

# Applications PSC-SNE

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## Cases

### Case 1

Sample on the  $(\mathbb{S}^1)^2$  where  $p = 1$  and  $r = 2$ .

Data:

```
# Sample on the  $(\mathbb{S}^1)^2$ 
n <- 200
vmf11 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 10)
vmf12 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi)), kappa = 10)
x1 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf11),
                             DirStats::to_rad(vmf12)))

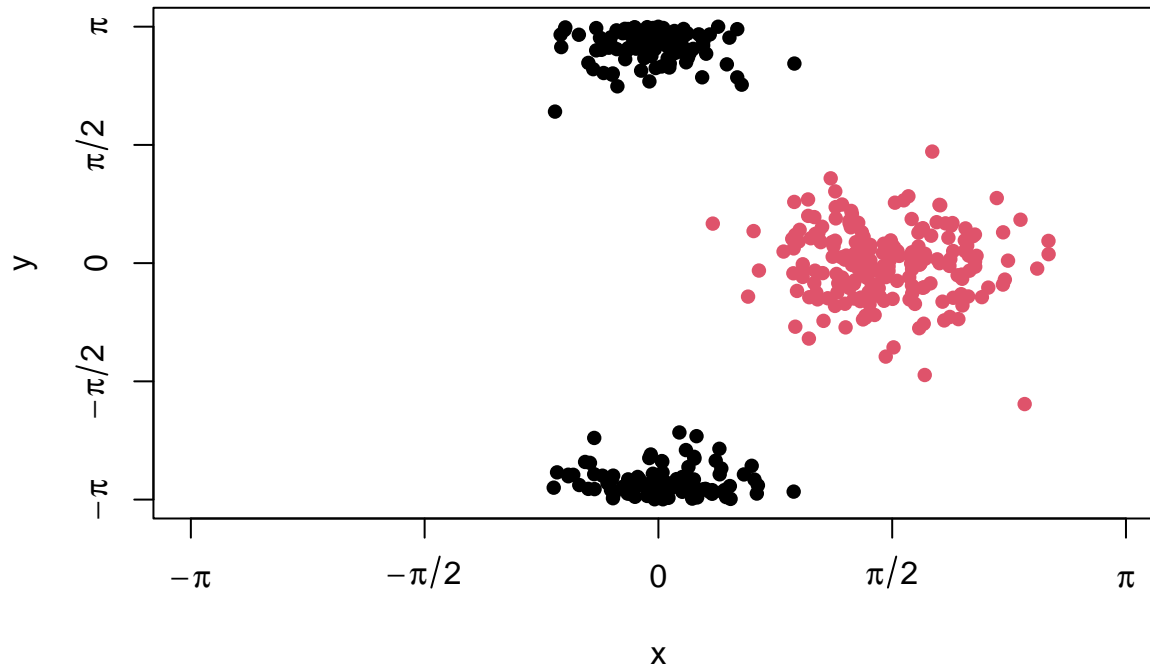
vmf21 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi / 2)), kappa = 5)
vmf22 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 5)
x2 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf21), DirStats::to_rad(vmf22)))

x <- rbind(x1, x2)

# Cartesian coordinates
n <- 400
x_array <- array(dim = c(n, 2, 2))
x_array[, , 1] <- DirStats::to_cir(x[, 1])
x_array[, , 2] <- DirStats::to_cir(x[, 2])

indexes <- sample(1:n)
x_array <- x_array[indexes,,]
colors <- rep(c(1, 2), each = n/2)[indexes]

plot(x, xlim = c(-pi, pi), ylim = c(-pi, pi), axes = FALSE,
     col = rep(c(1, 2), each = n/2), pch = 16,
     xlab = "x", ylab = "y")
sdetorus::torusAxis()
```



Let's calculate the rho parameters based on a perplexity of 20:

```
# Time difference of 27.58008 secs
rho_first_20 <- rho_optim_bst(x_array, 20)
```

```
## Time difference of 23.60194 secs
```

First, let's reduce to dimension  $\mathbb{S}^1$  then  $d = 1$  (circumference):

```
Y <- psc_sne(X=x_array, d=1, rho_psc_list = rho_first_20, num_iteration=125,
             colors=colors, visualize_prog = TRUE)
```

```
## It: 1; obj: 1.683e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.639e-01
## It: 2; obj: 1.600e+01; abs: 8.238e-01; rel: 4.895e-02; norm: 4.282e-01
## It: 3; obj: 1.559e+01; abs: 4.135e-01; rel: 2.584e-02; norm: 5.858e-01
## It: 4; obj: 1.536e+01; abs: 2.342e-01; rel: 1.502e-02; norm: 6.731e-01
## It: 5; obj: 1.521e+01; abs: 1.482e-01; rel: 9.650e-03; norm: 7.137e-01
## It: 6; obj: 1.508e+01; abs: 1.297e-01; rel: 8.529e-03; norm: 7.347e-01
## It: 7; obj: 1.497e+01; abs: 1.122e-01; rel: 7.444e-03; norm: 7.530e-01
## It: 8; obj: 1.488e+01; abs: 8.639e-02; rel: 5.772e-03; norm: 7.708e-01
## It: 9; obj: 1.475e+01; abs: 1.300e-01; rel: 8.740e-03; norm: 7.809e-01
## It: 10; obj: 1.463e+01; abs: 1.196e-01; rel: 8.112e-03; norm: 7.909e-01
## It: 11; obj: 1.453e+01; abs: 9.580e-02; rel: 6.548e-03; norm: 8.009e-01
## It: 12; obj: 1.444e+01; abs: 9.231e-02; rel: 6.351e-03; norm: 8.070e-01
## It: 13; obj: 1.436e+01; abs: 7.991e-02; rel: 5.533e-03; norm: 8.150e-01
## It: 14; obj: 1.430e+01; abs: 6.629e-02; rel: 4.616e-03; norm: 8.249e-01
## It: 15; obj: 1.424e+01; abs: 5.382e-02; rel: 3.764e-03; norm: 8.337e-01
## It: 16; obj: 1.420e+01; abs: 4.621e-02; rel: 3.245e-03; norm: 8.427e-01
## It: 17; obj: 1.415e+01; abs: 4.386e-02; rel: 3.090e-03; norm: 8.530e-01
## It: 18; obj: 1.411e+01; abs: 4.443e-02; rel: 3.140e-03; norm: 8.647e-01
## It: 19; obj: 1.406e+01; abs: 4.660e-02; rel: 3.303e-03; norm: 8.762e-01
## It: 20; obj: 1.401e+01; abs: 4.993e-02; rel: 3.551e-03; norm: 8.869e-01
## It: 21; obj: 1.396e+01; abs: 4.726e-02; rel: 3.373e-03; norm: 8.974e-01
## It: 22; obj: 1.392e+01; abs: 4.526e-02; rel: 3.241e-03; norm: 9.078e-01
```

```

## It: 23; obj: 1.388e+01; abs: 4.225e-02; rel: 3.036e-03; norm: 9.182e-01
## It: 24; obj: 1.384e+01; abs: 3.596e-02; rel: 2.592e-03; norm: 9.280e-01
## It: 25; obj: 1.381e+01; abs: 2.851e-02; rel: 2.060e-03; norm: 9.367e-01

## It: 26; obj: 1.379e+01; abs: 2.155e-02; rel: 1.560e-03; norm: 9.441e-01
## It: 27; obj: 1.377e+01; abs: 1.589e-02; rel: 1.153e-03; norm: 9.504e-01
## It: 28; obj: 1.376e+01; abs: 1.216e-02; rel: 8.830e-04; norm: 9.560e-01
## It: 29; obj: 1.375e+01; abs: 1.044e-02; rel: 7.587e-04; norm: 9.610e-01
## It: 30; obj: 1.374e+01; abs: 1.080e-02; rel: 7.850e-04; norm: 9.661e-01
## It: 31; obj: 1.373e+01; abs: 1.353e-02; rel: 9.846e-04; norm: 9.714e-01
## It: 32; obj: 1.371e+01; abs: 1.872e-02; rel: 1.364e-03; norm: 9.772e-01
## It: 33; obj: 1.368e+01; abs: 2.490e-02; rel: 1.816e-03; norm: 9.836e-01
## It: 34; obj: 1.366e+01; abs: 2.767e-02; rel: 2.022e-03; norm: 9.909e-01
## It: 35; obj: 1.363e+01; abs: 2.326e-02; rel: 1.703e-03; norm: 9.994e-01
## It: 36; obj: 1.362e+01; abs: 1.449e-02; rel: 1.063e-03; norm: 1.009e+00
## It: 37; obj: 1.361e+01; abs: 6.095e-03; rel: 4.475e-04; norm: 1.019e+00
## It: 38; obj: 1.361e+01; abs: 5.447e-04; rel: 4.001e-05; norm: 1.030e+00
## It: 39; obj: 1.362e+01; abs: 5.497e-03; rel: 4.038e-04; norm: 1.041e+00
## It: 40; obj: 1.363e+01; abs: 9.126e-03; rel: 6.701e-04; norm: 1.052e+00
## It: 41; obj: 1.364e+01; abs: 1.177e-02; rel: 8.639e-04; norm: 1.062e+00
## It: 42; obj: 1.365e+01; abs: 1.368e-02; rel: 1.003e-03; norm: 1.072e+00
## It: 43; obj: 1.367e+01; abs: 1.501e-02; rel: 1.100e-03; norm: 1.082e+00
## It: 44; obj: 1.368e+01; abs: 1.591e-02; rel: 1.164e-03; norm: 1.091e+00
## It: 45; obj: 1.370e+01; abs: 1.644e-02; rel: 1.202e-03; norm: 1.099e+00
## It: 46; obj: 1.372e+01; abs: 1.669e-02; rel: 1.219e-03; norm: 1.107e+00
## It: 47; obj: 1.373e+01; abs: 1.671e-02; rel: 1.218e-03; norm: 1.115e+00
## It: 48; obj: 1.375e+01; abs: 1.651e-02; rel: 1.202e-03; norm: 1.122e+00
## It: 49; obj: 1.377e+01; abs: 1.616e-02; rel: 1.175e-03; norm: 1.128e+00
## It: 50; obj: 1.378e+01; abs: 1.569e-02; rel: 1.140e-03; norm: 1.135e+00

## It: 51; obj: 1.380e+01; abs: 1.518e-02; rel: 1.101e-03; norm: 1.140e+00
## It: 52; obj: 1.381e+01; abs: 1.466e-02; rel: 1.062e-03; norm: 1.146e+00
## It: 53; obj: 1.383e+01; abs: 1.414e-02; rel: 1.024e-03; norm: 1.151e+00
## It: 54; obj: 1.384e+01; abs: 1.361e-02; rel: 9.846e-04; norm: 1.155e+00
## It: 55; obj: 1.385e+01; abs: 1.307e-02; rel: 9.447e-04; norm: 1.160e+00
## It: 56; obj: 1.387e+01; abs: 1.252e-02; rel: 9.040e-04; norm: 1.163e+00
## It: 57; obj: 1.388e+01; abs: 1.197e-02; rel: 8.629e-04; norm: 1.167e+00
## It: 58; obj: 1.389e+01; abs: 1.141e-02; rel: 8.219e-04; norm: 1.170e+00
## It: 59; obj: 1.390e+01; abs: 1.085e-02; rel: 7.813e-04; norm: 1.173e+00
## It: 60; obj: 1.391e+01; abs: 1.031e-02; rel: 7.414e-04; norm: 1.176e+00
## It: 61; obj: 1.392e+01; abs: 9.772e-03; rel: 7.025e-04; norm: 1.178e+00
## It: 62; obj: 1.393e+01; abs: 9.253e-03; rel: 6.647e-04; norm: 1.181e+00
## It: 63; obj: 1.394e+01; abs: 8.750e-03; rel: 6.282e-04; norm: 1.183e+00
## It: 64; obj: 1.395e+01; abs: 8.264e-03; rel: 5.929e-04; norm: 1.185e+00
## It: 65; obj: 1.395e+01; abs: 7.796e-03; rel: 5.590e-04; norm: 1.187e+00
## It: 66; obj: 1.396e+01; abs: 7.346e-03; rel: 5.265e-04; norm: 1.188e+00
## It: 67; obj: 1.397e+01; abs: 6.914e-03; rel: 4.953e-04; norm: 1.190e+00
## It: 68; obj: 1.397e+01; abs: 6.501e-03; rel: 4.654e-04; norm: 1.192e+00
## It: 69; obj: 1.398e+01; abs: 6.106e-03; rel: 4.369e-04; norm: 1.193e+00
## It: 70; obj: 1.399e+01; abs: 5.728e-03; rel: 4.097e-04; norm: 1.194e+00
## It: 71; obj: 1.399e+01; abs: 5.368e-03; rel: 3.838e-04; norm: 1.195e+00
## It: 72; obj: 1.400e+01; abs: 5.025e-03; rel: 3.591e-04; norm: 1.197e+00
## It: 73; obj: 1.400e+01; abs: 4.698e-03; rel: 3.356e-04; norm: 1.198e+00
## It: 74; obj: 1.401e+01; abs: 4.387e-03; rel: 3.133e-04; norm: 1.199e+00
## It: 75; obj: 1.401e+01; abs: 4.091e-03; rel: 2.921e-04; norm: 1.199e+00

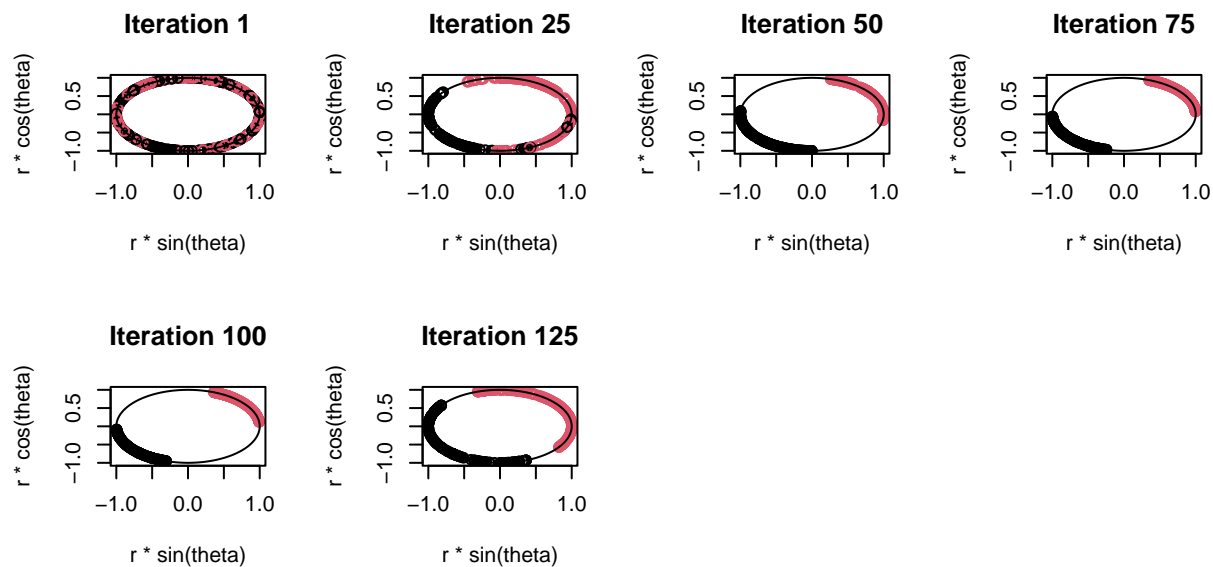
```

```

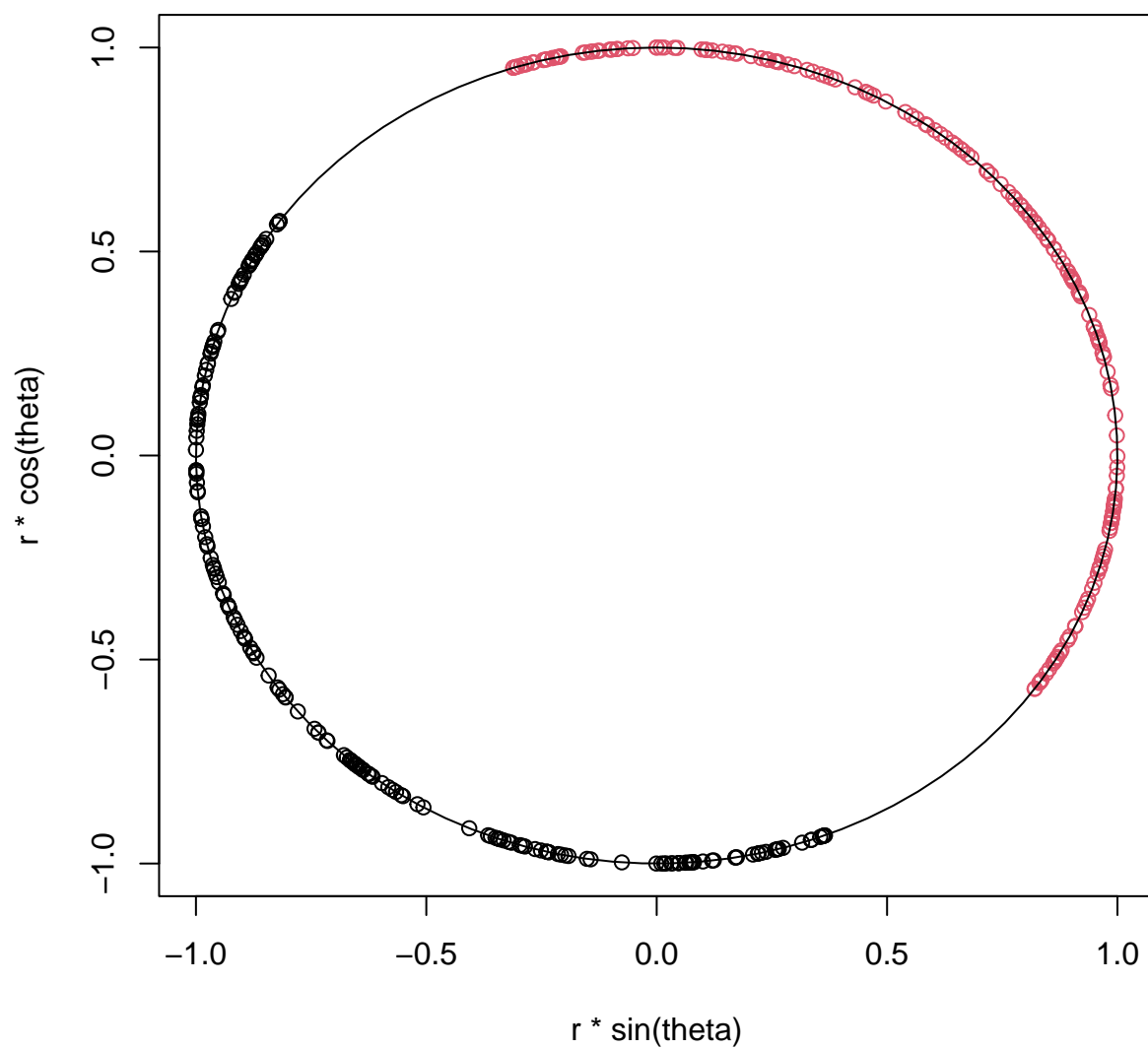
## It: 76; obj: 1.401e+01; abs: 3.810e-03; rel: 2.719e-04; norm: 1.200e+00
## It: 77; obj: 1.402e+01; abs: 3.543e-03; rel: 2.528e-04; norm: 1.201e+00
## It: 78; obj: 1.402e+01; abs: 3.288e-03; rel: 2.346e-04; norm: 1.202e+00
## It: 79; obj: 1.402e+01; abs: 3.046e-03; rel: 2.173e-04; norm: 1.202e+00
## It: 80; obj: 1.403e+01; abs: 2.816e-03; rel: 2.008e-04; norm: 1.203e+00
## It: 81; obj: 1.403e+01; abs: 2.597e-03; rel: 1.851e-04; norm: 1.204e+00
## It: 82; obj: 1.403e+01; abs: 2.388e-03; rel: 1.702e-04; norm: 1.204e+00
## It: 83; obj: 1.403e+01; abs: 2.188e-03; rel: 1.559e-04; norm: 1.205e+00
## It: 84; obj: 1.404e+01; abs: 1.997e-03; rel: 1.423e-04; norm: 1.205e+00
## It: 85; obj: 1.404e+01; abs: 1.814e-03; rel: 1.293e-04; norm: 1.206e+00
## It: 86; obj: 1.404e+01; abs: 1.639e-03; rel: 1.168e-04; norm: 1.206e+00
## It: 87; obj: 1.404e+01; abs: 1.471e-03; rel: 1.048e-04; norm: 1.206e+00
## It: 88; obj: 1.404e+01; abs: 1.308e-03; rel: 9.319e-05; norm: 1.207e+00
## It: 89; obj: 1.404e+01; abs: 1.152e-03; rel: 8.202e-05; norm: 1.207e+00
## It: 90; obj: 1.404e+01; abs: 1.000e-03; rel: 7.121e-05; norm: 1.207e+00
## It: 91; obj: 1.404e+01; abs: 8.528e-04; rel: 6.073e-05; norm: 1.208e+00
## It: 92; obj: 1.405e+01; abs: 7.097e-04; rel: 5.053e-05; norm: 1.208e+00
## It: 93; obj: 1.405e+01; abs: 5.701e-04; rel: 4.059e-05; norm: 1.208e+00
## It: 94; obj: 1.405e+01; abs: 4.337e-04; rel: 3.088e-05; norm: 1.208e+00
## It: 95; obj: 1.405e+01; abs: 3.003e-04; rel: 2.138e-05; norm: 1.209e+00
## It: 96; obj: 1.405e+01; abs: 1.695e-04; rel: 1.207e-05; norm: 1.209e+00
## It: 97; obj: 1.405e+01; abs: 4.120e-05; rel: 2.933e-06; norm: 1.209e+00
## It: 98; obj: 1.405e+01; abs: 8.468e-05; rel: 6.028e-06; norm: 1.209e+00
## It: 99; obj: 1.405e+01; abs: 2.081e-04; rel: 1.482e-05; norm: 1.209e+00
## It: 100; obj: 1.405e+01; abs: 3.290e-04; rel: 2.342e-05; norm: 1.209e+00

## It: 101; obj: 2.054e+00; abs: 1.199e+01; rel: 8.538e-01; norm: 1.189e-01
## It: 102; obj: 2.008e+00; abs: 4.568e-02; rel: 2.224e-02; norm: 1.239e-01
## It: 103; obj: 1.987e+00; abs: 2.142e-02; rel: 1.067e-02; norm: 1.289e-01
## It: 104; obj: 1.979e+00; abs: 7.283e-03; rel: 3.666e-03; norm: 1.316e-01
## It: 105; obj: 1.977e+00; abs: 2.661e-03; rel: 1.344e-03; norm: 1.325e-01
## It: 106; obj: 1.975e+00; abs: 1.767e-03; rel: 8.941e-04; norm: 1.329e-01
## It: 107; obj: 1.974e+00; abs: 5.447e-04; rel: 2.758e-04; norm: 1.333e-01
## It: 108; obj: 1.973e+00; abs: 1.114e-03; rel: 5.641e-04; norm: 1.334e-01
## It: 109; obj: 1.973e+00; abs: 4.954e-04; rel: 2.511e-04; norm: 1.335e-01
## It: 110; obj: 1.972e+00; abs: 2.415e-04; rel: 1.224e-04; norm: 1.335e-01
## It: 111; obj: 1.972e+00; abs: 6.829e-04; rel: 3.462e-04; norm: 1.334e-01
## It: 112; obj: 1.972e+00; abs: 2.350e-05; rel: 1.192e-05; norm: 1.335e-01
## It: 113; obj: 1.971e+00; abs: 4.391e-04; rel: 2.227e-04; norm: 1.336e-01
## It: 114; obj: 1.971e+00; abs: 3.362e-04; rel: 1.706e-04; norm: 1.336e-01
## It: 115; obj: 1.971e+00; abs: 5.011e-05; rel: 2.543e-05; norm: 1.336e-01
## It: 116; obj: 1.970e+00; abs: 3.933e-04; rel: 1.995e-04; norm: 1.335e-01
## It: 117; obj: 1.970e+00; abs: 6.176e-05; rel: 3.134e-05; norm: 1.336e-01
## It: 118; obj: 1.970e+00; abs: 1.760e-04; rel: 8.931e-05; norm: 1.336e-01
## It: 119; obj: 1.970e+00; abs: 2.874e-04; rel: 1.459e-04; norm: 1.337e-01
## It: 120; obj: 1.970e+00; abs: 1.884e-05; rel: 9.561e-06; norm: 1.337e-01
## It: 121; obj: 1.970e+00; abs: 2.270e-04; rel: 1.152e-04; norm: 1.336e-01
## It: 122; obj: 1.970e+00; abs: 1.017e-04; rel: 5.166e-05; norm: 1.336e-01
## It: 123; obj: 1.970e+00; abs: 8.819e-05; rel: 4.478e-05; norm: 1.337e-01
## It: 124; obj: 1.969e+00; abs: 2.278e-04; rel: 1.157e-04; norm: 1.337e-01
## It: 125; obj: 1.969e+00; abs: 3.016e-05; rel: 1.532e-05; norm: 1.337e-01

```



Iteration 125



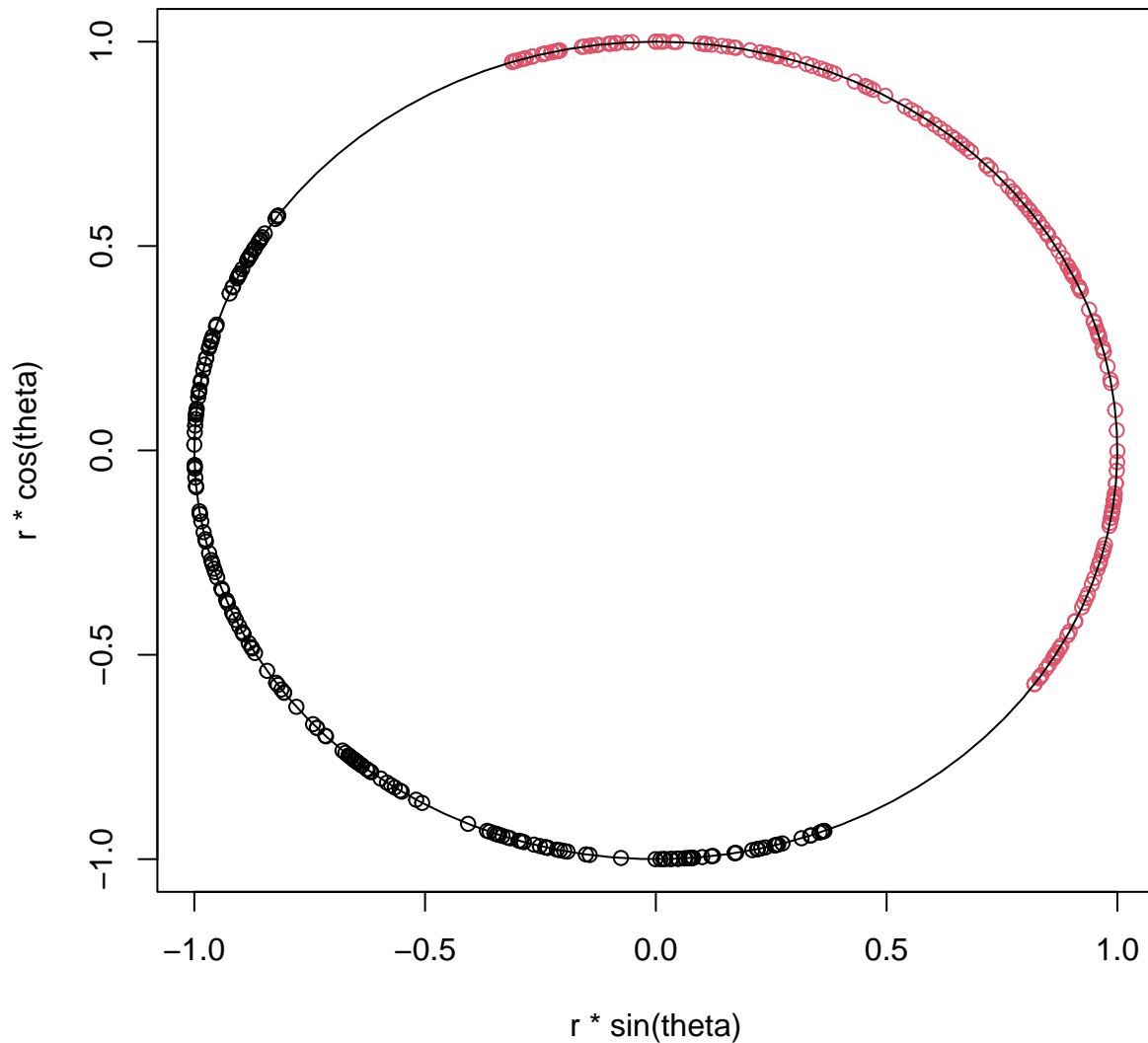
```

Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad

plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))

```



Now we are going to reduce to dimension  $\mathbb{S}^2$  then  $d = 2$  (sphere):

```

Y <- psc_sne(X=x_array, d=2, rho_psc_list = rho_first_20, num_iteration=125,
             colors=colors, visualize_prog = TRUE, eta=100, check=F)

```

```

## It: 1; obj: 1.694e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.993e-01
## It: 2; obj: 1.401e+01; abs: 2.931e+00; rel: 1.730e-01; norm: 5.234e-01
## It: 3; obj: 1.299e+01; abs: 1.021e+00; rel: 7.286e-02; norm: 8.625e-01

```

```

## It: 4; obj: 1.253e+01; abs: 4.641e-01; rel: 3.572e-02; norm: 1.013e+00
## It: 5; obj: 1.226e+01; abs: 2.708e-01; rel: 2.161e-02; norm: 1.093e+00
## It: 6; obj: 1.206e+01; abs: 1.990e-01; rel: 1.624e-02; norm: 1.143e+00
## It: 7; obj: 1.189e+01; abs: 1.690e-01; rel: 1.401e-02; norm: 1.180e+00
## It: 8; obj: 1.173e+01; abs: 1.624e-01; rel: 1.366e-02; norm: 1.208e+00
## It: 9; obj: 1.157e+01; abs: 1.557e-01; rel: 1.327e-02; norm: 1.234e+00
## It: 10; obj: 1.144e+01; abs: 1.291e-01; rel: 1.115e-02; norm: 1.260e+00
## It: 11; obj: 1.134e+01; abs: 1.064e-01; rel: 9.299e-03; norm: 1.287e+00
## It: 12; obj: 1.124e+01; abs: 9.079e-02; rel: 8.009e-03; norm: 1.316e+00
## It: 13; obj: 1.117e+01; abs: 7.844e-02; rel: 6.975e-03; norm: 1.345e+00
## It: 14; obj: 1.110e+01; abs: 6.258e-02; rel: 5.605e-03; norm: 1.374e+00
## It: 15; obj: 1.106e+01; abs: 4.288e-02; rel: 3.862e-03; norm: 1.401e+00
## It: 16; obj: 1.104e+01; abs: 2.424e-02; rel: 2.191e-03; norm: 1.427e+00
## It: 17; obj: 1.103e+01; abs: 9.986e-03; rel: 9.048e-04; norm: 1.453e+00
## It: 18; obj: 1.103e+01; abs: 4.857e-04; rel: 4.404e-05; norm: 1.481e+00
## It: 19; obj: 1.103e+01; abs: 5.843e-03; rel: 5.299e-04; norm: 1.512e+00
## It: 20; obj: 1.104e+01; abs: 1.085e-02; rel: 9.836e-04; norm: 1.544e+00
## It: 21; obj: 1.106e+01; abs: 1.545e-02; rel: 1.399e-03; norm: 1.578e+00
## It: 22; obj: 1.108e+01; abs: 1.924e-02; rel: 1.740e-03; norm: 1.613e+00
## It: 23; obj: 1.110e+01; abs: 2.158e-02; rel: 1.948e-03; norm: 1.648e+00
## It: 24; obj: 1.112e+01; abs: 2.250e-02; rel: 2.027e-03; norm: 1.682e+00
## It: 25; obj: 1.114e+01; abs: 2.251e-02; rel: 2.024e-03; norm: 1.715e+00

## It: 26; obj: 1.117e+01; abs: 2.212e-02; rel: 1.985e-03; norm: 1.746e+00
## It: 27; obj: 1.119e+01; abs: 2.164e-02; rel: 1.938e-03; norm: 1.777e+00
## It: 28; obj: 1.121e+01; abs: 2.122e-02; rel: 1.897e-03; norm: 1.807e+00
## It: 29; obj: 1.123e+01; abs: 2.093e-02; rel: 1.867e-03; norm: 1.835e+00
## It: 30; obj: 1.125e+01; abs: 2.084e-02; rel: 1.856e-03; norm: 1.862e+00
## It: 31; obj: 1.127e+01; abs: 2.097e-02; rel: 1.864e-03; norm: 1.887e+00
## It: 32; obj: 1.129e+01; abs: 2.133e-02; rel: 1.892e-03; norm: 1.912e+00
## It: 33; obj: 1.132e+01; abs: 2.186e-02; rel: 1.935e-03; norm: 1.935e+00
## It: 34; obj: 1.134e+01; abs: 2.250e-02; rel: 1.988e-03; norm: 1.956e+00
## It: 35; obj: 1.136e+01; abs: 2.318e-02; rel: 2.044e-03; norm: 1.976e+00
## It: 36; obj: 1.138e+01; abs: 2.382e-02; rel: 2.097e-03; norm: 1.995e+00
## It: 37; obj: 1.141e+01; abs: 2.438e-02; rel: 2.142e-03; norm: 2.013e+00
## It: 38; obj: 1.143e+01; abs: 2.481e-02; rel: 2.175e-03; norm: 2.030e+00
## It: 39; obj: 1.146e+01; abs: 2.509e-02; rel: 2.195e-03; norm: 2.046e+00
## It: 40; obj: 1.148e+01; abs: 2.522e-02; rel: 2.201e-03; norm: 2.060e+00
## It: 41; obj: 1.151e+01; abs: 2.518e-02; rel: 2.193e-03; norm: 2.074e+00
## It: 42; obj: 1.153e+01; abs: 2.499e-02; rel: 2.172e-03; norm: 2.087e+00
## It: 43; obj: 1.156e+01; abs: 2.467e-02; rel: 2.139e-03; norm: 2.098e+00
## It: 44; obj: 1.158e+01; abs: 2.422e-02; rel: 2.095e-03; norm: 2.109e+00
## It: 45; obj: 1.161e+01; abs: 2.366e-02; rel: 2.042e-03; norm: 2.120e+00
## It: 46; obj: 1.163e+01; abs: 2.300e-02; rel: 1.982e-03; norm: 2.129e+00
## It: 47; obj: 1.165e+01; abs: 2.227e-02; rel: 1.915e-03; norm: 2.138e+00
## It: 48; obj: 1.167e+01; abs: 2.148e-02; rel: 1.843e-03; norm: 2.146e+00
## It: 49; obj: 1.169e+01; abs: 2.063e-02; rel: 1.768e-03; norm: 2.154e+00
## It: 50; obj: 1.171e+01; abs: 1.976e-02; rel: 1.690e-03; norm: 2.161e+00

## It: 51; obj: 1.173e+01; abs: 1.886e-02; rel: 1.610e-03; norm: 2.168e+00
## It: 52; obj: 1.175e+01; abs: 1.795e-02; rel: 1.530e-03; norm: 2.174e+00
## It: 53; obj: 1.177e+01; abs: 1.703e-02; rel: 1.450e-03; norm: 2.179e+00
## It: 54; obj: 1.178e+01; abs: 1.613e-02; rel: 1.371e-03; norm: 2.184e+00
## It: 55; obj: 1.180e+01; abs: 1.523e-02; rel: 1.293e-03; norm: 2.189e+00
## It: 56; obj: 1.181e+01; abs: 1.436e-02; rel: 1.217e-03; norm: 2.193e+00

```

```

## It: 57; obj: 1.183e+01; abs: 1.351e-02; rel: 1.144e-03; norm: 2.197e+00
## It: 58; obj: 1.184e+01; abs: 1.269e-02; rel: 1.073e-03; norm: 2.201e+00
## It: 59; obj: 1.185e+01; abs: 1.190e-02; rel: 1.005e-03; norm: 2.205e+00
## It: 60; obj: 1.186e+01; abs: 1.114e-02; rel: 9.397e-04; norm: 2.208e+00
## It: 61; obj: 1.187e+01; abs: 1.041e-02; rel: 8.777e-04; norm: 2.211e+00
## It: 62; obj: 1.188e+01; abs: 9.723e-03; rel: 8.189e-04; norm: 2.213e+00
## It: 63; obj: 1.189e+01; abs: 9.070e-03; rel: 7.633e-04; norm: 2.216e+00
## It: 64; obj: 1.190e+01; abs: 8.451e-03; rel: 7.107e-04; norm: 2.218e+00
## It: 65; obj: 1.191e+01; abs: 7.868e-03; rel: 6.611e-04; norm: 2.220e+00
## It: 66; obj: 1.192e+01; abs: 7.318e-03; rel: 6.146e-04; norm: 2.222e+00
## It: 67; obj: 1.192e+01; abs: 6.802e-03; rel: 5.708e-04; norm: 2.224e+00
## It: 68; obj: 1.193e+01; abs: 6.318e-03; rel: 5.299e-04; norm: 2.225e+00
## It: 69; obj: 1.193e+01; abs: 5.864e-03; rel: 4.916e-04; norm: 2.227e+00
## It: 70; obj: 1.194e+01; abs: 5.439e-03; rel: 4.557e-04; norm: 2.228e+00
## It: 71; obj: 1.195e+01; abs: 5.042e-03; rel: 4.223e-04; norm: 2.229e+00
## It: 72; obj: 1.195e+01; abs: 4.672e-03; rel: 3.911e-04; norm: 2.231e+00
## It: 73; obj: 1.195e+01; abs: 4.327e-03; rel: 3.621e-04; norm: 2.232e+00
## It: 74; obj: 1.196e+01; abs: 4.005e-03; rel: 3.350e-04; norm: 2.233e+00
## It: 75; obj: 1.196e+01; abs: 3.706e-03; rel: 3.099e-04; norm: 2.234e+00

## It: 76; obj: 1.197e+01; abs: 3.427e-03; rel: 2.865e-04; norm: 2.234e+00
## It: 77; obj: 1.197e+01; abs: 3.168e-03; rel: 2.648e-04; norm: 2.235e+00
## It: 78; obj: 1.197e+01; abs: 2.927e-03; rel: 2.446e-04; norm: 2.236e+00
## It: 79; obj: 1.197e+01; abs: 2.704e-03; rel: 2.259e-04; norm: 2.237e+00
## It: 80; obj: 1.198e+01; abs: 2.497e-03; rel: 2.085e-04; norm: 2.237e+00
## It: 81; obj: 1.198e+01; abs: 2.304e-03; rel: 1.924e-04; norm: 2.238e+00
## It: 82; obj: 1.198e+01; abs: 2.126e-03; rel: 1.774e-04; norm: 2.239e+00
## It: 83; obj: 1.198e+01; abs: 1.960e-03; rel: 1.636e-04; norm: 2.239e+00
## It: 84; obj: 1.198e+01; abs: 1.807e-03; rel: 1.508e-04; norm: 2.240e+00
## It: 85; obj: 1.199e+01; abs: 1.665e-03; rel: 1.389e-04; norm: 2.240e+00
## It: 86; obj: 1.199e+01; abs: 1.533e-03; rel: 1.279e-04; norm: 2.240e+00
## It: 87; obj: 1.199e+01; abs: 1.411e-03; rel: 1.177e-04; norm: 2.241e+00
## It: 88; obj: 1.199e+01; abs: 1.298e-03; rel: 1.083e-04; norm: 2.241e+00
## It: 89; obj: 1.199e+01; abs: 1.194e-03; rel: 9.956e-05; norm: 2.242e+00
## It: 90; obj: 1.199e+01; abs: 1.097e-03; rel: 9.148e-05; norm: 2.242e+00
## It: 91; obj: 1.199e+01; abs: 1.008e-03; rel: 8.402e-05; norm: 2.242e+00
## It: 92; obj: 1.199e+01; abs: 9.248e-04; rel: 7.711e-05; norm: 2.243e+00
## It: 93; obj: 1.200e+01; abs: 8.483e-04; rel: 7.072e-05; norm: 2.243e+00
## It: 94; obj: 1.200e+01; abs: 7.775e-04; rel: 6.481e-05; norm: 2.243e+00
## It: 95; obj: 1.200e+01; abs: 7.120e-04; rel: 5.935e-05; norm: 2.243e+00
## It: 96; obj: 1.200e+01; abs: 6.515e-04; rel: 5.430e-05; norm: 2.244e+00
## It: 97; obj: 1.200e+01; abs: 5.955e-04; rel: 4.964e-05; norm: 2.244e+00
## It: 98; obj: 1.200e+01; abs: 5.438e-04; rel: 4.532e-05; norm: 2.244e+00
## It: 99; obj: 1.200e+01; abs: 4.960e-04; rel: 4.134e-05; norm: 2.244e+00
## It: 100; obj: 1.200e+01; abs: 4.519e-04; rel: 3.766e-05; norm: 2.244e+00

## It: 101; obj: 1.397e+00; abs: 1.060e+01; rel: 8.836e-01; norm: 1.861e-01
## It: 102; obj: 1.247e+00; abs: 1.500e-01; rel: 1.074e-01; norm: 1.728e-01
## It: 103; obj: 1.190e+00; abs: 5.677e-02; rel: 4.552e-02; norm: 1.852e-01
## It: 104; obj: 1.176e+00; abs: 1.434e-02; rel: 1.205e-02; norm: 1.920e-01
## It: 105; obj: 1.172e+00; abs: 3.680e-03; rel: 3.130e-03; norm: 1.929e-01
## It: 106; obj: 1.169e+00; abs: 2.928e-03; rel: 2.498e-03; norm: 1.937e-01
## It: 107; obj: 1.170e+00; abs: 3.637e-04; rel: 3.110e-04; norm: 1.941e-01
## It: 108; obj: 1.167e+00; abs: 2.854e-03; rel: 2.440e-03; norm: 1.938e-01
## It: 109; obj: 1.167e+00; abs: 2.474e-04; rel: 2.120e-04; norm: 1.936e-01

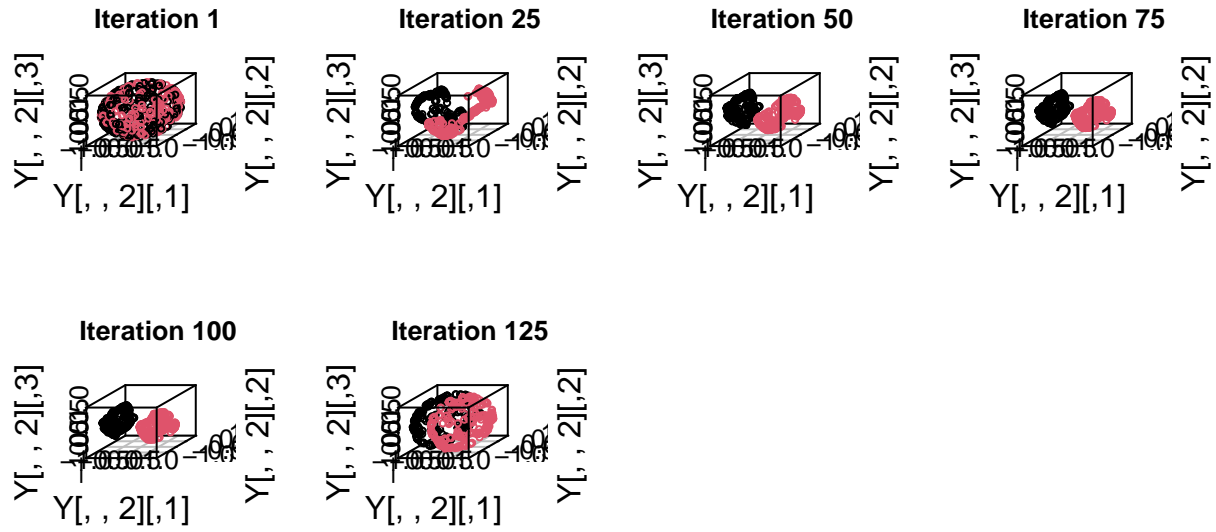
```



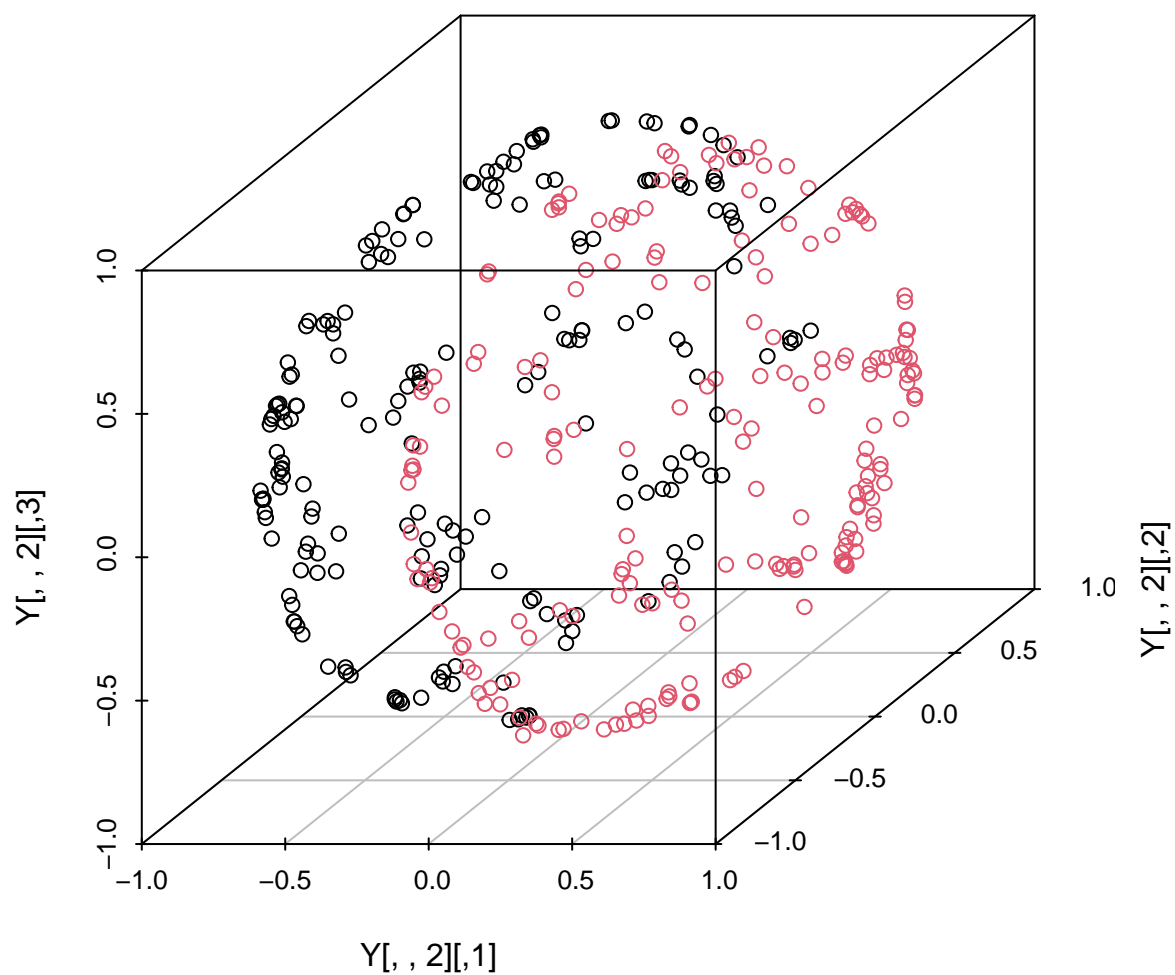
```

## It: 110; obj: 1.167e+00; abs: 2.720e-04; rel: 2.331e-04; norm: 1.935e-01
## It: 111; obj: 1.166e+00; abs: 1.416e-03; rel: 1.213e-03; norm: 1.933e-01
## It: 112; obj: 1.166e+00; abs: 3.838e-04; rel: 3.292e-04; norm: 1.936e-01
## It: 113; obj: 1.166e+00; abs: 4.678e-04; rel: 4.011e-04; norm: 1.936e-01
## It: 114; obj: 1.166e+00; abs: 3.310e-04; rel: 2.840e-04; norm: 1.935e-01
## It: 115; obj: 1.166e+00; abs: 1.657e-05; rel: 1.422e-05; norm: 1.934e-01
## It: 116; obj: 1.165e+00; abs: 5.669e-05; rel: 4.864e-05; norm: 1.933e-01
## It: 117; obj: 1.165e+00; abs: 1.506e-04; rel: 1.292e-04; norm: 1.934e-01
## It: 118; obj: 1.165e+00; abs: 1.208e-04; rel: 1.037e-04; norm: 1.934e-01
## It: 119; obj: 1.165e+00; abs: 1.318e-04; rel: 1.131e-04; norm: 1.935e-01
## It: 120; obj: 1.165e+00; abs: 3.133e-04; rel: 2.689e-04; norm: 1.934e-01
## It: 121; obj: 1.165e+00; abs: 5.174e-05; rel: 4.441e-05; norm: 1.934e-01
## It: 122; obj: 1.165e+00; abs: 3.115e-05; rel: 2.674e-05; norm: 1.934e-01
## It: 123; obj: 1.165e+00; abs: 1.942e-04; rel: 1.667e-04; norm: 1.934e-01
## It: 124; obj: 1.165e+00; abs: 9.806e-05; rel: 8.418e-05; norm: 1.934e-01
## It: 125; obj: 1.165e+00; abs: 5.120e-05; rel: 4.395e-05; norm: 1.934e-01

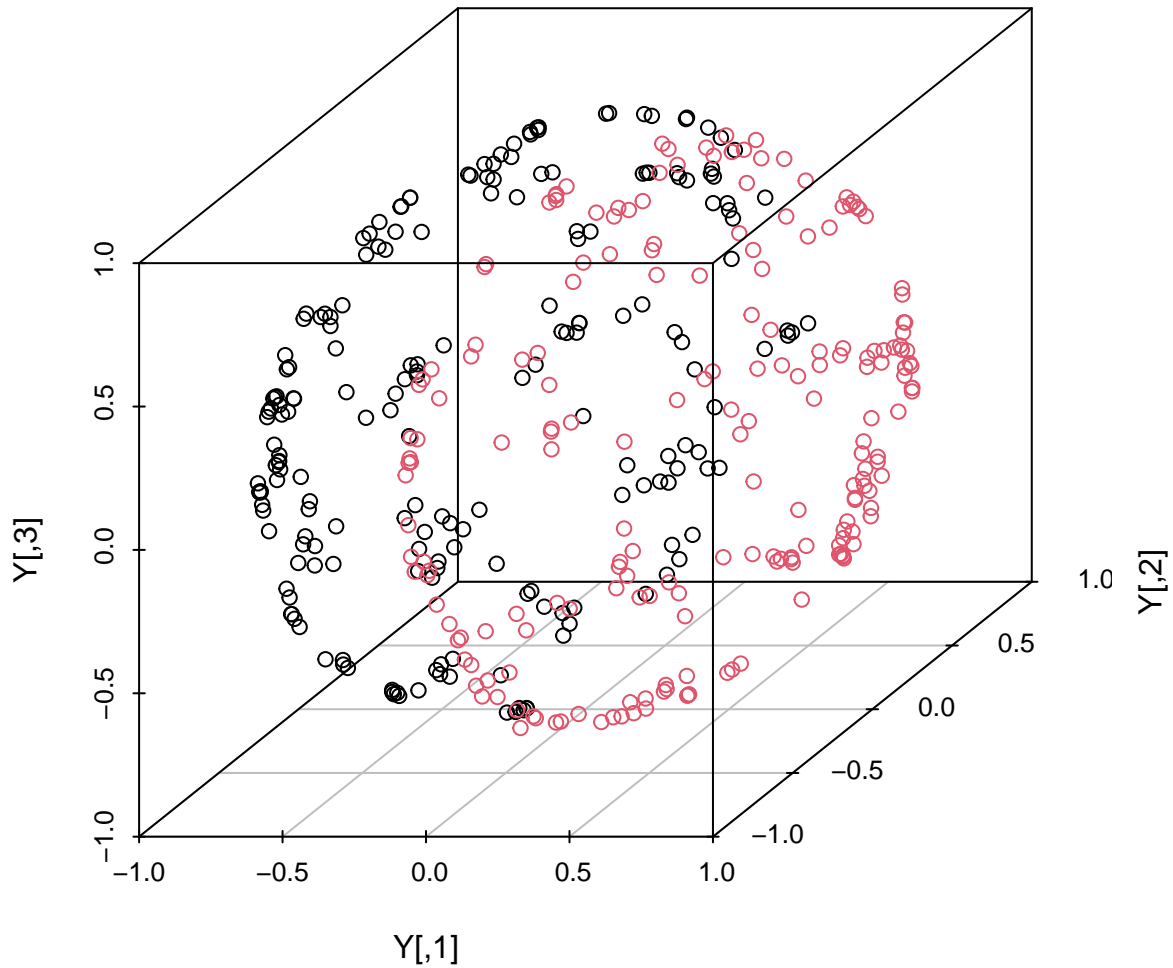
```



## Iteration 125



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1),  
                             zlim = c(-1, 1), color = colors)
```



It's clearly clustered the two groups we had in the original generated data.

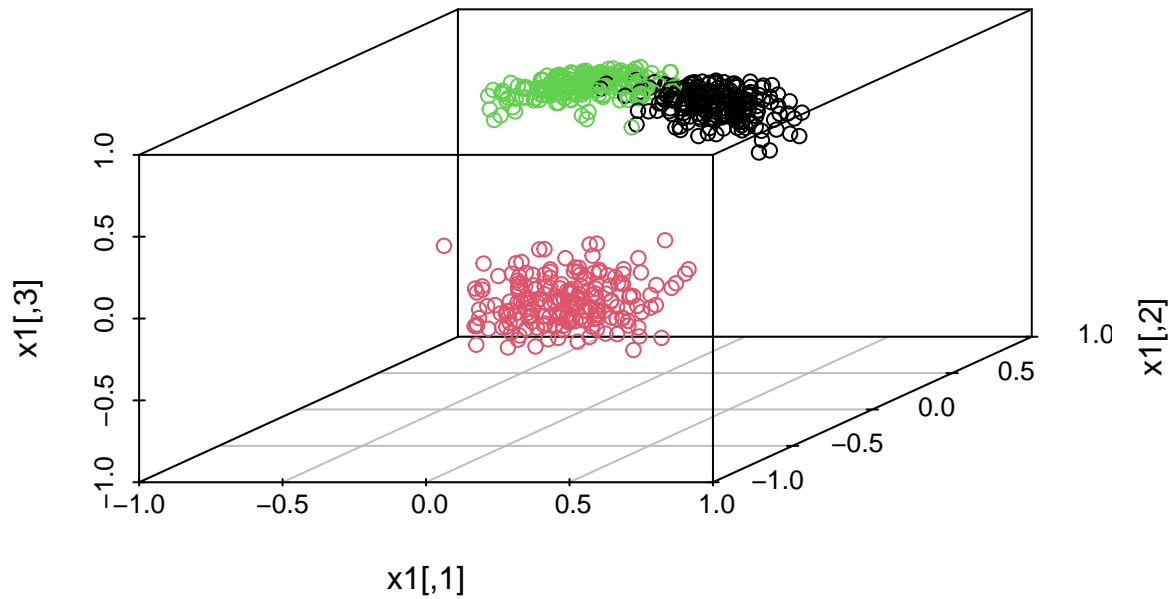
## Case 2

Sample on the  $(\mathbb{S}^2)^2$  where  $p = 2$  and  $r = 2$ .

```
n=200
d=2
r=2
samp1 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 0, ph = 0.5)),
                        kappa = 50)
samp2 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 2, ph = -1.5)),
                        kappa = 50)
samp3 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = -1, ph = 0)),
                        kappa = 50)
x1 <- rbind(samp1, samp2, samp3)

scatterplot3d::scatterplot3d(x1,
                             xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(1:3, each = n), main="Sphere 1")
```

## Sphere 1

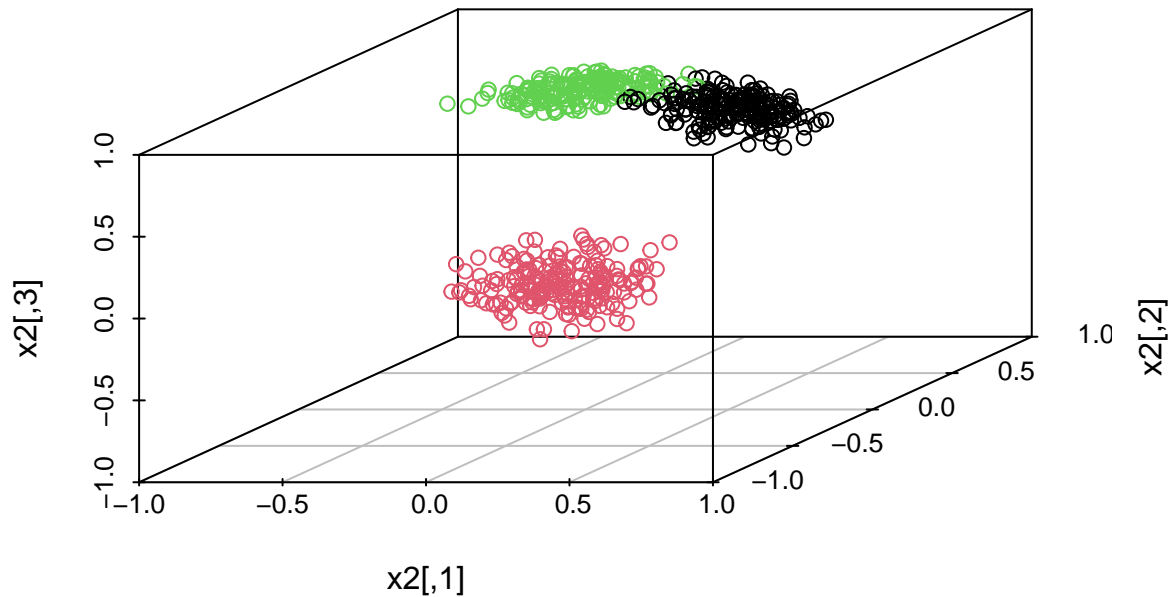


```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue", alpha = 0.25,
            lit = FALSE)
rgl::points3d(x1, col = rep(c(1,2,3), each=n))

samp4 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 0, ph = 0.55)),
                        kappa = 60)
samp5 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 2, ph = -1.41)),
                        kappa = 60)
samp6 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = -1.1, ph = 0.05)),
                        kappa = 60)
x2 <- rbind(samp4, samp5, samp6)

scatterplot3d::scatterplot3d(x2,
                              xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                              color = rep(1:3, each = n), main="Sphere 2")
```

## Sphere 2



```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue", alpha = 0.25,
            lit = FALSE)
rgl::points3d(x2, col = rep(c(1,2,3), each=n))

x_2 <- array(dim = c(n*3, 3, 2))
x_2[, , 1] <- x1
x_2[, , 2] <- x2

n <- nrow(x_2)
indexes <- sample(1:n)
x_2 <- x_2[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]
```

Let's calculate the rho parameters based on a perplexity of 20:

```
rho_second_perp20 <- rho_optim_bst(x_2, 20)
```

## Time difference of 54.93487 secs

First, let's reduce to dimension  $\mathbb{S}^1$  then  $d = 1$  (circumference):

```
Y <- psc_sne(X=x_2, d=1, rho_psc_list = rho_second_perp20, num_iteration=200,
            colors=colors, visualize_prog = TRUE)
```

```
## It: 1; obj: 1.851e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.424e-01
## It: 2; obj: 1.780e+01; abs: 7.079e-01; rel: 3.824e-02; norm: 3.051e-01
## It: 3; obj: 1.736e+01; abs: 4.462e-01; rel: 2.506e-02; norm: 3.981e-01
## It: 4; obj: 1.705e+01; abs: 3.080e-01; rel: 1.774e-02; norm: 4.613e-01
## It: 5; obj: 1.684e+01; abs: 2.072e-01; rel: 1.215e-02; norm: 5.055e-01
## It: 6; obj: 1.671e+01; abs: 1.366e-01; rel: 8.110e-03; norm: 5.439e-01
## It: 7; obj: 1.660e+01; abs: 1.036e-01; rel: 6.202e-03; norm: 5.759e-01
```

```

## It: 8; obj: 1.651e+01; abs: 9.426e-02; rel: 5.677e-03; norm: 6.044e-01
## It: 9; obj: 1.642e+01; abs: 8.671e-02; rel: 5.253e-03; norm: 6.277e-01
## It: 10; obj: 1.634e+01; abs: 8.590e-02; rel: 5.231e-03; norm: 6.454e-01
## It: 11; obj: 1.627e+01; abs: 7.035e-02; rel: 4.307e-03; norm: 6.618e-01
## It: 12; obj: 1.621e+01; abs: 5.669e-02; rel: 3.485e-03; norm: 6.757e-01
## It: 13; obj: 1.616e+01; abs: 4.801e-02; rel: 2.962e-03; norm: 6.875e-01
## It: 14; obj: 1.612e+01; abs: 4.343e-02; rel: 2.687e-03; norm: 6.977e-01
## It: 15; obj: 1.608e+01; abs: 4.070e-02; rel: 2.525e-03; norm: 7.072e-01
## It: 16; obj: 1.604e+01; abs: 3.751e-02; rel: 2.333e-03; norm: 7.165e-01
## It: 17; obj: 1.601e+01; abs: 3.248e-02; rel: 2.025e-03; norm: 7.257e-01
## It: 18; obj: 1.598e+01; abs: 2.763e-02; rel: 1.726e-03; norm: 7.349e-01
## It: 19; obj: 1.596e+01; abs: 2.409e-02; rel: 1.507e-03; norm: 7.442e-01
## It: 20; obj: 1.593e+01; abs: 2.115e-02; rel: 1.326e-03; norm: 7.535e-01
## It: 21; obj: 1.591e+01; abs: 2.108e-02; rel: 1.323e-03; norm: 7.627e-01
## It: 22; obj: 1.589e+01; abs: 2.210e-02; rel: 1.389e-03; norm: 7.715e-01
## It: 23; obj: 1.587e+01; abs: 2.116e-02; rel: 1.332e-03; norm: 7.801e-01
## It: 24; obj: 1.585e+01; abs: 1.707e-02; rel: 1.076e-03; norm: 7.885e-01
## It: 25; obj: 1.584e+01; abs: 1.220e-02; rel: 7.697e-04; norm: 7.968e-01

## It: 26; obj: 1.583e+01; abs: 1.027e-02; rel: 6.482e-04; norm: 8.052e-01
## It: 27; obj: 1.582e+01; abs: 1.106e-02; rel: 6.986e-04; norm: 8.137e-01
## It: 28; obj: 1.581e+01; abs: 1.198e-02; rel: 7.574e-04; norm: 8.223e-01
## It: 29; obj: 1.579e+01; abs: 1.555e-02; rel: 9.836e-04; norm: 8.309e-01
## It: 30; obj: 1.577e+01; abs: 2.304e-02; rel: 1.459e-03; norm: 8.395e-01
## It: 31; obj: 1.573e+01; abs: 3.372e-02; rel: 2.139e-03; norm: 8.484e-01
## It: 32; obj: 1.569e+01; abs: 4.522e-02; rel: 2.874e-03; norm: 8.576e-01
## It: 33; obj: 1.564e+01; abs: 5.318e-02; rel: 3.390e-03; norm: 8.671e-01
## It: 34; obj: 1.558e+01; abs: 5.403e-02; rel: 3.455e-03; norm: 8.763e-01
## It: 35; obj: 1.553e+01; abs: 4.851e-02; rel: 3.113e-03; norm: 8.840e-01
## It: 36; obj: 1.549e+01; abs: 4.013e-02; rel: 2.584e-03; norm: 8.895e-01
## It: 37; obj: 1.546e+01; abs: 3.312e-02; rel: 2.138e-03; norm: 8.929e-01
## It: 38; obj: 1.543e+01; abs: 3.233e-02; rel: 2.091e-03; norm: 8.947e-01
## It: 39; obj: 1.539e+01; abs: 3.905e-02; rel: 2.531e-03; norm: 8.960e-01
## It: 40; obj: 1.534e+01; abs: 4.593e-02; rel: 2.984e-03; norm: 8.974e-01
## It: 41; obj: 1.530e+01; abs: 3.992e-02; rel: 2.602e-03; norm: 8.991e-01
## It: 42; obj: 1.528e+01; abs: 2.172e-02; rel: 1.419e-03; norm: 8.999e-01
## It: 43; obj: 1.527e+01; abs: 7.161e-03; rel: 4.686e-04; norm: 8.990e-01
## It: 44; obj: 1.528e+01; abs: 8.423e-04; rel: 5.514e-05; norm: 8.973e-01
## It: 45; obj: 1.528e+01; abs: 4.683e-03; rel: 3.066e-04; norm: 8.954e-01
## It: 46; obj: 1.529e+01; abs: 6.187e-03; rel: 4.049e-04; norm: 8.936e-01
## It: 47; obj: 1.529e+01; abs: 6.440e-03; rel: 4.213e-04; norm: 8.919e-01
## It: 48; obj: 1.530e+01; abs: 6.064e-03; rel: 3.965e-04; norm: 8.902e-01
## It: 49; obj: 1.530e+01; abs: 5.408e-03; rel: 3.535e-04; norm: 8.887e-01
## It: 50; obj: 1.531e+01; abs: 4.660e-03; rel: 3.045e-04; norm: 8.872e-01

## It: 51; obj: 1.531e+01; abs: 3.916e-03; rel: 2.558e-04; norm: 8.858e-01
## It: 52; obj: 1.532e+01; abs: 3.223e-03; rel: 2.105e-04; norm: 8.844e-01
## It: 53; obj: 1.532e+01; abs: 2.599e-03; rel: 1.697e-04; norm: 8.830e-01
## It: 54; obj: 1.532e+01; abs: 2.047e-03; rel: 1.336e-04; norm: 8.817e-01
## It: 55; obj: 1.532e+01; abs: 1.563e-03; rel: 1.020e-04; norm: 8.803e-01
## It: 56; obj: 1.532e+01; abs: 1.141e-03; rel: 7.450e-05; norm: 8.790e-01
## It: 57; obj: 1.532e+01; abs: 7.746e-04; rel: 5.055e-05; norm: 8.777e-01
## It: 58; obj: 1.532e+01; abs: 4.549e-04; rel: 2.968e-05; norm: 8.763e-01
## It: 59; obj: 1.532e+01; abs: 1.756e-04; rel: 1.146e-05; norm: 8.750e-01
## It: 60; obj: 1.532e+01; abs: 6.929e-05; rel: 4.521e-06; norm: 8.736e-01

```

```

## It: 61; obj: 1.532e+01; abs: 2.850e-04; rel: 1.860e-05; norm: 8.723e-01
## It: 62; obj: 1.532e+01; abs: 4.759e-04; rel: 3.105e-05; norm: 8.709e-01
## It: 63; obj: 1.532e+01; abs: 6.457e-04; rel: 4.214e-05; norm: 8.696e-01
## It: 64; obj: 1.532e+01; abs: 7.977e-04; rel: 5.206e-05; norm: 8.682e-01
## It: 65; obj: 1.532e+01; abs: 9.343e-04; rel: 6.098e-05; norm: 8.668e-01
## It: 66; obj: 1.532e+01; abs: 1.058e-03; rel: 6.903e-05; norm: 8.654e-01
## It: 67; obj: 1.532e+01; abs: 1.170e-03; rel: 7.634e-05; norm: 8.641e-01
## It: 68; obj: 1.532e+01; abs: 1.271e-03; rel: 8.300e-05; norm: 8.627e-01
## It: 69; obj: 1.532e+01; abs: 1.364e-03; rel: 8.906e-05; norm: 8.613e-01
## It: 70; obj: 1.532e+01; abs: 1.449e-03; rel: 9.459e-05; norm: 8.599e-01
## It: 71; obj: 1.531e+01; abs: 1.526e-03; rel: 9.963e-05; norm: 8.585e-01
## It: 72; obj: 1.531e+01; abs: 1.596e-03; rel: 1.042e-04; norm: 8.571e-01
## It: 73; obj: 1.531e+01; abs: 1.659e-03; rel: 1.083e-04; norm: 8.557e-01
## It: 74; obj: 1.531e+01; abs: 1.716e-03; rel: 1.121e-04; norm: 8.543e-01
## It: 75; obj: 1.531e+01; abs: 1.766e-03; rel: 1.153e-04; norm: 8.529e-01

## It: 76; obj: 1.530e+01; abs: 1.810e-03; rel: 1.182e-04; norm: 8.515e-01
## It: 77; obj: 1.530e+01; abs: 1.847e-03; rel: 1.207e-04; norm: 8.501e-01
## It: 78; obj: 1.530e+01; abs: 1.878e-03; rel: 1.227e-04; norm: 8.487e-01
## It: 79; obj: 1.530e+01; abs: 1.903e-03; rel: 1.244e-04; norm: 8.473e-01
## It: 80; obj: 1.530e+01; abs: 1.922e-03; rel: 1.256e-04; norm: 8.459e-01
## It: 81; obj: 1.530e+01; abs: 1.934e-03; rel: 1.264e-04; norm: 8.444e-01
## It: 82; obj: 1.529e+01; abs: 1.941e-03; rel: 1.269e-04; norm: 8.430e-01
## It: 83; obj: 1.529e+01; abs: 1.942e-03; rel: 1.270e-04; norm: 8.416e-01
## It: 84; obj: 1.529e+01; abs: 1.937e-03; rel: 1.267e-04; norm: 8.402e-01
## It: 85; obj: 1.529e+01; abs: 1.927e-03; rel: 1.260e-04; norm: 8.388e-01
## It: 86; obj: 1.529e+01; abs: 1.912e-03; rel: 1.251e-04; norm: 8.374e-01
## It: 87; obj: 1.528e+01; abs: 1.893e-03; rel: 1.238e-04; norm: 8.359e-01
## It: 88; obj: 1.528e+01; abs: 1.869e-03; rel: 1.223e-04; norm: 8.345e-01
## It: 89; obj: 1.528e+01; abs: 1.842e-03; rel: 1.205e-04; norm: 8.331e-01
## It: 90; obj: 1.528e+01; abs: 1.812e-03; rel: 1.186e-04; norm: 8.317e-01
## It: 91; obj: 1.528e+01; abs: 1.778e-03; rel: 1.164e-04; norm: 8.303e-01
## It: 92; obj: 1.527e+01; abs: 1.742e-03; rel: 1.141e-04; norm: 8.289e-01
## It: 93; obj: 1.527e+01; abs: 1.705e-03; rel: 1.116e-04; norm: 8.275e-01
## It: 94; obj: 1.527e+01; abs: 1.666e-03; rel: 1.091e-04; norm: 8.261e-01
## It: 95; obj: 1.527e+01; abs: 1.625e-03; rel: 1.064e-04; norm: 8.247e-01
## It: 96; obj: 1.527e+01; abs: 1.584e-03; rel: 1.037e-04; norm: 8.233e-01
## It: 97; obj: 1.527e+01; abs: 1.543e-03; rel: 1.010e-04; norm: 8.219e-01
## It: 98; obj: 1.527e+01; abs: 1.501e-03; rel: 9.834e-05; norm: 8.206e-01
## It: 99; obj: 1.526e+01; abs: 1.460e-03; rel: 9.565e-05; norm: 8.192e-01
## It: 100; obj: 1.526e+01; abs: 1.419e-03; rel: 9.299e-05; norm: 8.178e-01

## It: 101; obj: 2.396e+00; abs: 1.287e+01; rel: 8.430e-01; norm: 8.464e-02
## It: 102; obj: 2.357e+00; abs: 3.857e-02; rel: 1.610e-02; norm: 9.032e-02
## It: 103; obj: 2.335e+00; abs: 2.259e-02; rel: 9.584e-03; norm: 9.841e-02
## It: 104; obj: 2.326e+00; abs: 8.875e-03; rel: 3.801e-03; norm: 1.039e-01
## It: 105; obj: 2.323e+00; abs: 3.200e-03; rel: 1.376e-03; norm: 1.064e-01
## It: 106; obj: 2.321e+00; abs: 1.388e-03; rel: 5.974e-04; norm: 1.074e-01
## It: 107; obj: 2.321e+00; abs: 8.611e-04; rel: 3.709e-04; norm: 1.078e-01
## It: 108; obj: 2.320e+00; abs: 5.271e-04; rel: 2.271e-04; norm: 1.080e-01
## It: 109; obj: 2.320e+00; abs: 3.547e-04; rel: 1.529e-04; norm: 1.082e-01
## It: 110; obj: 2.319e+00; abs: 2.641e-04; rel: 1.138e-04; norm: 1.084e-01
## It: 111; obj: 2.319e+00; abs: 2.173e-04; rel: 9.367e-05; norm: 1.085e-01
## It: 112; obj: 2.319e+00; abs: 1.931e-04; rel: 8.327e-05; norm: 1.087e-01
## It: 113; obj: 2.319e+00; abs: 1.781e-04; rel: 7.679e-05; norm: 1.088e-01

```

```

## It: 114; obj: 2.319e+00; abs: 1.739e-04; rel: 7.498e-05; norm: 1.089e-01
## It: 115; obj: 2.318e+00; abs: 1.652e-04; rel: 7.125e-05; norm: 1.090e-01
## It: 116; obj: 2.318e+00; abs: 1.220e-04; rel: 5.263e-05; norm: 1.091e-01
## It: 117; obj: 2.318e+00; abs: 1.080e-04; rel: 4.656e-05; norm: 1.092e-01
## It: 118; obj: 2.318e+00; abs: 1.258e-04; rel: 5.425e-05; norm: 1.092e-01
## It: 119; obj: 2.318e+00; abs: 8.552e-05; rel: 3.689e-05; norm: 1.093e-01
## It: 120; obj: 2.318e+00; abs: 6.061e-05; rel: 2.615e-05; norm: 1.093e-01
## It: 121; obj: 2.318e+00; abs: 9.370e-05; rel: 4.042e-05; norm: 1.094e-01
## It: 122; obj: 2.318e+00; abs: 6.601e-05; rel: 2.848e-05; norm: 1.094e-01
## It: 123; obj: 2.318e+00; abs: 6.734e-05; rel: 2.905e-05; norm: 1.094e-01
## It: 124; obj: 2.318e+00; abs: 5.680e-05; rel: 2.451e-05; norm: 1.094e-01
## It: 125; obj: 2.318e+00; abs: 4.701e-05; rel: 2.028e-05; norm: 1.094e-01

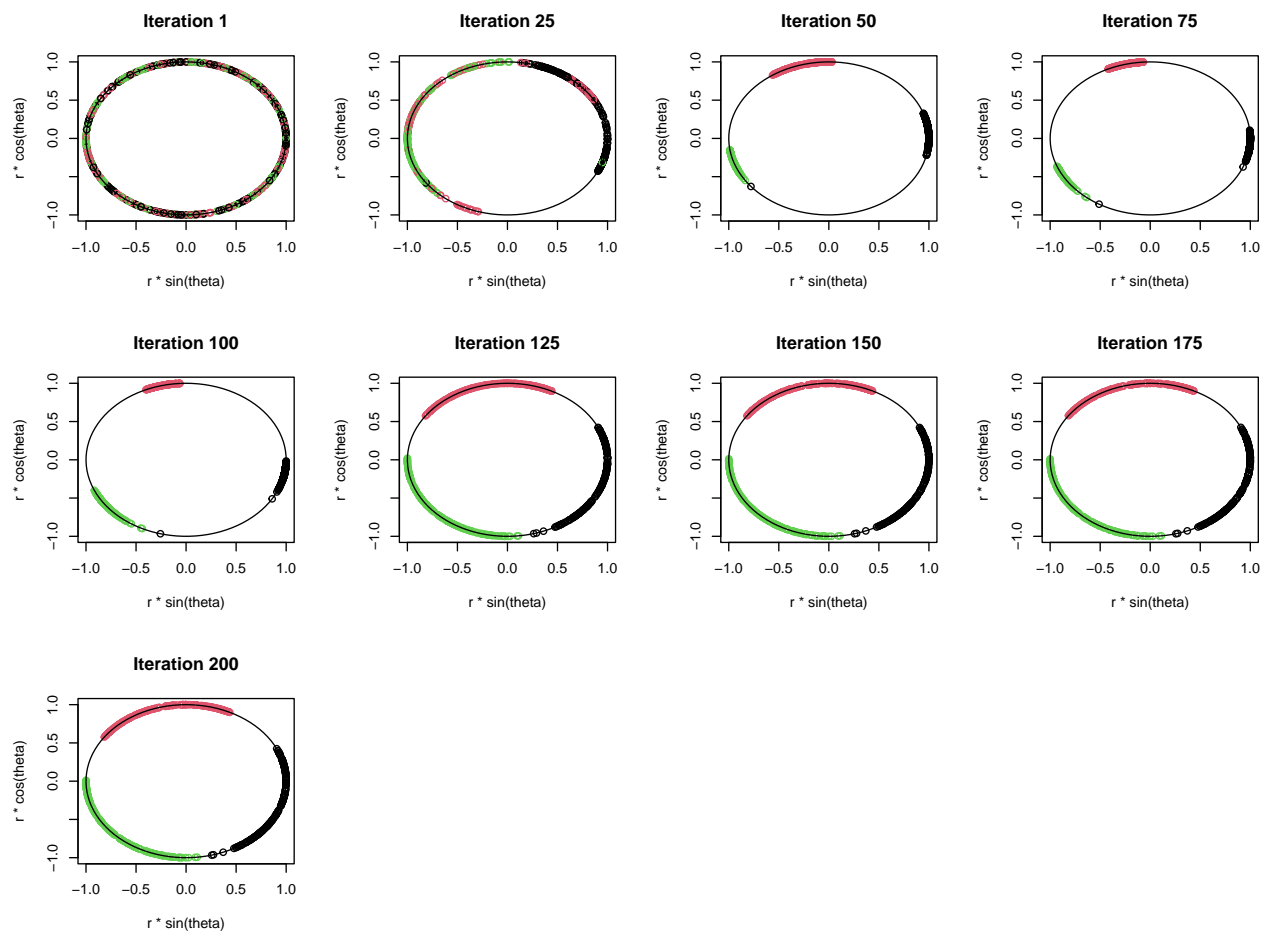
## It: 126; obj: 2.318e+00; abs: 4.543e-05; rel: 1.960e-05; norm: 1.094e-01
## It: 127; obj: 2.318e+00; abs: 4.240e-05; rel: 1.830e-05; norm: 1.094e-01
## It: 128; obj: 2.318e+00; abs: 3.941e-05; rel: 1.701e-05; norm: 1.094e-01
## It: 129; obj: 2.318e+00; abs: 3.623e-05; rel: 1.563e-05; norm: 1.094e-01
## It: 130; obj: 2.317e+00; abs: 3.401e-05; rel: 1.468e-05; norm: 1.094e-01
## It: 131; obj: 2.317e+00; abs: 3.260e-05; rel: 1.407e-05; norm: 1.094e-01
## It: 132; obj: 2.317e+00; abs: 3.059e-05; rel: 1.320e-05; norm: 1.094e-01
## It: 133; obj: 2.317e+00; abs: 2.846e-05; rel: 1.228e-05; norm: 1.094e-01
## It: 134; obj: 2.317e+00; abs: 2.642e-05; rel: 1.140e-05; norm: 1.094e-01
## It: 135; obj: 2.317e+00; abs: 2.462e-05; rel: 1.062e-05; norm: 1.094e-01
## It: 136; obj: 2.317e+00; abs: 2.303e-05; rel: 9.939e-06; norm: 1.094e-01
## It: 137; obj: 2.317e+00; abs: 2.146e-05; rel: 9.260e-06; norm: 1.094e-01
## It: 138; obj: 2.317e+00; abs: 1.989e-05; rel: 8.582e-06; norm: 1.094e-01
## It: 139; obj: 2.317e+00; abs: 1.837e-05; rel: 7.926e-06; norm: 1.094e-01
## It: 140; obj: 2.317e+00; abs: 1.691e-05; rel: 7.296e-06; norm: 1.094e-01
## It: 141; obj: 2.317e+00; abs: 1.549e-05; rel: 6.683e-06; norm: 1.094e-01
## It: 142; obj: 2.317e+00; abs: 1.411e-05; rel: 6.089e-06; norm: 1.094e-01
## It: 143; obj: 2.317e+00; abs: 1.281e-05; rel: 5.528e-06; norm: 1.094e-01
## It: 144; obj: 2.317e+00; abs: 1.161e-05; rel: 5.009e-06; norm: 1.094e-01
## It: 145; obj: 2.317e+00; abs: 1.051e-05; rel: 4.534e-06; norm: 1.094e-01
## It: 146; obj: 2.317e+00; abs: 9.498e-06; rel: 4.099e-06; norm: 1.094e-01
## It: 147; obj: 2.317e+00; abs: 8.577e-06; rel: 3.701e-06; norm: 1.094e-01
## It: 148; obj: 2.317e+00; abs: 7.735e-06; rel: 3.338e-06; norm: 1.094e-01
## It: 149; obj: 2.317e+00; abs: 6.971e-06; rel: 3.008e-06; norm: 1.094e-01
## It: 150; obj: 2.317e+00; abs: 6.277e-06; rel: 2.709e-06; norm: 1.094e-01

## It: 151; obj: 2.317e+00; abs: 5.649e-06; rel: 2.438e-06; norm: 1.094e-01
## It: 152; obj: 2.317e+00; abs: 5.082e-06; rel: 2.193e-06; norm: 1.094e-01
## It: 153; obj: 2.317e+00; abs: 4.573e-06; rel: 1.974e-06; norm: 1.094e-01
## It: 154; obj: 2.317e+00; abs: 4.115e-06; rel: 1.776e-06; norm: 1.094e-01
## It: 155; obj: 2.317e+00; abs: 3.703e-06; rel: 1.598e-06; norm: 1.094e-01
## It: 156; obj: 2.317e+00; abs: 3.332e-06; rel: 1.438e-06; norm: 1.093e-01
## It: 157; obj: 2.317e+00; abs: 2.998e-06; rel: 1.294e-06; norm: 1.093e-01
## It: 158; obj: 2.317e+00; abs: 2.696e-06; rel: 1.163e-06; norm: 1.093e-01
## It: 159; obj: 2.317e+00; abs: 2.423e-06; rel: 1.046e-06; norm: 1.093e-01
## It: 160; obj: 2.317e+00; abs: 2.177e-06; rel: 9.394e-07; norm: 1.093e-01
## It: 161; obj: 2.317e+00; abs: 1.954e-06; rel: 8.434e-07; norm: 1.093e-01
## It: 162; obj: 2.317e+00; abs: 1.753e-06; rel: 7.567e-07; norm: 1.093e-01
## It: 163; obj: 2.317e+00; abs: 1.572e-06; rel: 6.786e-07; norm: 1.093e-01
## It: 164; obj: 2.317e+00; abs: 1.409e-06; rel: 6.082e-07; norm: 1.093e-01
## It: 165; obj: 2.317e+00; abs: 1.262e-06; rel: 5.447e-07; norm: 1.093e-01
## It: 166; obj: 2.317e+00; abs: 1.130e-06; rel: 4.876e-07; norm: 1.093e-01

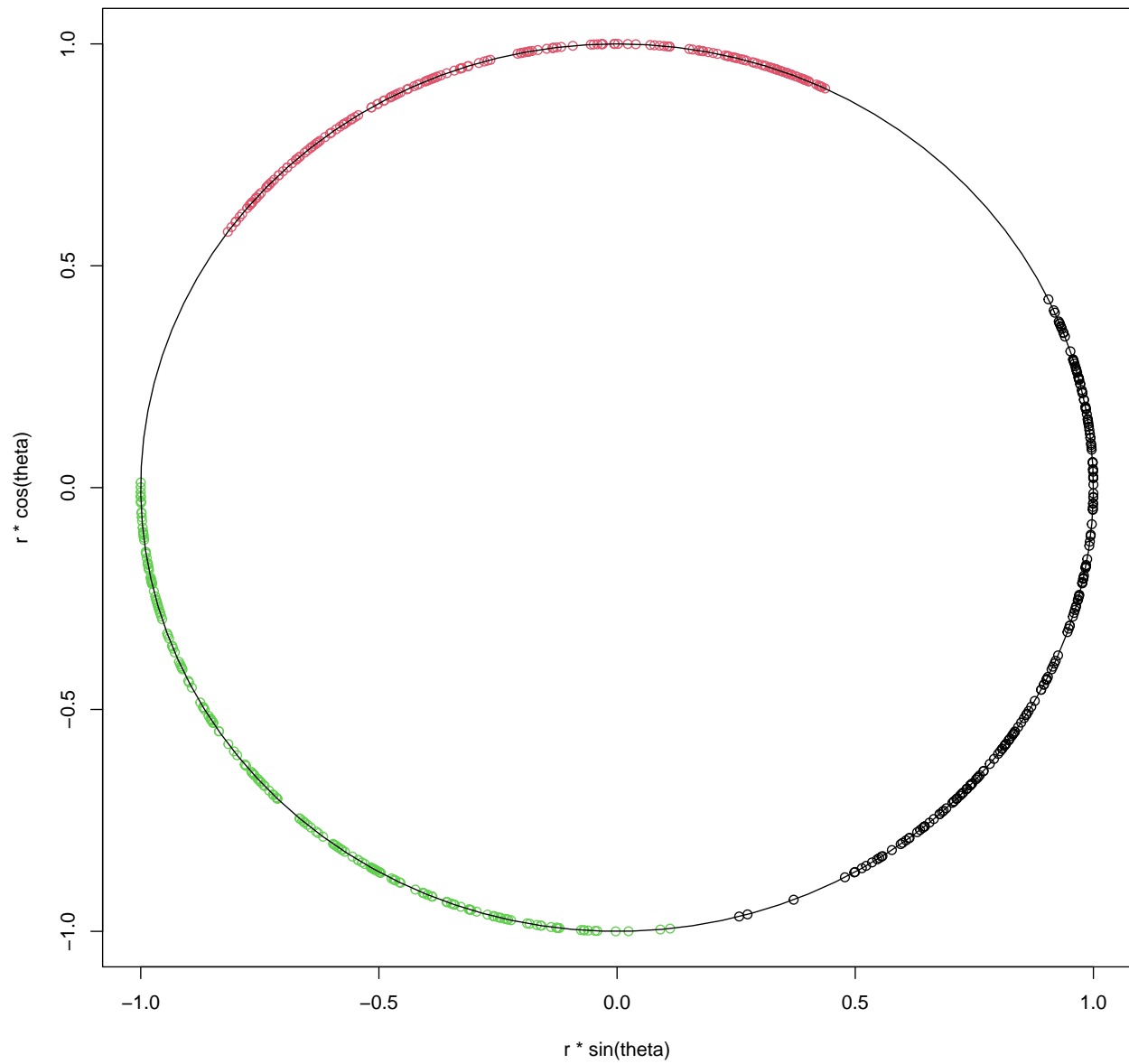
```



## It: 167; obj: 2.317e+00; abs: 1.011e-06; rel: 4.361e-07; norm: 1.093e-01  
## It: 168; obj: 2.317e+00; abs: 9.035e-07; rel: 3.899e-07; norm: 1.093e-01  
## It: 169; obj: 2.317e+00; abs: 8.073e-07; rel: 3.484e-07; norm: 1.093e-01  
## It: 170; obj: 2.317e+00; abs: 7.210e-07; rel: 3.112e-07; norm: 1.093e-01  
## It: 171; obj: 2.317e+00; abs: 6.437e-07; rel: 2.778e-07; norm: 1.093e-01  
## It: 172; obj: 2.317e+00; abs: 5.746e-07; rel: 2.480e-07; norm: 1.093e-01  
## It: 173; obj: 2.317e+00; abs: 5.127e-07; rel: 2.213e-07; norm: 1.093e-01  
## It: 174; obj: 2.317e+00; abs: 4.575e-07; rel: 1.974e-07; norm: 1.093e-01  
## It: 175; obj: 2.317e+00; abs: 4.082e-07; rel: 1.762e-07; norm: 1.093e-01  
  
## It: 176; obj: 2.317e+00; abs: 3.641e-07; rel: 1.572e-07; norm: 1.093e-01  
## It: 177; obj: 2.317e+00; abs: 3.249e-07; rel: 1.402e-07; norm: 1.093e-01  
## It: 178; obj: 2.317e+00; abs: 2.899e-07; rel: 1.251e-07; norm: 1.093e-01  
## It: 179; obj: 2.317e+00; abs: 2.587e-07; rel: 1.117e-07; norm: 1.093e-01  
## It: 180; obj: 2.317e+00; abs: 2.310e-07; rel: 9.969e-08; norm: 1.093e-01  
## It: 181; obj: 2.317e+00; abs: 2.063e-07; rel: 8.903e-08; norm: 1.093e-01  
## It: 182; obj: 2.317e+00; abs: 1.843e-07; rel: 7.953e-08; norm: 1.093e-01  
## It: 183; obj: 2.317e+00; abs: 1.647e-07; rel: 7.108e-08; norm: 1.093e-01  
## It: 184; obj: 2.317e+00; abs: 1.473e-07; rel: 6.355e-08; norm: 1.093e-01  
## It: 185; obj: 2.317e+00; abs: 1.317e-07; rel: 5.685e-08; norm: 1.093e-01  
## It: 186; obj: 2.317e+00; abs: 1.179e-07; rel: 5.088e-08; norm: 1.093e-01  
## It: 187; obj: 2.317e+00; abs: 1.056e-07; rel: 4.557e-08; norm: 1.093e-01  
## It: 188; obj: 2.317e+00; abs: 9.461e-08; rel: 4.083e-08; norm: 1.093e-01  
## It: 189; obj: 2.317e+00; abs: 8.483e-08; rel: 3.661e-08; norm: 1.093e-01  
## It: 190; obj: 2.317e+00; abs: 7.610e-08; rel: 3.284e-08; norm: 1.093e-01  
## It: 191; obj: 2.317e+00; abs: 6.831e-08; rel: 2.948e-08; norm: 1.093e-01  
## It: 192; obj: 2.317e+00; abs: 6.136e-08; rel: 2.648e-08; norm: 1.093e-01  
## It: 193; obj: 2.317e+00; abs: 5.516e-08; rel: 2.380e-08; norm: 1.093e-01  
## It: 194; obj: 2.317e+00; abs: 4.961e-08; rel: 2.141e-08; norm: 1.093e-01  
## It: 195; obj: 2.317e+00; abs: 4.465e-08; rel: 1.927e-08; norm: 1.093e-01  
## It: 196; obj: 2.317e+00; abs: 4.021e-08; rel: 1.735e-08; norm: 1.093e-01  
## It: 197; obj: 2.317e+00; abs: 3.623e-08; rel: 1.564e-08; norm: 1.093e-01  
## It: 198; obj: 2.317e+00; abs: 3.267e-08; rel: 1.410e-08; norm: 1.093e-01  
## It: 199; obj: 2.317e+00; abs: 2.948e-08; rel: 1.272e-08; norm: 1.093e-01  
## It: 200; obj: 2.317e+00; abs: 2.661e-08; rel: 1.149e-08; norm: 1.093e-01

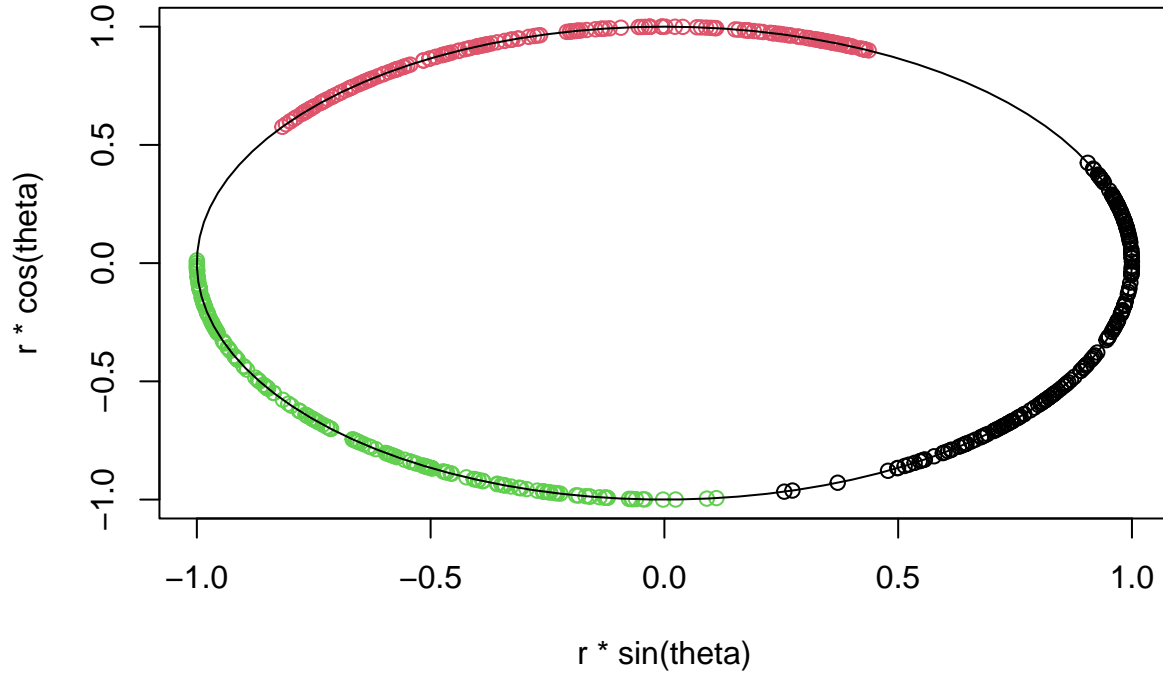


Iteration 200



```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension  $\mathbb{S}^2$  then  $d = 2$  (sphere):

```
Y <- psc_sne(X=x_2, d=2, rho_psc_list = rho_second_perp20, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

```
## It: 1; obj: 1.922e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.770e-01
## It: 2; obj: 1.585e+01; abs: 3.370e+00; rel: 1.753e-01; norm: 3.577e-01
## It: 3; obj: 1.434e+01; abs: 1.506e+00; rel: 9.502e-02; norm: 6.645e-01
## It: 4; obj: 1.373e+01; abs: 6.090e-01; rel: 4.246e-02; norm: 8.337e-01
## It: 5; obj: 1.340e+01; abs: 3.352e-01; rel: 2.441e-02; norm: 9.360e-01
## It: 6; obj: 1.321e+01; abs: 1.911e-01; rel: 1.426e-02; norm: 1.017e+00
## It: 7; obj: 1.311e+01; abs: 9.650e-02; rel: 7.306e-03; norm: 1.086e+00
## It: 8; obj: 1.307e+01; abs: 4.329e-02; rel: 3.301e-03; norm: 1.145e+00
## It: 9; obj: 1.306e+01; abs: 1.221e-02; rel: 9.343e-04; norm: 1.197e+00
## It: 10; obj: 1.306e+01; abs: 4.906e-03; rel: 3.757e-04; norm: 1.242e+00
## It: 11; obj: 1.308e+01; abs: 1.522e-02; rel: 1.166e-03; norm: 1.279e+00
## It: 12; obj: 1.309e+01; abs: 1.485e-02; rel: 1.136e-03; norm: 1.308e+00
## It: 13; obj: 1.310e+01; abs: 7.378e-03; rel: 5.636e-04; norm: 1.332e+00
## It: 14; obj: 1.309e+01; abs: 5.811e-03; rel: 4.436e-04; norm: 1.351e+00
## It: 15; obj: 1.308e+01; abs: 1.078e-02; rel: 8.231e-04; norm: 1.366e+00
## It: 16; obj: 1.308e+01; abs: 6.527e-03; rel: 4.989e-04; norm: 1.377e+00
## It: 17; obj: 1.307e+01; abs: 4.161e-03; rel: 3.183e-04; norm: 1.385e+00
## It: 18; obj: 1.307e+01; abs: 2.516e-03; rel: 1.925e-04; norm: 1.391e+00
## It: 19; obj: 1.307e+01; abs: 6.537e-04; rel: 5.002e-05; norm: 1.396e+00
## It: 20; obj: 1.308e+01; abs: 6.532e-03; rel: 4.998e-04; norm: 1.401e+00
## It: 21; obj: 1.309e+01; abs: 1.417e-02; rel: 1.084e-03; norm: 1.407e+00
## It: 22; obj: 1.311e+01; abs: 2.116e-02; rel: 1.617e-03; norm: 1.413e+00
## It: 23; obj: 1.314e+01; abs: 2.558e-02; rel: 1.951e-03; norm: 1.419e+00
## It: 24; obj: 1.316e+01; abs: 2.696e-02; rel: 2.052e-03; norm: 1.425e+00
## It: 25; obj: 1.319e+01; abs: 2.583e-02; rel: 1.962e-03; norm: 1.431e+00

## It: 26; obj: 1.321e+01; abs: 2.304e-02; rel: 1.747e-03; norm: 1.436e+00
## It: 27; obj: 1.323e+01; abs: 1.941e-02; rel: 1.469e-03; norm: 1.441e+00
```

```

## It: 28; obj: 1.325e+01; abs: 1.562e-02; rel: 1.180e-03; norm: 1.446e+00
## It: 29; obj: 1.326e+01; abs: 1.228e-02; rel: 9.273e-04; norm: 1.451e+00
## It: 30; obj: 1.327e+01; abs: 1.010e-02; rel: 7.619e-04; norm: 1.456e+00
## It: 31; obj: 1.328e+01; abs: 9.796e-03; rel: 7.382e-04; norm: 1.460e+00
## It: 32; obj: 1.329e+01; abs: 1.176e-02; rel: 8.854e-04; norm: 1.466e+00
## It: 33; obj: 1.331e+01; abs: 1.538e-02; rel: 1.157e-03; norm: 1.471e+00
## It: 34; obj: 1.333e+01; abs: 1.901e-02; rel: 1.428e-03; norm: 1.477e+00
## It: 35; obj: 1.335e+01; abs: 2.110e-02; rel: 1.584e-03; norm: 1.482e+00
## It: 36; obj: 1.337e+01; abs: 2.127e-02; rel: 1.594e-03; norm: 1.487e+00
## It: 37; obj: 1.339e+01; abs: 2.007e-02; rel: 1.501e-03; norm: 1.491e+00
## It: 38; obj: 1.341e+01; abs: 1.823e-02; rel: 1.362e-03; norm: 1.494e+00
## It: 39; obj: 1.342e+01; abs: 1.624e-02; rel: 1.211e-03; norm: 1.497e+00
## It: 40; obj: 1.344e+01; abs: 1.434e-02; rel: 1.068e-03; norm: 1.500e+00
## It: 41; obj: 1.345e+01; abs: 1.260e-02; rel: 9.381e-04; norm: 1.503e+00
## It: 42; obj: 1.346e+01; abs: 1.107e-02; rel: 8.229e-04; norm: 1.506e+00
## It: 43; obj: 1.347e+01; abs: 9.717e-03; rel: 7.218e-04; norm: 1.508e+00
## It: 44; obj: 1.348e+01; abs: 8.534e-03; rel: 6.335e-04; norm: 1.510e+00
## It: 45; obj: 1.349e+01; abs: 7.499e-03; rel: 5.563e-04; norm: 1.513e+00
## It: 46; obj: 1.349e+01; abs: 6.594e-03; rel: 4.889e-04; norm: 1.515e+00
## It: 47; obj: 1.350e+01; abs: 5.801e-03; rel: 4.299e-04; norm: 1.517e+00
## It: 48; obj: 1.350e+01; abs: 5.105e-03; rel: 3.782e-04; norm: 1.519e+00
## It: 49; obj: 1.351e+01; abs: 4.494e-03; rel: 3.328e-04; norm: 1.521e+00
## It: 50; obj: 1.351e+01; abs: 3.957e-03; rel: 2.929e-04; norm: 1.523e+00

## It: 51; obj: 1.352e+01; abs: 3.483e-03; rel: 2.577e-04; norm: 1.524e+00
## It: 52; obj: 1.352e+01; abs: 3.064e-03; rel: 2.267e-04; norm: 1.526e+00
## It: 53; obj: 1.352e+01; abs: 2.695e-03; rel: 1.993e-04; norm: 1.528e+00
## It: 54; obj: 1.352e+01; abs: 2.368e-03; rel: 1.751e-04; norm: 1.530e+00
## It: 55; obj: 1.353e+01; abs: 2.078e-03; rel: 1.537e-04; norm: 1.531e+00
## It: 56; obj: 1.353e+01; abs: 1.821e-03; rel: 1.346e-04; norm: 1.533e+00
## It: 57; obj: 1.353e+01; abs: 1.593e-03; rel: 1.178e-04; norm: 1.535e+00
## It: 58; obj: 1.353e+01; abs: 1.391e-03; rel: 1.028e-04; norm: 1.536e+00
## It: 59; obj: 1.353e+01; abs: 1.211e-03; rel: 8.952e-05; norm: 1.538e+00
## It: 60; obj: 1.353e+01; abs: 1.052e-03; rel: 7.771e-05; norm: 1.540e+00
## It: 61; obj: 1.353e+01; abs: 9.095e-04; rel: 6.721e-05; norm: 1.541e+00
## It: 62; obj: 1.354e+01; abs: 7.832e-04; rel: 5.787e-05; norm: 1.543e+00
## It: 63; obj: 1.354e+01; abs: 6.710e-04; rel: 4.957e-05; norm: 1.544e+00
## It: 64; obj: 1.354e+01; abs: 5.712e-04; rel: 4.220e-05; norm: 1.546e+00
## It: 65; obj: 1.354e+01; abs: 4.825e-04; rel: 3.565e-05; norm: 1.547e+00
## It: 66; obj: 1.354e+01; abs: 4.039e-04; rel: 2.984e-05; norm: 1.549e+00
## It: 67; obj: 1.354e+01; abs: 3.342e-04; rel: 2.469e-05; norm: 1.550e+00
## It: 68; obj: 1.354e+01; abs: 2.726e-04; rel: 2.013e-05; norm: 1.551e+00
## It: 69; obj: 1.354e+01; abs: 2.182e-04; rel: 1.612e-05; norm: 1.553e+00
## It: 70; obj: 1.354e+01; abs: 1.703e-04; rel: 1.258e-05; norm: 1.554e+00
## It: 71; obj: 1.354e+01; abs: 1.283e-04; rel: 9.477e-06; norm: 1.556e+00
## It: 72; obj: 1.354e+01; abs: 9.165e-05; rel: 6.769e-06; norm: 1.557e+00
## It: 73; obj: 1.354e+01; abs: 5.982e-05; rel: 4.419e-06; norm: 1.558e+00
## It: 74; obj: 1.354e+01; abs: 3.239e-05; rel: 2.392e-06; norm: 1.560e+00
## It: 75; obj: 1.354e+01; abs: 8.937e-06; rel: 6.602e-07; norm: 1.561e+00

## It: 76; obj: 1.354e+01; abs: 1.088e-05; rel: 8.037e-07; norm: 1.562e+00
## It: 77; obj: 1.354e+01; abs: 2.739e-05; rel: 2.023e-06; norm: 1.563e+00
## It: 78; obj: 1.354e+01; abs: 4.089e-05; rel: 3.020e-06; norm: 1.565e+00
## It: 79; obj: 1.354e+01; abs: 5.164e-05; rel: 3.815e-06; norm: 1.566e+00
## It: 80; obj: 1.354e+01; abs: 5.989e-05; rel: 4.424e-06; norm: 1.567e+00

```

```

## It: 81; obj: 1.354e+01; abs: 6.585e-05; rel: 4.864e-06; norm: 1.568e+00
## It: 82; obj: 1.354e+01; abs: 6.973e-05; rel: 5.151e-06; norm: 1.570e+00
## It: 83; obj: 1.354e+01; abs: 7.173e-05; rel: 5.298e-06; norm: 1.571e+00
## It: 84; obj: 1.354e+01; abs: 7.202e-05; rel: 5.319e-06; norm: 1.572e+00
## It: 85; obj: 1.354e+01; abs: 7.075e-05; rel: 5.226e-06; norm: 1.573e+00
## It: 86; obj: 1.354e+01; abs: 6.809e-05; rel: 5.030e-06; norm: 1.574e+00
## It: 87; obj: 1.354e+01; abs: 6.418e-05; rel: 4.740e-06; norm: 1.575e+00
## It: 88; obj: 1.354e+01; abs: 5.914e-05; rel: 4.369e-06; norm: 1.577e+00
## It: 89; obj: 1.354e+01; abs: 5.312e-05; rel: 3.924e-06; norm: 1.578e+00
## It: 90; obj: 1.354e+01; abs: 4.622e-05; rel: 3.414e-06; norm: 1.579e+00
## It: 91; obj: 1.354e+01; abs: 3.856e-05; rel: 2.848e-06; norm: 1.580e+00
## It: 92; obj: 1.354e+01; abs: 3.024e-05; rel: 2.234e-06; norm: 1.581e+00
## It: 93; obj: 1.354e+01; abs: 2.136e-05; rel: 1.578e-06; norm: 1.582e+00
## It: 94; obj: 1.354e+01; abs: 1.202e-05; rel: 8.877e-07; norm: 1.583e+00
## It: 95; obj: 1.354e+01; abs: 2.289e-06; rel: 1.691e-07; norm: 1.584e+00
## It: 96; obj: 1.354e+01; abs: 7.741e-06; rel: 5.718e-07; norm: 1.585e+00
## It: 97; obj: 1.354e+01; abs: 1.800e-05; rel: 1.329e-06; norm: 1.586e+00
## It: 98; obj: 1.354e+01; abs: 2.841e-05; rel: 2.099e-06; norm: 1.587e+00
## It: 99; obj: 1.354e+01; abs: 3.892e-05; rel: 2.875e-06; norm: 1.588e+00
## It: 100; obj: 1.354e+01; abs: 4.947e-05; rel: 3.654e-06; norm: 1.589e+00

## It: 101; obj: 1.921e+00; abs: 1.162e+01; rel: 8.581e-01; norm: 1.358e-01
## It: 102; obj: 1.769e+00; abs: 1.521e-01; rel: 7.918e-02; norm: 1.360e-01
## It: 103; obj: 1.730e+00; abs: 3.883e-02; rel: 2.195e-02; norm: 1.491e-01
## It: 104; obj: 1.722e+00; abs: 8.167e-03; rel: 4.721e-03; norm: 1.527e-01
## It: 105; obj: 1.707e+00; abs: 1.443e-02; rel: 8.382e-03; norm: 1.509e-01
## It: 106; obj: 1.708e+00; abs: 1.073e-03; rel: 6.286e-04; norm: 1.514e-01
## It: 107; obj: 1.730e+00; abs: 2.146e-02; rel: 1.256e-02; norm: 1.517e-01
## It: 108; obj: 1.721e+00; abs: 8.951e-03; rel: 5.175e-03; norm: 1.516e-01
## It: 109; obj: 1.730e+00; abs: 9.073e-03; rel: 5.272e-03; norm: 1.510e-01
## It: 110; obj: 1.731e+00; abs: 1.264e-03; rel: 7.306e-04; norm: 1.505e-01
## It: 111; obj: 1.722e+00; abs: 9.176e-03; rel: 5.300e-03; norm: 1.501e-01
## It: 112; obj: 1.733e+00; abs: 1.058e-02; rel: 6.143e-03; norm: 1.512e-01
## It: 113; obj: 1.712e+00; abs: 2.027e-02; rel: 1.170e-02; norm: 1.504e-01
## It: 114; obj: 1.706e+00; abs: 6.701e-03; rel: 3.913e-03; norm: 1.499e-01
## It: 115; obj: 1.716e+00; abs: 1.019e-02; rel: 5.977e-03; norm: 1.494e-01
## It: 116; obj: 1.727e+00; abs: 1.075e-02; rel: 6.263e-03; norm: 1.496e-01
## It: 117; obj: 1.748e+00; abs: 2.095e-02; rel: 1.213e-02; norm: 1.516e-01
## It: 118; obj: 1.771e+00; abs: 2.295e-02; rel: 1.313e-02; norm: 1.512e-01
## It: 119; obj: 1.755e+00; abs: 1.553e-02; rel: 8.772e-03; norm: 1.510e-01
## It: 120; obj: 1.746e+00; abs: 8.990e-03; rel: 5.122e-03; norm: 1.508e-01
## It: 121; obj: 1.736e+00; abs: 1.022e-02; rel: 5.852e-03; norm: 1.498e-01
## It: 122; obj: 1.751e+00; abs: 1.482e-02; rel: 8.540e-03; norm: 1.500e-01
## It: 123; obj: 1.760e+00; abs: 9.503e-03; rel: 5.428e-03; norm: 1.529e-01
## It: 124; obj: 1.753e+00; abs: 7.199e-03; rel: 4.090e-03; norm: 1.512e-01
## It: 125; obj: 1.771e+00; abs: 1.823e-02; rel: 1.040e-02; norm: 1.510e-01

## It: 126; obj: 1.769e+00; abs: 2.462e-03; rel: 1.390e-03; norm: 1.520e-01
## It: 127; obj: 1.742e+00; abs: 2.678e-02; rel: 1.514e-02; norm: 1.497e-01
## It: 128; obj: 1.741e+00; abs: 1.146e-03; rel: 6.580e-04; norm: 1.490e-01
## It: 129; obj: 1.742e+00; abs: 1.168e-03; rel: 6.712e-04; norm: 1.505e-01
## It: 130; obj: 1.742e+00; abs: 3.146e-04; rel: 1.806e-04; norm: 1.503e-01
## It: 131; obj: 1.737e+00; abs: 5.182e-03; rel: 2.974e-03; norm: 1.497e-01
## It: 132; obj: 1.745e+00; abs: 7.674e-03; rel: 4.418e-03; norm: 1.512e-01
## It: 133; obj: 1.755e+00; abs: 9.811e-03; rel: 5.623e-03; norm: 1.497e-01

```

```

## It: 134; obj: 1.740e+00; abs: 1.423e-02; rel: 8.109e-03; norm: 1.497e-01
## It: 135; obj: 1.751e+00; abs: 1.117e-02; rel: 6.419e-03; norm: 1.503e-01
## It: 136; obj: 1.730e+00; abs: 2.173e-02; rel: 1.241e-02; norm: 1.522e-01
## It: 137; obj: 1.739e+00; abs: 9.541e-03; rel: 5.516e-03; norm: 1.506e-01
## It: 138; obj: 1.766e+00; abs: 2.650e-02; rel: 1.524e-02; norm: 1.511e-01
## It: 139; obj: 1.742e+00; abs: 2.347e-02; rel: 1.329e-02; norm: 1.512e-01
## It: 140; obj: 1.720e+00; abs: 2.272e-02; rel: 1.304e-02; norm: 1.500e-01
## It: 141; obj: 1.737e+00; abs: 1.701e-02; rel: 9.893e-03; norm: 1.495e-01
## It: 142; obj: 1.755e+00; abs: 1.813e-02; rel: 1.044e-02; norm: 1.494e-01
## It: 143; obj: 1.757e+00; abs: 1.908e-03; rel: 1.087e-03; norm: 1.501e-01
## It: 144; obj: 1.755e+00; abs: 1.854e-03; rel: 1.055e-03; norm: 1.508e-01
## It: 145; obj: 1.751e+00; abs: 3.568e-03; rel: 2.033e-03; norm: 1.500e-01
## It: 146; obj: 1.757e+00; abs: 5.709e-03; rel: 3.260e-03; norm: 1.498e-01
## It: 147; obj: 1.756e+00; abs: 5.028e-04; rel: 2.862e-04; norm: 1.516e-01
## It: 148; obj: 1.752e+00; abs: 4.141e-03; rel: 2.358e-03; norm: 1.515e-01
## It: 149; obj: 1.757e+00; abs: 4.675e-03; rel: 2.668e-03; norm: 1.501e-01
## It: 150; obj: 1.745e+00; abs: 1.246e-02; rel: 7.091e-03; norm: 1.499e-01

## It: 151; obj: 1.755e+00; abs: 1.069e-02; rel: 6.126e-03; norm: 1.513e-01
## It: 152; obj: 1.756e+00; abs: 4.665e-04; rel: 2.658e-04; norm: 1.510e-01
## It: 153; obj: 1.755e+00; abs: 1.019e-03; rel: 5.806e-04; norm: 1.512e-01
## It: 154; obj: 1.750e+00; abs: 4.438e-03; rel: 2.529e-03; norm: 1.508e-01
## It: 155; obj: 1.783e+00; abs: 3.304e-02; rel: 1.888e-02; norm: 1.514e-01
## It: 156; obj: 1.795e+00; abs: 1.197e-02; rel: 6.715e-03; norm: 1.514e-01
## It: 157; obj: 1.777e+00; abs: 1.862e-02; rel: 1.037e-02; norm: 1.506e-01
## It: 158; obj: 1.782e+00; abs: 5.293e-03; rel: 2.979e-03; norm: 1.504e-01
## It: 159; obj: 1.789e+00; abs: 7.331e-03; rel: 4.114e-03; norm: 1.512e-01
## It: 160; obj: 1.776e+00; abs: 1.349e-02; rel: 7.541e-03; norm: 1.502e-01
## It: 161; obj: 1.756e+00; abs: 2.015e-02; rel: 1.135e-02; norm: 1.507e-01
## It: 162; obj: 1.764e+00; abs: 8.433e-03; rel: 4.804e-03; norm: 1.507e-01
## It: 163; obj: 1.763e+00; abs: 1.261e-03; rel: 7.149e-04; norm: 1.503e-01
## It: 164; obj: 1.787e+00; abs: 2.417e-02; rel: 1.371e-02; norm: 1.508e-01
## It: 165; obj: 1.790e+00; abs: 2.733e-03; rel: 1.529e-03; norm: 1.523e-01
## It: 166; obj: 1.744e+00; abs: 4.543e-02; rel: 2.539e-02; norm: 1.503e-01
## It: 167; obj: 1.778e+00; abs: 3.376e-02; rel: 1.936e-02; norm: 1.504e-01
## It: 168; obj: 1.790e+00; abs: 1.233e-02; rel: 6.934e-03; norm: 1.514e-01
## It: 169; obj: 1.749e+00; abs: 4.098e-02; rel: 2.289e-02; norm: 1.511e-01
## It: 170; obj: 1.743e+00; abs: 6.166e-03; rel: 3.525e-03; norm: 1.494e-01
## It: 171; obj: 1.777e+00; abs: 3.359e-02; rel: 1.927e-02; norm: 1.502e-01
## It: 172; obj: 1.777e+00; abs: 5.460e-04; rel: 3.073e-04; norm: 1.503e-01
## It: 173; obj: 1.766e+00; abs: 1.085e-02; rel: 6.104e-03; norm: 1.498e-01
## It: 174; obj: 1.771e+00; abs: 4.400e-03; rel: 2.491e-03; norm: 1.519e-01
## It: 175; obj: 1.759e+00; abs: 1.140e-02; rel: 6.436e-03; norm: 1.515e-01

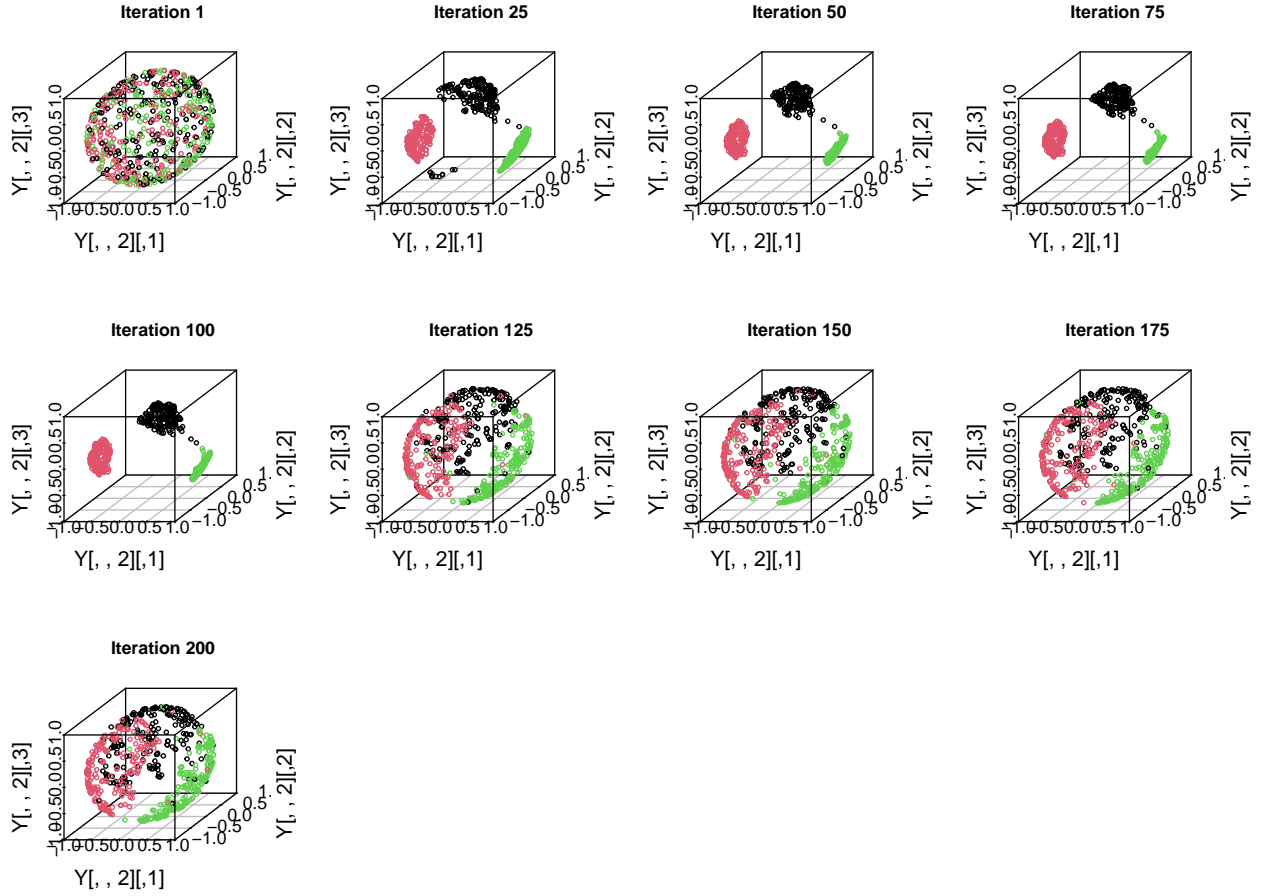
## It: 176; obj: 1.739e+00; abs: 2.042e-02; rel: 1.161e-02; norm: 1.493e-01
## It: 177; obj: 1.756e+00; abs: 1.736e-02; rel: 9.983e-03; norm: 1.499e-01
## It: 178; obj: 1.759e+00; abs: 2.230e-03; rel: 1.269e-03; norm: 1.500e-01
## It: 179; obj: 1.750e+00; abs: 9.128e-03; rel: 5.190e-03; norm: 1.514e-01
## It: 180; obj: 1.766e+00; abs: 1.663e-02; rel: 9.506e-03; norm: 1.516e-01
## It: 181; obj: 1.753e+00; abs: 1.334e-02; rel: 7.554e-03; norm: 1.506e-01
## It: 182; obj: 1.736e+00; abs: 1.691e-02; rel: 9.645e-03; norm: 1.491e-01
## It: 183; obj: 1.744e+00; abs: 8.153e-03; rel: 4.697e-03; norm: 1.507e-01
## It: 184; obj: 1.718e+00; abs: 2.571e-02; rel: 1.474e-02; norm: 1.502e-01
## It: 185; obj: 1.702e+00; abs: 1.631e-02; rel: 9.494e-03; norm: 1.503e-01
## It: 186; obj: 1.712e+00; abs: 9.797e-03; rel: 5.756e-03; norm: 1.507e-01

```

```

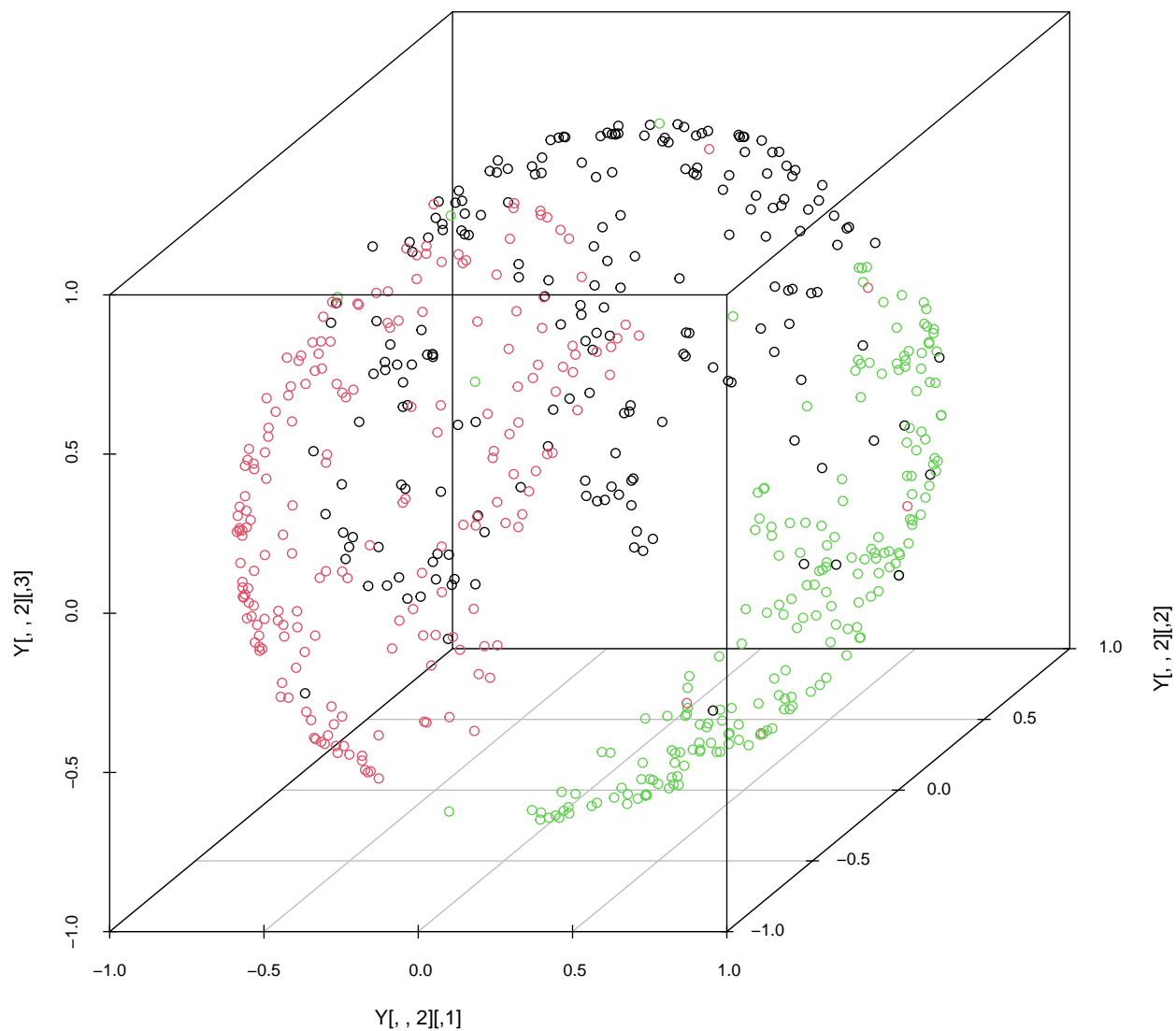
## It: 187; obj: 1.708e+00; abs: 3.940e-03; rel: 2.301e-03; norm: 1.502e-01
## It: 188; obj: 1.709e+00; abs: 1.598e-03; rel: 9.356e-04; norm: 1.500e-01
## It: 189; obj: 1.727e+00; abs: 1.733e-02; rel: 1.014e-02; norm: 1.500e-01
## It: 190; obj: 1.749e+00; abs: 2.183e-02; rel: 1.264e-02; norm: 1.499e-01
## It: 191; obj: 1.754e+00; abs: 5.544e-03; rel: 3.171e-03; norm: 1.502e-01
## It: 192; obj: 1.742e+00; abs: 1.261e-02; rel: 7.188e-03; norm: 1.519e-01
## It: 193; obj: 1.726e+00; abs: 1.520e-02; rel: 8.727e-03; norm: 1.495e-01
## It: 194; obj: 1.755e+00; abs: 2.835e-02; rel: 1.642e-02; norm: 1.520e-01
## It: 195; obj: 1.767e+00; abs: 1.213e-02; rel: 6.913e-03; norm: 1.514e-01
## It: 196; obj: 1.761e+00; abs: 5.951e-03; rel: 3.368e-03; norm: 1.497e-01
## It: 197; obj: 1.741e+00; abs: 2.010e-02; rel: 1.141e-02; norm: 1.505e-01
## It: 198; obj: 1.741e+00; abs: 1.908e-04; rel: 1.096e-04; norm: 1.508e-01
## It: 199; obj: 1.751e+00; abs: 1.083e-02; rel: 6.223e-03; norm: 1.503e-01
## It: 200; obj: 1.774e+00; abs: 2.214e-02; rel: 1.264e-02; norm: 1.513e-01

```

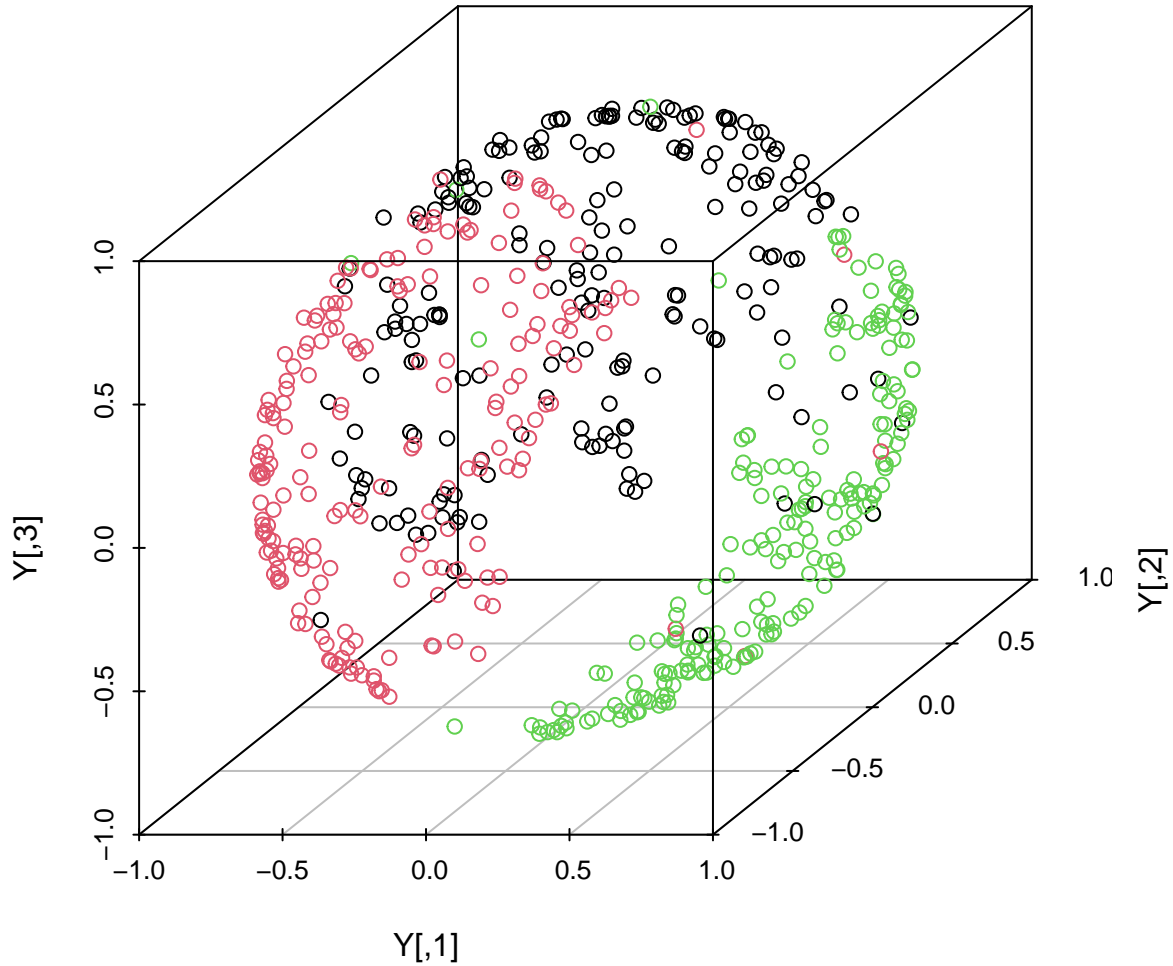




Iteration 200



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),  
                             color = colors)
```



It's clearly clustered the three groups we had in the original generated data.

### Case 3

Sample on the  $(S^1)^3$  where  $p = 1$  and  $r = 3$ .

```
# Sample on the  $(S^1)^2$ 
n_ori <- 200
vmf11 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 10)
vmf12 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 10)
vmf13 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi/2)), kappa = 10)
x1 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf11),
                             DirStats::to_rad(vmf12),
                             DirStats::to_rad(vmf13)))

vmf21 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi / 2)), kappa = 5)
vmf22 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 5)
vmf23 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 5)
x2 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf21), DirStats::to_rad(vmf22),
                             DirStats::to_rad(vmf23)))

vmf31 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 30)
vmf32 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 30)
```

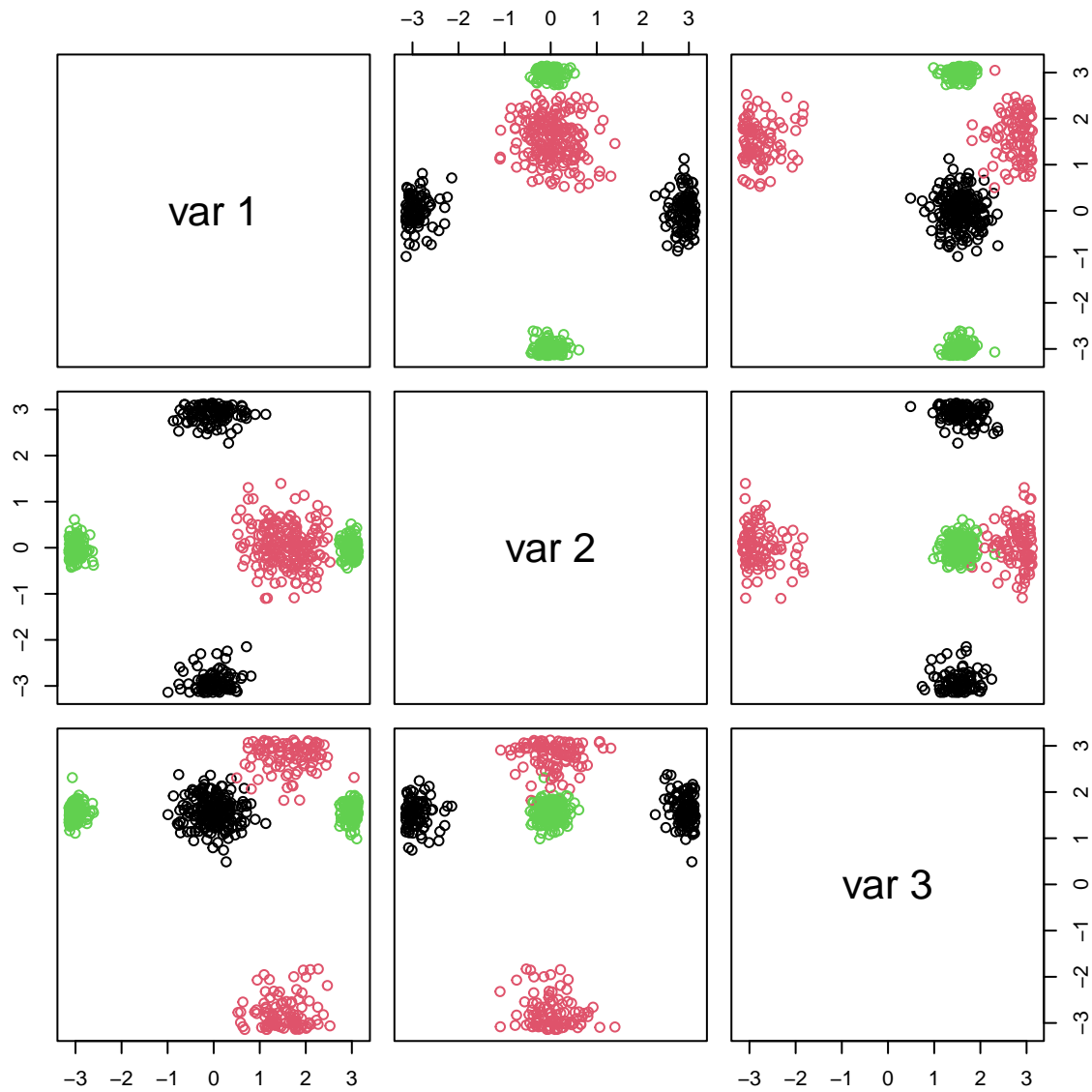
```
vmf33 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi/2)), kappa = 30)
x3 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf31), DirStats::to_rad(vmf32),
                             DirStats::to_rad(vmf33)))
```

Now, let's do the same visualization with the data on the torus:

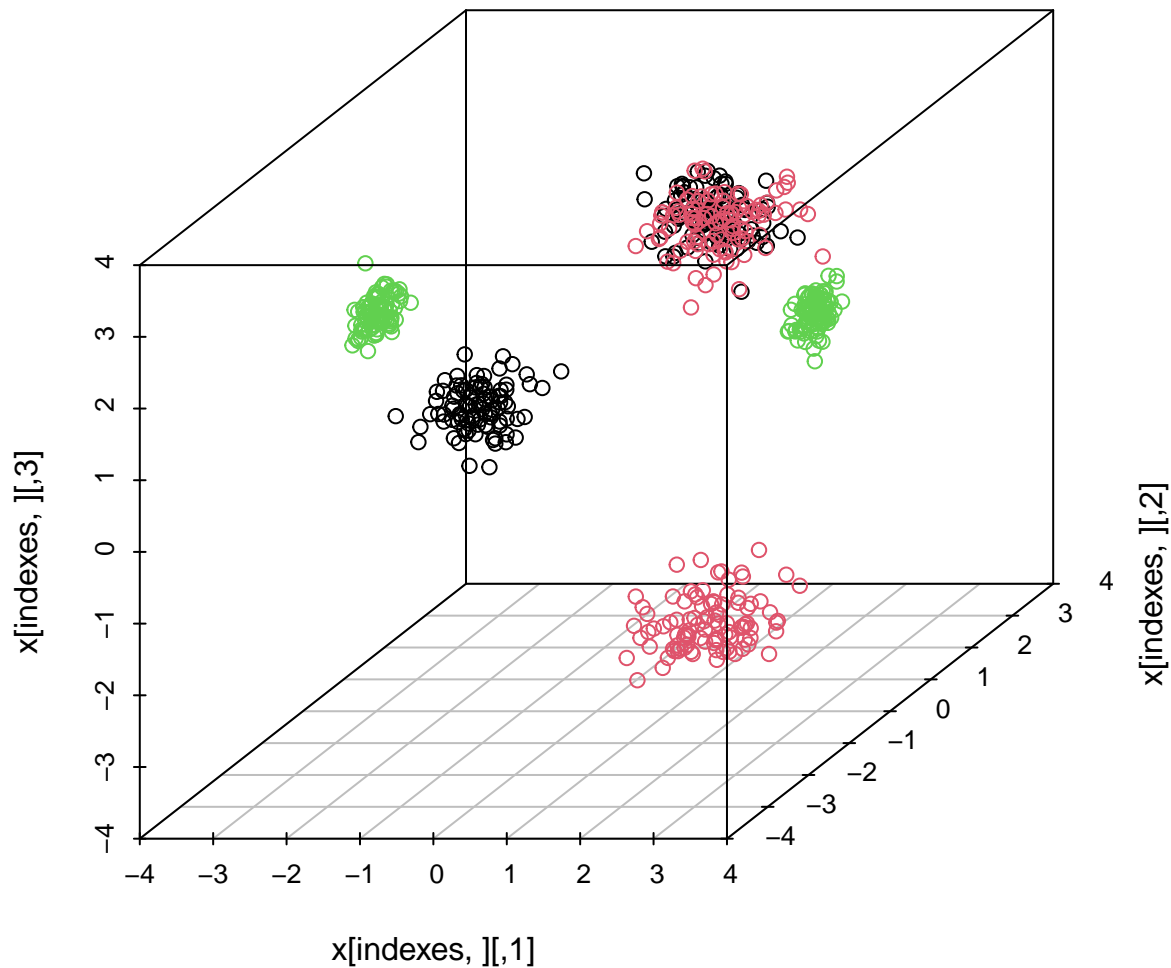
```
x <- rbind(x1, x2, x3)
n <- n_ori * 3
x_3 <- array(dim = c(n, 2, 3))
x_3[, , 1] <- DirStats::to_cir(x[, 1])
x_3[, , 2] <- DirStats::to_cir(x[, 2])
x_3[, , 3] <- DirStats::to_cir(x[, 3])

n <- nrow(x_3)
indexes <- sample(1:n)
x_3 <- x_3[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]

pairs(x[indexes,], col = colors)
```



```
scatterplot3d::scatterplot3d(x[indexes,], xlim = c(-pi, pi), ylim = c(-pi, pi), zlim = c(-pi, pi),
                             color = colors)
```



Let's calculate the rho parameters based on a perplexity of 25:

```
rho_third_perp25 <- rho_optim_bst(x_3, 25)
```

```
## Time difference of 1.233613 mins
```

First, let's reduce to dimension  $\mathbb{S}^1$  then  $d = 1$  (circumference):

```
Y <- psc_sne(X=x_3, d=1, rho_psc_list = rho_third_perp25, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

```
## It: 1; obj: 1.795e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.313e-01
## It: 2; obj: 1.728e+01; abs: 6.711e-01; rel: 3.738e-02; norm: 2.588e-01
## It: 3; obj: 1.688e+01; abs: 4.038e-01; rel: 2.336e-02; norm: 3.454e-01
## It: 4; obj: 1.659e+01; abs: 2.919e-01; rel: 1.729e-02; norm: 4.016e-01
## It: 5; obj: 1.637e+01; abs: 2.159e-01; rel: 1.302e-02; norm: 4.506e-01
## It: 6; obj: 1.620e+01; abs: 1.740e-01; rel: 1.063e-02; norm: 4.869e-01
## It: 7; obj: 1.605e+01; abs: 1.447e-01; rel: 8.934e-03; norm: 5.151e-01
## It: 8; obj: 1.592e+01; abs: 1.325e-01; rel: 8.251e-03; norm: 5.384e-01
## It: 9; obj: 1.580e+01; abs: 1.211e-01; rel: 7.606e-03; norm: 5.580e-01
## It: 10; obj: 1.568e+01; abs: 1.185e-01; rel: 7.498e-03; norm: 5.757e-01
## It: 11; obj: 1.557e+01; abs: 1.111e-01; rel: 7.086e-03; norm: 5.929e-01
## It: 12; obj: 1.546e+01; abs: 1.146e-01; rel: 7.359e-03; norm: 6.067e-01
## It: 13; obj: 1.534e+01; abs: 1.175e-01; rel: 7.603e-03; norm: 6.201e-01
```

```

## It: 14; obj: 1.522e+01; abs: 1.144e-01; rel: 7.461e-03; norm: 6.341e-01
## It: 15; obj: 1.512e+01; abs: 1.076e-01; rel: 7.067e-03; norm: 6.492e-01
## It: 16; obj: 1.503e+01; abs: 8.919e-02; rel: 5.901e-03; norm: 6.662e-01
## It: 17; obj: 1.496e+01; abs: 6.960e-02; rel: 4.632e-03; norm: 6.840e-01
## It: 18; obj: 1.490e+01; abs: 5.294e-02; rel: 3.540e-03; norm: 7.009e-01
## It: 19; obj: 1.486e+01; abs: 4.714e-02; rel: 3.163e-03; norm: 7.157e-01
## It: 20; obj: 1.481e+01; abs: 4.914e-02; rel: 3.307e-03; norm: 7.289e-01
## It: 21; obj: 1.476e+01; abs: 4.805e-02; rel: 3.245e-03; norm: 7.415e-01
## It: 22; obj: 1.472e+01; abs: 4.165e-02; rel: 2.822e-03; norm: 7.527e-01
## It: 23; obj: 1.469e+01; abs: 3.150e-02; rel: 2.140e-03; norm: 7.616e-01
## It: 24; obj: 1.467e+01; abs: 2.038e-02; rel: 1.388e-03; norm: 7.685e-01
## It: 25; obj: 1.466e+01; abs: 1.060e-02; rel: 7.230e-04; norm: 7.741e-01

## It: 26; obj: 1.465e+01; abs: 2.782e-03; rel: 1.898e-04; norm: 7.791e-01
## It: 27; obj: 1.466e+01; abs: 3.919e-03; rel: 2.675e-04; norm: 7.838e-01
## It: 28; obj: 1.467e+01; abs: 9.522e-03; rel: 6.497e-04; norm: 7.886e-01
## It: 29; obj: 1.468e+01; abs: 1.329e-02; rel: 9.064e-04; norm: 7.935e-01
## It: 30; obj: 1.469e+01; abs: 1.502e-02; rel: 1.024e-03; norm: 7.984e-01
## It: 31; obj: 1.471e+01; abs: 1.543e-02; rel: 1.050e-03; norm: 8.034e-01
## It: 32; obj: 1.472e+01; abs: 1.485e-02; rel: 1.010e-03; norm: 8.085e-01
## It: 33; obj: 1.474e+01; abs: 1.293e-02; rel: 8.784e-04; norm: 8.137e-01
## It: 34; obj: 1.475e+01; abs: 9.473e-03; rel: 6.428e-04; norm: 8.189e-01
## It: 35; obj: 1.475e+01; abs: 3.835e-03; rel: 2.601e-04; norm: 8.241e-01
## It: 36; obj: 1.475e+01; abs: 4.807e-03; rel: 3.259e-04; norm: 8.295e-01
## It: 37; obj: 1.473e+01; abs: 1.661e-02; rel: 1.126e-03; norm: 8.352e-01
## It: 38; obj: 1.470e+01; abs: 3.138e-02; rel: 2.130e-03; norm: 8.417e-01
## It: 39; obj: 1.465e+01; abs: 4.735e-02; rel: 3.221e-03; norm: 8.492e-01
## It: 40; obj: 1.459e+01; abs: 5.986e-02; rel: 4.086e-03; norm: 8.582e-01
## It: 41; obj: 1.453e+01; abs: 6.183e-02; rel: 4.237e-03; norm: 8.685e-01
## It: 42; obj: 1.448e+01; abs: 5.007e-02; rel: 3.446e-03; norm: 8.792e-01
## It: 43; obj: 1.445e+01; abs: 3.129e-02; rel: 2.161e-03; norm: 8.883e-01
## It: 44; obj: 1.443e+01; abs: 1.485e-02; rel: 1.028e-03; norm: 8.948e-01
## It: 45; obj: 1.443e+01; abs: 4.279e-03; rel: 2.965e-04; norm: 8.987e-01
## It: 46; obj: 1.443e+01; abs: 1.390e-03; rel: 9.632e-05; norm: 9.006e-01
## It: 47; obj: 1.443e+01; abs: 3.946e-03; rel: 2.735e-04; norm: 9.011e-01
## It: 48; obj: 1.444e+01; abs: 4.771e-03; rel: 3.306e-04; norm: 9.006e-01
## It: 49; obj: 1.444e+01; abs: 4.725e-03; rel: 3.273e-04; norm: 8.995e-01
## It: 50; obj: 1.445e+01; abs: 4.292e-03; rel: 2.972e-04; norm: 8.979e-01

## It: 51; obj: 1.445e+01; abs: 3.723e-03; rel: 2.577e-04; norm: 8.961e-01
## It: 52; obj: 1.445e+01; abs: 3.142e-03; rel: 2.174e-04; norm: 8.942e-01
## It: 53; obj: 1.446e+01; abs: 2.603e-03; rel: 1.801e-04; norm: 8.922e-01
## It: 54; obj: 1.446e+01; abs: 2.127e-03; rel: 1.471e-04; norm: 8.902e-01
## It: 55; obj: 1.446e+01; abs: 1.716e-03; rel: 1.187e-04; norm: 8.882e-01
## It: 56; obj: 1.446e+01; abs: 1.368e-03; rel: 9.457e-05; norm: 8.862e-01
## It: 57; obj: 1.446e+01; abs: 1.073e-03; rel: 7.422e-05; norm: 8.842e-01
## It: 58; obj: 1.446e+01; abs: 8.261e-04; rel: 5.712e-05; norm: 8.823e-01
## It: 59; obj: 1.446e+01; abs: 6.188e-04; rel: 4.278e-05; norm: 8.804e-01
## It: 60; obj: 1.447e+01; abs: 4.449e-04; rel: 3.076e-05; norm: 8.785e-01
## It: 61; obj: 1.447e+01; abs: 2.991e-04; rel: 2.067e-05; norm: 8.767e-01
## It: 62; obj: 1.447e+01; abs: 1.767e-04; rel: 1.221e-05; norm: 8.750e-01
## It: 63; obj: 1.447e+01; abs: 7.391e-05; rel: 5.109e-06; norm: 8.732e-01
## It: 64; obj: 1.447e+01; abs: 1.242e-05; rel: 8.586e-07; norm: 8.715e-01
## It: 65; obj: 1.447e+01; abs: 8.495e-05; rel: 5.872e-06; norm: 8.699e-01
## It: 66; obj: 1.447e+01; abs: 1.459e-04; rel: 1.008e-05; norm: 8.683e-01

```

```

## It: 67; obj: 1.447e+01; abs: 1.969e-04; rel: 1.361e-05; norm: 8.667e-01
## It: 68; obj: 1.447e+01; abs: 2.397e-04; rel: 1.657e-05; norm: 8.652e-01
## It: 69; obj: 1.446e+01; abs: 2.755e-04; rel: 1.905e-05; norm: 8.637e-01
## It: 70; obj: 1.446e+01; abs: 3.053e-04; rel: 2.110e-05; norm: 8.623e-01
## It: 71; obj: 1.446e+01; abs: 3.299e-04; rel: 2.281e-05; norm: 8.608e-01
## It: 72; obj: 1.446e+01; abs: 3.502e-04; rel: 2.421e-05; norm: 8.594e-01
## It: 73; obj: 1.446e+01; abs: 3.667e-04; rel: 2.535e-05; norm: 8.581e-01
## It: 74; obj: 1.446e+01; abs: 3.800e-04; rel: 2.627e-05; norm: 8.567e-01
## It: 75; obj: 1.446e+01; abs: 3.906e-04; rel: 2.701e-05; norm: 8.554e-01

## It: 76; obj: 1.446e+01; abs: 3.988e-04; rel: 2.757e-05; norm: 8.542e-01
## It: 77; obj: 1.446e+01; abs: 4.049e-04; rel: 2.800e-05; norm: 8.529e-01
## It: 78; obj: 1.446e+01; abs: 4.092e-04; rel: 2.830e-05; norm: 8.517e-01
## It: 79; obj: 1.446e+01; abs: 4.121e-04; rel: 2.849e-05; norm: 8.505e-01
## It: 80; obj: 1.446e+01; abs: 4.136e-04; rel: 2.860e-05; norm: 8.494e-01
## It: 81; obj: 1.446e+01; abs: 4.139e-04; rel: 2.863e-05; norm: 8.482e-01
## It: 82; obj: 1.446e+01; abs: 4.133e-04; rel: 2.858e-05; norm: 8.471e-01
## It: 83; obj: 1.446e+01; abs: 4.119e-04; rel: 2.848e-05; norm: 8.460e-01
## It: 84; obj: 1.446e+01; abs: 4.097e-04; rel: 2.833e-05; norm: 8.449e-01
## It: 85; obj: 1.446e+01; abs: 4.068e-04; rel: 2.814e-05; norm: 8.439e-01
## It: 86; obj: 1.446e+01; abs: 4.034e-04; rel: 2.790e-05; norm: 8.428e-01
## It: 87; obj: 1.446e+01; abs: 3.996e-04; rel: 2.764e-05; norm: 8.418e-01
## It: 88; obj: 1.446e+01; abs: 3.953e-04; rel: 2.734e-05; norm: 8.408e-01
## It: 89; obj: 1.446e+01; abs: 3.908e-04; rel: 2.703e-05; norm: 8.399e-01
## It: 90; obj: 1.446e+01; abs: 3.859e-04; rel: 2.669e-05; norm: 8.389e-01
## It: 91; obj: 1.446e+01; abs: 3.808e-04; rel: 2.634e-05; norm: 8.380e-01
## It: 92; obj: 1.446e+01; abs: 3.754e-04; rel: 2.597e-05; norm: 8.370e-01
## It: 93; obj: 1.446e+01; abs: 3.699e-04; rel: 2.559e-05; norm: 8.361e-01
## It: 94; obj: 1.446e+01; abs: 3.643e-04; rel: 2.520e-05; norm: 8.353e-01
## It: 95; obj: 1.445e+01; abs: 3.586e-04; rel: 2.481e-05; norm: 8.344e-01
## It: 96; obj: 1.445e+01; abs: 3.528e-04; rel: 2.440e-05; norm: 8.335e-01
## It: 97; obj: 1.445e+01; abs: 3.469e-04; rel: 2.400e-05; norm: 8.327e-01
## It: 98; obj: 1.445e+01; abs: 3.410e-04; rel: 2.359e-05; norm: 8.319e-01
## It: 99; obj: 1.445e+01; abs: 3.350e-04; rel: 2.318e-05; norm: 8.311e-01
## It: 100; obj: 1.445e+01; abs: 3.290e-04; rel: 2.277e-05; norm: 8.303e-01

## It: 101; obj: 2.200e+00; abs: 1.225e+01; rel: 8.478e-01; norm: 7.915e-02
## It: 102; obj: 2.161e+00; abs: 3.892e-02; rel: 1.769e-02; norm: 8.477e-02
## It: 103; obj: 2.133e+00; abs: 2.778e-02; rel: 1.285e-02; norm: 9.426e-02
## It: 104; obj: 2.121e+00; abs: 1.245e-02; rel: 5.837e-03; norm: 1.010e-01
## It: 105; obj: 2.116e+00; abs: 4.558e-03; rel: 2.149e-03; norm: 1.042e-01
## It: 106; obj: 2.115e+00; abs: 1.794e-03; rel: 8.477e-04; norm: 1.054e-01
## It: 107; obj: 2.114e+00; abs: 9.471e-04; rel: 4.479e-04; norm: 1.058e-01
## It: 108; obj: 2.113e+00; abs: 6.342e-04; rel: 3.001e-04; norm: 1.060e-01
## It: 109; obj: 2.113e+00; abs: 4.392e-04; rel: 2.078e-04; norm: 1.062e-01
## It: 110; obj: 2.112e+00; abs: 2.975e-04; rel: 1.408e-04; norm: 1.063e-01
## It: 111; obj: 2.112e+00; abs: 2.296e-04; rel: 1.087e-04; norm: 1.065e-01
## It: 112; obj: 2.112e+00; abs: 1.750e-04; rel: 8.286e-05; norm: 1.066e-01
## It: 113; obj: 2.112e+00; abs: 1.175e-04; rel: 5.563e-05; norm: 1.067e-01
## It: 114; obj: 2.112e+00; abs: 8.485e-05; rel: 4.018e-05; norm: 1.068e-01
## It: 115; obj: 2.112e+00; abs: 6.711e-05; rel: 3.178e-05; norm: 1.068e-01
## It: 116; obj: 2.112e+00; abs: 4.906e-05; rel: 2.323e-05; norm: 1.069e-01
## It: 117; obj: 2.112e+00; abs: 3.471e-05; rel: 1.644e-05; norm: 1.070e-01
## It: 118; obj: 2.112e+00; abs: 2.707e-05; rel: 1.282e-05; norm: 1.070e-01
## It: 119; obj: 2.112e+00; abs: 2.125e-05; rel: 1.006e-05; norm: 1.070e-01

```

```

## It: 120; obj: 2.111e+00; abs: 1.568e-05; rel: 7.426e-06; norm: 1.070e-01
## It: 121; obj: 2.111e+00; abs: 1.182e-05; rel: 5.598e-06; norm: 1.071e-01
## It: 122; obj: 2.111e+00; abs: 9.330e-06; rel: 4.419e-06; norm: 1.071e-01
## It: 123; obj: 2.111e+00; abs: 7.267e-06; rel: 3.442e-06; norm: 1.071e-01
## It: 124; obj: 2.111e+00; abs: 5.575e-06; rel: 2.640e-06; norm: 1.071e-01
## It: 125; obj: 2.111e+00; abs: 4.336e-06; rel: 2.054e-06; norm: 1.071e-01

## It: 126; obj: 2.111e+00; abs: 3.400e-06; rel: 1.610e-06; norm: 1.071e-01
## It: 127; obj: 2.111e+00; abs: 2.663e-06; rel: 1.261e-06; norm: 1.071e-01
## It: 128; obj: 2.111e+00; abs: 2.096e-06; rel: 9.927e-07; norm: 1.071e-01
## It: 129; obj: 2.111e+00; abs: 1.657e-06; rel: 7.846e-07; norm: 1.071e-01
## It: 130; obj: 2.111e+00; abs: 1.314e-06; rel: 6.223e-07; norm: 1.071e-01
## It: 131; obj: 2.111e+00; abs: 1.049e-06; rel: 4.970e-07; norm: 1.071e-01
## It: 132; obj: 2.111e+00; abs: 8.405e-07; rel: 3.981e-07; norm: 1.071e-01
## It: 133; obj: 2.111e+00; abs: 6.721e-07; rel: 3.183e-07; norm: 1.071e-01
## It: 134; obj: 2.111e+00; abs: 5.399e-07; rel: 2.557e-07; norm: 1.071e-01
## It: 135; obj: 2.111e+00; abs: 4.379e-07; rel: 2.074e-07; norm: 1.071e-01
## It: 136; obj: 2.111e+00; abs: 3.570e-07; rel: 1.691e-07; norm: 1.071e-01
## It: 137; obj: 2.111e+00; abs: 2.919e-07; rel: 1.382e-07; norm: 1.071e-01
## It: 138; obj: 2.111e+00; abs: 2.400e-07; rel: 1.137e-07; norm: 1.071e-01
## It: 139; obj: 2.111e+00; abs: 1.986e-07; rel: 9.404e-08; norm: 1.071e-01
## It: 140; obj: 2.111e+00; abs: 1.649e-07; rel: 7.810e-08; norm: 1.071e-01
## It: 141; obj: 2.111e+00; abs: 1.374e-07; rel: 6.507e-08; norm: 1.071e-01
## It: 142; obj: 2.111e+00; abs: 1.151e-07; rel: 5.451e-08; norm: 1.071e-01
## It: 143; obj: 2.111e+00; abs: 9.697e-08; rel: 4.592e-08; norm: 1.071e-01
## It: 144; obj: 2.111e+00; abs: 8.207e-08; rel: 3.887e-08; norm: 1.071e-01
## It: 145; obj: 2.111e+00; abs: 6.973e-08; rel: 3.303e-08; norm: 1.071e-01
## It: 146; obj: 2.111e+00; abs: 5.948e-08; rel: 2.817e-08; norm: 1.071e-01
## It: 147; obj: 2.111e+00; abs: 5.092e-08; rel: 2.412e-08; norm: 1.071e-01
## It: 148; obj: 2.111e+00; abs: 4.375e-08; rel: 2.072e-08; norm: 1.071e-01
## It: 149; obj: 2.111e+00; abs: 3.770e-08; rel: 1.785e-08; norm: 1.071e-01
## It: 150; obj: 2.111e+00; abs: 3.258e-08; rel: 1.543e-08; norm: 1.071e-01

## It: 151; obj: 2.111e+00; abs: 2.824e-08; rel: 1.338e-08; norm: 1.071e-01
## It: 152; obj: 2.111e+00; abs: 2.454e-08; rel: 1.162e-08; norm: 1.071e-01
## It: 153; obj: 2.111e+00; abs: 2.137e-08; rel: 1.012e-08; norm: 1.071e-01
## It: 154; obj: 2.111e+00; abs: 1.865e-08; rel: 8.833e-09; norm: 1.071e-01
## It: 155; obj: 2.111e+00; abs: 1.630e-08; rel: 7.722e-09; norm: 1.071e-01
## It: 156; obj: 2.111e+00; abs: 1.428e-08; rel: 6.762e-09; norm: 1.071e-01
## It: 157; obj: 2.111e+00; abs: 1.252e-08; rel: 5.930e-09; norm: 1.071e-01
## It: 158; obj: 2.111e+00; abs: 1.100e-08; rel: 5.208e-09; norm: 1.071e-01
## It: 159; obj: 2.111e+00; abs: 9.669e-09; rel: 4.579e-09; norm: 1.071e-01
## It: 160; obj: 2.111e+00; abs: 8.511e-09; rel: 4.031e-09; norm: 1.071e-01
## It: 161; obj: 2.111e+00; abs: 7.499e-09; rel: 3.552e-09; norm: 1.071e-01
## It: 162; obj: 2.111e+00; abs: 6.613e-09; rel: 3.132e-09; norm: 1.071e-01
## It: 163; obj: 2.111e+00; abs: 5.836e-09; rel: 2.764e-09; norm: 1.071e-01
## It: 164; obj: 2.111e+00; abs: 5.155e-09; rel: 2.441e-09; norm: 1.071e-01
## It: 165; obj: 2.111e+00; abs: 4.556e-09; rel: 2.158e-09; norm: 1.071e-01
## It: 166; obj: 2.111e+00; abs: 4.029e-09; rel: 1.908e-09; norm: 1.071e-01
## It: 167; obj: 2.111e+00; abs: 3.565e-09; rel: 1.688e-09; norm: 1.071e-01
## It: 168; obj: 2.111e+00; abs: 3.156e-09; rel: 1.494e-09; norm: 1.071e-01
## It: 169; obj: 2.111e+00; abs: 2.795e-09; rel: 1.324e-09; norm: 1.071e-01
## It: 170; obj: 2.111e+00; abs: 2.476e-09; rel: 1.173e-09; norm: 1.071e-01
## It: 171; obj: 2.111e+00; abs: 2.194e-09; rel: 1.039e-09; norm: 1.071e-01
## It: 172; obj: 2.111e+00; abs: 1.946e-09; rel: 9.214e-10; norm: 1.071e-01

```

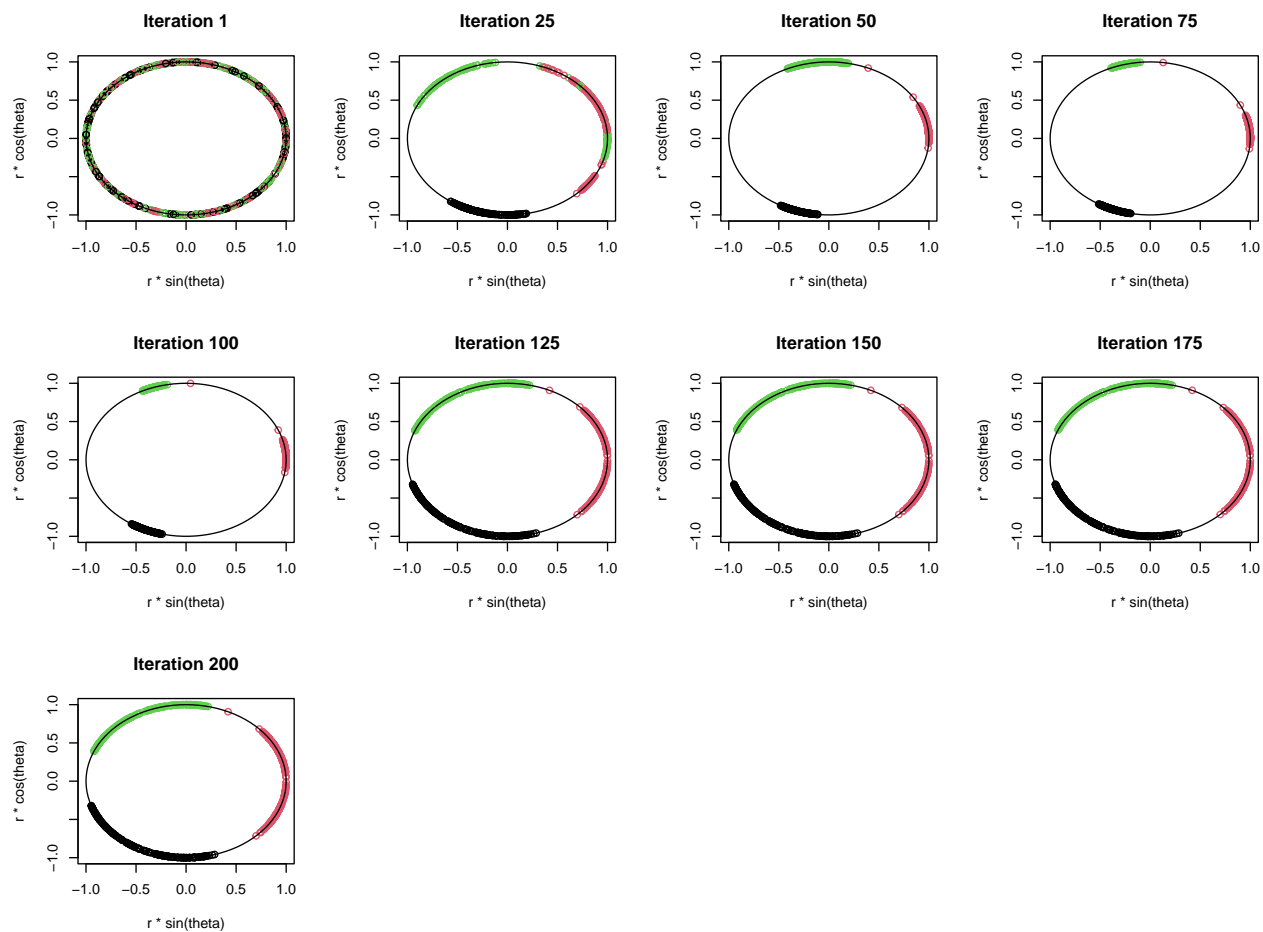


```

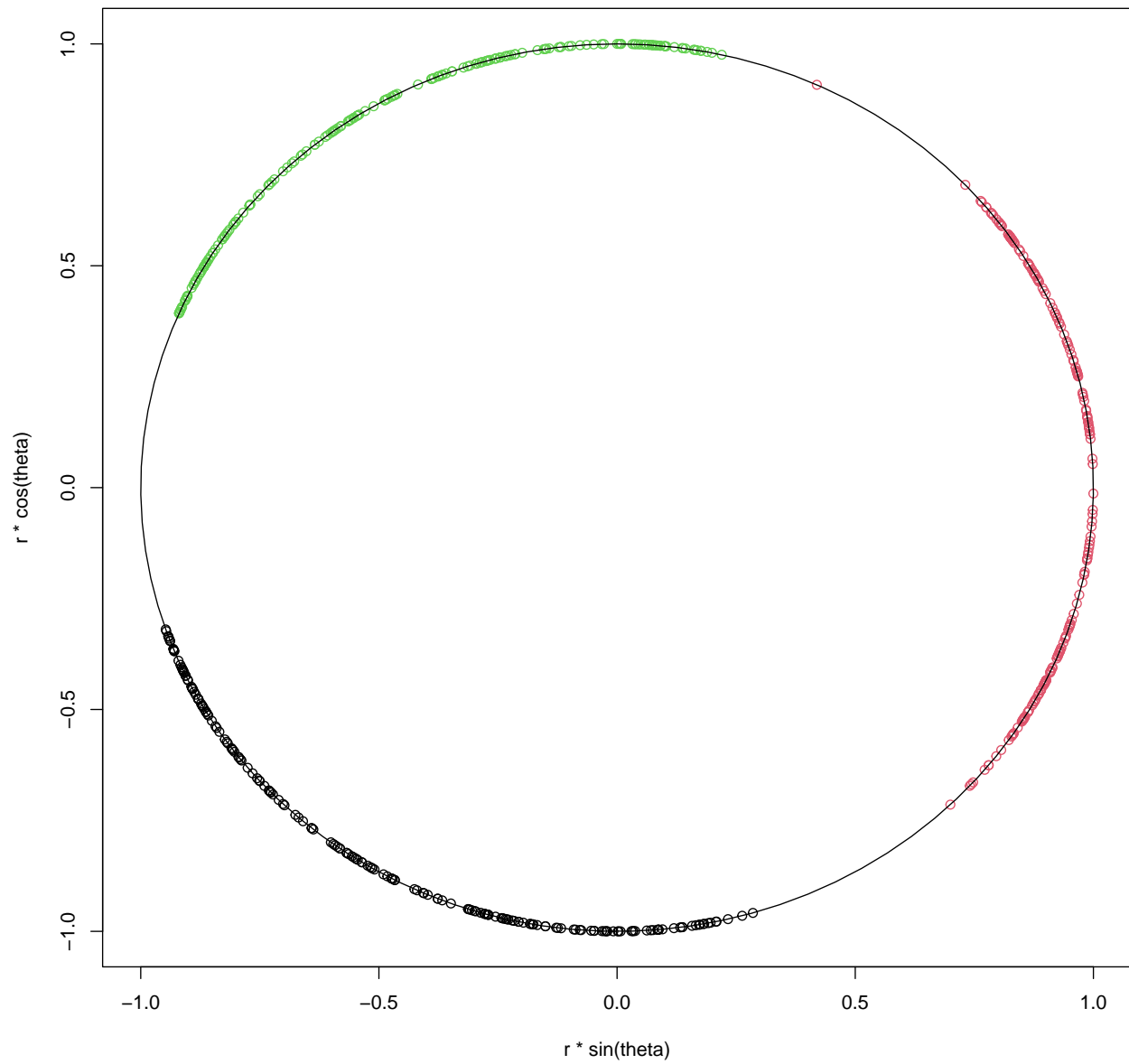
## It: 173; obj: 2.111e+00; abs: 1.726e-09; rel: 8.173e-10; norm: 1.071e-01
## It: 174; obj: 2.111e+00; abs: 1.531e-09; rel: 7.250e-10; norm: 1.071e-01
## It: 175; obj: 2.111e+00; abs: 1.359e-09; rel: 6.434e-10; norm: 1.071e-01

## It: 176; obj: 2.111e+00; abs: 1.206e-09; rel: 5.711e-10; norm: 1.071e-01
## It: 177; obj: 2.111e+00; abs: 1.071e-09; rel: 5.071e-10; norm: 1.071e-01
## It: 178; obj: 2.111e+00; abs: 9.507e-10; rel: 4.502e-10; norm: 1.071e-01
## It: 179; obj: 2.111e+00; abs: 8.446e-10; rel: 4.000e-10; norm: 1.071e-01
## It: 180; obj: 2.111e+00; abs: 7.503e-10; rel: 3.553e-10; norm: 1.071e-01
## It: 181; obj: 2.111e+00; abs: 6.668e-10; rel: 3.158e-10; norm: 1.071e-01
## It: 182; obj: 2.111e+00; abs: 5.926e-10; rel: 2.807e-10; norm: 1.071e-01
## It: 183; obj: 2.111e+00; abs: 5.267e-10; rel: 2.494e-10; norm: 1.071e-01
## It: 184; obj: 2.111e+00; abs: 4.685e-10; rel: 2.219e-10; norm: 1.071e-01
## It: 185; obj: 2.111e+00; abs: 4.165e-10; rel: 1.973e-10; norm: 1.071e-01
## It: 186; obj: 2.111e+00; abs: 3.704e-10; rel: 1.754e-10; norm: 1.071e-01
## It: 187; obj: 2.111e+00; abs: 3.296e-10; rel: 1.561e-10; norm: 1.071e-01
## It: 188; obj: 2.111e+00; abs: 2.931e-10; rel: 1.388e-10; norm: 1.071e-01
## It: 189; obj: 2.111e+00; abs: 2.610e-10; rel: 1.236e-10; norm: 1.071e-01
## It: 190; obj: 2.111e+00; abs: 2.324e-10; rel: 1.101e-10; norm: 1.071e-01
## It: 191; obj: 2.111e+00; abs: 2.066e-10; rel: 9.786e-11; norm: 1.071e-01
## It: 192; obj: 2.111e+00; abs: 1.840e-10; rel: 8.714e-11; norm: 1.071e-01
## It: 193; obj: 2.111e+00; abs: 1.639e-10; rel: 7.762e-11; norm: 1.071e-01
## It: 194; obj: 2.111e+00; abs: 1.461e-10; rel: 6.917e-11; norm: 1.071e-01
## It: 195; obj: 2.111e+00; abs: 1.299e-10; rel: 6.152e-11; norm: 1.071e-01
## It: 196; obj: 2.111e+00; abs: 1.158e-10; rel: 5.484e-11; norm: 1.071e-01
## It: 197; obj: 2.111e+00; abs: 1.033e-10; rel: 4.892e-11; norm: 1.071e-01
## It: 198; obj: 2.111e+00; abs: 9.184e-11; rel: 4.349e-11; norm: 1.071e-01
## It: 199; obj: 2.111e+00; abs: 8.196e-11; rel: 3.882e-11; norm: 1.071e-01
## It: 200; obj: 2.111e+00; abs: 7.301e-11; rel: 3.458e-11; norm: 1.071e-01

```

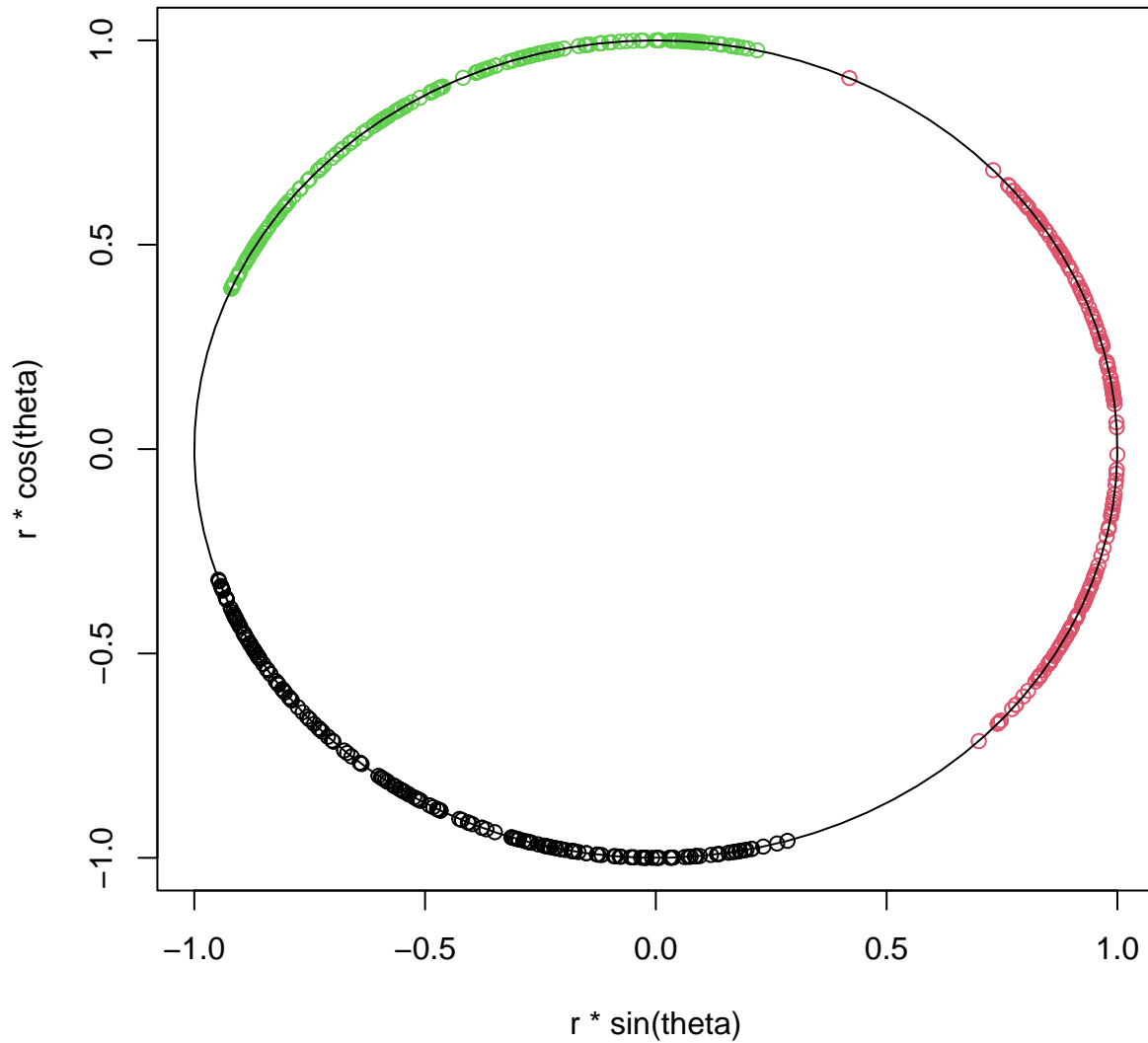


Iteration 200



```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension  $\mathbb{S}^2$  then  $d = 2$  (sphere):

```
Y <- psc_sne(X=x_3, d=2, rho_psc_list = rho_third_perp25, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

```
## It: 1; obj: 1.867e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.560e-01
## It: 2; obj: 1.573e+01; abs: 2.949e+00; rel: 1.579e-01; norm: 3.237e-01
## It: 3; obj: 1.462e+01; abs: 1.103e+00; rel: 7.017e-02; norm: 5.924e-01
## It: 4; obj: 1.411e+01; abs: 5.119e-01; rel: 3.501e-02; norm: 7.303e-01
## It: 5; obj: 1.382e+01; abs: 2.849e-01; rel: 2.019e-02; norm: 8.032e-01
## It: 6; obj: 1.365e+01; abs: 1.750e-01; rel: 1.266e-02; norm: 8.527e-01
## It: 7; obj: 1.349e+01; abs: 1.571e-01; rel: 1.151e-02; norm: 8.923e-01
## It: 8; obj: 1.333e+01; abs: 1.595e-01; rel: 1.182e-02; norm: 9.287e-01
## It: 9; obj: 1.319e+01; abs: 1.466e-01; rel: 1.100e-02; norm: 9.621e-01
## It: 10; obj: 1.305e+01; abs: 1.357e-01; rel: 1.029e-02; norm: 9.919e-01
## It: 11; obj: 1.293e+01; abs: 1.229e-01; rel: 9.418e-03; norm: 1.019e+00
## It: 12; obj: 1.282e+01; abs: 1.057e-01; rel: 8.173e-03; norm: 1.044e+00
## It: 13; obj: 1.273e+01; abs: 9.072e-02; rel: 7.075e-03; norm: 1.066e+00
## It: 14; obj: 1.265e+01; abs: 7.977e-02; rel: 6.266e-03; norm: 1.090e+00
## It: 15; obj: 1.258e+01; abs: 7.214e-02; rel: 5.702e-03; norm: 1.118e+00
```

```

## It: 16; obj: 1.251e+01; abs: 6.612e-02; rel: 5.256e-03; norm: 1.150e+00
## It: 17; obj: 1.245e+01; abs: 6.086e-02; rel: 4.863e-03; norm: 1.187e+00
## It: 18; obj: 1.240e+01; abs: 5.148e-02; rel: 4.134e-03; norm: 1.227e+00
## It: 19; obj: 1.237e+01; abs: 3.593e-02; rel: 2.897e-03; norm: 1.269e+00
## It: 20; obj: 1.235e+01; abs: 1.753e-02; rel: 1.418e-03; norm: 1.310e+00
## It: 21; obj: 1.235e+01; abs: 8.906e-04; rel: 7.212e-05; norm: 1.348e+00
## It: 22; obj: 1.236e+01; abs: 1.148e-02; rel: 9.297e-04; norm: 1.382e+00
## It: 23; obj: 1.238e+01; abs: 2.004e-02; rel: 1.621e-03; norm: 1.414e+00
## It: 24; obj: 1.241e+01; abs: 2.755e-02; rel: 2.226e-03; norm: 1.442e+00
## It: 25; obj: 1.244e+01; abs: 3.565e-02; rel: 2.874e-03; norm: 1.467e+00

## It: 26; obj: 1.248e+01; abs: 4.237e-02; rel: 3.405e-03; norm: 1.491e+00
## It: 27; obj: 1.253e+01; abs: 4.599e-02; rel: 3.684e-03; norm: 1.511e+00
## It: 28; obj: 1.258e+01; abs: 4.569e-02; rel: 3.646e-03; norm: 1.530e+00
## It: 29; obj: 1.262e+01; abs: 4.192e-02; rel: 3.333e-03; norm: 1.546e+00
## It: 30; obj: 1.266e+01; abs: 3.972e-02; rel: 3.148e-03; norm: 1.562e+00
## It: 31; obj: 1.270e+01; abs: 4.251e-02; rel: 3.359e-03; norm: 1.576e+00
## It: 32; obj: 1.275e+01; abs: 4.587e-02; rel: 3.612e-03; norm: 1.588e+00
## It: 33; obj: 1.279e+01; abs: 4.729e-02; rel: 3.710e-03; norm: 1.599e+00
## It: 34; obj: 1.284e+01; abs: 4.643e-02; rel: 3.630e-03; norm: 1.609e+00
## It: 35; obj: 1.288e+01; abs: 4.358e-02; rel: 3.394e-03; norm: 1.617e+00
## It: 36; obj: 1.292e+01; abs: 3.917e-02; rel: 3.041e-03; norm: 1.625e+00
## It: 37; obj: 1.296e+01; abs: 3.356e-02; rel: 2.597e-03; norm: 1.630e+00
## It: 38; obj: 1.298e+01; abs: 2.699e-02; rel: 2.083e-03; norm: 1.635e+00
## It: 39; obj: 1.300e+01; abs: 1.986e-02; rel: 1.530e-03; norm: 1.638e+00
## It: 40; obj: 1.302e+01; abs: 1.283e-02; rel: 9.865e-04; norm: 1.641e+00
## It: 41; obj: 1.302e+01; abs: 6.719e-03; rel: 5.162e-04; norm: 1.643e+00
## It: 42; obj: 1.302e+01; abs: 2.381e-03; rel: 1.828e-04; norm: 1.645e+00
## It: 43; obj: 1.303e+01; abs: 5.801e-04; rel: 4.454e-05; norm: 1.647e+00
## It: 44; obj: 1.303e+01; abs: 1.754e-03; rel: 1.346e-04; norm: 1.650e+00
## It: 45; obj: 1.303e+01; abs: 5.413e-03; rel: 4.155e-04; norm: 1.652e+00
## It: 46; obj: 1.304e+01; abs: 9.747e-03; rel: 7.479e-04; norm: 1.654e+00
## It: 47; obj: 1.305e+01; abs: 1.237e-02; rel: 9.487e-04; norm: 1.655e+00
## It: 48; obj: 1.307e+01; abs: 1.248e-02; rel: 9.560e-04; norm: 1.656e+00
## It: 49; obj: 1.308e+01; abs: 1.117e-02; rel: 8.547e-04; norm: 1.655e+00
## It: 50; obj: 1.309e+01; abs: 9.546e-03; rel: 7.299e-04; norm: 1.654e+00

## It: 51; obj: 1.310e+01; abs: 8.044e-03; rel: 6.146e-04; norm: 1.653e+00
## It: 52; obj: 1.310e+01; abs: 6.750e-03; rel: 5.154e-04; norm: 1.651e+00
## It: 53; obj: 1.311e+01; abs: 5.648e-03; rel: 4.311e-04; norm: 1.650e+00
## It: 54; obj: 1.311e+01; abs: 4.707e-03; rel: 3.591e-04; norm: 1.648e+00
## It: 55; obj: 1.312e+01; abs: 3.898e-03; rel: 2.973e-04; norm: 1.647e+00
## It: 56; obj: 1.312e+01; abs: 3.197e-03; rel: 2.437e-04; norm: 1.645e+00
## It: 57; obj: 1.312e+01; abs: 2.587e-03; rel: 1.971e-04; norm: 1.643e+00
## It: 58; obj: 1.312e+01; abs: 2.052e-03; rel: 1.564e-04; norm: 1.642e+00
## It: 59; obj: 1.313e+01; abs: 1.584e-03; rel: 1.207e-04; norm: 1.640e+00
## It: 60; obj: 1.313e+01; abs: 1.171e-03; rel: 8.923e-05; norm: 1.638e+00
## It: 61; obj: 1.313e+01; abs: 8.077e-04; rel: 6.153e-05; norm: 1.636e+00
## It: 62; obj: 1.313e+01; abs: 4.869e-04; rel: 3.709e-05; norm: 1.635e+00
## It: 63; obj: 1.313e+01; abs: 2.037e-04; rel: 1.551e-05; norm: 1.633e+00
## It: 64; obj: 1.313e+01; abs: 4.640e-05; rel: 3.535e-06; norm: 1.631e+00
## It: 65; obj: 1.313e+01; abs: 2.671e-04; rel: 2.035e-05; norm: 1.629e+00
## It: 66; obj: 1.313e+01; abs: 4.617e-04; rel: 3.517e-05; norm: 1.628e+00
## It: 67; obj: 1.313e+01; abs: 6.331e-04; rel: 4.823e-05; norm: 1.626e+00
## It: 68; obj: 1.313e+01; abs: 7.837e-04; rel: 5.970e-05; norm: 1.624e+00

```

```

## It: 69; obj: 1.313e+01; abs: 9.157e-04; rel: 6.976e-05; norm: 1.623e+00
## It: 70; obj: 1.312e+01; abs: 1.031e-03; rel: 7.854e-05; norm: 1.621e+00
## It: 71; obj: 1.312e+01; abs: 1.131e-03; rel: 8.619e-05; norm: 1.619e+00
## It: 72; obj: 1.312e+01; abs: 1.218e-03; rel: 9.280e-05; norm: 1.618e+00
## It: 73; obj: 1.312e+01; abs: 1.292e-03; rel: 9.849e-05; norm: 1.616e+00
## It: 74; obj: 1.312e+01; abs: 1.356e-03; rel: 1.033e-04; norm: 1.615e+00
## It: 75; obj: 1.312e+01; abs: 1.409e-03; rel: 1.074e-04; norm: 1.613e+00

## It: 76; obj: 1.312e+01; abs: 1.454e-03; rel: 1.108e-04; norm: 1.611e+00
## It: 77; obj: 1.312e+01; abs: 1.490e-03; rel: 1.136e-04; norm: 1.610e+00
## It: 78; obj: 1.311e+01; abs: 1.519e-03; rel: 1.158e-04; norm: 1.608e+00
## It: 79; obj: 1.311e+01; abs: 1.541e-03; rel: 1.175e-04; norm: 1.607e+00
## It: 80; obj: 1.311e+01; abs: 1.557e-03; rel: 1.187e-04; norm: 1.605e+00
## It: 81; obj: 1.311e+01; abs: 1.568e-03; rel: 1.196e-04; norm: 1.604e+00
## It: 82; obj: 1.311e+01; abs: 1.573e-03; rel: 1.200e-04; norm: 1.602e+00
## It: 83; obj: 1.311e+01; abs: 1.574e-03; rel: 1.201e-04; norm: 1.601e+00
## It: 84; obj: 1.310e+01; abs: 1.571e-03; rel: 1.199e-04; norm: 1.600e+00
## It: 85; obj: 1.310e+01; abs: 1.565e-03; rel: 1.194e-04; norm: 1.598e+00
## It: 86; obj: 1.310e+01; abs: 1.555e-03; rel: 1.187e-04; norm: 1.597e+00
## It: 87; obj: 1.310e+01; abs: 1.542e-03; rel: 1.177e-04; norm: 1.595e+00
## It: 88; obj: 1.310e+01; abs: 1.527e-03; rel: 1.166e-04; norm: 1.594e+00
## It: 89; obj: 1.310e+01; abs: 1.509e-03; rel: 1.152e-04; norm: 1.593e+00
## It: 90; obj: 1.310e+01; abs: 1.490e-03; rel: 1.138e-04; norm: 1.591e+00
## It: 91; obj: 1.309e+01; abs: 1.469e-03; rel: 1.122e-04; norm: 1.590e+00
## It: 92; obj: 1.309e+01; abs: 1.446e-03; rel: 1.104e-04; norm: 1.589e+00
## It: 93; obj: 1.309e+01; abs: 1.422e-03; rel: 1.086e-04; norm: 1.588e+00
## It: 94; obj: 1.309e+01; abs: 1.397e-03; rel: 1.067e-04; norm: 1.586e+00
## It: 95; obj: 1.309e+01; abs: 1.371e-03; rel: 1.048e-04; norm: 1.585e+00
## It: 96; obj: 1.309e+01; abs: 1.345e-03; rel: 1.027e-04; norm: 1.584e+00
## It: 97; obj: 1.309e+01; abs: 1.318e-03; rel: 1.007e-04; norm: 1.583e+00
## It: 98; obj: 1.308e+01; abs: 1.290e-03; rel: 9.860e-05; norm: 1.582e+00
## It: 99; obj: 1.308e+01; abs: 1.263e-03; rel: 9.650e-05; norm: 1.580e+00
## It: 100; obj: 1.308e+01; abs: 1.235e-03; rel: 9.438e-05; norm: 1.579e+00

## It: 101; obj: 1.838e+00; abs: 1.124e+01; rel: 8.595e-01; norm: 1.080e-01
## It: 102; obj: 1.651e+00; abs: 1.863e-01; rel: 1.014e-01; norm: 1.183e-01
## It: 103; obj: 1.538e+00; abs: 1.128e-01; rel: 6.831e-02; norm: 1.349e-01
## It: 104; obj: 1.541e+00; abs: 2.550e-03; rel: 1.658e-03; norm: 1.390e-01
## It: 105; obj: 1.541e+00; abs: 3.760e-04; rel: 2.440e-04; norm: 1.406e-01
## It: 106; obj: 1.500e+00; abs: 4.103e-02; rel: 2.662e-02; norm: 1.387e-01
## It: 107; obj: 1.538e+00; abs: 3.727e-02; rel: 2.484e-02; norm: 1.382e-01
## It: 108; obj: 1.554e+00; abs: 1.595e-02; rel: 1.038e-02; norm: 1.415e-01
## It: 109; obj: 1.514e+00; abs: 3.929e-02; rel: 2.529e-02; norm: 1.390e-01
## It: 110; obj: 1.524e+00; abs: 1.016e-02; rel: 6.707e-03; norm: 1.386e-01
## It: 111; obj: 1.539e+00; abs: 1.408e-02; rel: 9.235e-03; norm: 1.401e-01
## It: 112; obj: 1.523e+00; abs: 1.571e-02; rel: 1.021e-02; norm: 1.385e-01
## It: 113; obj: 1.532e+00; abs: 8.674e-03; rel: 5.696e-03; norm: 1.384e-01
## It: 114; obj: 1.536e+00; abs: 4.490e-03; rel: 2.932e-03; norm: 1.405e-01
## It: 115; obj: 1.509e+00; abs: 2.702e-02; rel: 1.759e-02; norm: 1.393e-01
## It: 116; obj: 1.526e+00; abs: 1.662e-02; rel: 1.102e-02; norm: 1.390e-01
## It: 117; obj: 1.534e+00; abs: 8.855e-03; rel: 5.804e-03; norm: 1.403e-01
## It: 118; obj: 1.510e+00; abs: 2.418e-02; rel: 1.576e-02; norm: 1.389e-01
## It: 119; obj: 1.521e+00; abs: 1.086e-02; rel: 7.191e-03; norm: 1.397e-01
## It: 120; obj: 1.522e+00; abs: 1.194e-03; rel: 7.853e-04; norm: 1.403e-01
## It: 121; obj: 1.515e+00; abs: 7.114e-03; rel: 4.673e-03; norm: 1.399e-01

```

```

## It: 122; obj: 1.526e+00; abs: 1.050e-02; rel: 6.931e-03; norm: 1.391e-01
## It: 123; obj: 1.520e+00; abs: 5.310e-03; rel: 3.481e-03; norm: 1.404e-01
## It: 124; obj: 1.519e+00; abs: 1.603e-03; rel: 1.054e-03; norm: 1.393e-01
## It: 125; obj: 1.502e+00; abs: 1.700e-02; rel: 1.119e-02; norm: 1.394e-01

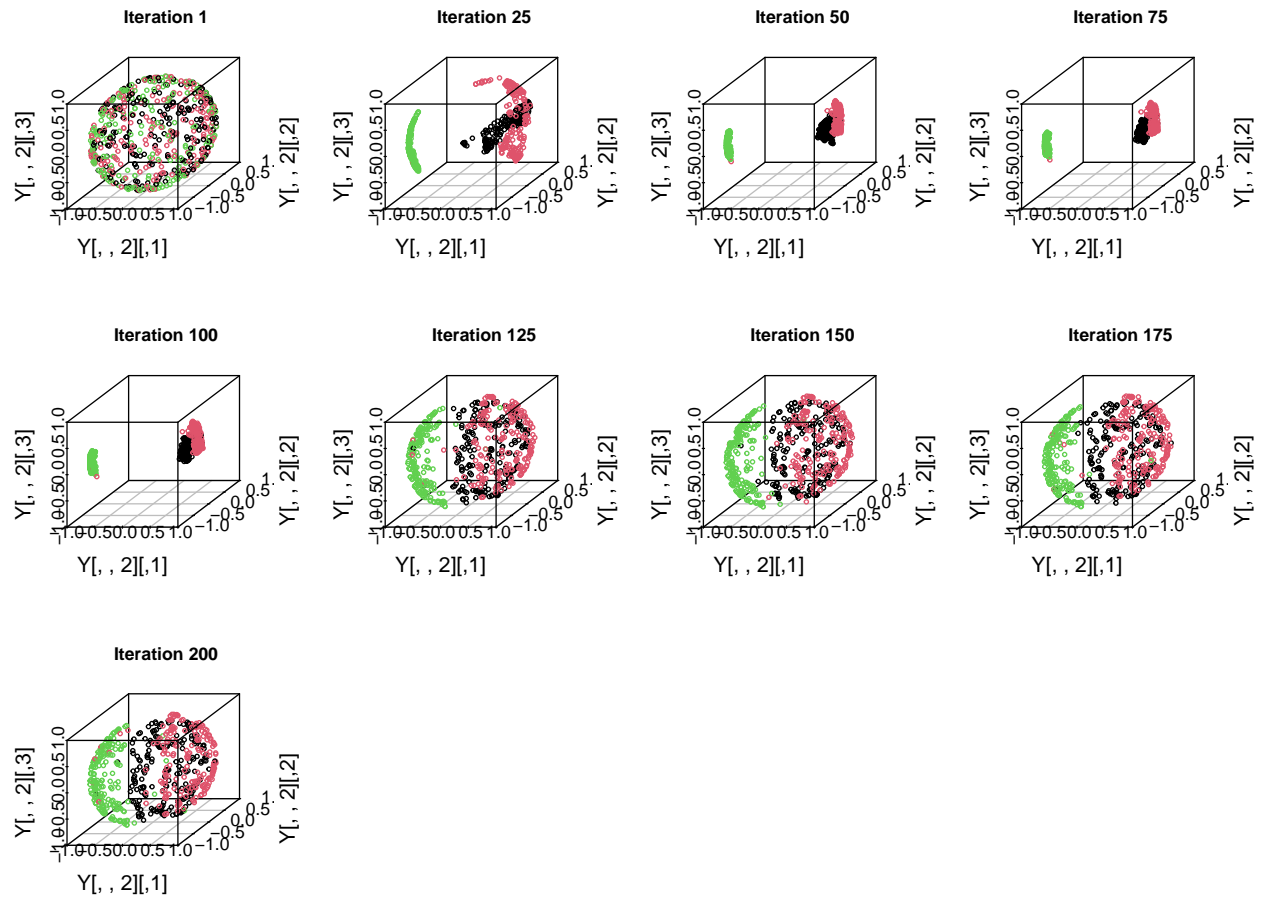
## It: 126; obj: 1.514e+00; abs: 1.247e-02; rel: 8.300e-03; norm: 1.399e-01
## It: 127; obj: 1.512e+00; abs: 2.240e-03; rel: 1.480e-03; norm: 1.396e-01
## It: 128; obj: 1.522e+00; abs: 9.959e-03; rel: 6.587e-03; norm: 1.402e-01
## It: 129; obj: 1.525e+00; abs: 2.927e-03; rel: 1.923e-03; norm: 1.415e-01
## It: 130; obj: 1.501e+00; abs: 2.419e-02; rel: 1.586e-02; norm: 1.393e-01
## It: 131; obj: 1.525e+00; abs: 2.462e-02; rel: 1.641e-02; norm: 1.396e-01
## It: 132; obj: 1.532e+00; abs: 6.953e-03; rel: 4.558e-03; norm: 1.408e-01
## It: 133; obj: 1.507e+00; abs: 2.483e-02; rel: 1.621e-02; norm: 1.400e-01
## It: 134; obj: 1.518e+00; abs: 1.079e-02; rel: 7.160e-03; norm: 1.392e-01
## It: 135; obj: 1.539e+00; abs: 2.029e-02; rel: 1.336e-02; norm: 1.410e-01
## It: 136; obj: 1.509e+00; abs: 2.941e-02; rel: 1.912e-02; norm: 1.392e-01
## It: 137; obj: 1.519e+00; abs: 1.007e-02; rel: 6.669e-03; norm: 1.385e-01
## It: 138; obj: 1.533e+00; abs: 1.382e-02; rel: 9.096e-03; norm: 1.397e-01
## It: 139; obj: 1.516e+00; abs: 1.746e-02; rel: 1.139e-02; norm: 1.390e-01
## It: 140; obj: 1.509e+00; abs: 6.169e-03; rel: 4.071e-03; norm: 1.390e-01
## It: 141; obj: 1.508e+00; abs: 1.247e-03; rel: 8.262e-04; norm: 1.400e-01
## It: 142; obj: 1.499e+00; abs: 8.980e-03; rel: 5.954e-03; norm: 1.401e-01
## It: 143; obj: 1.520e+00; abs: 2.085e-02; rel: 1.391e-02; norm: 1.398e-01
## It: 144; obj: 1.517e+00; abs: 2.800e-03; rel: 1.842e-03; norm: 1.401e-01
## It: 145; obj: 1.506e+00; abs: 1.136e-02; rel: 7.486e-03; norm: 1.408e-01
## It: 146; obj: 1.507e+00; abs: 1.495e-03; rel: 9.931e-04; norm: 1.392e-01
## It: 147; obj: 1.508e+00; abs: 6.212e-04; rel: 4.121e-04; norm: 1.395e-01
## It: 148; obj: 1.510e+00; abs: 1.546e-03; rel: 1.025e-03; norm: 1.397e-01
## It: 149; obj: 1.510e+00; abs: 9.492e-04; rel: 6.288e-04; norm: 1.399e-01
## It: 150; obj: 1.506e+00; abs: 4.896e-03; rel: 3.241e-03; norm: 1.396e-01

## It: 151; obj: 1.497e+00; abs: 8.549e-03; rel: 5.678e-03; norm: 1.398e-01
## It: 152; obj: 1.522e+00; abs: 2.538e-02; rel: 1.696e-02; norm: 1.398e-01
## It: 153; obj: 1.519e+00; abs: 3.093e-03; rel: 2.032e-03; norm: 1.411e-01
## It: 154; obj: 1.491e+00; abs: 2.825e-02; rel: 1.859e-02; norm: 1.397e-01
## It: 155; obj: 1.508e+00; abs: 1.701e-02; rel: 1.141e-02; norm: 1.396e-01
## It: 156; obj: 1.518e+00; abs: 9.522e-03; rel: 6.314e-03; norm: 1.413e-01
## It: 157; obj: 1.503e+00; abs: 1.423e-02; rel: 9.374e-03; norm: 1.394e-01
## It: 158; obj: 1.517e+00; abs: 1.316e-02; rel: 8.755e-03; norm: 1.396e-01
## It: 159; obj: 1.530e+00; abs: 1.314e-02; rel: 8.662e-03; norm: 1.411e-01
## It: 160; obj: 1.513e+00; abs: 1.687e-02; rel: 1.103e-02; norm: 1.406e-01
## It: 161; obj: 1.520e+00; abs: 6.954e-03; rel: 4.597e-03; norm: 1.396e-01
## It: 162; obj: 1.524e+00; abs: 4.145e-03; rel: 2.727e-03; norm: 1.399e-01
## It: 163; obj: 1.512e+00; abs: 1.203e-02; rel: 7.894e-03; norm: 1.398e-01
## It: 164; obj: 1.528e+00; abs: 1.625e-02; rel: 1.075e-02; norm: 1.394e-01
## It: 165; obj: 1.517e+00; abs: 1.087e-02; rel: 7.116e-03; norm: 1.393e-01
## It: 166; obj: 1.519e+00; abs: 1.526e-03; rel: 1.006e-03; norm: 1.396e-01
## It: 167; obj: 1.533e+00; abs: 1.470e-02; rel: 9.682e-03; norm: 1.405e-01
## It: 168; obj: 1.528e+00; abs: 5.381e-03; rel: 3.509e-03; norm: 1.396e-01
## It: 169; obj: 1.514e+00; abs: 1.456e-02; rel: 9.525e-03; norm: 1.389e-01
## It: 170; obj: 1.525e+00; abs: 1.112e-02; rel: 7.350e-03; norm: 1.406e-01
## It: 171; obj: 1.514e+00; abs: 1.022e-02; rel: 6.702e-03; norm: 1.399e-01
## It: 172; obj: 1.505e+00; abs: 9.898e-03; rel: 6.536e-03; norm: 1.396e-01
## It: 173; obj: 1.526e+00; abs: 2.186e-02; rel: 1.453e-02; norm: 1.417e-01
## It: 174; obj: 1.507e+00; abs: 1.910e-02; rel: 1.251e-02; norm: 1.408e-01

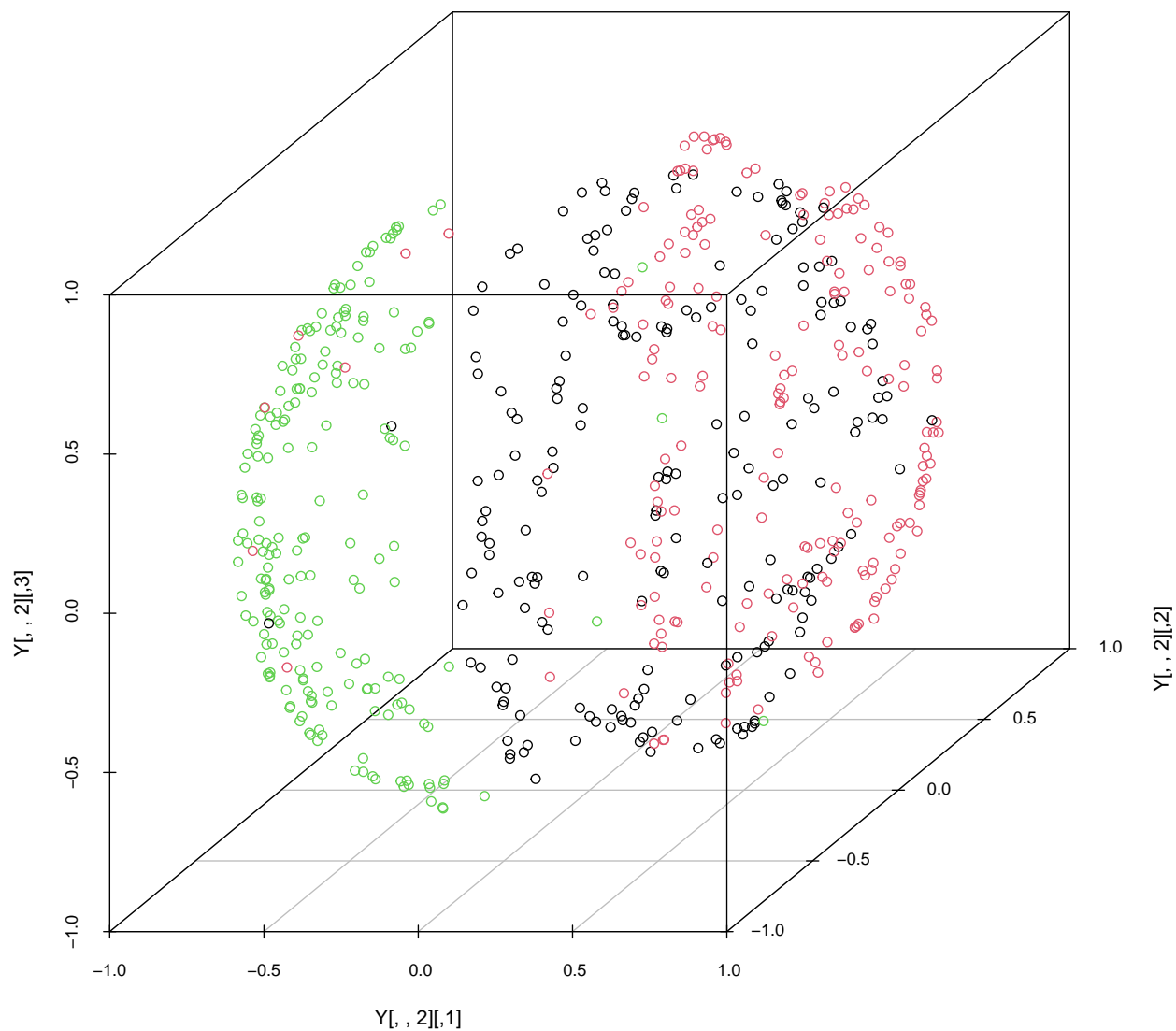
```

## It: 175; obj: 1.507e+00; abs: 2.693e-04; rel: 1.787e-04; norm: 1.403e-01  
## It: 176; obj: 1.532e+00; abs: 2.540e-02; rel: 1.685e-02; norm: 1.399e-01  
## It: 177; obj: 1.536e+00; abs: 3.521e-03; rel: 2.298e-03; norm: 1.407e-01  
## It: 178; obj: 1.518e+00; abs: 1.770e-02; rel: 1.152e-02; norm: 1.397e-01  
## It: 179; obj: 1.541e+00; abs: 2.239e-02; rel: 1.474e-02; norm: 1.400e-01  
## It: 180; obj: 1.543e+00; abs: 2.791e-03; rel: 1.811e-03; norm: 1.406e-01  
## It: 181; obj: 1.511e+00; abs: 3.293e-02; rel: 2.134e-02; norm: 1.393e-01  
## It: 182; obj: 1.534e+00; abs: 2.347e-02; rel: 1.554e-02; norm: 1.387e-01  
## It: 183; obj: 1.535e+00; abs: 1.362e-03; rel: 8.878e-04; norm: 1.407e-01  
## It: 184; obj: 1.516e+00; abs: 1.968e-02; rel: 1.282e-02; norm: 1.390e-01  
## It: 185; obj: 1.522e+00; abs: 6.162e-03; rel: 4.066e-03; norm: 1.392e-01  
## It: 186; obj: 1.524e+00; abs: 2.171e-03; rel: 1.426e-03; norm: 1.400e-01  
## It: 187; obj: 1.507e+00; abs: 1.722e-02; rel: 1.130e-02; norm: 1.394e-01  
## It: 188; obj: 1.524e+00; abs: 1.759e-02; rel: 1.167e-02; norm: 1.400e-01  
## It: 189; obj: 1.533e+00; abs: 8.365e-03; rel: 5.487e-03; norm: 1.399e-01  
## It: 190; obj: 1.526e+00; abs: 6.745e-03; rel: 4.401e-03; norm: 1.384e-01  
## It: 191; obj: 1.536e+00; abs: 1.049e-02; rel: 6.877e-03; norm: 1.396e-01  
## It: 192; obj: 1.522e+00; abs: 1.492e-02; rel: 9.713e-03; norm: 1.395e-01  
## It: 193; obj: 1.505e+00; abs: 1.659e-02; rel: 1.090e-02; norm: 1.383e-01  
## It: 194; obj: 1.525e+00; abs: 1.968e-02; rel: 1.307e-02; norm: 1.403e-01  
## It: 195; obj: 1.527e+00; abs: 1.907e-03; rel: 1.251e-03; norm: 1.403e-01  
## It: 196; obj: 1.504e+00; abs: 2.265e-02; rel: 1.484e-02; norm: 1.392e-01  
## It: 197; obj: 1.527e+00; abs: 2.312e-02; rel: 1.538e-02; norm: 1.397e-01  
## It: 198; obj: 1.519e+00; abs: 7.613e-03; rel: 4.985e-03; norm: 1.408e-01  
## It: 199; obj: 1.518e+00; abs: 1.104e-03; rel: 7.268e-04; norm: 1.400e-01  
## It: 200; obj: 1.535e+00; abs: 1.663e-02; rel: 1.096e-02; norm: 1.405e-01

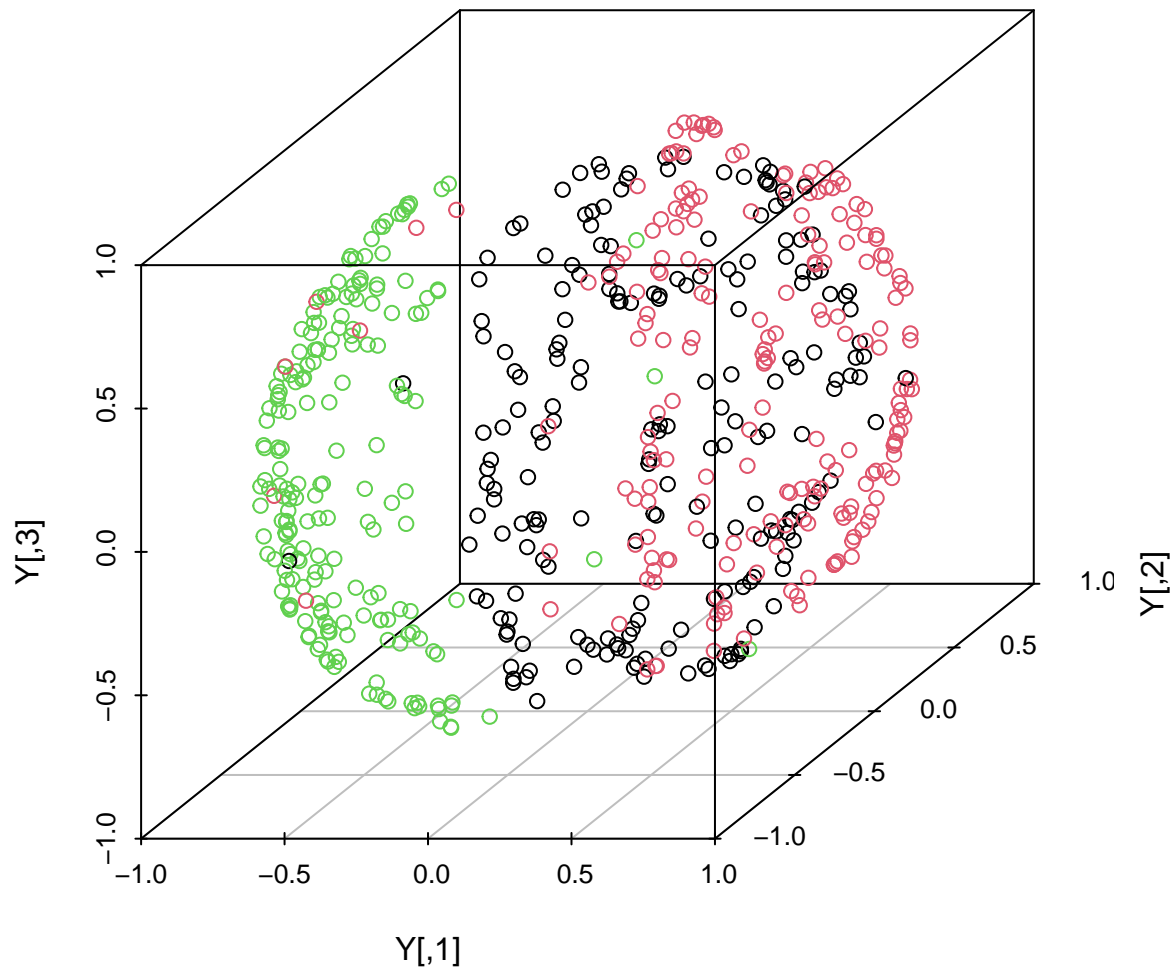




Iteration 200

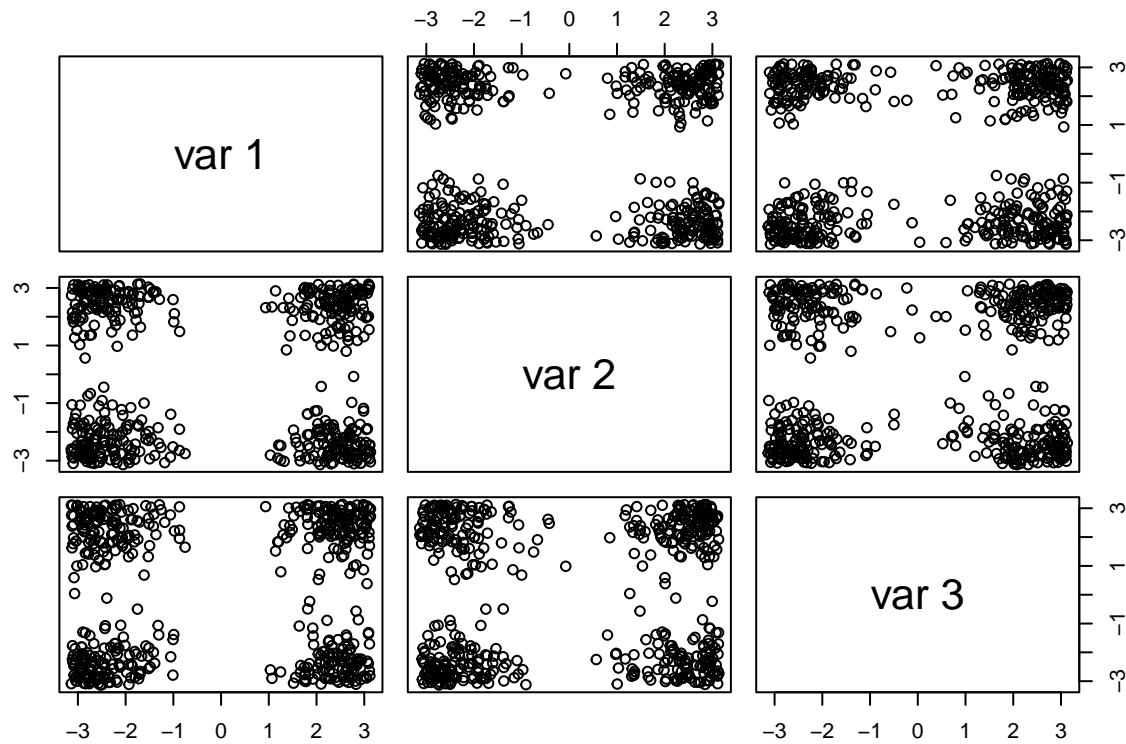


```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),  
                             color = colors)
```

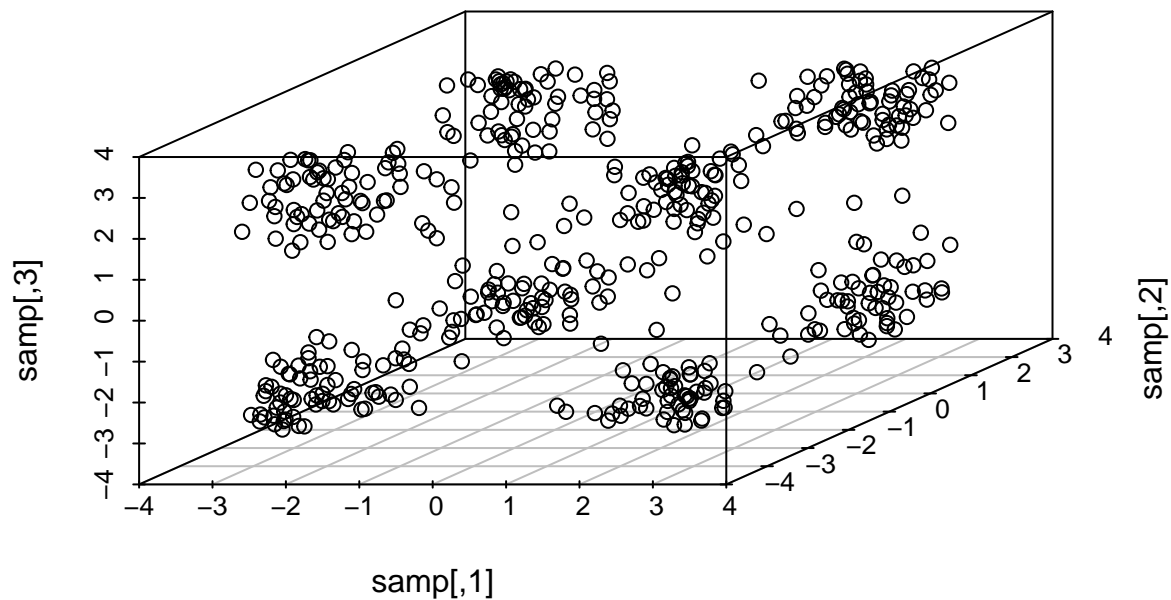


## Case 4

```
d <- 3
n <- 5e2
samp <- sdetorus::toPiInt(mvtnorm::rmvnorm(n = n, mean = rep(pi, d)))
pairs(samp)
```



```
scatterplot3d::scatterplot3d(samp, xlim = c(-pi, pi), ylim = c(-pi, pi), zlim = c(-pi, pi))
```



```
x_4 <- array(dim = c(n, 2, 3))
x_4[, , 1] <- DirStats::to_cir(samp[, 1])
x_4[, , 2] <- DirStats::to_cir(samp[, 2])
x_4[, , 3] <- DirStats::to_cir(samp[, 3])
```

Let's calculate the rho parameters based on a perplexity of 25:

```
rho_four_perp20 <- rho_optim_bst(x_4, 20)
```

```
## Time difference of 47.50501 secs
```

First, let's reduce to dimension  $\mathbb{S}^1$  then  $d = 1$  (circumference):

```
Y <- psc_sne(X=x_4, d=1, rho_psc_list = rho_four_perp20, num_iteration=200,  
            visualize_prog = TRUE)
```

```
## It: 1; obj: 1.771e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.564e-01  
## It: 2; obj: 1.709e+01; abs: 6.174e-01; rel: 3.486e-02; norm: 3.308e-01  
## It: 3; obj: 1.674e+01; abs: 3.529e-01; rel: 2.065e-02; norm: 4.190e-01  
## It: 4; obj: 1.655e+01; abs: 1.894e-01; rel: 1.131e-02; norm: 4.743e-01  
## It: 5; obj: 1.639e+01; abs: 1.558e-01; rel: 9.413e-03; norm: 5.061e-01  
## It: 6; obj: 1.628e+01; abs: 1.183e-01; rel: 7.216e-03; norm: 5.328e-01  
## It: 7; obj: 1.617e+01; abs: 1.089e-01; rel: 6.689e-03; norm: 5.561e-01  
## It: 8; obj: 1.607e+01; abs: 9.237e-02; rel: 5.713e-03; norm: 5.791e-01  
## It: 9; obj: 1.600e+01; abs: 7.697e-02; rel: 4.788e-03; norm: 5.993e-01  
## It: 10; obj: 1.593e+01; abs: 7.010e-02; rel: 4.382e-03; norm: 6.142e-01  
## It: 11; obj: 1.586e+01; abs: 6.761e-02; rel: 4.245e-03; norm: 6.246e-01  
## It: 12; obj: 1.580e+01; abs: 6.444e-02; rel: 4.063e-03; norm: 6.333e-01  
## It: 13; obj: 1.574e+01; abs: 5.582e-02; rel: 3.534e-03; norm: 6.416e-01  
## It: 14; obj: 1.569e+01; abs: 4.565e-02; rel: 2.900e-03; norm: 6.498e-01  
## It: 15; obj: 1.566e+01; abs: 3.774e-02; rel: 2.404e-03; norm: 6.571e-01  
## It: 16; obj: 1.562e+01; abs: 3.369e-02; rel: 2.152e-03; norm: 6.632e-01  
## It: 17; obj: 1.559e+01; abs: 3.135e-02; rel: 2.006e-03; norm: 6.684e-01  
## It: 18; obj: 1.556e+01; abs: 3.008e-02; rel: 1.929e-03; norm: 6.730e-01  
## It: 19; obj: 1.553e+01; abs: 3.008e-02; rel: 1.933e-03; norm: 6.773e-01  
## It: 20; obj: 1.550e+01; abs: 2.930e-02; rel: 1.886e-03; norm: 6.814e-01  
## It: 21; obj: 1.548e+01; abs: 2.602e-02; rel: 1.678e-03; norm: 6.853e-01  
## It: 22; obj: 1.545e+01; abs: 2.160e-02; rel: 1.396e-03; norm: 6.888e-01  
## It: 23; obj: 1.544e+01; abs: 1.834e-02; rel: 1.187e-03; norm: 6.914e-01  
## It: 24; obj: 1.542e+01; abs: 1.648e-02; rel: 1.068e-03; norm: 6.935e-01  
## It: 25; obj: 1.540e+01; abs: 1.556e-02; rel: 1.009e-03; norm: 6.954e-01  
  
## It: 26; obj: 1.539e+01; abs: 1.467e-02; rel: 9.525e-04; norm: 6.974e-01  
## It: 27; obj: 1.538e+01; abs: 1.310e-02; rel: 8.512e-04; norm: 6.998e-01  
## It: 28; obj: 1.537e+01; abs: 1.091e-02; rel: 7.098e-04; norm: 7.026e-01  
## It: 29; obj: 1.536e+01; abs: 8.920e-03; rel: 5.805e-04; norm: 7.059e-01  
## It: 30; obj: 1.535e+01; abs: 7.484e-03; rel: 4.873e-04; norm: 7.097e-01  
## It: 31; obj: 1.534e+01; abs: 6.652e-03; rel: 4.334e-04; norm: 7.137e-01  
## It: 32; obj: 1.534e+01; abs: 6.280e-03; rel: 4.093e-04; norm: 7.178e-01  
## It: 33; obj: 1.533e+01; abs: 6.234e-03; rel: 4.065e-04; norm: 7.219e-01  
## It: 34; obj: 1.532e+01; abs: 6.396e-03; rel: 4.172e-04; norm: 7.260e-01  
## It: 35; obj: 1.532e+01; abs: 6.570e-03; rel: 4.288e-04; norm: 7.301e-01  
## It: 36; obj: 1.531e+01; abs: 6.629e-03; rel: 4.328e-04; norm: 7.339e-01  
## It: 37; obj: 1.530e+01; abs: 6.650e-03; rel: 4.343e-04; norm: 7.376e-01  
## It: 38; obj: 1.530e+01; abs: 6.795e-03; rel: 4.440e-04; norm: 7.409e-01  
## It: 39; obj: 1.529e+01; abs: 7.176e-03; rel: 4.691e-04; norm: 7.438e-01  
## It: 40; obj: 1.528e+01; abs: 7.884e-03; rel: 5.156e-04; norm: 7.463e-01  
## It: 41; obj: 1.527e+01; abs: 9.056e-03; rel: 5.926e-04; norm: 7.485e-01  
## It: 42; obj: 1.526e+01; abs: 1.046e-02; rel: 6.852e-04; norm: 7.503e-01  
## It: 43; obj: 1.525e+01; abs: 1.086e-02; rel: 7.114e-04; norm: 7.517e-01  
## It: 44; obj: 1.524e+01; abs: 9.941e-03; rel: 6.518e-04; norm: 7.529e-01  
## It: 45; obj: 1.523e+01; abs: 8.363e-03; rel: 5.487e-04; norm: 7.537e-01  
## It: 46; obj: 1.523e+01; abs: 6.628e-03; rel: 4.351e-04; norm: 7.544e-01  
## It: 47; obj: 1.522e+01; abs: 5.512e-03; rel: 3.620e-04; norm: 7.547e-01  
## It: 48; obj: 1.522e+01; abs: 4.832e-03; rel: 3.175e-04; norm: 7.548e-01  
## It: 49; obj: 1.521e+01; abs: 3.982e-03; rel: 2.617e-04; norm: 7.548e-01
```

```

## It: 50; obj: 1.521e+01; abs: 2.876e-03; rel: 1.891e-04; norm: 7.547e-01
## It: 51; obj: 1.521e+01; abs: 1.867e-03; rel: 1.227e-04; norm: 7.545e-01
## It: 52; obj: 1.521e+01; abs: 1.121e-03; rel: 7.373e-05; norm: 7.541e-01
## It: 53; obj: 1.521e+01; abs: 6.125e-04; rel: 4.028e-05; norm: 7.537e-01
## It: 54; obj: 1.521e+01; abs: 2.657e-04; rel: 1.747e-05; norm: 7.532e-01
## It: 55; obj: 1.521e+01; abs: 2.063e-05; rel: 1.356e-06; norm: 7.526e-01
## It: 56; obj: 1.521e+01; abs: 1.585e-04; rel: 1.043e-05; norm: 7.520e-01
## It: 57; obj: 1.521e+01; abs: 2.905e-04; rel: 1.911e-05; norm: 7.514e-01
## It: 58; obj: 1.521e+01; abs: 3.879e-04; rel: 2.551e-05; norm: 7.507e-01
## It: 59; obj: 1.521e+01; abs: 4.672e-04; rel: 3.072e-05; norm: 7.499e-01
## It: 60; obj: 1.521e+01; abs: 5.516e-04; rel: 3.627e-05; norm: 7.492e-01
## It: 61; obj: 1.521e+01; abs: 6.603e-04; rel: 4.342e-05; norm: 7.484e-01
## It: 62; obj: 1.521e+01; abs: 7.948e-04; rel: 5.227e-05; norm: 7.477e-01
## It: 63; obj: 1.521e+01; abs: 9.409e-04; rel: 6.187e-05; norm: 7.470e-01
## It: 64; obj: 1.521e+01; abs: 1.084e-03; rel: 7.125e-05; norm: 7.462e-01
## It: 65; obj: 1.521e+01; abs: 1.217e-03; rel: 8.000e-05; norm: 7.455e-01
## It: 66; obj: 1.521e+01; abs: 1.342e-03; rel: 8.820e-05; norm: 7.449e-01
## It: 67; obj: 1.521e+01; abs: 1.462e-03; rel: 9.610e-05; norm: 7.442e-01
## It: 68; obj: 1.522e+01; abs: 1.582e-03; rel: 1.039e-04; norm: 7.436e-01
## It: 69; obj: 1.522e+01; abs: 1.703e-03; rel: 1.119e-04; norm: 7.430e-01
## It: 70; obj: 1.522e+01; abs: 1.827e-03; rel: 1.200e-04; norm: 7.424e-01
## It: 71; obj: 1.522e+01; abs: 1.954e-03; rel: 1.284e-04; norm: 7.419e-01
## It: 72; obj: 1.522e+01; abs: 2.083e-03; rel: 1.368e-04; norm: 7.413e-01
## It: 73; obj: 1.523e+01; abs: 2.213e-03; rel: 1.454e-04; norm: 7.408e-01
## It: 74; obj: 1.523e+01; abs: 2.344e-03; rel: 1.540e-04; norm: 7.403e-01
## It: 75; obj: 1.523e+01; abs: 2.475e-03; rel: 1.625e-04; norm: 7.399e-01

## It: 76; obj: 1.523e+01; abs: 2.605e-03; rel: 1.710e-04; norm: 7.394e-01
## It: 77; obj: 1.524e+01; abs: 2.734e-03; rel: 1.794e-04; norm: 7.390e-01
## It: 78; obj: 1.524e+01; abs: 2.860e-03; rel: 1.877e-04; norm: 7.386e-01
## It: 79; obj: 1.524e+01; abs: 2.985e-03; rel: 1.959e-04; norm: 7.382e-01
## It: 80; obj: 1.525e+01; abs: 3.108e-03; rel: 2.039e-04; norm: 7.378e-01
## It: 81; obj: 1.525e+01; abs: 3.229e-03; rel: 2.118e-04; norm: 7.374e-01
## It: 82; obj: 1.525e+01; abs: 3.348e-03; rel: 2.196e-04; norm: 7.371e-01
## It: 83; obj: 1.526e+01; abs: 3.466e-03; rel: 2.273e-04; norm: 7.367e-01
## It: 84; obj: 1.526e+01; abs: 3.584e-03; rel: 2.349e-04; norm: 7.364e-01
## It: 85; obj: 1.526e+01; abs: 3.701e-03; rel: 2.426e-04; norm: 7.361e-01
## It: 86; obj: 1.527e+01; abs: 3.819e-03; rel: 2.503e-04; norm: 7.358e-01
## It: 87; obj: 1.527e+01; abs: 3.939e-03; rel: 2.580e-04; norm: 7.355e-01
## It: 88; obj: 1.527e+01; abs: 4.061e-03; rel: 2.659e-04; norm: 7.353e-01
## It: 89; obj: 1.528e+01; abs: 4.186e-03; rel: 2.741e-04; norm: 7.350e-01
## It: 90; obj: 1.528e+01; abs: 4.318e-03; rel: 2.826e-04; norm: 7.348e-01
## It: 91; obj: 1.529e+01; abs: 4.460e-03; rel: 2.918e-04; norm: 7.345e-01
## It: 92; obj: 1.529e+01; abs: 4.616e-03; rel: 3.020e-04; norm: 7.343e-01
## It: 93; obj: 1.530e+01; abs: 4.789e-03; rel: 3.132e-04; norm: 7.341e-01
## It: 94; obj: 1.530e+01; abs: 4.877e-03; rel: 3.188e-04; norm: 7.339e-01
## It: 95; obj: 1.531e+01; abs: 4.071e-03; rel: 2.661e-04; norm: 7.338e-01
## It: 96; obj: 1.531e+01; abs: 4.310e-03; rel: 2.816e-04; norm: 7.340e-01
## It: 97; obj: 1.531e+01; abs: 4.983e-03; rel: 3.255e-04; norm: 7.345e-01
## It: 98; obj: 1.532e+01; abs: 5.161e-03; rel: 3.370e-04; norm: 7.344e-01
## It: 99; obj: 1.533e+01; abs: 5.319e-03; rel: 3.472e-04; norm: 7.343e-01
## It: 100; obj: 1.533e+01; abs: 5.448e-03; rel: 3.555e-04; norm: 7.341e-01

## It: 101; obj: 2.429e+00; abs: 1.290e+01; rel: 8.416e-01; norm: 1.007e-01

```

```

## It: 102; obj: 2.414e+00; abs: 1.460e-02; rel: 6.011e-03; norm: 9.630e-02
## It: 103; obj: 2.402e+00; abs: 1.179e-02; rel: 4.883e-03; norm: 9.524e-02
## It: 104; obj: 2.393e+00; abs: 9.655e-03; rel: 4.019e-03; norm: 9.526e-02
## It: 105; obj: 2.389e+00; abs: 4.111e-03; rel: 1.718e-03; norm: 9.547e-02
## It: 106; obj: 2.386e+00; abs: 2.809e-03; rel: 1.176e-03; norm: 9.608e-02
## It: 107; obj: 2.383e+00; abs: 2.424e-03; rel: 1.016e-03; norm: 9.667e-02
## It: 108; obj: 2.381e+00; abs: 2.464e-03; rel: 1.034e-03; norm: 9.665e-02
## It: 109; obj: 2.380e+00; abs: 9.339e-04; rel: 3.922e-04; norm: 9.658e-02
## It: 110; obj: 2.381e+00; abs: 1.370e-03; rel: 5.754e-04; norm: 9.651e-02
## It: 111; obj: 2.380e+00; abs: 1.131e-03; rel: 4.750e-04; norm: 9.663e-02
## It: 112; obj: 2.379e+00; abs: 1.193e-03; rel: 5.012e-04; norm: 9.669e-02
## It: 113; obj: 2.379e+00; abs: 1.413e-04; rel: 5.941e-05; norm: 9.675e-02
## It: 114; obj: 2.380e+00; abs: 5.909e-04; rel: 2.484e-04; norm: 9.686e-02
## It: 115; obj: 2.380e+00; abs: 2.373e-04; rel: 9.973e-05; norm: 9.699e-02
## It: 116; obj: 2.379e+00; abs: 6.197e-04; rel: 2.604e-04; norm: 9.686e-02
## It: 117; obj: 2.378e+00; abs: 5.594e-04; rel: 2.351e-04; norm: 9.685e-02
## It: 118; obj: 2.378e+00; abs: 2.202e-04; rel: 9.259e-05; norm: 9.692e-02
## It: 119; obj: 2.379e+00; abs: 4.081e-04; rel: 1.716e-04; norm: 9.700e-02
## It: 120; obj: 2.377e+00; abs: 1.911e-03; rel: 8.032e-04; norm: 9.723e-02
## It: 121; obj: 2.377e+00; abs: 6.573e-04; rel: 2.765e-04; norm: 9.697e-02
## It: 122; obj: 2.378e+00; abs: 8.460e-04; rel: 3.559e-04; norm: 9.675e-02
## It: 123; obj: 2.378e+00; abs: 3.537e-05; rel: 1.487e-05; norm: 9.714e-02
## It: 124; obj: 2.377e+00; abs: 1.418e-03; rel: 5.961e-04; norm: 9.711e-02
## It: 125; obj: 2.377e+00; abs: 2.152e-04; rel: 9.056e-05; norm: 9.688e-02

## It: 126; obj: 2.377e+00; abs: 4.815e-04; rel: 2.026e-04; norm: 9.696e-02
## It: 127; obj: 2.377e+00; abs: 1.450e-04; rel: 6.101e-05; norm: 9.714e-02
## It: 128; obj: 2.376e+00; abs: 9.747e-04; rel: 4.101e-04; norm: 9.697e-02
## It: 129; obj: 2.376e+00; abs: 8.071e-05; rel: 3.397e-05; norm: 9.689e-02
## It: 130; obj: 2.376e+00; abs: 3.259e-04; rel: 1.372e-04; norm: 9.677e-02
## It: 131; obj: 2.375e+00; abs: 3.517e-05; rel: 1.481e-05; norm: 9.684e-02
## It: 132; obj: 2.377e+00; abs: 1.494e-03; rel: 6.291e-04; norm: 9.674e-02
## It: 133; obj: 2.375e+00; abs: 1.903e-03; rel: 8.008e-04; norm: 9.678e-02
## It: 134; obj: 2.374e+00; abs: 9.556e-04; rel: 4.023e-04; norm: 9.704e-02
## It: 135; obj: 2.376e+00; abs: 1.403e-03; rel: 5.909e-04; norm: 9.681e-02
## It: 136; obj: 2.376e+00; abs: 5.218e-04; rel: 2.197e-04; norm: 9.656e-02
## It: 137; obj: 2.375e+00; abs: 6.672e-04; rel: 2.808e-04; norm: 9.657e-02
## It: 138; obj: 2.376e+00; abs: 7.538e-04; rel: 3.173e-04; norm: 9.659e-02
## It: 139; obj: 2.375e+00; abs: 1.180e-03; rel: 4.968e-04; norm: 9.659e-02
## It: 140; obj: 2.374e+00; abs: 1.286e-03; rel: 5.414e-04; norm: 9.677e-02
## It: 141; obj: 2.374e+00; abs: 8.391e-04; rel: 3.535e-04; norm: 9.686e-02
## It: 142; obj: 2.374e+00; abs: 3.525e-04; rel: 1.484e-04; norm: 9.679e-02
## It: 143; obj: 2.375e+00; abs: 8.692e-04; rel: 3.661e-04; norm: 9.678e-02
## It: 144; obj: 2.376e+00; abs: 8.046e-04; rel: 3.388e-04; norm: 9.699e-02
## It: 145; obj: 2.375e+00; abs: 8.194e-04; rel: 3.449e-04; norm: 9.704e-02
## It: 146; obj: 2.374e+00; abs: 1.291e-03; rel: 5.437e-04; norm: 9.724e-02
## It: 147; obj: 2.374e+00; abs: 5.685e-04; rel: 2.395e-04; norm: 9.717e-02
## It: 148; obj: 2.374e+00; abs: 1.177e-04; rel: 4.956e-05; norm: 9.706e-02
## It: 149; obj: 2.375e+00; abs: 4.571e-04; rel: 1.925e-04; norm: 9.707e-02
## It: 150; obj: 2.374e+00; abs: 4.478e-04; rel: 1.885e-04; norm: 9.696e-02

## It: 151; obj: 2.376e+00; abs: 1.289e-03; rel: 5.428e-04; norm: 9.697e-02
## It: 152; obj: 2.373e+00; abs: 2.888e-03; rel: 1.216e-03; norm: 9.726e-02
## It: 153; obj: 2.373e+00; abs: 9.124e-05; rel: 3.845e-05; norm: 9.736e-02
## It: 154; obj: 2.374e+00; abs: 7.997e-04; rel: 3.370e-04; norm: 9.715e-02

```

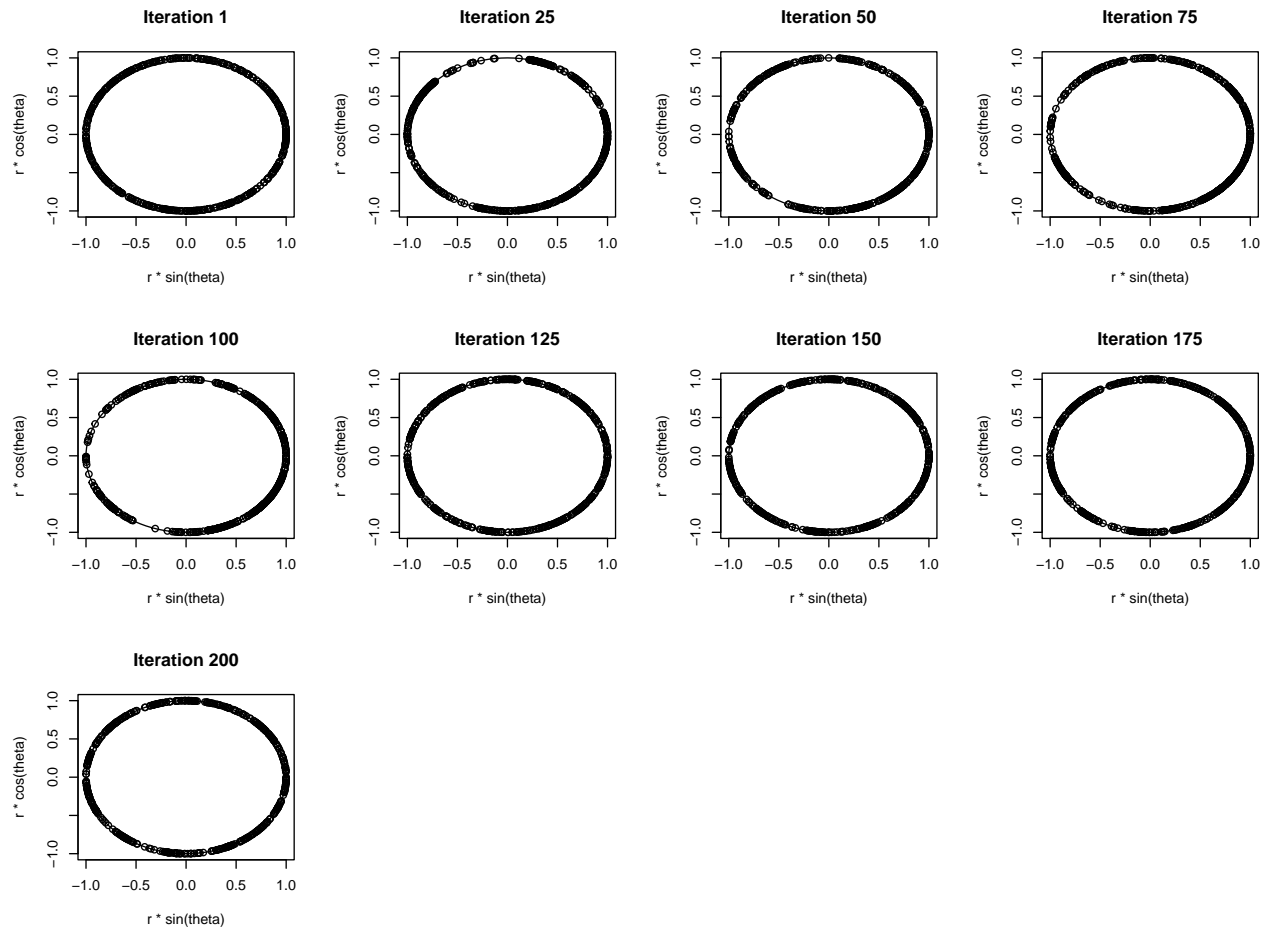
```

## It: 155; obj: 2.374e+00; abs: 7.428e-04; rel: 3.129e-04; norm: 9.699e-02
## It: 156; obj: 2.375e+00; abs: 5.353e-04; rel: 2.255e-04; norm: 9.716e-02
## It: 157; obj: 2.374e+00; abs: 6.207e-04; rel: 2.614e-04; norm: 9.707e-02
## It: 158; obj: 2.374e+00; abs: 7.682e-04; rel: 3.235e-04; norm: 9.714e-02
## It: 159; obj: 2.373e+00; abs: 6.350e-04; rel: 2.675e-04; norm: 9.749e-02
## It: 160; obj: 2.373e+00; abs: 5.285e-04; rel: 2.227e-04; norm: 9.729e-02
## It: 161; obj: 2.374e+00; abs: 7.863e-04; rel: 3.313e-04; norm: 9.711e-02
## It: 162; obj: 2.375e+00; abs: 6.887e-04; rel: 2.901e-04; norm: 9.722e-02
## It: 163; obj: 2.374e+00; abs: 7.443e-04; rel: 3.134e-04; norm: 9.718e-02
## It: 164; obj: 2.373e+00; abs: 9.146e-04; rel: 3.852e-04; norm: 9.723e-02
## It: 165; obj: 2.373e+00; abs: 2.764e-04; rel: 1.165e-04; norm: 9.751e-02
## It: 166; obj: 2.374e+00; abs: 6.376e-04; rel: 2.687e-04; norm: 9.747e-02
## It: 167; obj: 2.374e+00; abs: 3.207e-04; rel: 1.351e-04; norm: 9.721e-02
## It: 168; obj: 2.375e+00; abs: 9.614e-04; rel: 4.050e-04; norm: 9.736e-02
## It: 169; obj: 2.374e+00; abs: 1.246e-03; rel: 5.245e-04; norm: 9.735e-02
## It: 170; obj: 2.373e+00; abs: 5.007e-04; rel: 2.110e-04; norm: 9.740e-02
## It: 171; obj: 2.373e+00; abs: 3.660e-04; rel: 1.542e-04; norm: 9.756e-02
## It: 172; obj: 2.373e+00; abs: 1.048e-04; rel: 4.416e-05; norm: 9.733e-02
## It: 173; obj: 2.374e+00; abs: 1.218e-03; rel: 5.131e-04; norm: 9.734e-02
## It: 174; obj: 2.375e+00; abs: 5.913e-04; rel: 2.491e-04; norm: 9.751e-02
## It: 175; obj: 2.374e+00; abs: 2.901e-04; rel: 1.222e-04; norm: 9.728e-02

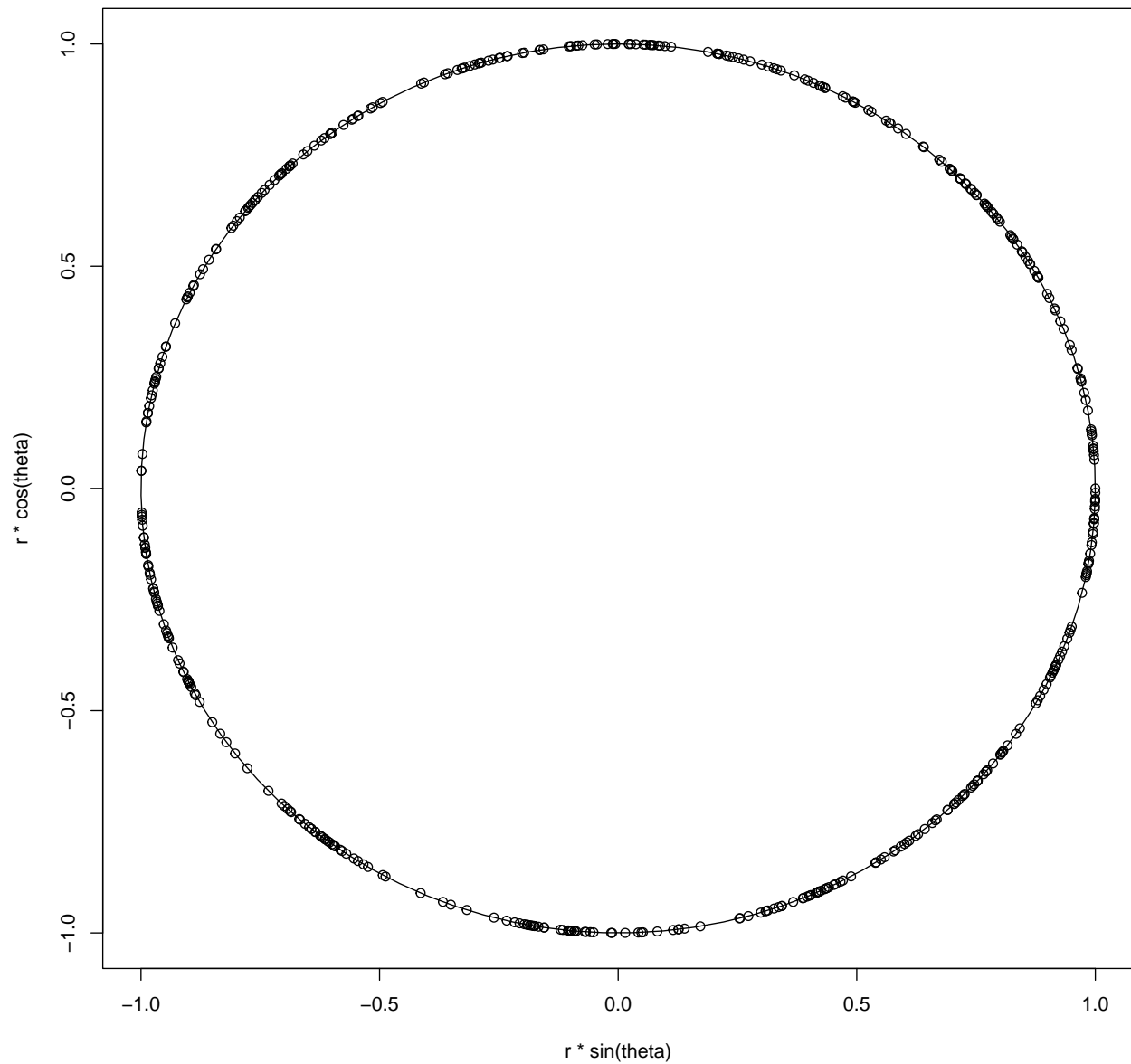
## It: 176; obj: 2.373e+00; abs: 1.243e-03; rel: 5.236e-04; norm: 9.753e-02
## It: 177; obj: 2.373e+00; abs: 4.236e-04; rel: 1.785e-04; norm: 9.767e-02
## It: 178; obj: 2.373e+00; abs: 3.062e-04; rel: 1.290e-04; norm: 9.739e-02
## It: 179; obj: 2.374e+00; abs: 1.098e-03; rel: 4.629e-04; norm: 9.750e-02
## It: 180; obj: 2.374e+00; abs: 1.733e-04; rel: 7.300e-05; norm: 9.761e-02
## It: 181; obj: 2.373e+00; abs: 1.195e-03; rel: 5.035e-04; norm: 9.753e-02
## It: 182; obj: 2.373e+00; abs: 4.422e-06; rel: 1.863e-06; norm: 9.767e-02
## It: 183; obj: 2.372e+00; abs: 1.064e-03; rel: 4.483e-04; norm: 9.759e-02
## It: 184; obj: 2.373e+00; abs: 7.040e-04; rel: 2.968e-04; norm: 9.758e-02
## It: 185; obj: 2.374e+00; abs: 9.809e-04; rel: 4.134e-04; norm: 9.762e-02
## It: 186; obj: 2.374e+00; abs: 5.025e-04; rel: 2.117e-04; norm: 9.752e-02
## It: 187; obj: 2.375e+00; abs: 8.036e-04; rel: 3.385e-04; norm: 9.768e-02
## It: 188; obj: 2.373e+00; abs: 1.696e-03; rel: 7.139e-04; norm: 9.771e-02
## It: 189; obj: 2.372e+00; abs: 1.156e-03; rel: 4.872e-04; norm: 9.773e-02
## It: 190; obj: 2.373e+00; abs: 5.033e-04; rel: 2.121e-04; norm: 9.780e-02
## It: 191; obj: 2.373e+00; abs: 5.867e-04; rel: 2.473e-04; norm: 9.771e-02
## It: 192; obj: 2.374e+00; abs: 8.017e-04; rel: 3.378e-04; norm: 9.772e-02
## It: 193; obj: 2.375e+00; abs: 7.374e-04; rel: 3.106e-04; norm: 9.780e-02
## It: 194; obj: 2.373e+00; abs: 1.795e-03; rel: 7.558e-04; norm: 9.767e-02
## It: 195; obj: 2.372e+00; abs: 1.000e-03; rel: 4.215e-04; norm: 9.790e-02
## It: 196; obj: 2.373e+00; abs: 4.675e-04; rel: 1.971e-04; norm: 9.794e-02
## It: 197; obj: 2.373e+00; abs: 5.346e-04; rel: 2.253e-04; norm: 9.768e-02
## It: 198; obj: 2.374e+00; abs: 9.027e-04; rel: 3.804e-04; norm: 9.783e-02
## It: 199; obj: 2.374e+00; abs: 1.549e-04; rel: 6.524e-05; norm: 9.791e-02
## It: 200; obj: 2.373e+00; abs: 1.516e-03; rel: 6.386e-04; norm: 9.776e-02

```



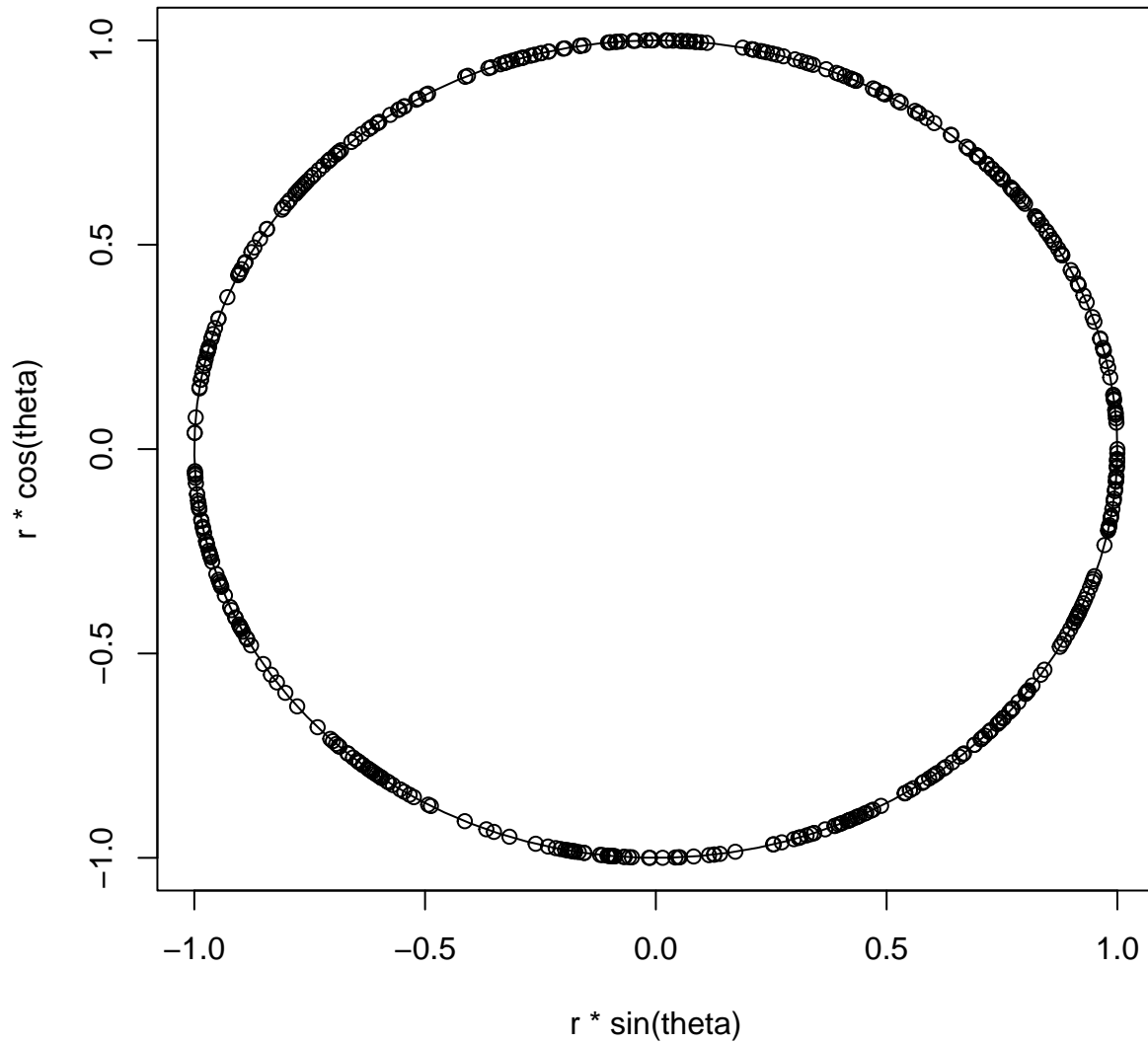


Iteration 200



```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension  $\mathbb{S}^2$  then  $d = 2$  (sphere):

```
Y <- psc_sne(X=x_4, d=2, rho_psc_list = rho_four_perp20, num_iteration=200,
             visualize_prog = TRUE)
```

```
## It: 1; obj: 1.964e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 2.002e-01
## It: 2; obj: 1.689e+01; abs: 2.743e+00; rel: 1.397e-01; norm: 3.746e-01
## It: 3; obj: 1.533e+01; abs: 1.565e+00; rel: 9.261e-02; norm: 6.768e-01
## It: 4; obj: 1.466e+01; abs: 6.731e-01; rel: 4.391e-02; norm: 8.654e-01
## It: 5; obj: 1.427e+01; abs: 3.897e-01; rel: 2.659e-02; norm: 9.624e-01
## It: 6; obj: 1.401e+01; abs: 2.520e-01; rel: 1.766e-02; norm: 1.008e+00
## It: 7; obj: 1.385e+01; abs: 1.652e-01; rel: 1.179e-02; norm: 1.037e+00
## It: 8; obj: 1.373e+01; abs: 1.235e-01; rel: 8.916e-03; norm: 1.054e+00
## It: 9; obj: 1.362e+01; abs: 1.039e-01; rel: 7.570e-03; norm: 1.065e+00
## It: 10; obj: 1.353e+01; abs: 9.436e-02; rel: 6.927e-03; norm: 1.072e+00
## It: 11; obj: 1.344e+01; abs: 8.359e-02; rel: 6.179e-03; norm: 1.079e+00
## It: 12; obj: 1.337e+01; abs: 7.912e-02; rel: 5.885e-03; norm: 1.086e+00
## It: 13; obj: 1.330e+01; abs: 6.630e-02; rel: 4.961e-03; norm: 1.097e+00
## It: 14; obj: 1.324e+01; abs: 5.922e-02; rel: 4.453e-03; norm: 1.112e+00
## It: 15; obj: 1.319e+01; abs: 4.852e-02; rel: 3.665e-03; norm: 1.131e+00
```

```

## It: 16; obj: 1.315e+01; abs: 4.001e-02; rel: 3.033e-03; norm: 1.150e+00
## It: 17; obj: 1.313e+01; abs: 2.599e-02; rel: 1.976e-03; norm: 1.168e+00
## It: 18; obj: 1.311e+01; abs: 1.513e-02; rel: 1.152e-03; norm: 1.184e+00
## It: 19; obj: 1.310e+01; abs: 5.611e-03; rel: 4.280e-04; norm: 1.199e+00
## It: 20; obj: 1.310e+01; abs: 2.447e-04; rel: 1.867e-05; norm: 1.211e+00
## It: 21; obj: 1.311e+01; abs: 3.531e-03; rel: 2.694e-04; norm: 1.222e+00
## It: 22; obj: 1.311e+01; abs: 5.165e-03; rel: 3.941e-04; norm: 1.232e+00
## It: 23; obj: 1.312e+01; abs: 6.256e-03; rel: 4.771e-04; norm: 1.240e+00
## It: 24; obj: 1.313e+01; abs: 6.007e-03; rel: 4.579e-04; norm: 1.247e+00
## It: 25; obj: 1.313e+01; abs: 5.346e-03; rel: 4.073e-04; norm: 1.253e+00

## It: 26; obj: 1.313e+01; abs: 3.230e-03; rel: 2.460e-04; norm: 1.258e+00
## It: 27; obj: 1.313e+01; abs: 1.108e-03; rel: 8.437e-05; norm: 1.261e+00
## It: 28; obj: 1.313e+01; abs: 3.153e-03; rel: 2.401e-04; norm: 1.264e+00
## It: 29; obj: 1.312e+01; abs: 7.585e-03; rel: 5.776e-04; norm: 1.266e+00
## It: 30; obj: 1.311e+01; abs: 1.180e-02; rel: 8.988e-04; norm: 1.267e+00
## It: 31; obj: 1.310e+01; abs: 8.609e-03; rel: 6.566e-04; norm: 1.268e+00
## It: 32; obj: 1.309e+01; abs: 9.589e-03; rel: 7.318e-04; norm: 1.269e+00
## It: 33; obj: 1.308e+01; abs: 1.113e-02; rel: 8.501e-04; norm: 1.269e+00
## It: 34; obj: 1.307e+01; abs: 1.252e-02; rel: 9.573e-04; norm: 1.269e+00
## It: 35; obj: 1.306e+01; abs: 1.314e-02; rel: 1.006e-03; norm: 1.269e+00
## It: 36; obj: 1.304e+01; abs: 1.377e-02; rel: 1.055e-03; norm: 1.268e+00
## It: 37; obj: 1.303e+01; abs: 1.464e-02; rel: 1.123e-03; norm: 1.267e+00
## It: 38; obj: 1.301e+01; abs: 1.635e-02; rel: 1.255e-03; norm: 1.266e+00
## It: 39; obj: 1.299e+01; abs: 1.829e-02; rel: 1.406e-03; norm: 1.265e+00
## It: 40; obj: 1.297e+01; abs: 1.958e-02; rel: 1.507e-03; norm: 1.264e+00
## It: 41; obj: 1.296e+01; abs: 1.970e-02; rel: 1.518e-03; norm: 1.263e+00
## It: 42; obj: 1.294e+01; abs: 1.895e-02; rel: 1.463e-03; norm: 1.262e+00
## It: 43; obj: 1.292e+01; abs: 1.759e-02; rel: 1.360e-03; norm: 1.260e+00
## It: 44; obj: 1.290e+01; abs: 1.590e-02; rel: 1.231e-03; norm: 1.258e+00
## It: 45; obj: 1.289e+01; abs: 1.400e-02; rel: 1.085e-03; norm: 1.255e+00
## It: 46; obj: 1.288e+01; abs: 1.195e-02; rel: 9.271e-04; norm: 1.252e+00
## It: 47; obj: 1.287e+01; abs: 9.809e-03; rel: 7.618e-04; norm: 1.249e+00
## It: 48; obj: 1.286e+01; abs: 7.660e-03; rel: 5.954e-04; norm: 1.245e+00
## It: 49; obj: 1.285e+01; abs: 5.579e-03; rel: 4.338e-04; norm: 1.242e+00
## It: 50; obj: 1.285e+01; abs: 3.615e-03; rel: 2.813e-04; norm: 1.238e+00

## It: 51; obj: 1.285e+01; abs: 1.779e-03; rel: 1.384e-04; norm: 1.235e+00
## It: 52; obj: 1.285e+01; abs: 4.300e-05; rel: 3.346e-06; norm: 1.232e+00
## It: 53; obj: 1.285e+01; abs: 1.638e-03; rel: 1.275e-04; norm: 1.229e+00
## It: 54; obj: 1.285e+01; abs: 3.310e-03; rel: 2.576e-04; norm: 1.227e+00
## It: 55; obj: 1.286e+01; abs: 5.007e-03; rel: 3.895e-04; norm: 1.225e+00
## It: 56; obj: 1.286e+01; abs: 6.742e-03; rel: 5.243e-04; norm: 1.224e+00
## It: 57; obj: 1.287e+01; abs: 8.510e-03; rel: 6.615e-04; norm: 1.223e+00
## It: 58; obj: 1.288e+01; abs: 1.029e-02; rel: 7.997e-04; norm: 1.223e+00
## It: 59; obj: 1.290e+01; abs: 1.207e-02; rel: 9.370e-04; norm: 1.224e+00
## It: 60; obj: 1.291e+01; abs: 1.382e-02; rel: 1.072e-03; norm: 1.225e+00
## It: 61; obj: 1.293e+01; abs: 1.553e-02; rel: 1.203e-03; norm: 1.227e+00
## It: 62; obj: 1.294e+01; abs: 1.718e-02; rel: 1.329e-03; norm: 1.229e+00
## It: 63; obj: 1.296e+01; abs: 1.878e-02; rel: 1.451e-03; norm: 1.232e+00
## It: 64; obj: 1.298e+01; abs: 2.032e-02; rel: 1.567e-03; norm: 1.236e+00
## It: 65; obj: 1.300e+01; abs: 2.180e-02; rel: 1.679e-03; norm: 1.240e+00
## It: 66; obj: 1.303e+01; abs: 2.322e-02; rel: 1.786e-03; norm: 1.245e+00
## It: 67; obj: 1.305e+01; abs: 2.459e-02; rel: 1.887e-03; norm: 1.250e+00
## It: 68; obj: 1.308e+01; abs: 2.590e-02; rel: 1.984e-03; norm: 1.256e+00

```

```

## It: 69; obj: 1.310e+01; abs: 2.716e-02; rel: 2.077e-03; norm: 1.262e+00
## It: 70; obj: 1.313e+01; abs: 2.837e-02; rel: 2.165e-03; norm: 1.268e+00
## It: 71; obj: 1.316e+01; abs: 2.954e-02; rel: 2.249e-03; norm: 1.275e+00
## It: 72; obj: 1.319e+01; abs: 3.067e-02; rel: 2.330e-03; norm: 1.282e+00
## It: 73; obj: 1.322e+01; abs: 3.176e-02; rel: 2.408e-03; norm: 1.289e+00
## It: 74; obj: 1.326e+01; abs: 3.284e-02; rel: 2.483e-03; norm: 1.296e+00
## It: 75; obj: 1.329e+01; abs: 3.391e-02; rel: 2.557e-03; norm: 1.304e+00

## It: 76; obj: 1.333e+01; abs: 3.497e-02; rel: 2.631e-03; norm: 1.313e+00
## It: 77; obj: 1.336e+01; abs: 3.604e-02; rel: 2.705e-03; norm: 1.321e+00
## It: 78; obj: 1.340e+01; abs: 3.713e-02; rel: 2.779e-03; norm: 1.330e+00
## It: 79; obj: 1.344e+01; abs: 3.826e-02; rel: 2.855e-03; norm: 1.340e+00
## It: 80; obj: 1.348e+01; abs: 3.943e-02; rel: 2.934e-03; norm: 1.350e+00
## It: 81; obj: 1.352e+01; abs: 4.064e-02; rel: 3.016e-03; norm: 1.361e+00
## It: 82; obj: 1.356e+01; abs: 4.192e-02; rel: 3.101e-03; norm: 1.372e+00
## It: 83; obj: 1.360e+01; abs: 4.326e-02; rel: 3.190e-03; norm: 1.384e+00
## It: 84; obj: 1.365e+01; abs: 4.466e-02; rel: 3.283e-03; norm: 1.397e+00
## It: 85; obj: 1.369e+01; abs: 4.613e-02; rel: 3.380e-03; norm: 1.411e+00
## It: 86; obj: 1.374e+01; abs: 4.766e-02; rel: 3.480e-03; norm: 1.425e+00
## It: 87; obj: 1.379e+01; abs: 4.924e-02; rel: 3.583e-03; norm: 1.441e+00
## It: 88; obj: 1.384e+01; abs: 5.085e-02; rel: 3.688e-03; norm: 1.457e+00
## It: 89; obj: 1.389e+01; abs: 5.249e-02; rel: 3.792e-03; norm: 1.474e+00
## It: 90; obj: 1.395e+01; abs: 5.414e-02; rel: 3.896e-03; norm: 1.492e+00
## It: 91; obj: 1.400e+01; abs: 5.577e-02; rel: 3.998e-03; norm: 1.510e+00
## It: 92; obj: 1.406e+01; abs: 5.736e-02; rel: 4.096e-03; norm: 1.530e+00
## It: 93; obj: 1.412e+01; abs: 5.891e-02; rel: 4.189e-03; norm: 1.550e+00
## It: 94; obj: 1.418e+01; abs: 6.038e-02; rel: 4.276e-03; norm: 1.570e+00
## It: 95; obj: 1.424e+01; abs: 6.177e-02; rel: 4.356e-03; norm: 1.591e+00
## It: 96; obj: 1.431e+01; abs: 6.304e-02; rel: 4.426e-03; norm: 1.612e+00
## It: 97; obj: 1.437e+01; abs: 6.419e-02; rel: 4.487e-03; norm: 1.634e+00
## It: 98; obj: 1.443e+01; abs: 6.519e-02; rel: 4.537e-03; norm: 1.655e+00
## It: 99; obj: 1.450e+01; abs: 6.603e-02; rel: 4.575e-03; norm: 1.677e+00
## It: 100; obj: 1.457e+01; abs: 6.669e-02; rel: 4.599e-03; norm: 1.699e+00

## It: 101; obj: 2.198e+00; abs: 1.237e+01; rel: 8.491e-01; norm: 1.648e-01
## It: 102; obj: 2.052e+00; abs: 1.468e-01; rel: 6.677e-02; norm: 1.115e-01
## It: 103; obj: 1.979e+00; abs: 7.270e-02; rel: 3.543e-02; norm: 1.154e-01
## It: 104; obj: 2.017e+00; abs: 3.816e-02; rel: 1.928e-02; norm: 1.208e-01
## It: 105; obj: 2.057e+00; abs: 3.992e-02; rel: 1.979e-02; norm: 1.300e-01
## It: 106; obj: 2.139e+00; abs: 8.154e-02; rel: 3.964e-02; norm: 1.332e-01
## It: 107; obj: 2.249e+00; abs: 1.103e-01; rel: 5.157e-02; norm: 1.429e-01
## It: 108; obj: 2.343e+00; abs: 9.392e-02; rel: 4.176e-02; norm: 1.514e-01
## It: 109; obj: 2.428e+00; abs: 8.497e-02; rel: 3.627e-02; norm: 1.627e-01
## It: 110; obj: 2.538e+00; abs: 1.100e-01; rel: 4.532e-02; norm: 1.657e-01
## It: 111; obj: 2.642e+00; abs: 1.039e-01; rel: 4.094e-02; norm: 1.689e-01
## It: 112; obj: 2.776e+00; abs: 1.340e-01; rel: 5.071e-02; norm: 1.693e-01
## It: 113; obj: 2.849e+00; abs: 7.371e-02; rel: 2.656e-02; norm: 1.755e-01
## It: 114; obj: 3.027e+00; abs: 1.774e-01; rel: 6.227e-02; norm: 2.021e-01
## It: 115; obj: 3.060e+00; abs: 3.358e-02; rel: 1.109e-02; norm: 2.217e-01
## It: 116; obj: 3.113e+00; abs: 5.232e-02; rel: 1.710e-02; norm: 2.563e-01
## It: 117; obj: 3.105e+00; abs: 7.542e-03; rel: 2.423e-03; norm: 2.743e-01
## It: 118; obj: 3.106e+00; abs: 1.114e-03; rel: 3.586e-04; norm: 3.116e-01
## It: 119; obj: 3.092e+00; abs: 1.398e-02; rel: 4.502e-03; norm: 3.252e-01
## It: 120; obj: 3.073e+00; abs: 1.906e-02; rel: 6.164e-03; norm: 3.529e-01
## It: 121; obj: 3.087e+00; abs: 1.407e-02; rel: 4.579e-03; norm: 3.738e-01

```

```

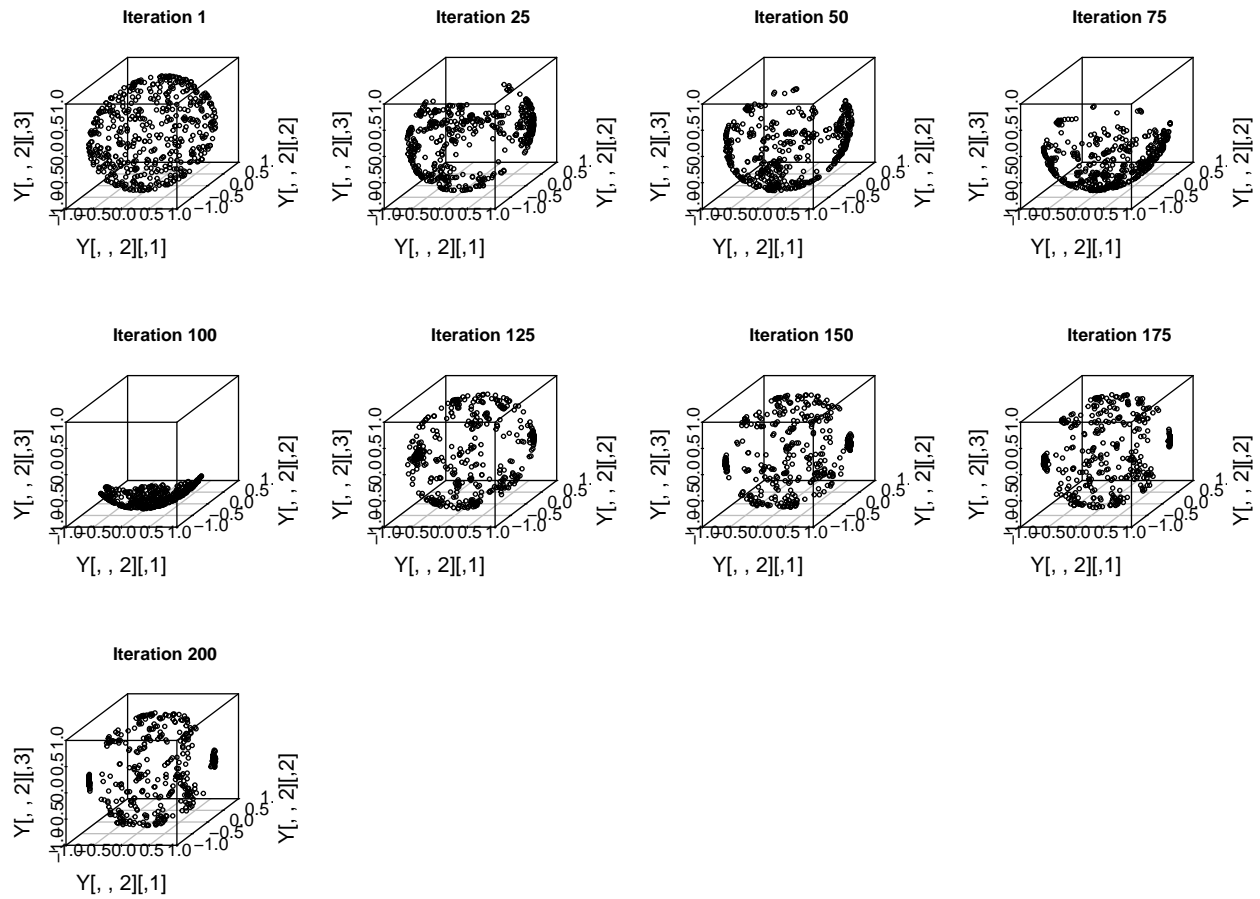
## It: 122; obj: 3.048e+00; abs: 3.940e-02; rel: 1.276e-02; norm: 3.922e-01
## It: 123; obj: 3.037e+00; abs: 1.082e-02; rel: 3.549e-03; norm: 3.986e-01
## It: 124; obj: 3.025e+00; abs: 1.225e-02; rel: 4.034e-03; norm: 4.118e-01
## It: 125; obj: 3.019e+00; abs: 5.599e-03; rel: 1.851e-03; norm: 4.195e-01

## It: 126; obj: 2.991e+00; abs: 2.813e-02; rel: 9.316e-03; norm: 4.313e-01
## It: 127; obj: 2.966e+00; abs: 2.462e-02; rel: 8.231e-03; norm: 4.383e-01
## It: 128; obj: 2.928e+00; abs: 3.828e-02; rel: 1.291e-02; norm: 4.402e-01
## It: 129; obj: 2.920e+00; abs: 8.382e-03; rel: 2.862e-03; norm: 4.405e-01
## It: 130; obj: 2.912e+00; abs: 7.804e-03; rel: 2.673e-03; norm: 4.452e-01
## It: 131; obj: 2.918e+00; abs: 6.157e-03; rel: 2.115e-03; norm: 4.522e-01
## It: 132; obj: 2.905e+00; abs: 1.310e-02; rel: 4.489e-03; norm: 4.599e-01
## It: 133; obj: 2.885e+00; abs: 2.045e-02; rel: 7.038e-03; norm: 4.612e-01
## It: 134; obj: 2.872e+00; abs: 1.242e-02; rel: 4.305e-03; norm: 4.611e-01
## It: 135; obj: 2.851e+00; abs: 2.137e-02; rel: 7.441e-03; norm: 4.632e-01
## It: 136; obj: 2.873e+00; abs: 2.187e-02; rel: 7.671e-03; norm: 4.664e-01
## It: 137; obj: 2.868e+00; abs: 4.810e-03; rel: 1.674e-03; norm: 4.734e-01
## It: 138; obj: 2.859e+00; abs: 9.233e-03; rel: 3.220e-03; norm: 4.778e-01
## It: 139; obj: 2.845e+00; abs: 1.385e-02; rel: 4.846e-03; norm: 4.786e-01
## It: 140; obj: 2.851e+00; abs: 6.492e-03; rel: 2.282e-03; norm: 4.801e-01
## It: 141; obj: 2.837e+00; abs: 1.470e-02; rel: 5.156e-03; norm: 4.815e-01
## It: 142; obj: 2.835e+00; abs: 1.081e-03; rel: 3.811e-04; norm: 4.825e-01
## It: 143; obj: 2.830e+00; abs: 5.163e-03; rel: 1.821e-03; norm: 4.837e-01
## It: 144; obj: 2.837e+00; abs: 6.430e-03; rel: 2.272e-03; norm: 4.827e-01
## It: 145; obj: 2.820e+00; abs: 1.687e-02; rel: 5.947e-03; norm: 4.809e-01
## It: 146; obj: 2.839e+00; abs: 1.945e-02; rel: 6.899e-03; norm: 4.820e-01
## It: 147; obj: 2.831e+00; abs: 8.144e-03; rel: 2.868e-03; norm: 4.821e-01
## It: 148; obj: 2.848e+00; abs: 1.728e-02; rel: 6.103e-03; norm: 4.851e-01
## It: 149; obj: 2.834e+00; abs: 1.462e-02; rel: 5.132e-03; norm: 4.836e-01
## It: 150; obj: 2.841e+00; abs: 7.619e-03; rel: 2.689e-03; norm: 4.857e-01

## It: 151; obj: 2.823e+00; abs: 1.874e-02; rel: 6.596e-03; norm: 4.843e-01
## It: 152; obj: 2.826e+00; abs: 3.171e-03; rel: 1.123e-03; norm: 4.844e-01
## It: 153; obj: 2.814e+00; abs: 1.144e-02; rel: 4.049e-03; norm: 4.822e-01
## It: 154; obj: 2.818e+00; abs: 3.152e-03; rel: 1.120e-03; norm: 4.826e-01
## It: 155; obj: 2.810e+00; abs: 7.865e-03; rel: 2.791e-03; norm: 4.807e-01
## It: 156; obj: 2.821e+00; abs: 1.089e-02; rel: 3.874e-03; norm: 4.799e-01
## It: 157; obj: 2.810e+00; abs: 1.074e-02; rel: 3.809e-03; norm: 4.781e-01
## It: 158; obj: 2.830e+00; abs: 2.064e-02; rel: 7.346e-03; norm: 4.793e-01
## It: 159; obj: 2.816e+00; abs: 1.438e-02; rel: 5.079e-03; norm: 4.784e-01
## It: 160; obj: 2.821e+00; abs: 4.471e-03; rel: 1.588e-03; norm: 4.794e-01
## It: 161; obj: 2.816e+00; abs: 4.783e-03; rel: 1.696e-03; norm: 4.816e-01
## It: 162; obj: 2.826e+00; abs: 1.065e-02; rel: 3.783e-03; norm: 4.829e-01
## It: 163; obj: 2.821e+00; abs: 5.484e-03; rel: 1.940e-03; norm: 4.843e-01
## It: 164; obj: 2.825e+00; abs: 4.520e-03; rel: 1.602e-03; norm: 4.857e-01
## It: 165; obj: 2.817e+00; abs: 8.568e-03; rel: 3.032e-03; norm: 4.855e-01
## It: 166; obj: 2.819e+00; abs: 1.952e-03; rel: 6.931e-04; norm: 4.843e-01
## It: 167; obj: 2.801e+00; abs: 1.778e-02; rel: 6.307e-03; norm: 4.826e-01
## It: 168; obj: 2.813e+00; abs: 1.181e-02; rel: 4.216e-03; norm: 4.817e-01
## It: 169; obj: 2.793e+00; abs: 2.039e-02; rel: 7.249e-03; norm: 4.792e-01
## It: 170; obj: 2.813e+00; abs: 2.045e-02; rel: 7.322e-03; norm: 4.807e-01
## It: 171; obj: 2.792e+00; abs: 2.090e-02; rel: 7.430e-03; norm: 4.784e-01
## It: 172; obj: 2.811e+00; abs: 1.860e-02; rel: 6.661e-03; norm: 4.804e-01
## It: 173; obj: 2.787e+00; abs: 2.398e-02; rel: 8.532e-03; norm: 4.778e-01
## It: 174; obj: 2.803e+00; abs: 1.587e-02; rel: 5.693e-03; norm: 4.789e-01

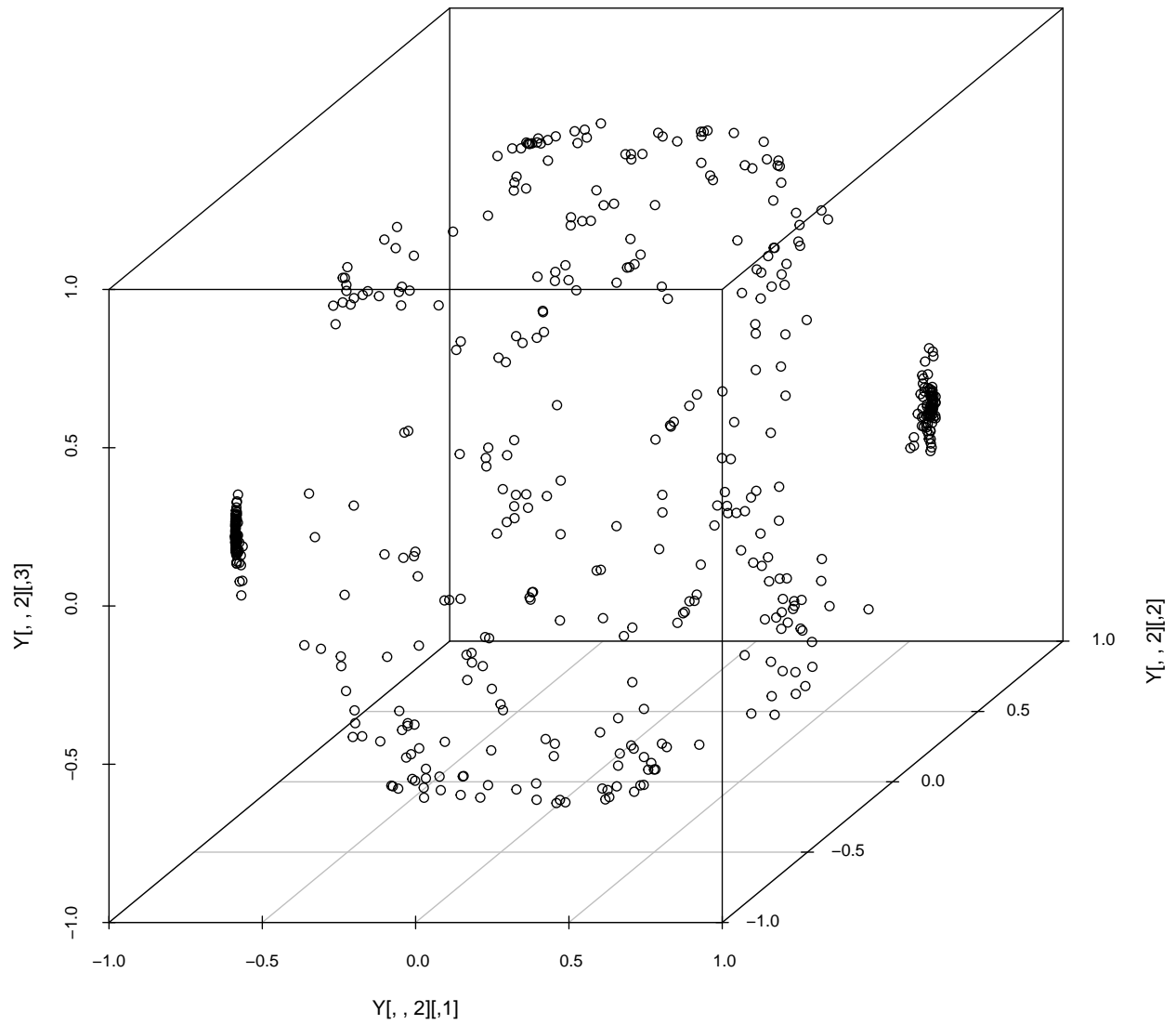
```

## It: 175; obj: 2.785e+00; abs: 1.791e-02; rel: 6.391e-03; norm: 4.769e-01  
## It: 176; obj: 2.799e+00; abs: 1.476e-02; rel: 5.300e-03; norm: 4.781e-01  
## It: 177; obj: 2.784e+00; abs: 1.543e-02; rel: 5.514e-03; norm: 4.765e-01  
## It: 178; obj: 2.800e+00; abs: 1.598e-02; rel: 5.738e-03; norm: 4.778e-01  
## It: 179; obj: 2.785e+00; abs: 1.512e-02; rel: 5.402e-03; norm: 4.763e-01  
## It: 180; obj: 2.801e+00; abs: 1.653e-02; rel: 5.937e-03; norm: 4.777e-01  
## It: 181; obj: 2.786e+00; abs: 1.498e-02; rel: 5.348e-03; norm: 4.762e-01  
## It: 182; obj: 2.804e+00; abs: 1.793e-02; rel: 6.433e-03; norm: 4.777e-01  
## It: 183; obj: 2.791e+00; abs: 1.300e-02; rel: 4.637e-03; norm: 4.762e-01  
## It: 184; obj: 2.812e+00; abs: 2.117e-02; rel: 7.583e-03; norm: 4.789e-01  
## It: 185; obj: 2.800e+00; abs: 1.242e-02; rel: 4.416e-03; norm: 4.785e-01  
## It: 186; obj: 2.816e+00; abs: 1.636e-02; rel: 5.841e-03; norm: 4.822e-01  
## It: 187; obj: 2.804e+00; abs: 1.244e-02; rel: 4.418e-03; norm: 4.811e-01  
## It: 188; obj: 2.823e+00; abs: 1.951e-02; rel: 6.959e-03; norm: 4.830e-01  
## It: 189; obj: 2.811e+00; abs: 1.290e-02; rel: 4.568e-03; norm: 4.826e-01  
## It: 190; obj: 2.825e+00; abs: 1.410e-02; rel: 5.016e-03; norm: 4.849e-01  
## It: 191; obj: 2.811e+00; abs: 1.325e-02; rel: 4.690e-03; norm: 4.838e-01  
## It: 192; obj: 2.826e+00; abs: 1.433e-02; rel: 5.097e-03; norm: 4.863e-01  
## It: 193; obj: 2.812e+00; abs: 1.392e-02; rel: 4.926e-03; norm: 4.850e-01  
## It: 194; obj: 2.826e+00; abs: 1.400e-02; rel: 4.978e-03; norm: 4.867e-01  
## It: 195; obj: 2.812e+00; abs: 1.405e-02; rel: 4.971e-03; norm: 4.851e-01  
## It: 196; obj: 2.826e+00; abs: 1.402e-02; rel: 4.987e-03; norm: 4.867e-01  
## It: 197; obj: 2.812e+00; abs: 1.404e-02; rel: 4.969e-03; norm: 4.850e-01  
## It: 198; obj: 2.826e+00; abs: 1.402e-02; rel: 4.987e-03; norm: 4.866e-01  
## It: 199; obj: 2.812e+00; abs: 1.403e-02; rel: 4.966e-03; norm: 4.849e-01  
## It: 200; obj: 2.826e+00; abs: 1.403e-02; rel: 4.989e-03; norm: 4.865e-01

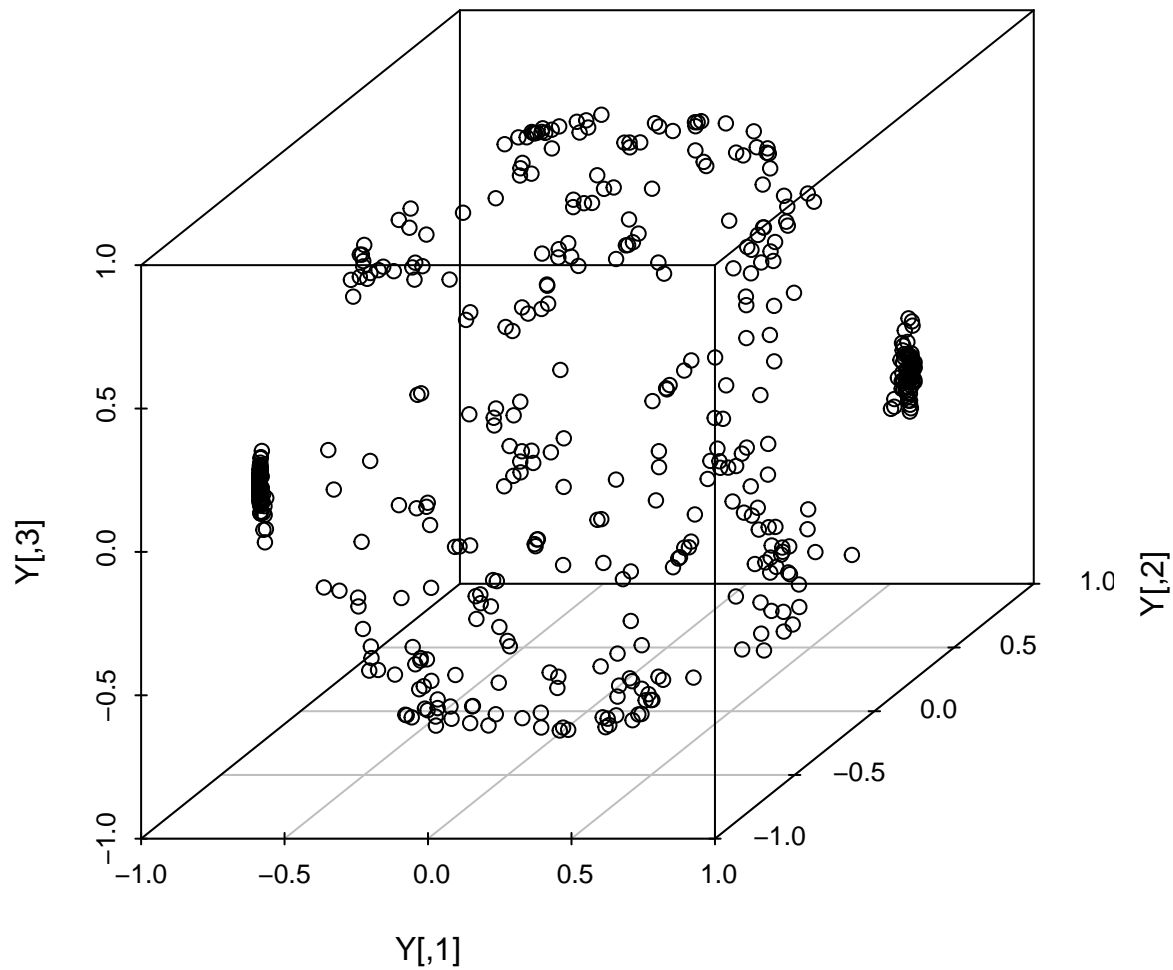




Iteration 200



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1))
```



## Case 4

Playing with belts of points onto the sphere. First, let's generate the data

```
rotate_matrix_z_axis <- function(alpha) {
  deg2rad <- function(deg) {(deg * pi) / (180)}
  rads <- deg2rad(alpha)
  matrix(c(cos(rads), -sin(rads), 0,
           sin(rads), cos(rads), 0,
           0,          0, 1),
         byrow = T, nrow = 3)
}

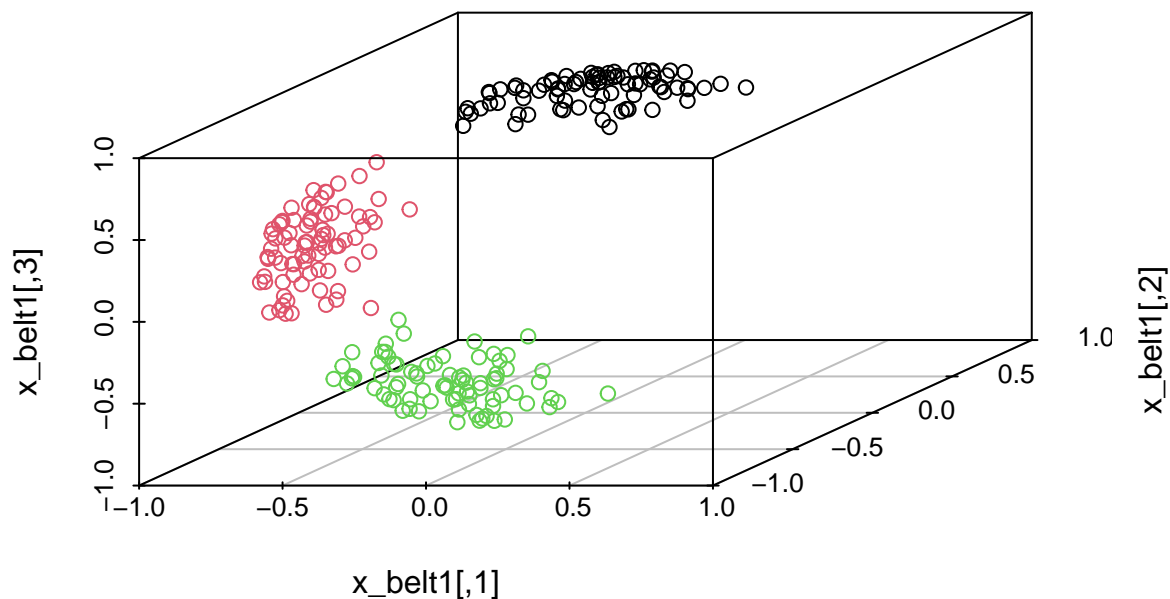
rotate_matrix_x_axis <- function(alpha) {
  deg2rad <- function(deg) {(deg * pi) / (180)}
  rads <- deg2rad(alpha)
  matrix(c(1,          0, 0,
           0, cos(rads), -sin(rads),
           0, sin(rads), cos(rads)),
         byrow = T, nrow = 3)
}
```

```

rotate_matrix_y_axis <- function(alpha) {
  deg2rad <- function(deg) {(deg * pi) / (180)}
  rads <- deg2rad(alpha)
  matrix(c(cos(rads), 0, sin(rads),
           0, 1, 0,
           -sin(rads), 0, cos(rads)),
         byrow = T, nrow = 3)
}

p = 2
set.seed(2)
# A <- matrix(runif((p + 1)^2)*2-0.5, ncol = p + 1)
# sigma <- t(A) %*% A
sigma <- diag(c(10, 2, .5))
n <- 80
# x_1_belt1 <- rotasym::r_ACG(n, sigma)
x_1_belt1 <- sphunif::r_alt(n = n, p = 3, alt = "SC", kappa = 200, nu = 0.9999)[,1]
rotate_mat_90 <- rotate_matrix_y_axis(90)
x_2_belt1 <- x_1_belt1 %*% rotate_mat_90
rotate_mat_45 <- rotate_matrix_y_axis(60)
x_3_belt1 <- x_2_belt1 %*% rotate_mat_45
x_belt1 <- rbind(x_1_belt1, x_2_belt1, x_3_belt1)
scatterplot3d::scatterplot3d(x_belt1, xlim = c(-1, 1),
                             ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(c(1,2,3), each=n))

```



```

rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue",
            lit = FALSE)
rgl::points3d(x_belt1, col = rep(c(1,2,3), each=n))

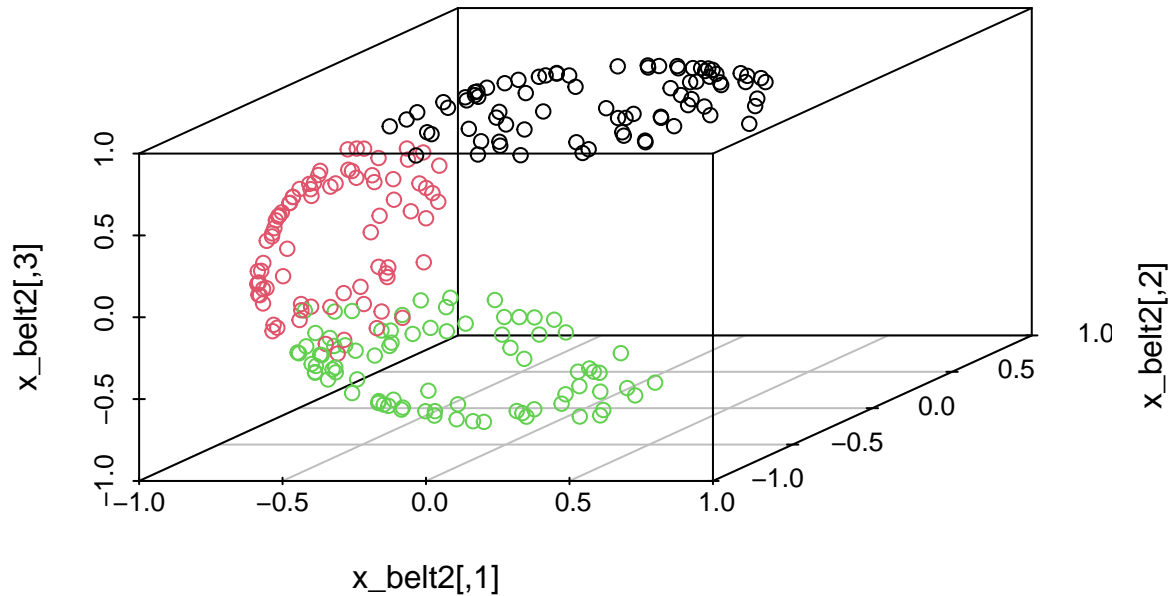
# x_1_belt2 <- rotasym::r_ACG(n, sigma)
x_1_belt2 <- sphunif::r_alt(n = n, p = 3, alt = "SC", kappa = 200, nu = 0.89)[,1]
x_2_belt2 <- x_1_belt2 %*% rotate_mat_90

```

```

x_3_belt2 <- x_2_belt2 %*% rotate_mat_45
x_belt2 <- rbind(x_1_belt2, x_2_belt2, x_3_belt2)
scatterplot3d::scatterplot3d(x_belt2, xlim = c(-1, 1),
                             ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(c(1,2,3), each=n))

```

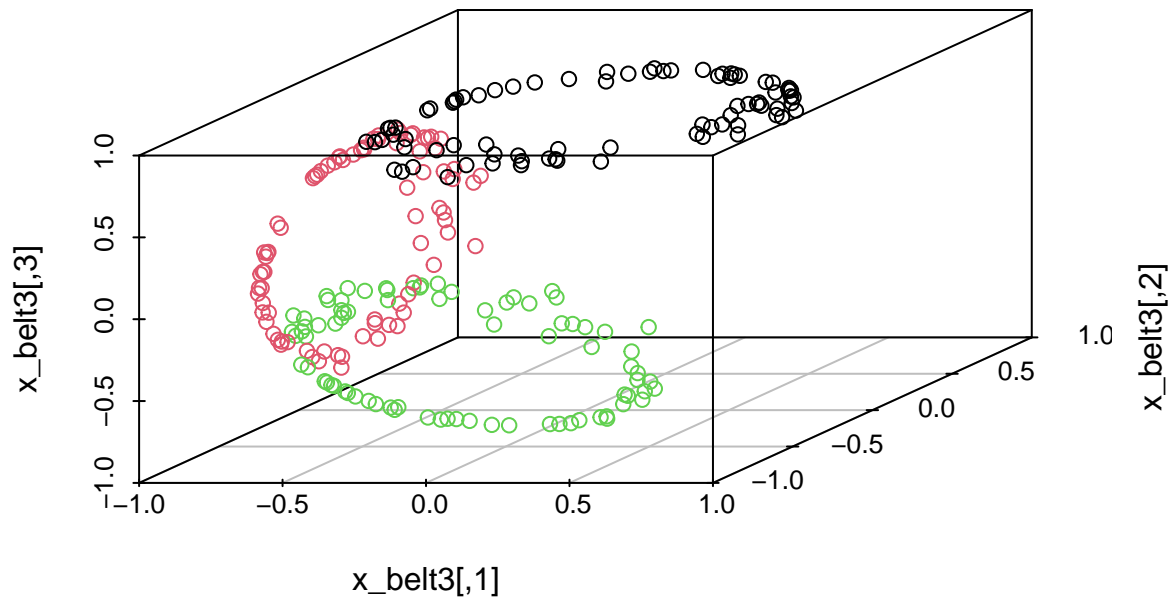


```

rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue",
            lit = FALSE)
rgl::points3d(x_belt2, col = rep(c(1,2,3), each=n))

# x_1_belt3 <- rotasym::r_ACG(n, sigma)
x_1_belt3 <- sphunif::r_alt(n = n, p = 3, alt = "SC", kappa = 400, nu = 0.80)[,1]
x_2_belt3 <- x_1_belt3 %*% rotate_mat_90
x_3_belt3 <- x_2_belt3 %*% rotate_mat_45
x_belt3 <- rbind(x_1_belt3, x_2_belt3, x_3_belt3)
scatterplot3d::scatterplot3d(x_belt3, xlim = c(-1, 1),
                             ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(c(1,2,3), each=n))

```



```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue",
            lit = FALSE)
rgl::points3d(x_belt3, col = rep(c(1,2,3), each=n))

n <- n * 3
x_5 <- array(dim = c(n, 3, 3))
x_5[, , 1] <- x_belt1
x_5[, , 2] <- x_belt2
x_5[, , 3] <- x_belt3

indexes <- sample(1:n)
x_5 <- x_5[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]
```

Let's calculate the rho parameters based on a perplexity of 25:

```
rho_five_perp20 <- rho_optim_bst(x_5, 50)
```

## Time difference of 11.81187 secs

First, let's reduce to dimension  $\mathbb{S}^1$  then  $d = 1$  (circumference):

```
Y <- psc_sne(X=x_5, d=1, rho_psc_list = rho_five_perp20, num_iteration=200, eta = 20,
            visualize_prog = TRUE, colors = colors)
```

```
## It: 1; obj: 1.284e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.250e-01
## It: 2; obj: 1.270e+01; abs: 1.404e-01; rel: 1.094e-02; norm: 1.504e-01
## It: 3; obj: 1.250e+01; abs: 1.964e-01; rel: 1.547e-02; norm: 1.919e-01
## It: 4; obj: 1.226e+01; abs: 2.438e-01; rel: 1.950e-02; norm: 2.466e-01
## It: 5; obj: 1.199e+01; abs: 2.679e-01; rel: 2.186e-02; norm: 3.205e-01
## It: 6; obj: 1.172e+01; abs: 2.715e-01; rel: 2.265e-02; norm: 4.018e-01
## It: 7; obj: 1.145e+01; abs: 2.655e-01; rel: 2.265e-02; norm: 4.753e-01
## It: 8; obj: 1.114e+01; abs: 3.151e-01; rel: 2.751e-02; norm: 5.501e-01
## It: 9; obj: 1.072e+01; abs: 4.142e-01; rel: 3.719e-02; norm: 6.349e-01
## It: 10; obj: 1.027e+01; abs: 4.589e-01; rel: 4.279e-02; norm: 7.405e-01
```

```

## It: 11; obj: 9.886e+00; abs: 3.798e-01; rel: 3.700e-02; norm: 8.754e-01
## It: 12; obj: 9.593e+00; abs: 2.923e-01; rel: 2.957e-02; norm: 1.006e+00
## It: 13; obj: 9.319e+00; abs: 2.747e-01; rel: 2.863e-02; norm: 1.086e+00
## It: 14; obj: 9.120e+00; abs: 1.986e-01; rel: 2.131e-02; norm: 1.142e+00
## It: 15; obj: 9.035e+00; abs: 8.490e-02; rel: 9.309e-03; norm: 1.195e+00
## It: 16; obj: 8.996e+00; abs: 3.929e-02; rel: 4.348e-03; norm: 1.238e+00
## It: 17; obj: 8.989e+00; abs: 6.909e-03; rel: 7.680e-04; norm: 1.263e+00
## It: 18; obj: 8.995e+00; abs: 5.526e-03; rel: 6.148e-04; norm: 1.278e+00
## It: 19; obj: 8.980e+00; abs: 1.460e-02; rel: 1.624e-03; norm: 1.285e+00
## It: 20; obj: 8.972e+00; abs: 8.419e-03; rel: 9.375e-04; norm: 1.290e+00
## It: 21; obj: 8.971e+00; abs: 1.172e-04; rel: 1.307e-05; norm: 1.293e+00
## It: 22; obj: 8.971e+00; abs: 7.463e-04; rel: 8.319e-05; norm: 1.295e+00
## It: 23; obj: 8.969e+00; abs: 1.588e-03; rel: 1.770e-04; norm: 1.296e+00
## It: 24; obj: 8.967e+00; abs: 2.073e-03; rel: 2.312e-04; norm: 1.297e+00
## It: 25; obj: 8.965e+00; abs: 2.429e-03; rel: 2.709e-04; norm: 1.298e+00

## It: 26; obj: 8.962e+00; abs: 2.676e-03; rel: 2.985e-04; norm: 1.299e+00
## It: 27; obj: 8.959e+00; abs: 2.829e-03; rel: 3.157e-04; norm: 1.299e+00
## It: 28; obj: 8.956e+00; abs: 2.887e-03; rel: 3.223e-04; norm: 1.300e+00
## It: 29; obj: 8.953e+00; abs: 2.843e-03; rel: 3.175e-04; norm: 1.301e+00
## It: 30; obj: 8.951e+00; abs: 2.697e-03; rel: 3.012e-04; norm: 1.303e+00
## It: 31; obj: 8.948e+00; abs: 2.459e-03; rel: 2.747e-04; norm: 1.304e+00
## It: 32; obj: 8.946e+00; abs: 2.144e-03; rel: 2.396e-04; norm: 1.305e+00
## It: 33; obj: 8.944e+00; abs: 1.763e-03; rel: 1.971e-04; norm: 1.307e+00
## It: 34; obj: 8.943e+00; abs: 1.319e-03; rel: 1.475e-04; norm: 1.309e+00
## It: 35; obj: 8.942e+00; abs: 8.084e-04; rel: 9.040e-05; norm: 1.311e+00
## It: 36; obj: 8.942e+00; abs: 2.202e-04; rel: 2.463e-05; norm: 1.313e+00
## It: 37; obj: 8.942e+00; abs: 4.585e-04; rel: 5.128e-05; norm: 1.316e+00
## It: 38; obj: 8.944e+00; abs: 1.242e-03; rel: 1.389e-04; norm: 1.319e+00
## It: 39; obj: 8.946e+00; abs: 2.143e-03; rel: 2.396e-04; norm: 1.322e+00
## It: 40; obj: 8.949e+00; abs: 3.165e-03; rel: 3.538e-04; norm: 1.326e+00
## It: 41; obj: 8.953e+00; abs: 4.278e-03; rel: 4.780e-04; norm: 1.329e+00
## It: 42; obj: 8.958e+00; abs: 5.185e-03; rel: 5.791e-04; norm: 1.333e+00
## It: 43; obj: 8.962e+00; abs: 3.666e-03; rel: 4.092e-04; norm: 1.338e+00
## It: 44; obj: 8.951e+00; abs: 1.076e-02; rel: 1.200e-03; norm: 1.343e+00
## It: 45; obj: 8.931e+00; abs: 2.072e-02; rel: 2.315e-03; norm: 1.349e+00
## It: 46; obj: 8.929e+00; abs: 1.362e-03; rel: 1.525e-04; norm: 1.356e+00
## It: 47; obj: 8.936e+00; abs: 6.615e-03; rel: 7.408e-04; norm: 1.363e+00
## It: 48; obj: 8.944e+00; abs: 8.425e-03; rel: 9.428e-04; norm: 1.369e+00
## It: 49; obj: 8.954e+00; abs: 9.816e-03; rel: 1.097e-03; norm: 1.375e+00
## It: 50; obj: 8.965e+00; abs: 1.129e-02; rel: 1.260e-03; norm: 1.382e+00

## It: 51; obj: 8.978e+00; abs: 1.284e-02; rel: 1.432e-03; norm: 1.389e+00
## It: 52; obj: 8.993e+00; abs: 1.452e-02; rel: 1.617e-03; norm: 1.397e+00
## It: 53; obj: 9.009e+00; abs: 1.632e-02; rel: 1.814e-03; norm: 1.404e+00
## It: 54; obj: 9.027e+00; abs: 1.824e-02; rel: 2.024e-03; norm: 1.412e+00
## It: 55; obj: 9.048e+00; abs: 2.027e-02; rel: 2.245e-03; norm: 1.420e+00
## It: 56; obj: 9.070e+00; abs: 2.239e-02; rel: 2.475e-03; norm: 1.429e+00
## It: 57; obj: 9.094e+00; abs: 2.457e-02; rel: 2.708e-03; norm: 1.437e+00
## It: 58; obj: 9.121e+00; abs: 2.672e-02; rel: 2.938e-03; norm: 1.446e+00
## It: 59; obj: 9.150e+00; abs: 2.879e-02; rel: 3.156e-03; norm: 1.455e+00
## It: 60; obj: 9.181e+00; abs: 3.080e-02; rel: 3.366e-03; norm: 1.464e+00
## It: 61; obj: 9.214e+00; abs: 3.288e-02; rel: 3.581e-03; norm: 1.474e+00
## It: 62; obj: 9.249e+00; abs: 3.501e-02; rel: 3.800e-03; norm: 1.483e+00
## It: 63; obj: 9.286e+00; abs: 3.701e-02; rel: 4.001e-03; norm: 1.493e+00

```

```

## It: 64; obj: 9.324e+00; abs: 3.862e-02; rel: 4.159e-03; norm: 1.502e+00
## It: 65; obj: 9.364e+00; abs: 3.955e-02; rel: 4.242e-03; norm: 1.512e+00
## It: 66; obj: 9.403e+00; abs: 3.934e-02; rel: 4.201e-03; norm: 1.522e+00
## It: 67; obj: 9.440e+00; abs: 3.725e-02; rel: 3.961e-03; norm: 1.531e+00
## It: 68; obj: 9.473e+00; abs: 3.232e-02; rel: 3.423e-03; norm: 1.541e+00
## It: 69; obj: 9.496e+00; abs: 2.353e-02; rel: 2.484e-03; norm: 1.551e+00
## It: 70; obj: 9.507e+00; abs: 1.020e-02; rel: 1.074e-03; norm: 1.562e+00
## It: 71; obj: 9.499e+00; abs: 7.316e-03; rel: 7.696e-04; norm: 1.572e+00
## It: 72; obj: 9.473e+00; abs: 2.659e-02; rel: 2.799e-03; norm: 1.582e+00
## It: 73; obj: 9.429e+00; abs: 4.340e-02; rel: 4.582e-03; norm: 1.591e+00
## It: 74; obj: 9.375e+00; abs: 5.418e-02; rel: 5.746e-03; norm: 1.597e+00
## It: 75; obj: 9.317e+00; abs: 5.826e-02; rel: 6.214e-03; norm: 1.600e+00

## It: 76; obj: 9.259e+00; abs: 5.736e-02; rel: 6.157e-03; norm: 1.600e+00
## It: 77; obj: 9.206e+00; abs: 5.368e-02; rel: 5.797e-03; norm: 1.598e+00
## It: 78; obj: 9.157e+00; abs: 4.882e-02; rel: 5.303e-03; norm: 1.594e+00
## It: 79; obj: 9.113e+00; abs: 4.372e-02; rel: 4.775e-03; norm: 1.589e+00
## It: 80; obj: 9.074e+00; abs: 3.886e-02; rel: 4.264e-03; norm: 1.584e+00
## It: 81; obj: 9.040e+00; abs: 3.443e-02; rel: 3.794e-03; norm: 1.579e+00
## It: 82; obj: 9.009e+00; abs: 3.050e-02; rel: 3.374e-03; norm: 1.573e+00
## It: 83; obj: 8.982e+00; abs: 2.706e-02; rel: 3.004e-03; norm: 1.568e+00
## It: 84; obj: 8.958e+00; abs: 2.407e-02; rel: 2.680e-03; norm: 1.562e+00
## It: 85; obj: 8.937e+00; abs: 2.148e-02; rel: 2.398e-03; norm: 1.557e+00
## It: 86; obj: 8.918e+00; abs: 1.923e-02; rel: 2.152e-03; norm: 1.552e+00
## It: 87; obj: 8.900e+00; abs: 1.728e-02; rel: 1.938e-03; norm: 1.547e+00
## It: 88; obj: 8.885e+00; abs: 1.559e-02; rel: 1.751e-03; norm: 1.542e+00
## It: 89; obj: 8.871e+00; abs: 1.411e-02; rel: 1.588e-03; norm: 1.538e+00
## It: 90; obj: 8.858e+00; abs: 1.281e-02; rel: 1.444e-03; norm: 1.533e+00
## It: 91; obj: 8.846e+00; abs: 1.167e-02; rel: 1.318e-03; norm: 1.529e+00
## It: 92; obj: 8.835e+00; abs: 1.067e-02; rel: 1.206e-03; norm: 1.525e+00
## It: 93; obj: 8.826e+00; abs: 9.778e-03; rel: 1.107e-03; norm: 1.522e+00
## It: 94; obj: 8.817e+00; abs: 8.987e-03; rel: 1.018e-03; norm: 1.518e+00
## It: 95; obj: 8.808e+00; abs: 8.282e-03; rel: 9.393e-04; norm: 1.515e+00
## It: 96; obj: 8.801e+00; abs: 7.651e-03; rel: 8.686e-04; norm: 1.511e+00
## It: 97; obj: 8.794e+00; abs: 7.085e-03; rel: 8.050e-04; norm: 1.508e+00
## It: 98; obj: 8.787e+00; abs: 6.575e-03; rel: 7.477e-04; norm: 1.505e+00
## It: 99; obj: 8.781e+00; abs: 6.115e-03; rel: 6.959e-04; norm: 1.502e+00
## It: 100; obj: 8.775e+00; abs: 5.698e-03; rel: 6.489e-04; norm: 1.500e+00

## It: 101; obj: 7.979e-01; abs: 7.977e+00; rel: 9.091e-01; norm: 1.166e-01
## It: 102; obj: 7.872e-01; abs: 1.068e-02; rel: 1.339e-02; norm: 1.145e-01
## It: 103; obj: 7.779e-01; abs: 9.305e-03; rel: 1.182e-02; norm: 1.120e-01
## It: 104; obj: 7.701e-01; abs: 7.766e-03; rel: 9.983e-03; norm: 1.097e-01
## It: 105; obj: 7.635e-01; abs: 6.652e-03; rel: 8.638e-03; norm: 1.080e-01
## It: 106; obj: 7.575e-01; abs: 5.926e-03; rel: 7.762e-03; norm: 1.069e-01
## It: 107; obj: 7.522e-01; abs: 5.386e-03; rel: 7.110e-03; norm: 1.062e-01
## It: 108; obj: 7.473e-01; abs: 4.854e-03; rel: 6.454e-03; norm: 1.059e-01
## It: 109; obj: 7.430e-01; abs: 4.250e-03; rel: 5.687e-03; norm: 1.058e-01
## It: 110; obj: 7.394e-01; abs: 3.597e-03; rel: 4.841e-03; norm: 1.057e-01
## It: 111; obj: 7.365e-01; abs: 2.973e-03; rel: 4.020e-03; norm: 1.056e-01
## It: 112; obj: 7.340e-01; abs: 2.447e-03; rel: 3.322e-03; norm: 1.055e-01
## It: 113; obj: 7.320e-01; abs: 2.054e-03; rel: 2.798e-03; norm: 1.054e-01
## It: 114; obj: 7.302e-01; abs: 1.804e-03; rel: 2.464e-03; norm: 1.055e-01
## It: 115; obj: 7.285e-01; abs: 1.696e-03; rel: 2.323e-03; norm: 1.059e-01
## It: 116; obj: 7.268e-01; abs: 1.726e-03; rel: 2.370e-03; norm: 1.067e-01

```

```

## It: 117; obj: 7.249e-01; abs: 1.875e-03; rel: 2.580e-03; norm: 1.075e-01
## It: 118; obj: 7.228e-01; abs: 2.080e-03; rel: 2.870e-03; norm: 1.082e-01
## It: 119; obj: 7.206e-01; abs: 2.230e-03; rel: 3.085e-03; norm: 1.089e-01
## It: 120; obj: 7.184e-01; abs: 2.205e-03; rel: 3.060e-03; norm: 1.096e-01
## It: 121; obj: 7.164e-01; abs: 1.969e-03; rel: 2.741e-03; norm: 1.106e-01
## It: 122; obj: 7.148e-01; abs: 1.594e-03; rel: 2.224e-03; norm: 1.119e-01
## It: 123; obj: 7.136e-01; abs: 1.197e-03; rel: 1.674e-03; norm: 1.134e-01
## It: 124; obj: 7.127e-01; abs: 8.623e-04; rel: 1.208e-03; norm: 1.146e-01
## It: 125; obj: 7.121e-01; abs: 6.169e-04; rel: 8.656e-04; norm: 1.155e-01

## It: 126; obj: 7.117e-01; abs: 4.502e-04; rel: 6.322e-04; norm: 1.162e-01
## It: 127; obj: 7.113e-01; abs: 3.399e-04; rel: 4.776e-04; norm: 1.167e-01
## It: 128; obj: 7.111e-01; abs: 2.658e-04; rel: 3.737e-04; norm: 1.170e-01
## It: 129; obj: 7.109e-01; abs: 2.140e-04; rel: 3.010e-04; norm: 1.172e-01
## It: 130; obj: 7.107e-01; abs: 1.758e-04; rel: 2.473e-04; norm: 1.173e-01
## It: 131; obj: 7.105e-01; abs: 1.462e-04; rel: 2.057e-04; norm: 1.173e-01
## It: 132; obj: 7.104e-01; abs: 1.225e-04; rel: 1.723e-04; norm: 1.174e-01
## It: 133; obj: 7.103e-01; abs: 1.029e-04; rel: 1.449e-04; norm: 1.174e-01
## It: 134; obj: 7.102e-01; abs: 8.671e-05; rel: 1.221e-04; norm: 1.173e-01
## It: 135; obj: 7.101e-01; abs: 7.314e-05; rel: 1.030e-04; norm: 1.173e-01
## It: 136; obj: 7.101e-01; abs: 6.177e-05; rel: 8.698e-05; norm: 1.173e-01
## It: 137; obj: 7.100e-01; abs: 5.222e-05; rel: 7.354e-05; norm: 1.172e-01
## It: 138; obj: 7.100e-01; abs: 4.421e-05; rel: 6.227e-05; norm: 1.172e-01
## It: 139; obj: 7.099e-01; abs: 3.749e-05; rel: 5.280e-05; norm: 1.172e-01
## It: 140; obj: 7.099e-01; abs: 3.184e-05; rel: 4.484e-05; norm: 1.171e-01
## It: 141; obj: 7.099e-01; abs: 2.708e-05; rel: 3.814e-05; norm: 1.171e-01
## It: 142; obj: 7.099e-01; abs: 2.307e-05; rel: 3.249e-05; norm: 1.171e-01
## It: 143; obj: 7.098e-01; abs: 1.968e-05; rel: 2.772e-05; norm: 1.170e-01
## It: 144; obj: 7.098e-01; abs: 1.682e-05; rel: 2.369e-05; norm: 1.170e-01
## It: 145; obj: 7.098e-01; abs: 1.439e-05; rel: 2.027e-05; norm: 1.170e-01
## It: 146; obj: 7.098e-01; abs: 1.233e-05; rel: 1.737e-05; norm: 1.169e-01
## It: 147; obj: 7.098e-01; abs: 1.058e-05; rel: 1.490e-05; norm: 1.169e-01
## It: 148; obj: 7.098e-01; abs: 9.087e-06; rel: 1.280e-05; norm: 1.169e-01
## It: 149; obj: 7.098e-01; abs: 7.814e-06; rel: 1.101e-05; norm: 1.169e-01
## It: 150; obj: 7.098e-01; abs: 6.727e-06; rel: 9.477e-06; norm: 1.168e-01

## It: 151; obj: 7.098e-01; abs: 5.797e-06; rel: 8.167e-06; norm: 1.168e-01
## It: 152; obj: 7.098e-01; abs: 4.999e-06; rel: 7.044e-06; norm: 1.168e-01
## It: 153; obj: 7.098e-01; abs: 4.316e-06; rel: 6.080e-06; norm: 1.168e-01
## It: 154; obj: 7.098e-01; abs: 3.728e-06; rel: 5.253e-06; norm: 1.168e-01
## It: 155; obj: 7.097e-01; abs: 3.223e-06; rel: 4.541e-06; norm: 1.168e-01
## It: 156; obj: 7.097e-01; abs: 2.788e-06; rel: 3.928e-06; norm: 1.168e-01
## It: 157; obj: 7.097e-01; abs: 2.413e-06; rel: 3.400e-06; norm: 1.168e-01
## It: 158; obj: 7.097e-01; abs: 2.090e-06; rel: 2.944e-06; norm: 1.167e-01
## It: 159; obj: 7.097e-01; abs: 1.811e-06; rel: 2.551e-06; norm: 1.167e-01
## It: 160; obj: 7.097e-01; abs: 1.570e-06; rel: 2.212e-06; norm: 1.167e-01
## It: 161; obj: 7.097e-01; abs: 1.361e-06; rel: 1.918e-06; norm: 1.167e-01
## It: 162; obj: 7.097e-01; abs: 1.181e-06; rel: 1.664e-06; norm: 1.167e-01
## It: 163; obj: 7.097e-01; abs: 1.025e-06; rel: 1.444e-06; norm: 1.167e-01
## It: 164; obj: 7.097e-01; abs: 8.900e-07; rel: 1.254e-06; norm: 1.167e-01
## It: 165; obj: 7.097e-01; abs: 7.729e-07; rel: 1.089e-06; norm: 1.167e-01
## It: 166; obj: 7.097e-01; abs: 6.715e-07; rel: 9.461e-07; norm: 1.167e-01
## It: 167; obj: 7.097e-01; abs: 5.835e-07; rel: 8.221e-07; norm: 1.167e-01
## It: 168; obj: 7.097e-01; abs: 5.071e-07; rel: 7.145e-07; norm: 1.167e-01
## It: 169; obj: 7.097e-01; abs: 4.409e-07; rel: 6.212e-07; norm: 1.167e-01

```

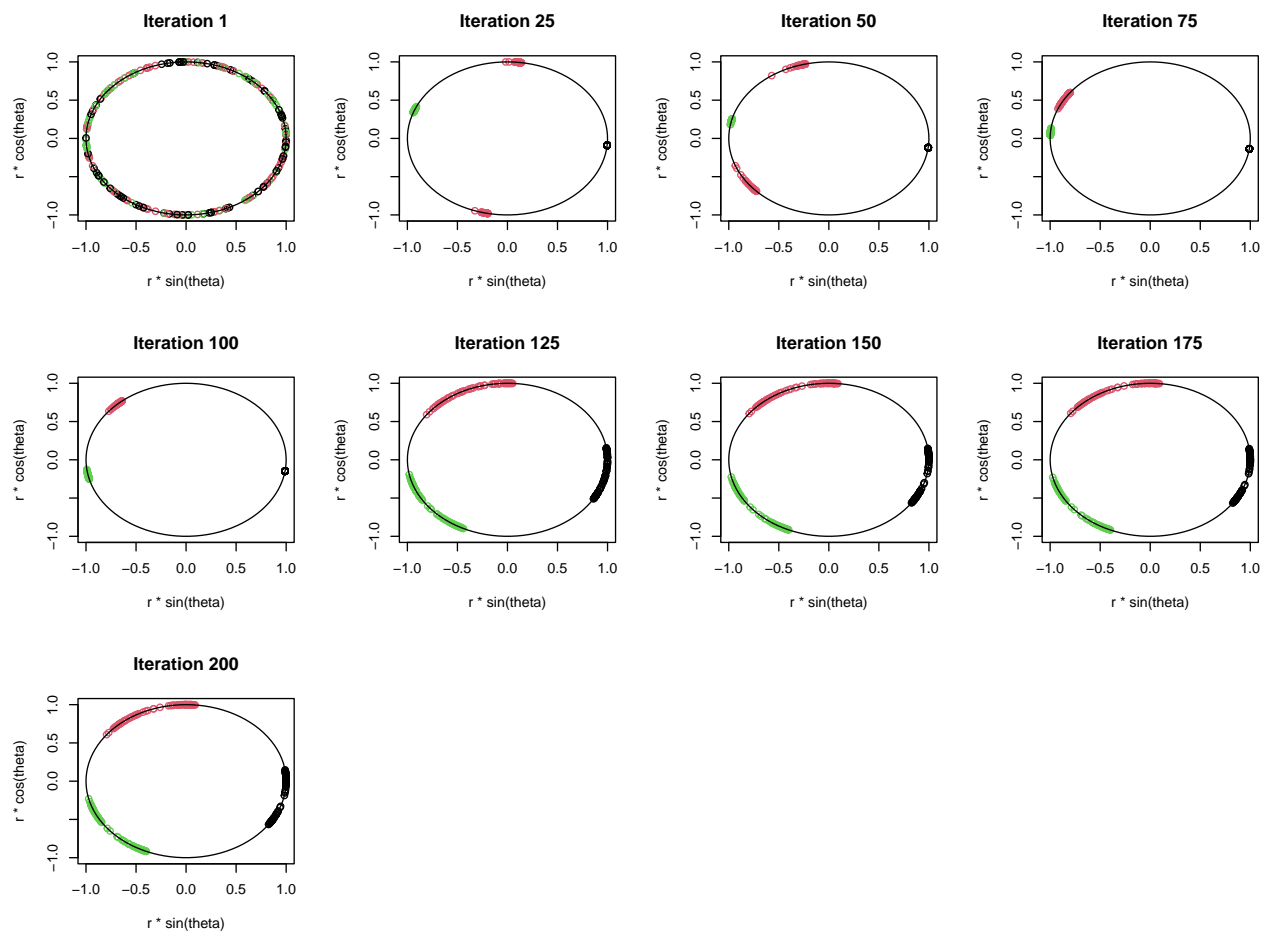


```

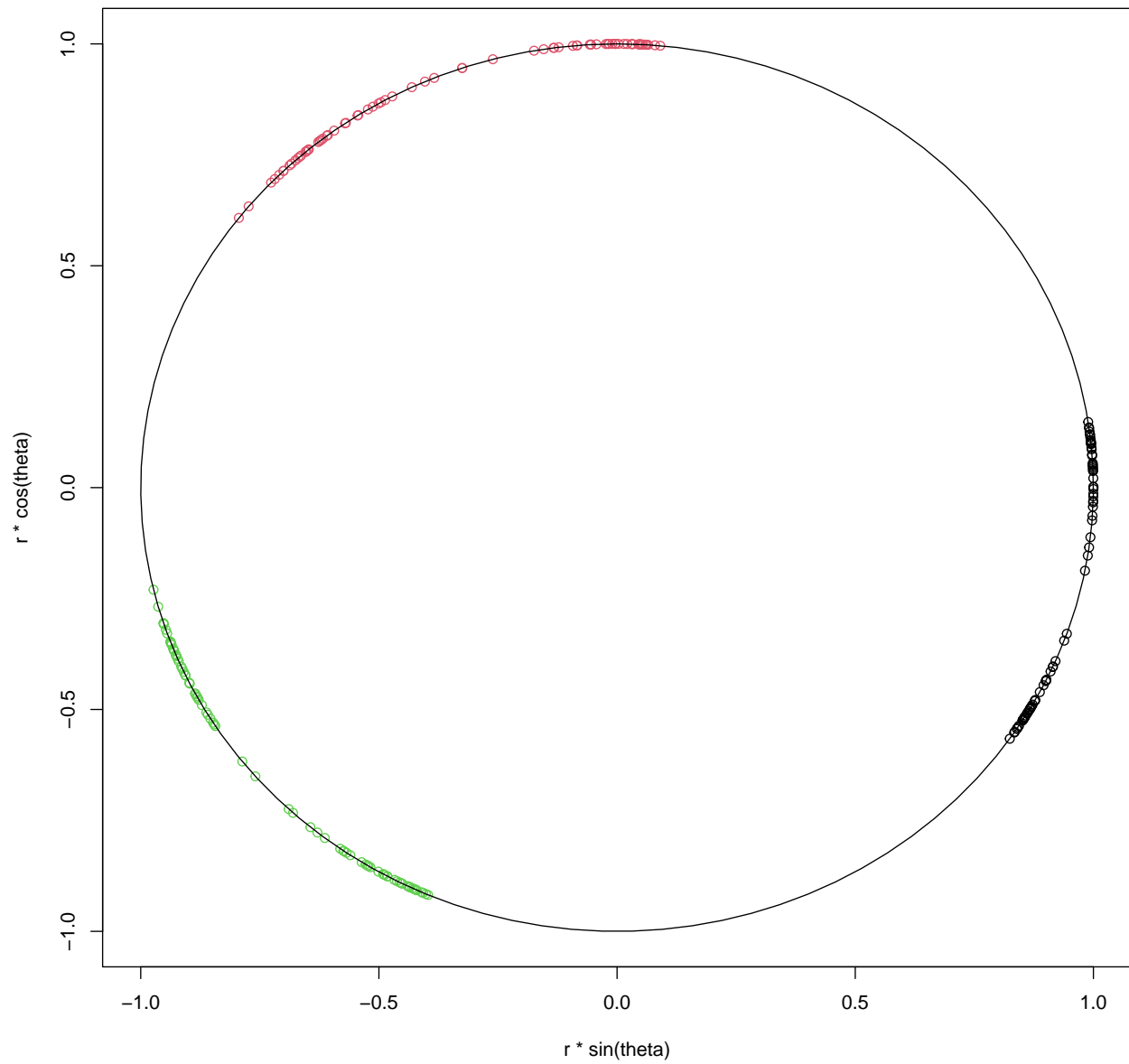
## It: 170; obj: 7.097e-01; abs: 3.834e-07; rel: 5.401e-07; norm: 1.167e-01
## It: 171; obj: 7.097e-01; abs: 3.334e-07; rel: 4.698e-07; norm: 1.167e-01
## It: 172; obj: 7.097e-01; abs: 2.900e-07; rel: 4.086e-07; norm: 1.167e-01
## It: 173; obj: 7.097e-01; abs: 2.523e-07; rel: 3.555e-07; norm: 1.167e-01
## It: 174; obj: 7.097e-01; abs: 2.195e-07; rel: 3.093e-07; norm: 1.167e-01
## It: 175; obj: 7.097e-01; abs: 1.911e-07; rel: 2.692e-07; norm: 1.167e-01

## It: 176; obj: 7.097e-01; abs: 1.663e-07; rel: 2.343e-07; norm: 1.167e-01
## It: 177; obj: 7.097e-01; abs: 1.448e-07; rel: 2.040e-07; norm: 1.167e-01
## It: 178; obj: 7.097e-01; abs: 1.260e-07; rel: 1.776e-07; norm: 1.167e-01
## It: 179; obj: 7.097e-01; abs: 1.097e-07; rel: 1.546e-07; norm: 1.167e-01
## It: 180; obj: 7.097e-01; abs: 9.554e-08; rel: 1.346e-07; norm: 1.167e-01
## It: 181; obj: 7.097e-01; abs: 8.320e-08; rel: 1.172e-07; norm: 1.167e-01
## It: 182; obj: 7.097e-01; abs: 7.246e-08; rel: 1.021e-07; norm: 1.167e-01
## It: 183; obj: 7.097e-01; abs: 6.312e-08; rel: 8.893e-08; norm: 1.167e-01
## It: 184; obj: 7.097e-01; abs: 5.498e-08; rel: 7.746e-08; norm: 1.167e-01
## It: 185; obj: 7.097e-01; abs: 4.789e-08; rel: 6.748e-08; norm: 1.167e-01
## It: 186; obj: 7.097e-01; abs: 4.172e-08; rel: 5.879e-08; norm: 1.167e-01
## It: 187; obj: 7.097e-01; abs: 3.635e-08; rel: 5.122e-08; norm: 1.167e-01
## It: 188; obj: 7.097e-01; abs: 3.167e-08; rel: 4.463e-08; norm: 1.167e-01
## It: 189; obj: 7.097e-01; abs: 2.760e-08; rel: 3.888e-08; norm: 1.167e-01
## It: 190; obj: 7.097e-01; abs: 2.405e-08; rel: 3.388e-08; norm: 1.167e-01
## It: 191; obj: 7.097e-01; abs: 2.096e-08; rel: 2.953e-08; norm: 1.167e-01
## It: 192; obj: 7.097e-01; abs: 1.826e-08; rel: 2.573e-08; norm: 1.167e-01
## It: 193; obj: 7.097e-01; abs: 1.592e-08; rel: 2.243e-08; norm: 1.167e-01
## It: 194; obj: 7.097e-01; abs: 1.387e-08; rel: 1.954e-08; norm: 1.167e-01
## It: 195; obj: 7.097e-01; abs: 1.209e-08; rel: 1.703e-08; norm: 1.167e-01
## It: 196; obj: 7.097e-01; abs: 1.054e-08; rel: 1.485e-08; norm: 1.167e-01
## It: 197; obj: 7.097e-01; abs: 9.185e-09; rel: 1.294e-08; norm: 1.167e-01
## It: 198; obj: 7.097e-01; abs: 8.007e-09; rel: 1.128e-08; norm: 1.166e-01
## It: 199; obj: 7.097e-01; abs: 6.979e-09; rel: 9.834e-09; norm: 1.166e-01
## It: 200; obj: 7.097e-01; abs: 6.084e-09; rel: 8.572e-09; norm: 1.166e-01

```

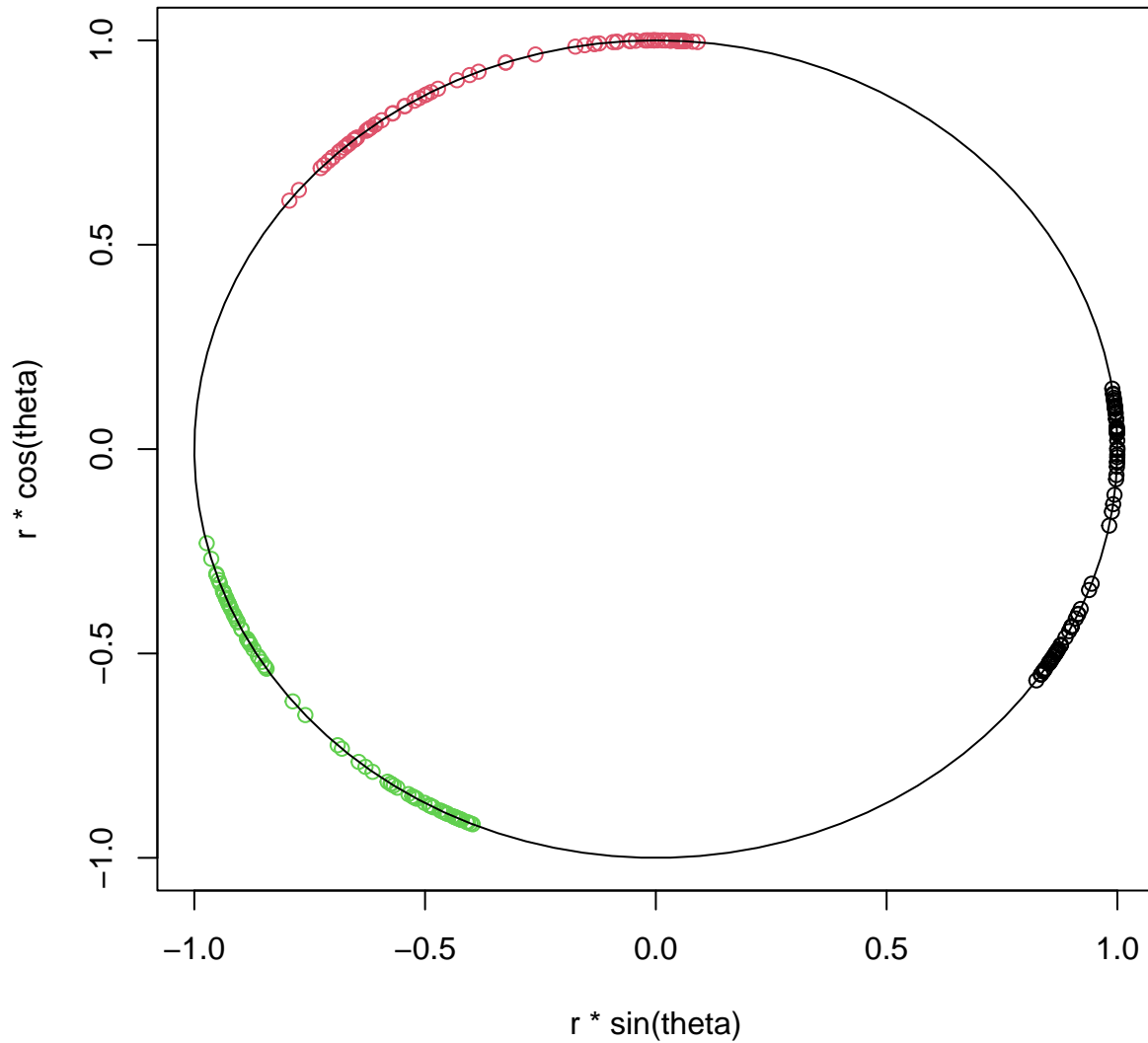


Iteration 200



```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension  $\mathbb{S}^2$  then  $d = 2$  (sphere):

```
Y <- psc_sne(X=x_5, d=2, rho_psc_list = rho_five_perp20, num_iteration=200, eta = 400,
             visualize_prog = TRUE, colors = colors)
```

```
## It: 1; obj: 1.275e+01; abs: 0.000e+00; rel: 0.000e+00; norm: 1.405e-01
## It: 2; obj: 8.117e+00; abs: 4.630e+00; rel: 3.632e-01; norm: 5.028e-01
## It: 3; obj: 7.320e+00; abs: 7.966e-01; rel: 9.815e-02; norm: 1.492e+00
## It: 4; obj: 7.384e+00; abs: 6.401e-02; rel: 8.745e-03; norm: 2.017e+00
## It: 5; obj: 7.522e+00; abs: 1.383e-01; rel: 1.872e-02; norm: 2.249e+00
## It: 6; obj: 7.655e+00; abs: 1.332e-01; rel: 1.770e-02; norm: 2.382e+00
## It: 7; obj: 7.756e+00; abs: 1.005e-01; rel: 1.313e-02; norm: 2.461e+00
## It: 8; obj: 7.821e+00; abs: 6.475e-02; rel: 8.348e-03; norm: 2.506e+00
## It: 9; obj: 7.860e+00; abs: 3.893e-02; rel: 4.978e-03; norm: 2.535e+00
## It: 10; obj: 7.882e+00; abs: 2.276e-02; rel: 2.896e-03; norm: 2.555e+00
## It: 11; obj: 7.895e+00; abs: 1.308e-02; rel: 1.659e-03; norm: 2.571e+00
## It: 12; obj: 7.903e+00; abs: 7.397e-03; rel: 9.368e-04; norm: 2.585e+00
## It: 13; obj: 7.907e+00; abs: 4.125e-03; rel: 5.220e-04; norm: 2.598e+00
## It: 14; obj: 7.909e+00; abs: 2.304e-03; rel: 2.915e-04; norm: 2.610e+00
## It: 15; obj: 7.911e+00; abs: 1.358e-03; rel: 1.717e-04; norm: 2.622e+00
```

```

## It: 16; obj: 7.912e+00; abs: 9.370e-04; rel: 1.184e-04; norm: 2.633e+00
## It: 17; obj: 7.912e+00; abs: 8.306e-04; rel: 1.050e-04; norm: 2.645e+00
## It: 18; obj: 7.913e+00; abs: 9.111e-04; rel: 1.151e-04; norm: 2.656e+00
## It: 19; obj: 7.914e+00; abs: 1.101e-03; rel: 1.391e-04; norm: 2.667e+00
## It: 20; obj: 7.916e+00; abs: 1.354e-03; rel: 1.710e-04; norm: 2.678e+00
## It: 21; obj: 7.917e+00; abs: 1.640e-03; rel: 2.072e-04; norm: 2.689e+00
## It: 22; obj: 7.919e+00; abs: 1.944e-03; rel: 2.456e-04; norm: 2.700e+00
## It: 23; obj: 7.922e+00; abs: 2.254e-03; rel: 2.847e-04; norm: 2.710e+00
## It: 24; obj: 7.924e+00; abs: 2.564e-03; rel: 3.236e-04; norm: 2.721e+00
## It: 25; obj: 7.927e+00; abs: 2.867e-03; rel: 3.618e-04; norm: 2.731e+00

## It: 26; obj: 7.930e+00; abs: 3.161e-03; rel: 3.988e-04; norm: 2.741e+00
## It: 27; obj: 7.934e+00; abs: 3.442e-03; rel: 4.340e-04; norm: 2.751e+00
## It: 28; obj: 7.937e+00; abs: 3.707e-03; rel: 4.673e-04; norm: 2.761e+00
## It: 29; obj: 7.941e+00; abs: 3.954e-03; rel: 4.982e-04; norm: 2.770e+00
## It: 30; obj: 7.945e+00; abs: 4.181e-03; rel: 5.264e-04; norm: 2.779e+00
## It: 31; obj: 7.950e+00; abs: 4.384e-03; rel: 5.517e-04; norm: 2.788e+00
## It: 32; obj: 7.954e+00; abs: 4.561e-03; rel: 5.738e-04; norm: 2.796e+00
## It: 33; obj: 7.959e+00; abs: 4.712e-03; rel: 5.924e-04; norm: 2.804e+00
## It: 34; obj: 7.964e+00; abs: 4.834e-03; rel: 6.074e-04; norm: 2.812e+00
## It: 35; obj: 7.969e+00; abs: 4.927e-03; rel: 6.187e-04; norm: 2.820e+00
## It: 36; obj: 7.974e+00; abs: 4.991e-03; rel: 6.263e-04; norm: 2.827e+00
## It: 37; obj: 7.979e+00; abs: 5.025e-03; rel: 6.302e-04; norm: 2.834e+00
## It: 38; obj: 7.984e+00; abs: 5.031e-03; rel: 6.305e-04; norm: 2.841e+00
## It: 39; obj: 7.989e+00; abs: 5.008e-03; rel: 6.273e-04; norm: 2.847e+00
## It: 40; obj: 7.994e+00; abs: 4.960e-03; rel: 6.208e-04; norm: 2.853e+00
## It: 41; obj: 7.999e+00; abs: 4.887e-03; rel: 6.113e-04; norm: 2.858e+00
## It: 42; obj: 8.004e+00; abs: 4.791e-03; rel: 5.990e-04; norm: 2.864e+00
## It: 43; obj: 8.008e+00; abs: 4.677e-03; rel: 5.843e-04; norm: 2.869e+00
## It: 44; obj: 8.013e+00; abs: 4.544e-03; rel: 5.675e-04; norm: 2.874e+00
## It: 45; obj: 8.017e+00; abs: 4.398e-03; rel: 5.488e-04; norm: 2.878e+00
## It: 46; obj: 8.021e+00; abs: 4.239e-03; rel: 5.287e-04; norm: 2.882e+00
## It: 47; obj: 8.025e+00; abs: 4.071e-03; rel: 5.075e-04; norm: 2.886e+00
## It: 48; obj: 8.029e+00; abs: 3.895e-03; rel: 4.854e-04; norm: 2.890e+00
## It: 49; obj: 8.033e+00; abs: 3.715e-03; rel: 4.627e-04; norm: 2.893e+00
## It: 50; obj: 8.037e+00; abs: 3.532e-03; rel: 4.397e-04; norm: 2.896e+00

## It: 51; obj: 8.040e+00; abs: 3.349e-03; rel: 4.167e-04; norm: 2.899e+00
## It: 52; obj: 8.043e+00; abs: 3.166e-03; rel: 3.938e-04; norm: 2.902e+00
## It: 53; obj: 8.046e+00; abs: 2.986e-03; rel: 3.712e-04; norm: 2.905e+00
## It: 54; obj: 8.049e+00; abs: 2.809e-03; rel: 3.491e-04; norm: 2.907e+00
## It: 55; obj: 8.052e+00; abs: 2.636e-03; rel: 3.275e-04; norm: 2.909e+00
## It: 56; obj: 8.054e+00; abs: 2.470e-03; rel: 3.067e-04; norm: 2.911e+00
## It: 57; obj: 8.056e+00; abs: 2.309e-03; rel: 2.867e-04; norm: 2.913e+00
## It: 58; obj: 8.058e+00; abs: 2.155e-03; rel: 2.674e-04; norm: 2.915e+00
## It: 59; obj: 8.060e+00; abs: 2.007e-03; rel: 2.491e-04; norm: 2.916e+00
## It: 60; obj: 8.062e+00; abs: 1.867e-03; rel: 2.316e-04; norm: 2.918e+00
## It: 61; obj: 8.064e+00; abs: 1.734e-03; rel: 2.151e-04; norm: 2.919e+00
## It: 62; obj: 8.066e+00; abs: 1.609e-03; rel: 1.995e-04; norm: 2.921e+00
## It: 63; obj: 8.067e+00; abs: 1.490e-03; rel: 1.848e-04; norm: 2.922e+00
## It: 64; obj: 8.069e+00; abs: 1.379e-03; rel: 1.710e-04; norm: 2.923e+00
## It: 65; obj: 8.070e+00; abs: 1.275e-03; rel: 1.580e-04; norm: 2.924e+00
## It: 66; obj: 8.071e+00; abs: 1.177e-03; rel: 1.459e-04; norm: 2.925e+00
## It: 67; obj: 8.072e+00; abs: 1.086e-03; rel: 1.345e-04; norm: 2.926e+00
## It: 68; obj: 8.073e+00; abs: 1.001e-03; rel: 1.240e-04; norm: 2.926e+00

```

```

## It: 69; obj: 8.074e+00; abs: 9.217e-04; rel: 1.142e-04; norm: 2.927e+00
## It: 70; obj: 8.075e+00; abs: 8.481e-04; rel: 1.050e-04; norm: 2.928e+00
## It: 71; obj: 8.076e+00; abs: 7.798e-04; rel: 9.658e-05; norm: 2.929e+00
## It: 72; obj: 8.076e+00; abs: 7.165e-04; rel: 8.873e-05; norm: 2.929e+00
## It: 73; obj: 8.077e+00; abs: 6.579e-04; rel: 8.146e-05; norm: 2.930e+00
## It: 74; obj: 8.078e+00; abs: 6.037e-04; rel: 7.474e-05; norm: 2.930e+00
## It: 75; obj: 8.078e+00; abs: 5.536e-04; rel: 6.853e-05; norm: 2.931e+00

## It: 76; obj: 8.079e+00; abs: 5.073e-04; rel: 6.280e-05; norm: 2.931e+00
## It: 77; obj: 8.079e+00; abs: 4.646e-04; rel: 5.751e-05; norm: 2.931e+00
## It: 78; obj: 8.080e+00; abs: 4.253e-04; rel: 5.264e-05; norm: 2.932e+00
## It: 79; obj: 8.080e+00; abs: 3.890e-04; rel: 4.814e-05; norm: 2.932e+00
## It: 80; obj: 8.080e+00; abs: 3.556e-04; rel: 4.401e-05; norm: 2.933e+00
## It: 81; obj: 8.081e+00; abs: 3.249e-04; rel: 4.021e-05; norm: 2.933e+00
## It: 82; obj: 8.081e+00; abs: 2.966e-04; rel: 3.671e-05; norm: 2.933e+00
## It: 83; obj: 8.081e+00; abs: 2.706e-04; rel: 3.349e-05; norm: 2.933e+00
## It: 84; obj: 8.081e+00; abs: 2.468e-04; rel: 3.054e-05; norm: 2.934e+00
## It: 85; obj: 8.082e+00; abs: 2.248e-04; rel: 2.782e-05; norm: 2.934e+00
## It: 86; obj: 8.082e+00; abs: 2.047e-04; rel: 2.533e-05; norm: 2.934e+00
## It: 87; obj: 8.082e+00; abs: 1.862e-04; rel: 2.304e-05; norm: 2.934e+00
## It: 88; obj: 8.082e+00; abs: 1.692e-04; rel: 2.094e-05; norm: 2.935e+00
## It: 89; obj: 8.082e+00; abs: 1.537e-04; rel: 1.901e-05; norm: 2.935e+00
## It: 90; obj: 8.083e+00; abs: 1.394e-04; rel: 1.725e-05; norm: 2.935e+00
## It: 91; obj: 8.083e+00; abs: 1.263e-04; rel: 1.563e-05; norm: 2.935e+00
## It: 92; obj: 8.083e+00; abs: 1.143e-04; rel: 1.414e-05; norm: 2.935e+00
## It: 93; obj: 8.083e+00; abs: 1.033e-04; rel: 1.278e-05; norm: 2.935e+00
## It: 94; obj: 8.083e+00; abs: 9.325e-05; rel: 1.154e-05; norm: 2.935e+00
## It: 95; obj: 8.083e+00; abs: 8.402e-05; rel: 1.039e-05; norm: 2.936e+00
## It: 96; obj: 8.083e+00; abs: 7.556e-05; rel: 9.348e-06; norm: 2.936e+00
## It: 97; obj: 8.083e+00; abs: 6.781e-05; rel: 8.390e-06; norm: 2.936e+00
## It: 98; obj: 8.083e+00; abs: 6.072e-05; rel: 7.512e-06; norm: 2.936e+00
## It: 99; obj: 8.083e+00; abs: 5.423e-05; rel: 6.709e-06; norm: 2.936e+00
## It: 100; obj: 8.083e+00; abs: 4.828e-05; rel: 5.973e-06; norm: 2.936e+00

## It: 101; obj: 1.211e+00; abs: 6.873e+00; rel: 8.502e-01; norm: 1.152e-01
## It: 102; obj: 1.399e+00; abs: 1.887e-01; rel: 1.559e-01; norm: 3.786e-01
## It: 103; obj: 1.314e+00; abs: 8.485e-02; rel: 6.064e-02; norm: 3.968e-01
## It: 104; obj: 1.276e+00; abs: 3.891e-02; rel: 2.960e-02; norm: 4.484e-01
## It: 105; obj: 1.285e+00; abs: 9.096e-03; rel: 7.132e-03; norm: 4.287e-01
## It: 106; obj: 1.394e+00; abs: 1.094e-01; rel: 8.518e-02; norm: 4.251e-01
## It: 107; obj: 1.471e+00; abs: 7.745e-02; rel: 5.556e-02; norm: 4.635e-01
## It: 108; obj: 1.652e+00; abs: 1.806e-01; rel: 1.227e-01; norm: 5.133e-01
## It: 109; obj: 1.673e+00; abs: 2.114e-02; rel: 1.280e-02; norm: 5.922e-01
## It: 110; obj: 1.692e+00; abs: 1.852e-02; rel: 1.107e-02; norm: 6.409e-01
## It: 111; obj: 1.596e+00; abs: 9.579e-02; rel: 5.662e-02; norm: 6.692e-01
## It: 112; obj: 1.594e+00; abs: 2.218e-03; rel: 1.389e-03; norm: 6.579e-01
## It: 113; obj: 1.562e+00; abs: 3.145e-02; rel: 1.973e-02; norm: 6.504e-01
## It: 114; obj: 1.555e+00; abs: 7.167e-03; rel: 4.588e-03; norm: 6.498e-01
## It: 115; obj: 1.550e+00; abs: 5.081e-03; rel: 3.267e-03; norm: 6.490e-01
## It: 116; obj: 1.529e+00; abs: 2.062e-02; rel: 1.330e-02; norm: 6.468e-01
## It: 117; obj: 1.543e+00; abs: 1.389e-02; rel: 9.083e-03; norm: 6.434e-01
## It: 118; obj: 1.535e+00; abs: 8.173e-03; rel: 5.296e-03; norm: 6.406e-01
## It: 119; obj: 1.552e+00; abs: 1.691e-02; rel: 1.102e-02; norm: 6.390e-01
## It: 120; obj: 1.553e+00; abs: 1.320e-03; rel: 8.502e-04; norm: 6.363e-01
## It: 121; obj: 1.525e+00; abs: 2.791e-02; rel: 1.797e-02; norm: 6.516e-01

```

```

## It: 122; obj: 1.515e+00; abs: 1.028e-02; rel: 6.738e-03; norm: 6.360e-01
## It: 123; obj: 1.543e+00; abs: 2.817e-02; rel: 1.859e-02; norm: 6.372e-01
## It: 124; obj: 1.523e+00; abs: 2.041e-02; rel: 1.322e-02; norm: 6.269e-01
## It: 125; obj: 1.537e+00; abs: 1.384e-02; rel: 9.084e-03; norm: 6.361e-01

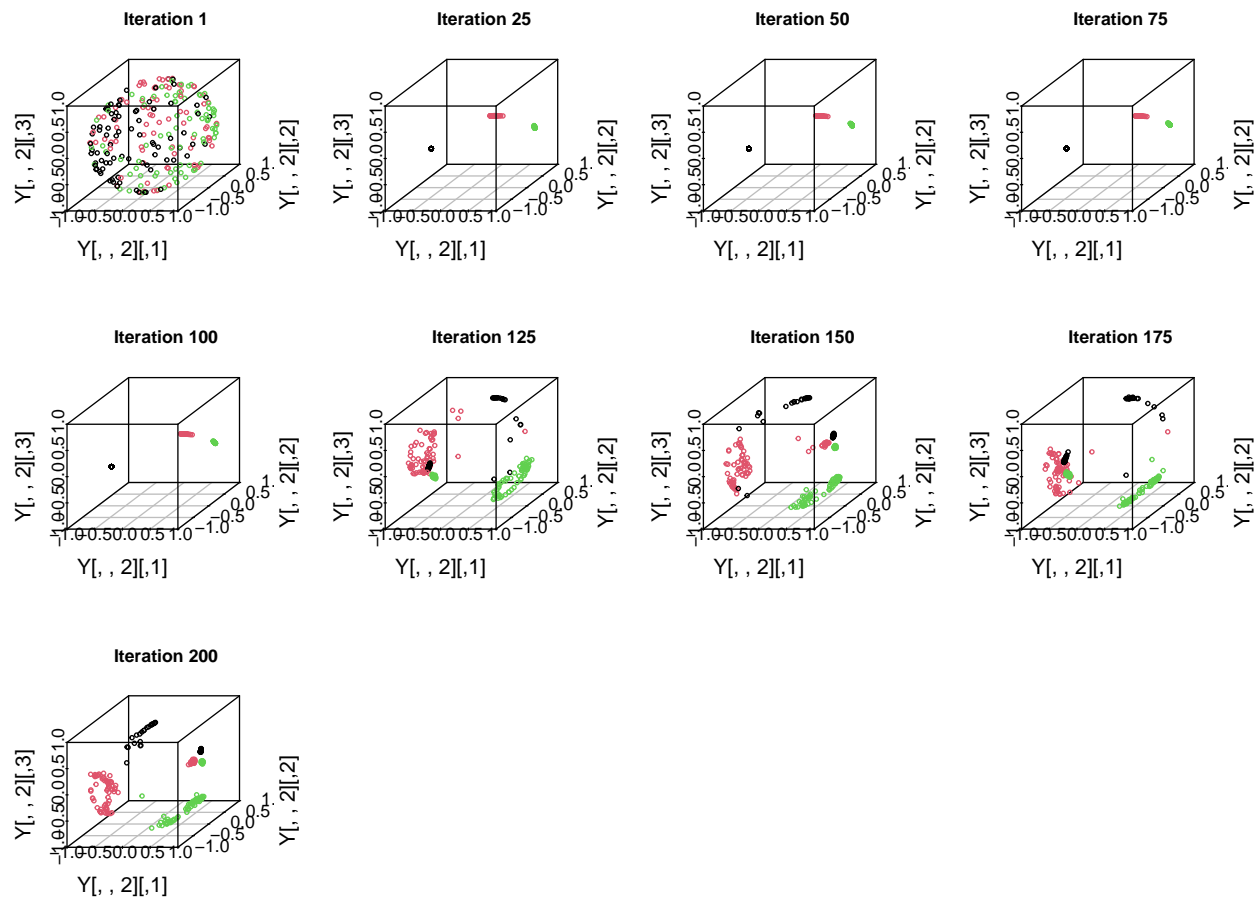
## It: 126; obj: 1.523e+00; abs: 1.403e-02; rel: 9.126e-03; norm: 6.313e-01
## It: 127; obj: 1.529e+00; abs: 5.780e-03; rel: 3.796e-03; norm: 6.384e-01
## It: 128; obj: 1.488e+00; abs: 4.062e-02; rel: 2.657e-02; norm: 6.285e-01
## It: 129; obj: 1.517e+00; abs: 2.947e-02; rel: 1.981e-02; norm: 6.370e-01
## It: 130; obj: 1.490e+00; abs: 2.777e-02; rel: 1.830e-02; norm: 6.195e-01
## It: 131; obj: 1.521e+00; abs: 3.089e-02; rel: 2.074e-02; norm: 6.302e-01
## It: 132; obj: 1.472e+00; abs: 4.829e-02; rel: 3.176e-02; norm: 6.240e-01
## It: 133; obj: 1.517e+00; abs: 4.498e-02; rel: 3.056e-02; norm: 6.294e-01
## It: 134; obj: 1.473e+00; abs: 4.412e-02; rel: 2.908e-02; norm: 6.140e-01
## It: 135; obj: 1.516e+00; abs: 4.315e-02; rel: 2.929e-02; norm: 6.295e-01
## It: 136; obj: 1.477e+00; abs: 3.950e-02; rel: 2.605e-02; norm: 6.129e-01
## It: 137; obj: 1.510e+00; abs: 3.321e-02; rel: 2.249e-02; norm: 6.187e-01
## It: 138; obj: 1.460e+00; abs: 4.961e-02; rel: 3.285e-02; norm: 6.081e-01
## It: 139; obj: 1.515e+00; abs: 5.474e-02; rel: 3.749e-02; norm: 6.210e-01
## It: 140; obj: 1.442e+00; abs: 7.304e-02; rel: 4.821e-02; norm: 6.073e-01
## It: 141; obj: 1.529e+00; abs: 8.731e-02; rel: 6.055e-02; norm: 6.071e-01
## It: 142; obj: 1.506e+00; abs: 2.362e-02; rel: 1.544e-02; norm: 5.979e-01
## It: 143; obj: 1.549e+00; abs: 4.341e-02; rel: 2.883e-02; norm: 6.292e-01
## It: 144; obj: 1.493e+00; abs: 5.571e-02; rel: 3.596e-02; norm: 6.205e-01
## It: 145; obj: 1.527e+00; abs: 3.388e-02; rel: 2.268e-02; norm: 6.339e-01
## It: 146; obj: 1.539e+00; abs: 1.155e-02; rel: 7.564e-03; norm: 6.067e-01
## It: 147; obj: 1.563e+00; abs: 2.458e-02; rel: 1.597e-02; norm: 6.269e-01
## It: 148; obj: 1.518e+00; abs: 4.546e-02; rel: 2.908e-02; norm: 6.229e-01
## It: 149; obj: 1.559e+00; abs: 4.129e-02; rel: 2.720e-02; norm: 6.405e-01
## It: 150; obj: 1.494e+00; abs: 6.548e-02; rel: 4.200e-02; norm: 6.214e-01

## It: 151; obj: 1.540e+00; abs: 4.650e-02; rel: 3.113e-02; norm: 6.188e-01
## It: 152; obj: 1.524e+00; abs: 1.654e-02; rel: 1.074e-02; norm: 6.017e-01
## It: 153; obj: 1.559e+00; abs: 3.540e-02; rel: 2.323e-02; norm: 6.203e-01
## It: 154; obj: 1.518e+00; abs: 4.086e-02; rel: 2.621e-02; norm: 6.235e-01
## It: 155; obj: 1.526e+00; abs: 7.855e-03; rel: 5.173e-03; norm: 6.358e-01
## It: 156; obj: 1.505e+00; abs: 2.088e-02; rel: 1.368e-02; norm: 6.103e-01
## It: 157; obj: 1.512e+00; abs: 7.082e-03; rel: 4.705e-03; norm: 6.135e-01
## It: 158; obj: 1.512e+00; abs: 4.252e-04; rel: 2.812e-04; norm: 6.049e-01
## It: 159; obj: 1.539e+00; abs: 2.734e-02; rel: 1.808e-02; norm: 6.100e-01
## It: 160; obj: 1.528e+00; abs: 1.114e-02; rel: 7.234e-03; norm: 6.057e-01
## It: 161; obj: 1.543e+00; abs: 1.439e-02; rel: 9.415e-03; norm: 6.164e-01
## It: 162; obj: 1.508e+00; abs: 3.485e-02; rel: 2.259e-02; norm: 6.107e-01
## It: 163; obj: 1.513e+00; abs: 5.459e-03; rel: 3.621e-03; norm: 6.149e-01
## It: 164; obj: 1.542e+00; abs: 2.911e-02; rel: 1.924e-02; norm: 5.952e-01
## It: 165; obj: 1.520e+00; abs: 2.187e-02; rel: 1.418e-02; norm: 6.132e-01
## It: 166; obj: 1.510e+00; abs: 1.052e-02; rel: 6.919e-03; norm: 6.150e-01
## It: 167; obj: 1.512e+00; abs: 1.766e-03; rel: 1.170e-03; norm: 6.148e-01
## It: 168; obj: 1.522e+00; abs: 1.055e-02; rel: 6.977e-03; norm: 5.964e-01
## It: 169; obj: 1.509e+00; abs: 1.360e-02; rel: 8.932e-03; norm: 6.069e-01
## It: 170; obj: 1.533e+00; abs: 2.443e-02; rel: 1.620e-02; norm: 6.027e-01
## It: 171; obj: 1.526e+00; abs: 6.862e-03; rel: 4.476e-03; norm: 6.043e-01
## It: 172; obj: 1.559e+00; abs: 3.245e-02; rel: 2.126e-02; norm: 5.969e-01
## It: 173; obj: 1.520e+00; abs: 3.845e-02; rel: 2.467e-02; norm: 6.126e-01
## It: 174; obj: 1.501e+00; abs: 1.913e-02; rel: 1.258e-02; norm: 6.102e-01

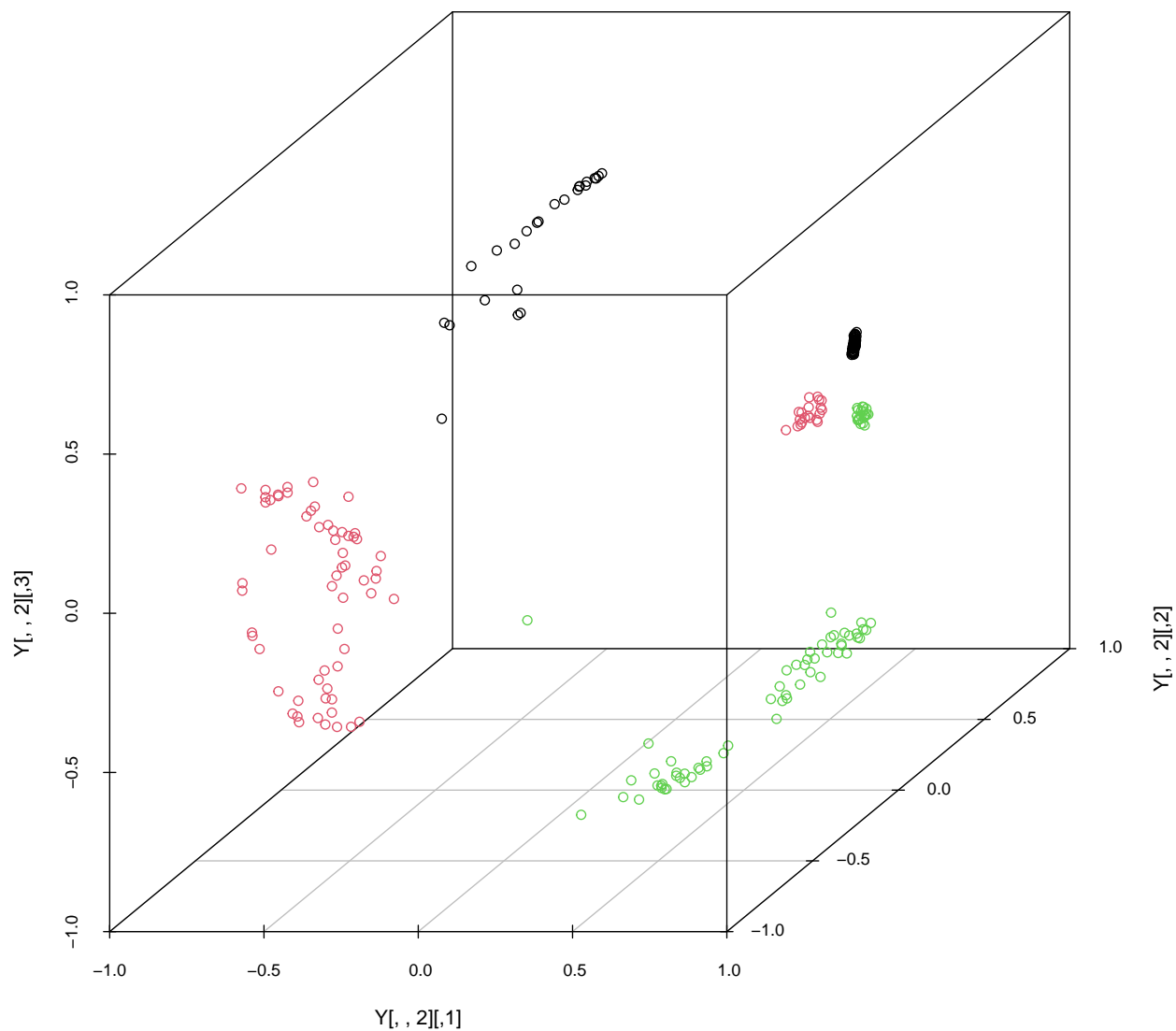
```

## It: 175; obj: 1.495e+00; abs: 5.712e-03; rel: 3.805e-03; norm: 6.059e-01  
## It: 176; obj: 1.526e+00; abs: 3.109e-02; rel: 2.079e-02; norm: 5.878e-01  
## It: 177; obj: 1.522e+00; abs: 4.789e-03; rel: 3.138e-03; norm: 5.984e-01  
## It: 178; obj: 1.539e+00; abs: 1.699e-02; rel: 1.116e-02; norm: 6.055e-01  
## It: 179; obj: 1.541e+00; abs: 2.368e-03; rel: 1.539e-03; norm: 6.131e-01  
## It: 180; obj: 1.546e+00; abs: 5.484e-03; rel: 3.559e-03; norm: 6.091e-01  
## It: 181; obj: 1.551e+00; abs: 4.517e-03; rel: 2.921e-03; norm: 6.159e-01  
## It: 182; obj: 1.546e+00; abs: 5.294e-03; rel: 3.414e-03; norm: 6.130e-01  
## It: 183; obj: 1.532e+00; abs: 1.370e-02; rel: 8.860e-03; norm: 6.146e-01  
## It: 184; obj: 1.563e+00; abs: 3.106e-02; rel: 2.027e-02; norm: 6.107e-01  
## It: 185; obj: 1.551e+00; abs: 1.206e-02; rel: 7.715e-03; norm: 6.149e-01  
## It: 186; obj: 1.540e+00; abs: 1.061e-02; rel: 6.841e-03; norm: 6.156e-01  
## It: 187; obj: 1.544e+00; abs: 3.404e-03; rel: 2.210e-03; norm: 6.212e-01  
## It: 188; obj: 1.561e+00; abs: 1.687e-02; rel: 1.093e-02; norm: 6.090e-01  
## It: 189; obj: 1.536e+00; abs: 2.414e-02; rel: 1.547e-02; norm: 6.160e-01  
## It: 190; obj: 1.555e+00; abs: 1.861e-02; rel: 1.211e-02; norm: 6.145e-01  
## It: 191; obj: 1.573e+00; abs: 1.831e-02; rel: 1.177e-02; norm: 6.153e-01  
## It: 192; obj: 1.553e+00; abs: 2.079e-02; rel: 1.321e-02; norm: 6.118e-01  
## It: 193; obj: 1.556e+00; abs: 3.081e-03; rel: 1.984e-03; norm: 6.228e-01  
## It: 194; obj: 1.580e+00; abs: 2.474e-02; rel: 1.590e-02; norm: 6.171e-01  
## It: 195; obj: 1.586e+00; abs: 5.143e-03; rel: 3.254e-03; norm: 6.140e-01  
## It: 196; obj: 1.567e+00; abs: 1.861e-02; rel: 1.174e-02; norm: 6.334e-01  
## It: 197; obj: 1.587e+00; abs: 2.026e-02; rel: 1.293e-02; norm: 6.269e-01  
## It: 198; obj: 1.574e+00; abs: 1.324e-02; rel: 8.340e-03; norm: 6.223e-01  
## It: 199; obj: 1.549e+00; abs: 2.545e-02; rel: 1.617e-02; norm: 6.325e-01  
## It: 200; obj: 1.532e+00; abs: 1.641e-02; rel: 1.059e-02; norm: 6.231e-01





Iteration 200



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),  
                             color = colors)
```

