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A simple iterative method for pricing American-style options

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2016

Resumo

Esta tese tem como objectivo a análise e implementação do método iterativo para avaliação de opções Americanas proposto por Kim, Jang and Kim (2013) assumindo que o activo subjacente segue um movimento Browniano geométrico.

Desde dos famosos estudos de Back-Scholes (1973) e de Merton (1973), são conhecidas as soluções para as fórmulas de avaliação de opções Europeias. Estas equações são fórmulas fechadas bem definidas e facilmente calculáveis.

Contudo, o mesmo não se verifica para as opções Americanas, uma vez que este tipo de opção dá ao seu detentor a possibilidade de exercer o direito da opção antes da sua maturidade. Todavia, Merton (1973) afirma que se o ativo subjacente não pagar dividendos, não existe qualquer benefício em exercer a opção antecipadamente. Por consequência, estes contratos podem ser avaliados como se de opções Europeias se tratassem. Não obstante, Whaley (1981) prova que se o ativo subjacente pagar dividendos discretos no tempo, é possível definir uma fórmula analítica para avaliação de opções Americanas, visto que o momento ótimo para exercer a opção ocorre imediatamente antes do pagamento de um dividendo.

A principal dificuldade matemática em encontrar uma fórmula analítica para avaliar opções Americanas deve-se ao facto da barreira de exercício ótimo ter de ser determinada como parte da solução da equação de Black-Scholes. Por este motivo, recorre-se a métodos numéricos e de aproximação para calcular o valor deste tipo de opções. Alguns dos métodos mais comuns são baseados no método binomial de Cox et al. (1979) e no método de diferenças finitas de Brennan e Schwartz (1976). Este tipo de métodos acarretam largos tempos computacionais e devido a sua natureza recursiva, podem gerar erros significativos quando se avaliam opções com longos períodos de maturidade. São também usados métodos de caracter aleatório, isto é, métodos que assumem que um parâmetro se comporta de forma aleatória e é lhe atribuída uma distribuição de probabilidade conhecida (Carr (1998)). A simplicidade destes métodos deve-se essencialmente ao comportamento da barreira de exercício ótimo – a barreira torna-se independente do tempo e o problema reduz-se a encontrar um único ponto crítico do valor do ativo subjacente. Apesar deste tipo de métodos gerar fórmulas de avaliação mais simples e com tempos de computação inferiores, é perdida alguma precisão no cálculo dos coeficientes de sensibilidade.

Kim (1990) desenvolveu um novo capítulo na literatura ao introduzir os métodos de representações integrais para a avaliação de opções Americanas. A eficiência deste género de métodos encontra-se intrinsecamente ligada à abordagem adotada para representar a barreira de exercício ótimo. Kim (1990) utiliza um método recursivo para determinar esta barreira, contudo o seu método não demonstra vantagens computacionais sobre outros métodos numéricos devido à sua complexidade e ao elevado número de equações necessárias no seu cálculo. Little et al. (2000) baseia-se no método de Kim (1990), no entanto oferece uma representação mais simples da barreira, reduzindo o grau dos integrais, e não sendo necessário recorrer a uma aproximação da função cumulativa da distribuição Normal. Huang et al. (1996) desenvolve um método que elimina a necessidade de discretizar a barreira de exercício ótimo, usando uma extrapolação de Richard que utiliza apenas três pontos da barreira de exercício ótimo. No mesmo espirito, Ju (1998) apresenta um esquema de idêntico de extrapolação. Todavia, Ju (1998) usa como base funções exponenciais para a representação da barreira, o que torna este método mais eficiente do ponto de vista computacional.

Outro método com relevo na literatura, é o método desenvolvido por Zhu (2006), no qual é apresentado pela primeira vez uma solução explícita da equação de Black-Scholes. Apesar deste feito notável, este método não é atrativo do ponto de vista computacional, devido a sua complexidade numérica – a solução é baseada numa expansão em series de Taylor a qual contem uma infinidade de termos, e em que cada termo é composto por três integrais simples e dois integrais duplos.

O método sugerido por Kim et al. (2013) pretende oferecer uma solução viável para algumas das limitações dos métodos correntes. Para isso, Kim et al. (2013) propõe uma solução não recursiva e uma representação menos complexa para a barreira de exercício ótimo, contendo somente um integral. Este método explora a ideia abordada por Little et al. (2000) em que o preço do ativo subjacente é expresso como função da barreira de exercício ótimo. Similarmente a Little et al. (2000), Kim et al. (2013) deriva uma equação em que ambos os membros contêm a barreira de exercício ótimo, no entanto, Kim et al. (2013) interpreta esta equação por outra perspetiva: Kim et al. (2013) utiliza o segundo membro da equação como sendo uma aproximação do primeiro membro. Isto possibilita

a criação de um algoritmo iterativo, conduzindo a uma aproximação da função da barreira de exercício ótimo. Esta aproximação é tanto melhor quanto maior o numero de iterações executadas pelo método. Contudo, o método mostra-se bastante eficiente, na medida que ao fim de quatro iterações, o valor da barreira de exercício ótimo não se altera significativamente. Uma vez obtida a função da barreira de exercício ótimo, o valor da opção Americana é facilmente calculado.

O método iterativo estudado mostrou ser competitivo para a avaliação de opções Americanas, no entanto a sua eficácia depende das circunstâncias em que está a ser utilizado. O fator determinante para a eficácia deste método é o número de vezes que a barreira de exercício ótimo tem de ser calculada. A barreira depende de quatro variáveis: preço de exercício da opção, volatilidade, taxa de juro e taxa de rendimento de dividendos (K, σ, r e q, respetivamente). Estes fatores dependem diretamente das opções que estamos a avaliar: se estes quatro parâmetros forem iguais em todas as opções, a barreira de exercício ótimo é a mesma, e portanto apenas necessita de ser calculada uma vez. Se algum dos parâmetros for diferente em alguma das opções que estejamos a avaliar, a barreira de exercício ótimo precisa de ser recalculada. No entanto, mesmo que todas as opções tenham barreiras de exercício ótimo diferentes, o método de Kim et al. (2013) é ainda assim mais rápido (6 a 8 vezes) que o método binomial. A grande vantagem deste método está no cálculo dos coeficientes de sensibilidade. Contrariamente aos métodos numéricos, em que é necessário o cálculo de duas opções para determinar um coeficiente de sensibilidade, o método de Kim et al. (2013) apresenta uma fórmula exata para o cálculo dos mesmos.

A dissertação está organizada da seguinte forma. No Capitulo 1 são apresentados alguns dos métodos mais comuns na avaliação de opções Americanas. No Capitulo 2 define-se o modelo e os pressupostos relativos ao ativo subjacente nos quais nos vamos debruçar ao longo do estudo. São também revistos vários conceitos e resultados teóricos conhecidos da literatura. No Capitulo 3 é explorado o trabalho de Little et al. (2000) e apresentada uma nova equação para a barreira de exercício ótimo. No Capitulo 4, define-se como podemos utilizar esta nova equação para criar um algoritmo iterativo capaz de calcular o valor de opções Americanas. No Capitulo 5 são apresentados os resultados do algoritmo e é feita uma comparação em relação à precisão e velocidade do método iterativo contra outros algoritmos usados na literatura. No Capitulo 6 estendemos o estudo de Kim et al. (2013) a ativos com dividendos. No Capitulo 7 são apresentados as conclusões do estudo.

Palavras-chave: Opções Americanas; Barreira de exercício antecipada; Aproximação numérica; Método iterativo

Abstract

In this thesis it is analysed and implemented the iterative method for the valuation of American options proposed by Kim, Jang and Kim (2013) assuming the underlying asset price follow a geometric Brownian motion.

The method suggested by Kim et al. (2013) intends to offer a viable solution from some of the limitations found on the traditional methods. In order to reach this objective, Kim et al. (2013) proposes a non-recursive solution and a simpler representation of the optimal exercise boundary, containing only a single integral. This method explores the Little et al.'s (2000) idea in which the price of the underlying asset is defined as a function of the optimal exercise boundary. Similarly to Little et al. (2000), Kim et al. (2013) derives an equation in which both sides are dependable of the optimal exercise boundary. However, Kim et al. (2013) takes a different approach by considering the right-hand side term as an approximation of the left-hand side of the equation. With this in mind, one can create an iterative algorithm that leads to an approximation of the early exercise boundary. This approximation can be more accurate if the number of the iterations is increased. Once the optimal exercise boundary is achieved, the value of the American option is easily obtained. Furthermore, founding the value of the Greeks is equally effortless, as this method offers an exact formula for this sensitivity factors. This is a huge advantage over the conventional numeric methods, which often needs to calculate the value of 2 options. Overall this method is more efficient than other traditional numerical methods. However his performance is directly dependable on the options that are being evaluated, as calculating the optimal exercise boundary is the most time consuming step on this method (different options may have different optimal exercise boundaries).

Keywords: American option; Early exercise boundary; Numerical approach; Iterative method

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Chapter 1

Introduction

Since the path-breaking papers of Black-Scholes (1973) and Merton (1973) it is known that European options have closed-form solutions that can be easily calculated.

However, for American options this is not the case, because there is the possibility of early exercise. Merton (1973) states that if the underlying asset does not pay dividends, there is no benefit of exercising an American call before maturity, which means that these contracts could be valuated as European calls. Even if the underlying asset pays discrete dividends, it is possible to derive an analytical formula for the American call, as the optimum exercise moment will occur immediately before a dividend payment (Whaley, 1981). Nevertheless, for assets that pay continuous dividends, there are no closed-form valuation formulas. The same applies to American puts.

This mathematical difficulty of finding an analytical formula for American options is due to the fact that the optimal exercise boundary must be determined as a part of the solution of the Black-Scholes equation, and for that reason American options are usually computed via numerical and approximation methods. Some of the most usual methods are based on the binomial method (Cox et al. 1979) and the finite difference method (Brennan and Schwartz, 1976). These are time-recursive methods and not only involve a large number of calculations and time, but also accumulate errors that can be substantial when considering long periods to the option's maturity. Other relevant methods include the randomization method which is based on randomizing a parameter and assuming a reasonable distribution for it (maturity date – Carr, 1998). For American options the simplicity is mainly due to the taming of the behaviour of the exercise boundary - the boundary becomes independent of time and the problem is reduced to finding a single critical stock price. Although this lead to a simpler valuation formula and quick computation times, when calculating the sensitivity coefficients (Greeks) some accuracy is lost.

Kim (1990) added another chapter on the option price literature by introducing the so-called integral representation methods. However, the numerical efficiency of this approach depends on the specification that is adopted for the early exercise boundary. Kim (1990) uses a time recursive method for calculating the optimal exercise boundary. Focused on the same representation, Little et al. (2000) offers a simpler method by reducing the complexity of the integral equation proposed by Kim (1990). Huang et al. (1996) adopt a time consuming step function approximation, while Ju (1998) proposes a multipiece exponential representation of the early exercise boundary.

Another method was presented by Zhu (2006) containing an explicit solution of the Black-Scholes equation for an American put. A closed-form solution for the Black-Scholes equation was a remarkable accomplishment, however due to his complexity, it is hard to implement this method numerically - the solution is based on Taylor series expansion with infinitely many terms, where each term contains three single integrals and two double integrals.

The method presented by Kim et al. (2013) intends to offer a solution for some problems of the current methods by offering a non-time-recursive method and a simpler representation of the optimal exercise boundary containing only a single integral. Once the optimal exercise boundary is achieved, the value of the American option is easily obtained.

This thesis is organized as follows. In the following chapter we will setup the framework and walk through some background theoretical results. Chapter 3 explores the Little et al.'s (2000) idea and presents the optimal exercise boundary equation. Chapter 4 explains how these results can be implemented into the algorithm and how the algorithm can be programmed to calculate the American option's value and delta measures. Chapter 5 presents the computational result of the algorithm against other popular methods. Chapter 6 studies the extension of Kim et al. (2013) method for assets with dividends. Chapter 7 concludes the thesis.

Chapter 2

Background

2.1 General setup of the model

In this section we will present some well known theoretical results associated with the Black-Scholes model and later explore the new approach offered by Kim et al. (2013). Whenever possible, we will adopt the same notation as on the original paper.

We will denote the underlying asset by S and is value at time t by S_t with $0 < t < T$ where T stands for the maturity of the put option. We will assume the usual conditions: the value of the asset must be positive $0 < S_t < \infty$ and markets are perfect.

Our analysis will fall under the geometric Brownian motion assumption which assumes that the underlying asset price (S) follows a diffusion process under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = (r - q) S_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{2.1}$$

where r is the risk-free interest rate, q the dividend yeld, while $\sigma \in \mathbb{R}_+$ represents the volatility of the asset. The model's filtration is generated by a standard Brownian motion $\left\{ dW_t^{\mathbb{Q}}; 0 \leq u \leq t \right\}$ which is assumed to be initialized at zero and to be defined under a suitable probability space[1] the martingale measure $\mathbb{Q}$. During our study we will consider r and q constants. [2]

2.2 From Black-Scholes to Kim Optimal exercise boundary

As stated before, the difference between European and American style options is that on the second one the option can be exercised at any moment before maturity. This lead us to an uncertainty of when the holder should or should not exercise his right on the underlying asset. The optimal exercise boundary should be seen as a function of time and its output should be compared against the price of the underlying asset at the exact same moment.

For the American call, this boundary is irrelevant (unless the underlying asset pays dividends), so we will focus on the put option.

Following McKean (1965) and Myneni (1992) lemmas, we will assume during this study that the optimal exercise boundary function B_t, defined in $[0, T] \to [0, \infty[$, is unique and continuously differentiable.[3]

Once the function B_t is known, we can conclude at each point if the option is worth exercising: it is optimal to exercise the option when the stock price is below the optimal exercise boundary. In that case, at the time t, the option pays off:

$$P_\tau = K - S_\tau$$

[1] Consider the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ where the filtration $(\mathcal{F}_\sqcup)$ is the completation of σ-algebra generated by the Brownian motions and $(\mathcal{F}_t) = \mathcal{F}$

[2] We will consider $q = 0$ during the first part of our study, and get back to it on chapter 7 when studying assets with dividends.

[3] For the put option, the optimal exercise boundary it is a non-decreasing function, which means that B_τ is non-increasing, where $\tau = T - t$

We can now write the valuation of the American put $P(S,\tau)$ as a function defined in $[B_\tau,\infty[\times[0,T]\rightarrow[0,\infty[$, which is $C^{2,1}$ and it is the solution of the Black-Scholes partial differential equation (PDE)[4]

$$\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}+rS\frac{\partial P}{\partial S}-rP=\frac{\partial P}{\partial \tau} \tag{2.2}$$

subject to a terminal condition:

$$P(S,0)=max\{0,K-S\}\text{ for all }S\geq B_0$$

and boundary conditions:

$$\lim_{S_\tau\uparrow\infty}P(S,\tau)=0$$

$$\lim_{S_\tau\downarrow B_\tau}P(S,\tau)=K-B_\tau$$

$$\lim_{S\downarrow B_\tau}\frac{\partial P(S,\tau)}{\partial S}=-1$$

for all $\tau\in]0,T]$. Kim (1990) derives a valuation formula for American options that contains an optimal exercise boundary as a function of time to expiration, and an implicit-form integral equation with respect to the optimal exercise boundary. The valuation formula for a live American put is, for $S>B_\tau$

$$P(S,\tau)=p(S,\tau)+\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-d_2\left(S,\tau-\xi,B_\xi\right)\right)d\xi \tag{2.3}$$

where $\aleph$ is the cumulative distribution function for the standard normal distribution with:

$$d_1(S,\tau,B)=\frac{ln\left(\frac{S}{B}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2(S,\tau,B)=d_1(S,\tau,B)-\sigma\sqrt{\tau}$$

and $p(S,\tau)$ stands for the Black-Scholes formula for the European put option:

$$p(S,\tau)=Ke^{-r\tau}\aleph\left(-d_2(S,\tau,K)\right)-S\aleph\left(-d_1(S,\tau,K)\right)$$

Equation (2.2) can be interpreted as a sum of two components: a regular European put option, and the discounted expectation of an American put being exercised at time τ.

2.3 Calculating the optimal exercise boundary

It is now clear that the main difficulty on valuing American option is to determine the optimal exercise boundary. Once the optimal exercise boundary is known, the value of the American put is easily obtained using equation (2.3). This is a critical step for valuing American options, as there is no analytical solution for the optimal exercise boundary. Different approaches have been proposed through the years to resolve accurately and efficiently this numerical problem.

2.3.1 Kim (1990)

Kim (1990) suggests a discretization of the optimal exercise boundary by dividing τ into n subintervals τ_i, for $i=1,2,...,n$ where $\tau_n=\tau$. Then, considering the payoff of the option $P(B_\tau,\tau)=B_\tau-K$, we need to solve numerically n integral equations:

$$K-B_{\tau_i}=p\left(B_{\tau_i},\tau_i\right)+\int_0^{\tau_i} rKe^{-r(\tau_i-\xi)}\aleph\left(-d_2\left(B_{\tau_i},\tau_i-\xi,B_\xi\right)\right)d\xi \tag{2.4}$$

[4]Note that by defining $P(S,\tau)$ on $[B_\tau,\infty[\times[0,T]$ implies that $S\in[B_\tau,\infty]$

These integral equations are solved recursively and are easy and simple to implement for short-term options, where only a few points of the optimal exercise boundary need to be calculated. For long-term options, this method looses some speed as it needs more and more points to calculate a reasonable approximation of the optimal exercise boundary.

2.3.2 Huang et al. (1996)

Huang et al. (1996) suggest a method that solves the need of discretization of the optimal exercise boundary. This method follows the same idea of Geske and Johnson (1984), and his main advantage over Kim's (1990) method is that it only uses 3 points of the optimal exercise boundary. Let P_1 denote a put option that can only be exercised at T (i.e. an European put option with maturity T). Let P_2 denote a put option that can only be exercised at $T/2$ or T. Let P_3 denote a put option that can only be exercised at $T/3$, $2T/3$ or T. It follows from equation (2.3) that:

$$P_1 = p(S,T)$$

$$P_2 = p(S,T) + \frac{rKT}{2} e^{-\frac{rT}{2}} \aleph\left(-d_2\left(S, \frac{T}{2}, B_{\frac{T}{2}}\right)\right)$$

$$P_3 = p(S,T) + \frac{rKT}{3} \times \left[e^{-\frac{rT}{3}} \aleph\left(-d_2\left(S, \frac{T}{3}, B_{\frac{T}{3}}\right)\right) + e^{-\frac{2rT}{3}} \aleph\left(-d_2\left(S, \frac{2T}{3}, B_{\frac{2T}{3}}\right)\right)\right]$$

Then, using the same Richard extrapolation scheme as Geske and Johnson (1984) we can calculate an approximation of the value of the American put option using the following:

$$P \approx \frac{P_1 - 8P_2 + 9P_3}{2}$$

Note that for calculating P_1,P_2 and P_3 we only need 3 points of the optimal exercise boundary:

$$B_{\frac{T}{3}}, B_{\frac{T}{2}}, B_{\frac{2T}{3}}$$

This points can be calculated solving 3 integral equations using (2.4).

2.3.3 Ju (1998)

Ju (1998) noted that B_τ only appears as $ln(S/B_\tau)$ on $d_\bullet(\bullet,\bullet,\bullet)$ formulas. Therefore, equation (2.4) does not depend on the exact values of B_τ critically. Assuming that for each interval $[\tau_1, \tau_2]$ the optimal exercise boundary B_τ can be expressed as an exponential function $Bexp(b\tau)$, it is possible to find an analytical formula for the integral of equation (2.4) - this formula is composed by 3 piece-wise exponential functions which from a computational point of view are fairly easy to calculate. Then, Ju (1998) uses a three point Richard extrapolation to calculate the value of the American option: [5]

$$P \approx \frac{P_1 - 8P_2 + 9P_3}{2}$$

From a computationally point of view, this method is very efficient as it avoids the numerical difficulties of calculating integrals.

[5]The definitions of P_1, P_2 and P_3 can be found on Ju (1998)

Chapter 3

Exploring Little et al.'s (2000) idea

3.1 Little et al.'s (2000) representation of the optimal exercise boundary

Little et al.'s (2000) approach on calculating the optimal exercise boundary differs from the other methods on the interpretation of the value of S_τ. An adequate choice of S_τ will allow to represent (2.3) in a one-dimensional integral equation[1] without recurring to an approximation of the cumulative normal distribution. They start by focusing on the scenario where $S_t \leq B_\tau$ which by the definition of the optimal exercise boundary is optimal to exercise the option. Then at time τ the option payoff is equal to:

$$P_\tau = K - S_\tau$$

Since $S_t \leq B_\tau$ we can express S_τ as ϵB_τ where $\epsilon \in]0, 1]$.

$$P(S, \tau) = K - S_\tau = K - \epsilon B_\tau$$

Making this substitution on equation (2.3) we have:

$$K - \epsilon B_\tau = p(\epsilon B_\tau, \tau) + \int_0^\tau rKe^{-r(\tau-\xi)} \aleph\left(-d_2\left(\epsilon B_\tau, \tau - \xi, B_\xi\right)\right) d\xi$$

Differentiating the equation above with respect to ϵ:

$$B_\tau \aleph\left(d_1\left(\epsilon B_\tau, \tau, K\right)\right) + B_\tau \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{1}{2} d_1\left(\epsilon B_\tau, \tau, K\right)^2\right\} = K \frac{1}{\epsilon\sigma\sqrt{2\pi\tau}} \exp\left\{-r\tau - \frac{1}{2} d_2\left(\epsilon B_\tau, \tau, K\right)^2\right\}$$

$$+ \frac{rK}{\epsilon\sigma\sqrt{2\pi}} \int_0^\tau \frac{1}{\sqrt{\tau-\xi}} \exp\left\{-r(\tau-\xi) - \frac{1}{2} d_2\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi$$

Differentiating again with respect to ϵ and rearranging the equation:

$$B_\tau = -\frac{Kr\sqrt{\tau}}{\epsilon\sigma} \exp\left\{\frac{1}{2} d_1\left(\epsilon B_\tau, \tau, K\right)^2\right\} \times \int_0^\tau \frac{d_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)}{\tau-\xi} \exp\left\{-r(\tau-\xi) - \frac{1}{2} d_2\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi$$

Taking the limit $\epsilon \to 1$, Little et al. (2000) conclude that the optimal exercise boundary is equal to:

$$B_\tau = Kr \exp\left\{\frac{1}{2} d_1\left(B_\tau, \tau, K\right)^2\right\} \sqrt{2\pi\tau}$$

$$-Kr \exp\left\{\frac{1}{2} d_1\left(B_\tau, \tau, K\right)^2\right\} \sqrt{\tau} \int_0^\tau \frac{d_1\left(B_\tau, \tau-\xi, B_\xi\right)}{\tau-\xi} \exp\left\{-r(\tau-\xi) - \frac{1}{2} d_2\left(B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi \tag{3.1}$$

[1] Note that equation (2.3) is a two-dimension integral as $\aleph$ is a one-dimensional integral.

Note that both sides of the equation are still dependable of B_τ, so B_τ cannot be calculated directly. To attack this problem, Little et al. (2000) divide the interval $[0, \tau]$ into N equal subintervals of length τ/N. This discretization yields N implicit integral equations defining the exercise boundary at points $0, \tau/N, 2\tau/N, ..., \tau$. Since we know that at the maturity $B_0 = K$, the boundary at the other points can be computed recursively by solving a nonlinear equation for the boundary value at each of these points. This method produces a good approximation to the early exercise boundary - note that equation (3.1) does not involve the cumulative normal function, which reduces the numerical errors used by approximating this function. In terms of speed, this method is a lot slower than other methods mentioned earlier.

3.2 Kim et al.'s (2013) approach

Kim et al. (2013) offered a new method of calculating the optimal exercise boundary in line with the logic used by Little et al. (2000). Kim et al. (2013) representation of the boundary also emphasis on the scenario where $S_t \leq B_\tau$, so at time τ the option payoff is equal to:

$$P_\tau = K - S_\tau$$

As seen on Little et al.'s (2000) work, we can express S_τ as ϵB_τ where $\epsilon \in]0, 1]$.

$$P(S, \tau) = K - S_\tau = K - \epsilon B_\tau$$

Making this substitution on equation (2.3) we have:

$$K - \epsilon B_\tau = p(\epsilon B_\tau, \tau) + \int_0^\tau rKe^{-r(\tau-\xi)} \aleph \left(-d_2 \left(\epsilon B_\tau, \tau - \xi, B_\xi\right)\right) d\xi$$

And differentiating both sides with respect to ϵ and taking the limit $\epsilon \to 1$,[2]

$$B_\tau = \left[\aleph\left(d_1\left(B_\tau, \tau, K\right)\right) + \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{1}{2} d_1\left(B_\tau, \tau, K\right)^2\right\}\right]^{-1} \times$$
$$\times \left[Ke^{-r\tau} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{1}{2} d_2\left(B_\tau, \tau, K\right)^2\right\} + \frac{rK}{\sigma\sqrt{2\pi}} \int_0^\tau \frac{1}{\sqrt{\tau - \xi}} \exp\left\{-r(\tau - \xi) - \frac{1}{2} d_2\left(B_\tau, \tau - \xi, B_\xi\right)^2\right\} d\xi\right] \tag{3.2}$$

This is a very similar approach to the one proposed by Little et al. (2000), the main difference between the two methods is the process for calculating the optimal exercise boundary. Note that both sides of the equation are still dependable of B_τ, however Kim et al. (2013) propose a non-recursive method that allows to calculate the boundary more efficiently. Computationally this method may loose some accuracy (as it will use an approximation for the cumulative normal distribution function), but in terms of computational speed the gains are tremendous. This point will be analysed on the next chapter.

[2]See appendix B for the detailed steps.

Chapter 4

The iterative method

4.1 Calculating the optimal exercise boundary

The idea behind the iterative method is to consider the right hand side of the equation (3.1) as an approximation of B_τ. Denoting B_τ^k as the k – order approximation of B_τ, the iterative method can be described as follow:

$$B_\tau^{k+1} = \left[\aleph\left(d_1\left(B_\tau^k, \tau, K\right)\right) + \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{1}{2} d_1\left(B_\tau^k, \tau, K\right)^2\right\}\right]^{-1} \times$$

$$\times \left[K e^{-r\tau} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{1}{2} d_2\left(B_\tau^k, \tau, K\right)^2\right\} + \frac{rK}{\sigma\sqrt{2\pi}} \int_0^\tau \frac{1}{\sqrt{\tau-\xi}} \exp\left\{-r\left(\tau-\xi\right) - \frac{1}{2} d_2\left(B_\tau^k, \tau, B_\xi^k\right)^2\right\} d\xi \right] \tag{4.1}$$

In order to respect the terminal condition of the Black-Scholes PDE, we must impose that:

$$B_0 = K$$

which implies:

$$B_0^k = K$$

for all k.

Due to the fact that the American put will never be exercised if the asset value is greater than the exercise price ($S > K$) we can choose K for the first round approximation of the optimal exercise boundary.

$$B_\tau^0 = K$$

Making this substitution on the right-hand side of equation (4.1), will permit to derive the first-round approximation B_τ^1 explicitly:

$$B_\tau^1 = \left[\aleph\left(d_1\left(K, \tau, K\right)\right) + \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{1}{2} d_1\left(K, \tau, K\right)^2\right\}\right]^{-1} \times$$

$$\times \left[K e^{-r\tau} \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{-\frac{1}{2} d_2\left(K, \tau, K\right)^2\right\} + \frac{K\sqrt{r}}{\sigma\sqrt{2}} erf\left(\sqrt{r\tau}\right) \exp\left\{-\frac{\tau\left(\sigma^2 - 2r\right)^2}{8\sigma^2}\right\}\right] \tag{4.2}$$

This method is repeated until convergence is obtained. Note that equation (4.1) contains only one integral (instead of the double integral of equation (2.3)). To calculate this integral we have used the *Gauss-Kronrod rule* as the original paper suggest.[1] We also followed the proposed method to determine B_ξ at each iteration, via

[1] This method comes detailed on Kronrod (1964) and Gautschi (1999).

polynomial interpolation.

4.2 Calculating delta hedging

In order to calculate the delta hedging (or any other Greek) we take a step back and remember the standard formula for the *live* American put:

$$P(S,\tau) = p(S,\tau) + \int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-d_2\left(S,\tau-\xi,B_\xi\right)\right)d\xi$$

As this formula can be interpreted as a sum of an European put option and premium for early exercising the option, we can obtain the delta factor by calculating the delta of the European option and deriving the last term of the equation above.[2]

$$\Delta = \frac{\partial}{\partial S}P(S,\tau) = -\aleph\left(-d_1\left(S,\tau,K\right)\right) - \frac{rK}{\sqrt{2\pi}\sigma S}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi) - \frac{1}{2}d_2\left(S,\tau-\xi,B_\xi\right)^2\right\}d\xi \tag{4.3}$$

4.3 Computing the iterative method

Valuing an American option or any of the corresponding Greeks, requires that one must start by calculating the optimal exercise boundary B_τ.

This boundary can be calculated recursively through equation (3.2):

$$B_\tau^{k+1} = \left[\aleph\left(d_1\left(B_\tau^k,\tau,K\right)\right) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1\left(B_\tau^k,\tau,K\right)^2\right\}\right]^{-1}\times$$

$$\times\left[Ke^{-r\tau}\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_2\left(B_\tau^k,\tau,K\right)^2\right\} + \frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi) - \frac{1}{2}d_2\left(B_\tau^k,\tau,B_\xi^k\right)^2\right\}d\xi\right]$$

Computationally this can be programmed by the following steps:

- **Step 1**: set $(n+1)$ to be the number of nodes for time to maturity, and k for the number of iterations in line with the accuracy needed - the higher the value of n and k more accurate will be our estimative of the boundary;
- **Step 2**: set $B_\tau^0 = K$ and use equation (4.2) to calculate B_τ^1;
- **Step 3**: for each $k = 2, 3, ...$:
  - **Step 3a**: create the function B_ξ^{k-1} by approximating, via polynomial of degree n, the function B_τ^{k-1};
  - **Step 3b**: calculate B_τ^k by substituting B_τ^{k-1} on the right-hand side of equation (4.2);
- **Step 4**: calculate the value of option using equation (2.3) (or calculate the *delta* using (4.3)).

[2] See appendix B for detailed steps.

Chapter 5

Results

5.1 Calculating the optimal exercise boundary

In this section we will test how Kim et al.'s (2013) method behaves when calculating the optimal exercise boundary and later compare it against some traditional and well-known numerical methods.

Across this section we will consider the value of the binomial tree method (Cox et al. (1979)) with 10000 time steps to be the exact value of the put option and consider this value as our benchmark to compare it against the iterative method.

All routines were programmed on MATLAB R2012a and run on a 2.4 GHz Intel i7-4700MQ processor.

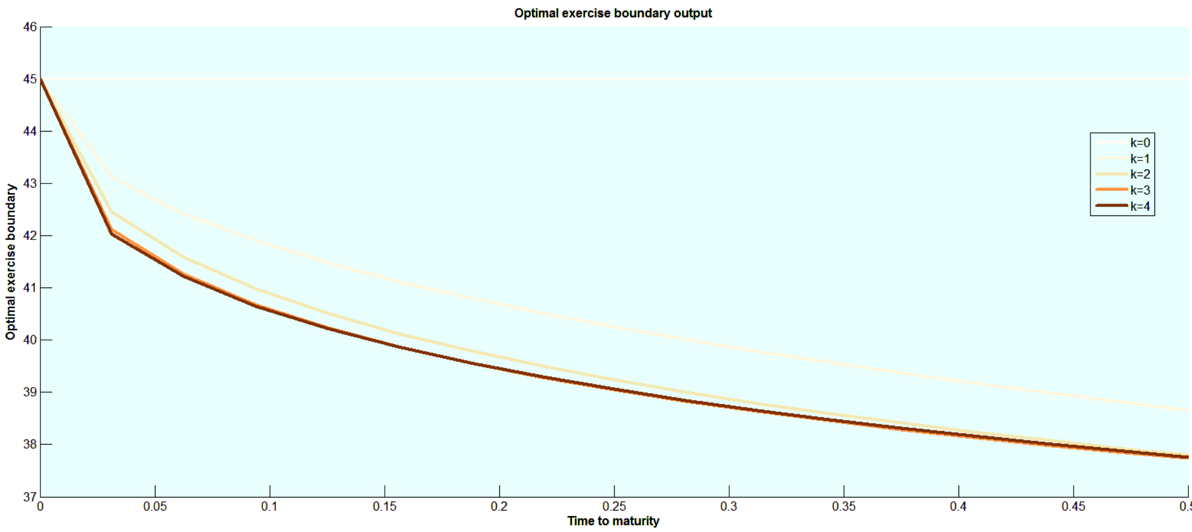

Figure 5.1: The convergence of the optimal exercise boundary

Figure 5.1 shows a few iterations of the iterative method with parameters $r = 0.05$, $\sigma = 0.2$, $K = 45$, $T = 0.5$ and the specific parameters of accuracy $n = 16$ nodes and $k = 0, 1, 2, ...$ The convergence of the method for calculating the optimal exercise boundary is rather quick, as the optimal exercise boundary does not change much after three iterations. The computational time for calculating 4 iterations was 0.03 seconds.

5.2 Calculating American put options

As seen in the previous sections, the iterative method converges quickly to a stable optimal exercise boundary, although for better accuracy we will increase the number of iterations and node points. Table 5.1 shows both value of the optimal exercise boundary and American put value.

Parameters	n	B_T^{10}	Put value	Benchmark
I. Baseline case				
$\sigma = 0.2, K =$US45, T = 1$ year	4	36.3704	2.7524	
	8	36.3881	2.7438	
	16	36.3922	2.7415	
	32	36.3933	2.7409	2.7406
II. Change in σ				
$\sigma = 0.15, K =$ US45, T = 1$ year	4	39.0978	1.9152	
	8	39.1124	1.9074	
	16	39.1160	1.9054	
	32	39.1170	1.9049	1.9047
$\sigma = 0.25, K =$ US45, T = 1$ year	4	33.6764	3.6014	
	8	33.6949	3.5919	
	16	33.6991	3.5894	
	32	33.7002	3.5888	3.5885
III. Change in K				
$\sigma = 0.2, K =$ US43, T = 1$ year	4	34.7539	1.9078	
	8	34.7708	1.9003	
	16	34.7748	1.8982	
	32	34.7759	1.8976	1.8975
$\sigma = 0.2, K =$ US47, T = 1$ year	4	37.9868	3.7998	
	8	38.0053	3.7908	
	16	38.0097	3.7885	
	32	38.0108	3.7879	3.7876
IV. Change in T				
$\sigma = 0.2, K =$ US45, T = 0.5$ year	4	37.7490	2.1016	
	8	37.7602	2.0968	
	16	37.7629	2.0955	
	32	37.7635	2.0952	2.0950
$\sigma = 0.2, K =$ US45, T = 3$ year	4	34.2922	3.9479	
	8	34.3191	3.9271	
	16	34.3256	3.9216	
	32	34.3274	3.9203	3.9197

Table 5.1: The convergence of the calculated values of the optimal exercise boundary using 10 iterations and various number of node points n. The defaults parameters are $r = 0.05$ ans $S = 45$.

As expected the accuracy of the method is better as the number of nodes points increases. Note that with only 10 iterations, the error between the method and the benchmark value is bellow $1.00e - 3$.

There is a slightly difference between the put value and the ones presented in the original paper. The major difference is that, for $n = 32$ the values presented on the paper are generally below the benchmark value, which suggest that if we choose $n > 32$ the error against the benchmark could increase. This can be due to various reasons: different software rounding approximations; bad interpolation method (for small T and high n the value of consecutive node points are almost identical). This problem was not found on our computation, preserving our initial belief: the higher the number of node points, the better accuracy of the method.

On the following tables we show the results of the iterative method calculating a large number of options and compare it against the binomial tree method (Cox et al. (1979)) and Smart Binomial method. [1]

[1]The Smart Binomial method is in theory the same method as the binomial tree method. The difference between them is the way they are implemented: the Smart Binomial method is programmed on a single vector opposed to binomial tree which is based on a matrix. This allows to the Smart Binomial method to be significantly quicker while maintaining the same numerical results

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	0.65	6.30e-3	100	0.12	6.30e-3	3	6	0.29	6.40e-3
	1000	13.02	6.15e-4	1000	10.11	6.15e-4	4	8	0.31	2.80e-3
	2000	41.85	2.98e-4	2000	40.38	2.98e-4	6	10	0.32	2.00e-3
	5000	232.35	1.15e-4	5000	254.28	1.15e-4	8	16	0.35	8.23e-4
	10000	868.15	0	10000	981.85	0	10	32	0.45	2.10e-4
$\sigma = 0.4$										
	100	0.65	1.74e-2	100	0.11	1.74e-2	3	6	0.30	1.68e-2
	1000	13.56	1.70e-2	1000	10.10	1.70e-2	4	8	0.31	5.00e-3
	2000	43.30	8.19e-4	2000	39.75	8.19e-4	6	10	0.32	3.00e-3
	5000	251.55	3.17e-4	5000	255.31	3.17e-4	8	16	0.35	1.20e-4
	10000	975.81	0	10000	1,160.96	0	10	32	0.46	2.98e-4

Table 5.2: The results are the computation times and RMSEs for the values of 601 American put option contracts with 601 underlying stock prices (from S = US\$90 to US\$120, in US\$ 0.05 steps). The default parameters are $r = 0.05$, $K = 100$ and $T = 0.5$.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	1.71	1.06e-2	100	0.31	1.06e-2	3	6	0.76	2.92e-2
	1000	35.48	8.95e-4	1000	27.27	8.95e-4	4	8	0.77	1.83e-2
	2000	118.31	4.19e-4	2000	107.48	4.19e-4	6	10	0.77	1.20e-2
	5000	694.94	1.56e-4	5000	677.66	1.56e-4	8	16	0.81	4.80e-3
	10000	2,316.64	0	10000	2,669.94	0	10	32	0.92	1.20e-3
$\sigma = 0.4$										
	100	1.80	2.65e-2	100	0.3	2.65e-2	3	6	0.75	4.54e-2
	1000	36.99	2.50e-3	1000	26.70	2.50e-3	4	8	0.77	2.87e-2
	2000	117.69	1.30e-3	2000	106.42	1.3e-3	6	10	0.78	2.02e-2
	5000	706.33	5.42e-4	5000	714.51	5.42e-4	8	16	0.80	8.00e-3
	10000	2,642.20	0	10000	2,964.22	0	10	32	1.00	2.10e-3

Table 5.3: The results are the computation times and RMSEs for the values of 1601 American put option contracts with 1601 underlying stock prices (from S = US\$80 to US\$160, in US\$0.05 steps). The default parameters are $r = 0.05$, $K = 100$ and $T = 5$. The results of the iterative method using 6 iterations and 25 node points ($n = 6$, $k = 25$) are RMSE = 2.00e-3 in 0.84 seconds, considering $\sigma = 0.2$ and RMSE = 3.40e-3 in 1.03 seconds, considering $\sigma = 0.4$.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	0.58	1.80e-3	100	0.10	1.80e-3	3	6	0.52	2.10e-3
	1000	12.01	1.95e-4	1000	8.79	1.95e-4	4	8	0.68	7.90e-4
	2000	37.61	9.38e-5	2000	34.95	9.38e-5	6	10	1.08	5.76e-4
	5000	200.11	3.61e-5	5000	220.45	3.61e-5	8	16	1.91	2.40e-4
	10000	824.13	0	10000	877.46	0	10	32	4.26	6.55e-5
$\sigma = 0.4$										
	100	0.58	6.30e-3	100	0.10	6.30e-3	3	6	0.51	5.50e-3
	1000	12.18	6.08e-4	1000	8.71	6.08e-4	4	8	0.69	1.70e-3
	2000	37.83	2.82e-4	2000	34.36	2.82e-4	6	10	1.08	9.45e-4
	5000	211.94	1.13e-4	5000	231.15	1.13e-4	8	16	1.93	3.97e-4
	10000	940.51	0	10000	1,013.08	0	10	32	4.30	1.09e-4

Table 5.4: The results are the computation times and RMSEs for the values of 520 American put option contracts with 20 underlying stock prices (from $S =$ US\$41 to US\$60, in US\$ 1 steps) and 26 maturities (from $T = 0.25$ to 0.5 years, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45$.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	4.55	4.10e-3	100	0.73	4.10e-3	3	6	3.55	1.30e-2
	1000	93.01	3.73e-4	1000	64.84	3.73e-4	4	8	5.52	7.90e-3
	2000	291.23	1.75e-4	2000	261.35	1.75e-4	6	10	8.38	5.20e-3
	5000	1,541.34	7.64e-5	5000	1,610.43	7.64e-5	8	16	15.09	2.10e-3
	10000	5,807.96	0	10000	6,769.09	0	10	32	33.35	5.35e-4
$\sigma = 0.4$										
	100	4.55	1.28e-2	100	0.73	1.28e-2	3	6	4.08	1.89e-2
	1000	93.01	1.20e-3	1000	64.76	1.20e-3	4	8	5.51	1.11e-2
	2000	292.71	5.96e-4	2000	258.23	5.96e-4	6	10	8.40	7.90e-3
	5000	1,609.60	2.44e-5	5000	1,710.30	2.44e-4	8	16	15.40	3.10e-3
	10000	6,745.37	0	10000	7,743.37	0	10	32	32.49	8.05e-4

Table 5.5: The results are the computation times and RMSEs for the values of 4020 American put option contracts with 20 underlying stock prices (from $S =$ US\$41 to US\$60, in US\$ 1 steps) and 201 maturities (from $T = 3$ to 5 years, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45$. The results of the iterative method using 6 iterations and 25 node points ($n = 6$, $k = 25$) are RMSE = 8.62e-4 in 16.42 seconds, considering $\sigma = 0.2$, and RMSE = 1.30e-3 in 16.71 seconds, considering $\sigma = 0.4$.

5.3 Calculating the delta hedge ratios

To calculate the delta hedge ratio we can use equation (4.3) and apply the iterative method making the respective adjustment to Step 4 detailed on the previous chapter.
For the numerical methods, we will use the *central difference* form with change in $\Delta S = 0.01$:

$$\Delta = \frac{\partial}{\partial S} P\left(S, \tau\right) = \frac{P\left(S + 0.01, \tau\right) - P\left(S - 0.01, \tau\right)}{0.02}$$

On the following tables we show the results of the iterative method calculating the delta hedging ratios for a large number of options and compare it against the binomial tree method (Cox et al. (1979)) and Smart Binomial method.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	0.98	1.03e-2	100	0.15	1.03e-2	3	6	0.19	9.73e-4
	1000	19.53	3.30e-3	1000	13.41	3.30e-3	4	8	0.19	9.05e-4
	2000	62.07	2.30e-3	2000	53.24	2.30e-3	6	10	0.22	8.92e-4
	5000	315.77	1.60e-3	5000	334.86	1.60e-3	8	16	0.24	8.87e-4
	10000	1,201.52	0	10000	1,336.61	0	10	32	0.33	8.87e-4
$\sigma = 0.4$										
	100	0.91	1.58e-2	100	0.15	1.58e-2	3	6	0.19	1.50e-3
	1000	18.09	5.00e-3	1000	13.53	5.00e-3	4	8	0.20	1.45e-3
	2000	61.14	3.70e-3	2000	53.83	3.70e-3	6	10	0.21	1.45e-3
	5000	343.55	2.60e-3	5000	356.12	2.60e-3	8	16	0.24	1.45e-3
	10000	1,302.39	0	10000	1,555.64	0	10	32	0.33	1.45e-3

Table 5.6: The results are the computation times and RMSEs for the hedge ratios of 401 American put option contracts with 401 underlying stock prices (from $S =$ US\$40 to US\$60, in US\$ 0.05 steps). The default parameters are $r = 0.05$, $K = 45$ and $T = 0.5$.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	1.72	9.10e-3	100	0.27	9.10e-3	3	6	0.33	1.60e-3
	1000	34.92	2.20e-3	1000	24.18	2.20e-3	4	8	0.35	8.19e-4
	2000	108.22	1.50e-3	2000	95.17	1.50e-3	6	10	0.36	6.72e-4
	5000	579.11	9.21e-4	5000	594.60	9.21e-4	8	16	0.38	5.48e-4
	10000	2,389.62	0	10000	2,472.62	0	10	32	0.47	5.24e-4
$\sigma = 0.4$										
	100	1.71	9.00e-3	100	0.27	9.00e-3	3	6	0.32	1.00e-3
	1000	34.70	2.80e-3	1000	24.05	2.80e-3	4	8	0.33	8.50e-4
	2000	108.24	2.10e-3	2000	95.09	2.10e-3	6	10	0.34	8.34e-4
	5000	610.30	1.40e-3	5000	633.77	1.40e-4	8	16	0.37	8.29e-4
	10000	2,440.37	0	10000	2,760.74	0	10	32	0.46	8.28e-4

Table 5.7: The results are the computation times and RMSEs for the hedge ratios of 701 American put option contracts with 701 underlying stock prices (from $S =$ US\$35 to US\$70, in US\$ 0.05 steps). The default parameters are $r = 0.05$, $K = 45$ and $T = 5$. The results of the iterative method using 6 iterations and 25 node points ($n = 6$, $k = 25$) are RMSE = 5.23e-4 in 0.40 seconds, considering $\sigma = 0.2$, and RMSE = 8.28e-4 in 0.41 seconds, considering $\sigma = 0.4$.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	1.20	1.12e-2	100	0.20	1.12e-2	3	6	0.45	8.64e-4
	1000	24.16	3.00e-3	1000	17.17	3.00e-3	4	8	0.61	8.25e-4
	2000	77.39	2.10e-3	2000	69.63	2.10e-3	6	10	0.97	8.14e-4
	5000	413.12	1.50e-3	5000	436.06	1.50e-3	8	16	1.74	8.11e-4
	10000	1,717.31	0	10000	1,796.79	0	10	32	3.90	8.10e-4
$\sigma = 0.4$										
	100	1.23	1.48e-2	100	0.20	1.48e-2	3	6	0.45	1.43e-3
	1000	25.81	4.60e-3	1000	16.77	4.60e-3	4	8	0.61	1.39e-3
	2000	85.34	3.60e-3	2000	67.95	3.60e-3	6	10	0.97	1.39e-3
	5000	453.47	2.50e-3	5000	459.05	2.50e-3	8	16	1.76	1.39e-3
	10000	1,740.13	0	10000	1,839.03	0	10	32	3.81	1.39e-3

Table 5.8: The results are the computation times and RMSEs for the hedge ratios of 520 American put option contracts with 20 underlying stock prices (from S = US\$41 to US\$60, in US\$ 1 steps) and 26 maturities (from $T = 0.25$ to 0.5 years, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45$.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	9.84	6.50e-3	100	1.52	6.50e-3	3	6	3.44	7.24e-4
	1000	190.91	2.10e-3	1000	131.53	2.10e-3	4	8	4.78	6.22e-4
	2000	588.04	1.40e-3	2000	532.18	1.40e-3	6	10	7.29	5.99e-4
	5000	3,148.90	1.00e-3	5000	3,291.28	1.00e-3	8	16	13.26	5.88e-4
	10000	12,894.04	0	10000	13,579.72	0	10	32	29.61	5.87e-4
$\sigma = 0.4$										
	100	9.91	9.20e-3	100	1.55	9.20e-3	3	6	3.47	9.94e-4
	1000	200.80	3.10e-3	1000	136.49	3.10e-3	4	8	4.80	9.23e-4
	2000	617.27	2.40e-3	2000	531.63	2.40e-3	6	10	7.48	9.16e-4
	5000	3,635.95	1.80e-3	5000	3,740.55	1.80e-3	8	16	13.25	9.14e-4
	10000	15,508.95	0	10000	16,752.85	0	10	32	29.64	9.13e-4

Table 5.9: The results are the computation times and RMSEs for the hedge ratios of 4020 American put option contracts with 20 underlying stock prices (from S = US\$41 to US\$60, in US\$ 1 steps) and 201 maturities (from $T = 3$ to 5 years, in 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45$. The results of the iterative method using 6 iterations and 25 node points ($n = 6$, $k = 25$) are RMSE = 5.87e-4 in 14.55 seconds, considering $\sigma = 0.2$, and RMSE = 9.14-4 in 14.64 seconds, considering $\sigma = 0.4$.

Chapter 6

Extension to American option with dividends

6.1 Calculating the optimal exercise boundary

The iterative method can also be applied to value American options whose underlying asset pays dividends. To make it possible we must adjust our background and derive a new formula for the optimal exercise boundary. Considering the underlying asset S pays continuous proportional dividends at a rate $q > 0$, the asset price follows the diffusion process represented on equation (2.1):

$$dS_t = (r - q)\, S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

The Black-Scholes PDE for the American put takes the form:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - q)\, S \frac{\partial P}{\partial S} - rP = \frac{\partial P}{\partial \tau}$$

with the same terminal and boundary conditions as equation (2.2). Following Kim (1990), the valuation formula for the American put, for $S > B_\tau$, is:

$$\tilde{P}(S, \tau) = \tilde{p}(S, \tau) + \int_0^\tau rKe^{-r(\tau-\xi)} \aleph\left(-\tilde{d}_2(S, \tau - \xi, B_\xi)\right) - qSe^{-q(\tau-\xi)} \aleph\left(-\tilde{d}_1(S, \tau - \xi, B_\xi)\right) d\xi \quad (6.1)$$

where $\aleph$ is the cumulative distribution function for the standard normal distribution with:

$$\tilde{d}_1(S, \tau, B) = \frac{ln\left(\frac{S}{B}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$\tilde{d}_2(S, \tau, K) = \tilde{d}_1(S, \tau, B) - \sigma\sqrt{\tau}$$

and $\tilde{p}(S, \tau)$ stands for the Black-Scholes formula for the European put option:

$$\tilde{p}(S, \tau) = Ke^{-r\tau} \aleph\left(-\tilde{d}_2(S, \tau, K)\right) - Se^{-q\tau} \aleph\left(-\tilde{d}_1(S, \tau, K)\right)$$

Following the same approach as in Chapter 3 we can express S_τ as ϵB_τ where $\epsilon \in]0, 1]$:

$$\tilde{P}(S, \tau) = K - S_\tau = K - \epsilon B_\tau$$

Making this substitution on equation (6.1) we have:

$$K - \epsilon B_\tau = \tilde{p}(S,\tau) + \int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-\tilde{d}_2(S,\tau-\xi,B_\xi)\right) - qSe^{-q(\tau-\xi)}\aleph\left(-\tilde{d}_1(\epsilon B_\tau,\tau-\xi,B_\xi)\right)d\xi$$

and differentiating both sides with respect to ϵ and taking the limit $\epsilon \to 1$:

$$\begin{aligned} B_\tau \; = & \left[Ke^{-r\tau}\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}\tilde{d}_2(B_\tau,\tau,K)^2\right\} + \frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi)-\frac{1}{2}\tilde{d}_2(B_\tau,\tau,K)^2\right\}d\xi\right] \\ & \times\left[e^{-q\tau}\aleph\left(\tilde{d}_1(B_\tau,\tau,K)\right) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-q\tau-\frac{1}{2}\tilde{d}_1(B_\tau,\tau,K)^2\right\}\right. \\ & \left. + q\int_0^\tau e^{-r(\tau-\xi)}\aleph\left(-\tilde{d}_1(B_\tau,\tau-\xi,B_\xi)\right) + \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}r\exp\left\{-r(\tau-\xi)-\frac{1}{2}\tilde{d}_1(B_\tau,\tau-\xi,B_\xi)^2\right\}d\xi\right]^{-1} \end{aligned} \tag{6.2}$$

6.2 Calculating American put options

Using the same method as shown on section (4.3) we can use the equation (6.2) to determine iteratively a reliable approximation of the optimal exercise boundary. Once the optimal exercise boundary is achieved, the value of the option can be calculated using equation (6.1). On the following tables we show the results of the iterative method calculating a large number of options and compare it against the binomial tree method (Cox et al. (1979)) and Smart Binomial method.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	0.62	1.90e-3	100	0.10	1.90e-3	3	6	1.04	3.30e-3
	1000	12.04	2.07e-4	1000	8.82	2.07e-4	4	8	1.22	1.60e-3
	2000	39.90	9.96e-5	2000	221.57	9.96e-5	6	10	1.98	1.40e-3
	5000	201.13	3.84e-5	5000	254.28	3.84e-5	8	16	3.84	1.10e-3
	10000	842.64	0	10000	891.72	0	10	32	8.53	9.92e-4
$\sigma = 0.4$										
	100	0.60	6.40e-3	100	0.11	6.40e-3	3	6	0.93	5.60e-3
	1000	12.79	6.17e-4	1000	8.79	6.17e-4	4	8	1.21	1.80e-3
	2000	38.55	2.87e-4	2000	35.02	2.87e-4	6	10	1.94	1.10e-3
	5000	216.12	1.16e-4	5000	239.36	1.16e-4	8	16	3.65	6.10e-4
	10000	949.54	0	10000	1,025.14	0	10	32	8.11	3.66e-4

Table 6.1: The results are the computation times and RMSEs for the values of 520 American put option contracts with 20 underlying stock prices (from $S =$ US\$41 to US\$60, in US\$ 1 steps) and 26 maturities (from $T = 0.25$ to 0.5 years, in 0.01 year steps). The default parameters are $r = 0.05$, $q = 0.01$ and $K = 45$.

	Binomial			Smart Binomial			Kim			
	n	Time (sec)	RMSE	n	Time (sec)	RMSE	k	n	Time (sec)	RMSE
$\sigma = 0.2$										
	100	0.50	4.40e-3	100	0.08	4.40e-3	3	6	0.65	3.36e-2
	1000	10.05	4.21e-4	1000	7.17	4.21e-4	4	8	0.94	2.83e-2
	2000	31.42	2.10e-4	2000	28.37	2.10e-4	6	10	1.50	2.58e-2
	5000	179.66	8.87e-5	5000	176.41	8.87e-5	8	16	2.84	2.29e-2
	10000	633.07	0	10000	702.07	0	10	32	6.81	2.16e-2
$\sigma = 0.4$										
	100	0.52	1.37e-2	100	0.08	1.37e-2	3	6	0.65	2.20e-2
	1000	12.33	1.30e-3	1000	6.97	1.30e-3	4	8	0.94	1.18e-2
	2000	37.33	6.37e-4	2000	28.38	6.37e-4	6	10	1.53	8.60e-3
	5000	196.40	2.60e-4	5000	189.79	2.60e-4	8	16	2.90	4.20e-3
	10000	724.07	0	10000	830.36	0	10	32	6.93	2.00e-3

Table 6.2: The results are the computation times and RMSEs for the values of 420 American put option contracts with 20 underlying stock prices (from $S =$ US\$41 to US\$60, in US\$ 1 steps) and 21 maturities (from $T = 3$ to 5 years, in 0.1 year steps). The default parameters are $r = 0.05$, $q = 0.01$ and $K = 45$.

Chapter 7

Conclusion

In this thesis we analysed and tested the recursive formula for the optimal exercise boundary derived by Kim et al. (2013), which can be used iteratively to value American put options and their Greeks. Compared to some well-known numerical methods, like the binomial tree, Kim et al.'s (2013) method proved to be very competitive both in terms of precision and effectiveness.

Table 5.1 shows how solid this method is when changing the various parameters of the American option contract: the method managed to reach values within $1e-03$ of the benchmark through all tests.

Although the method is not perfect, and depending on the circumstances in which it is being used, it might be slower or lose some accuracy compared to the binomial method (on his *Smart* programming descendent).

The crucial factor is how many times the iterative method needs to calculate the optimal exercise boundary.

The optimal exercise boundary is calculated using the exercise price of the option, maturity, volatility, risk-free interest rate and dividend yield (K, T, σ, r and q respectively). This depends directly on the options we are valuing: if those 4 parameters are equal on all options, the optimal exercise boundary is the same, making the Kim et al.'s (2013) method extremely quick and precise; if one those parameters change within consecutive options the optimal exercise boundary needs to be recalculated. On the "worst" case scenario, i.e. when all options have different optimal exercise boundaries, there is the need to calculate an increased number of integrals which from a computational point of view requires some additional time and accumulates some numerical approximations. Even so, Kim et al.'s (2013) method is still faster than the binomial method (6 to 8 times faster).

This problem can be minimized by meticulously choosing the order of the options we want to value: for instance, if we need to value the 520 options on table 5.4, we should put together all the 20 options with maturity $T = 0.25$ first, and then another 20 options with maturity $T = 0.26$, and repeat successively.

One of the main advantages of the integral representation methods over the numerical ones is the calculation of the Greeks. Using a numerical method, one must value 2 options to calculate the wanted sensitivity factor. On the other hand, integral representation methods like the one presented by Kim et al. (2013), offer an exact formula to calculate the Greeks, which hugely reduces computational time required. Once again, Kim et al.'s (2013) method proved to be extremely fast and precise.

Overall, Kim et al.'s (2013) method proves to be a valid and competitive alternative to the current methods.

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Appendix A

Useful formulas

$$\frac{\partial d_{\bullet}\left(S,t,B\right)}{\partial S}=\frac{1}{S\sigma\sqrt{t}}$$

$$\aleph\left(-x\right)=1-\aleph\left(x\right)$$

$$\frac{\partial\aleph\left(x\right)}{\partial x}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^{2}}$$

$$\frac{\partial\aleph\left(d_{\bullet}\left(S,t,B\right)\right)}{\partial S}=\frac{\partial\aleph\left(d_{\bullet}\left(S,t,B\right)\right)}{\partial d_{\bullet}\left(S,t,B\right)}\frac{\partial d_{\bullet}\left(S,t,B\right)}{\partial S}=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_{\bullet}(S,t,B)^{2}}\frac{1}{S\sigma\sqrt{t}}=\frac{1}{S\sigma\sqrt{2\pi t}}e^{-\frac{1}{2}d_{\bullet}(S,t,B)^{2}}$$

Appendix B

Optimal exercise boundary implicit formula

$$\frac{\partial}{\partial \epsilon}\left[K-\epsilon B_\tau\right]=\frac{\partial}{\partial \epsilon}\left[p\left(\epsilon B_\tau\right)+\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

$$\Leftrightarrow \frac{\partial}{\partial \epsilon}\left[K-\epsilon B_\tau\right]=\frac{\partial}{\partial \epsilon}\left[Ke^{-r\tau}\aleph\left(-d_2\left(\epsilon B_\tau,\tau,K\right)\right)-\epsilon B_\tau\aleph\left(-d_1\left(\epsilon B_\tau,\tau,K\right)\right)\right]+$$

$$+\frac{\partial}{\partial \epsilon}\left[\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

The left-hand side of the equation:

$$\frac{\partial}{\partial \epsilon}\left[K-\epsilon B_\tau\right]=-B_\tau$$

The first term of the right-hand side of the equation:

$$\frac{\partial}{\partial \epsilon}\left[p\left(\epsilon B_\tau\right)\right]=\frac{\partial}{\partial \epsilon}\left[Ke^{-r\tau}\aleph\left(-d_2\left(\epsilon B_\tau,\tau,K\right)\right)-\epsilon B_\tau\aleph\left(-d_1\left(\epsilon B_\tau,\tau,K\right)\right)\right]$$

$$=Ke^{-r\tau}\frac{\partial}{\partial \epsilon}\left[1-\aleph\left(d_2\left(\epsilon B_\tau,\tau,K\right)\right)\right]-B_\tau\frac{\partial}{\partial \epsilon}\left[\epsilon\left(1-\aleph\left(d_1\left(\epsilon B_\tau,\tau,K\right)\right)\right)\right]$$

$$=Ke^{-r\tau}\frac{\partial}{\partial \epsilon}\left[1-\aleph\left(d_2\left(\epsilon B_\tau,\tau,K\right)\right)\right]-B_\tau\frac{\partial}{\partial \epsilon}\left[\epsilon\right]+B_\tau\frac{\partial}{\partial \epsilon}\left[\epsilon\aleph\left(d_1\left(\epsilon B_\tau,\tau,K\right)\right)\right]$$

$$=-Ke^{-r\tau}\frac{1}{\epsilon B_\tau\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau,K\right)^2\right\}B_\tau-B_\tau$$

$$+B_\tau\left[\aleph\left(d_1\left(\epsilon B_\tau,\tau,K\right)\right)+\epsilon\frac{1}{\epsilon B_\tau\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1\left(\epsilon B_\tau,\tau,K\right)^2\right\}B_\tau\right]$$

Rearranging:

$$\frac{\partial}{\partial \epsilon}\left[p\left(\epsilon B_\tau\right)\right]=-K\frac{1}{\epsilon\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau,K\right)^2\right\}-B_\tau$$

$$+B_\tau\aleph\left(d_1\left(\epsilon B_\tau,\tau,K\right)\right)+B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1\left(\epsilon B_\tau,\tau,K\right)^2\right\}$$

The second term of the right side of equation:

$$\frac{\partial}{\partial \epsilon}\left[\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]=rK\int_0^\tau e^{-r(\tau-\xi)}\frac{\partial}{\partial \epsilon}\left[1-\aleph\left(d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)\right]d\xi$$

$$= -rK\int_0^\tau e^{-r(\tau-\xi)}\left[\frac{1}{\epsilon B_\tau \sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)^2\right\}B_\tau\right]d\xi$$

Rearranging:

$$\frac{\partial}{\partial\epsilon}\left[\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

$$= -\frac{rK}{\epsilon\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi)-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)^2\right\}d\xi$$

Grouping all together:

$$-B_\tau = -K\frac{1}{\epsilon\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau,K\right)^2\right\} - B_\tau + B_\tau\aleph\left(d_1\left(\epsilon B_\tau,\tau,K\right)\right)$$

$$+B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1\left(\epsilon B_\tau,\tau,K\right)^2\right\} - \frac{rK}{\epsilon\sigma\sqrt{2\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi)-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)^2\right\}d\xi$$

Rearranging:

$$B_\tau\aleph\left(d_1\left(\epsilon B_\tau,\tau,K\right)\right) + B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1\left(\epsilon B_\tau,\tau,K\right)^2\right\} = K\frac{1}{\epsilon\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau,K\right)^2\right\}$$

$$+\frac{rK}{\epsilon\sigma\sqrt{2\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi)-\frac{1}{2}d_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)^2\right\}d\xi$$

Taking the limit $\epsilon\to 1$:

$$B_\tau\aleph\left(d_1\left(B_\tau,\tau,K\right)\right) + B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1\left(B_\tau,\tau,K\right)^2\right\} = K\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau-\frac{1}{2}d_2\left(B_\tau,\tau,K\right)^2\right\}$$

$$+\frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi)-\frac{1}{2}d_2\left(B_\tau,\tau-\xi,B_\xi\right)^2\right\}d\xi$$

Rearranging the equation:

$$B_\tau = \left[\aleph\left(d_1\left(B_\tau,\tau,K\right)\right)+\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1\left(B_\tau,\tau,K\right)^2\right\}\right]^{-1}$$

$$\left[\frac{K}{\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau-\frac{1}{2}d_2\left(B_\tau,\tau,K\right)^2\right\}+\frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi)-\frac{1}{2}d_2\left(B_\tau,\tau-\xi,B_\xi\right)^2\right\}d\xi\right]$$

Appendix C

Delta hedging formula

$$\Delta = \frac{\partial}{\partial S} P(S,\tau) = -\aleph\left(-d_1(S,\tau,K)\right) - \frac{rK}{S\sigma\sqrt{2\pi}} \int_0^\tau \frac{1}{\sqrt{\tau-\xi}} \exp\left\{-r(\tau-\xi) - \frac{1}{2} d_2\left(S,\tau-\xi,B_\xi\right)^2\right\} d\xi$$

Remembering that:

$$P(S,\tau) = p(S,\tau) + \int_0^\tau rKe^{-r(\tau-\xi)} \aleph\left(-d_2\left(S,\tau-\xi,B_\xi\right)\right) d\xi$$

$$\Delta = \frac{\partial P(S,\tau)}{\partial S} = \frac{\partial}{\partial S}\left[p(S,\tau)\right] + \frac{\partial}{\partial S}\left[\int_0^\tau rKe^{-r(\tau-\xi)} \aleph\left(-d_2\left(S,\tau-\xi,B_\xi\right)\right) d\xi\right]$$

The first member is the delta of a regular European put option. [1]

$$\frac{\partial}{\partial S}\left[p(S,\tau)\right] = -\aleph\left(-d_1(S,\tau,K)\right)$$

The second member can be derived as follows:

$$\frac{\partial}{\partial S}\left[\int_0^\tau rKe^{-r(\tau-\xi)} \aleph\left(-d_2\left(S,\tau-\xi,B_\xi\right)\right) d\xi\right] = \frac{\partial}{\partial S}\left[rK\int_0^\tau e^{-r(\tau-\xi)}\left[1-\aleph\left(d_2\left(S,\tau-\xi,B_\xi\right)\right)\right] d\xi\right]$$

$$= -rK\int_0^\tau e^{-r(\tau-\xi)}\left[\frac{1}{S\sigma\sqrt{2\pi(\tau-\xi)}}\exp\left\{-\frac{1}{2}d_2\left(S,\tau-\xi,B_\xi\right)^2\right\}\right] d\xi$$

$$= -\frac{rK}{S\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r(\tau-\xi)-\frac{1}{2}d_2\left(S,\tau-\xi,B_\xi\right)^2\right\} d\xi$$

[1] See Huang et Subrahmanyam (1996)

Appendix D

Optimal exercise boundary implicit formula with dividends

$$\frac{\partial}{\partial \epsilon}\left[K-\epsilon B_\tau\right]=$$

$$\frac{\partial}{\partial \epsilon}\left[\tilde{p}\left(\epsilon B_\tau,\tau\right)+\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-\tilde{d}_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)-q\epsilon B_\tau e^{-q(\tau-\xi)}\aleph\left(-\tilde{d}_1\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

$$\Leftrightarrow \frac{\partial}{\partial \epsilon}\left[K-\epsilon B_\tau\right]=\frac{\partial}{\partial \epsilon}\left[\tilde{p}\left(\epsilon B_\tau,\tau\right)\right]+\frac{\partial}{\partial \epsilon}\left[\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-\tilde{d}_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

$$-\frac{\partial}{\partial \epsilon}\left[\int_0^\tau q\epsilon B_\tau e^{-q(\tau-\xi)}\aleph\left(-\tilde{d}_1\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

The left-hand side of the equation:

$$\frac{\partial}{\partial \epsilon}\left[K-\epsilon B_\tau\right]=-B_\tau$$

Noting that $\frac{\partial}{\partial \epsilon}\tilde{d}_\bullet=\frac{\partial}{\partial \epsilon}d_\bullet$, the first term of the right-hand side of the equation can be easily obtained by making an adjustment to the calculation of Appendix B:

$$\frac{\partial}{\partial \epsilon}\left[\tilde{p}\left(\epsilon B_\tau,\tau\right)\right]=\frac{\partial}{\partial \epsilon}\left[Ke^{-r\tau}\aleph\left(-\tilde{d}_2\left(\epsilon B_\tau,\tau,K\right)\right)-\epsilon B_\tau e^{-q\tau}\aleph\left(-\tilde{d}_1\left(\epsilon B_\tau,\tau,K\right)\right)\right]$$

$$=-K\frac{1}{\epsilon\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau-\frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau,\tau,K\right)^2\right\}-B_\tau e^{-q\tau}+e^{-q\tau}B_\tau\aleph\left(\tilde{d}_1\left(\epsilon B_\tau,\tau,K\right)\right)$$

$$+B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau,\tau,K\right)^2\right\}$$

Using the same argument we can easily derive the second term of the right side of equation:

$$\frac{\partial}{\partial \epsilon}\left[\int_0^\tau rKe^{-r(\tau-\xi)}\aleph\left(-\tilde{d}_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

$$=-\frac{rK}{\epsilon\sigma\sqrt{2\pi}}\int_0^\tau\frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r\left(\tau-\xi\right)-\frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)^2\right\}d\xi$$

The third term of the right-hand side of the equation can be derived as:

$$\frac{\partial}{\partial \epsilon}\left[\int_0^\tau q\epsilon B_\tau e^{-q(\tau-\xi)}\aleph\left(-\tilde{d}_1\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)d\xi\right]$$

$$=qB_\tau\frac{\partial}{\partial \epsilon}\left[\int_0^\tau e^{-q(\tau-\xi)}\epsilon\left[1-\aleph\left(\tilde{d}_1\left(\epsilon B_\tau,\tau-\xi,B_\xi\right)\right)\right]d\xi\right]$$

$$= qB_\tau \times$$

$$\times \int_0^\tau e^{-q(\tau-\xi)}\epsilon \left[1 - \aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)\right) - \epsilon \frac{1}{\epsilon B_\tau \sigma\sqrt{2\pi}\left(\tau-\xi\right)} \exp\left\{-\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} B_\tau\right] d\xi$$

Rearranging:

$$\frac{\partial}{\partial\epsilon}\left[\int_0^\tau q\epsilon B_\tau e^{-q(\tau-\xi)}\aleph\left(-\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)\right) d\xi\right] =$$

$$\frac{qB_\tau}{\sigma\sqrt{2\pi}}\int_0^\tau e^{-q(\tau-\xi)} - e^{-q(\tau-\xi)}\aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)\right) - \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-q\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} B_\tau d\xi$$

Grouping all together:

$$-B_\tau = -K\frac{1}{\epsilon\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau - \frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau, \tau, K\right)^2\right\} - B_\tau e^{-q\tau} + e^{-q\tau}B_\tau\aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau, K\right)\right)$$

$$+B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau, K\right)^2\right\} - \frac{rK}{\epsilon\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi$$

$$-\frac{qB_\tau}{\sigma\sqrt{2\pi}}\int_0^\tau e^{-q(\tau-\xi)} - e^{-q(\tau-\xi)}\aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)\right) - \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-q\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} B_\tau d\xi$$

Rearranging:

$$-B_\tau + B_\tau e^{-q\tau} - e^{-q\tau}B_\tau\aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau, K\right)\right) - B_\tau\frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-q\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau, K\right)^2\right\} +$$

$$\frac{qB_\tau}{\sigma\sqrt{2\pi}}\int_0^\tau e^{-q(\tau-\xi)} - e^{-q(\tau-\xi)}\aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)\right) - \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-q\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi$$

$$= -K\frac{1}{\epsilon\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau - \frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau, \tau, K\right)^2\}\right) - \frac{rK}{\epsilon\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi$$

Rearranging the equation:

$$-B_\tau\left[1 - e^{-q\tau} + e^{-q\tau}\aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau, K\right)\right) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-q\tau - \frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau, K\right)^2\right\}\right.$$

$$\left.-\frac{q}{\sigma\sqrt{2\pi}}\int_0^\tau e^{-q(\tau-\xi)} - e^{-q(\tau-\xi)}\aleph\left(\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)\right) - \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-q\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_1\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi\right]$$

$$= \left[-\frac{K}{\epsilon\sigma\sqrt{2\pi\tau}}\exp\left\{-r\tau - \frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau, \tau, K\right)^2\}\right) - \frac{rK}{\epsilon\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_2\left(\epsilon B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi\right]$$

Taking the limit $\epsilon \to 1$:

$$B_\tau = \left[1 - e^{-q\tau} + e^{-q\tau}\aleph\left(\tilde{d}_1\left(B_\tau, \tau, K\right)\right) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-q\tau - \frac{1}{2}\tilde{d}_1\left(B_\tau, \tau, K\right)^2\right\}\right.$$

$$\left.-\frac{q}{\sigma\sqrt{2\pi}}\int_0^\tau e^{-q(\tau-\xi)} - e^{-q(\tau-\xi)}\aleph\left(\tilde{d}_1\left(B_\tau, \tau-\xi, B_\xi\right)\right) - \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-q\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_1\left(B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi\right]^{-1}$$

$$\times\left[\frac{K}{\sigma\sqrt{2\pi\tau}}\exp\left\{+r\tau - \frac{1}{2}\tilde{d}_2\left(B_\tau, \tau, K\right)^2\}\right) - \frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}\exp\left\{-r\left(\tau-\xi\right) - \frac{1}{2}\tilde{d}_2\left(B_\tau, \tau-\xi, B_\xi\right)^2\right\} d\xi\right]$$