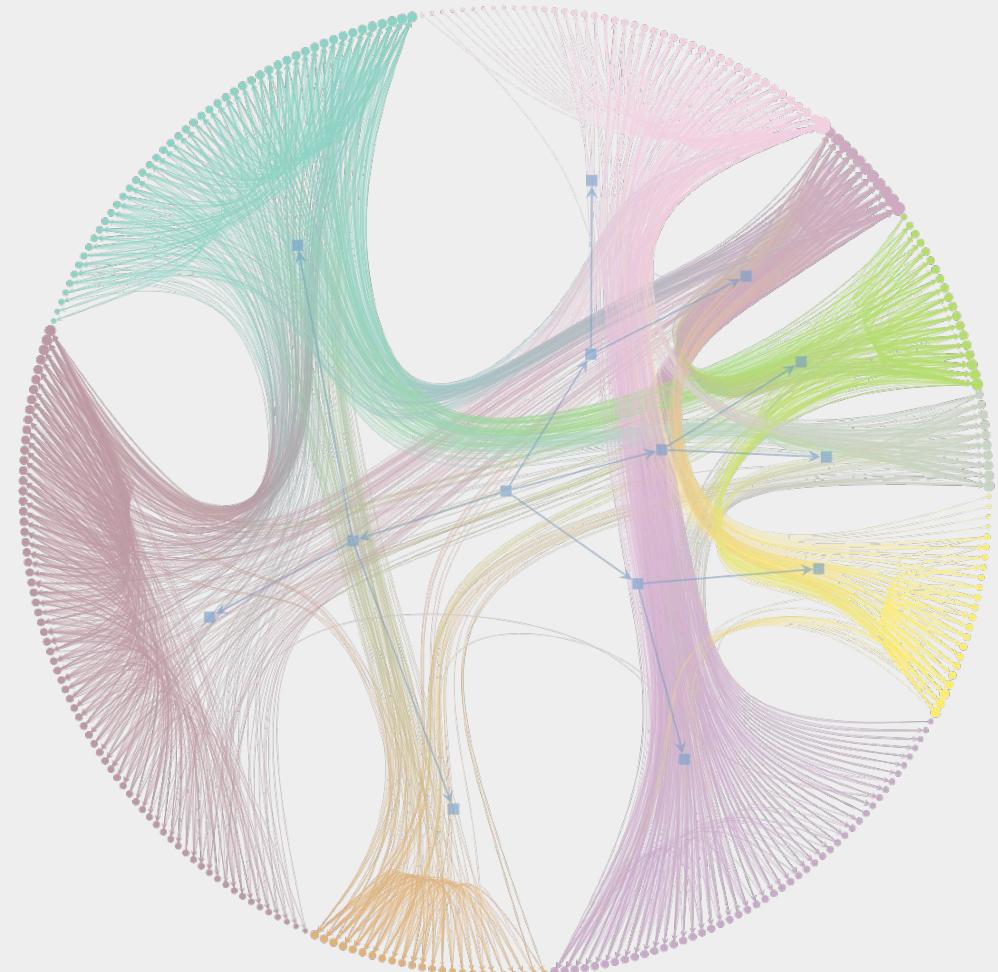


Network Elements

IMD 1155 - Network Analysis
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[@ivanovitchm](https://twitter.com/ivanovitchm)



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Basic Definitions

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Extended Graphs

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Density and Sparsity,
Subnetworks

Network Science is a Data Science

04

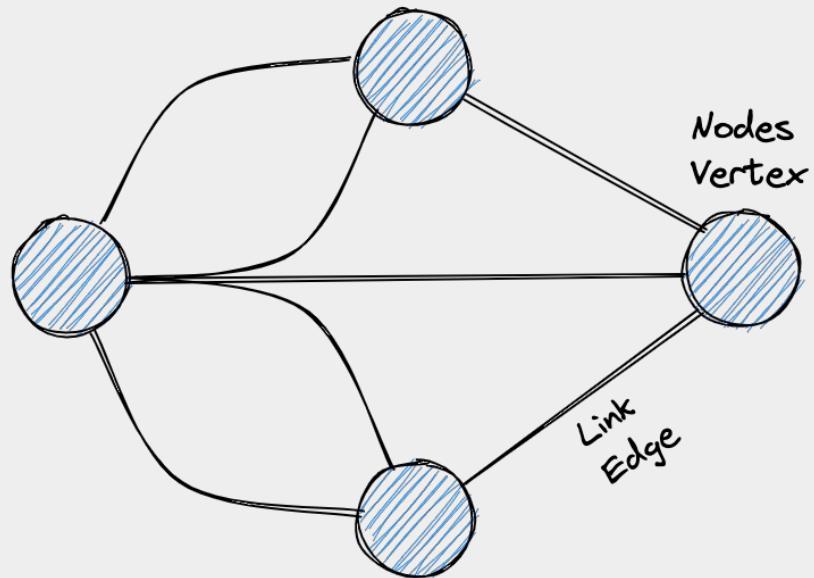
Degree and Network
Representations

05

Further Reading

Basic Definition

Network or Graph



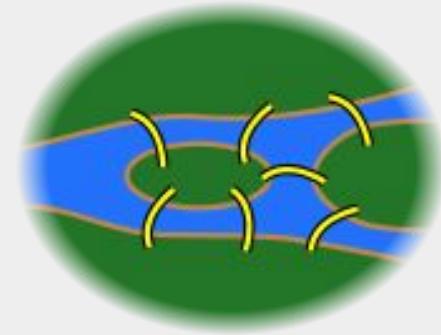
In very general terms a network, or graph, is a set of elements, which we call **nodes**, along with a set of connections between pairs of nodes, which we call **links**. The links represent the presence of a relation among the elements represented by the nodes.

Leonhard Euler, 1735



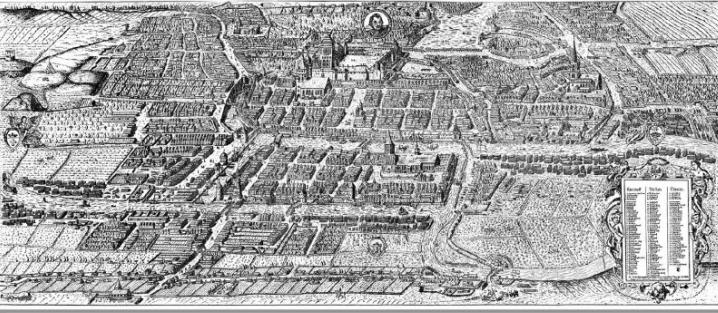
Seven Bridges of Königsberg

Can one walk across all seven bridges and never cross the same one twice?

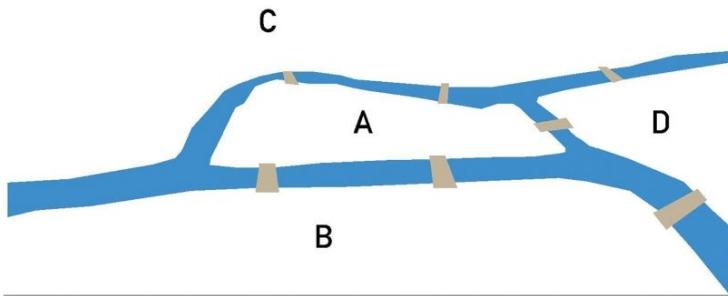


The rigorous language for the description of networks is found in graph theory, a field of mathematics that can be traced back to the pioneering work of Leonhard Euler in the eighteenth century.

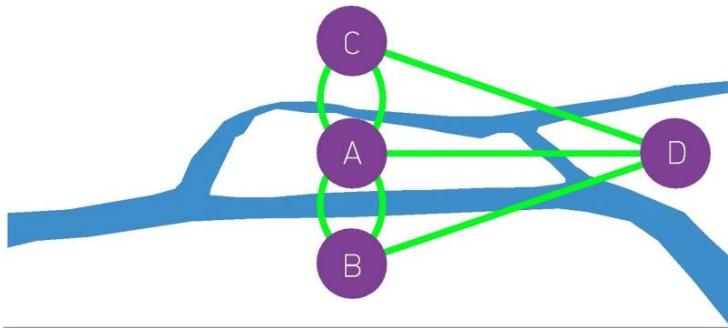
a.



b.



c.

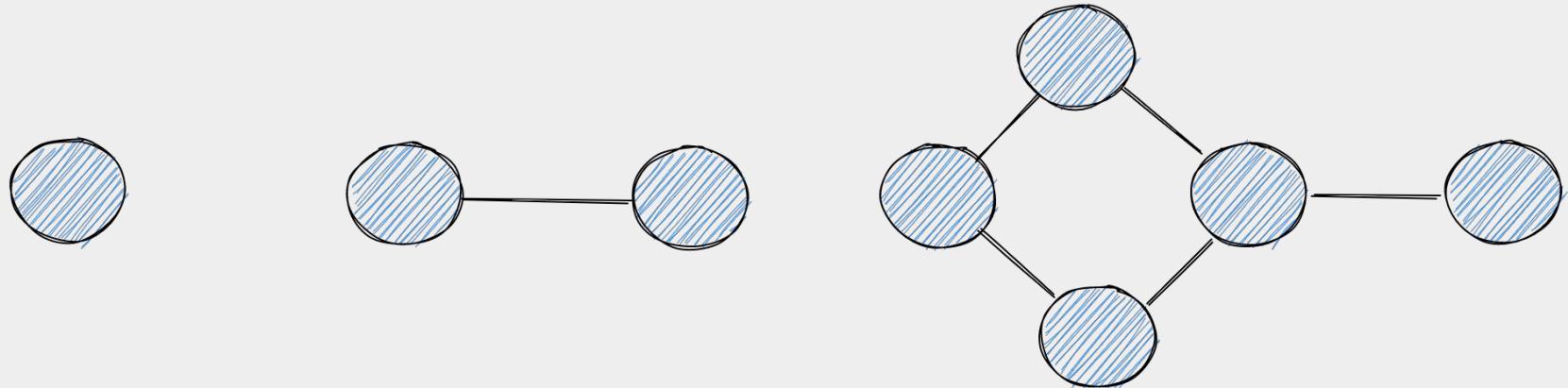


Despite many attempts, no one could find such path. The problem remained unsolved until 1735, when Leonhard Euler, offered a rigorous mathematical proof that such path does not exist.



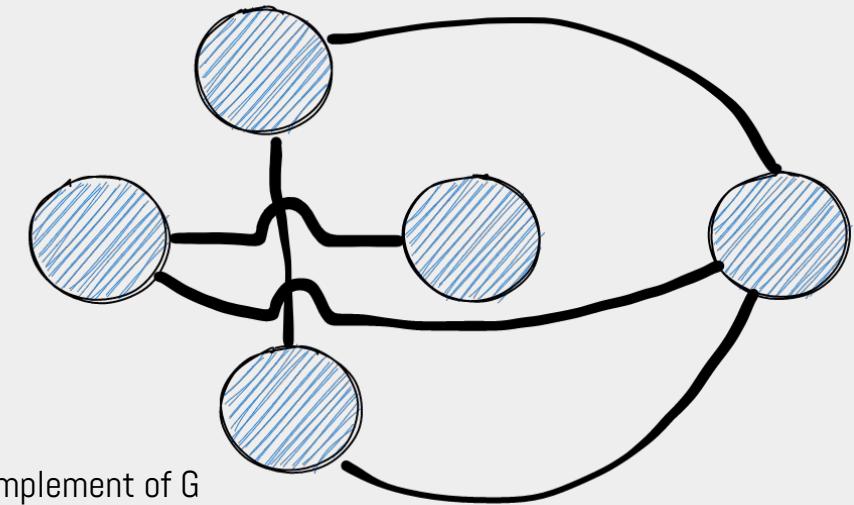
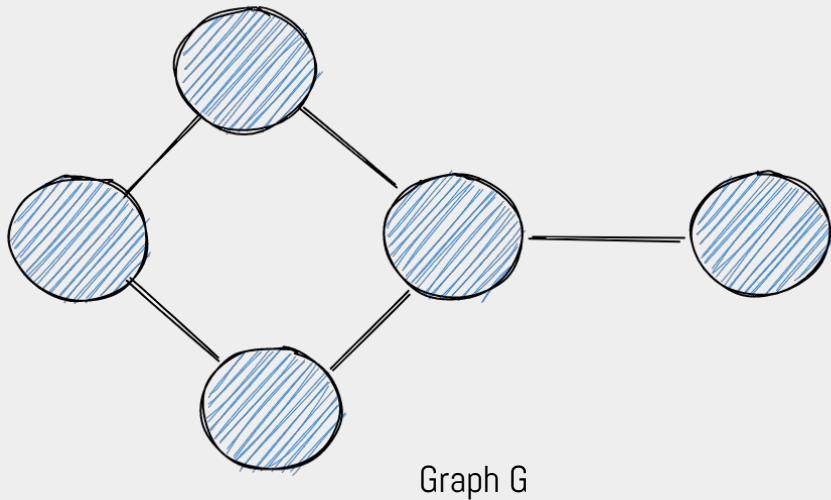
A more rigorous definition of a network

$$G = (V, E), \text{with } E \subseteq V \times V$$



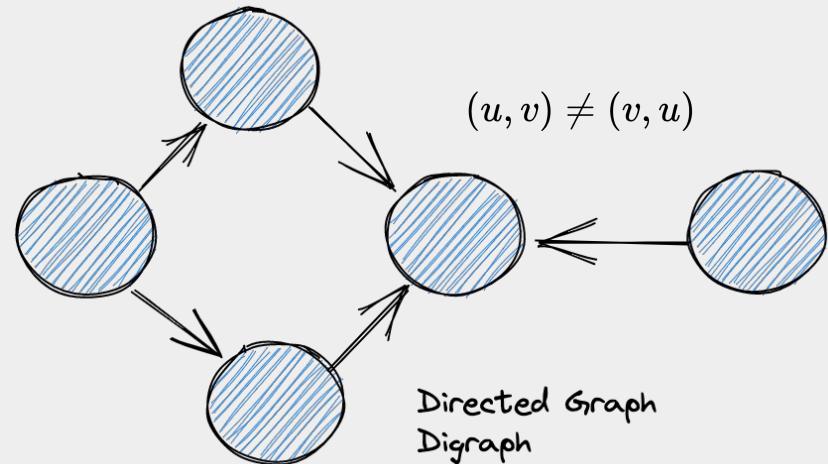
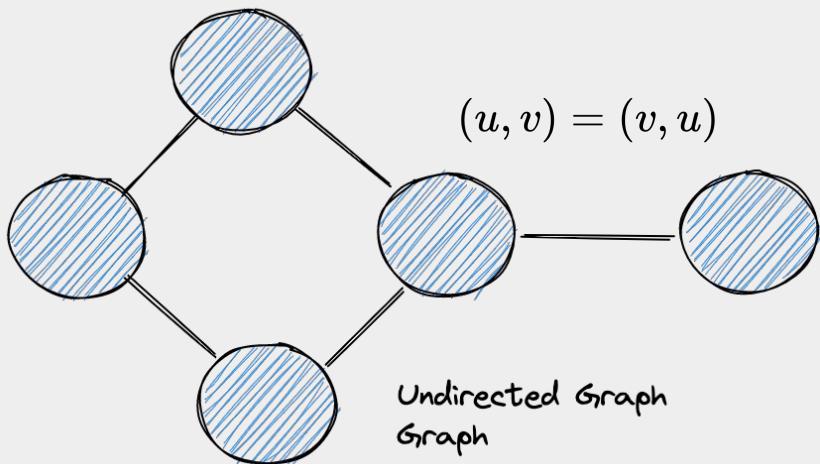
We can derive a series of special graphs

For instance, we can derive the complement of G . This is equivalent to remove all of the original edges of G , and then connect all the unconnected pairs of nodes in G .



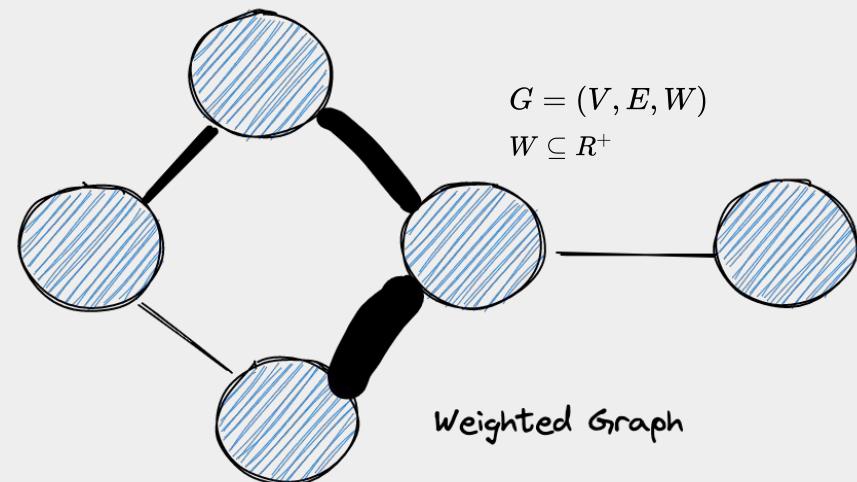
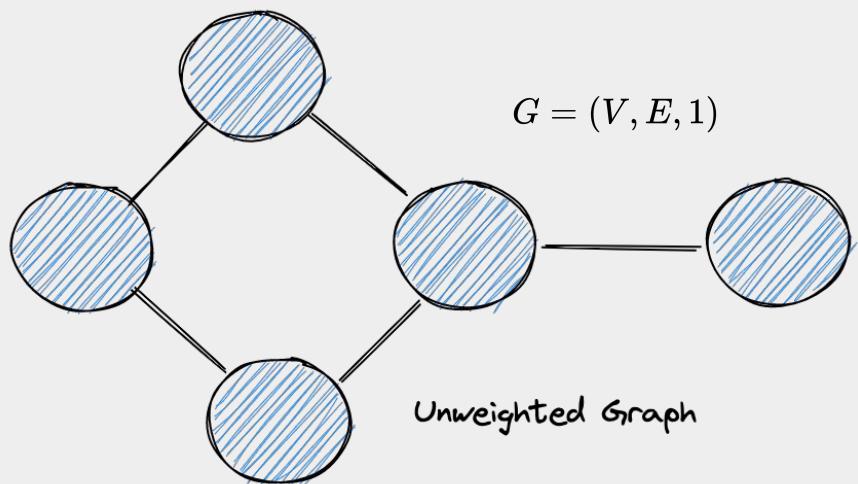
Sometimes you really need to complicate stuff

A first thing we will do is realizing that **not all relations are reciprocal**. We can introduce this asymmetry in the graph model.

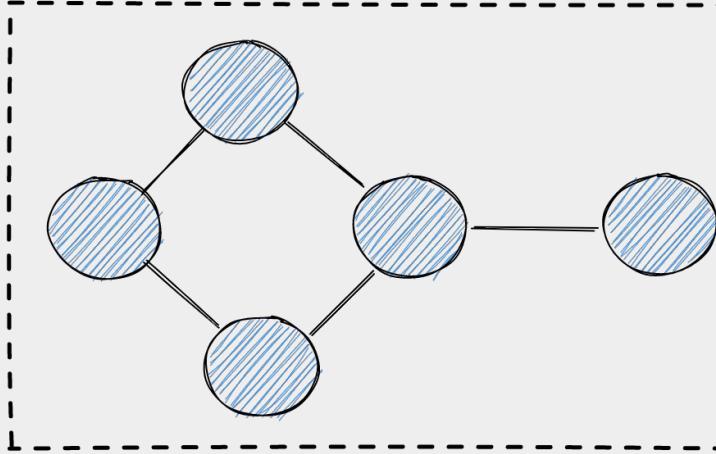


Sometimes you really need to complicate stuff

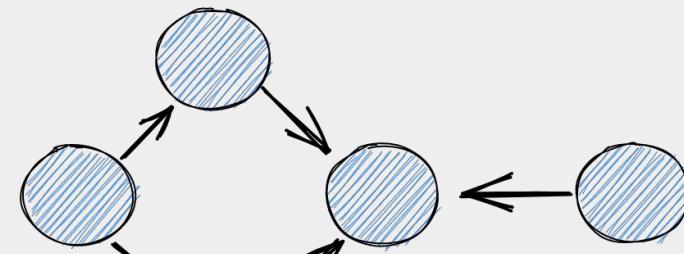
Another way to make edges more interesting is realizing that **two connections are not necessarily equally important** in the network.



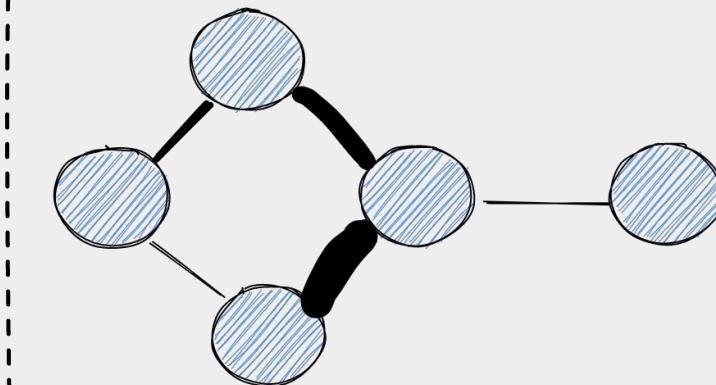
Undirected



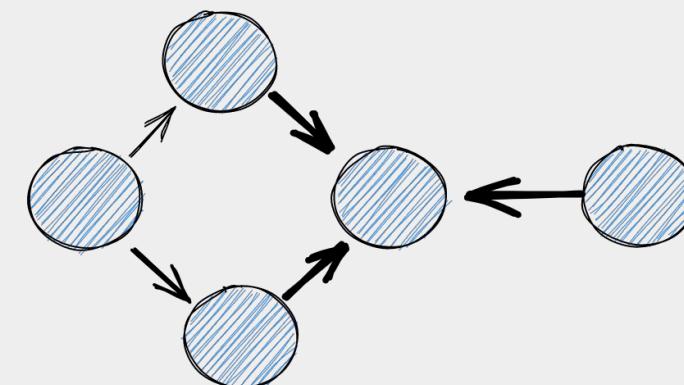
Directed



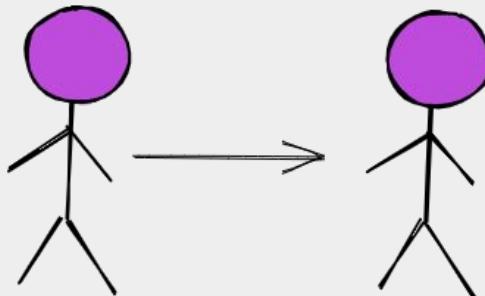
Unweighted



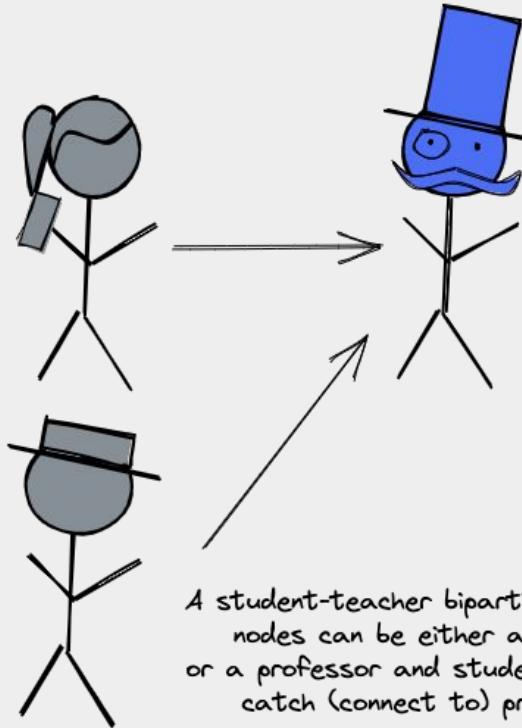
Weighted



Extended Graphs (bipartite network)

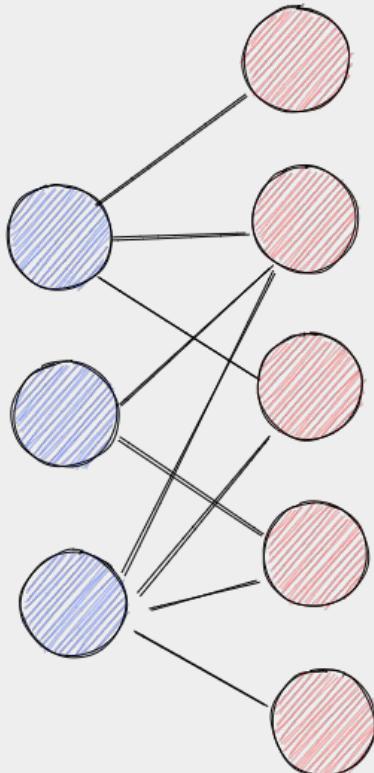


A simple graph representing
a social network with
no additional constraints.



A student-teacher bipartite network:
nodes can be either a student
or a professor and students can only
connect to professors.

Extended Graphs (bipartite network)



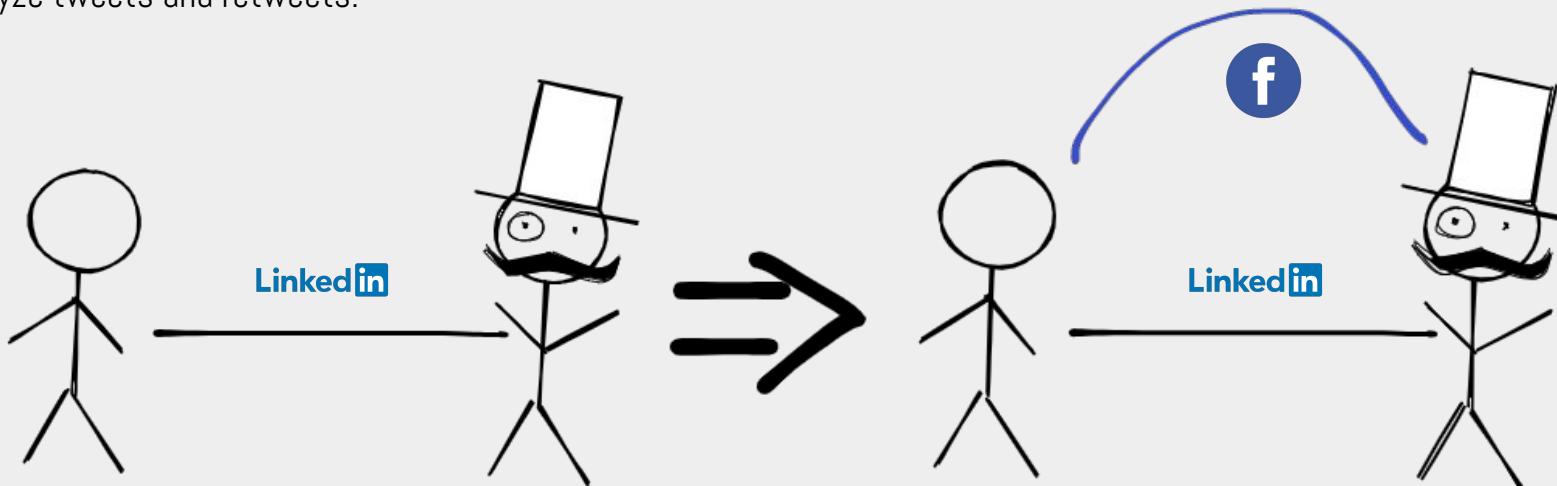
Bipartite networks are used for countless things, connecting:

1. countries to the products they export
2. hosts to guest in symbiotic relationships
3. users to the social media items they tag
4. bank-firm relationships in financial networks
5. players-bands in jazz
6. listener-band in music consumption
7. plant-pollinators in ecosystems

Extended Graphs (multilayer graph)

Traditionally, network scientists try to focus on one thing at a time. For instance, they will download a sample of the LinkedIn graph. Or they will analyze tweets and retweets.

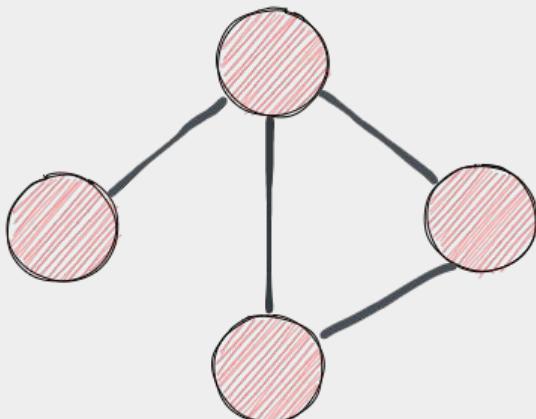
Two people might have started working in the same company and thus first connected on LinkedIn, and then became friends and connected on Facebook. Such scenario could not be captured by simply looking at one of the two networks.



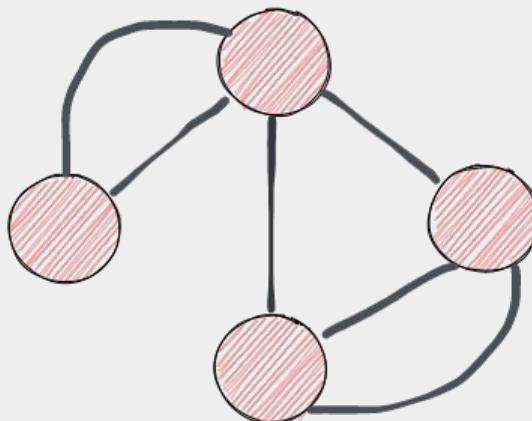
Extended Graphs (multilayer graph)

$$G = (V, E, L)$$

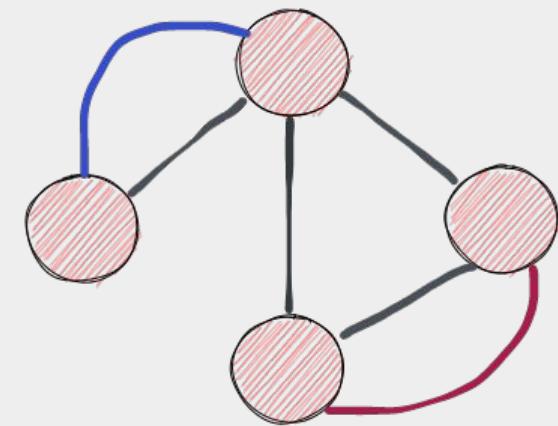
$(u, v, l) \in E$, with $u, v \in V$ and $l \in L$



A simple graph



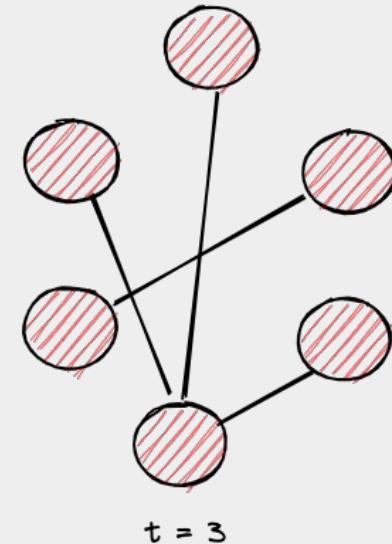
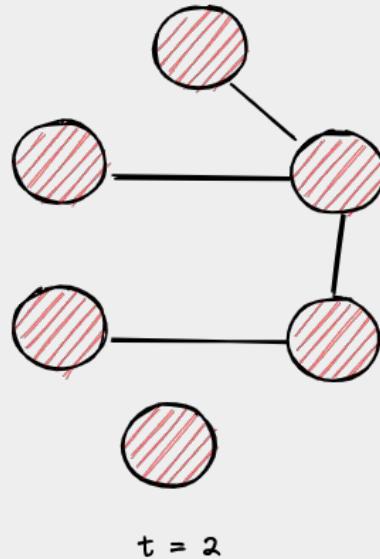
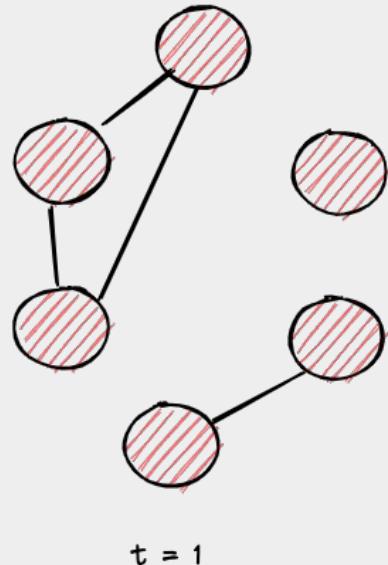
A multigraph, with multiple edges between the same node pairs



A multilayer network, where each edge has a type (represented by its color)

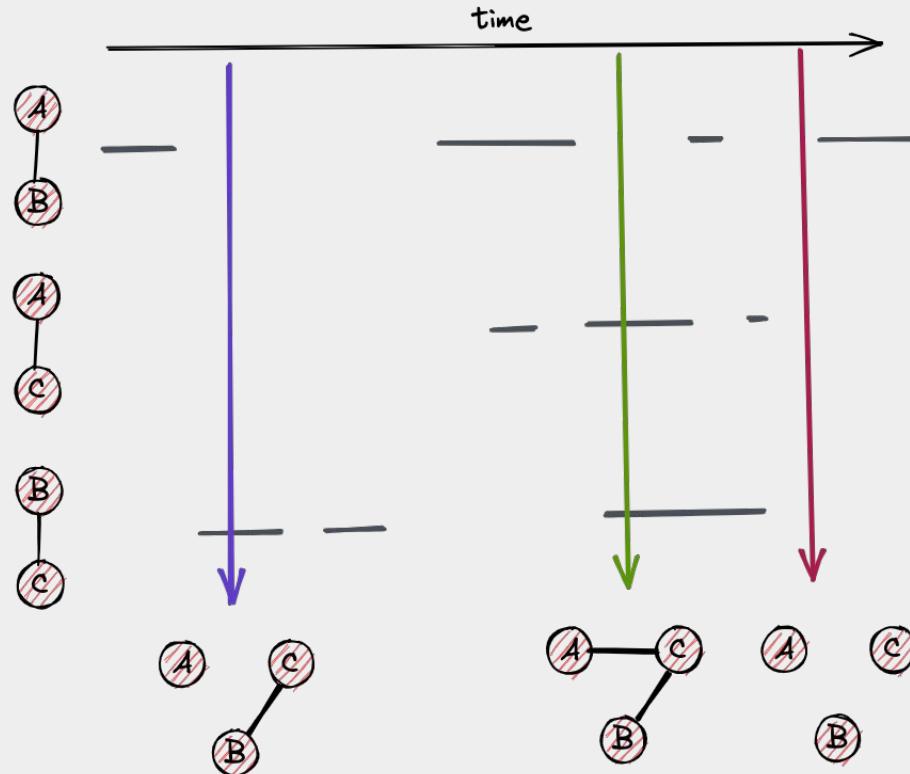
Extended Graphs (dynamic graph)

Imagine that your network represents a road graph. Nodes are intersections, and edges are stretches of the street connecting them. Roadworks might cut off a segment for a few days. If your network model cannot take this into account, you would end up telling drivers to use a road that is blocked, creating traffic jams and a lot of discomfort



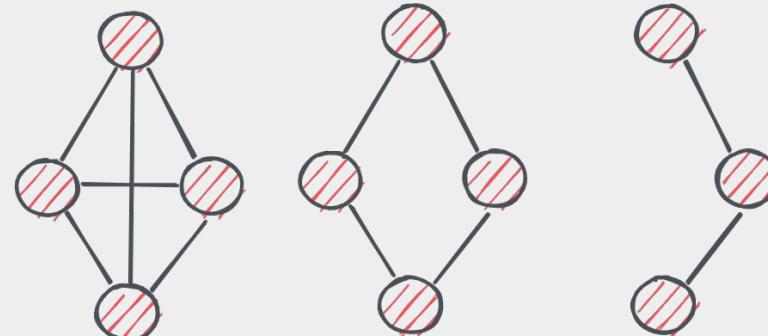
Extended Graphs (dynamic graph)

$$G = (G_1, G_2, \dots, G_n)$$
$$G_i = (V_i, E_i)$$



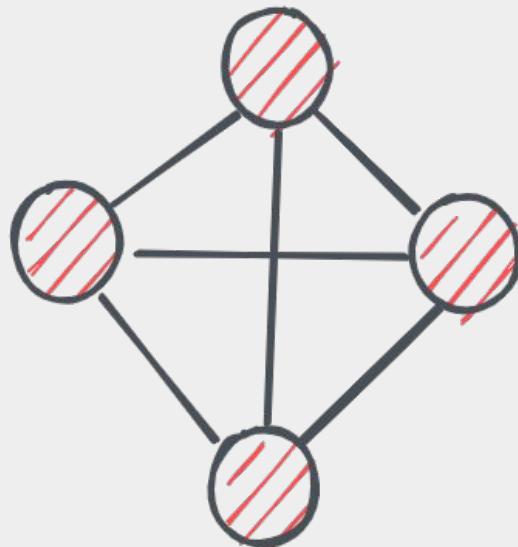


Density and Sparsity, Subnetworks



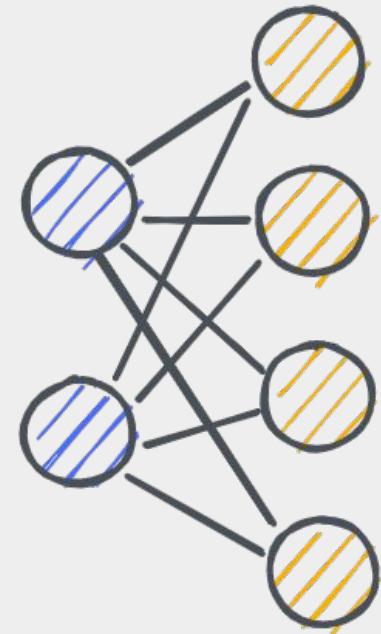
Density and Sparsity

The maximum number of links in a network is bounded by the possible number of distinct connections among the nodes of the system. The maximum number of links is therefore given by the number of pairs of nodes. A network with the maximum number of links, in which all possible pairs of nodes are connected by links, is called a **complete network**.



$$L_{max} = \binom{N}{2} = \frac{N(N - 1)}{2}$$

$$L_{max}^{directed} = N(N - 1)$$



$$L_{max} = N_1 \times N_2$$

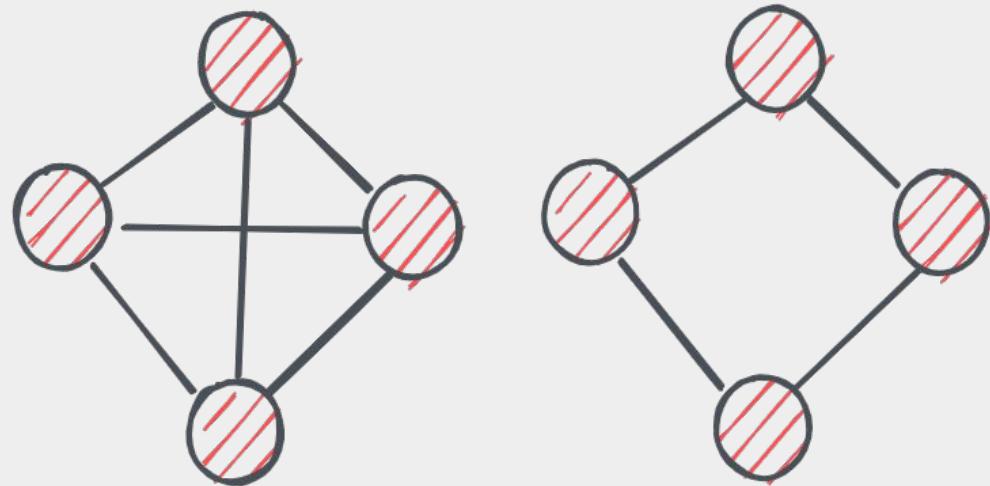
Density and Sparsity

$$d = \frac{L}{L_{max}} = \frac{2L}{N(N - 1)}$$

$$d = \frac{L}{L_{max}^{directed}} = \frac{L}{N(N - 1)}$$

The fraction of possible links that actually exist, which is the same as the fraction of pairs of nodes that are actually connected, is called the **density of the network**.

$$d = \frac{L}{L_{max}}$$

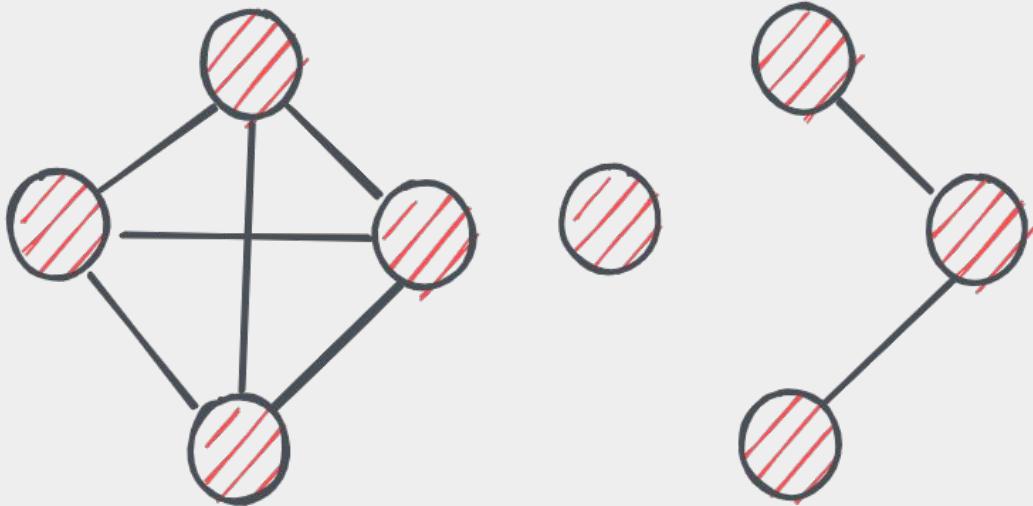


$$d = 1$$

$$d < 1$$

Density and Sparsity

We say that the **network is sparse** if the number of links grows proportionally to the number of nodes ($L \sim N$), or even slower. If instead the number of links grows faster, e.g. quadratically with network size ($L \sim N^2$), then we say that the **network is dense**.

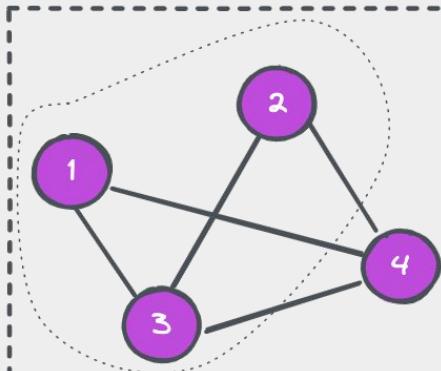


$$L \ll L_{max} \rightarrow d \ll 1$$

Network	Type	Nodes (N)	Links (L)	Density (d)	Average degree ($\langle k \rangle$)
Facebook Northwestern Univ.		10,567	488,337	0.009	92.4
IMDB movies and stars		563,443	921,160	0.000006	3.3
IMDB co-stars	W	252,999	1,015,187	0.00003	8.0
Twitter US politics	DW	18,470	48,365	0.0001	2.6
Enron email	DW	87,273	321,918	0.00004	3.7
Wikipedia math	D	15,220	194,103	0.0008	12.8
Internet routers		190,914	607,610	0.00003	6.4
US air transportation		546	2,781	0.02	10.2
World air transportation		3,179	18,617	0.004	11.7
Yeast protein interactions		1,870	2,277	0.001	2.4
<i>C. elegans</i> brain	DW	297	2,345	0.03	7.9
Everglades ecological food web	DW	69	916	0.2	13.3

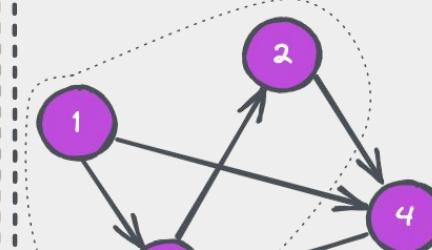
Subnetwork

Undirected



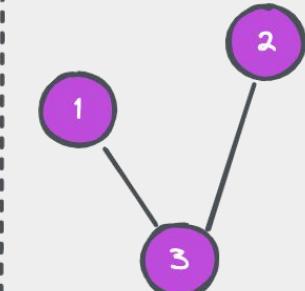
	1	2	3	4
1	0	0	1	1
2	0	0	1	1
3	1	1	0	1
4	1	1	1	0

Directed



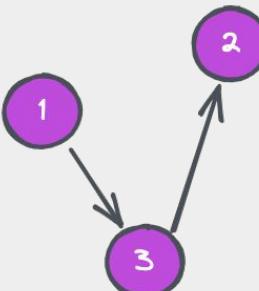
	1	2	3	4
1	0	0	1	1
2	0	0	0	1
3	0	1	0	0
4	0	0	1	0

Original



	1	2	3
1	0	0	1
2	0	0	1
3	1	1	0

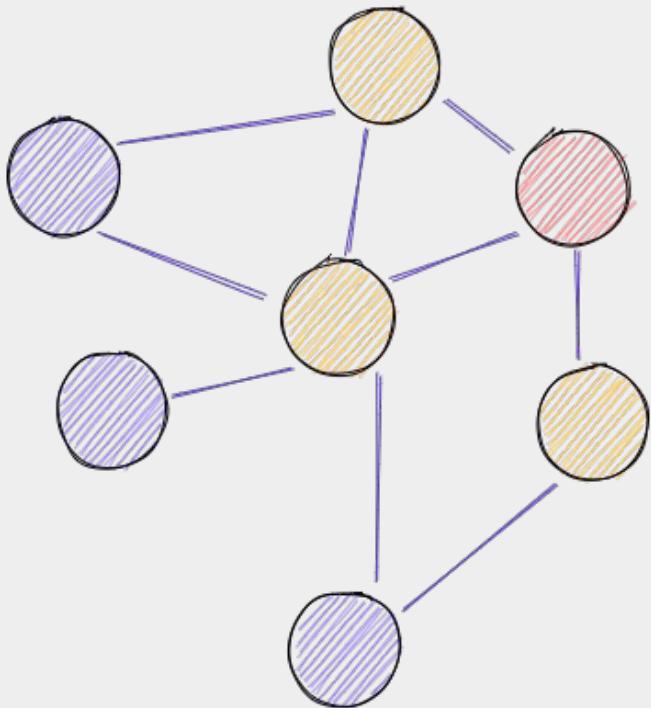
Subnetwork



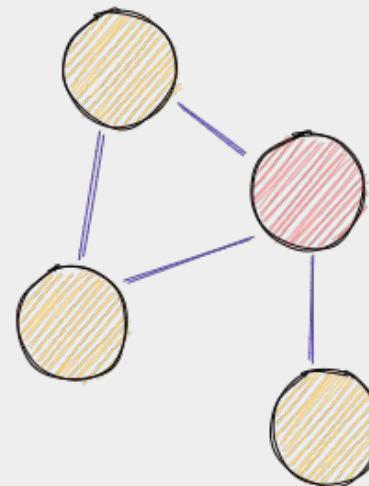
	1	2	3
1	0	0	1
2	0	0	0
3	0	1	0

Subnetwork

A special type of subnetwork is the ego network of a node, which is the subnetwork consisting of the chosen node — called the ego — and its neighbors. Ego networks are often studied in social network analysis.

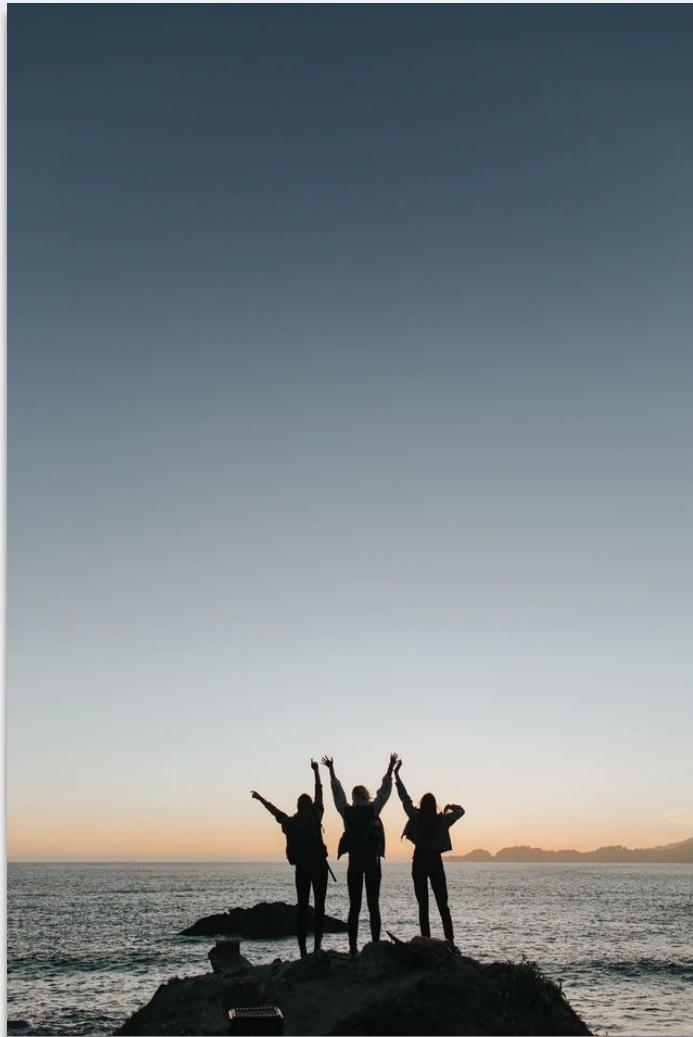


Original Network

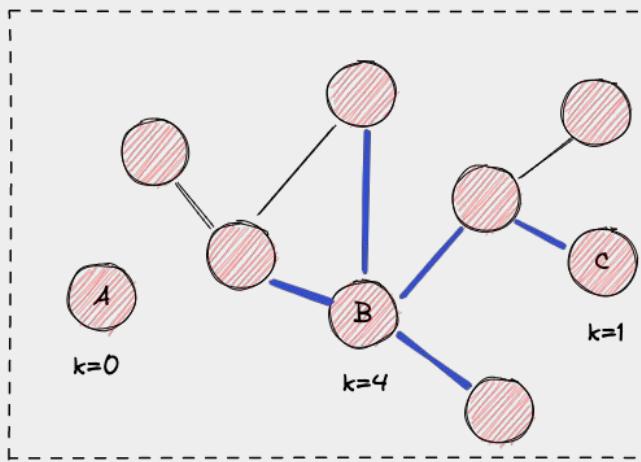


Ego Network

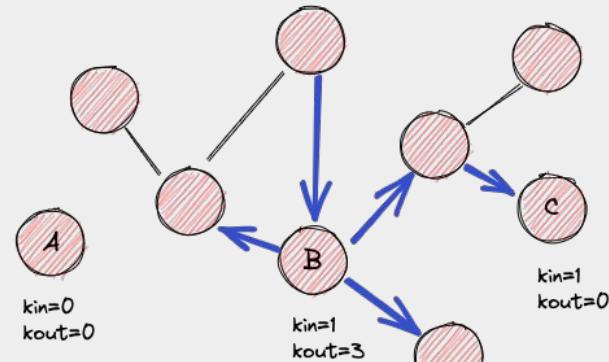
Degree and Network Representation



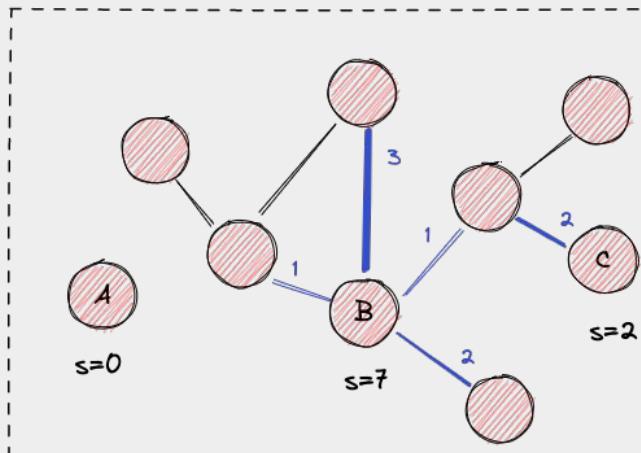
Undirected



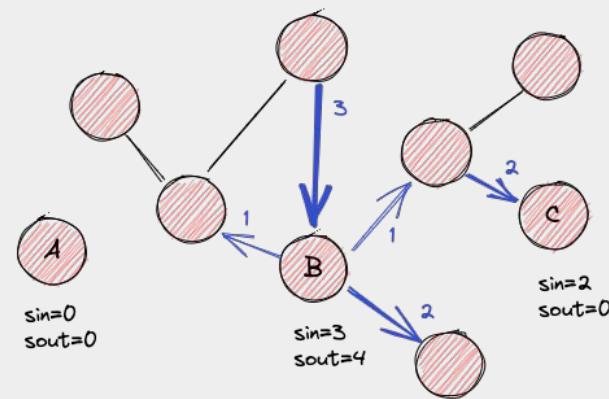
Directed



Unweighted

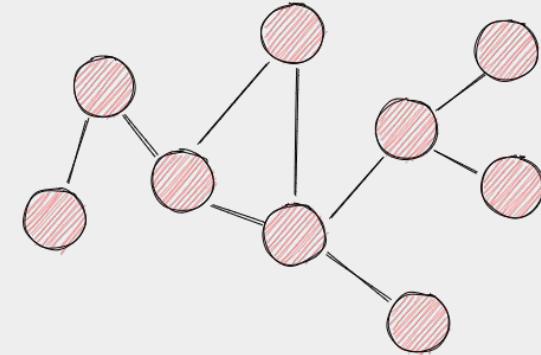


Weighted



The average degree of a network is defined as:

$$\langle k \rangle = \frac{\sum_i k_i}{N}$$



Since each link contributes to the degree of two nodes in an undirected network:

$$\langle k \rangle = \frac{2L}{N} = d(N - 1)$$

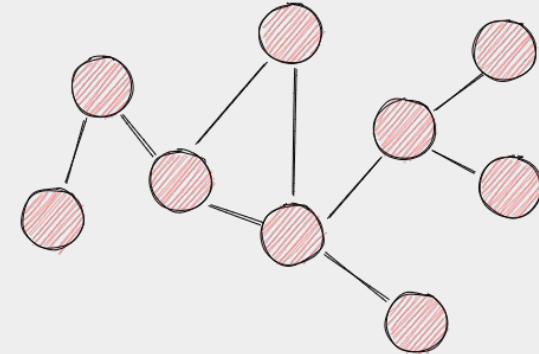
$$d = \frac{2L}{N(N - 1)}$$

The average degree of a network is defined as:

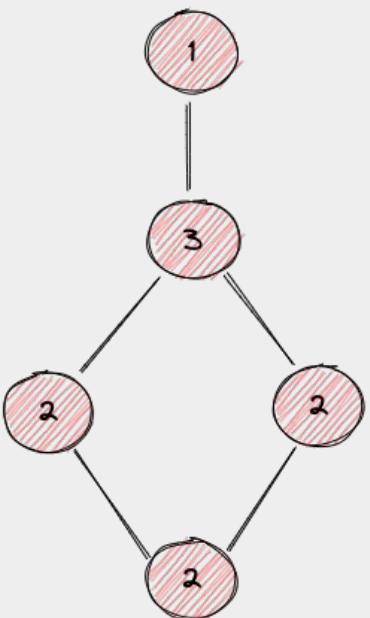
$$\langle k \rangle = \frac{2L}{N} = d(N - 1)$$

$$d = \frac{\langle k \rangle}{N - 1}$$

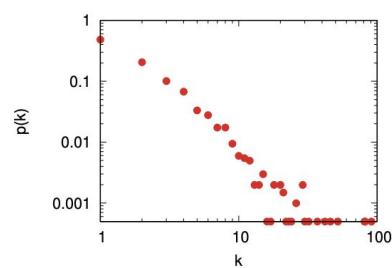
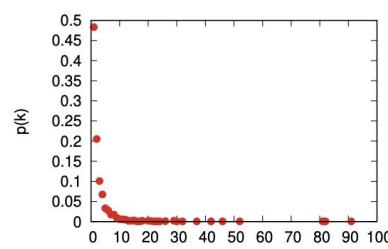
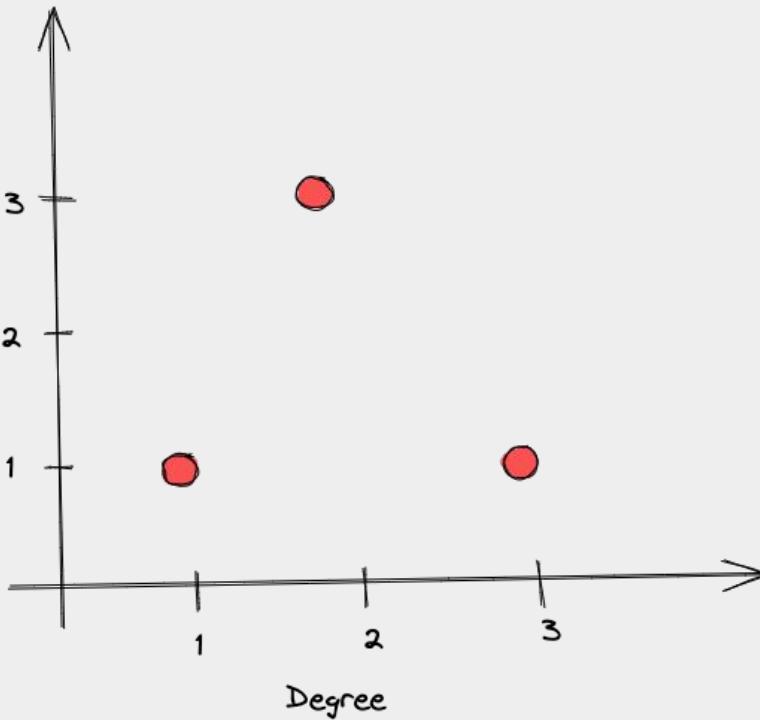
This makes sense because the **maximum possible degree** of a node is $k_{\max} = N - 1$, obtained when the node is connected to every other node. Intuitively, the **density is the ratio between the average and maximum degree**.



Degree Distribution

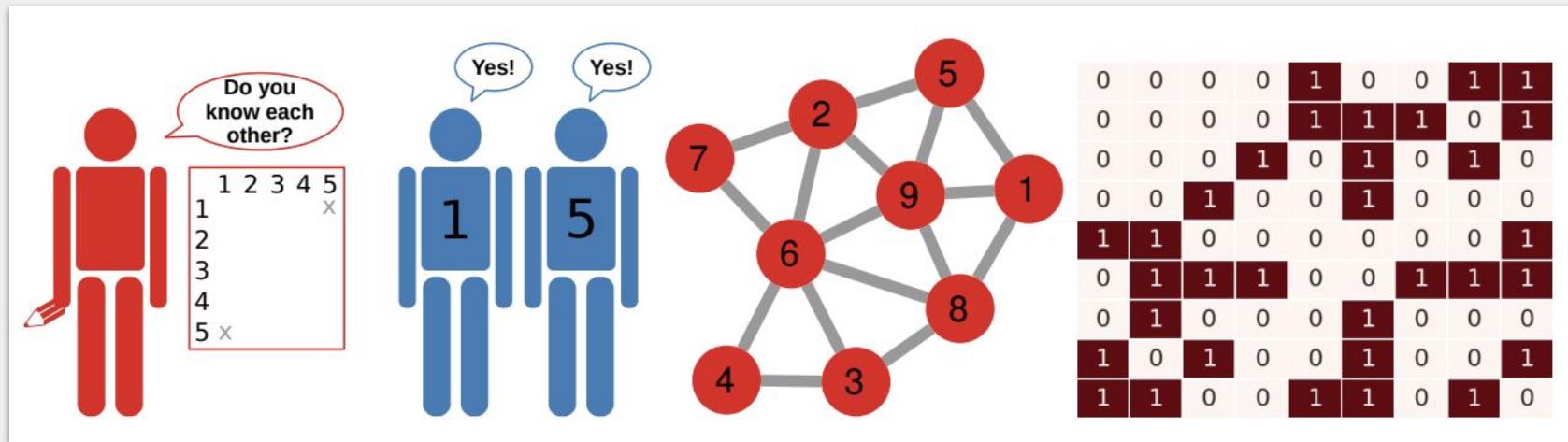


Nodes



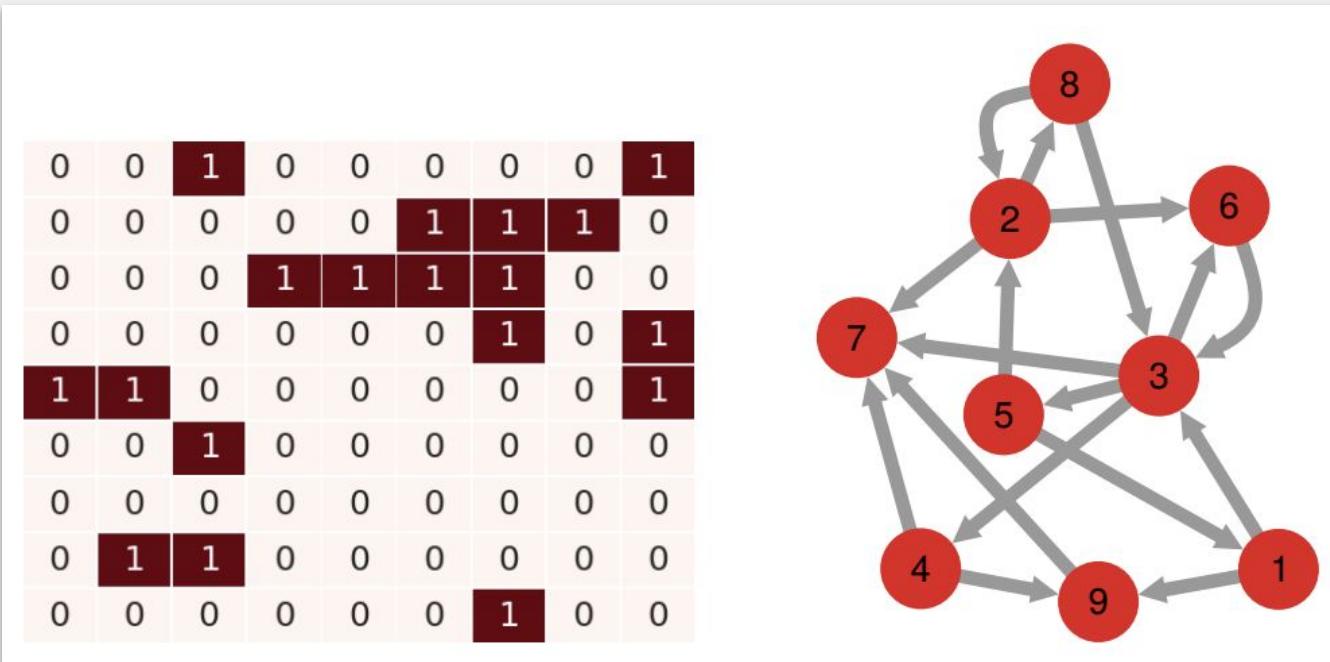
Network Representation

The **adjacency matrix** is the basic representation of a graph as a **matrix**. Each row/column corresponds to a node. Each cell represents an edge, set to one if the edge exists, and zero otherwise.



Network Representation

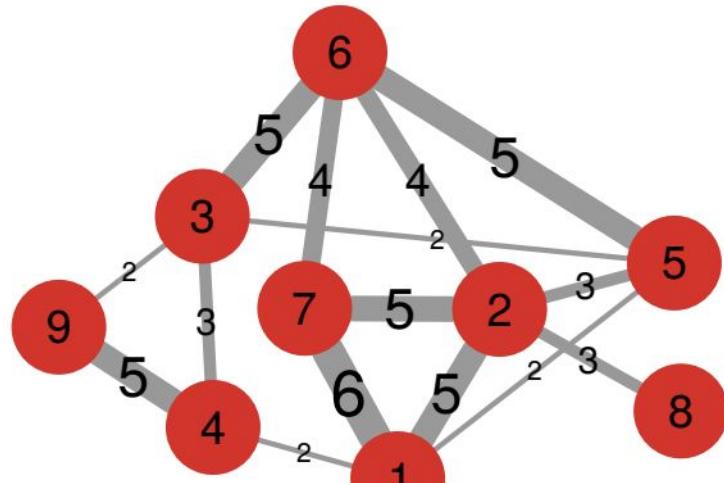
A non-symmetric adjacency matrix corresponding to a directed graph



Network Representation

A non-binary adjacency matrix corresponding a weighted graph

0	5	0	2	2	0	6	0	0
5	0	0	0	3	4	5	3	0
0	0	0	3	2	5	0	0	2
2	0	3	0	0	0	0	0	5
2	3	2	0	0	5	0	0	0
0	4	5	0	5	0	4	0	0
6	5	0	0	0	4	0	0	0
0	3	0	0	0	0	0	0	0
0	0	2	5	0	0	0	0	0

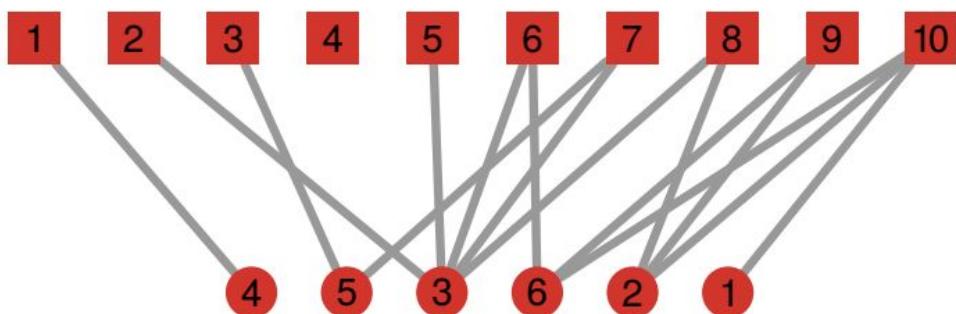


Network Representation

A non-square adjacency matrix corresponding a bipartite graph

$$\begin{bmatrix} |V_1| & |V_2| \\ |V_1| & 0 & A \\ |V_2| & A^T & 0 \end{bmatrix}$$

0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	1	1
0	1	0	0	1	1	1	1	0	0
1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0	0	0
0	0	0	0	0	1	0	0	1	1



Further Reading



1 *Introduction* 9

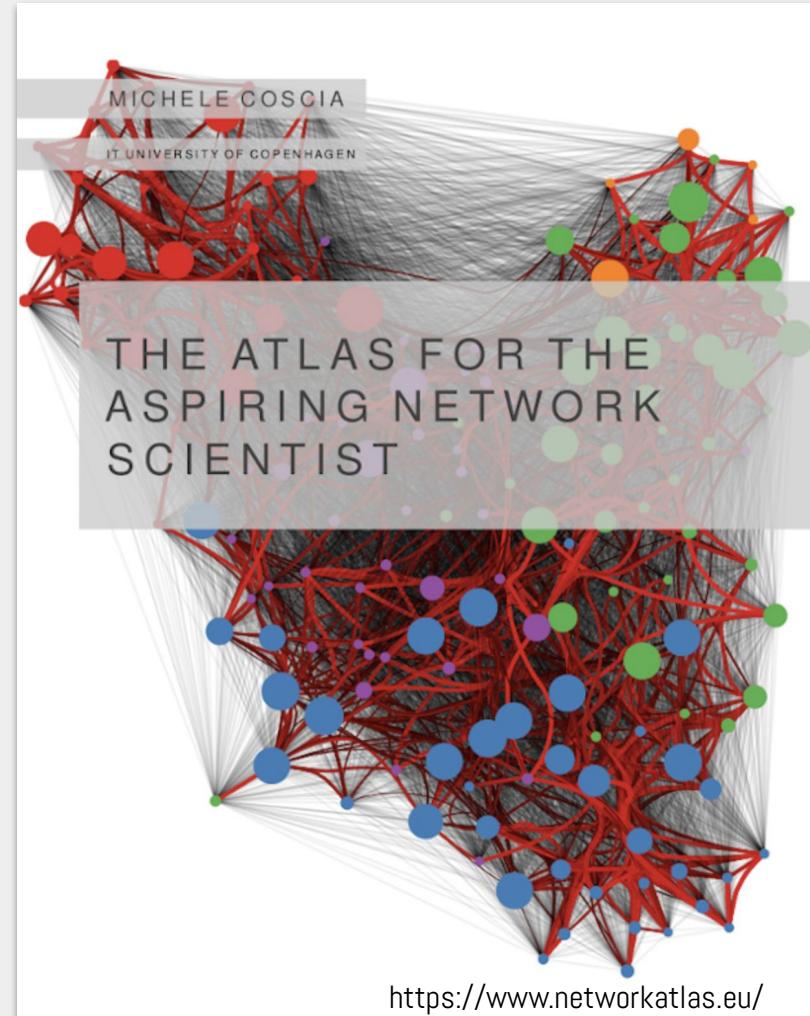
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Releases

Stable (notes)

2.5 – August 2020

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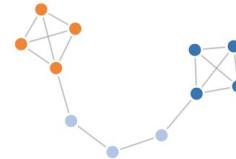
Latest (notes)

2.6 development

[github](#) | [doc](#) | [pdf](#)

Archive

NetworkX is a Python package for the creation, manipulation, and study of the structure, dynamics, and functions of complex networks.



Software for complex networks

- Data structures for graphs, digraphs, and multigraphs
- Many standard graph algorithms
- Network structure and analysis measures
- Generators for classic graphs, random graphs, and synthetic networks
- Nodes can be "anything" (e.g., text, images, XML records)
- Edges can hold arbitrary data (e.g., weights, time-series)
- Open source [3-clause BSD license](#)
- Well tested with over 90% code coverage
- Additional benefits from Python include fast prototyping, easy to teach, and multi-platform



What is graph-tool?

Graph-tool is an efficient [Python](#) module for manipulation and statistical analysis of [graphs](#) (a.k.a. [networks](#)). Contrary to most other Python modules with similar functionality, the core data structures and algorithms are implemented in [C++](#), making extensive use of [template metaprogramming](#), based heavily on the [Boost Graph Library](#). This confers it a level of [performance](#) that is comparable (both in memory usage and computation time) to that of a pure C/C++ library.

▶ It is *Fast!*

Despite its nice, soft outer appearance of a regular Python module, the core algorithms and data structures of graph-tool are written in C++, with performance in mind. Most of the time, you can expect the algorithms to run just as fast as if graph-tool were a pure C/C++ library. See a [performance comparison](#).

➊ OpenMP Support

Many algorithms are implemented in parallel using [OpenMP](#), which

⌚ Extensive Features

An extensive array of features is included, such as support for arbitrary vertex, edge or graph [properties](#), efficient "on the fly" [filtering](#) of vertices and edges, powerful graph I/O using the [GraphML](#), [GML](#) and [dot](#) file formats, graph [pickling](#), [graph statistics](#) (degree/property histogram, vertex correlations, average shortest distance, etc.), [centrality measures](#), standard [topological algorithms](#) (isomorphism, minimum spanning tree, connected components, dominator tree, [maximum flow](#), etc.), [generation of random graphs](#) with arbitrary degrees and correlations, [detection of](#)

Download version 2.38

[Installation instructions](#) | [Changelog](#)

Conda installation (GNU/Linux|MacOS)

```
conda create --name gt -c conda-forge graph-tool  
conda activate gt
```



👁 Powerful Visualization

Conveniently [draw](#) your graphs, using a variety of algorithms and output formats (including to the screen). Graph-tool has its own layout algorithms and versatile, interactive drawing routines based on [cairo](#) and [GTK+](#), but it can also work as a very comfortable interface to the excellent [graphviz](#) package.

📘 Fully Documented

Every single function in the module is documented in the docstrings and in the online [documentation](#), which is full of