Applications PSC-SNE

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Cases

Case 1

Sample on the $(\mathbb{S}^1)^2$ where $p = 1$ and $r = 2$.

Data:

```
# Sample on the (S^1)^2
n <- 200
vmf11 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 10)
vmf12 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi)), kappa = 10)
x1 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf11),
                              DirStats::to_rad(vmf12)))

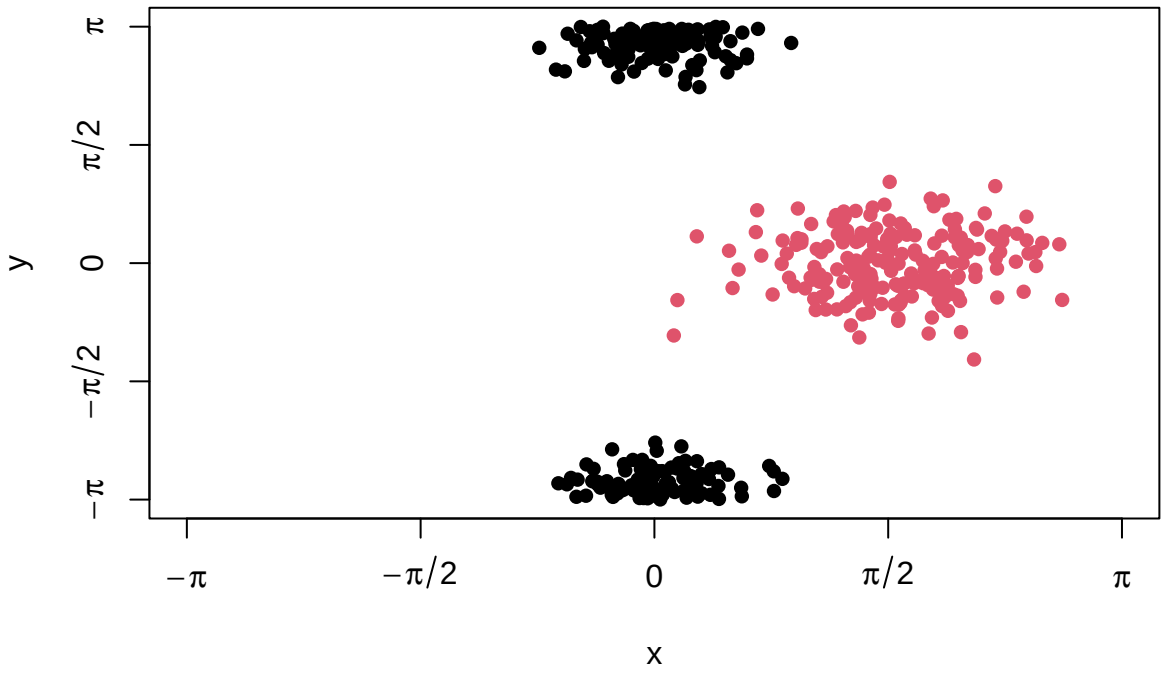
vmf21 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = pi / 2)), kappa = 5)
vmf22 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_cir(th = 0)), kappa = 5)
x2 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf21),DirStats::to_rad(vmf22)))

x <- rbind(x1, x2)

# Cartesian coordinates
n <- 400
x_array <- array(dim = c(n, 2, 2))
x_array[, , 1] <- DirStats::to_cir(x[, 1])
x_array[, , 2] <- DirStats::to_cir(x[, 2])

indexes <- sample(1:n)
x_array <- x_array[indexes,,]
colors <- rep(c(1, 2), each = n/2)[indexes]

plot(x, xlim = c(-pi, pi), ylim = c(-pi, pi), axes = FALSE,
     col = rep(c(1, 2), each = n/2), pch = 16,
     xlab = "x", ylab = "y")
sdetorus::torusAxis()
```



Let's calculate the rho parameters based on a perplexity of 20:

```
# Time difference of 2.582784 mins
rho_first_20 <- rho_optim_bst(x_array, 20)
```

```
## Time difference of 28.62258 secs
```

First, let's reduce to dimension $\mathbb{S}^1$ then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_array, d=1, rho_psc_list = rho_first_20, num_iteration=125,
             colors=colors, visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 16.765664, abs 0.000000, rel 0.000000, norm 0.163919"
```

```
## [1] "Iter 2, obj 15.882874, abs 0.882790, rel 0.052655, norm 0.410459"
## [1] "Iter 3, obj 15.482195, abs 0.400679, rel 0.025227, norm 0.563344"
## [1] "Iter 4, obj 15.261955, abs 0.220240, rel 0.014225, norm 0.629375"
## [1] "Iter 5, obj 15.140320, abs 0.121635, rel 0.007970, norm 0.671015"
## [1] "Iter 6, obj 15.061308, abs 0.079012, rel 0.005219, norm 0.699682"
## [1] "Iter 7, obj 14.998592, abs 0.062716, rel 0.004164, norm 0.722008"
## [1] "Iter 8, obj 14.939754, abs 0.058837, rel 0.003923, norm 0.738623"
## [1] "Iter 9, obj 14.876455, abs 0.063299, rel 0.004237, norm 0.751395"
## [1] "Iter 10, obj 14.811554, abs 0.064902, rel 0.004363, norm 0.763057"
## [1] "Iter 11, obj 14.753039, abs 0.058514, rel 0.003951, norm 0.774705"
## [1] "Iter 12, obj 14.693697, abs 0.059343, rel 0.004022, norm 0.786673"
## [1] "Iter 13, obj 14.641535, abs 0.052162, rel 0.003550, norm 0.798606"
## [1] "Iter 14, obj 14.597755, abs 0.043781, rel 0.002990, norm 0.809611"
## [1] "Iter 15, obj 14.561423, abs 0.036332, rel 0.002489, norm 0.818503"
## [1] "Iter 16, obj 14.528381, abs 0.033042, rel 0.002269, norm 0.824890"
## [1] "Iter 17, obj 14.488249, abs 0.040131, rel 0.002762, norm 0.829280"
## [1] "Iter 18, obj 14.435306, abs 0.052943, rel 0.003654, norm 0.832405"
## [1] "Iter 19, obj 14.402666, abs 0.032640, rel 0.002261, norm 0.836927"
## [1] "Iter 20, obj 14.374921, abs 0.027745, rel 0.001926, norm 0.839178"
## [1] "Iter 21, obj 14.349300, abs 0.025622, rel 0.001782, norm 0.840903"
## [1] "Iter 22, obj 14.323372, abs 0.025928, rel 0.001807, norm 0.842551"
```

```
## [1] "Iter 23, obj 14.295213, abs 0.028159, rel 0.001966, norm 0.844284"
## [1] "Iter 24, obj 14.263621, abs 0.031592, rel 0.002210, norm 0.846171"
## [1] "Iter 25, obj 14.231163, abs 0.032458, rel 0.002276, norm 0.848499"
## [1] "Iter 26, obj 14.201912, abs 0.029251, rel 0.002055, norm 0.851397"
## [1] "Iter 27, obj 14.176404, abs 0.025507, rel 0.001796, norm 0.854050"
## [1] "Iter 28, obj 14.154882, abs 0.021522, rel 0.001518, norm 0.856499"
## [1] "Iter 29, obj 14.136373, abs 0.018510, rel 0.001308, norm 0.858286"
## [1] "Iter 30, obj 14.119995, abs 0.016378, rel 0.001159, norm 0.859193"
## [1] "Iter 31, obj 14.105252, abs 0.014743, rel 0.001044, norm 0.859479"
## [1] "Iter 32, obj 14.091668, abs 0.013584, rel 0.000963, norm 0.859389"
## [1] "Iter 33, obj 14.078306, abs 0.013361, rel 0.000948, norm 0.859072"
## [1] "Iter 34, obj 14.064216, abs 0.014091, rel 0.001001, norm 0.858699"
## [1] "Iter 35, obj 14.050059, abs 0.014156, rel 0.001007, norm 0.858475"
## [1] "Iter 36, obj 14.037143, abs 0.012916, rel 0.000919, norm 0.858437"
## [1] "Iter 37, obj 14.025227, abs 0.011916, rel 0.000849, norm 0.858507"
## [1] "Iter 38, obj 14.014355, abs 0.010872, rel 0.000775, norm 0.858798"
## [1] "Iter 39, obj 14.004810, abs 0.009546, rel 0.000681, norm 0.859506"
## [1] "Iter 40, obj 13.995750, abs 0.009060, rel 0.000647, norm 0.860702"
## [1] "Iter 41, obj 13.985835, abs 0.009915, rel 0.000708, norm 0.862369"
## [1] "Iter 42, obj 13.973872, abs 0.011963, rel 0.000855, norm 0.864526"
## [1] "Iter 43, obj 13.959078, abs 0.014794, rel 0.001059, norm 0.867237"
## [1] "Iter 44, obj 13.941541, abs 0.017537, rel 0.001256, norm 0.870567"
## [1] "Iter 45, obj 13.922293, abs 0.019248, rel 0.001381, norm 0.874550"
## [1] "Iter 46, obj 13.902645, abs 0.019648, rel 0.001411, norm 0.879164"
## [1] "Iter 47, obj 13.883724, abs 0.018921, rel 0.001361, norm 0.884354"
## [1] "Iter 48, obj 13.866262, abs 0.017462, rel 0.001258, norm 0.890070"
## [1] "Iter 49, obj 13.850654, abs 0.015608, rel 0.001126, norm 0.896261"
## [1] "Iter 50, obj 13.837199, abs 0.013455, rel 0.000971, norm 0.902868"
## [1] "Iter 51, obj 13.826212, abs 0.010987, rel 0.000794, norm 0.909812"
## [1] "Iter 52, obj 13.817840, abs 0.008372, rel 0.000605, norm 0.917008"
## [1] "Iter 53, obj 13.811830, abs 0.006010, rel 0.000435, norm 0.924368"
## [1] "Iter 54, obj 13.807585, abs 0.004245, rel 0.000307, norm 0.931814"
## [1] "Iter 55, obj 13.804345, abs 0.003240, rel 0.000235, norm 0.939295"
## [1] "Iter 56, obj 13.801218, abs 0.003127, rel 0.000227, norm 0.946796"
## [1] "Iter 57, obj 13.797141, abs 0.004077, rel 0.000295, norm 0.954336"
## [1] "Iter 58, obj 13.790942, abs 0.006199, rel 0.000449, norm 0.961960"
## [1] "Iter 59, obj 13.781589, abs 0.009352, rel 0.000678, norm 0.969728"
## [1] "Iter 60, obj 13.768616, abs 0.012974, rel 0.000941, norm 0.977719"
## [1] "Iter 61, obj 13.752503, abs 0.016113, rel 0.001170, norm 0.986009"
## [1] "Iter 62, obj 13.734657, abs 0.017846, rel 0.001298, norm 0.994663"
## [1] "Iter 63, obj 13.717294, abs 0.017363, rel 0.001264, norm 1.003703"
## [1] "Iter 64, obj 13.703379, abs 0.013915, rel 0.001014, norm 1.013056"
## [1] "Iter 65, obj 13.695449, abs 0.007930, rel 0.000579, norm 1.022555"
## [1] "Iter 66, obj 13.694073, abs 0.001376, rel 0.000100, norm 1.032024"
## [1] "Iter 67, obj 13.698160, abs 0.004087, rel 0.000298, norm 1.041343"
## [1] "Iter 68, obj 13.706202, abs 0.008042, rel 0.000587, norm 1.050436"
## [1] "Iter 69, obj 13.716923, abs 0.010721, rel 0.000782, norm 1.059250"
## [1] "Iter 70, obj 13.729388, abs 0.012464, rel 0.000909, norm 1.067745"
## [1] "Iter 71, obj 13.742937, abs 0.013550, rel 0.000987, norm 1.075888"
## [1] "Iter 72, obj 13.757115, abs 0.014178, rel 0.001032, norm 1.083659"
## [1] "Iter 73, obj 13.771602, abs 0.014487, rel 0.001053, norm 1.091041"
## [1] "Iter 74, obj 13.786175, abs 0.014572, rel 0.001058, norm 1.098028"
## [1] "Iter 75, obj 13.800676, abs 0.014501, rel 0.001052, norm 1.104623"
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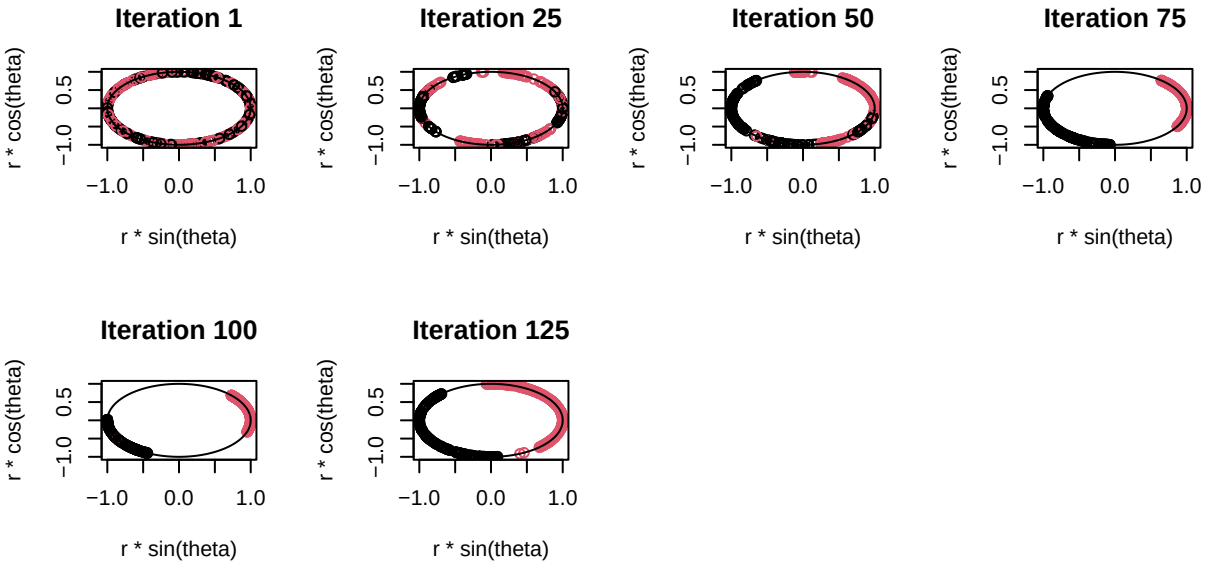
```
## [1] "Iter 76, obj 13.814995, abs 0.014319, rel 0.001038, norm 1.110834"
## [1] "Iter 77, obj 13.829054, abs 0.014059, rel 0.001018, norm 1.116676"
## [1] "Iter 78, obj 13.842798, abs 0.013744, rel 0.000994, norm 1.122167"
## [1] "Iter 79, obj 13.856187, abs 0.013389, rel 0.000967, norm 1.127327"
## [1] "Iter 80, obj 13.869194, abs 0.013007, rel 0.000939, norm 1.132178"
## [1] "Iter 81, obj 13.881800, abs 0.012605, rel 0.000909, norm 1.136738"
## [1] "Iter 82, obj 13.893988, abs 0.012189, rel 0.000878, norm 1.141026"
## [1] "Iter 83, obj 13.905750, abs 0.011762, rel 0.000847, norm 1.145059"
## [1] "Iter 84, obj 13.917077, abs 0.011327, rel 0.000815, norm 1.148854"
## [1] "Iter 85, obj 13.927963, abs 0.010886, rel 0.000782, norm 1.152424"
## [1] "Iter 86, obj 13.938403, abs 0.010440, rel 0.000750, norm 1.155783"
## [1] "Iter 87, obj 13.948393, abs 0.009990, rel 0.000717, norm 1.158943"
## [1] "Iter 88, obj 13.957931, abs 0.009538, rel 0.000684, norm 1.161916"
## [1] "Iter 89, obj 13.967015, abs 0.009084, rel 0.000651, norm 1.164712"
## [1] "Iter 90, obj 13.975644, abs 0.008629, rel 0.000618, norm 1.167341"
## [1] "Iter 91, obj 13.983819, abs 0.008175, rel 0.000585, norm 1.169811"
## [1] "Iter 92, obj 13.991542, abs 0.007723, rel 0.000552, norm 1.172132"
## [1] "Iter 93, obj 13.998818, abs 0.007276, rel 0.000520, norm 1.174310"
## [1] "Iter 94, obj 14.005652, abs 0.006834, rel 0.000488, norm 1.176352"
## [1] "Iter 95, obj 14.012052, abs 0.006400, rel 0.000457, norm 1.178267"
## [1] "Iter 96, obj 14.018029, abs 0.005977, rel 0.000427, norm 1.180060"
## [1] "Iter 97, obj 14.023594, abs 0.005565, rel 0.000397, norm 1.181736"
## [1] "Iter 98, obj 14.028761, abs 0.005167, rel 0.000368, norm 1.183303"
## [1] "Iter 99, obj 14.033546, abs 0.004784, rel 0.000341, norm 1.184765"
## [1] "Iter 100, obj 14.037964, abs 0.004418, rel 0.000315, norm 1.186129"

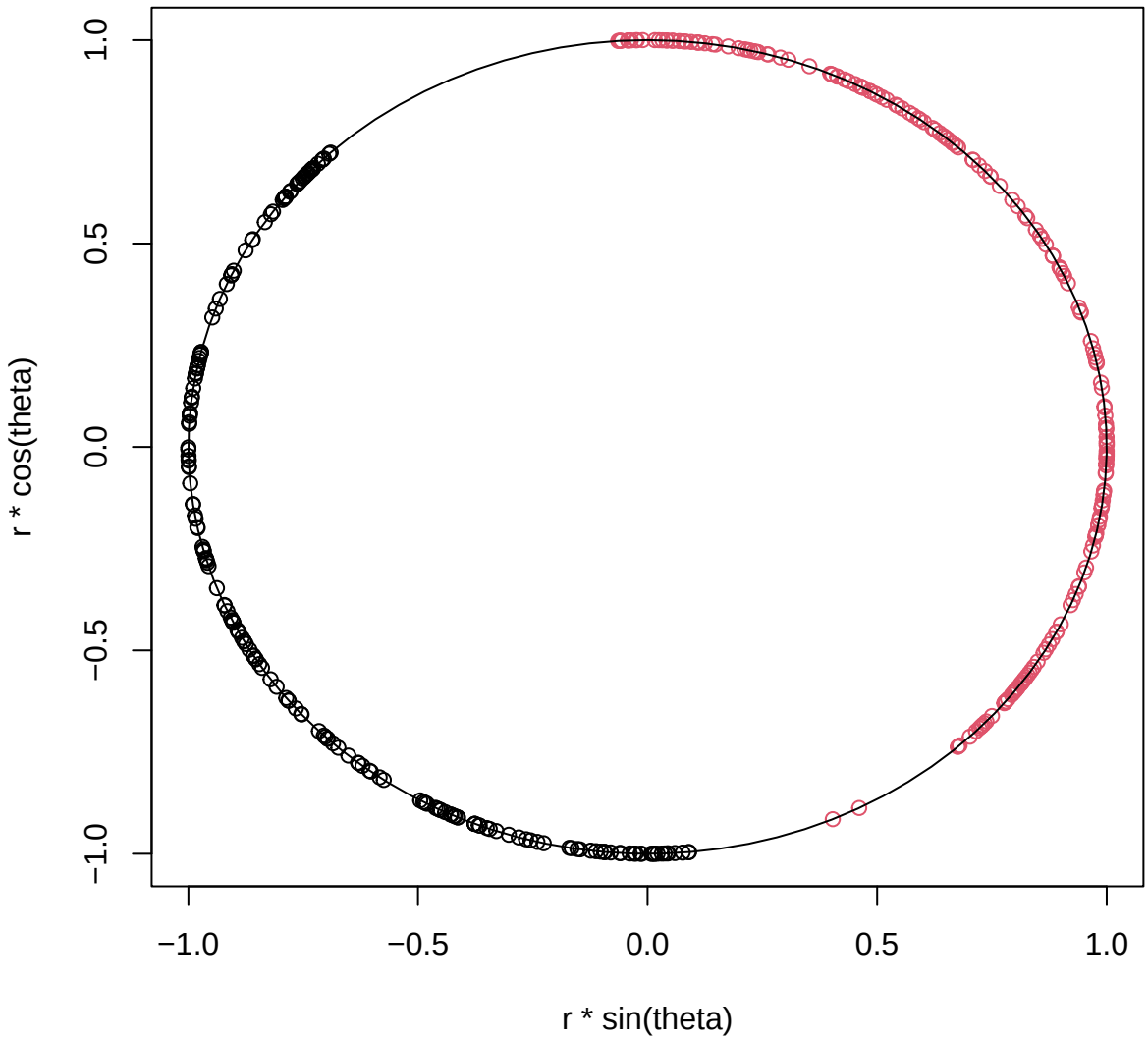
## [1] "Iter 101, obj 2.062265, abs 11.975699, rel 0.853094, norm 0.122160"
## [1] "Iter 102, obj 2.024416, abs 0.037849, rel 0.018353, norm 0.125378"
## [1] "Iter 103, obj 2.005789, abs 0.018627, rel 0.009201, norm 0.128718"
## [1] "Iter 104, obj 1.998704, abs 0.007085, rel 0.003532, norm 0.130512"
## [1] "Iter 105, obj 1.996439, abs 0.002265, rel 0.001133, norm 0.131117"
## [1] "Iter 106, obj 1.991945, abs 0.004494, rel 0.002251, norm 0.130864"
## [1] "Iter 107, obj 1.989230, abs 0.002715, rel 0.001363, norm 0.130802"
## [1] "Iter 108, obj 1.990009, abs 0.000780, rel 0.000392, norm 0.131535"
## [1] "Iter 109, obj 1.994050, abs 0.004040, rel 0.002030, norm 0.131367"
## [1] "Iter 110, obj 1.989877, abs 0.004172, rel 0.002092, norm 0.131455"
## [1] "Iter 111, obj 1.986696, abs 0.003182, rel 0.001599, norm 0.131219"
## [1] "Iter 112, obj 1.989170, abs 0.002474, rel 0.001245, norm 0.131232"
## [1] "Iter 113, obj 1.988067, abs 0.001103, rel 0.000555, norm 0.131275"
## [1] "Iter 114, obj 1.986159, abs 0.001908, rel 0.000960, norm 0.131325"
## [1] "Iter 115, obj 1.988784, abs 0.002625, rel 0.001322, norm 0.131298"
## [1] "Iter 116, obj 1.986386, abs 0.002398, rel 0.001206, norm 0.130860"
## [1] "Iter 117, obj 1.985458, abs 0.000928, rel 0.000467, norm 0.131134"
## [1] "Iter 118, obj 1.983766, abs 0.001692, rel 0.000852, norm 0.131126"
## [1] "Iter 119, obj 1.983908, abs 0.000142, rel 0.000072, norm 0.131118"
## [1] "Iter 120, obj 1.983380, abs 0.000528, rel 0.000266, norm 0.131136"
## [1] "Iter 121, obj 1.984158, abs 0.000778, rel 0.000392, norm 0.131079"
## [1] "Iter 122, obj 1.981425, abs 0.002733, rel 0.001377, norm 0.131097"
## [1] "Iter 123, obj 1.981211, abs 0.000215, rel 0.000108, norm 0.131225"
## [1] "Iter 124, obj 1.981347, abs 0.000137, rel 0.000069, norm 0.131173"
## [1] "Iter 125, obj 1.982696, abs 0.001349, rel 0.000681, norm 0.131173"
```




```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <- Y_rad

plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension $\mathbb{S}^2$ then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_array, d=2, rho_psc_list = rho_first_20, num_iteration=125,
             colors=colors, visualize_prog = TRUE, eta=100, check=F)
```

```
## [1] "Iter 1, obj 16.806936, abs 0.000000, rel 0.000000, norm 0.197196"
## [1] "Iter 2, obj 14.096472, abs 2.710464, rel 0.161271, norm 0.539132"
## [1] "Iter 3, obj 13.133573, abs 0.962899, rel 0.068308, norm 0.907912"
## [1] "Iter 4, obj 12.688972, abs 0.444602, rel 0.033852, norm 1.047142"
## [1] "Iter 5, obj 12.436486, abs 0.252486, rel 0.019898, norm 1.135266"
## [1] "Iter 6, obj 12.228627, abs 0.207859, rel 0.016714, norm 1.216777"
## [1] "Iter 7, obj 12.051381, abs 0.177246, rel 0.014494, norm 1.296698"
## [1] "Iter 8, obj 11.921517, abs 0.129864, rel 0.010776, norm 1.372584"
## [1] "Iter 9, obj 11.835904, abs 0.085613, rel 0.007181, norm 1.439638"
## [1] "Iter 10, obj 11.776389, abs 0.059514, rel 0.005028, norm 1.499936"
## [1] "Iter 11, obj 11.732253, abs 0.044137, rel 0.003748, norm 1.557267"
## [1] "Iter 12, obj 11.692793, abs 0.039460, rel 0.003363, norm 1.611359"
## [1] "Iter 13, obj 11.655479, abs 0.037313, rel 0.003191, norm 1.661078"
## [1] "Iter 14, obj 11.615365, abs 0.040114, rel 0.003442, norm 1.705829"
## [1] "Iter 15, obj 11.580213, abs 0.035152, rel 0.003026, norm 1.746369"
```

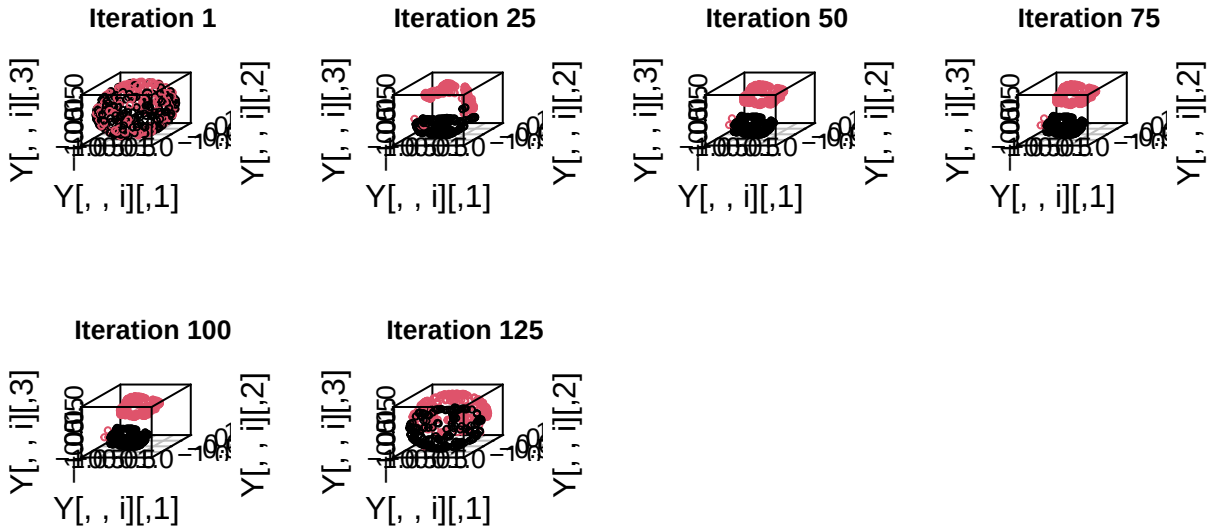
```
## [1] "Iter 16, obj 11.555491, abs 0.024721, rel 0.002135, norm 1.783009"
## [1] "Iter 17, obj 11.541189, abs 0.014302, rel 0.001238, norm 1.815999"
## [1] "Iter 18, obj 11.536500, abs 0.004690, rel 0.000406, norm 1.846425"
## [1] "Iter 19, obj 11.539600, abs 0.003101, rel 0.000269, norm 1.874954"
## [1] "Iter 20, obj 11.548656, abs 0.009056, rel 0.000785, norm 1.901935"
## [1] "Iter 21, obj 11.562244, abs 0.013588, rel 0.001177, norm 1.927497"
## [1] "Iter 22, obj 11.579304, abs 0.017060, rel 0.001475, norm 1.951665"
## [1] "Iter 23, obj 11.599012, abs 0.019708, rel 0.001702, norm 1.974447"
## [1] "Iter 24, obj 11.620679, abs 0.021667, rel 0.001868, norm 1.995871"
## [1] "Iter 25, obj 11.643647, abs 0.022968, rel 0.001976, norm 2.015990"

## [1] "Iter 26, obj 11.667231, abs 0.023584, rel 0.002025, norm 2.034868"
## [1] "Iter 27, obj 11.690699, abs 0.023467, rel 0.002011, norm 2.052572"
## [1] "Iter 28, obj 11.713305, abs 0.022606, rel 0.001934, norm 2.069175"
## [1] "Iter 29, obj 11.734375, abs 0.021071, rel 0.001799, norm 2.084758"
## [1] "Iter 30, obj 11.753414, abs 0.019039, rel 0.001623, norm 2.099396"
## [1] "Iter 31, obj 11.770182, abs 0.016767, rel 0.001427, norm 2.113148"
## [1] "Iter 32, obj 11.784702, abs 0.014521, rel 0.001234, norm 2.126032"
## [1] "Iter 33, obj 11.797209, abs 0.012506, rel 0.001061, norm 2.138037"
## [1] "Iter 34, obj 11.808070, abs 0.010861, rel 0.000921, norm 2.149144"
## [1] "Iter 35, obj 11.817735, abs 0.009665, rel 0.000819, norm 2.159343"
## [1] "Iter 36, obj 11.826667, abs 0.008932, rel 0.000756, norm 2.168636"
## [1] "Iter 37, obj 11.835273, abs 0.008606, rel 0.000728, norm 2.177028"
## [1] "Iter 38, obj 11.843844, abs 0.008570, rel 0.000724, norm 2.184533"
## [1] "Iter 39, obj 11.852525, abs 0.008681, rel 0.000733, norm 2.191180"
## [1] "Iter 40, obj 11.861336, abs 0.008811, rel 0.000743, norm 2.197020"
## [1] "Iter 41, obj 11.870210, abs 0.008874, rel 0.000748, norm 2.202122"
## [1] "Iter 42, obj 11.879039, abs 0.008829, rel 0.000744, norm 2.206566"
## [1] "Iter 43, obj 11.887711, abs 0.008672, rel 0.000730, norm 2.210439"
## [1] "Iter 44, obj 11.896130, abs 0.008419, rel 0.000708, norm 2.213822"
## [1] "Iter 45, obj 11.904221, abs 0.008091, rel 0.000680, norm 2.216787"
## [1] "Iter 46, obj 11.911932, abs 0.007711, rel 0.000648, norm 2.219399"
## [1] "Iter 47, obj 11.919232, abs 0.007300, rel 0.000613, norm 2.221709"
## [1] "Iter 48, obj 11.926106, abs 0.006874, rel 0.000577, norm 2.223761"
## [1] "Iter 49, obj 11.932551, abs 0.006445, rel 0.000540, norm 2.225593"
## [1] "Iter 50, obj 11.938573, abs 0.006022, rel 0.000505, norm 2.227233"

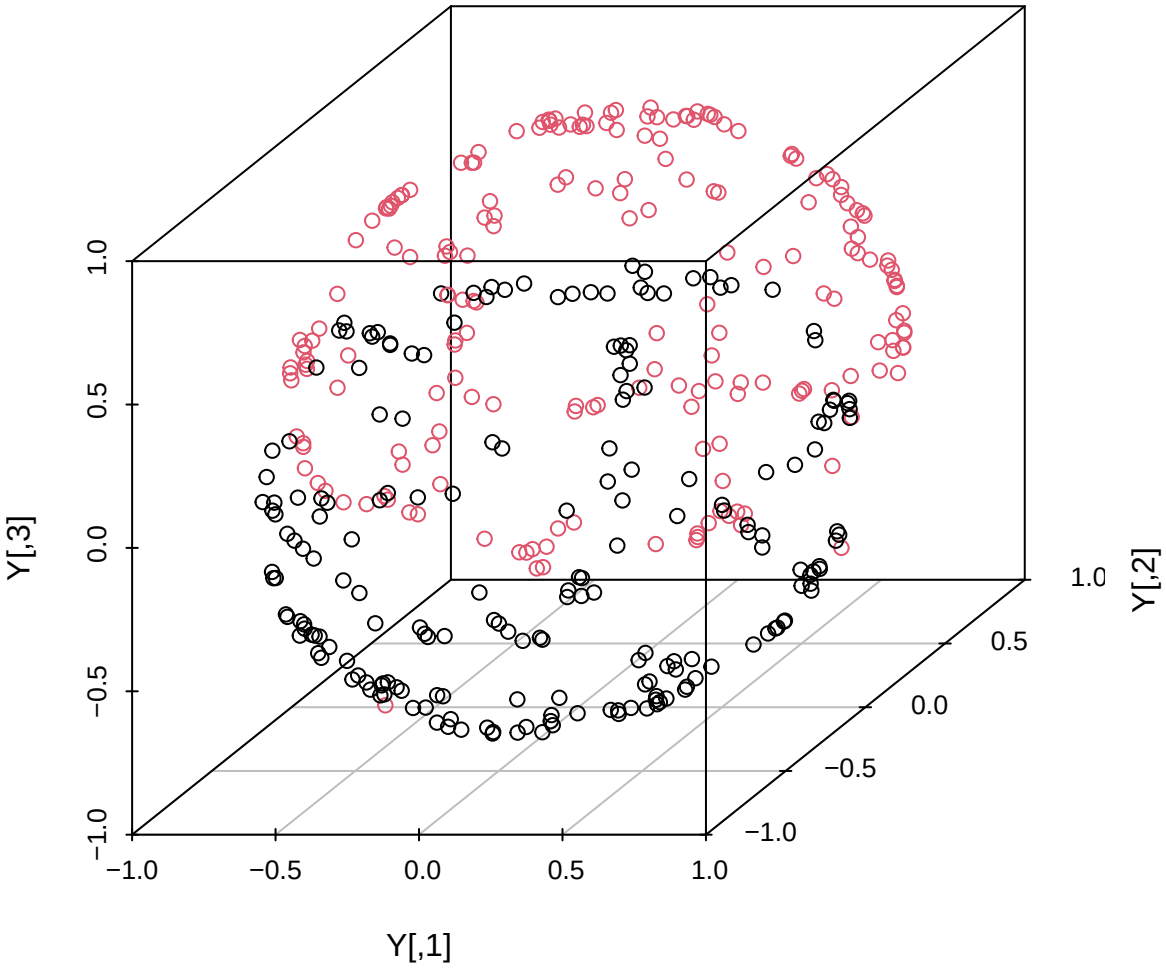
## [1] "Iter 51, obj 11.944185, abs 0.005612, rel 0.000470, norm 2.228707"
## [1] "Iter 52, obj 11.949402, abs 0.005218, rel 0.000437, norm 2.230035"
## [1] "Iter 53, obj 11.954245, abs 0.004843, rel 0.000405, norm 2.231235"
## [1] "Iter 54, obj 11.958734, abs 0.004489, rel 0.000376, norm 2.232322"
## [1] "Iter 55, obj 11.962891, abs 0.004156, rel 0.000348, norm 2.233308"
## [1] "Iter 56, obj 11.966736, abs 0.003845, rel 0.000321, norm 2.234204"
## [1] "Iter 57, obj 11.970290, abs 0.003555, rel 0.000297, norm 2.235019"
## [1] "Iter 58, obj 11.973575, abs 0.003285, rel 0.000274, norm 2.235762"
## [1] "Iter 59, obj 11.976609, abs 0.003034, rel 0.000253, norm 2.236440"
## [1] "Iter 60, obj 11.979410, abs 0.002801, rel 0.000234, norm 2.237058"
## [1] "Iter 61, obj 11.981995, abs 0.002585, rel 0.000216, norm 2.237623"
## [1] "Iter 62, obj 11.984381, abs 0.002386, rel 0.000199, norm 2.238139"
## [1] "Iter 63, obj 11.986583, abs 0.002202, rel 0.000184, norm 2.238610"
## [1] "Iter 64, obj 11.988614, abs 0.002031, rel 0.000169, norm 2.239042"
## [1] "Iter 65, obj 11.990488, abs 0.001874, rel 0.000156, norm 2.239436"
## [1] "Iter 66, obj 11.992217, abs 0.001729, rel 0.000144, norm 2.239797"
## [1] "Iter 67, obj 11.993811, abs 0.001594, rel 0.000133, norm 2.240127"
## [1] "Iter 68, obj 11.995282, abs 0.001471, rel 0.000123, norm 2.240429"
```

```
## [1] "Iter 69, obj 11.996638, abs 0.001356, rel 0.000113, norm 2.240704"
## [1] "Iter 70, obj 11.997889, abs 0.001251, rel 0.000104, norm 2.240956"
## [1] "Iter 71, obj 11.999043, abs 0.001153, rel 0.000096, norm 2.241187"
## [1] "Iter 72, obj 12.000106, abs 0.001064, rel 0.000089, norm 2.241397"
## [1] "Iter 73, obj 12.001087, abs 0.000981, rel 0.000082, norm 2.241588"
## [1] "Iter 74, obj 12.001991, abs 0.000904, rel 0.000075, norm 2.241763"
## [1] "Iter 75, obj 12.002825, abs 0.000834, rel 0.000069, norm 2.241922"
## [1] "Iter 76, obj 12.003594, abs 0.000769, rel 0.000064, norm 2.242066"
## [1] "Iter 77, obj 12.004302, abs 0.000708, rel 0.000059, norm 2.242198"
## [1] "Iter 78, obj 12.004955, abs 0.000653, rel 0.000054, norm 2.242317"
## [1] "Iter 79, obj 12.005557, abs 0.000602, rel 0.000050, norm 2.242425"
## [1] "Iter 80, obj 12.006111, abs 0.000554, rel 0.000046, norm 2.242522"
## [1] "Iter 81, obj 12.006622, abs 0.000511, rel 0.000043, norm 2.242610"
## [1] "Iter 82, obj 12.007093, abs 0.000471, rel 0.000039, norm 2.242689"
## [1] "Iter 83, obj 12.007526, abs 0.000433, rel 0.000036, norm 2.242759"
## [1] "Iter 84, obj 12.007925, abs 0.000399, rel 0.000033, norm 2.242822"
## [1] "Iter 85, obj 12.008293, abs 0.000367, rel 0.000031, norm 2.242878"
## [1] "Iter 86, obj 12.008631, abs 0.000338, rel 0.000028, norm 2.242928"
## [1] "Iter 87, obj 12.008942, abs 0.000311, rel 0.000026, norm 2.242972"
## [1] "Iter 88, obj 12.009229, abs 0.000286, rel 0.000024, norm 2.243010"
## [1] "Iter 89, obj 12.009492, abs 0.000263, rel 0.000022, norm 2.243043"
## [1] "Iter 90, obj 12.009734, abs 0.000242, rel 0.000020, norm 2.243071"
## [1] "Iter 91, obj 12.009957, abs 0.000223, rel 0.000019, norm 2.243095"
## [1] "Iter 92, obj 12.010161, abs 0.000204, rel 0.000017, norm 2.243115"
## [1] "Iter 93, obj 12.010349, abs 0.000188, rel 0.000016, norm 2.243131"
## [1] "Iter 94, obj 12.010521, abs 0.000172, rel 0.000014, norm 2.243143"
## [1] "Iter 95, obj 12.010679, abs 0.000158, rel 0.000013, norm 2.243153"
## [1] "Iter 96, obj 12.010824, abs 0.000145, rel 0.000012, norm 2.243159"
## [1] "Iter 97, obj 12.010957, abs 0.000133, rel 0.000011, norm 2.243162"
## [1] "Iter 98, obj 12.011078, abs 0.000122, rel 0.000010, norm 2.243163"
## [1] "Iter 99, obj 12.011189, abs 0.000111, rel 0.000009, norm 2.243162"
## [1] "Iter 100, obj 12.011291, abs 0.000102, rel 0.000008, norm 2.243158"
## [1] "Iter 101, obj 1.414561, abs 10.596730, rel 0.882231, norm 0.192696"
## [1] "Iter 102, obj 1.264461, abs 0.150099, rel 0.106110, norm 0.174525"
## [1] "Iter 103, obj 1.205263, abs 0.059198, rel 0.046817, norm 0.183489"
## [1] "Iter 104, obj 1.196680, abs 0.008583, rel 0.007121, norm 0.190212"
## [1] "Iter 105, obj 1.202157, abs 0.005476, rel 0.004576, norm 0.192105"
## [1] "Iter 106, obj 1.206507, abs 0.004350, rel 0.003619, norm 0.193511"
## [1] "Iter 107, obj 1.195127, abs 0.011379, rel 0.009432, norm 0.192483"
## [1] "Iter 108, obj 1.194677, abs 0.000450, rel 0.000377, norm 0.191969"
## [1] "Iter 109, obj 1.195720, abs 0.001043, rel 0.000873, norm 0.192090"
## [1] "Iter 110, obj 1.199895, abs 0.004175, rel 0.003491, norm 0.192222"
## [1] "Iter 111, obj 1.208298, abs 0.008404, rel 0.007004, norm 0.192681"
## [1] "Iter 112, obj 1.206023, abs 0.002275, rel 0.001883, norm 0.192368"
## [1] "Iter 113, obj 1.199126, abs 0.006897, rel 0.005719, norm 0.192379"
## [1] "Iter 114, obj 1.201696, abs 0.002569, rel 0.002143, norm 0.191767"
## [1] "Iter 115, obj 1.197788, abs 0.003908, rel 0.003252, norm 0.192759"
## [1] "Iter 116, obj 1.196839, abs 0.000949, rel 0.000793, norm 0.192051"
## [1] "Iter 117, obj 1.210506, abs 0.013667, rel 0.011419, norm 0.191247"
## [1] "Iter 118, obj 1.214780, abs 0.004275, rel 0.003531, norm 0.191576"
## [1] "Iter 119, obj 1.197625, abs 0.017155, rel 0.014122, norm 0.192447"
## [1] "Iter 120, obj 1.195942, abs 0.001683, rel 0.001406, norm 0.192419"
## [1] "Iter 121, obj 1.207201, abs 0.011259, rel 0.009414, norm 0.191975"
```

```
## [1] "Iter 122, obj 1.210010, abs 0.002809, rel 0.002327, norm 0.192202"
## [1] "Iter 123, obj 1.200080, abs 0.009930, rel 0.008207, norm 0.192219"
## [1] "Iter 124, obj 1.202419, abs 0.002340, rel 0.001950, norm 0.192133"
## [1] "Iter 125, obj 1.197175, abs 0.005244, rel 0.004362, norm 0.192465"
```




```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1),
                             zlim = c(-1, 1), color = colors)
```



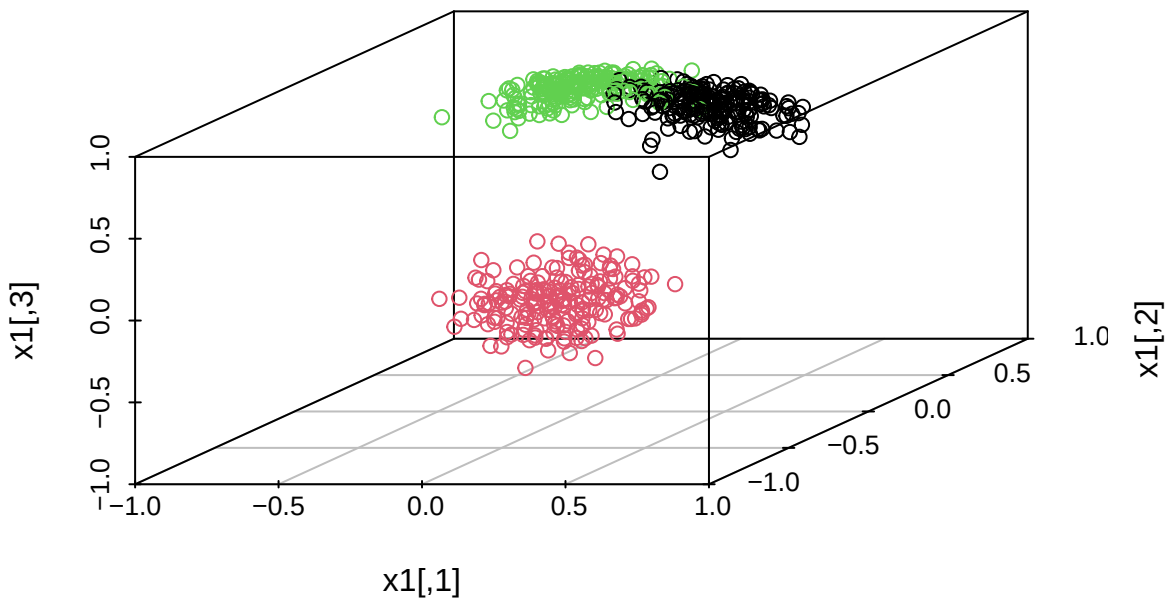
It's clearly clustered the two groups we had in the original generated data.

Case 2

Sample on the $(\mathbb{S}^2)^2$ where $p = 2$ and $r = 2$.

```
n=200
d=2
r=2
samp1 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 0, ph = 0.5)),
                        kappa = 50)
samp2 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 2, ph = -1.5)),
                        kappa = 50)
samp3 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = -1, ph = 0)),
                        kappa = 50)
x1 <- rbind(samp1, samp2, samp3)

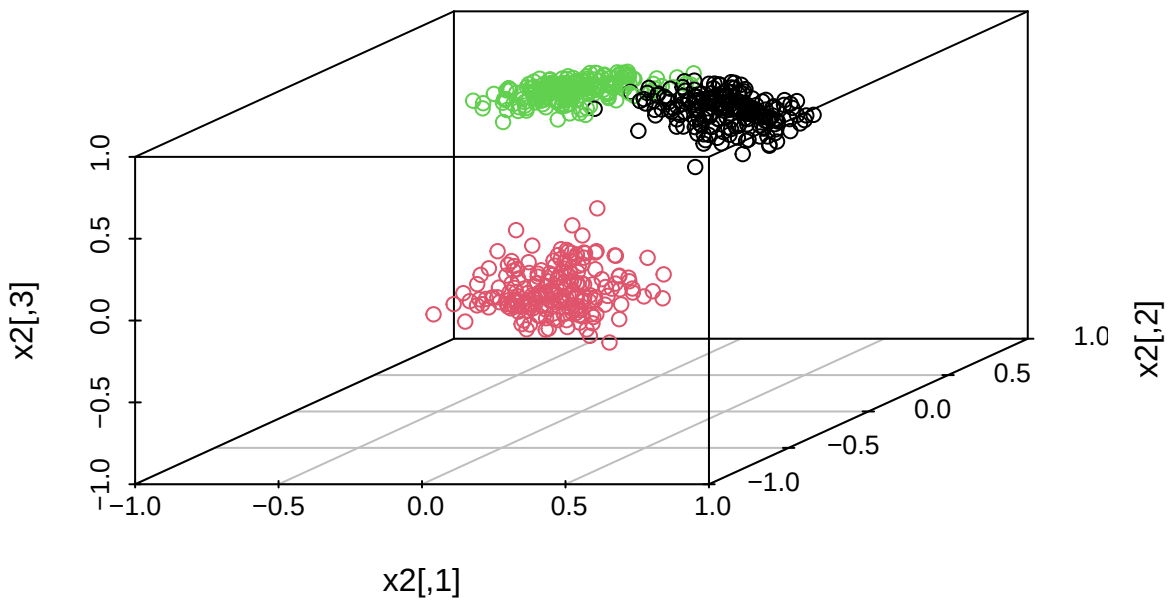
scatterplot3d::scatterplot3d(x1,
                             xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(1:3, each = n), main="Sphere 1")
```




```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue", alpha = 0.25,
            lit = FALSE)
rgl::points3d(x1, col = rep(c(1,2,3), each=n))

samp4 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 0, ph = 0.55)),
                        kappa = 60)
samp5 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = 2, ph = -1.41)),
                        kappa = 60)
samp6 <- rotasym::r_vMF(n = n, mu = drop(DirStats::to_sph(th = -1.1, ph = 0.05)),
                        kappa = 60)
x2 <- rbind(samp4, samp5, samp6)

scatterplot3d::scatterplot3d(x2,
                             xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                             color = rep(1:3, each = n), main="Sphere 2")
```




```
rgl::plot3d(0, 0, 0, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
            radius = 1, type = "s", col = "lightblue", alpha = 0.25,
            lit = FALSE)
rgl::points3d(x2, col = rep(c(1,2,3), each=n))


x_2 <- array(dim = c(n*3, 3, 2))
x_2[, , 1] <- x1
x_2[, , 2] <- x2

n <- nrow(x_2)
indexes <- sample(1:n)
x_2 <- x_2[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]
```

Let's calculate the rho parameters based on a perplexity of 20:

```
rho_second_perp20 <- rho_optim_bst(x_2, 20)
```

```
## Time difference of 1.119833 mins
```

First, let's reduce to dimension $\mathbb{S}^1$ then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_2, d=1, rho_psc_list = rho_second_perp20, num_iteration=200,
            colors=colors, visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 18.512789, abs 0.000000, rel 0.000000, norm 0.142405"
## [1] "Iter 2, obj 17.794721, abs 0.718068, rel 0.038788, norm 0.287251"
## [1] "Iter 3, obj 17.345704, abs 0.449017, rel 0.025233, norm 0.384736"
## [1] "Iter 4, obj 17.046358, abs 0.299346, rel 0.017258, norm 0.445220"
## [1] "Iter 5, obj 16.811012, abs 0.235346, rel 0.013806, norm 0.492131"
## [1] "Iter 6, obj 16.610812, abs 0.200199, rel 0.011909, norm 0.535521"
## [1] "Iter 7, obj 16.460026, abs 0.150787, rel 0.009078, norm 0.579321"
```

```
## [1] "Iter 8, obj 16.343929, abs 0.116097, rel 0.007053, norm 0.618233"
## [1] "Iter 9, obj 16.238742, abs 0.105186, rel 0.006436, norm 0.648687"
## [1] "Iter 10, obj 16.133176, abs 0.105566, rel 0.006501, norm 0.675448"
## [1] "Iter 11, obj 16.025631, abs 0.107545, rel 0.006666, norm 0.700719"
## [1] "Iter 12, obj 15.920380, abs 0.105252, rel 0.006568, norm 0.724751"
## [1] "Iter 13, obj 15.821490, abs 0.098890, rel 0.006212, norm 0.747021"
## [1] "Iter 14, obj 15.738722, abs 0.082768, rel 0.005231, norm 0.767610"
## [1] "Iter 15, obj 15.670520, abs 0.068202, rel 0.004333, norm 0.787399"
## [1] "Iter 16, obj 15.619181, abs 0.051340, rel 0.003276, norm 0.805339"
## [1] "Iter 17, obj 15.585099, abs 0.034082, rel 0.002182, norm 0.821641"
## [1] "Iter 18, obj 15.563071, abs 0.022028, rel 0.001413, norm 0.834917"
## [1] "Iter 19, obj 15.547589, abs 0.015482, rel 0.000995, norm 0.845095"
## [1] "Iter 20, obj 15.535779, abs 0.011810, rel 0.000760, norm 0.853213"
## [1] "Iter 21, obj 15.527026, abs 0.008753, rel 0.000563, norm 0.859511"
## [1] "Iter 22, obj 15.520361, abs 0.006665, rel 0.000429, norm 0.863900"
## [1] "Iter 23, obj 15.515842, abs 0.004519, rel 0.000291, norm 0.866957"
## [1] "Iter 24, obj 15.513867, abs 0.001975, rel 0.000127, norm 0.869279"
## [1] "Iter 25, obj 15.514305, abs 0.000438, rel 0.000028, norm 0.871218"
## [1] "Iter 26, obj 15.516397, abs 0.002092, rel 0.000135, norm 0.872975"
## [1] "Iter 27, obj 15.518576, abs 0.002179, rel 0.000140, norm 0.874643"
## [1] "Iter 28, obj 15.518626, abs 0.000051, rel 0.000003, norm 0.876311"
## [1] "Iter 29, obj 15.514966, abs 0.003660, rel 0.000236, norm 0.878167"
## [1] "Iter 30, obj 15.508476, abs 0.006490, rel 0.000418, norm 0.880499"
## [1] "Iter 31, obj 15.501409, abs 0.007067, rel 0.000456, norm 0.883442"
## [1] "Iter 32, obj 15.497040, abs 0.004369, rel 0.000282, norm 0.886836"
## [1] "Iter 33, obj 15.497832, abs 0.000791, rel 0.000051, norm 0.890165"
## [1] "Iter 34, obj 15.502161, abs 0.004330, rel 0.000279, norm 0.892616"
## [1] "Iter 35, obj 15.507724, abs 0.005563, rel 0.000359, norm 0.894075"
## [1] "Iter 36, obj 15.513468, abs 0.005744, rel 0.000370, norm 0.894934"
## [1] "Iter 37, obj 15.519094, abs 0.005626, rel 0.000363, norm 0.895524"
## [1] "Iter 38, obj 15.524568, abs 0.005474, rel 0.000353, norm 0.896025"
## [1] "Iter 39, obj 15.529932, abs 0.005364, rel 0.000345, norm 0.896522"
## [1] "Iter 40, obj 15.535235, abs 0.005303, rel 0.000341, norm 0.897054"
## [1] "Iter 41, obj 15.540512, abs 0.005278, rel 0.000340, norm 0.897638"
## [1] "Iter 42, obj 15.545782, abs 0.005270, rel 0.000339, norm 0.898282"
## [1] "Iter 43, obj 15.551055, abs 0.005272, rel 0.000339, norm 0.898987"
## [1] "Iter 44, obj 15.556345, abs 0.005290, rel 0.000340, norm 0.899751"
## [1] "Iter 45, obj 15.561680, abs 0.005335, rel 0.000343, norm 0.900569"
## [1] "Iter 46, obj 15.567099, abs 0.005419, rel 0.000348, norm 0.901439"
## [1] "Iter 47, obj 15.572645, abs 0.005546, rel 0.000356, norm 0.902357"
## [1] "Iter 48, obj 15.578344, abs 0.005699, rel 0.000366, norm 0.903316"
## [1] "Iter 49, obj 15.584021, abs 0.005676, rel 0.000364, norm 0.904310"
## [1] "Iter 50, obj 15.587724, abs 0.003704, rel 0.000238, norm 0.905334"
## [1] "Iter 51, obj 15.584539, abs 0.003185, rel 0.000204, norm 0.906505"
## [1] "Iter 52, obj 15.582040, abs 0.002499, rel 0.000160, norm 0.908333"
## [1] "Iter 53, obj 15.578538, abs 0.003502, rel 0.000225, norm 0.910124"
## [1] "Iter 54, obj 15.580941, abs 0.002403, rel 0.000154, norm 0.912075"
## [1] "Iter 55, obj 15.585147, abs 0.004206, rel 0.000270, norm 0.913377"
## [1] "Iter 56, obj 15.589699, abs 0.004552, rel 0.000292, norm 0.914340"
## [1] "Iter 57, obj 15.594308, abs 0.004609, rel 0.000296, norm 0.915209"
## [1] "Iter 58, obj 15.598922, abs 0.004614, rel 0.000296, norm 0.916065"
## [1] "Iter 59, obj 15.603551, abs 0.004628, rel 0.000297, norm 0.916933"
## [1] "Iter 60, obj 15.608215, abs 0.004665, rel 0.000299, norm 0.917819"
```

```
## [1] "Iter 61, obj 15.612938, abs 0.004723, rel 0.000303, norm 0.918728"
## [1] "Iter 62, obj 15.617740, abs 0.004802, rel 0.000308, norm 0.919658"
## [1] "Iter 63, obj 15.622644, abs 0.004903, rel 0.000314, norm 0.920611"
## [1] "Iter 64, obj 15.627669, abs 0.005026, rel 0.000322, norm 0.921587"
## [1] "Iter 65, obj 15.632841, abs 0.005172, rel 0.000331, norm 0.922587"
## [1] "Iter 66, obj 15.638185, abs 0.005344, rel 0.000342, norm 0.923609"
## [1] "Iter 67, obj 15.643730, abs 0.005545, rel 0.000355, norm 0.924654"
## [1] "Iter 68, obj 15.649507, abs 0.005777, rel 0.000369, norm 0.925723"
## [1] "Iter 69, obj 15.655550, abs 0.006042, rel 0.000386, norm 0.926814"
## [1] "Iter 70, obj 15.661882, abs 0.006333, rel 0.000404, norm 0.927928"
## [1] "Iter 71, obj 15.668487, abs 0.006605, rel 0.000422, norm 0.929062"
## [1] "Iter 72, obj 15.675169, abs 0.006682, rel 0.000426, norm 0.930220"
## [1] "Iter 73, obj 15.681040, abs 0.005871, rel 0.000375, norm 0.931410"
## [1] "Iter 74, obj 15.684876, abs 0.003836, rel 0.000245, norm 0.932692"
## [1] "Iter 75, obj 15.689814, abs 0.004938, rel 0.000315, norm 0.934186"

## [1] "Iter 76, obj 15.693158, abs 0.003344, rel 0.000213, norm 0.935726"
## [1] "Iter 77, obj 15.693921, abs 0.000763, rel 0.000049, norm 0.937277"
## [1] "Iter 78, obj 15.694420, abs 0.000499, rel 0.000032, norm 0.939016"
## [1] "Iter 79, obj 15.691867, abs 0.002553, rel 0.000163, norm 0.940972"
## [1] "Iter 80, obj 15.684583, abs 0.007284, rel 0.000464, norm 0.943438"
## [1] "Iter 81, obj 15.679912, abs 0.004671, rel 0.000298, norm 0.946513"
## [1] "Iter 82, obj 15.679399, abs 0.000513, rel 0.000033, norm 0.949067"
## [1] "Iter 83, obj 15.680949, abs 0.001550, rel 0.000099, norm 0.950659"
## [1] "Iter 84, obj 15.683190, abs 0.002241, rel 0.000143, norm 0.951626"
## [1] "Iter 85, obj 15.686182, abs 0.002992, rel 0.000191, norm 0.952274"
## [1] "Iter 86, obj 15.690033, abs 0.003850, rel 0.000245, norm 0.952655"
## [1] "Iter 87, obj 15.694701, abs 0.004668, rel 0.000298, norm 0.952788"
## [1] "Iter 88, obj 15.699719, abs 0.005018, rel 0.000320, norm 0.952757"
## [1] "Iter 89, obj 15.704719, abs 0.005000, rel 0.000318, norm 0.952678"
## [1] "Iter 90, obj 15.709446, abs 0.004726, rel 0.000301, norm 0.952636"
## [1] "Iter 91, obj 15.713873, abs 0.004428, rel 0.000282, norm 0.952669"
## [1] "Iter 92, obj 15.718751, abs 0.004878, rel 0.000310, norm 0.952771"
## [1] "Iter 93, obj 15.723826, abs 0.005076, rel 0.000323, norm 0.952851"
## [1] "Iter 94, obj 15.729146, abs 0.005319, rel 0.000338, norm 0.952936"
## [1] "Iter 95, obj 15.733064, abs 0.003918, rel 0.000249, norm 0.953012"
## [1] "Iter 96, obj 15.732755, abs 0.000308, rel 0.000020, norm 0.953206"
## [1] "Iter 97, obj 15.725554, abs 0.007202, rel 0.000458, norm 0.953777"
## [1] "Iter 98, obj 15.707696, abs 0.017857, rel 0.001136, norm 0.955014"
## [1] "Iter 99, obj 15.677571, abs 0.030126, rel 0.001918, norm 0.957201"
## [1] "Iter 100, obj 15.642715, abs 0.034855, rel 0.002223, norm 0.960147"

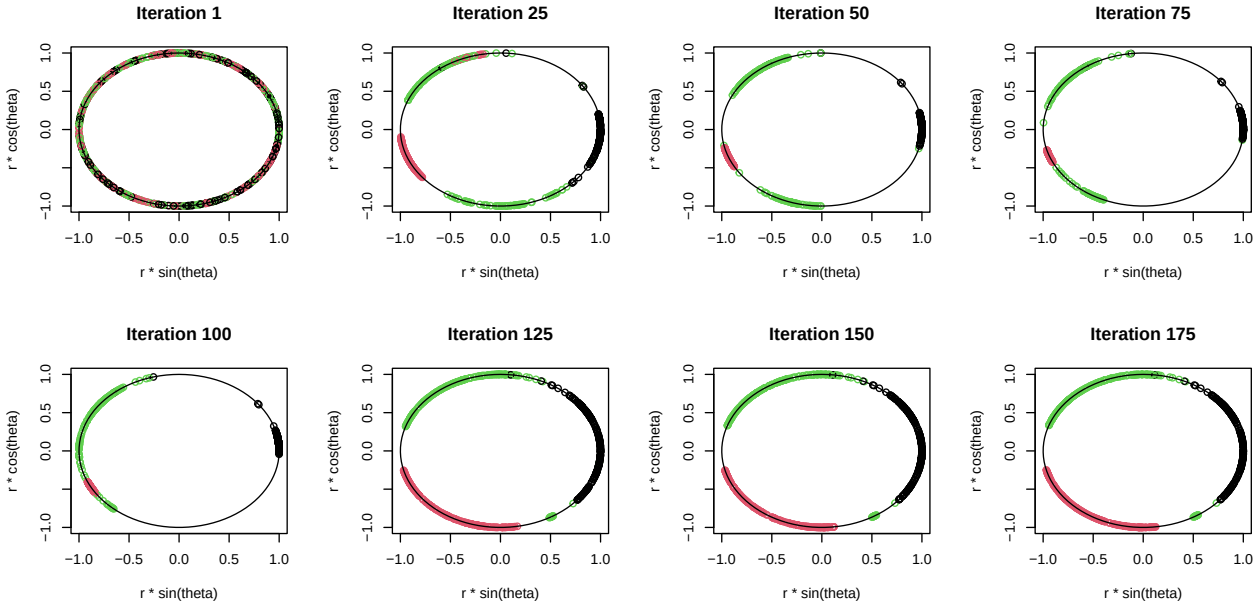
## [1] "Iter 101, obj 2.473622, abs 13.169093, rel 0.841867, norm 0.100550"
## [1] "Iter 102, obj 2.429065, abs 0.044557, rel 0.018013, norm 0.098681"
## [1] "Iter 103, obj 2.394706, abs 0.034359, rel 0.014145, norm 0.101626"
## [1] "Iter 104, obj 2.374289, abs 0.020417, rel 0.008526, norm 0.106420"
## [1] "Iter 105, obj 2.363901, abs 0.010388, rel 0.004375, norm 0.109228"
## [1] "Iter 106, obj 2.357542, abs 0.006359, rel 0.002690, norm 0.110299"
## [1] "Iter 107, obj 2.354171, abs 0.003371, rel 0.001430, norm 0.110450"
## [1] "Iter 108, obj 2.351884, abs 0.002287, rel 0.000971, norm 0.110496"
## [1] "Iter 109, obj 2.350664, abs 0.001220, rel 0.000519, norm 0.110569"
## [1] "Iter 110, obj 2.349732, abs 0.000932, rel 0.000396, norm 0.110522"
## [1] "Iter 111, obj 2.349118, abs 0.000615, rel 0.000262, norm 0.110431"
## [1] "Iter 112, obj 2.348626, abs 0.000492, rel 0.000209, norm 0.110345"
## [1] "Iter 113, obj 2.348227, abs 0.000399, rel 0.000170, norm 0.110295"
```

```
## [1] "Iter 114, obj 2.347886, abs 0.000342, rel 0.000146, norm 0.110269"
## [1] "Iter 115, obj 2.347587, abs 0.000299, rel 0.000127, norm 0.110256"
## [1] "Iter 116, obj 2.347316, abs 0.000271, rel 0.000115, norm 0.110252"
## [1] "Iter 117, obj 2.347066, abs 0.000249, rel 0.000106, norm 0.110255"
## [1] "Iter 118, obj 2.346834, abs 0.000232, rel 0.000099, norm 0.110264"
## [1] "Iter 119, obj 2.346615, abs 0.000220, rel 0.000094, norm 0.110275"
## [1] "Iter 120, obj 2.346407, abs 0.000208, rel 0.000088, norm 0.110287"
## [1] "Iter 121, obj 2.346212, abs 0.000195, rel 0.000083, norm 0.110298"
## [1] "Iter 122, obj 2.346030, abs 0.000182, rel 0.000077, norm 0.110308"
## [1] "Iter 123, obj 2.345862, abs 0.000167, rel 0.000071, norm 0.110317"
## [1] "Iter 124, obj 2.345709, abs 0.000153, rel 0.000065, norm 0.110325"
## [1] "Iter 125, obj 2.345568, abs 0.000141, rel 0.000060, norm 0.110332"

## [1] "Iter 126, obj 2.345437, abs 0.000131, rel 0.000056, norm 0.110338"
## [1] "Iter 127, obj 2.345313, abs 0.000124, rel 0.000053, norm 0.110342"
## [1] "Iter 128, obj 2.345193, abs 0.000120, rel 0.000051, norm 0.110345"
## [1] "Iter 129, obj 2.345076, abs 0.000118, rel 0.000050, norm 0.110347"
## [1] "Iter 130, obj 2.344959, abs 0.000117, rel 0.000050, norm 0.110347"
## [1] "Iter 131, obj 2.344842, abs 0.000117, rel 0.000050, norm 0.110346"
## [1] "Iter 132, obj 2.344724, abs 0.000117, rel 0.000050, norm 0.110343"
## [1] "Iter 133, obj 2.344609, abs 0.000116, rel 0.000049, norm 0.110339"
## [1] "Iter 134, obj 2.344497, abs 0.000112, rel 0.000048, norm 0.110333"
## [1] "Iter 135, obj 2.344390, abs 0.000106, rel 0.000045, norm 0.110327"
## [1] "Iter 136, obj 2.344291, abs 0.000099, rel 0.000042, norm 0.110320"
## [1] "Iter 137, obj 2.344200, abs 0.000091, rel 0.000039, norm 0.110314"
## [1] "Iter 138, obj 2.344118, abs 0.000082, rel 0.000035, norm 0.110308"
## [1] "Iter 139, obj 2.344045, abs 0.000074, rel 0.000031, norm 0.110302"
## [1] "Iter 140, obj 2.343978, abs 0.000066, rel 0.000028, norm 0.110298"
## [1] "Iter 141, obj 2.343919, abs 0.000059, rel 0.000025, norm 0.110293"
## [1] "Iter 142, obj 2.343865, abs 0.000054, rel 0.000023, norm 0.110289"
## [1] "Iter 143, obj 2.343817, abs 0.000048, rel 0.000021, norm 0.110286"
## [1] "Iter 144, obj 2.343773, abs 0.000044, rel 0.000019, norm 0.110283"
## [1] "Iter 145, obj 2.343733, abs 0.000040, rel 0.000017, norm 0.110281"
## [1] "Iter 146, obj 2.343696, abs 0.000037, rel 0.000016, norm 0.110278"
## [1] "Iter 147, obj 2.343662, abs 0.000034, rel 0.000015, norm 0.110276"
## [1] "Iter 148, obj 2.343630, abs 0.000031, rel 0.000013, norm 0.110274"
## [1] "Iter 149, obj 2.343601, abs 0.000029, rel 0.000012, norm 0.110272"
## [1] "Iter 150, obj 2.343574, abs 0.000027, rel 0.000012, norm 0.110271"

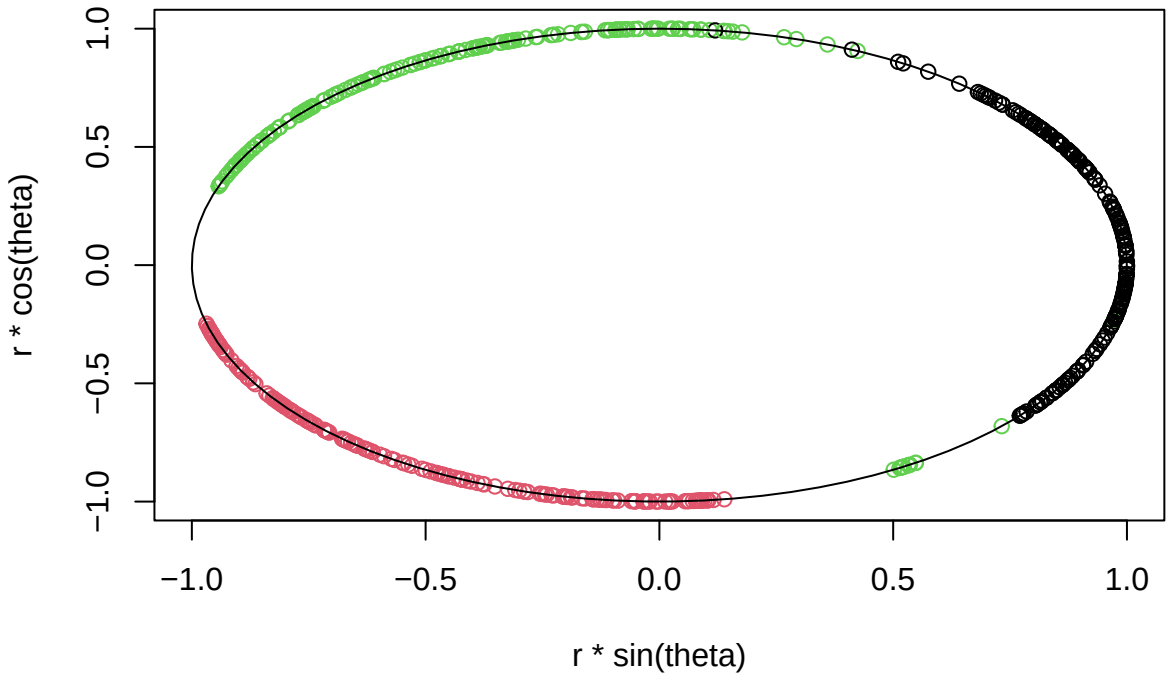
## [1] "Iter 151, obj 2.343549, abs 0.000025, rel 0.000011, norm 0.110269"
## [1] "Iter 152, obj 2.343526, abs 0.000023, rel 0.000010, norm 0.110267"
## [1] "Iter 153, obj 2.343504, abs 0.000022, rel 0.000009, norm 0.110265"
## [1] "Iter 154, obj 2.343483, abs 0.000021, rel 0.000009, norm 0.110262"
## [1] "Iter 155, obj 2.343464, abs 0.000019, rel 0.000008, norm 0.110260"
## [1] "Iter 156, obj 2.343446, abs 0.000018, rel 0.000008, norm 0.110258"
## [1] "Iter 157, obj 2.343429, abs 0.000017, rel 0.000007, norm 0.110255"
## [1] "Iter 158, obj 2.343412, abs 0.000016, rel 0.000007, norm 0.110252"
## [1] "Iter 159, obj 2.343397, abs 0.000015, rel 0.000007, norm 0.110250"
## [1] "Iter 160, obj 2.343382, abs 0.000015, rel 0.000006, norm 0.110247"
## [1] "Iter 161, obj 2.343368, abs 0.000014, rel 0.000006, norm 0.110244"
## [1] "Iter 162, obj 2.343355, abs 0.000013, rel 0.000006, norm 0.110241"
## [1] "Iter 163, obj 2.343342, abs 0.000013, rel 0.000005, norm 0.110238"
## [1] "Iter 164, obj 2.343330, abs 0.000012, rel 0.000005, norm 0.110235"
## [1] "Iter 165, obj 2.343319, abs 0.000011, rel 0.000005, norm 0.110231"
## [1] "Iter 166, obj 2.343308, abs 0.000011, rel 0.000005, norm 0.110228"
```

```
## [1] "Iter 167, obj 2.343298, abs 0.000010, rel 0.000004, norm 0.110224"
## [1] "Iter 168, obj 2.343288, abs 0.000010, rel 0.000004, norm 0.110221"
## [1] "Iter 169, obj 2.343279, abs 0.000009, rel 0.000004, norm 0.110217"
## [1] "Iter 170, obj 2.343270, abs 0.000009, rel 0.000004, norm 0.110214"
## [1] "Iter 171, obj 2.343262, abs 0.000008, rel 0.000003, norm 0.110210"
## [1] "Iter 172, obj 2.343255, abs 0.000007, rel 0.000003, norm 0.110206"
## [1] "Iter 173, obj 2.343248, abs 0.000007, rel 0.000003, norm 0.110203"
## [1] "Iter 174, obj 2.343242, abs 0.000006, rel 0.000003, norm 0.110199"
## [1] "Iter 175, obj 2.343236, abs 0.000006, rel 0.000002, norm 0.110195"
```




```
## [1] "Iter 176, obj 2.343231, abs 0.000005, rel 0.000002, norm 0.110191"
## [1] "Iter 177, obj 2.343227, abs 0.000004, rel 0.000002, norm 0.110188"
## [1] "Iter 178, obj 2.343223, abs 0.000004, rel 0.000002, norm 0.110184"
## [1] "Iter 179, obj 2.343219, abs 0.000004, rel 0.000002, norm 0.110180"
## [1] "Iter 180, obj 2.343216, abs 0.000003, rel 0.000001, norm 0.110177"
## [1] "Iter 181, obj 2.343214, abs 0.000003, rel 0.000001, norm 0.110173"
## [1] "Iter 182, obj 2.343211, abs 0.000002, rel 0.000001, norm 0.110170"
## [1] "Iter 183, obj 2.343209, abs 0.000002, rel 0.000001, norm 0.110167"
```

```r
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <<- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension $\mathbb{S}^2$ then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_2, d=2, rho_psc_list = rho_second_perp20, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

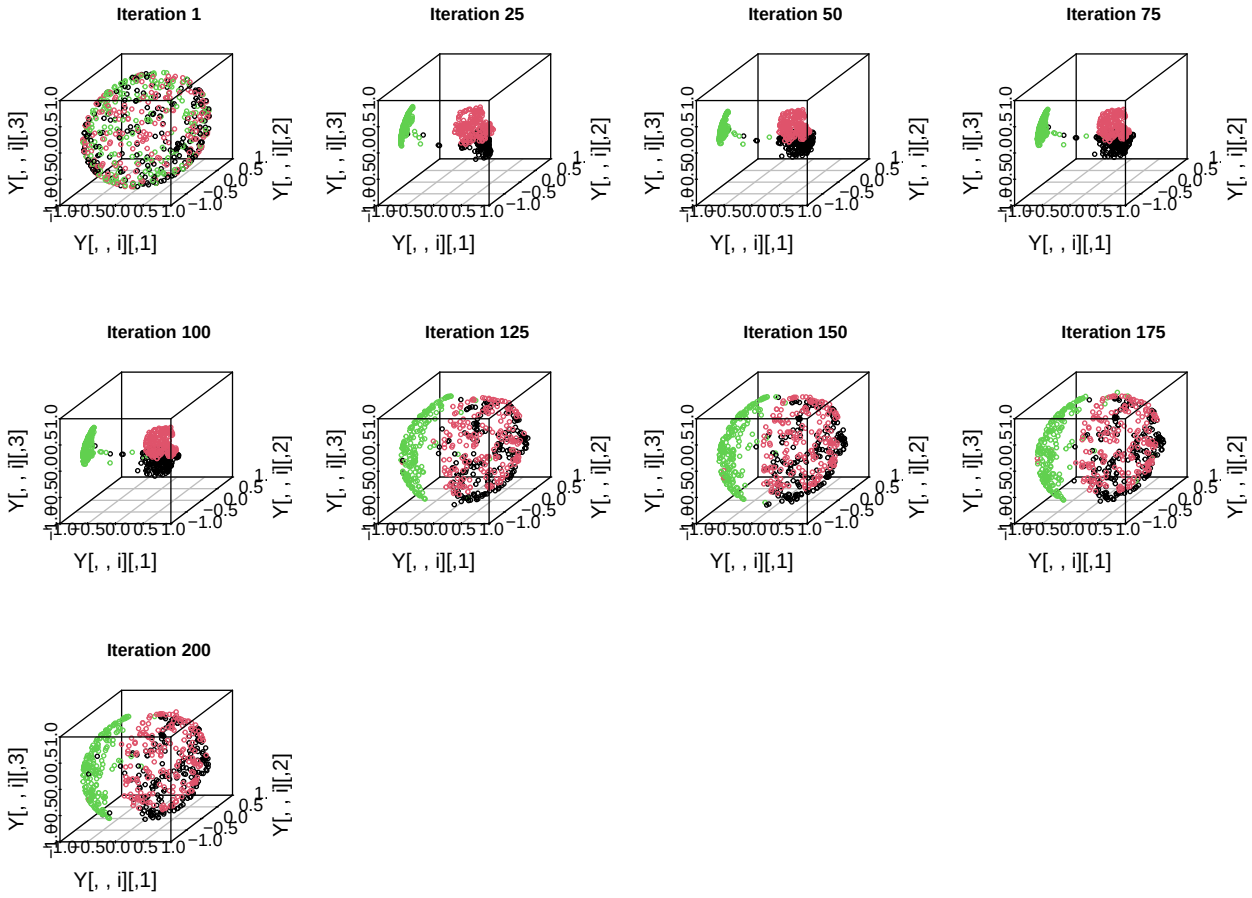
```
## [1] "Iter 1, obj 19.371617, abs 0.000000, rel 0.000000, norm 0.177009"

## [1] "Iter 2, obj 15.735606, abs 3.636011, rel 0.187698, norm 0.371972"
## [1] "Iter 3, obj 14.380501, abs 1.355104, rel 0.086117, norm 0.716452"
## [1] "Iter 4, obj 13.773186, abs 0.607315, rel 0.042232, norm 0.923003"
## [1] "Iter 5, obj 13.443191, abs 0.329995, rel 0.023959, norm 1.054549"
## [1] "Iter 6, obj 13.233084, abs 0.210106, rel 0.015629, norm 1.150439"
## [1] "Iter 7, obj 13.093265, abs 0.139819, rel 0.010566, norm 1.227003"
## [1] "Iter 8, obj 13.012637, abs 0.080628, rel 0.006158, norm 1.290091"
## [1] "Iter 9, obj 12.968887, abs 0.043750, rel 0.003362, norm 1.341504"
## [1] "Iter 10, obj 12.966789, abs 0.002098, rel 0.000162, norm 1.383718"
## [1] "Iter 11, obj 12.986374, abs 0.019585, rel 0.001510, norm 1.419228"
## [1] "Iter 12, obj 13.018717, abs 0.032343, rel 0.002491, norm 1.448713"
## [1] "Iter 13, obj 13.060352, abs 0.041636, rel 0.003198, norm 1.473436"
## [1] "Iter 14, obj 13.109877, abs 0.049525, rel 0.003792, norm 1.494355"
## [1] "Iter 15, obj 13.166316, abs 0.056439, rel 0.004305, norm 1.512147"
## [1] "Iter 16, obj 13.228011, abs 0.061695, rel 0.004686, norm 1.527261"
## [1] "Iter 17, obj 13.292846, abs 0.064835, rel 0.004901, norm 1.540146"
## [1] "Iter 18, obj 13.358776, abs 0.065930, rel 0.004960, norm 1.551310"
## [1] "Iter 19, obj 13.423654, abs 0.064878, rel 0.004857, norm 1.561120"
## [1] "Iter 20, obj 13.485321, abs 0.061667, rel 0.004594, norm 1.569742"
## [1] "Iter 21, obj 13.542035, abs 0.056714, rel 0.004206, norm 1.577234"
## [1] "Iter 22, obj 13.592729, abs 0.050694, rel 0.003743, norm 1.583623"
## [1] "Iter 23, obj 13.636935, abs 0.044206, rel 0.003252, norm 1.588980"
## [1] "Iter 24, obj 13.674667, abs 0.037732, rel 0.002767, norm 1.593440"
## [1] "Iter 25, obj 13.706283, abs 0.031616, rel 0.002312, norm 1.597158"

## [1] "Iter 26, obj 13.732325, abs 0.026042, rel 0.001900, norm 1.600271"
## [1] "Iter 27, obj 13.753393, abs 0.021069, rel 0.001534, norm 1.602901"
```

```
## [1] "Iter 28, obj 13.770099, abs 0.016705, rel 0.001215, norm 1.605143"
## [1] "Iter 29, obj 13.783048, abs 0.012950, rel 0.000940, norm 1.607074"
## [1] "Iter 30, obj 13.792836, abs 0.009788, rel 0.000710, norm 1.608751"
## [1] "Iter 31, obj 13.800017, abs 0.007181, rel 0.000521, norm 1.610215"
## [1] "Iter 32, obj 13.805086, abs 0.005069, rel 0.000367, norm 1.611495"
## [1] "Iter 33, obj 13.808467, abs 0.003382, rel 0.000245, norm 1.612614"
## [1] "Iter 34, obj 13.810514, abs 0.002047, rel 0.000148, norm 1.613592"
## [1] "Iter 35, obj 13.811512, abs 0.000997, rel 0.000072, norm 1.614448"
## [1] "Iter 36, obj 13.811686, abs 0.000175, rel 0.000013, norm 1.615200"
## [1] "Iter 37, obj 13.811217, abs 0.000470, rel 0.000034, norm 1.615864"
## [1] "Iter 38, obj 13.810239, abs 0.000977, rel 0.000071, norm 1.616455"
## [1] "Iter 39, obj 13.808860, abs 0.001380, rel 0.000100, norm 1.616985"
## [1] "Iter 40, obj 13.807158, abs 0.001702, rel 0.000123, norm 1.617464"
## [1] "Iter 41, obj 13.805193, abs 0.001964, rel 0.000142, norm 1.617902"
## [1] "Iter 42, obj 13.803012, abs 0.002182, rel 0.000158, norm 1.618305"
## [1] "Iter 43, obj 13.800647, abs 0.002365, rel 0.000171, norm 1.618679"
## [1] "Iter 44, obj 13.798125, abs 0.002522, rel 0.000183, norm 1.619028"
## [1] "Iter 45, obj 13.795466, abs 0.002659, rel 0.000193, norm 1.619356"
## [1] "Iter 46, obj 13.792688, abs 0.002778, rel 0.000201, norm 1.619665"
## [1] "Iter 47, obj 13.789807, abs 0.002881, rel 0.000209, norm 1.619958"
## [1] "Iter 48, obj 13.786837, abs 0.002970, rel 0.000215, norm 1.620238"
## [1] "Iter 49, obj 13.783791, abs 0.003046, rel 0.000221, norm 1.620507"
## [1] "Iter 50, obj 13.780684, abs 0.003107, rel 0.000225, norm 1.620766"
## [1] "Iter 51, obj 13.777529, abs 0.003155, rel 0.000229, norm 1.621017"
## [1] "Iter 52, obj 13.774340, abs 0.003188, rel 0.000231, norm 1.621263"
## [1] "Iter 53, obj 13.771132, abs 0.003208, rel 0.000233, norm 1.621504"
## [1] "Iter 54, obj 13.767918, abs 0.003214, rel 0.000233, norm 1.621743"
## [1] "Iter 55, obj 13.764711, abs 0.003207, rel 0.000233, norm 1.621981"
## [1] "Iter 56, obj 13.761524, abs 0.003187, rel 0.000232, norm 1.622219"
## [1] "Iter 57, obj 13.758368, abs 0.003156, rel 0.000229, norm 1.622459"
## [1] "Iter 58, obj 13.755254, abs 0.003114, rel 0.000226, norm 1.622702"
## [1] "Iter 59, obj 13.752192, abs 0.003062, rel 0.000223, norm 1.622949"
## [1] "Iter 60, obj 13.749190, abs 0.003002, rel 0.000218, norm 1.623201"
## [1] "Iter 61, obj 13.746254, abs 0.002935, rel 0.000213, norm 1.623459"
## [1] "Iter 62, obj 13.743392, abs 0.002863, rel 0.000208, norm 1.623723"
## [1] "Iter 63, obj 13.740607, abs 0.002785, rel 0.000203, norm 1.623994"
## [1] "Iter 64, obj 13.737902, abs 0.002705, rel 0.000197, norm 1.624273"
## [1] "Iter 65, obj 13.735279, abs 0.002623, rel 0.000191, norm 1.624559"
## [1] "Iter 66, obj 13.732740, abs 0.002539, rel 0.000185, norm 1.624852"
## [1] "Iter 67, obj 13.730284, abs 0.002456, rel 0.000179, norm 1.625153"
## [1] "Iter 68, obj 13.727910, abs 0.002374, rel 0.000173, norm 1.625460"
## [1] "Iter 69, obj 13.725617, abs 0.002293, rel 0.000167, norm 1.625775"
## [1] "Iter 70, obj 13.723403, abs 0.002215, rel 0.000161, norm 1.626096"
## [1] "Iter 71, obj 13.721263, abs 0.002140, rel 0.000156, norm 1.626424"
## [1] "Iter 72, obj 13.719195, abs 0.002068, rel 0.000151, norm 1.626757"
## [1] "Iter 73, obj 13.717195, abs 0.002000, rel 0.000146, norm 1.627096"
## [1] "Iter 74, obj 13.715259, abs 0.001936, rel 0.000141, norm 1.627440"
## [1] "Iter 75, obj 13.713382, abs 0.001877, rel 0.000137, norm 1.627787"
## [1] "Iter 76, obj 13.711560, abs 0.001822, rel 0.000133, norm 1.628139"
## [1] "Iter 77, obj 13.709788, abs 0.001772, rel 0.000129, norm 1.628493"
## [1] "Iter 78, obj 13.708061, abs 0.001727, rel 0.000126, norm 1.628851"
## [1] "Iter 79, obj 13.706374, abs 0.001687, rel 0.000123, norm 1.629210"
## [1] "Iter 80, obj 13.704722, abs 0.001651, rel 0.000120, norm 1.629571"
```

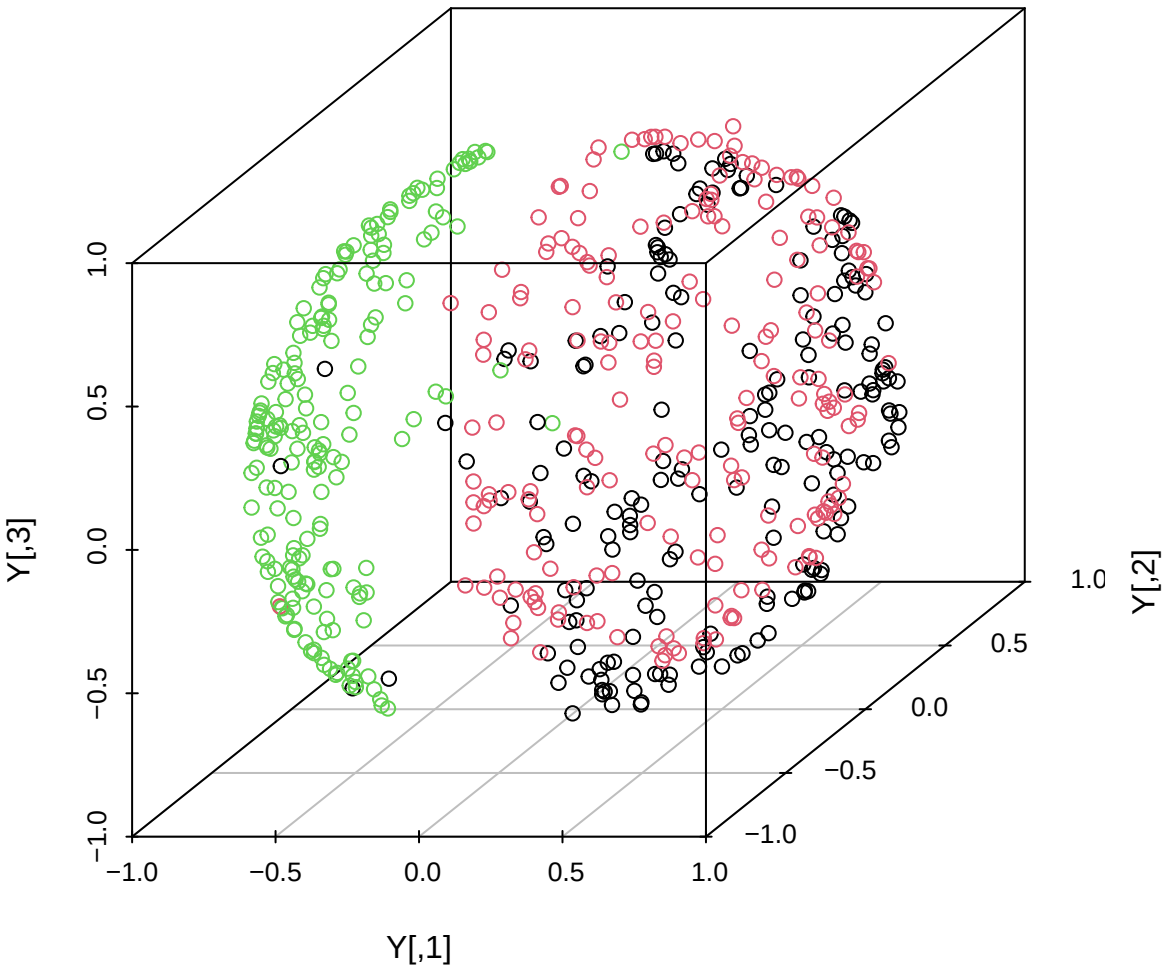
```
## [1] "Iter 81, obj 13.703101, abs 0.001621, rel 0.000118, norm 1.629933"
## [1] "Iter 82, obj 13.701506, abs 0.001595, rel 0.000116, norm 1.630295"
## [1] "Iter 83, obj 13.699933, abs 0.001574, rel 0.000115, norm 1.630658"
## [1] "Iter 84, obj 13.698376, abs 0.001557, rel 0.000114, norm 1.631020"
## [1] "Iter 85, obj 13.696831, abs 0.001544, rel 0.000113, norm 1.631381"
## [1] "Iter 86, obj 13.695295, abs 0.001536, rel 0.000112, norm 1.631741"
## [1] "Iter 87, obj 13.693763, abs 0.001532, rel 0.000112, norm 1.632100"
## [1] "Iter 88, obj 13.692232, abs 0.001531, rel 0.000112, norm 1.632456"
## [1] "Iter 89, obj 13.690699, abs 0.001533, rel 0.000112, norm 1.632809"
## [1] "Iter 90, obj 13.689160, abs 0.001539, rel 0.000112, norm 1.633160"
## [1] "Iter 91, obj 13.687613, abs 0.001547, rel 0.000113, norm 1.633507"
## [1] "Iter 92, obj 13.686055, abs 0.001558, rel 0.000114, norm 1.633851"
## [1] "Iter 93, obj 13.684484, abs 0.001570, rel 0.000115, norm 1.634190"
## [1] "Iter 94, obj 13.682900, abs 0.001584, rel 0.000116, norm 1.634525"
## [1] "Iter 95, obj 13.681301, abs 0.001599, rel 0.000117, norm 1.634856"
## [1] "Iter 96, obj 13.679686, abs 0.001615, rel 0.000118, norm 1.635182"
## [1] "Iter 97, obj 13.678056, abs 0.001630, rel 0.000119, norm 1.635503"
## [1] "Iter 98, obj 13.676411, abs 0.001645, rel 0.000120, norm 1.635818"
## [1] "Iter 99, obj 13.674752, abs 0.001659, rel 0.000121, norm 1.636128"
## [1] "Iter 100, obj 13.673081, abs 0.001672, rel 0.000122, norm 1.636432"

## [1] "Iter 101, obj 1.998239, abs 11.674841, rel 0.853856, norm 0.132571"
## [1] "Iter 102, obj 1.799793, abs 0.198447, rel 0.099311, norm 0.134891"
## [1] "Iter 103, obj 1.745285, abs 0.054507, rel 0.030285, norm 0.144784"
## [1] "Iter 104, obj 1.751790, abs 0.006505, rel 0.003727, norm 0.148404"
## [1] "Iter 105, obj 1.736050, abs 0.015740, rel 0.008985, norm 0.147973"
## [1] "Iter 106, obj 1.691687, abs 0.044363, rel 0.025554, norm 0.145893"
## [1] "Iter 107, obj 1.696000, abs 0.004312, rel 0.002549, norm 0.146529"
## [1] "Iter 108, obj 1.696119, abs 0.000119, rel 0.000070, norm 0.148754"
## [1] "Iter 109, obj 1.701048, abs 0.004929, rel 0.002906, norm 0.147850"
## [1] "Iter 110, obj 1.708458, abs 0.007409, rel 0.004356, norm 0.149545"
## [1] "Iter 111, obj 1.703508, abs 0.004950, rel 0.002897, norm 0.149556"
## [1] "Iter 112, obj 1.675565, abs 0.027943, rel 0.016403, norm 0.149359"
## [1] "Iter 113, obj 1.678400, abs 0.002836, rel 0.001692, norm 0.151611"
## [1] "Iter 114, obj 1.678069, abs 0.000331, rel 0.000197, norm 0.149501"
## [1] "Iter 115, obj 1.666621, abs 0.011448, rel 0.006822, norm 0.149931"
## [1] "Iter 116, obj 1.671967, abs 0.005345, rel 0.003207, norm 0.150884"
## [1] "Iter 117, obj 1.665582, abs 0.006384, rel 0.003819, norm 0.150049"
## [1] "Iter 118, obj 1.656656, abs 0.008926, rel 0.005359, norm 0.150703"
## [1] "Iter 119, obj 1.659550, abs 0.002894, rel 0.001747, norm 0.151145"
## [1] "Iter 120, obj 1.668014, abs 0.008463, rel 0.005100, norm 0.151254"
## [1] "Iter 121, obj 1.690933, abs 0.022919, rel 0.013740, norm 0.151711"
## [1] "Iter 122, obj 1.675581, abs 0.015352, rel 0.009079, norm 0.152160"
## [1] "Iter 123, obj 1.646342, abs 0.029238, rel 0.017450, norm 0.151449"
## [1] "Iter 124, obj 1.663893, abs 0.017550, rel 0.010660, norm 0.151957"
## [1] "Iter 125, obj 1.668356, abs 0.004463, rel 0.002682, norm 0.151669"

## [1] "Iter 126, obj 1.666224, abs 0.002131, rel 0.001278, norm 0.152438"
## [1] "Iter 127, obj 1.665463, abs 0.000761, rel 0.000457, norm 0.152570"
## [1] "Iter 128, obj 1.649620, abs 0.015843, rel 0.009513, norm 0.151786"
## [1] "Iter 129, obj 1.634455, abs 0.015165, rel 0.009193, norm 0.151024"
## [1] "Iter 130, obj 1.655434, abs 0.020979, rel 0.012836, norm 0.153127"
## [1] "Iter 131, obj 1.652643, abs 0.002791, rel 0.001686, norm 0.152641"
## [1] "Iter 132, obj 1.657878, abs 0.005235, rel 0.003168, norm 0.151641"
## [1] "Iter 133, obj 1.660122, abs 0.002244, rel 0.001353, norm 0.152027"
```

```
## [1] "Iter 134, obj 1.659162, abs 0.000960, rel 0.000578, norm 0.151886"
## [1] "Iter 135, obj 1.650738, abs 0.008424, rel 0.005077, norm 0.153050"
## [1] "Iter 136, obj 1.663830, abs 0.013092, rel 0.007931, norm 0.152442"
## [1] "Iter 137, obj 1.660726, abs 0.003104, rel 0.001866, norm 0.152860"
## [1] "Iter 138, obj 1.640571, abs 0.020155, rel 0.012136, norm 0.153911"
## [1] "Iter 139, obj 1.655826, abs 0.015255, rel 0.009298, norm 0.152514"
## [1] "Iter 140, obj 1.658984, abs 0.003158, rel 0.001907, norm 0.152349"
## [1] "Iter 141, obj 1.642928, abs 0.016056, rel 0.009678, norm 0.152593"
## [1] "Iter 142, obj 1.658047, abs 0.015119, rel 0.009203, norm 0.152498"
## [1] "Iter 143, obj 1.647497, abs 0.010550, rel 0.006363, norm 0.152691"
## [1] "Iter 144, obj 1.636772, abs 0.010726, rel 0.006510, norm 0.153152"
## [1] "Iter 145, obj 1.660070, abs 0.023299, rel 0.014235, norm 0.152744"
## [1] "Iter 146, obj 1.645565, abs 0.014505, rel 0.008738, norm 0.153040"
## [1] "Iter 147, obj 1.639137, abs 0.006428, rel 0.003906, norm 0.153205"
## [1] "Iter 148, obj 1.665314, abs 0.026177, rel 0.015970, norm 0.152808"
## [1] "Iter 149, obj 1.665711, abs 0.000396, rel 0.000238, norm 0.152029"
## [1] "Iter 150, obj 1.654682, abs 0.011029, rel 0.006621, norm 0.152601"

## [1] "Iter 151, obj 1.664938, abs 0.010256, rel 0.006198, norm 0.152433"
## [1] "Iter 152, obj 1.660388, abs 0.004550, rel 0.002733, norm 0.153065"
## [1] "Iter 153, obj 1.649498, abs 0.010890, rel 0.006559, norm 0.152707"
## [1] "Iter 154, obj 1.659574, abs 0.010077, rel 0.006109, norm 0.152827"
## [1] "Iter 155, obj 1.657715, abs 0.001859, rel 0.001120, norm 0.152419"
## [1] "Iter 156, obj 1.653841, abs 0.003874, rel 0.002337, norm 0.153317"
## [1] "Iter 157, obj 1.666808, abs 0.012968, rel 0.007841, norm 0.152890"
## [1] "Iter 158, obj 1.659502, abs 0.007306, rel 0.004383, norm 0.153091"
## [1] "Iter 159, obj 1.644564, abs 0.014938, rel 0.009002, norm 0.153195"
## [1] "Iter 160, obj 1.648739, abs 0.004175, rel 0.002539, norm 0.152293"
## [1] "Iter 161, obj 1.660653, abs 0.011913, rel 0.007226, norm 0.152072"
## [1] "Iter 162, obj 1.657934, abs 0.002719, rel 0.001637, norm 0.153658"
## [1] "Iter 163, obj 1.658615, abs 0.000681, rel 0.000411, norm 0.153419"
## [1] "Iter 164, obj 1.649208, abs 0.009407, rel 0.005671, norm 0.152638"
## [1] "Iter 165, obj 1.641990, abs 0.007218, rel 0.004376, norm 0.153933"
## [1] "Iter 166, obj 1.663019, abs 0.021029, rel 0.012807, norm 0.152658"
## [1] "Iter 167, obj 1.667479, abs 0.004460, rel 0.002682, norm 0.153379"
## [1] "Iter 168, obj 1.643131, abs 0.024348, rel 0.014602, norm 0.153602"
## [1] "Iter 169, obj 1.656402, abs 0.013271, rel 0.008077, norm 0.153541"
## [1] "Iter 170, obj 1.646806, abs 0.009596, rel 0.005793, norm 0.153009"
## [1] "Iter 171, obj 1.639432, abs 0.007373, rel 0.004477, norm 0.152728"
## [1] "Iter 172, obj 1.660493, abs 0.021060, rel 0.012846, norm 0.152562"
## [1] "Iter 173, obj 1.655256, abs 0.005237, rel 0.003154, norm 0.153912"
## [1] "Iter 174, obj 1.645456, abs 0.009800, rel 0.005920, norm 0.152943"
## [1] "Iter 175, obj 1.659577, abs 0.014121, rel 0.008582, norm 0.153282"

## [1] "Iter 176, obj 1.646416, abs 0.013161, rel 0.007930, norm 0.154075"
## [1] "Iter 177, obj 1.642075, abs 0.004341, rel 0.002637, norm 0.152957"
## [1] "Iter 178, obj 1.662980, abs 0.020905, rel 0.012731, norm 0.153377"
## [1] "Iter 179, obj 1.647389, abs 0.015591, rel 0.009375, norm 0.153498"
## [1] "Iter 180, obj 1.648952, abs 0.001563, rel 0.000949, norm 0.153165"
## [1] "Iter 181, obj 1.655963, abs 0.007011, rel 0.004252, norm 0.153165"
## [1] "Iter 182, obj 1.640674, abs 0.015289, rel 0.009233, norm 0.153541"
## [1] "Iter 183, obj 1.625626, abs 0.015048, rel 0.009172, norm 0.153157"
## [1] "Iter 184, obj 1.650025, abs 0.024399, rel 0.015009, norm 0.152940"
## [1] "Iter 185, obj 1.649180, abs 0.000845, rel 0.000512, norm 0.153059"
## [1] "Iter 186, obj 1.652681, abs 0.003501, rel 0.002123, norm 0.153411"
```

```
## [1] "Iter 187, obj 1.669631, abs 0.016950, rel 0.010256, norm 0.153313"
## [1] "Iter 188, obj 1.665906, abs 0.003725, rel 0.002231, norm 0.153650"
## [1] "Iter 189, obj 1.651190, abs 0.014716, rel 0.008834, norm 0.153185"
## [1] "Iter 190, obj 1.657882, abs 0.006692, rel 0.004053, norm 0.153023"
## [1] "Iter 191, obj 1.647423, abs 0.010459, rel 0.006309, norm 0.151791"
## [1] "Iter 192, obj 1.643510, abs 0.003913, rel 0.002375, norm 0.153153"
## [1] "Iter 193, obj 1.675609, abs 0.032099, rel 0.019531, norm 0.152817"
## [1] "Iter 194, obj 1.663079, abs 0.012531, rel 0.007478, norm 0.153721"
## [1] "Iter 195, obj 1.648191, abs 0.014888, rel 0.008952, norm 0.152753"
## [1] "Iter 196, obj 1.647046, abs 0.001145, rel 0.000694, norm 0.152528"
## [1] "Iter 197, obj 1.631163, abs 0.015883, rel 0.009644, norm 0.153089"
## [1] "Iter 198, obj 1.644620, abs 0.013457, rel 0.008250, norm 0.153064"
## [1] "Iter 199, obj 1.662339, abs 0.017720, rel 0.010774, norm 0.152442"
## [1] "Iter 200, obj 1.653922, abs 0.008418, rel 0.005064, norm 0.153784"
```




```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                             color = colors)
```



It's clearly clustered the three groups we had in the original generated data.

Case 3

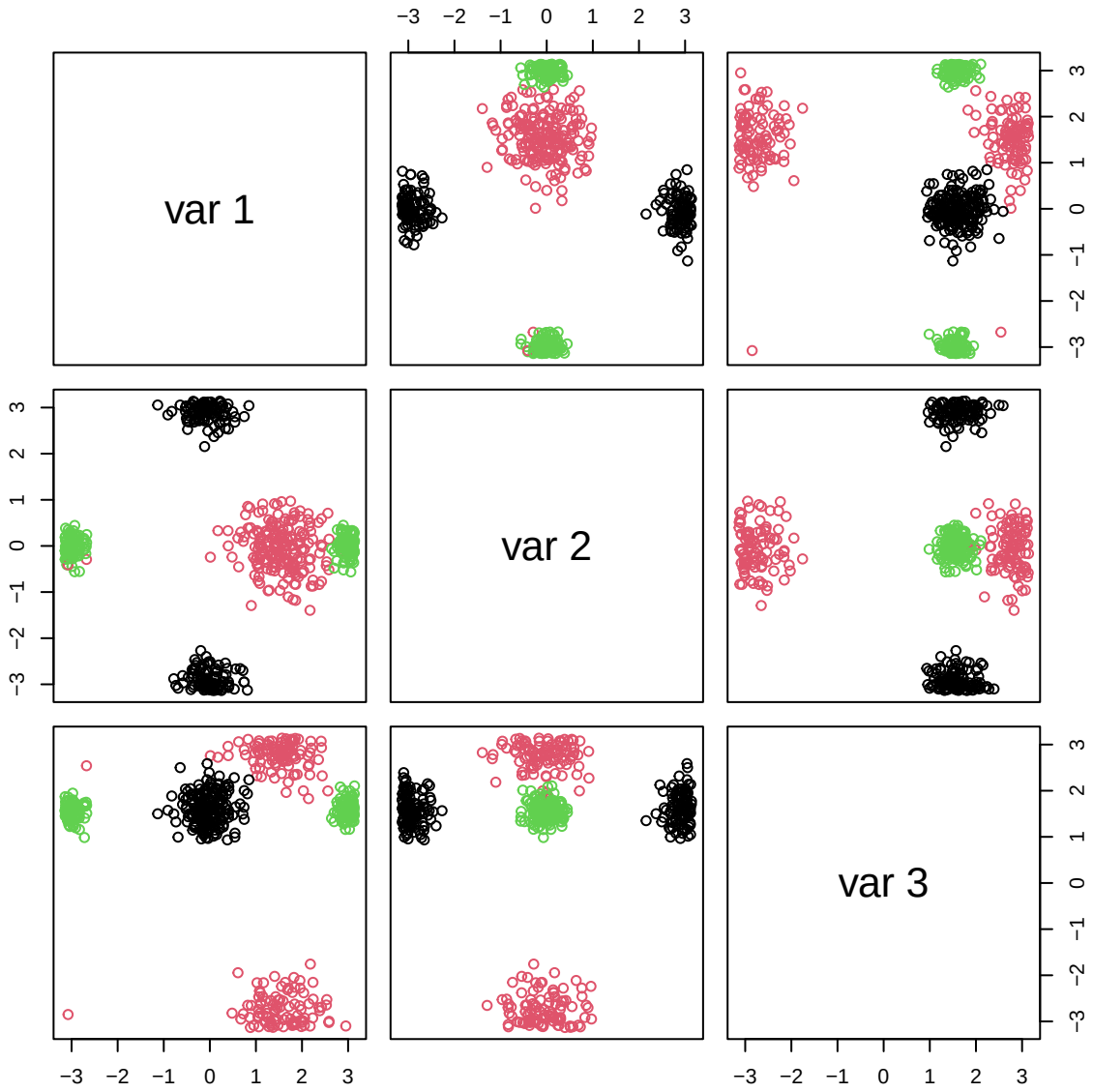
Sample on the $(\mathbb{S}^1)^3$ where $p = 1$ and $r = 3$.

```
# Sample on the (S^1)^2
n_ori <- 200
vmf11 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 10)
vmf12 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 10)
vmf13 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi/2)), kappa = 10)
x1 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf11),
                              DirStats::to_rad(vmf12),
                              DirStats::to_rad(vmf13)))

vmf21 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi / 2)), kappa = 5)
vmf22 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 5)
vmf23 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 5)
x2 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf21),DirStats::to_rad(vmf22),
                              DirStats::to_rad(vmf23)))

vmf31 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi)), kappa = 30)
vmf32 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = 0)), kappa = 30)
```

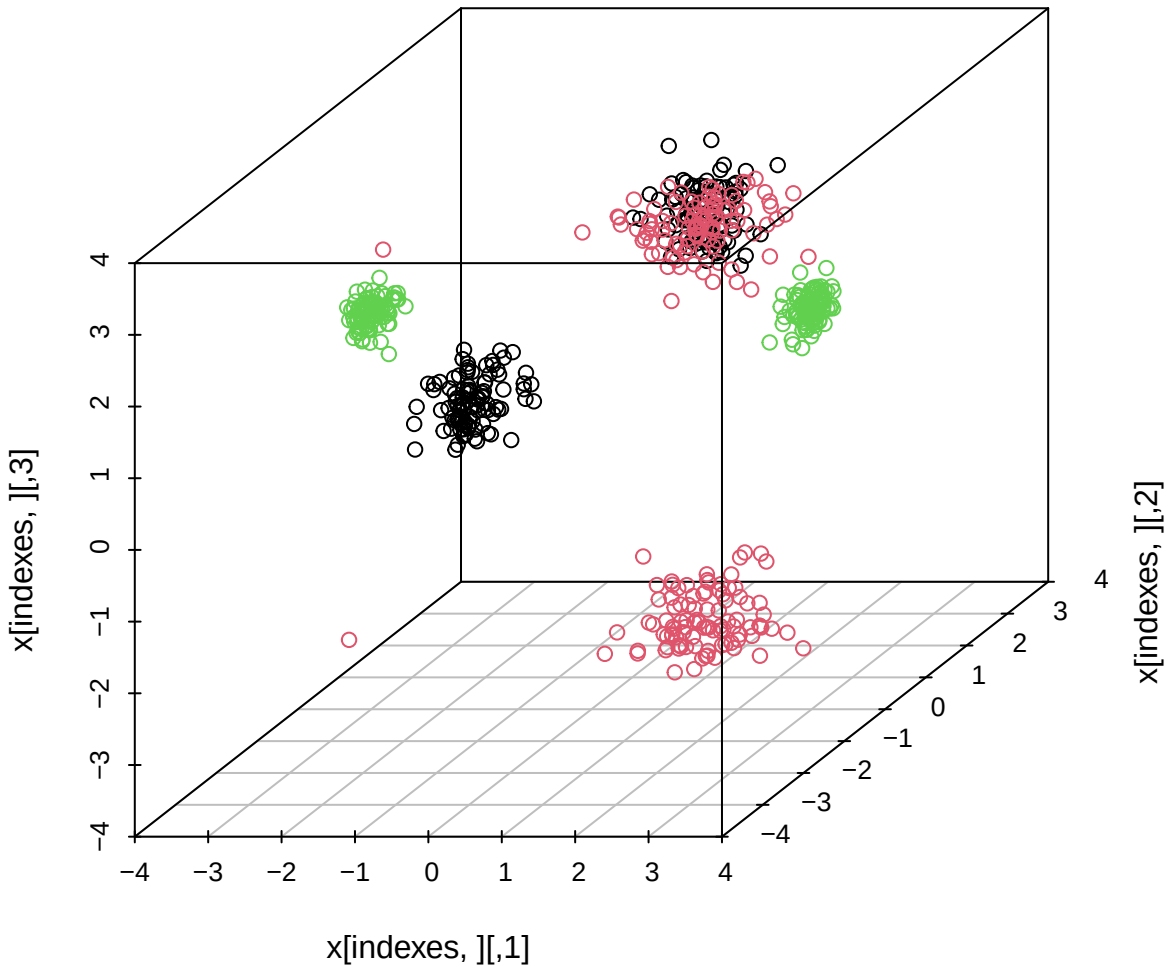
```
vmf33 <- rotasym::r_vMF(n = n_ori, mu = drop(DirStats::to_cir(th = pi/2)), kappa = 30)
x3 <- sdetorus::toPiInt(cbind(DirStats::to_rad(vmf31),DirStats::to_rad(vmf32),
                              DirStats::to_rad(vmf33)))
```

Now, let's do the same visualization with the data on the torus:

```
x <- rbind(x1, x2, x3)
n <- n_ori * 3
x_3 <- array(dim = c(n, 2, 3))
x_3[, , 1] <- DirStats::to_cir(x[, 1])
x_3[, , 2] <- DirStats::to_cir(x[, 2])
x_3[, , 3] <- DirStats::to_cir(x[, 3])

n <- nrow(x_3)
indexes <- sample(1:n)
x_3 <- x_3[indexes,,]
colors <- rep(c(1, 2, 3), each = n / 3)[indexes]


pairs(x[indexes,], col = colors)
```




```
scatterplot3d::scatterplot3d(x[indexes,], xlim = c(-pi, pi), ylim = c(-pi, pi), zlim = c(-pi, pi),
                             color = colors)
```



Let's calculate the rho parameters based on a perplexity of 25:

```
rho_third_perp25 <- rho_optim_bst(x_3, 25)
```

```
## Time difference of 1.404378 mins
```

First, let's reduce to dimension $\mathbb{S}^1$ then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_3, d=1, rho_psc_list = rho_third_perp25, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

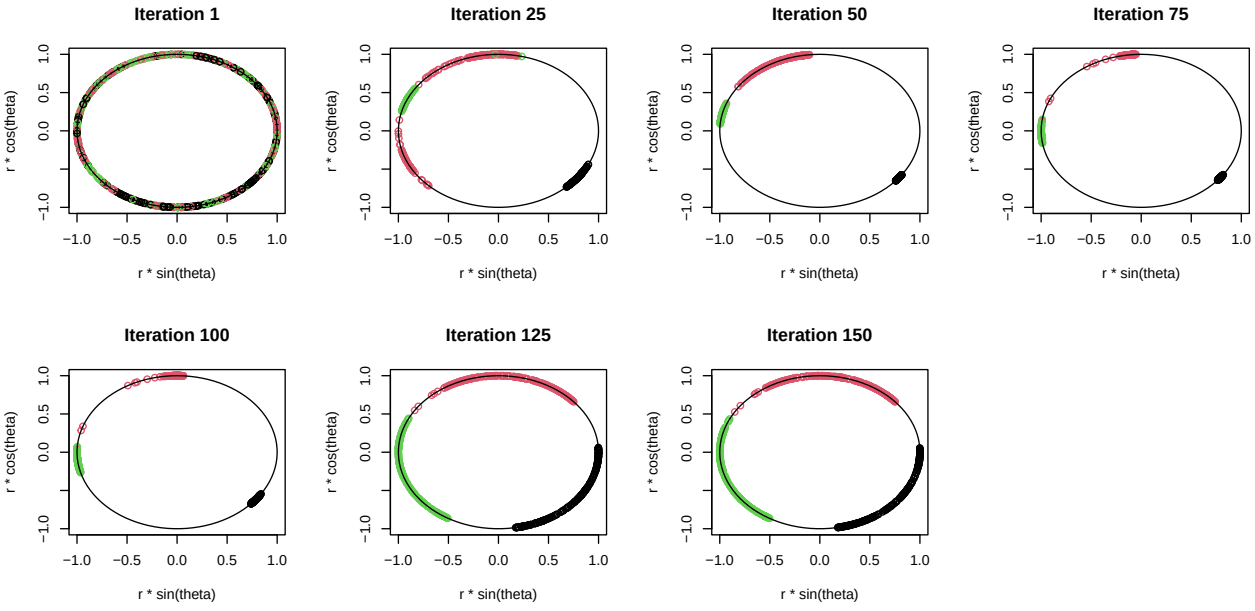
```
## [1] "Iter 1, obj 18.063813, abs 0.000000, rel 0.000000, norm 0.115721"
```

```
## [1] "Iter 2, obj 17.342037, abs 0.721776, rel 0.039957, norm 0.225727"
## [1] "Iter 3, obj 16.786851, abs 0.555185, rel 0.032014, norm 0.321782"
## [1] "Iter 4, obj 16.356705, abs 0.430146, rel 0.025624, norm 0.403094"
## [1] "Iter 5, obj 16.038423, abs 0.318282, rel 0.019459, norm 0.477257"
## [1] "Iter 6, obj 15.792807, abs 0.245616, rel 0.015314, norm 0.537977"
## [1] "Iter 7, obj 15.611138, abs 0.181669, rel 0.011503, norm 0.594696"
## [1] "Iter 8, obj 15.476698, abs 0.134440, rel 0.008612, norm 0.643609"
## [1] "Iter 9, obj 15.367074, abs 0.109624, rel 0.007083, norm 0.684705"
## [1] "Iter 10, obj 15.288270, abs 0.078804, rel 0.005128, norm 0.719966"
## [1] "Iter 11, obj 15.217645, abs 0.070625, rel 0.004620, norm 0.749044"
## [1] "Iter 12, obj 15.146354, abs 0.071291, rel 0.004685, norm 0.774835"
## [1] "Iter 13, obj 15.081052, abs 0.065303, rel 0.004311, norm 0.798975"
```

```
## [1] "Iter 14, obj 15.015992, abs 0.065060, rel 0.004314, norm 0.821035"
## [1] "Iter 15, obj 14.954722, abs 0.061270, rel 0.004080, norm 0.841378"
## [1] "Iter 16, obj 14.897431, abs 0.057291, rel 0.003831, norm 0.857861"
## [1] "Iter 17, obj 14.857571, abs 0.039861, rel 0.002676, norm 0.872964"
## [1] "Iter 18, obj 14.834121, abs 0.023450, rel 0.001578, norm 0.885794"
## [1] "Iter 19, obj 14.820685, abs 0.013436, rel 0.000906, norm 0.894678"
## [1] "Iter 20, obj 14.814170, abs 0.006514, rel 0.000440, norm 0.900920"
## [1] "Iter 21, obj 14.813121, abs 0.001049, rel 0.000071, norm 0.905832"
## [1] "Iter 22, obj 14.813819, abs 0.000698, rel 0.000047, norm 0.910082"
## [1] "Iter 23, obj 14.819430, abs 0.005611, rel 0.000379, norm 0.914379"
## [1] "Iter 24, obj 14.826998, abs 0.007568, rel 0.000511, norm 0.918222"
## [1] "Iter 25, obj 14.835251, abs 0.008253, rel 0.000557, norm 0.921816"

## [1] "Iter 26, obj 14.843573, abs 0.008322, rel 0.000561, norm 0.925680"
## [1] "Iter 27, obj 14.849829, abs 0.006256, rel 0.000421, norm 0.929890"
## [1] "Iter 28, obj 14.845995, abs 0.003834, rel 0.000258, norm 0.934293"
## [1] "Iter 29, obj 14.846983, abs 0.000988, rel 0.000067, norm 0.939435"
## [1] "Iter 30, obj 14.854487, abs 0.007503, rel 0.000505, norm 0.944431"
## [1] "Iter 31, obj 14.866159, abs 0.011672, rel 0.000786, norm 0.948386"
## [1] "Iter 32, obj 14.877309, abs 0.011150, rel 0.000750, norm 0.951055"
## [1] "Iter 33, obj 14.888528, abs 0.011219, rel 0.000754, norm 0.953241"
## [1] "Iter 34, obj 14.902212, abs 0.013684, rel 0.000919, norm 0.955283"
## [1] "Iter 35, obj 14.916199, abs 0.013987, rel 0.000939, norm 0.956979"
## [1] "Iter 36, obj 14.929971, abs 0.013772, rel 0.000923, norm 0.958638"
## [1] "Iter 37, obj 14.942581, abs 0.012610, rel 0.000845, norm 0.960345"
## [1] "Iter 38, obj 14.952357, abs 0.009776, rel 0.000654, norm 0.962188"
## [1] "Iter 39, obj 14.957945, abs 0.005588, rel 0.000374, norm 0.964327"
## [1] "Iter 40, obj 14.960237, abs 0.002291, rel 0.000153, norm 0.966869"
## [1] "Iter 41, obj 14.961139, abs 0.000902, rel 0.000060, norm 0.969577"
## [1] "Iter 42, obj 14.960865, abs 0.000274, rel 0.000018, norm 0.972055"
## [1] "Iter 43, obj 14.958455, abs 0.002410, rel 0.000161, norm 0.974246"
## [1] "Iter 44, obj 14.952000, abs 0.006456, rel 0.000432, norm 0.976305"
## [1] "Iter 45, obj 14.939523, abs 0.012477, rel 0.000834, norm 0.978426"
## [1] "Iter 46, obj 14.919450, abs 0.020073, rel 0.001344, norm 0.980720"
## [1] "Iter 47, obj 14.888561, abs 0.030889, rel 0.002070, norm 0.983245"
## [1] "Iter 48, obj 14.843754, abs 0.044807, rel 0.003010, norm 0.986171"
## [1] "Iter 49, obj 14.793434, abs 0.050320, rel 0.003390, norm 0.989537"
## [1] "Iter 50, obj 14.751836, abs 0.041598, rel 0.002812, norm 0.992001"

## [1] "Iter 51, obj 14.721613, abs 0.030223, rel 0.002049, norm 0.991835"
## [1] "Iter 52, obj 14.698945, abs 0.022667, rel 0.001540, norm 0.989201"
## [1] "Iter 53, obj 14.680718, abs 0.018227, rel 0.001240, norm 0.985211"
## [1] "Iter 54, obj 14.665278, abs 0.015441, rel 0.001052, norm 0.980638"
## [1] "Iter 55, obj 14.651766, abs 0.013512, rel 0.000921, norm 0.975871"
## [1] "Iter 56, obj 14.639705, abs 0.012061, rel 0.000823, norm 0.971096"
## [1] "Iter 57, obj 14.628802, abs 0.010902, rel 0.000745, norm 0.966403"
## [1] "Iter 58, obj 14.618866, abs 0.009936, rel 0.000679, norm 0.961836"
## [1] "Iter 59, obj 14.609757, abs 0.009109, rel 0.000623, norm 0.957416"
## [1] "Iter 60, obj 14.601372, abs 0.008385, rel 0.000574, norm 0.953149"
## [1] "Iter 61, obj 14.593628, abs 0.007744, rel 0.000530, norm 0.949039"
## [1] "Iter 62, obj 14.586457, abs 0.007171, rel 0.000491, norm 0.945082"
## [1] "Iter 63, obj 14.579801, abs 0.006656, rel 0.000456, norm 0.941273"
## [1] "Iter 64, obj 14.573608, abs 0.006193, rel 0.000425, norm 0.937608"
## [1] "Iter 65, obj 14.567832, abs 0.005776, rel 0.000396, norm 0.934080"
## [1] "Iter 66, obj 14.562431, abs 0.005401, rel 0.000371, norm 0.930682"
```

```
## [1] "Iter 67, obj 14.557370, abs 0.005061, rel 0.000348, norm 0.927409"
## [1] "Iter 68, obj 14.552616, abs 0.004754, rel 0.000327, norm 0.924254"
## [1] "Iter 69, obj 14.548141, abs 0.004475, rel 0.000307, norm 0.921212"
## [1] "Iter 70, obj 14.543920, abs 0.004221, rel 0.000290, norm 0.918277"
## [1] "Iter 71, obj 14.539932, abs 0.003988, rel 0.000274, norm 0.915444"
## [1] "Iter 72, obj 14.536157, abs 0.003775, rel 0.000260, norm 0.912708"
## [1] "Iter 73, obj 14.532577, abs 0.003580, rel 0.000246, norm 0.910064"
## [1] "Iter 74, obj 14.529178, abs 0.003400, rel 0.000234, norm 0.907508"
## [1] "Iter 75, obj 14.525944, abs 0.003234, rel 0.000223, norm 0.905036"

## [1] "Iter 76, obj 14.522864, abs 0.003080, rel 0.000212, norm 0.902644"
## [1] "Iter 77, obj 14.519926, abs 0.002938, rel 0.000202, norm 0.900329"
## [1] "Iter 78, obj 14.517119, abs 0.002806, rel 0.000193, norm 0.898087"
## [1] "Iter 79, obj 14.514435, abs 0.002684, rel 0.000185, norm 0.895915"
## [1] "Iter 80, obj 14.511865, abs 0.002570, rel 0.000177, norm 0.893810"
## [1] "Iter 81, obj 14.509400, abs 0.002465, rel 0.000170, norm 0.891769"
## [1] "Iter 82, obj 14.507034, abs 0.002366, rel 0.000163, norm 0.889790"
## [1] "Iter 83, obj 14.504760, abs 0.002274, rel 0.000157, norm 0.887870"
## [1] "Iter 84, obj 14.502572, abs 0.002188, rel 0.000151, norm 0.886007"
## [1] "Iter 85, obj 14.500464, abs 0.002108, rel 0.000145, norm 0.884198"
## [1] "Iter 86, obj 14.498432, abs 0.002033, rel 0.000140, norm 0.882442"
## [1] "Iter 87, obj 14.496469, abs 0.001963, rel 0.000135, norm 0.880737"
## [1] "Iter 88, obj 14.494572, abs 0.001897, rel 0.000131, norm 0.879080"
## [1] "Iter 89, obj 14.492736, abs 0.001836, rel 0.000127, norm 0.877469"
## [1] "Iter 90, obj 14.490957, abs 0.001778, rel 0.000123, norm 0.875904"
## [1] "Iter 91, obj 14.489233, abs 0.001725, rel 0.000119, norm 0.874382"
## [1] "Iter 92, obj 14.487558, abs 0.001675, rel 0.000116, norm 0.872903"
## [1] "Iter 93, obj 14.485930, abs 0.001628, rel 0.000112, norm 0.871463"
## [1] "Iter 94, obj 14.484345, abs 0.001584, rel 0.000109, norm 0.870063"
## [1] "Iter 95, obj 14.482802, abs 0.001544, rel 0.000107, norm 0.868701"
## [1] "Iter 96, obj 14.481296, abs 0.001506, rel 0.000104, norm 0.867375"
## [1] "Iter 97, obj 14.479825, abs 0.001471, rel 0.000102, norm 0.866084"
## [1] "Iter 98, obj 14.478388, abs 0.001438, rel 0.000099, norm 0.864827"
## [1] "Iter 99, obj 14.476980, abs 0.001407, rel 0.000097, norm 0.863603"
## [1] "Iter 100, obj 14.475601, abs 0.001379, rel 0.000095, norm 0.862412"

## [1] "Iter 101, obj 2.214522, abs 12.261079, rel 0.847017, norm 0.080707"
## [1] "Iter 102, obj 2.181098, abs 0.033424, rel 0.015093, norm 0.082323"
## [1] "Iter 103, obj 2.145833, abs 0.035264, rel 0.016168, norm 0.089407"
## [1] "Iter 104, obj 2.124724, abs 0.021109, rel 0.009837, norm 0.098646"
## [1] "Iter 105, obj 2.115729, abs 0.008995, rel 0.004234, norm 0.104272"
## [1] "Iter 106, obj 2.112160, abs 0.003569, rel 0.001687, norm 0.106483"
## [1] "Iter 107, obj 2.110597, abs 0.001563, rel 0.000740, norm 0.107201"
## [1] "Iter 108, obj 2.109757, abs 0.000841, rel 0.000398, norm 0.107417"
## [1] "Iter 109, obj 2.109250, abs 0.000507, rel 0.000240, norm 0.107469"
## [1] "Iter 110, obj 2.108882, abs 0.000368, rel 0.000174, norm 0.107440"
## [1] "Iter 111, obj 2.108600, abs 0.000282, rel 0.000134, norm 0.107374"
## [1] "Iter 112, obj 2.108365, abs 0.000235, rel 0.000111, norm 0.107296"
## [1] "Iter 113, obj 2.108146, abs 0.000219, rel 0.000104, norm 0.107220"
## [1] "Iter 114, obj 2.107937, abs 0.000209, rel 0.000099, norm 0.107147"
## [1] "Iter 115, obj 2.107738, abs 0.000199, rel 0.000094, norm 0.107077"
## [1] "Iter 116, obj 2.107540, abs 0.000198, rel 0.000094, norm 0.107014"
## [1] "Iter 117, obj 2.107339, abs 0.000201, rel 0.000096, norm 0.106959"
## [1] "Iter 118, obj 2.107141, abs 0.000198, rel 0.000094, norm 0.106913"
## [1] "Iter 119, obj 2.106948, abs 0.000193, rel 0.000092, norm 0.106876"
```

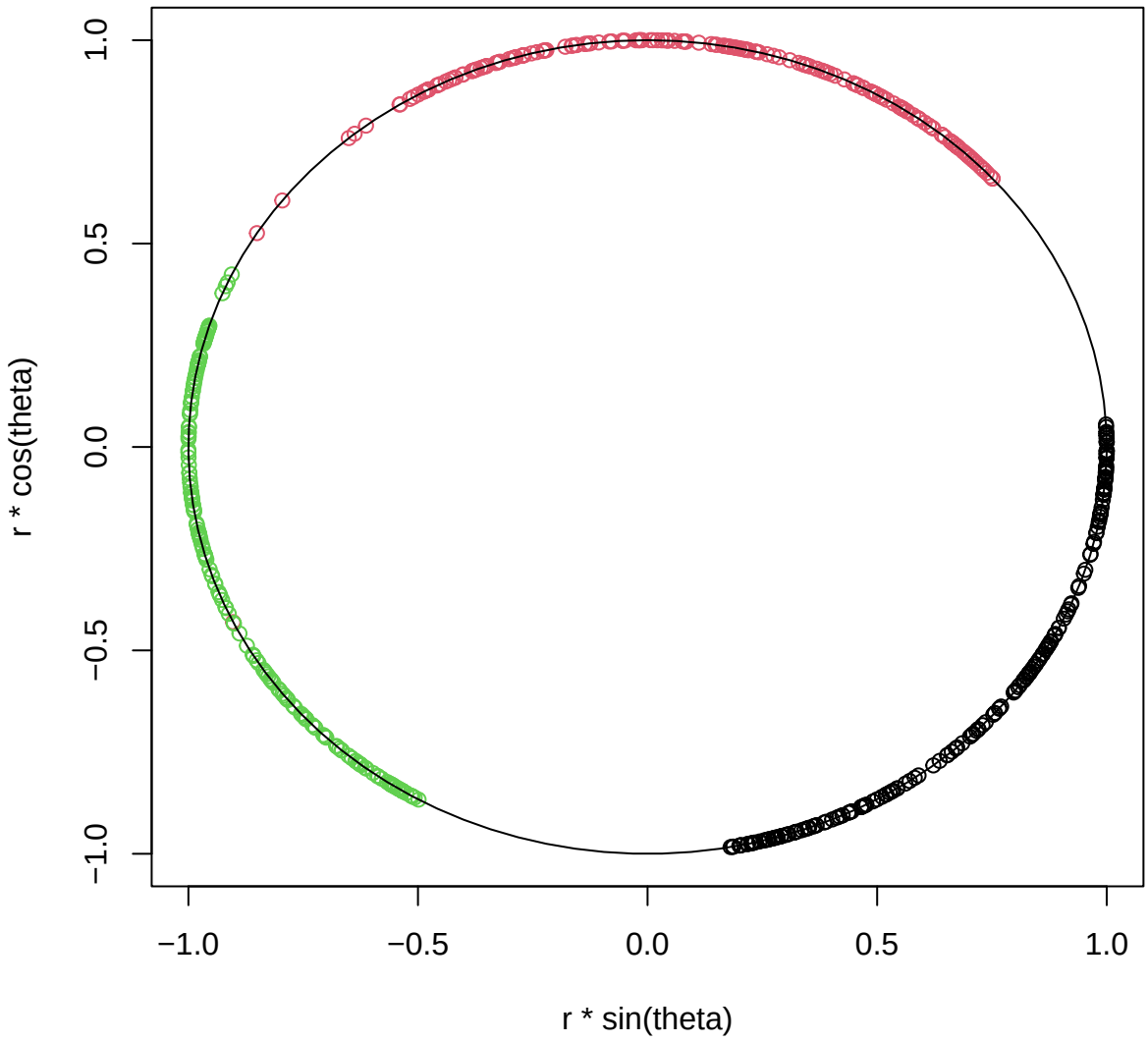
```
## [1] "Iter 120, obj 2.106757, abs 0.000191, rel 0.000091, norm 0.106845"
## [1] "Iter 121, obj 2.106571, abs 0.000186, rel 0.000088, norm 0.106819"
## [1] "Iter 122, obj 2.106397, abs 0.000174, rel 0.000083, norm 0.106794"
## [1] "Iter 123, obj 2.106235, abs 0.000161, rel 0.000077, norm 0.106774"
## [1] "Iter 124, obj 2.106087, abs 0.000149, rel 0.000071, norm 0.106759"
## [1] "Iter 125, obj 2.105954, abs 0.000133, rel 0.000063, norm 0.106753"

## [1] "Iter 126, obj 2.105838, abs 0.000116, rel 0.000055, norm 0.106755"
## [1] "Iter 127, obj 2.105734, abs 0.000104, rel 0.000049, norm 0.106764"
## [1] "Iter 128, obj 2.105640, abs 0.000093, rel 0.000044, norm 0.106777"
## [1] "Iter 129, obj 2.105557, abs 0.000083, rel 0.000039, norm 0.106792"
## [1] "Iter 130, obj 2.105482, abs 0.000075, rel 0.000036, norm 0.106810"
## [1] "Iter 131, obj 2.105413, abs 0.000070, rel 0.000033, norm 0.106831"
## [1] "Iter 132, obj 2.105348, abs 0.000064, rel 0.000031, norm 0.106856"
## [1] "Iter 133, obj 2.105287, abs 0.000061, rel 0.000029, norm 0.106883"
## [1] "Iter 134, obj 2.105226, abs 0.000061, rel 0.000029, norm 0.106911"
## [1] "Iter 135, obj 2.105156, abs 0.000070, rel 0.000033, norm 0.106939"
## [1] "Iter 136, obj 2.105057, abs 0.000099, rel 0.000047, norm 0.106971"
## [1] "Iter 137, obj 2.104973, abs 0.000083, rel 0.000040, norm 0.107022"
## [1] "Iter 138, obj 2.104951, abs 0.000022, rel 0.000010, norm 0.107103"
## [1] "Iter 139, obj 2.104842, abs 0.000109, rel 0.000052, norm 0.107157"
## [1] "Iter 140, obj 2.104759, abs 0.000083, rel 0.000039, norm 0.107187"
## [1] "Iter 141, obj 2.104721, abs 0.000038, rel 0.000018, norm 0.107200"
## [1] "Iter 142, obj 2.104661, abs 0.000060, rel 0.000029, norm 0.107225"
## [1] "Iter 143, obj 2.104603, abs 0.000058, rel 0.000027, norm 0.107270"
## [1] "Iter 144, obj 2.104571, abs 0.000032, rel 0.000015, norm 0.107326"
## [1] "Iter 145, obj 2.104530, abs 0.000041, rel 0.000020, norm 0.107369"
## [1] "Iter 146, obj 2.104495, abs 0.000035, rel 0.000017, norm 0.107397"
## [1] "Iter 147, obj 2.104472, abs 0.000023, rel 0.000011, norm 0.107415"
## [1] "Iter 148, obj 2.104448, abs 0.000024, rel 0.000011, norm 0.107436"
## [1] "Iter 149, obj 2.104428, abs 0.000019, rel 0.000009, norm 0.107463"
## [1] "Iter 150, obj 2.104414, abs 0.000015, rel 0.000007, norm 0.107489"
```




```
## [1] "Iter 151, obj 2.104400, abs 0.000014, rel 0.000007, norm 0.107509"
## [1] "Iter 152, obj 2.104388, abs 0.000011, rel 0.000005, norm 0.107522"
```

```
## [1] "Iter 153, obj 2.104380, abs 0.000009, rel 0.000004, norm 0.107532"
## [1] "Iter 154, obj 2.104372, abs 0.000008, rel 0.000004, norm 0.107543"
## [1] "Iter 155, obj 2.104365, abs 0.000006, rel 0.000003, norm 0.107555"
## [1] "Iter 156, obj 2.104360, abs 0.000005, rel 0.000003, norm 0.107566"
## [1] "Iter 157, obj 2.104356, abs 0.000004, rel 0.000002, norm 0.107573"
## [1] "Iter 158, obj 2.104352, abs 0.000004, rel 0.000002, norm 0.107579"
## [1] "Iter 159, obj 2.104349, abs 0.000003, rel 0.000001, norm 0.107585"
## [1] "Iter 160, obj 2.104346, abs 0.000003, rel 0.000001, norm 0.107590"
## [1] "Iter 161, obj 2.104344, abs 0.000002, rel 0.000001, norm 0.107595"
## [1] "Iter 162, obj 2.104342, abs 0.000002, rel 0.000001, norm 0.107600"
```

```
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <<- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     col=colors,
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension $\mathbb{S}^2$ then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_3, d=2, rho_psc_list = rho_third_perp25, num_iteration=200,
             colors=colors, visualize_prog = TRUE)
```

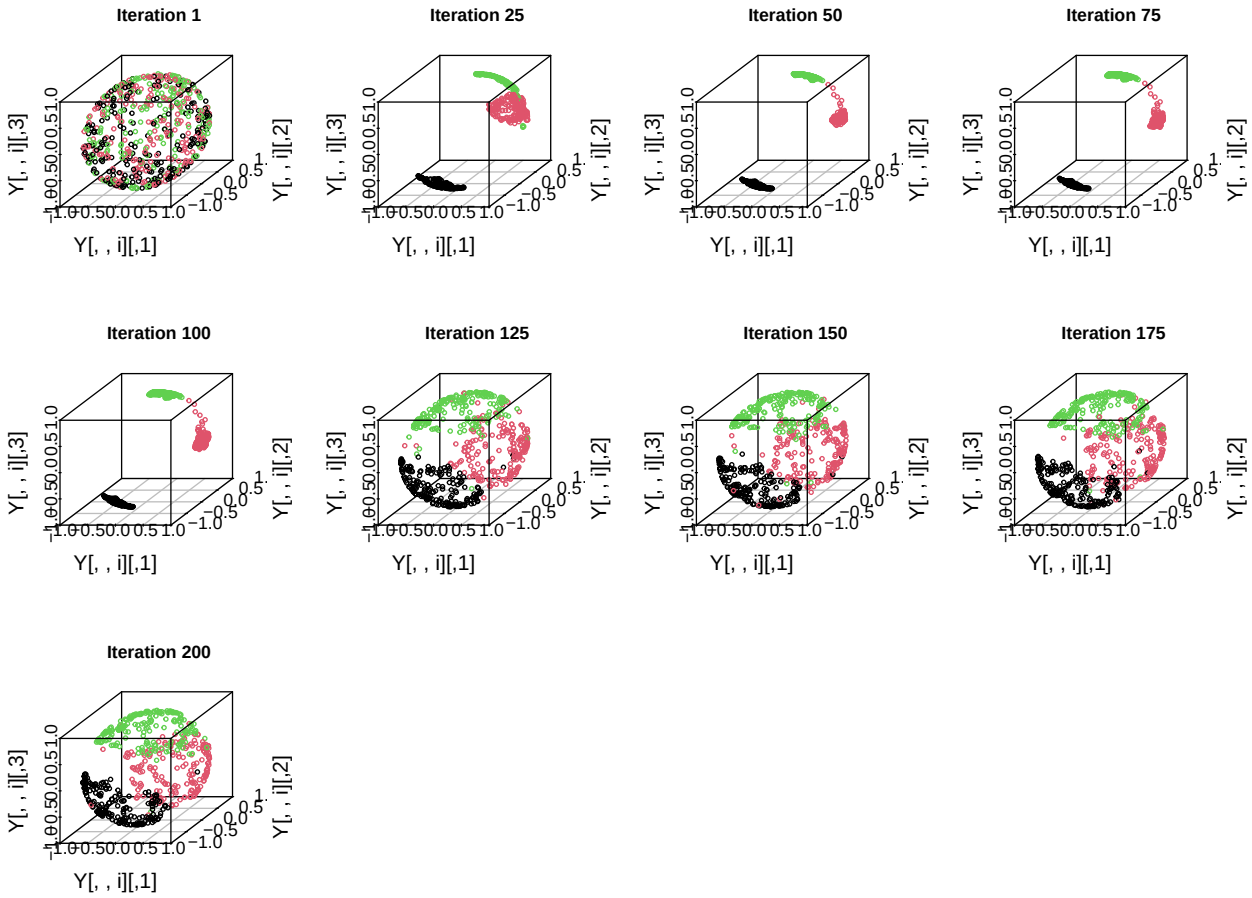
```
## [1] "Iter 1, obj 18.414509, abs 0.000000, rel 0.000000, norm 0.152510"
## [1] "Iter 2, obj 15.593380, abs 2.821129, rel 0.153201, norm 0.414414"
## [1] "Iter 3, obj 14.474232, abs 1.119147, rel 0.071771, norm 0.724148"
## [1] "Iter 4, obj 14.016106, abs 0.458126, rel 0.031651, norm 0.864717"
## [1] "Iter 5, obj 13.754339, abs 0.261768, rel 0.018676, norm 0.936997"
## [1] "Iter 6, obj 13.560583, abs 0.193756, rel 0.014087, norm 0.986281"
## [1] "Iter 7, obj 13.380244, abs 0.180339, rel 0.013299, norm 1.029919"
## [1] "Iter 8, obj 13.212788, abs 0.167456, rel 0.012515, norm 1.072909"
## [1] "Iter 9, obj 13.060316, abs 0.152471, rel 0.011540, norm 1.116460"
## [1] "Iter 10, obj 12.925761, abs 0.134555, rel 0.010303, norm 1.161212"
## [1] "Iter 11, obj 12.804249, abs 0.121512, rel 0.009401, norm 1.207936"
## [1] "Iter 12, obj 12.695088, abs 0.109161, rel 0.008525, norm 1.256790"
## [1] "Iter 13, obj 12.604461, abs 0.090627, rel 0.007139, norm 1.306226"
## [1] "Iter 14, obj 12.544584, abs 0.059877, rel 0.004750, norm 1.354070"
## [1] "Iter 15, obj 12.516075, abs 0.028509, rel 0.002273, norm 1.399280"
```

```
## [1] "Iter 16, obj 12.515374, abs 0.000702, rel 0.000056, norm 1.441160"
## [1] "Iter 17, obj 12.535351, abs 0.019978, rel 0.001596, norm 1.479739"
## [1] "Iter 18, obj 12.569280, abs 0.033929, rel 0.002707, norm 1.515909"
## [1] "Iter 19, obj 12.612098, abs 0.042818, rel 0.003407, norm 1.550304"
## [1] "Iter 20, obj 12.660729, abs 0.048631, rel 0.003856, norm 1.582664"
## [1] "Iter 21, obj 12.712279, abs 0.051549, rel 0.004072, norm 1.612352"
## [1] "Iter 22, obj 12.763563, abs 0.051284, rel 0.004034, norm 1.638971"
## [1] "Iter 23, obj 12.811642, abs 0.048079, rel 0.003767, norm 1.662640"
## [1] "Iter 24, obj 12.854693, abs 0.043051, rel 0.003360, norm 1.683745"
## [1] "Iter 25, obj 12.891998, abs 0.037305, rel 0.002902, norm 1.702616"

## [1] "Iter 26, obj 12.923372, abs 0.031373, rel 0.002434, norm 1.719418"
## [1] "Iter 27, obj 12.948872, abs 0.025500, rel 0.001973, norm 1.734222"
## [1] "Iter 28, obj 12.968826, abs 0.019955, rel 0.001541, norm 1.747136"
## [1] "Iter 29, obj 12.984457, abs 0.015630, rel 0.001205, norm 1.758334"
## [1] "Iter 30, obj 12.998957, abs 0.014500, rel 0.001117, norm 1.767926"
## [1] "Iter 31, obj 13.015717, abs 0.016760, rel 0.001289, norm 1.775804"
## [1] "Iter 32, obj 13.035282, abs 0.019565, rel 0.001503, norm 1.781803"
## [1] "Iter 33, obj 13.056438, abs 0.021156, rel 0.001623, norm 1.786042"
## [1] "Iter 34, obj 13.077954, abs 0.021516, rel 0.001648, norm 1.788802"
## [1] "Iter 35, obj 13.098866, abs 0.020912, rel 0.001599, norm 1.790332"
## [1] "Iter 36, obj 13.118444, abs 0.019578, rel 0.001495, norm 1.790841"
## [1] "Iter 37, obj 13.136205, abs 0.017761, rel 0.001354, norm 1.790528"
## [1] "Iter 38, obj 13.151906, abs 0.015701, rel 0.001195, norm 1.789580"
## [1] "Iter 39, obj 13.165493, abs 0.013588, rel 0.001033, norm 1.788161"
## [1] "Iter 40, obj 13.177045, abs 0.011551, rel 0.000877, norm 1.786407"
## [1] "Iter 41, obj 13.186710, abs 0.009665, rel 0.000733, norm 1.784421"
## [1] "Iter 42, obj 13.194670, abs 0.007961, rel 0.000604, norm 1.782285"
## [1] "Iter 43, obj 13.201117, abs 0.006446, rel 0.000489, norm 1.780056"
## [1] "Iter 44, obj 13.206231, abs 0.005114, rel 0.000387, norm 1.777778"
## [1] "Iter 45, obj 13.210180, abs 0.003949, rel 0.000299, norm 1.775482"
## [1] "Iter 46, obj 13.213115, abs 0.002935, rel 0.000222, norm 1.773189"
## [1] "Iter 47, obj 13.215169, abs 0.002054, rel 0.000155, norm 1.770915"
## [1] "Iter 48, obj 13.216458, abs 0.001289, rel 0.000098, norm 1.768671"
## [1] "Iter 49, obj 13.217083, abs 0.000625, rel 0.000047, norm 1.766465"
## [1] "Iter 50, obj 13.217134, abs 0.000051, rel 0.000004, norm 1.764302"

## [1] "Iter 51, obj 13.216688, abs 0.000446, rel 0.000034, norm 1.762185"
## [1] "Iter 52, obj 13.215813, abs 0.000875, rel 0.000066, norm 1.760115"
## [1] "Iter 53, obj 13.214569, abs 0.001244, rel 0.000094, norm 1.758095"
## [1] "Iter 54, obj 13.213008, abs 0.001561, rel 0.000118, norm 1.756124"
## [1] "Iter 55, obj 13.211177, abs 0.001831, rel 0.000139, norm 1.754203"
## [1] "Iter 56, obj 13.209116, abs 0.002061, rel 0.000156, norm 1.752330"
## [1] "Iter 57, obj 13.206862, abs 0.002254, rel 0.000171, norm 1.750506"
## [1] "Iter 58, obj 13.204447, abs 0.002415, rel 0.000183, norm 1.748728"
## [1] "Iter 59, obj 13.201898, abs 0.002548, rel 0.000193, norm 1.746998"
## [1] "Iter 60, obj 13.199242, abs 0.002656, rel 0.000201, norm 1.745312"
## [1] "Iter 61, obj 13.196500, abs 0.002742, rel 0.000208, norm 1.743670"
## [1] "Iter 62, obj 13.193692, abs 0.002808, rel 0.000213, norm 1.742072"
## [1] "Iter 63, obj 13.190836, abs 0.002857, rel 0.000217, norm 1.740516"
## [1] "Iter 64, obj 13.187946, abs 0.002890, rel 0.000219, norm 1.739000"
## [1] "Iter 65, obj 13.185036, abs 0.002910, rel 0.000221, norm 1.737524"
## [1] "Iter 66, obj 13.182117, abs 0.002918, rel 0.000221, norm 1.736087"
## [1] "Iter 67, obj 13.179201, abs 0.002916, rel 0.000221, norm 1.734687"
## [1] "Iter 68, obj 13.176295, abs 0.002906, rel 0.000220, norm 1.733324"
```

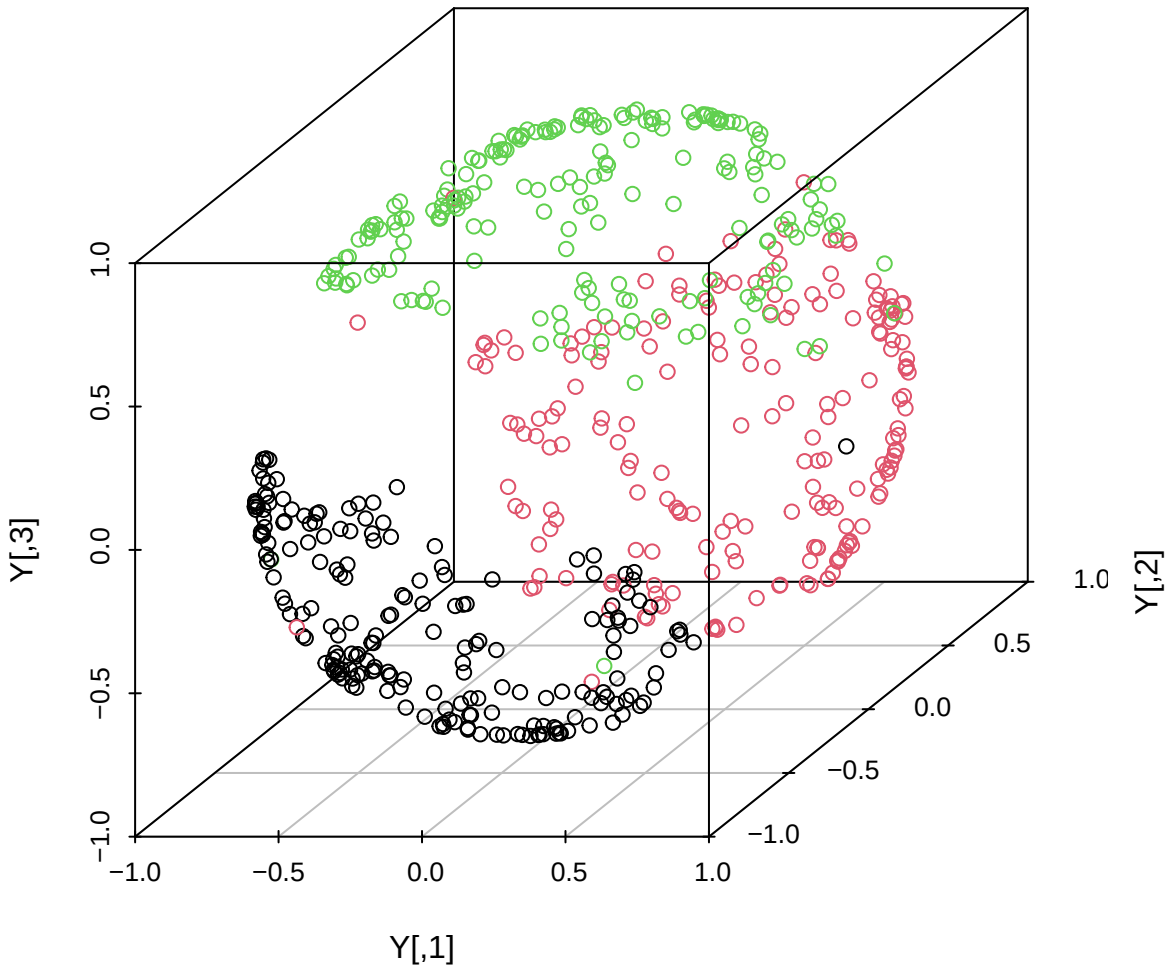
```
## [1] "Iter 69, obj 13.173408, abs 0.002888, rel 0.000219, norm 1.731996"
## [1] "Iter 70, obj 13.170545, abs 0.002863, rel 0.000217, norm 1.730702"
## [1] "Iter 71, obj 13.167713, abs 0.002832, rel 0.000215, norm 1.729443"
## [1] "Iter 72, obj 13.164915, abs 0.002797, rel 0.000212, norm 1.728215"
## [1] "Iter 73, obj 13.162156, abs 0.002759, rel 0.000210, norm 1.727019"
## [1] "Iter 74, obj 13.159440, abs 0.002717, rel 0.000206, norm 1.725854"
## [1] "Iter 75, obj 13.156768, abs 0.002672, rel 0.000203, norm 1.724719"
## [1] "Iter 76, obj 13.154142, abs 0.002625, rel 0.000200, norm 1.723612"
## [1] "Iter 77, obj 13.151565, abs 0.002577, rel 0.000196, norm 1.722534"
## [1] "Iter 78, obj 13.149037, abs 0.002528, rel 0.000192, norm 1.721483"
## [1] "Iter 79, obj 13.146559, abs 0.002478, rel 0.000188, norm 1.720458"
## [1] "Iter 80, obj 13.144132, abs 0.002427, rel 0.000185, norm 1.719459"
## [1] "Iter 81, obj 13.141755, abs 0.002377, rel 0.000181, norm 1.718484"
## [1] "Iter 82, obj 13.139429, abs 0.002326, rel 0.000177, norm 1.717534"
## [1] "Iter 83, obj 13.137153, abs 0.002276, rel 0.000173, norm 1.716608"
## [1] "Iter 84, obj 13.134926, abs 0.002226, rel 0.000169, norm 1.715704"
## [1] "Iter 85, obj 13.132749, abs 0.002177, rel 0.000166, norm 1.714822"
## [1] "Iter 86, obj 13.130621, abs 0.002129, rel 0.000162, norm 1.713961"
## [1] "Iter 87, obj 13.128539, abs 0.002081, rel 0.000159, norm 1.713121"
## [1] "Iter 88, obj 13.126504, abs 0.002035, rel 0.000155, norm 1.712301"
## [1] "Iter 89, obj 13.124515, abs 0.001989, rel 0.000152, norm 1.711501"
## [1] "Iter 90, obj 13.122570, abs 0.001945, rel 0.000148, norm 1.710720"
## [1] "Iter 91, obj 13.120669, abs 0.001902, rel 0.000145, norm 1.709957"
## [1] "Iter 92, obj 13.118809, abs 0.001860, rel 0.000142, norm 1.709212"
## [1] "Iter 93, obj 13.116990, abs 0.001819, rel 0.000139, norm 1.708484"
## [1] "Iter 94, obj 13.115211, abs 0.001779, rel 0.000136, norm 1.707773"
## [1] "Iter 95, obj 13.113471, abs 0.001740, rel 0.000133, norm 1.707078"
## [1] "Iter 96, obj 13.111768, abs 0.001703, rel 0.000130, norm 1.706399"
## [1] "Iter 97, obj 13.110101, abs 0.001667, rel 0.000127, norm 1.705736"
## [1] "Iter 98, obj 13.108468, abs 0.001632, rel 0.000125, norm 1.705087"
## [1] "Iter 99, obj 13.106869, abs 0.001599, rel 0.000122, norm 1.704452"
## [1] "Iter 100, obj 13.105303, abs 0.001566, rel 0.000119, norm 1.703832"
## [1] "Iter 101, obj 1.838502, abs 11.266802, rel 0.859713, norm 0.122625"
## [1] "Iter 102, obj 1.654550, abs 0.183951, rel 0.100055, norm 0.127722"
## [1] "Iter 103, obj 1.602375, abs 0.052175, rel 0.031534, norm 0.145909"
## [1] "Iter 104, obj 1.592316, abs 0.010059, rel 0.006278, norm 0.153351"
## [1] "Iter 105, obj 1.558562, abs 0.033754, rel 0.021198, norm 0.152375"
## [1] "Iter 106, obj 1.550815, abs 0.007747, rel 0.004971, norm 0.151486"
## [1] "Iter 107, obj 1.546437, abs 0.004378, rel 0.002823, norm 0.149915"
## [1] "Iter 108, obj 1.540021, abs 0.006416, rel 0.004149, norm 0.147161"
## [1] "Iter 109, obj 1.538240, abs 0.001782, rel 0.001157, norm 0.146966"
## [1] "Iter 110, obj 1.529381, abs 0.008859, rel 0.005759, norm 0.147136"
## [1] "Iter 111, obj 1.541588, abs 0.012207, rel 0.007982, norm 0.147508"
## [1] "Iter 112, obj 1.525786, abs 0.015801, rel 0.010250, norm 0.146502"
## [1] "Iter 113, obj 1.527787, abs 0.002000, rel 0.001311, norm 0.146262"
## [1] "Iter 114, obj 1.535825, abs 0.008039, rel 0.005262, norm 0.146809"
## [1] "Iter 115, obj 1.523264, abs 0.012561, rel 0.008179, norm 0.146281"
## [1] "Iter 116, obj 1.512451, abs 0.010813, rel 0.007099, norm 0.146557"
## [1] "Iter 117, obj 1.517740, abs 0.005289, rel 0.003497, norm 0.146280"
## [1] "Iter 118, obj 1.515255, abs 0.002485, rel 0.001637, norm 0.146324"
## [1] "Iter 119, obj 1.503275, abs 0.011980, rel 0.007906, norm 0.145975"
## [1] "Iter 120, obj 1.501598, abs 0.001677, rel 0.001116, norm 0.146284"
## [1] "Iter 121, obj 1.513725, abs 0.012127, rel 0.008076, norm 0.145645"
```

```
## [1] "Iter 122, obj 1.503134, abs 0.010591, rel 0.006997, norm 0.145427"
## [1] "Iter 123, obj 1.484804, abs 0.018331, rel 0.012195, norm 0.146060"
## [1] "Iter 124, obj 1.481042, abs 0.003762, rel 0.002534, norm 0.144906"
## [1] "Iter 125, obj 1.493281, abs 0.012239, rel 0.008264, norm 0.145527"
## [1] "Iter 126, obj 1.510788, abs 0.017507, rel 0.011724, norm 0.146075"
## [1] "Iter 127, obj 1.498795, abs 0.011994, rel 0.007939, norm 0.145100"
## [1] "Iter 128, obj 1.478446, abs 0.020349, rel 0.013577, norm 0.144265"
## [1] "Iter 129, obj 1.488382, abs 0.009936, rel 0.006720, norm 0.145899"
## [1] "Iter 130, obj 1.470979, abs 0.017403, rel 0.011693, norm 0.144625"
## [1] "Iter 131, obj 1.479945, abs 0.008966, rel 0.006095, norm 0.143257"
## [1] "Iter 132, obj 1.501053, abs 0.021108, rel 0.014263, norm 0.144971"
## [1] "Iter 133, obj 1.512960, abs 0.011907, rel 0.007932, norm 0.146060"
## [1] "Iter 134, obj 1.494842, abs 0.018118, rel 0.011975, norm 0.145028"
## [1] "Iter 135, obj 1.491264, abs 0.003577, rel 0.002393, norm 0.145009"
## [1] "Iter 136, obj 1.492333, abs 0.001069, rel 0.000717, norm 0.144822"
## [1] "Iter 137, obj 1.484032, abs 0.008301, rel 0.005562, norm 0.145053"
## [1] "Iter 138, obj 1.493313, abs 0.009281, rel 0.006254, norm 0.144842"
## [1] "Iter 139, obj 1.496192, abs 0.002879, rel 0.001928, norm 0.146050"
## [1] "Iter 140, obj 1.493668, abs 0.002523, rel 0.001686, norm 0.144952"
## [1] "Iter 141, obj 1.503953, abs 0.010285, rel 0.006886, norm 0.145848"
## [1] "Iter 142, obj 1.493146, abs 0.010808, rel 0.007186, norm 0.145643"
## [1] "Iter 143, obj 1.494948, abs 0.001802, rel 0.001207, norm 0.144955"
## [1] "Iter 144, obj 1.486758, abs 0.008190, rel 0.005478, norm 0.145663"
## [1] "Iter 145, obj 1.481052, abs 0.005706, rel 0.003838, norm 0.146589"
## [1] "Iter 146, obj 1.491867, abs 0.010815, rel 0.007302, norm 0.146305"
## [1] "Iter 147, obj 1.478161, abs 0.013706, rel 0.009187, norm 0.146666"
## [1] "Iter 148, obj 1.462393, abs 0.015768, rel 0.010668, norm 0.146546"
## [1] "Iter 149, obj 1.476288, abs 0.013894, rel 0.009501, norm 0.146266"
## [1] "Iter 150, obj 1.486595, abs 0.010307, rel 0.006982, norm 0.146747"
## [1] "Iter 151, obj 1.486721, abs 0.000126, rel 0.000085, norm 0.146139"
## [1] "Iter 152, obj 1.488006, abs 0.001286, rel 0.000865, norm 0.146539"
## [1] "Iter 153, obj 1.484170, abs 0.003836, rel 0.002578, norm 0.146481"
## [1] "Iter 154, obj 1.482877, abs 0.001293, rel 0.000871, norm 0.146035"
## [1] "Iter 155, obj 1.498094, abs 0.015216, rel 0.010261, norm 0.146227"
## [1] "Iter 156, obj 1.503899, abs 0.005805, rel 0.003875, norm 0.147247"
## [1] "Iter 157, obj 1.478188, abs 0.025711, rel 0.017096, norm 0.146583"
## [1] "Iter 158, obj 1.470931, abs 0.007257, rel 0.004909, norm 0.145376"
## [1] "Iter 159, obj 1.489687, abs 0.018756, rel 0.012751, norm 0.146650"
## [1] "Iter 160, obj 1.487386, abs 0.002301, rel 0.001545, norm 0.145704"
## [1] "Iter 161, obj 1.470433, abs 0.016953, rel 0.011398, norm 0.145505"
## [1] "Iter 162, obj 1.468620, abs 0.001812, rel 0.001232, norm 0.146310"
## [1] "Iter 163, obj 1.469642, abs 0.001022, rel 0.000696, norm 0.144150"
## [1] "Iter 164, obj 1.462030, abs 0.007612, rel 0.005180, norm 0.144922"
## [1] "Iter 165, obj 1.479966, abs 0.017936, rel 0.012268, norm 0.145685"
## [1] "Iter 166, obj 1.480709, abs 0.000743, rel 0.000502, norm 0.144665"
## [1] "Iter 167, obj 1.492313, abs 0.011604, rel 0.007837, norm 0.145692"
## [1] "Iter 168, obj 1.463820, abs 0.028493, rel 0.019093, norm 0.145555"
## [1] "Iter 169, obj 1.474493, abs 0.010673, rel 0.007291, norm 0.145226"
## [1] "Iter 170, obj 1.498631, abs 0.024138, rel 0.016370, norm 0.145883"
## [1] "Iter 171, obj 1.483720, abs 0.014910, rel 0.009949, norm 0.144573"
## [1] "Iter 172, obj 1.471678, abs 0.012043, rel 0.008116, norm 0.145711"
## [1] "Iter 173, obj 1.484355, abs 0.012677, rel 0.008614, norm 0.144183"
## [1] "Iter 174, obj 1.490011, abs 0.005656, rel 0.003811, norm 0.145217"
```

```
## [1] "Iter 175, obj 1.484988, abs 0.005023, rel 0.003371, norm 0.145191"
## [1] "Iter 176, obj 1.489299, abs 0.004311, rel 0.002903, norm 0.144817"
## [1] "Iter 177, obj 1.503339, abs 0.014041, rel 0.009428, norm 0.144686"
## [1] "Iter 178, obj 1.499196, abs 0.004143, rel 0.002756, norm 0.144963"
## [1] "Iter 179, obj 1.487397, abs 0.011799, rel 0.007871, norm 0.145369"
## [1] "Iter 180, obj 1.503590, abs 0.016193, rel 0.010887, norm 0.145484"
## [1] "Iter 181, obj 1.491428, abs 0.012162, rel 0.008089, norm 0.144493"
## [1] "Iter 182, obj 1.481349, abs 0.010079, rel 0.006758, norm 0.144707"
## [1] "Iter 183, obj 1.499371, abs 0.018022, rel 0.012166, norm 0.146288"
## [1] "Iter 184, obj 1.499360, abs 0.000010, rel 0.000007, norm 0.144532"
## [1] "Iter 185, obj 1.487713, abs 0.011648, rel 0.007769, norm 0.144640"
## [1] "Iter 186, obj 1.508889, abs 0.021176, rel 0.014234, norm 0.145542"
## [1] "Iter 187, obj 1.488590, abs 0.020298, rel 0.013452, norm 0.145255"
## [1] "Iter 188, obj 1.493257, abs 0.004667, rel 0.003135, norm 0.145163"
## [1] "Iter 189, obj 1.480033, abs 0.013224, rel 0.008856, norm 0.145152"
## [1] "Iter 190, obj 1.469417, abs 0.010616, rel 0.007173, norm 0.145335"
## [1] "Iter 191, obj 1.475326, abs 0.005909, rel 0.004021, norm 0.144773"
## [1] "Iter 192, obj 1.479679, abs 0.004353, rel 0.002951, norm 0.144540"
## [1] "Iter 193, obj 1.471027, abs 0.008653, rel 0.005848, norm 0.145646"
## [1] "Iter 194, obj 1.455385, abs 0.015642, rel 0.010633, norm 0.144150"
## [1] "Iter 195, obj 1.470575, abs 0.015190, rel 0.010437, norm 0.145333"
## [1] "Iter 196, obj 1.464787, abs 0.005788, rel 0.003936, norm 0.144804"
## [1] "Iter 197, obj 1.462460, abs 0.002327, rel 0.001589, norm 0.144770"
## [1] "Iter 198, obj 1.482076, abs 0.019616, rel 0.013413, norm 0.145184"
## [1] "Iter 199, obj 1.476695, abs 0.005381, rel 0.003631, norm 0.145246"
## [1] "Iter 200, obj 1.460362, abs 0.016333, rel 0.011060, norm 0.144687"
```

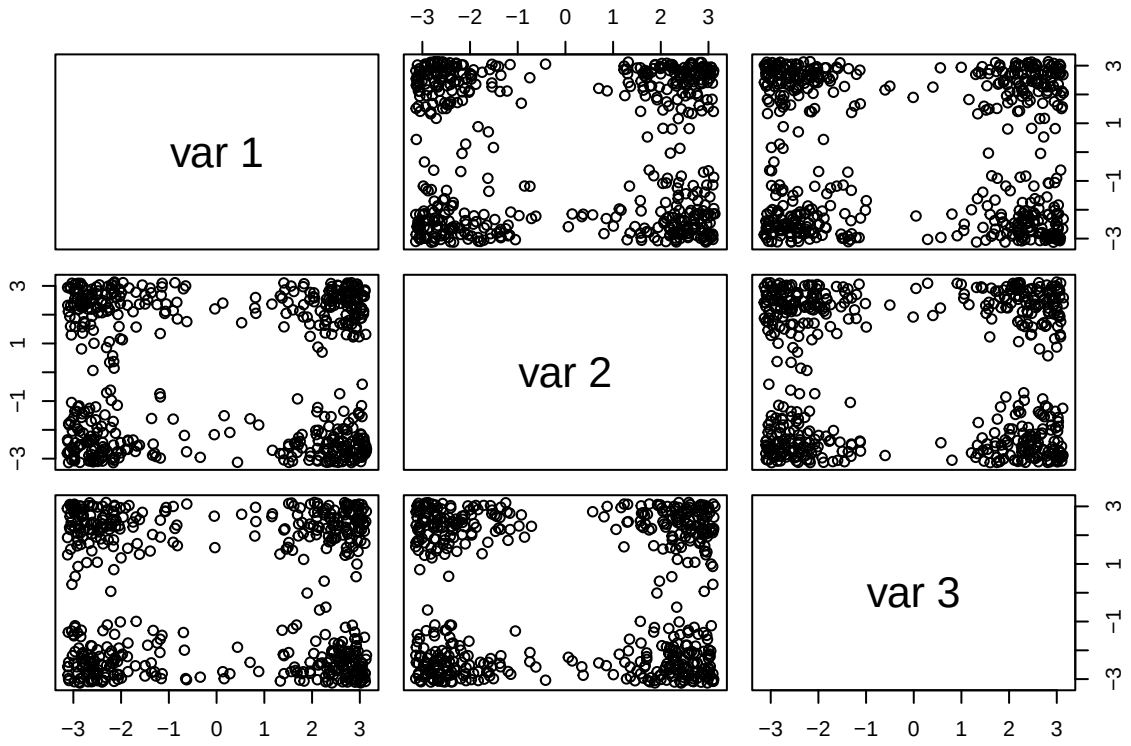



```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1),
                             color = colors)
```

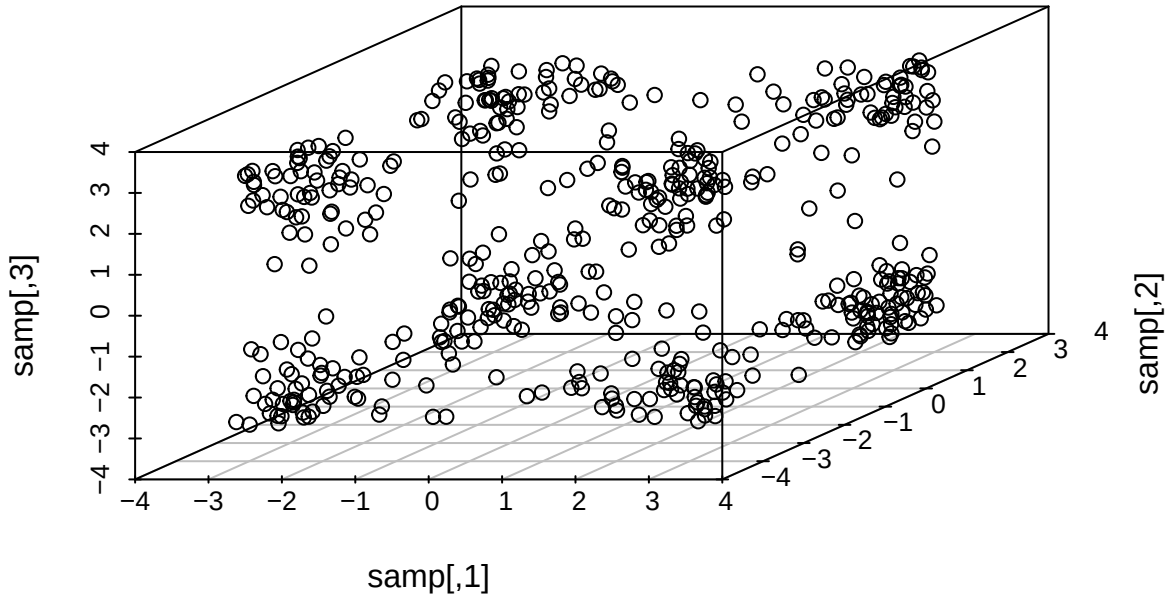


Case 4

```
d <- 3
n <- 5e2
samp <- sdetorus::toPiInt(mvtnorm::rmvnorm(n = n, mean = rep(pi, d)))
pairs(samp)
```




```
scatterplot3d::scatterplot3d(samp, xlim = c(-pi, pi), ylim = c(-pi, pi), zlim = c(-pi, pi))
```




```
x_4 <- array(dim = c(n, 2, 3))
x_4[, , 1] <- DirStats::to_cir(samp[, 1])
x_4[, , 2] <- DirStats::to_cir(samp[, 2])
x_4[, , 3] <- DirStats::to_cir(samp[, 3])
```

Let's calculate the rho parameters based on a perplexity of 25:

```
rho_four_perp20 <- rho_optim_bst(x_4, 20)
```

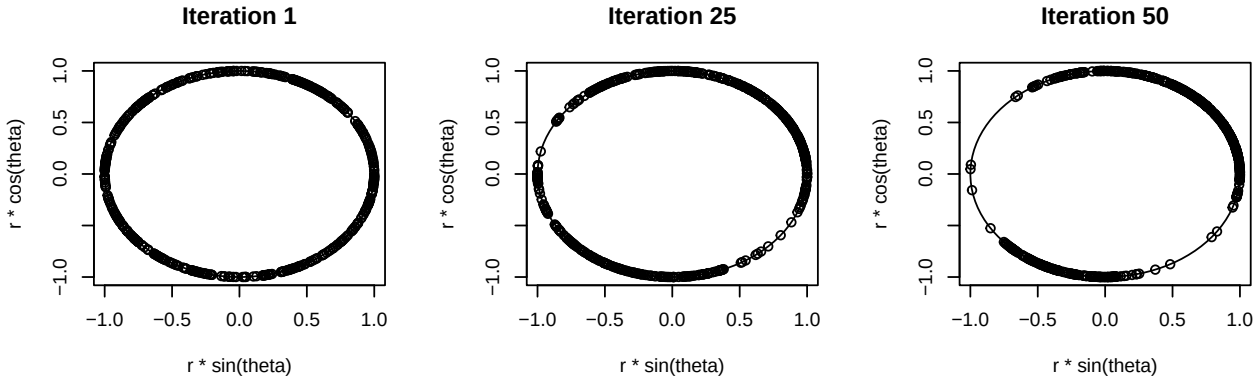
```
## Time difference of 1.02358 mins
```

First, let's reduce to dimension $\mathbb{S}^1$ then $d = 1$ (circumference):

```
Y <- psc_sne(X=x_4, d=1, rho_psc_list = rho_four_perp20, num_iteration=200,
             visualize_prog = TRUE)
```

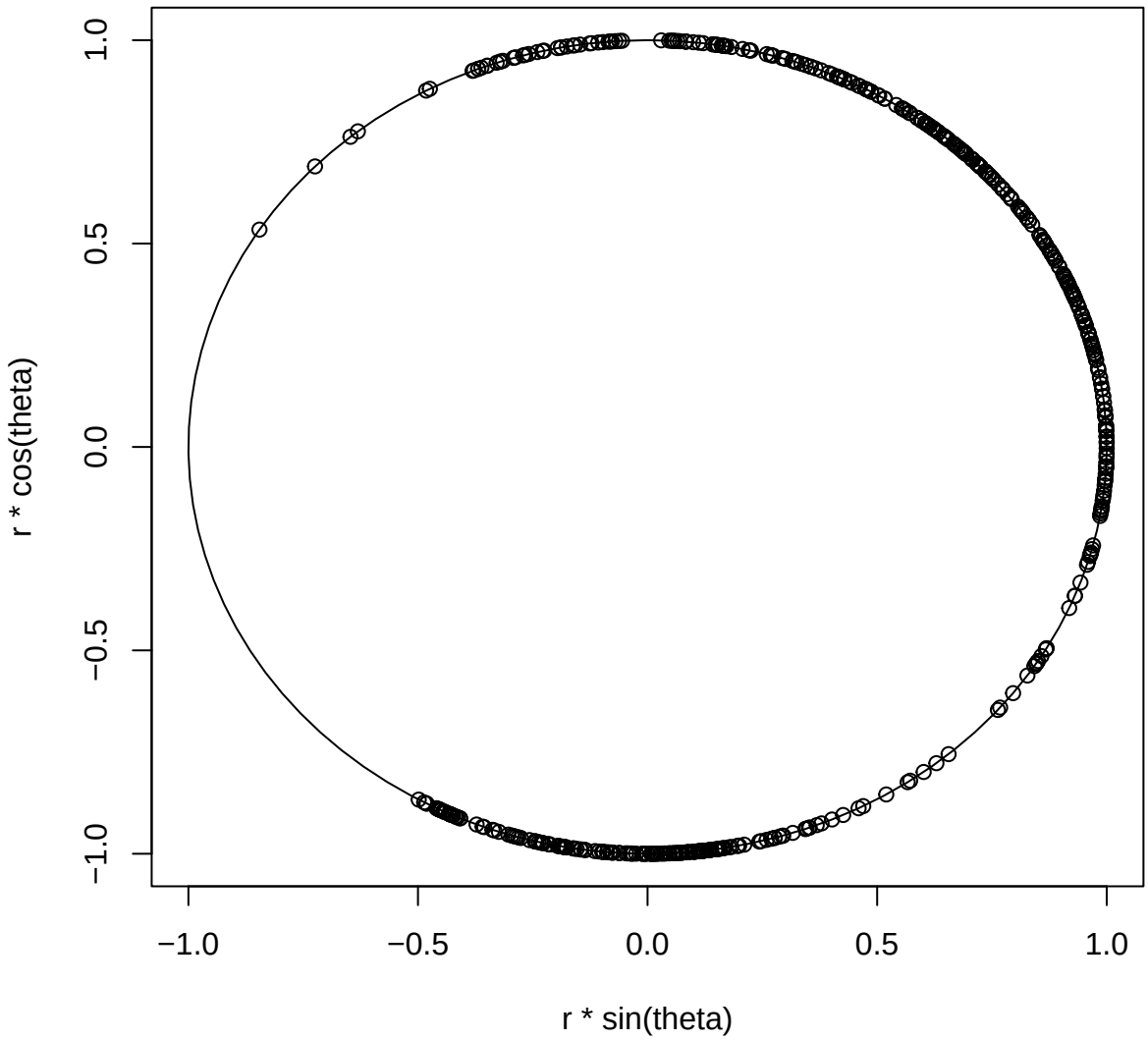
```
## [1] "Iter 1, obj 17.767040, abs 0.000000, rel 0.000000, norm 0.157928"
## [1] "Iter 2, obj 17.120901, abs 0.646139, rel 0.036367, norm 0.307608"
## [1] "Iter 3, obj 16.829121, abs 0.291779, rel 0.017042, norm 0.404195"
## [1] "Iter 4, obj 16.655518, abs 0.173603, rel 0.010316, norm 0.443175"
## [1] "Iter 5, obj 16.516471, abs 0.139048, rel 0.008348, norm 0.465753"
## [1] "Iter 6, obj 16.413421, abs 0.103049, rel 0.006239, norm 0.482790"
## [1] "Iter 7, obj 16.317255, abs 0.096166, rel 0.005859, norm 0.497138"
## [1] "Iter 8, obj 16.227795, abs 0.089460, rel 0.005483, norm 0.511084"
## [1] "Iter 9, obj 16.151800, abs 0.075995, rel 0.004683, norm 0.524683"
## [1] "Iter 10, obj 16.088035, abs 0.063765, rel 0.003948, norm 0.536100"
## [1] "Iter 11, obj 16.029547, abs 0.058487, rel 0.003635, norm 0.546008"
## [1] "Iter 12, obj 15.973520, abs 0.056027, rel 0.003495, norm 0.555845"
## [1] "Iter 13, obj 15.918447, abs 0.055074, rel 0.003448, norm 0.565922"
## [1] "Iter 14, obj 15.864790, abs 0.053657, rel 0.003371, norm 0.574627"
## [1] "Iter 15, obj 15.818398, abs 0.046391, rel 0.002924, norm 0.583187"
## [1] "Iter 16, obj 15.777491, abs 0.040908, rel 0.002586, norm 0.591857"
## [1] "Iter 17, obj 15.742493, abs 0.034997, rel 0.002218, norm 0.600809"
## [1] "Iter 18, obj 15.714744, abs 0.027749, rel 0.001763, norm 0.610577"
## [1] "Iter 19, obj 15.693074, abs 0.021670, rel 0.001379, norm 0.620842"
## [1] "Iter 20, obj 15.675810, abs 0.017264, rel 0.001100, norm 0.631140"
## [1] "Iter 21, obj 15.662211, abs 0.013599, rel 0.000868, norm 0.641451"
## [1] "Iter 22, obj 15.651172, abs 0.011039, rel 0.000705, norm 0.652211"
## [1] "Iter 23, obj 15.644876, abs 0.006296, rel 0.000402, norm 0.663779"
## [1] "Iter 24, obj 15.642796, abs 0.002080, rel 0.000133, norm 0.675496"
## [1] "Iter 25, obj 15.642889, abs 0.000093, rel 0.000006, norm 0.686900"
## [1] "Iter 26, obj 15.646082, abs 0.003193, rel 0.000204, norm 0.698258"
## [1] "Iter 27, obj 15.651869, abs 0.005786, rel 0.000370, norm 0.708968"
## [1] "Iter 28, obj 15.659227, abs 0.007359, rel 0.000470, norm 0.718737"
## [1] "Iter 29, obj 15.666263, abs 0.007036, rel 0.000449, norm 0.727991"
## [1] "Iter 30, obj 15.672175, abs 0.005912, rel 0.000377, norm 0.737158"
## [1] "Iter 31, obj 15.677313, abs 0.005139, rel 0.000328, norm 0.746068"
## [1] "Iter 32, obj 15.682328, abs 0.005015, rel 0.000320, norm 0.754601"
## [1] "Iter 33, obj 15.687776, abs 0.005448, rel 0.000347, norm 0.762755"
## [1] "Iter 34, obj 15.692795, abs 0.005019, rel 0.000320, norm 0.770238"
## [1] "Iter 35, obj 15.696704, abs 0.003909, rel 0.000249, norm 0.777043"
## [1] "Iter 36, obj 15.699326, abs 0.002621, rel 0.000167, norm 0.783198"
## [1] "Iter 37, obj 15.700769, abs 0.001443, rel 0.000092, norm 0.788704"
## [1] "Iter 38, obj 15.701171, abs 0.000402, rel 0.000026, norm 0.793533"
## [1] "Iter 39, obj 15.700112, abs 0.001058, rel 0.000067, norm 0.797693"
## [1] "Iter 40, obj 15.695653, abs 0.004459, rel 0.000284, norm 0.801273"
## [1] "Iter 41, obj 15.692667, abs 0.002987, rel 0.000190, norm 0.804412"
## [1] "Iter 42, obj 15.691093, abs 0.001573, rel 0.000100, norm 0.807050"
## [1] "Iter 43, obj 15.689998, abs 0.001096, rel 0.000070, norm 0.809297"
## [1] "Iter 44, obj 15.689214, abs 0.000784, rel 0.000050, norm 0.811235"
## [1] "Iter 45, obj 15.688538, abs 0.000676, rel 0.000043, norm 0.812875"
## [1] "Iter 46, obj 15.687771, abs 0.000768, rel 0.000049, norm 0.814196"
## [1] "Iter 47, obj 15.686742, abs 0.001029, rel 0.000066, norm 0.815175"
## [1] "Iter 48, obj 15.685329, abs 0.001412, rel 0.000090, norm 0.815799"
## [1] "Iter 49, obj 15.683472, abs 0.001858, rel 0.000118, norm 0.816069"
```

```
## [1] "Iter 50, obj 15.681159, abs 0.002313, rel 0.000147, norm 0.815995"
```




```
## [1] "Iter 51, obj 15.678413, abs 0.002746, rel 0.000175, norm 0.815591"
## [1] "Iter 52, obj 15.675267, abs 0.003146, rel 0.000201, norm 0.814878"
## [1] "Iter 53, obj 15.671751, abs 0.003515, rel 0.000224, norm 0.813882"
## [1] "Iter 54, obj 15.667909, abs 0.003842, rel 0.000245, norm 0.812637"
## [1] "Iter 55, obj 15.663825, abs 0.004084, rel 0.000261, norm 0.811185"
## [1] "Iter 56, obj 15.659664, abs 0.004161, rel 0.000266, norm 0.809588"
## [1] "Iter 57, obj 15.655680, abs 0.003984, rel 0.000254, norm 0.807923"
## [1] "Iter 58, obj 15.652087, abs 0.003593, rel 0.000229, norm 0.806259"
## [1] "Iter 59, obj 15.648965, abs 0.003122, rel 0.000199, norm 0.804612"
## [1] "Iter 60, obj 15.646490, abs 0.002475, rel 0.000158, norm 0.802937"
## [1] "Iter 61, obj 15.645052, abs 0.001438, rel 0.000092, norm 0.801206"
## [1] "Iter 62, obj 15.645049, abs 0.000003, rel 0.000000, norm 0.799406"
```

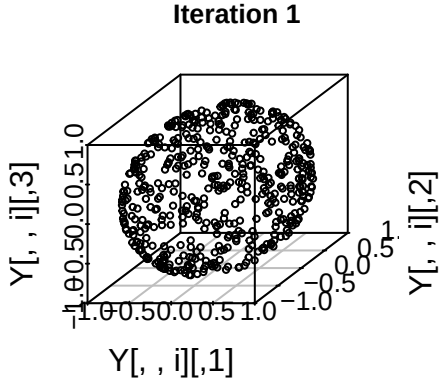
```r
Y_rad <- DirStats::to_rad(Y)
r <- 1
theta <<- Y_rad
plot(r*sin(theta),
     r*cos(theta),
     xlim=c(-max(r),max(r)),
     ylim=c(-max(r),max(r)))

polygon(max(r)*sin(seq(0,2*pi,length.out=100)),max(r)*cos(seq(0,2*pi,length.out=100)))
```



Now we are going to reduce to dimension $\mathbb{S}^2$ then $d = 2$ (sphere):

```
Y <- psc_sne(X=x_4, d=2, rho_psc_list = rho_four_perp20, num_iteration=200,
             visualize_prog = TRUE)
```

```
## [1] "Iter 1, obj 19.364487, abs 0.000000, rel 0.000000, norm 0.198669"
```




```
## [1] "Iter 2, obj 16.866188, abs 2.498298, rel 0.129014, norm 0.332528"
## [1] "Iter 3, obj 15.438307, abs 1.427881, rel 0.084659, norm 0.595939"
```

```
## [1] "Iter 4, obj 14.839717, abs 0.598590, rel 0.038773, norm 0.765845"
## [1] "Iter 5, obj 14.491651, abs 0.348066, rel 0.023455, norm 0.849938"
## [1] "Iter 6, obj 14.263590, abs 0.228061, rel 0.015737, norm 0.907512"
## [1] "Iter 7, obj 14.130766, abs 0.132825, rel 0.009312, norm 0.944548"
## [1] "Iter 8, obj 14.030340, abs 0.100425, rel 0.007107, norm 0.970364"
## [1] "Iter 9, obj 13.967466, abs 0.062874, rel 0.004481, norm 0.991111"
## [1] "Iter 10, obj 13.912294, abs 0.055172, rel 0.003950, norm 1.009228"
## [1] "Iter 11, obj 13.872653, abs 0.039642, rel 0.002849, norm 1.026089"
## [1] "Iter 12, obj 13.839976, abs 0.032676, rel 0.002355, norm 1.042185"
## [1] "Iter 13, obj 13.816282, abs 0.023694, rel 0.001712, norm 1.057741"
## [1] "Iter 14, obj 13.794988, abs 0.021294, rel 0.001541, norm 1.072928"
## [1] "Iter 15, obj 13.777032, abs 0.017957, rel 0.001302, norm 1.087916"
## [1] "Iter 16, obj 13.759148, abs 0.017883, rel 0.001298, norm 1.102776"
## [1] "Iter 17, obj 13.743710, abs 0.015438, rel 0.001122, norm 1.118008"
## [1] "Iter 18, obj 13.729551, abs 0.014159, rel 0.001030, norm 1.133205"
## [1] "Iter 19, obj 13.718612, abs 0.010939, rel 0.000797, norm 1.148912"
## [1] "Iter 20, obj 13.710095, abs 0.008517, rel 0.000621, norm 1.164189"
## [1] "Iter 21, obj 13.704714, abs 0.005382, rel 0.000393, norm 1.179620"
## [1] "Iter 22, obj 13.701804, abs 0.002909, rel 0.000212, norm 1.193888"
## [1] "Iter 23, obj 13.701797, abs 0.000008, rel 0.000001, norm 1.207757"
```

```
scatterplot3d::scatterplot3d(Y, xlim = c(-1, 1), ylim = c(-1, 1), zlim = c(-1, 1))
```