003519, norm 0.475410"

```

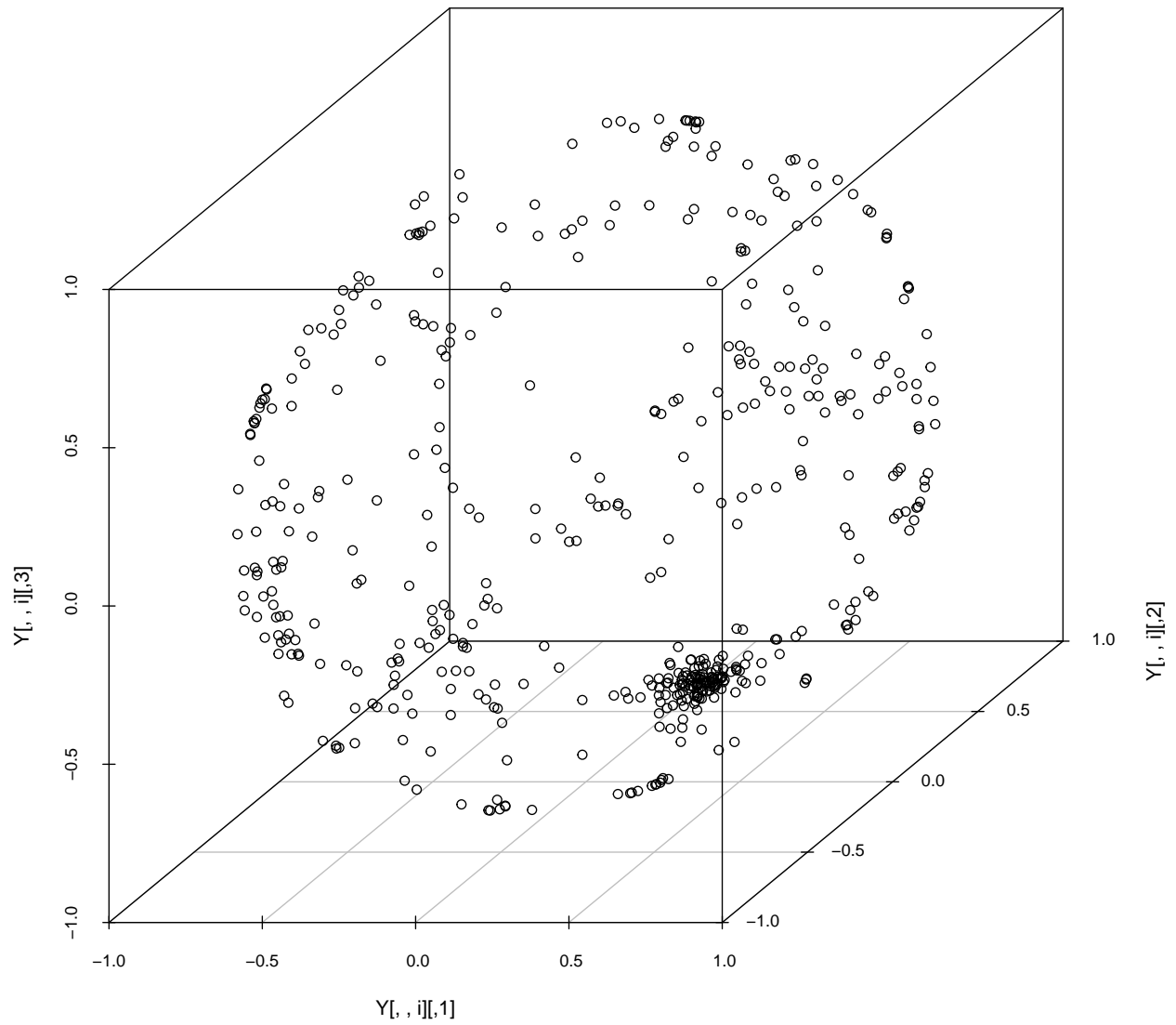
```

## [1] "Iter 175, obj 2.525064, abs 0.010986, rel 0.004332, norm 0.475316"
## [1] "Iter 176, obj 2.537034, abs 0.011969, rel 0.004740, norm 0.483834"
## [1] "Iter 177, obj 2.522772, abs 0.014262, rel 0.005622, norm 0.479986"
## [1] "Iter 178, obj 2.530679, abs 0.007908, rel 0.003135, norm 0.485448"
## [1] "Iter 179, obj 2.522367, abs 0.008312, rel 0.003284, norm 0.484598"
## [1] "Iter 180, obj 2.512582, abs 0.009785, rel 0.003879, norm 0.488443"
## [1] "Iter 181, obj 2.505014, abs 0.007569, rel 0.003012, norm 0.481963"
## [1] "Iter 182, obj 2.514887, abs 0.009873, rel 0.003941, norm 0.479730"
## [1] "Iter 183, obj 2.500949, abs 0.013938, rel 0.005542, norm 0.476326"
## [1] "Iter 184, obj 2.514660, abs 0.013711, rel 0.005482, norm 0.477355"
## [1] "Iter 185, obj 2.512047, abs 0.002613, rel 0.001039, norm 0.475268"
## [1] "Iter 186, obj 2.526436, abs 0.014390, rel 0.005728, norm 0.476642"
## [1] "Iter 187, obj 2.512876, abs 0.013561, rel 0.005368, norm 0.476810"
## [1] "Iter 188, obj 2.524128, abs 0.011253, rel 0.004478, norm 0.484523"
## [1] "Iter 189, obj 2.513408, abs 0.010721, rel 0.004247, norm 0.482606"
## [1] "Iter 190, obj 2.511593, abs 0.001814, rel 0.000722, norm 0.482154"
## [1] "Iter 191, obj 2.521811, abs 0.010218, rel 0.004068, norm 0.476297"
## [1] "Iter 192, obj 2.505440, abs 0.016371, rel 0.006492, norm 0.480392"
## [1] "Iter 193, obj 2.516602, abs 0.011163, rel 0.004455, norm 0.476443"
## [1] "Iter 194, obj 2.516051, abs 0.000552, rel 0.000219, norm 0.481274"
## [1] "Iter 195, obj 2.507244, abs 0.008807, rel 0.003500, norm 0.478324"
## [1] "Iter 196, obj 2.525531, abs 0.018287, rel 0.007294, norm 0.479873"
## [1] "Iter 197, obj 2.508373, abs 0.017158, rel 0.006794, norm 0.479782"
## [1] "Iter 198, obj 2.512258, abs 0.003884, rel 0.001549, norm 0.479667"
## [1] "Iter 199, obj 2.532932, abs 0.020675, rel 0.008229, norm 0.478900"
## [1] "Iter 200, obj 2.531431, abs 0.001501, rel 0.000593, norm 0.485108"

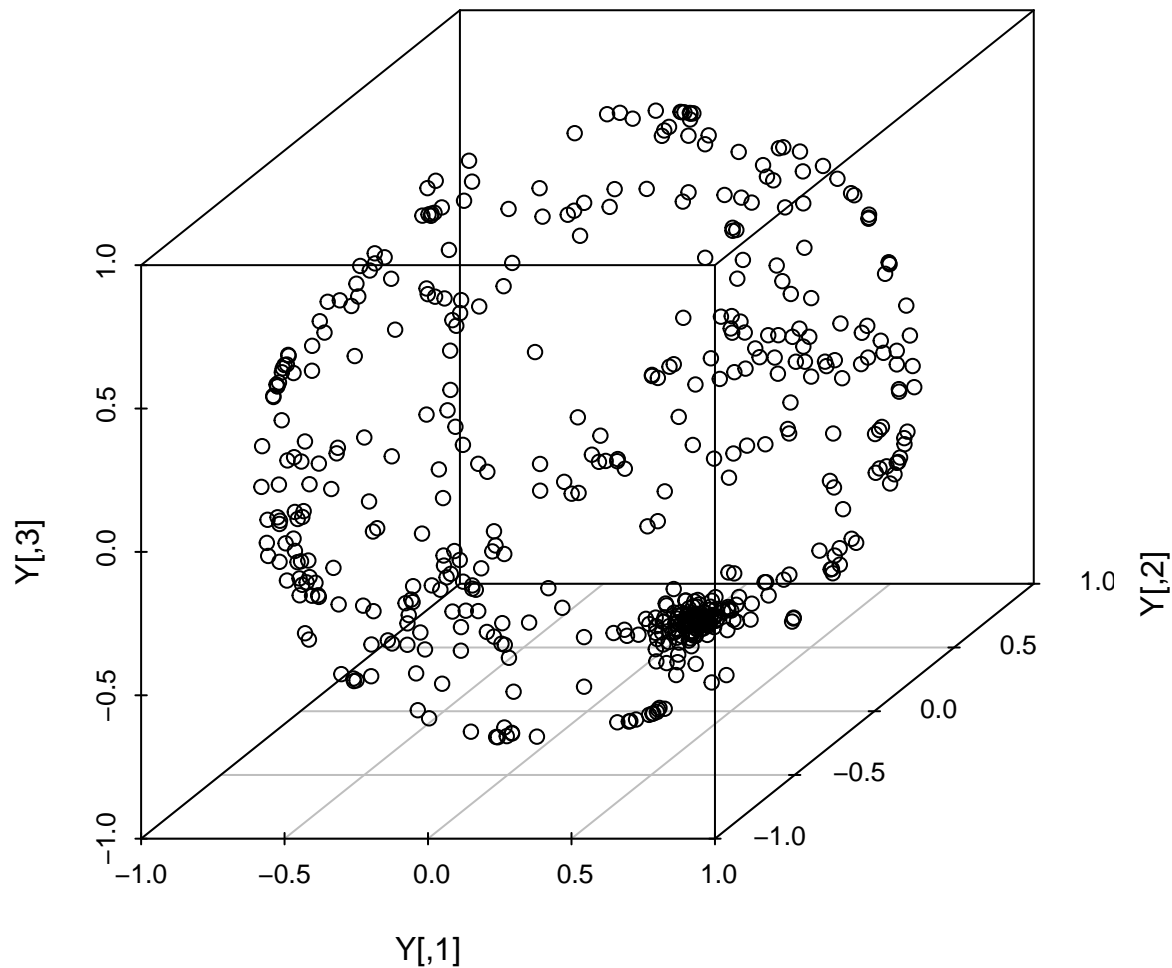
```



Iteration 200



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1))
```

Case 4

Playing with belts of points onto the sphere. First, let's generate the data

```
rotate_matrix_z_axis <- function(alpha) {
  deg2rad <- function(deg) {(deg * pi) / (180)}
  rads <- deg2rad(alpha)
  matrix(c(cos(rads), -sin(rads), 0,
           sin(rads),  cos(rads), 0,
           0,          0, 1),
         byrow = T, nrow = 3)
}

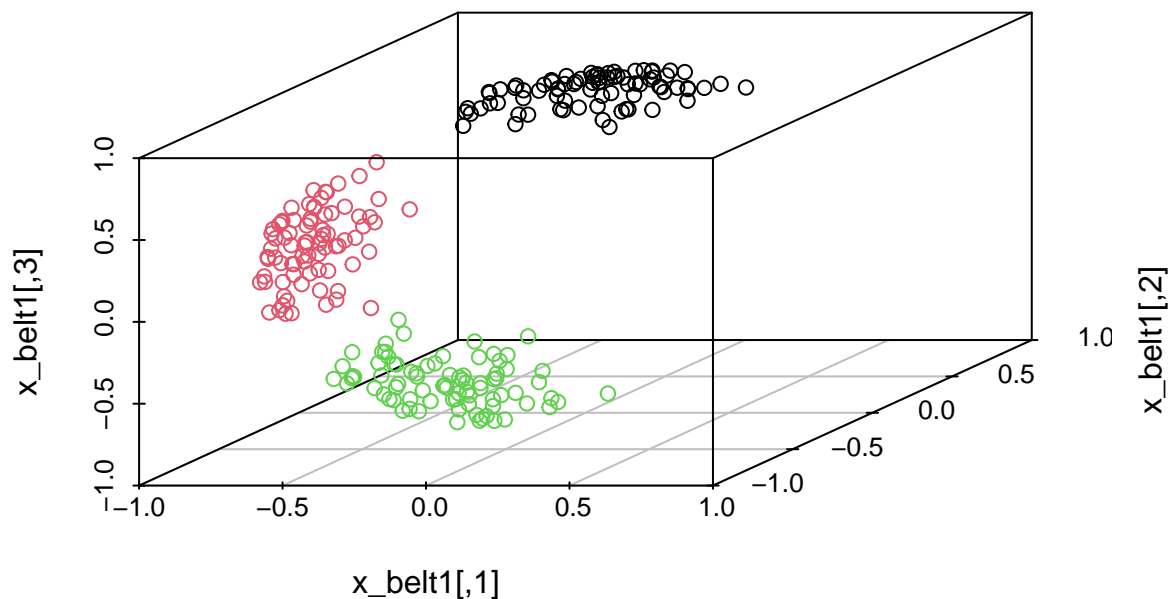
rotate_matrix_x_axis <- function(alpha) {
  deg2rad <- function(deg) {(deg * pi) / (180)}
  rads <- deg2rad(alpha)
  matrix(c(1, 0, 0,
           0, cos(rads), -sin(rads),
           0, sin(rads),  cos(rads)),
         byrow = T, nrow = 3)
}
```

```

rotate_matrix_y_axis <- function(alpha) {
  deg2rad <- function(deg) {(deg * pi) / (180)}
  rads <- deg2rad(alpha)
  matrix(c(cos(rads), 0, sin(rads),
           0, 1, 0,
           -sin(rads), 0, cos(rads)),
         byrow = T, nrow = 3)
}

p = 2
set.seed(2)
# A <- matrix(runif((p + 1)^2)*2-0.5, ncol = p + 1)
# sigma <- t(A) %*% A
sigma <- diag(c(10, 2, .5))
n <- 80
# x_1_belt1 <- rotasym::r_ACG(n, sigma)
x_1_belt1 <- sphunif::r_alt(n = n, p = 3, alt = "SC", kappa = 200, nu = 0.9999)[,1]
rotate_mat_90 <- rotate_matrix_y_axis(90)
x_2_belt1 <- x_1_belt1 %*% rotate_mat_90
rotate_mat_45 <- rotate_matrix_y_axis(60)
x_3_belt1 <- x_2_belt1 %*% rotate_mat_45
x_belt1 <- rbind(x_1_belt1, x_2_belt1, x_3_belt1)
scatterplot3d::scatterplot3d(x_belt1, xlim = c(-1, 1),
                             ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(c(1,2,3), each=n))

```



```

rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue",
            lit = FALSE)
rgl::points3d(x_belt1, col = rep(c(1,2,3), each=n))

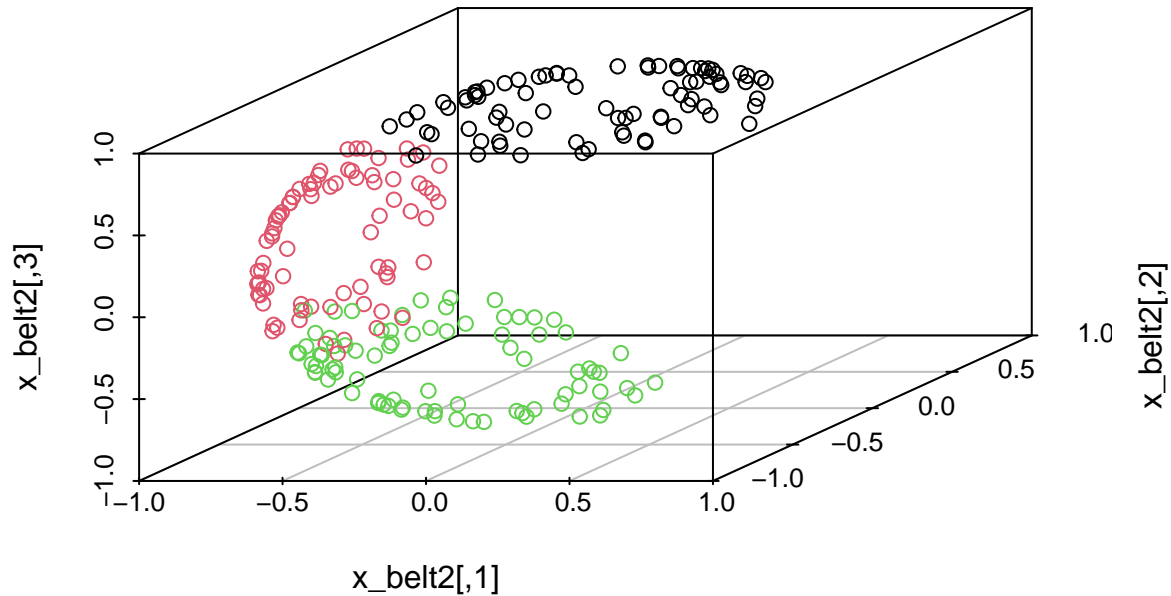
# x_1_belt2 <- rotasym::r_ACG(n, sigma)
x_1_belt2 <- sphunif::r_alt(n = n, p = 3, alt = "SC", kappa = 200, nu = 0.89)[,1]
x_2_belt2 <- x_1_belt2 %*% rotate_mat_90

```

```

x_3_belt2 <- x_2_belt2 %*% rotate_mat_45
x_belt2 <- rbind(x_1_belt2, x_2_belt2, x_3_belt2)
scatterplot3d::scatterplot3d(x_belt2, xlim = c(-1, 1),
                             ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(c(1,2,3), each=n))

```

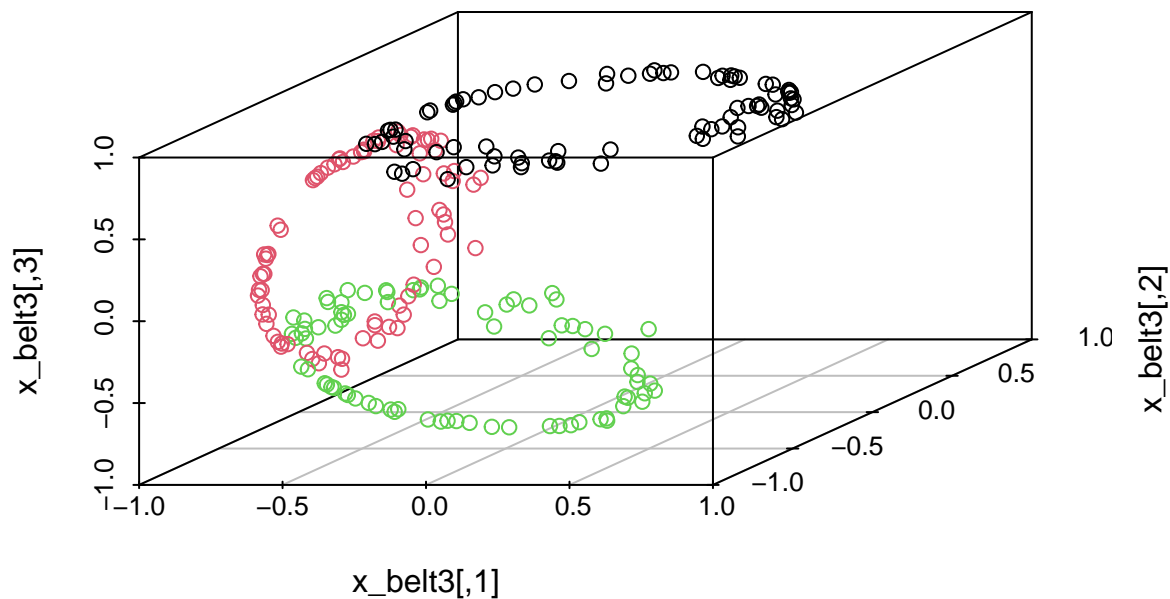


```

rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue",
            lit = FALSE)
rgl::points3d(x_belt2, col = rep(c(1,2,3), each=n))

# x_1_belt3 <- rotasym::r_ACG(n, sigma)
x_1_belt3 <- sphunif::r_alt(n = n, p = 3, alt = "SC", kappa = 400, nu = 0.80)[,1]
x_2_belt3 <- x_1_belt3 %*% rotate_mat_90
x_3_belt3 <- x_2_belt3 %*% rotate_mat_45
x_belt3 <- rbind(x_1_belt3, x_2_belt3, x_3_belt3)
scatterplot3d::scatterplot3d(x_belt3, xlim = c(-1, 1),
                             ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(c(1,2,3), each=n))

```



```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue",
            lit = FALSE)
rgl::points3d(x_belt3, col = rep(c(1,2,3), each=n))

n <- n * 3
x_5 <- array(dim = c(n, 3, 3))
x_5[, , 1] <- x_belt1
x_5[, , 2] <- x_belt2
x_5[, , 3] <- x_belt3

indexes <- sample(1:n)
x_5 <- x_5[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]
```

Let's calculate the rho parameters based on a perplexity of 25:

```
rho_five_perp20 <- rho_optim_bst(x_5, 50)
```

Time difference of 11.70569 secs

First, let's reduce to dimension \mathbb{S}^1 then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_5, d=1, rho_psc_list = rho_five_perp20, num_iteration=200, eta = 20,
            visualize_prog = TRUE, colors = colors)
```

```
## [1] "Iter 1, obj 12.839227, abs 0.000000, rel 0.000000, norm 0.125004"
## [1] "Iter 2, obj 12.698820, abs 0.140407, rel 0.010936, norm 0.150446"
## [1] "Iter 3, obj 12.502384, abs 0.196436, rel 0.015469, norm 0.191926"
## [1] "Iter 4, obj 12.258573, abs 0.243811, rel 0.019501, norm 0.246560"
## [1] "Iter 5, obj 11.990660, abs 0.267914, rel 0.021855, norm 0.320516"
## [1] "Iter 6, obj 11.719116, abs 0.271544, rel 0.022646, norm 0.401822"
## [1] "Iter 7, obj 11.453628, abs 0.265488, rel 0.022654, norm 0.475343"
## [1] "Iter 8, obj 11.138541, abs 0.315087, rel 0.027510, norm 0.550137"
## [1] "Iter 9, obj 10.724339, abs 0.414202, rel 0.037186, norm 0.634870"
## [1] "Iter 10, obj 10.265487, abs 0.458852, rel 0.042786, norm 0.740468"
```

```

## [1] "Iter 11, obj 9.885677, abs 0.379810, rel 0.036999, norm 0.875408"
## [1] "Iter 12, obj 9.593375, abs 0.292302, rel 0.029568, norm 1.005723"
## [1] "Iter 13, obj 9.318708, abs 0.274667, rel 0.028631, norm 1.086434"
## [1] "Iter 14, obj 9.120116, abs 0.198592, rel 0.021311, norm 1.141824"
## [1] "Iter 15, obj 9.035217, abs 0.084899, rel 0.009309, norm 1.195350"
## [1] "Iter 16, obj 8.995928, abs 0.039289, rel 0.004348, norm 1.237954"
## [1] "Iter 17, obj 8.989019, abs 0.006909, rel 0.000768, norm 1.263414"
## [1] "Iter 18, obj 8.994545, abs 0.005526, rel 0.000615, norm 1.277708"
## [1] "Iter 19, obj 8.979940, abs 0.014605, rel 0.001624, norm 1.285217"
## [1] "Iter 20, obj 8.971521, abs 0.008419, rel 0.000938, norm 1.290101"
## [1] "Iter 21, obj 8.971404, abs 0.000117, rel 0.000013, norm 1.293028"
## [1] "Iter 22, obj 8.970657, abs 0.000746, rel 0.000083, norm 1.294662"
## [1] "Iter 23, obj 8.969070, abs 0.001588, rel 0.000177, norm 1.295835"
## [1] "Iter 24, obj 8.966996, abs 0.002073, rel 0.000231, norm 1.296805"
## [1] "Iter 25, obj 8.964567, abs 0.002429, rel 0.000271, norm 1.297682"

## [1] "Iter 26, obj 8.961891, abs 0.002676, rel 0.000299, norm 1.298542"
## [1] "Iter 27, obj 8.959062, abs 0.002829, rel 0.000316, norm 1.299431"
## [1] "Iter 28, obj 8.956175, abs 0.002887, rel 0.000322, norm 1.300383"
## [1] "Iter 29, obj 8.953332, abs 0.002843, rel 0.000317, norm 1.301428"
## [1] "Iter 30, obj 8.950635, abs 0.002697, rel 0.000301, norm 1.302591"
## [1] "Iter 31, obj 8.948176, abs 0.002459, rel 0.000275, norm 1.303898"
## [1] "Iter 32, obj 8.946032, abs 0.002144, rel 0.000240, norm 1.305373"
## [1] "Iter 33, obj 8.944269, abs 0.001763, rel 0.000197, norm 1.307036"
## [1] "Iter 34, obj 8.942950, abs 0.001319, rel 0.000148, norm 1.308908"
## [1] "Iter 35, obj 8.942141, abs 0.000808, rel 0.000090, norm 1.311008"
## [1] "Iter 36, obj 8.941921, abs 0.000220, rel 0.000025, norm 1.313353"
## [1] "Iter 37, obj 8.942379, abs 0.000459, rel 0.000051, norm 1.315961"
## [1] "Iter 38, obj 8.943621, abs 0.001242, rel 0.000139, norm 1.318847"
## [1] "Iter 39, obj 8.945764, abs 0.002143, rel 0.000240, norm 1.322026"
## [1] "Iter 40, obj 8.948929, abs 0.003165, rel 0.000354, norm 1.325508"
## [1] "Iter 41, obj 8.953207, abs 0.004278, rel 0.000478, norm 1.329299"
## [1] "Iter 42, obj 8.958392, abs 0.005185, rel 0.000579, norm 1.333400"
## [1] "Iter 43, obj 8.962057, abs 0.003666, rel 0.000409, norm 1.337821"
## [1] "Iter 44, obj 8.951299, abs 0.010759, rel 0.001200, norm 1.342724"
## [1] "Iter 45, obj 8.930574, abs 0.020724, rel 0.002315, norm 1.349057"
## [1] "Iter 46, obj 8.929213, abs 0.001362, rel 0.000152, norm 1.356444"
## [1] "Iter 47, obj 8.935827, abs 0.006615, rel 0.000741, norm 1.363004"
## [1] "Iter 48, obj 8.944252, abs 0.008425, rel 0.000943, norm 1.369119"
## [1] "Iter 49, obj 8.954068, abs 0.009816, rel 0.001097, norm 1.375490"
## [1] "Iter 50, obj 8.965354, abs 0.011286, rel 0.001260, norm 1.382193"

## [1] "Iter 51, obj 8.978195, abs 0.012842, rel 0.001432, norm 1.389210"
## [1] "Iter 52, obj 8.992712, abs 0.014516, rel 0.001617, norm 1.396531"
## [1] "Iter 53, obj 9.009028, abs 0.016316, rel 0.001814, norm 1.404139"
## [1] "Iter 54, obj 9.027265, abs 0.018237, rel 0.002024, norm 1.412018"
## [1] "Iter 55, obj 9.047535, abs 0.020269, rel 0.002245, norm 1.420152"
## [1] "Iter 56, obj 9.069927, abs 0.022393, rel 0.002475, norm 1.428522"
## [1] "Iter 57, obj 9.094492, abs 0.024565, rel 0.002708, norm 1.437114"
## [1] "Iter 58, obj 9.121210, abs 0.026718, rel 0.002938, norm 1.445921"
## [1] "Iter 59, obj 9.149997, abs 0.028787, rel 0.003156, norm 1.454951"
## [1] "Iter 60, obj 9.180800, abs 0.030802, rel 0.003366, norm 1.464215"
## [1] "Iter 61, obj 9.213681, abs 0.032881, rel 0.003581, norm 1.473684"
## [1] "Iter 62, obj 9.248694, abs 0.035013, rel 0.003800, norm 1.483255"
## [1] "Iter 63, obj 9.285699, abs 0.037005, rel 0.004001, norm 1.492824"

```

```

## [1] "Iter 64, obj 9.324318, abs 0.038618, rel 0.004159, norm 1.502363"
## [1] "Iter 65, obj 9.363872, abs 0.039554, rel 0.004242, norm 1.511903"
## [1] "Iter 66, obj 9.403212, abs 0.039340, rel 0.004201, norm 1.521504"
## [1] "Iter 67, obj 9.440460, abs 0.037248, rel 0.003961, norm 1.531235"
## [1] "Iter 68, obj 9.472780, abs 0.032319, rel 0.003423, norm 1.541180"
## [1] "Iter 69, obj 9.496314, abs 0.023534, rel 0.002484, norm 1.551402"
## [1] "Iter 70, obj 9.506511, abs 0.010197, rel 0.001074, norm 1.561865"
## [1] "Iter 71, obj 9.499194, abs 0.007316, rel 0.000770, norm 1.572318"
## [1] "Iter 72, obj 9.472605, abs 0.026589, rel 0.002799, norm 1.582214"
## [1] "Iter 73, obj 9.429204, abs 0.043402, rel 0.004582, norm 1.590668"
## [1] "Iter 74, obj 9.375027, abs 0.054176, rel 0.005746, norm 1.596674"
## [1] "Iter 75, obj 9.316771, abs 0.058257, rel 0.006214, norm 1.599648"

## [1] "Iter 76, obj 9.259410, abs 0.057361, rel 0.006157, norm 1.599746"
## [1] "Iter 77, obj 9.205731, abs 0.053679, rel 0.005797, norm 1.597611"
## [1] "Iter 78, obj 9.156911, abs 0.048820, rel 0.005303, norm 1.593950"
## [1] "Iter 79, obj 9.113189, abs 0.043722, rel 0.004775, norm 1.589335"
## [1] "Iter 80, obj 9.074332, abs 0.038857, rel 0.004264, norm 1.584171"
## [1] "Iter 81, obj 9.039902, abs 0.034429, rel 0.003794, norm 1.578728"
## [1] "Iter 82, obj 9.009403, abs 0.030500, rel 0.003374, norm 1.573188"
## [1] "Iter 83, obj 8.982342, abs 0.027061, rel 0.003004, norm 1.567669"
## [1] "Iter 84, obj 8.958271, abs 0.024071, rel 0.002680, norm 1.562248"
## [1] "Iter 85, obj 8.936792, abs 0.021480, rel 0.002398, norm 1.556972"
## [1] "Iter 86, obj 8.917558, abs 0.019234, rel 0.002152, norm 1.551872"
## [1] "Iter 87, obj 8.900274, abs 0.017284, rel 0.001938, norm 1.546962"
## [1] "Iter 88, obj 8.884686, abs 0.015588, rel 0.001751, norm 1.542250"
## [1] "Iter 89, obj 8.870578, abs 0.014108, rel 0.001588, norm 1.537738"
## [1] "Iter 90, obj 8.857766, abs 0.012812, rel 0.001444, norm 1.533421"
## [1] "Iter 91, obj 8.846093, abs 0.011672, rel 0.001318, norm 1.529296"
## [1] "Iter 92, obj 8.835426, abs 0.010667, rel 0.001206, norm 1.525356"
## [1] "Iter 93, obj 8.825649, abs 0.009778, rel 0.001107, norm 1.521592"
## [1] "Iter 94, obj 8.816662, abs 0.008987, rel 0.001018, norm 1.517998"
## [1] "Iter 95, obj 8.808380, abs 0.008282, rel 0.000939, norm 1.514564"
## [1] "Iter 96, obj 8.800729, abs 0.007651, rel 0.000869, norm 1.511283"
## [1] "Iter 97, obj 8.793644, abs 0.007085, rel 0.000805, norm 1.508147"
## [1] "Iter 98, obj 8.787069, abs 0.006575, rel 0.000748, norm 1.505149"
## [1] "Iter 99, obj 8.780954, abs 0.006115, rel 0.000696, norm 1.502281"
## [1] "Iter 100, obj 8.775257, abs 0.005698, rel 0.000649, norm 1.499536"

## [1] "Iter 101, obj 0.797866, abs 7.977391, rel 0.909078, norm 0.116599"
## [1] "Iter 102, obj 0.787185, abs 0.010682, rel 0.013388, norm 0.114518"
## [1] "Iter 103, obj 0.777880, abs 0.009305, rel 0.011820, norm 0.111990"
## [1] "Iter 104, obj 0.770114, abs 0.007766, rel 0.009983, norm 0.109722"
## [1] "Iter 105, obj 0.763462, abs 0.006652, rel 0.008638, norm 0.107996"
## [1] "Iter 106, obj 0.757536, abs 0.005926, rel 0.007762, norm 0.106857"
## [1] "Iter 107, obj 0.752150, abs 0.005386, rel 0.007110, norm 0.106219"
## [1] "Iter 108, obj 0.747296, abs 0.004854, rel 0.006454, norm 0.105929"
## [1] "Iter 109, obj 0.743046, abs 0.004250, rel 0.005687, norm 0.105813"
## [1] "Iter 110, obj 0.739450, abs 0.003597, rel 0.004841, norm 0.105728"
## [1] "Iter 111, obj 0.736477, abs 0.002973, rel 0.004020, norm 0.105603"
## [1] "Iter 112, obj 0.734030, abs 0.002447, rel 0.003322, norm 0.105451"
## [1] "Iter 113, obj 0.731976, abs 0.002054, rel 0.002798, norm 0.105358"
## [1] "Iter 114, obj 0.730172, abs 0.001804, rel 0.002464, norm 0.105477"
## [1] "Iter 115, obj 0.728476, abs 0.001696, rel 0.002323, norm 0.105941"
## [1] "Iter 116, obj 0.726750, abs 0.001726, rel 0.002370, norm 0.106680"

```

```

## [1] "Iter 117, obj 0.724875, abs 0.001875, rel 0.002580, norm 0.107487"
## [1] "Iter 118, obj 0.722795, abs 0.002080, rel 0.002870, norm 0.108231"
## [1] "Iter 119, obj 0.720565, abs 0.002230, rel 0.003085, norm 0.108897"
## [1] "Iter 120, obj 0.718360, abs 0.002205, rel 0.003060, norm 0.109596"
## [1] "Iter 121, obj 0.716391, abs 0.001969, rel 0.002741, norm 0.110580"
## [1] "Iter 122, obj 0.714797, abs 0.001594, rel 0.002224, norm 0.111932"
## [1] "Iter 123, obj 0.713600, abs 0.001197, rel 0.001674, norm 0.113356"
## [1] "Iter 124, obj 0.712738, abs 0.000862, rel 0.001208, norm 0.114582"
## [1] "Iter 125, obj 0.712121, abs 0.000617, rel 0.000866, norm 0.115524"

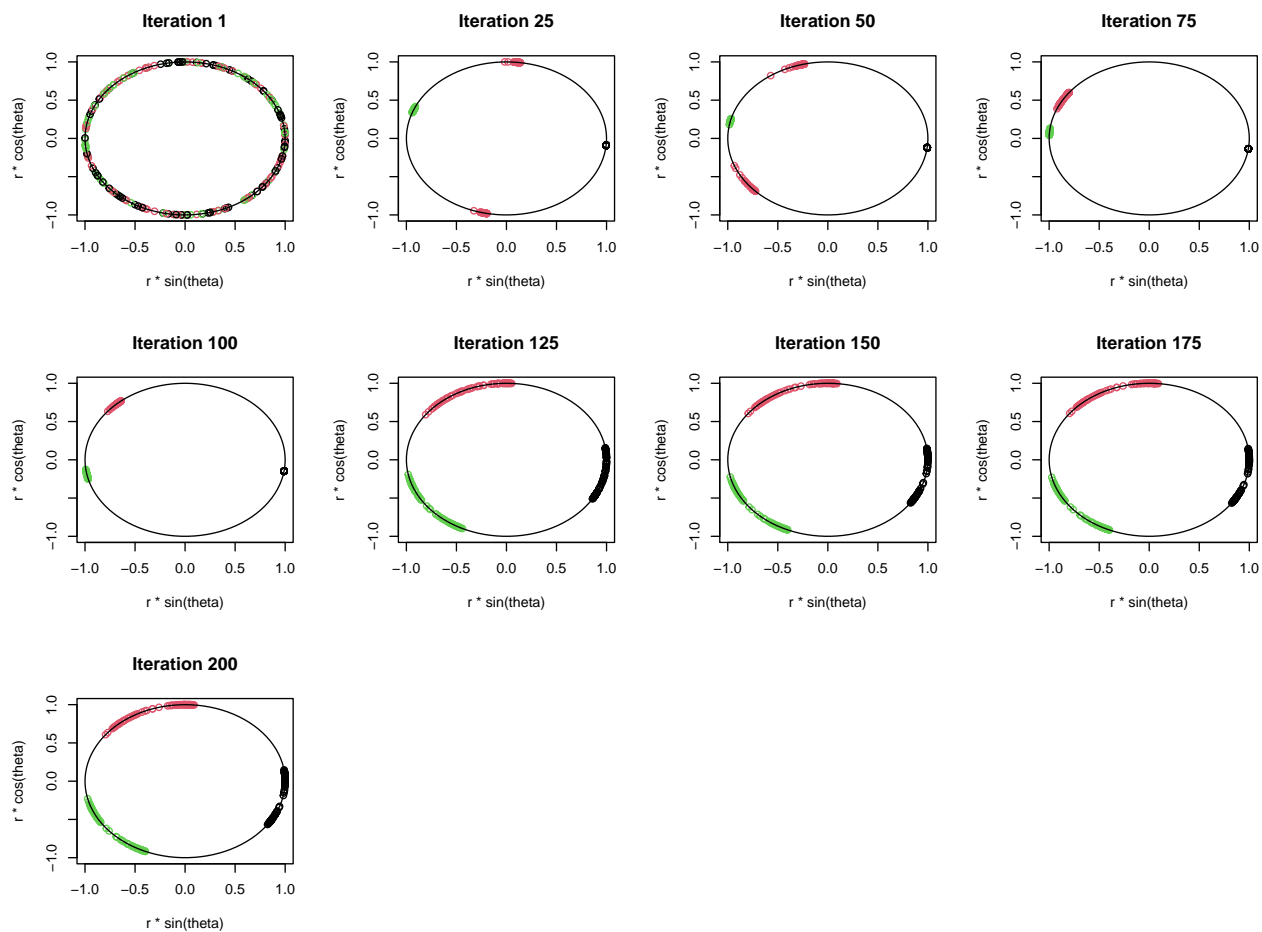
## [1] "Iter 126, obj 0.711671, abs 0.000450, rel 0.000632, norm 0.116200"
## [1] "Iter 127, obj 0.711331, abs 0.000340, rel 0.000478, norm 0.116663"
## [1] "Iter 128, obj 0.711065, abs 0.000266, rel 0.000374, norm 0.116968"
## [1] "Iter 129, obj 0.710851, abs 0.000214, rel 0.000301, norm 0.117160"
## [1] "Iter 130, obj 0.710675, abs 0.000176, rel 0.000247, norm 0.117273"
## [1] "Iter 131, obj 0.710529, abs 0.000146, rel 0.000206, norm 0.117332"
## [1] "Iter 132, obj 0.710407, abs 0.000122, rel 0.000172, norm 0.117355"
## [1] "Iter 133, obj 0.710304, abs 0.000103, rel 0.000145, norm 0.117354"
## [1] "Iter 134, obj 0.710217, abs 0.000087, rel 0.000122, norm 0.117336"
## [1] "Iter 135, obj 0.710144, abs 0.000073, rel 0.000103, norm 0.117309"
## [1] "Iter 136, obj 0.710082, abs 0.000062, rel 0.000087, norm 0.117275"
## [1] "Iter 137, obj 0.710030, abs 0.000052, rel 0.000074, norm 0.117238"
## [1] "Iter 138, obj 0.709986, abs 0.000044, rel 0.000062, norm 0.117200"
## [1] "Iter 139, obj 0.709948, abs 0.000037, rel 0.000053, norm 0.117161"
## [1] "Iter 140, obj 0.709916, abs 0.000032, rel 0.000045, norm 0.117123"
## [1] "Iter 141, obj 0.709889, abs 0.000027, rel 0.000038, norm 0.117087"
## [1] "Iter 142, obj 0.709866, abs 0.000023, rel 0.000032, norm 0.117052"
## [1] "Iter 143, obj 0.709847, abs 0.000020, rel 0.000028, norm 0.117019"
## [1] "Iter 144, obj 0.709830, abs 0.000017, rel 0.000024, norm 0.116988"
## [1] "Iter 145, obj 0.709815, abs 0.000014, rel 0.000020, norm 0.116960"
## [1] "Iter 146, obj 0.709803, abs 0.000012, rel 0.000017, norm 0.116933"
## [1] "Iter 147, obj 0.709792, abs 0.000011, rel 0.000015, norm 0.116908"
## [1] "Iter 148, obj 0.709783, abs 0.000009, rel 0.000013, norm 0.116886"
## [1] "Iter 149, obj 0.709776, abs 0.000008, rel 0.000011, norm 0.116865"
## [1] "Iter 150, obj 0.709769, abs 0.000007, rel 0.000009, norm 0.116846"

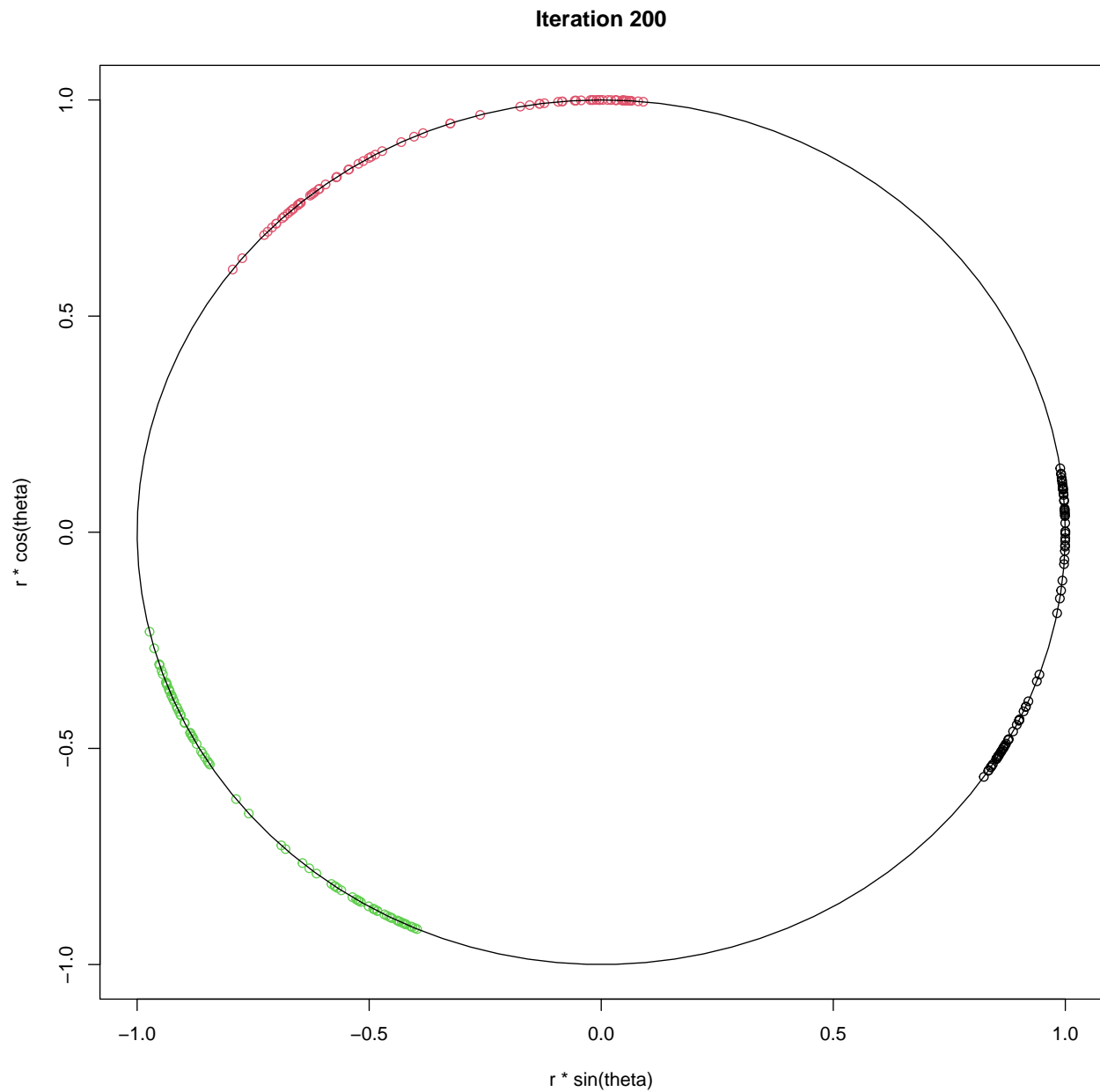
## [1] "Iter 151, obj 0.709763, abs 0.000006, rel 0.000008, norm 0.116829"
## [1] "Iter 152, obj 0.709758, abs 0.000005, rel 0.000007, norm 0.116813"
## [1] "Iter 153, obj 0.709754, abs 0.000004, rel 0.000006, norm 0.116798"
## [1] "Iter 154, obj 0.709750, abs 0.000004, rel 0.000005, norm 0.116785"
## [1] "Iter 155, obj 0.709747, abs 0.000003, rel 0.000005, norm 0.116773"
## [1] "Iter 156, obj 0.709744, abs 0.000003, rel 0.000004, norm 0.116762"
## [1] "Iter 157, obj 0.709742, abs 0.000002, rel 0.000003, norm 0.116752"
## [1] "Iter 158, obj 0.709740, abs 0.000002, rel 0.000003, norm 0.116743"
## [1] "Iter 159, obj 0.709738, abs 0.000002, rel 0.000003, norm 0.116734"
## [1] "Iter 160, obj 0.709736, abs 0.000002, rel 0.000002, norm 0.116727"
## [1] "Iter 161, obj 0.709735, abs 0.000001, rel 0.000002, norm 0.116720"
## [1] "Iter 162, obj 0.709734, abs 0.000001, rel 0.000002, norm 0.116714"
## [1] "Iter 163, obj 0.709733, abs 0.000001, rel 0.000001, norm 0.116708"
## [1] "Iter 164, obj 0.709732, abs 0.000001, rel 0.000001, norm 0.116703"
## [1] "Iter 165, obj 0.709731, abs 0.000001, rel 0.000001, norm 0.116698"
## [1] "Iter 166, obj 0.709730, abs 0.000001, rel 0.000001, norm 0.116693"
## [1] "Iter 167, obj 0.709730, abs 0.000001, rel 0.000001, norm 0.116690"
## [1] "Iter 168, obj 0.709729, abs 0.000001, rel 0.000001, norm 0.116686"
## [1] "Iter 169, obj 0.709729, abs 0.000000, rel 0.000001, norm 0.116683"

```

```
## [1] "Iter 170, obj 0.709728, abs 0.000000, rel 0.000001, norm 0.116680"
## [1] "Iter 171, obj 0.709728, abs 0.000000, rel 0.000000, norm 0.116677"
## [1] "Iter 172, obj 0.709728, abs 0.000000, rel 0.000000, norm 0.116674"
## [1] "Iter 173, obj 0.709727, abs 0.000000, rel 0.000000, norm 0.116672"
## [1] "Iter 174, obj 0.709727, abs 0.000000, rel 0.000000, norm 0.116670"
## [1] "Iter 175, obj 0.709727, abs 0.000000, rel 0.000000, norm 0.116668"

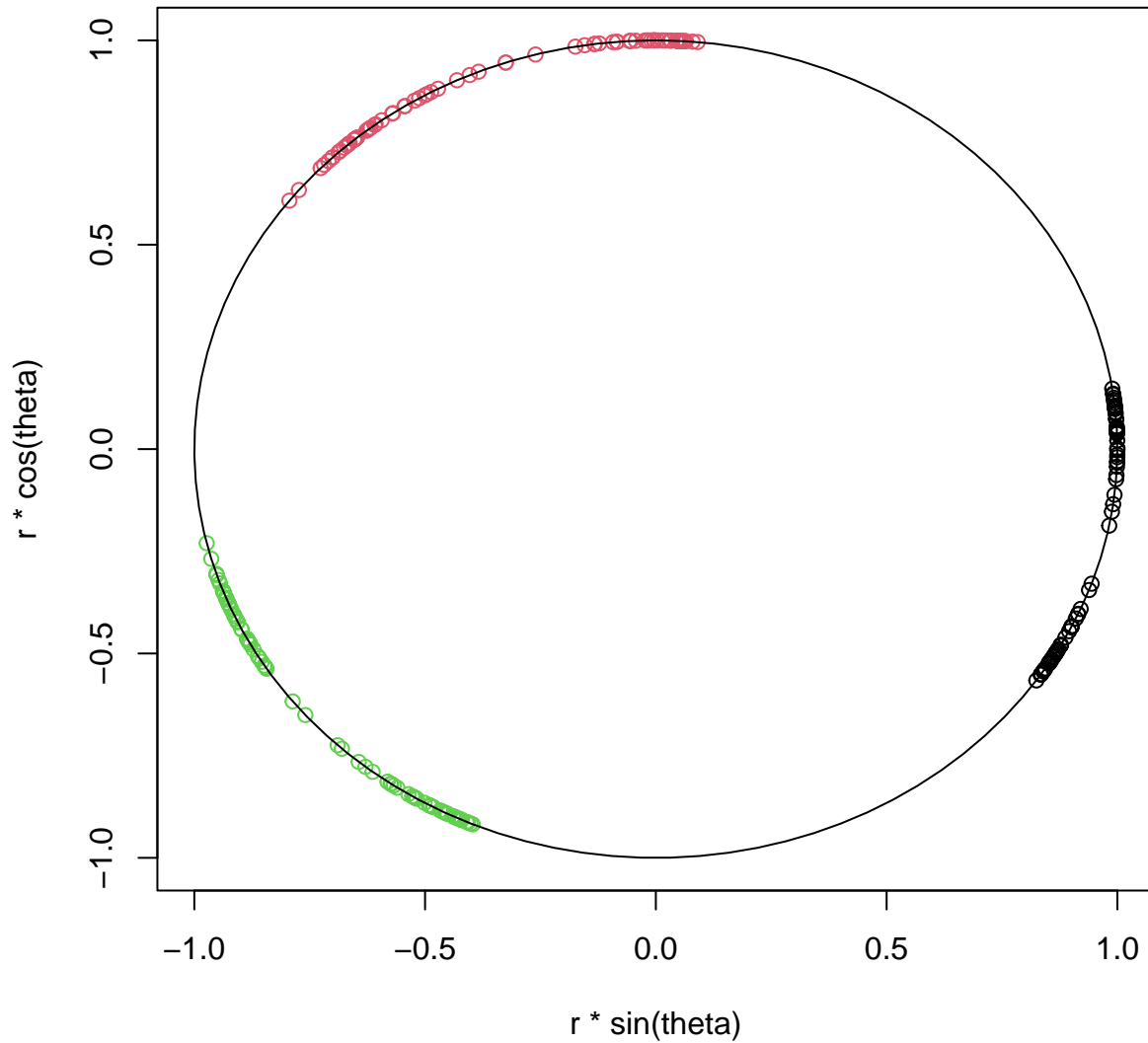
## [1] "Iter 176, obj 0.709727, abs 0.000000, rel 0.000000, norm 0.116666"
## [1] "Iter 177, obj 0.709727, abs 0.000000, rel 0.000000, norm 0.116665"
## [1] "Iter 178, obj 0.709727, abs 0.000000, rel 0.000000, norm 0.116663"
## [1] "Iter 179, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116662"
## [1] "Iter 180, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116661"
## [1] "Iter 181, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116659"
## [1] "Iter 182, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116658"
## [1] "Iter 183, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116657"
## [1] "Iter 184, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116657"
## [1] "Iter 185, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116656"
## [1] "Iter 186, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116655"
## [1] "Iter 187, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116654"
## [1] "Iter 188, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116654"
## [1] "Iter 189, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116653"
## [1] "Iter 190, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116653"
## [1] "Iter 191, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116652"
## [1] "Iter 192, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116652"
## [1] "Iter 193, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116651"
## [1] "Iter 194, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116651"
## [1] "Iter 195, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116651"
## [1] "Iter 196, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116650"
## [1] "Iter 197, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116650"
## [1] "Iter 198, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116650"
## [1] "Iter 199, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116650"
## [1] "Iter 200, obj 0.709726, abs 0.000000, rel 0.000000, norm 0.116649"
```



```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension \mathbb{S}^2 then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_5, d=2, rho_psc_list = rho_five_perp20, num_iteration=200, eta = 400,
             visualize_prog = TRUE, colors = colors)
```

```
## [1] "Iter 1, obj 12.746545, abs 0.000000, rel 0.000000, norm 0.140546"
## [1] "Iter 2, obj 8.116595, abs 4.629950, rel 0.363232, norm 0.502767"
## [1] "Iter 3, obj 7.319947, abs 0.796648, rel 0.098150, norm 1.491514"
## [1] "Iter 4, obj 7.383959, abs 0.064012, rel 0.008745, norm 2.016622"
## [1] "Iter 5, obj 7.522218, abs 0.138259, rel 0.018724, norm 2.248831"
## [1] "Iter 6, obj 7.655387, abs 0.133169, rel 0.017703, norm 2.382042"
## [1] "Iter 7, obj 7.755869, abs 0.100482, rel 0.013126, norm 2.460757"
## [1] "Iter 8, obj 7.820617, abs 0.064748, rel 0.008348, norm 2.506457"
## [1] "Iter 9, obj 7.859546, abs 0.038929, rel 0.004978, norm 2.534940"
## [1] "Iter 10, obj 7.882305, abs 0.022759, rel 0.002896, norm 2.555163"
## [1] "Iter 11, obj 7.895385, abs 0.013080, rel 0.001659, norm 2.571280"
## [1] "Iter 12, obj 7.902782, abs 0.007397, rel 0.000937, norm 2.585246"
## [1] "Iter 13, obj 7.906907, abs 0.004125, rel 0.000522, norm 2.598038"
## [1] "Iter 14, obj 7.909211, abs 0.002304, rel 0.000291, norm 2.610168"
## [1] "Iter 15, obj 7.910569, abs 0.001358, rel 0.000172, norm 2.621911"
```

```

## [1] "Iter 16, obj 7.911506, abs 0.000937, rel 0.000118, norm 2.633415"
## [1] "Iter 17, obj 7.912337, abs 0.000831, rel 0.000105, norm 2.644760"
## [1] "Iter 18, obj 7.913248, abs 0.000911, rel 0.000115, norm 2.655982"
## [1] "Iter 19, obj 7.914349, abs 0.001101, rel 0.000139, norm 2.667094"
## [1] "Iter 20, obj 7.915702, abs 0.001354, rel 0.000171, norm 2.678097"
## [1] "Iter 21, obj 7.917343, abs 0.001640, rel 0.000207, norm 2.688981"
## [1] "Iter 22, obj 7.919287, abs 0.001944, rel 0.000246, norm 2.699734"
## [1] "Iter 23, obj 7.921541, abs 0.002254, rel 0.000285, norm 2.710337"
## [1] "Iter 24, obj 7.924105, abs 0.002564, rel 0.000324, norm 2.720775"
## [1] "Iter 25, obj 7.926972, abs 0.002867, rel 0.000362, norm 2.731029"

## [1] "Iter 26, obj 7.930133, abs 0.003161, rel 0.000399, norm 2.741080"
## [1] "Iter 27, obj 7.933575, abs 0.003442, rel 0.000434, norm 2.750912"
## [1] "Iter 28, obj 7.937282, abs 0.003707, rel 0.000467, norm 2.760506"
## [1] "Iter 29, obj 7.941237, abs 0.003954, rel 0.000498, norm 2.769848"
## [1] "Iter 30, obj 7.945417, abs 0.004181, rel 0.000526, norm 2.778922"
## [1] "Iter 31, obj 7.949801, abs 0.004384, rel 0.000552, norm 2.787715"
## [1] "Iter 32, obj 7.954362, abs 0.004561, rel 0.000574, norm 2.796216"
## [1] "Iter 33, obj 7.959074, abs 0.004712, rel 0.000592, norm 2.804415"
## [1] "Iter 34, obj 7.963908, abs 0.004834, rel 0.000607, norm 2.812302"
## [1] "Iter 35, obj 7.968836, abs 0.004927, rel 0.000619, norm 2.819871"
## [1] "Iter 36, obj 7.973827, abs 0.004991, rel 0.000626, norm 2.827119"
## [1] "Iter 37, obj 7.978852, abs 0.005025, rel 0.000630, norm 2.834042"
## [1] "Iter 38, obj 7.983883, abs 0.005031, rel 0.000630, norm 2.840639"
## [1] "Iter 39, obj 7.988891, abs 0.005008, rel 0.000627, norm 2.846912"
## [1] "Iter 40, obj 7.993851, abs 0.004960, rel 0.000621, norm 2.852862"
## [1] "Iter 41, obj 7.998737, abs 0.004887, rel 0.000611, norm 2.858496"
## [1] "Iter 42, obj 8.003529, abs 0.004791, rel 0.000599, norm 2.863818"
## [1] "Iter 43, obj 8.008205, abs 0.004677, rel 0.000584, norm 2.868836"
## [1] "Iter 44, obj 8.012750, abs 0.004544, rel 0.000567, norm 2.873559"
## [1] "Iter 45, obj 8.017147, abs 0.004398, rel 0.000549, norm 2.877995"
## [1] "Iter 46, obj 8.021386, abs 0.004239, rel 0.000529, norm 2.882156"
## [1] "Iter 47, obj 8.025457, abs 0.004071, rel 0.000507, norm 2.886051"
## [1] "Iter 48, obj 8.029352, abs 0.003895, rel 0.000485, norm 2.889693"
## [1] "Iter 49, obj 8.033067, abs 0.003715, rel 0.000463, norm 2.893094"
## [1] "Iter 50, obj 8.036599, abs 0.003532, rel 0.000440, norm 2.896264"

## [1] "Iter 51, obj 8.039948, abs 0.003349, rel 0.000417, norm 2.899218"
## [1] "Iter 52, obj 8.043114, abs 0.003166, rel 0.000394, norm 2.901965"
## [1] "Iter 53, obj 8.046100, abs 0.002986, rel 0.000371, norm 2.904518"
## [1] "Iter 54, obj 8.048908, abs 0.002809, rel 0.000349, norm 2.906890"
## [1] "Iter 55, obj 8.051545, abs 0.002636, rel 0.000328, norm 2.909090"
## [1] "Iter 56, obj 8.054014, abs 0.002470, rel 0.000307, norm 2.911130"
## [1] "Iter 57, obj 8.056323, abs 0.002309, rel 0.000287, norm 2.913020"
## [1] "Iter 58, obj 8.058478, abs 0.002155, rel 0.000267, norm 2.914771"
## [1] "Iter 59, obj 8.060485, abs 0.002007, rel 0.000249, norm 2.916391"
## [1] "Iter 60, obj 8.062352, abs 0.001867, rel 0.000232, norm 2.917891"
## [1] "Iter 61, obj 8.064086, abs 0.001734, rel 0.000215, norm 2.919278"
## [1] "Iter 62, obj 8.065695, abs 0.001609, rel 0.000199, norm 2.920562"
## [1] "Iter 63, obj 8.067186, abs 0.001490, rel 0.000185, norm 2.921748"
## [1] "Iter 64, obj 8.068565, abs 0.001379, rel 0.000171, norm 2.922845"
## [1] "Iter 65, obj 8.069840, abs 0.001275, rel 0.000158, norm 2.923860"
## [1] "Iter 66, obj 8.071017, abs 0.001177, rel 0.000146, norm 2.924798"
## [1] "Iter 67, obj 8.072103, abs 0.001086, rel 0.000135, norm 2.925666"
## [1] "Iter 68, obj 8.073103, abs 0.001001, rel 0.000124, norm 2.926468"

```

```

## [1] "Iter 69, obj 8.074025, abs 0.000922, rel 0.000114, norm 2.927211"
## [1] "Iter 70, obj 8.074873, abs 0.000848, rel 0.000105, norm 2.927898"
## [1] "Iter 71, obj 8.075653, abs 0.000780, rel 0.000097, norm 2.928535"
## [1] "Iter 72, obj 8.076370, abs 0.000717, rel 0.000089, norm 2.929124"
## [1] "Iter 73, obj 8.077028, abs 0.000658, rel 0.000081, norm 2.929671"
## [1] "Iter 74, obj 8.077631, abs 0.000604, rel 0.000075, norm 2.930178"
## [1] "Iter 75, obj 8.078185, abs 0.000554, rel 0.000069, norm 2.930648"

## [1] "Iter 76, obj 8.078692, abs 0.000507, rel 0.000063, norm 2.931085"
## [1] "Iter 77, obj 8.079157, abs 0.000465, rel 0.000058, norm 2.931491"
## [1] "Iter 78, obj 8.079582, abs 0.000425, rel 0.000053, norm 2.931869"
## [1] "Iter 79, obj 8.079971, abs 0.000389, rel 0.000048, norm 2.932221"
## [1] "Iter 80, obj 8.080327, abs 0.000356, rel 0.000044, norm 2.932549"
## [1] "Iter 81, obj 8.080652, abs 0.000325, rel 0.000040, norm 2.932856"
## [1] "Iter 82, obj 8.080948, abs 0.000297, rel 0.000037, norm 2.933142"
## [1] "Iter 83, obj 8.081219, abs 0.000271, rel 0.000033, norm 2.933410"
## [1] "Iter 84, obj 8.081466, abs 0.000247, rel 0.000031, norm 2.933660"
## [1] "Iter 85, obj 8.081690, abs 0.000225, rel 0.000028, norm 2.933896"
## [1] "Iter 86, obj 8.081895, abs 0.000205, rel 0.000025, norm 2.934117"
## [1] "Iter 87, obj 8.082081, abs 0.000186, rel 0.000023, norm 2.934325"
## [1] "Iter 88, obj 8.082250, abs 0.000169, rel 0.000021, norm 2.934521"
## [1] "Iter 89, obj 8.082404, abs 0.000154, rel 0.000019, norm 2.934705"
## [1] "Iter 90, obj 8.082544, abs 0.000139, rel 0.000017, norm 2.934880"
## [1] "Iter 91, obj 8.082670, abs 0.000126, rel 0.000016, norm 2.935046"
## [1] "Iter 92, obj 8.082784, abs 0.000114, rel 0.000014, norm 2.935203"
## [1] "Iter 93, obj 8.082888, abs 0.000103, rel 0.000013, norm 2.935352"
## [1] "Iter 94, obj 8.082981, abs 0.000093, rel 0.000012, norm 2.935494"
## [1] "Iter 95, obj 8.083065, abs 0.000084, rel 0.000010, norm 2.935630"
## [1] "Iter 96, obj 8.083140, abs 0.000076, rel 0.000009, norm 2.935759"
## [1] "Iter 97, obj 8.083208, abs 0.000068, rel 0.000008, norm 2.935883"
## [1] "Iter 98, obj 8.083269, abs 0.000061, rel 0.000008, norm 2.936002"
## [1] "Iter 99, obj 8.083323, abs 0.000054, rel 0.000007, norm 2.936116"
## [1] "Iter 100, obj 8.083371, abs 0.000048, rel 0.000006, norm 2.936225"

## [1] "Iter 101, obj 1.210593, abs 6.872778, rel 0.850237, norm 0.115165"
## [1] "Iter 102, obj 1.399271, abs 0.188677, rel 0.155855, norm 0.378621"
## [1] "Iter 103, obj 1.314425, abs 0.084845, rel 0.060635, norm 0.396849"
## [1] "Iter 104, obj 1.275512, abs 0.038913, rel 0.029604, norm 0.448352"
## [1] "Iter 105, obj 1.284609, abs 0.009096, rel 0.007132, norm 0.428748"
## [1] "Iter 106, obj 1.394037, abs 0.109428, rel 0.085184, norm 0.425139"
## [1] "Iter 107, obj 1.471487, abs 0.077450, rel 0.055558, norm 0.463542"
## [1] "Iter 108, obj 1.652091, abs 0.180604, rel 0.122736, norm 0.513292"
## [1] "Iter 109, obj 1.673236, abs 0.021145, rel 0.012799, norm 0.592234"
## [1] "Iter 110, obj 1.691757, abs 0.018522, rel 0.011069, norm 0.640918"
## [1] "Iter 111, obj 1.595964, abs 0.095794, rel 0.056624, norm 0.669245"
## [1] "Iter 112, obj 1.593746, abs 0.002218, rel 0.001389, norm 0.657853"
## [1] "Iter 113, obj 1.562301, abs 0.031446, rel 0.019731, norm 0.650375"
## [1] "Iter 114, obj 1.555133, abs 0.007167, rel 0.004588, norm 0.649809"
## [1] "Iter 115, obj 1.550052, abs 0.005081, rel 0.003267, norm 0.648988"
## [1] "Iter 116, obj 1.529436, abs 0.020617, rel 0.013301, norm 0.646819"
## [1] "Iter 117, obj 1.543327, abs 0.013891, rel 0.009083, norm 0.643431"
## [1] "Iter 118, obj 1.535154, abs 0.008173, rel 0.005296, norm 0.640574"
## [1] "Iter 119, obj 1.552068, abs 0.016914, rel 0.011018, norm 0.638981"
## [1] "Iter 120, obj 1.553387, abs 0.001320, rel 0.000850, norm 0.636340"
## [1] "Iter 121, obj 1.525475, abs 0.027913, rel 0.017969, norm 0.651596"

```

```

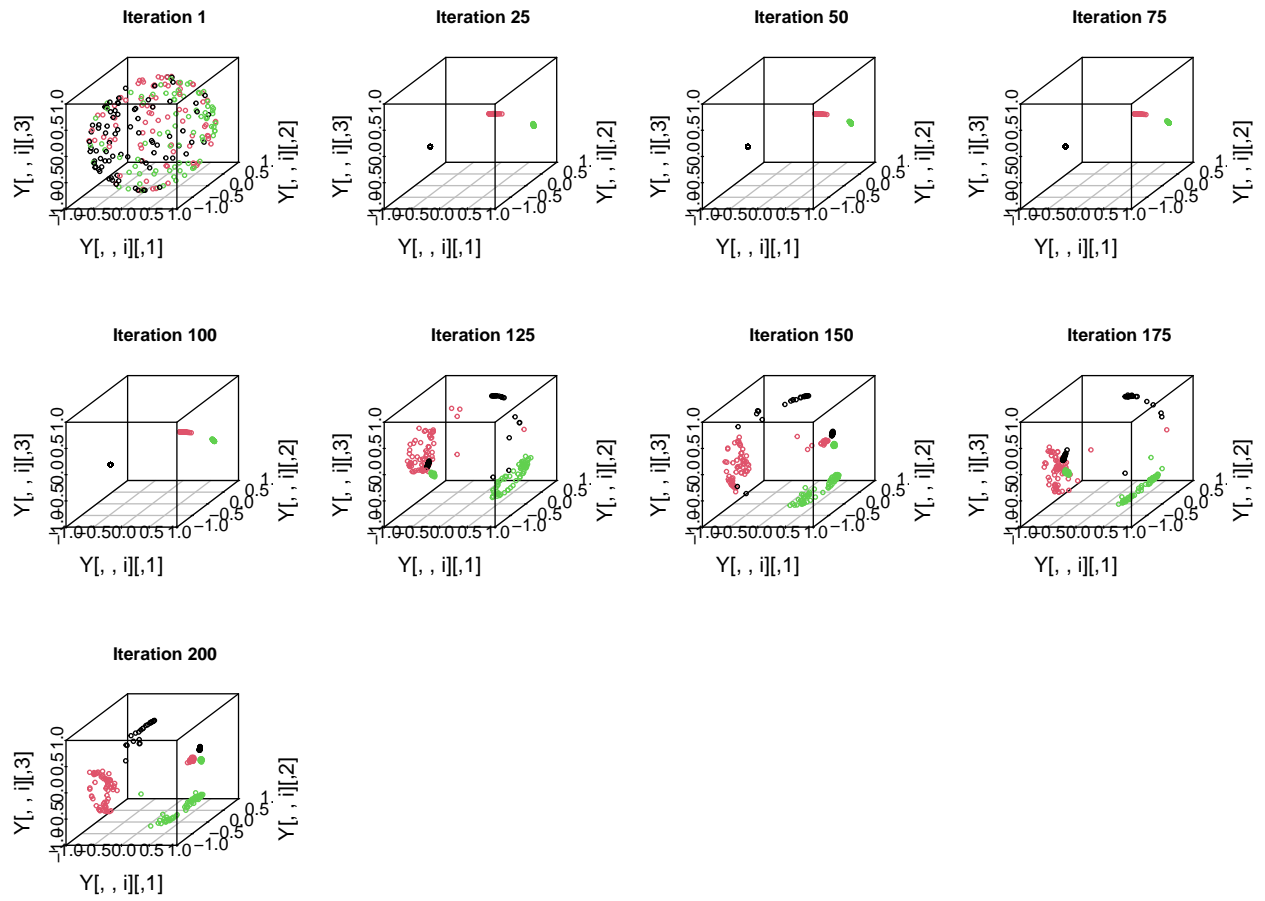
## [1] "Iter 122, obj 1.515195, abs 0.010279, rel 0.006738, norm 0.635975"
## [1] "Iter 123, obj 1.543368, abs 0.028172, rel 0.018593, norm 0.637211"
## [1] "Iter 124, obj 1.522962, abs 0.020406, rel 0.013222, norm 0.626903"
## [1] "Iter 125, obj 1.536797, abs 0.013835, rel 0.009084, norm 0.636069"

## [1] "Iter 126, obj 1.522772, abs 0.014025, rel 0.009126, norm 0.631293"
## [1] "Iter 127, obj 1.528552, abs 0.005780, rel 0.003796, norm 0.638373"
## [1] "Iter 128, obj 1.487931, abs 0.040621, rel 0.026575, norm 0.628513"
## [1] "Iter 129, obj 1.517402, abs 0.029472, rel 0.019807, norm 0.637021"
## [1] "Iter 130, obj 1.489630, abs 0.027772, rel 0.018303, norm 0.619540"
## [1] "Iter 131, obj 1.520518, abs 0.030889, rel 0.020736, norm 0.630197"
## [1] "Iter 132, obj 1.472228, abs 0.048291, rel 0.031759, norm 0.624028"
## [1] "Iter 133, obj 1.517212, abs 0.044985, rel 0.030555, norm 0.629412"
## [1] "Iter 134, obj 1.473092, abs 0.044120, rel 0.029079, norm 0.614026"
## [1] "Iter 135, obj 1.516242, abs 0.043150, rel 0.029292, norm 0.629467"
## [1] "Iter 136, obj 1.476744, abs 0.039499, rel 0.026050, norm 0.612945"
## [1] "Iter 137, obj 1.509955, abs 0.033212, rel 0.022490, norm 0.618688"
## [1] "Iter 138, obj 1.460346, abs 0.049609, rel 0.032855, norm 0.608067"
## [1] "Iter 139, obj 1.515090, abs 0.054744, rel 0.037487, norm 0.621047"
## [1] "Iter 140, obj 1.442050, abs 0.073040, rel 0.048208, norm 0.607294"
## [1] "Iter 141, obj 1.529361, abs 0.087311, rel 0.060547, norm 0.607123"
## [1] "Iter 142, obj 1.505744, abs 0.023617, rel 0.015442, norm 0.597925"
## [1] "Iter 143, obj 1.549158, abs 0.043414, rel 0.028832, norm 0.629167"
## [1] "Iter 144, obj 1.493449, abs 0.055709, rel 0.035961, norm 0.620496"
## [1] "Iter 145, obj 1.527327, abs 0.033878, rel 0.022684, norm 0.633922"
## [1] "Iter 146, obj 1.538880, abs 0.011553, rel 0.007564, norm 0.606660"
## [1] "Iter 147, obj 1.563456, abs 0.024576, rel 0.015970, norm 0.626931"
## [1] "Iter 148, obj 1.517996, abs 0.045460, rel 0.029076, norm 0.622924"
## [1] "Iter 149, obj 1.559289, abs 0.041293, rel 0.027202, norm 0.640475"
## [1] "Iter 150, obj 1.493805, abs 0.065484, rel 0.041996, norm 0.621433"

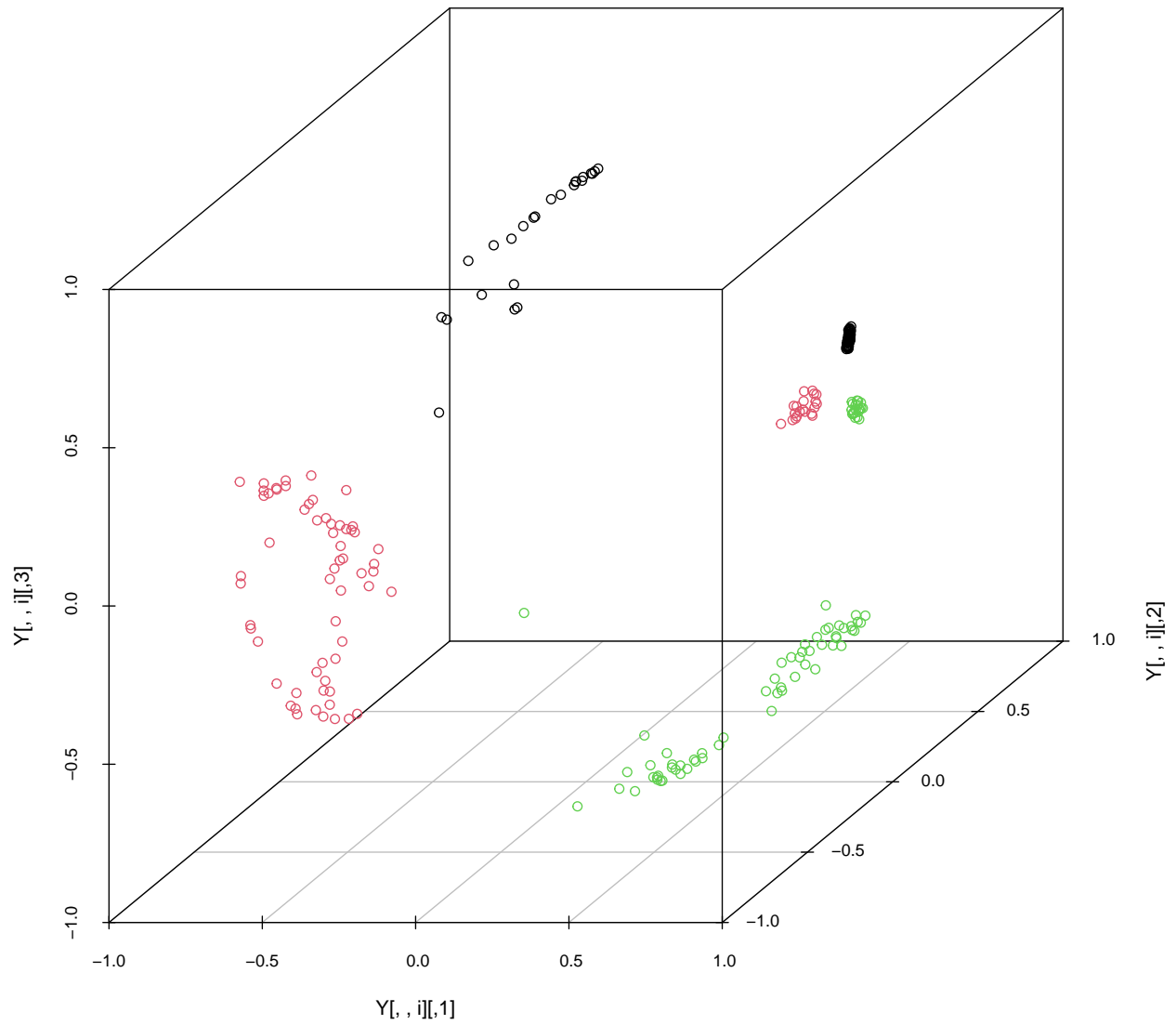
## [1] "Iter 151, obj 1.540308, abs 0.046502, rel 0.031130, norm 0.618760"
## [1] "Iter 152, obj 1.523765, abs 0.016542, rel 0.010740, norm 0.601665"
## [1] "Iter 153, obj 1.559168, abs 0.035403, rel 0.023234, norm 0.620288"
## [1] "Iter 154, obj 1.518308, abs 0.040860, rel 0.026206, norm 0.623513"
## [1] "Iter 155, obj 1.526163, abs 0.007855, rel 0.005173, norm 0.635771"
## [1] "Iter 156, obj 1.505285, abs 0.020877, rel 0.013680, norm 0.610326"
## [1] "Iter 157, obj 1.512367, abs 0.007082, rel 0.004705, norm 0.613487"
## [1] "Iter 158, obj 1.511942, abs 0.000425, rel 0.000281, norm 0.604911"
## [1] "Iter 159, obj 1.539279, abs 0.027337, rel 0.018081, norm 0.610024"
## [1] "Iter 160, obj 1.528144, abs 0.011135, rel 0.007234, norm 0.605690"
## [1] "Iter 161, obj 1.542531, abs 0.014387, rel 0.009415, norm 0.616380"
## [1] "Iter 162, obj 1.507682, abs 0.034849, rel 0.022592, norm 0.610683"
## [1] "Iter 163, obj 1.513141, abs 0.005459, rel 0.003621, norm 0.614853"
## [1] "Iter 164, obj 1.542250, abs 0.029109, rel 0.019238, norm 0.595217"
## [1] "Iter 165, obj 1.520377, abs 0.021874, rel 0.014183, norm 0.613238"
## [1] "Iter 166, obj 1.509856, abs 0.010520, rel 0.006919, norm 0.614977"
## [1] "Iter 167, obj 1.511623, abs 0.001766, rel 0.001170, norm 0.614819"
## [1] "Iter 168, obj 1.522169, abs 0.010546, rel 0.006977, norm 0.596406"
## [1] "Iter 169, obj 1.508573, abs 0.013596, rel 0.008932, norm 0.606940"
## [1] "Iter 170, obj 1.533005, abs 0.024432, rel 0.016196, norm 0.602675"
## [1] "Iter 171, obj 1.526144, abs 0.006862, rel 0.004476, norm 0.604350"
## [1] "Iter 172, obj 1.558593, abs 0.032450, rel 0.021263, norm 0.596871"
## [1] "Iter 173, obj 1.520146, abs 0.038447, rel 0.024668, norm 0.612572"
## [1] "Iter 174, obj 1.501017, abs 0.019129, rel 0.012584, norm 0.610200"

```

[1] "Iter 175, obj 1.495305, abs 0.005712, rel 0.003805, norm 0.605916"
[1] "Iter 176, obj 1.526400, abs 0.031095, rel 0.020795, norm 0.587753"
[1] "Iter 177, obj 1.521611, abs 0.004789, rel 0.003138, norm 0.598440"
[1] "Iter 178, obj 1.538597, abs 0.016986, rel 0.011163, norm 0.605542"
[1] "Iter 179, obj 1.540965, abs 0.002368, rel 0.001539, norm 0.613090"
[1] "Iter 180, obj 1.546449, abs 0.005484, rel 0.003559, norm 0.609077"
[1] "Iter 181, obj 1.550966, abs 0.004517, rel 0.002921, norm 0.615936"
[1] "Iter 182, obj 1.545672, abs 0.005294, rel 0.003414, norm 0.613024"
[1] "Iter 183, obj 1.531976, abs 0.013695, rel 0.008860, norm 0.614643"
[1] "Iter 184, obj 1.563033, abs 0.031057, rel 0.020272, norm 0.610655"
[1] "Iter 185, obj 1.550974, abs 0.012059, rel 0.007715, norm 0.614921"
[1] "Iter 186, obj 1.540364, abs 0.010611, rel 0.006841, norm 0.615580"
[1] "Iter 187, obj 1.543767, abs 0.003404, rel 0.002210, norm 0.621202"
[1] "Iter 188, obj 1.560634, abs 0.016867, rel 0.010926, norm 0.608961"
[1] "Iter 189, obj 1.536497, abs 0.024137, rel 0.015466, norm 0.615999"
[1] "Iter 190, obj 1.555106, abs 0.018608, rel 0.012111, norm 0.614488"
[1] "Iter 191, obj 1.573413, abs 0.018308, rel 0.011773, norm 0.615312"
[1] "Iter 192, obj 1.552623, abs 0.020790, rel 0.013213, norm 0.611811"
[1] "Iter 193, obj 1.555704, abs 0.003081, rel 0.001984, norm 0.622755"
[1] "Iter 194, obj 1.580446, abs 0.024741, rel 0.015904, norm 0.617125"
[1] "Iter 195, obj 1.585588, abs 0.005143, rel 0.003254, norm 0.614020"
[1] "Iter 196, obj 1.566981, abs 0.018608, rel 0.011735, norm 0.633428"
[1] "Iter 197, obj 1.587242, abs 0.020261, rel 0.012930, norm 0.626853"
[1] "Iter 198, obj 1.574005, abs 0.013237, rel 0.008340, norm 0.622309"
[1] "Iter 199, obj 1.548554, abs 0.025451, rel 0.016170, norm 0.632489"
[1] "Iter 200, obj 1.532147, abs 0.016407, rel 0.010595, norm 0.623084"



Iteration 200



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),  
                             color = colors)
```

