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An Alternative Approach to the Valuation of American Options and Applications

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Abstract. In this paper we examine the structure of American option valuation problems and derive the analytic valuation formulas under general underlying security price processes by an alternative but intuitive method. For alternative diffusion processes, we derive closed-form analytic valuation formulas and analyze the implications of asset price dynamics on the early exercise premiums of American options. In this regard, we introduce useful and interesting diffusion processes into American option-pricing literature, thus providing a wide range of choices of pricing models for various American-type derivative assets. This work offers a useful analytic framework for empirical testing and practical applications such as the valuation of corporate securities and examining the impact of options trading on market micro-structure.

1. Introduction

The valuation of American options has drawn growing interest from the finance community. Recently, some theoretical research has generated analytic valuation formulas for American options by formulating the valuation problems as free-boundary problems, not only making the structure of American option values more transparent but also providing computational advantages for the option value and option hedge parameters. Kim (1990) obtains the analytic valuation formulas for American options by taking the limits of Geske and Johnson (1984) discrete valuation formulas. Carr, Jarrow, and Myneni (1992) rigorously prove these valuation formulas by studying the trading strategies that convert American options into the corresponding European options and using the generalized Ito's calculus.[1] Although ingenious, their proofs are not only insufficient to improve our intuition but also specific to the lognormal diffusion assumption on the underlying security price.

Although the lognormal assumption of security price has been a tradition in option pricing theory, it is rather restrictive. Previous empirical studies on the behaviors of stock prices, such as Black (1976), Christie (1982), and MacBeth and Merville (1980), have uniformly presented evidence against the lognormal postulation, but supportive of changing stock price volatility. Studies on the current issues, such as the effects of trading mechanisms and various market imperfections, have also articulated further supports for the nonlognormal stock price dynamics. Therefore, to better reconcile the option pricing model with the actual stock price dynamics so as to facilitate empirical testing and to study option price behaviors, it is also necessary to develop American option pricing models for alternative stock price movement assumptions.

The current paper makes two contributions on the valuation of American options. First, we take an alternative but intuitive approach to derive the analytic valuation formulas of American options for general underlying security price processes. By doing so, we generalize the literature on American options in an important way. Second, in complement to the analysis of Cox and Ross (1976) on European option pricing, we derive and analyze the analytic valuation formulas of American options under alternative diffusion processes. These valuation models can have many potential applications in various finance problems.

Our intuitive derivation of the analytic valuation formulas for American options comes in an analogy to the study of Black and Cox (1976) on the valuation of corporate fixed-income securities. Such corporate securities can be decomposed into three mutually independent sources of value: the security's value at the reorganization boundary, its value at maturity given that the reorganization boundary has not been reached, and its value from cash flows to be received until maturity or the reorganization boundary, whichever comes first, is reached. The American option values can be analyzed in the same manner. For example, the value of an American put would be determined by its value at the optimal exercise boundary and its value at expiration given that the optimal exercise boundary has not been reached. However, it is important to understand two main differences between the valuation of corporate fixed-income securities as analyzed in Black and Cox (1976) and the valuation of American options. First, the optimal exercise boundary for American options must be determined endogenously as a part of the solution, while the reorganization boundary for corporate fixed-income securities is exogenously specified. Second, the reorganization of the fixed-income securities at the boundary are generally not self-financing, normally with some loss of the security values.[2] On the other hand, the "reorganization" of American options at the early exercise boundary are always self-financing as guaranteed by the optimal early exercise conditions—the so-called smooth pasting conditions. For an American option, these two points reflect the balance of costs and benefits of early exercise decisions. This is the reason that it is important to recognize the American option-valuation problems as free-boundary problems, a convention first adopted by McKean (1965) and Merton (1973).

We derive the analytic valuation formulas by applying the value decomposition result mentioned above and the costs and benefits of early exercising an American option. Therefore, naturally, our intuitive but rigorous proof of the analytic valuation formulas directly complements the Kim's (1990) logic and the dynamic trading strategy of Carr, Jarrow, and Myneni (1992) to convert an American option into its European counterpart. Moreover, since our deriviation is straightforward and involves simple mathematics, we introduce the intuition and the valuation methodology of American options to a wider audience.

Furthermore, since our approach does not depend on a specific distributional assumption on the underlying security price, our valuation formulas are more general than the existing option pricing models. This enables us to obtain closed-form analytic valuation formulas for diffusion processes that incorporate changing volatility and some special characteristics of stock price dynamics. Specifically, we derive valuation formulas for several typical diffusion processes studied in finance literature: the absorbing Gaussian diffusion, the residual volatility diffusion, and the CEV diffusion. These diffusion processes offer the alternatives to the lognormal diffusion process in different ways. For the lognormal diffusion itself, we also generalize the option pricing literature by introducing absorbing barriers, for which important applications can be perceived. These valuation formulas provide us with a convenient workhorse to examine how asset price dynamics affects option values and the early exercise premiums of American options. For instance, the effect of absorption at a barrier can have important implications for the dynamic hedging of options. Moreover, these valuation formulas offer a useful analytic framework for empirical testing and practical applications such as the valuation of some corporate securities with American option feature and the impact of options trading on market microstructure, for examples, for examining the efficiency of options markets and the options price behavior under noise trading and capital structure reorganizations.

The paper is organized as follows. We prove in Section 2 the analytic valuation formulas for American options for a general diffusion process. Then in Section 3 we present the analytic valuation formulas for some typical diffusion processes that have been studied in finance literature. Conclusions follow in Section 4.

2. Risk-neutral valuation of American options

In this section, we offer an alternative yet intuitive derivation of the analytic valuation formulas for American options. For expositional convenience, the presentation in this section will be exclusively on *live* American put options on common stocks, although this approach is well applicable to any underlying financial asset.

Consider an American put that has an exercise price of K and expires at time T. Assume the option is written on a common stock paying continuous proportional dividends at a rate of $\alpha > 0$. Let S_t denote the stock price at time t. Also assume perfect markets and continuous trading and no arbitrage opportunities. Finally, assume the interest rate $r > 0$ is constant.

We invoke the risk-neutral valuation framework of Cox and Ross (1976) and Harrision and Kreps (1979). Assume the risk-neutral stock price process follows a diffusion process (generated by a Brownian motion on a complete filtration, and) with well-defined local drift and variance parameters, respectively,

$$dS_t = (r-\alpha)S_t dt + \sigma(S,t)\,dZ_t, \tag{1}$$

where $\sigma(S,t)$ is a continuous and positive function of S_t and t, and Z_t is a standard Brownian motion. The drift parameter of this diffusion process that specifies the local expected return of the security is $r-\alpha$. Let $\Psi(S_\xi; S_t)$ be the risk-neutral transition density function of the stock price S_ξ for the time $\xi > t$ given that the time t stock price is S_t. It is well known that $\Psi(S_\xi; S_t)$ satisfies

$$e^{-\alpha(\xi-t)}S_t = \int_0^\infty e^{-r(\xi-t)}S_\xi \Psi(S_\xi; S_t)\,dS_\xi. \tag{2}$$

As in McKean (1965) and Merton (1973), we define the American put value as the solution to a free-boundary problem.[3] Assume the early exercise boundary for this American put is well-defined and is unique and has continuous sample path. Let $\mathcal{B} \equiv \{B_t, t \in [0, T]\}$ denote the optimal early exercise boundary of the put, and denote the American put value at time t by $P(S_t, t)$, which is differentiable with respect to t and twice differentiable with respect to S_t, defined on the domain $\{(S, t); 0 \leq B_t \leq S_t \leq \infty, 0 \leq t \leq T\}$, where $S_t \geq 0$ is assumed to incorporate the limited liability of stock. In order to derive the analytic formula for the put price, we (as conventions) temporarily treat the early exercise boundary as known. Then the American put price, $P(S_t, t)$, satisfy the following boundary conditions:

$$P(S_T, T) = \max[0, K - S_T], \tag{3}$$

$$\lim_{S_t \uparrow \infty} P(S_t, t) = 0, \tag{4}$$

$$\lim_{S_t \downarrow B_t} P(S_t, t) = K - B_t, \tag{5}$$

$$\lim_{S_t \downarrow B_t} \frac{\partial P(S_t t)}{\partial S_t} = -1. \tag{6}$$

Condition (5) is called the *value matching* condition and condition (6) is called the *super-contact* condition, and they are jointly referred to as the *smooth pasting* conditions. They jointly ensure that the premature exercise of the put option on the *endogenously determined* early exercise boundary, $\mathcal{B}$, will be optimal and self-financing.

As discussed in the introduction, following a direct analogy to the work of Black and Cox (1976), we recognize that an American put has two sources of value: the immediate exercise value at the exercise boundary and its payoff at expiration if the American put has not been exercised prematurely. Let s be the time when the stock price reaches the early exercise boundary $\mathcal{B}$ for the first time, or the first passage time of the stock price to boundary $\mathcal{B}$. Define $p(s, \mathcal{B}; S_t)$ as the risk-neutral density function of the first passage time s given that the current stock price is S_t.[4] Then, the first part of the American put value is the discounted expected value of $K - B_s$, with expectation taken with respect to the density function, $p(s, \mathcal{B}; S_t)$. Let $\phi(S_T; S_t)$ be the risk-neutral density function of the terminal stock price S_T given that the stock price has not reached the exercise boundary until expiration. Then the second component of the value is the discounted expected value of $\max[K - S_T, 0]$ under the density function $\phi(S_T; S_t)$. Thus, we have the following equation for the put price:

$$\begin{aligned} P(S_t, t) &= \int_t^T e^{-r(s-t)}(K - B_s)p(s, \mathcal{B}; S_t)\, ds \\ &\quad + \int_{B_T}^K e^{-r(T-t)}(K - S_T)\phi(S_T; S_t)\, dS_T, \end{aligned} \tag{7}$$

where B_T is the ex-expiration-date early exercise boundary. We will further characterize B_T later.

To simplify equation (7), consider the balance of costs and benefits associated with the early exercise of the American put. Suppose the investor has a covered put position. By exercising the put, the investor can earn interest on the exercise proceeds—that is, the exercise price. The costs associated with early exercise is that the investor has to forego dividends from stock and may regret early exercise if the stock price declines further at a later time. Clearly, early exercise of the American put option would be optimal if the benefits outweigh the costs associated with early exercise.

Because of the smooth pasting conditions (5) and (6), the put buyer can implement the following dynamic trading strategy starting with the optimal exercise policy at time s. He optimally exercises the put at s by shorting one share of the stock and investing the proceeds of $\$K$ into riskless bonds. As time passes, whenever the stock price crosses the exercise boundary from below, the investor instantaneously establishes anew a put position and holds this put position as long as the stock price is above the exercise boundary. On the other hand, whenever the stock price falls below the early exercise boundary, the investor instantaneously converts his option position into a portfolio of shorting one share of stock and investing $\$K$ in riskless bonds. Since the investor carries out these transactions only on the early exercise boundary, the smooth pasting conditions (5) and (6) ensure that these transactions are self-financing and optimal. At the option's expiration date, the investor liquidates all his positions.

Consider the payoffs of implementing this trading strategy. Before expiration, the investor continuously pays dividends on the short stock position and continuously receives riskless interests on the long bond positions whenever the stock price is below the early exercise boundary. By following the trading strategy, the investor is guaranteed the payoff of $\max[K - S_T, 0]$ at the option's expiration date, T. Since the initiation cost of this trading strategy is $K - B_s$, the following equation must hold to prevent arbitrage:

$$\begin{aligned} K - B_s &= \int_s^T e^{-r(\xi - s)}\, d\xi \int_0^{B_\xi} (rK - \alpha S_\xi)\Psi(S_\xi, B_s)\, dS_\xi \\ &\quad + \int_0^K e^{-r(T-s)}(K - S_T)\Psi(S_T; B_s)\, dS_T. \end{aligned} \tag{8}$$

Clearly, equation (8) is expressed in the way to show the terms representing the benefits and costs associated with early exercise. It is similar to the dynamic trading strategy of Carr, Jarrow, and Myneni (1992) to converting an American option into the corresponding European option. Notice that equation (8) is also equivalent to the "value matching" condition (5) defining the optimal early exercise price at time s, B_s. Over time, equation (8) implicitly defines the optimal early exercise boundary, $\mathcal{B}$.

We use equation (8) to evaluate the first term of equation (7). By straightforward simplification, we have

$$\begin{aligned} \int_t^T e^{-r(s-t)}(K - B_s)p(s, \mathcal{B}; S_t)\, ds &= e^{-r(T-t)} \int_0^k (K - S_T)\, dS_T \\ &\quad \times \int_t^T p(s, \mathcal{B}; S_t)\Psi(S_T; B_s)\, ds \end{aligned}$$

$$+ \int_t^T e^{-r(\xi-t)}\, d\xi \int_0^{B_\xi} (rK - \alpha S_\xi)\, dS_\xi$$
$$\times \int_t^\xi p(s, \mathcal{B}; S_t)\Psi(S_\xi; B_s)\, ds. \tag{9}$$

Notice that $\int_t^\xi p(s, \mathcal{B}; S_t)\Psi(S_\xi; B_s)\, ds$, $t \le s \le \xi \le T$, is the transition density function of the stock price at time ξ given that at time s the stock price had crossed the boundary from above. For $S_\xi \le B_\xi$ such transition density function is identical to the unconditional transition density $\Psi(S_\xi, S_t)$. Combining (9) with (7), we have the following expression for the American put price:

$$P(S_t, t) = p(S_t, t) + \int_t^T e^{-r(\xi-t)}\, d\xi \int_0^{B_\xi} (rK - \alpha S_\xi)\Psi(S_\xi; S_t)\, dS_\xi, \tag{10}$$

where

$$p(S_t, t) = e^{-r(T-t)} \int_0^K (K - S_T)\Psi(S_T; S_t)\, dS_T, \tag{11}$$

is the price of a European put with the same terms as the American put option under consideration. Naturally, the second term is the early exercise premium of the American put.

Thus, the American put value can be expressed as the sum of corresponding European put value and the early exercise premium. While the European put value stems from the minimum guaranteed payoff of the American put, the early exercise premium represents the value attributable to the right of exercising the option early. From the aforementioned trading strategy, this value comes from the excess of expected benefits (interests earned on the exercise proceeds) over the expected losses (losses of dividend payouts and better exercise opportunities in the future) associated with early exercise.

Finally, we characterize the ex-expiration-date early exercise boundary, B_T, also in a simple and intuitive manner.[5] Clearly $B_T \le K$. If the option is exercised at the ex-expiration-date, say at $T - dt$, then the instantaneous value that accrues to the investor over $[T - dt, T]$ is $(rK - \alpha B_T)dt$, in addition to the expiration date payoff $\max[K - S_T, 0]$. To justify early exercise of the option, we must have $(rK - \alpha B_T)dt \ge 0$, implying $B_T \le rK/\alpha$. In conclusion, the ex-expiration-date early exercise boundary for the American put is

$$B_T = \min\left(\frac{rK}{\alpha}, K\right). \tag{12}$$

Therefore, when $\alpha \le r$, since $B_T = K$, the American put will have a zero payoff at expiration even if it has not been exercised early. This is because it is not possible for the stock price at expiration to fall below K without crossing the exercise boundary at an earlier time. When $\alpha > r$, $B_T < K$, then the American put can have a positive value at expiration given that it has not been exercised early. This indicates large dividend payouts reduce the incentives of early exercise.

Following an analogous logic, one can decompose the American call price, $C(S_t, t)$, in the same manner:

$$C(S_t, t) = c(S_t, t) + \int_t^T e^{-r(\xi - t)}\, d\xi \int_{B_\xi}^{\infty} (\alpha S_\xi - rK)\Psi(S_\xi; S_t)\, dS_\xi, \tag{13}$$

where $c(S_t, t) = e^{-r(T-t)} \int_0^K (S_T - K)\Psi(S_T; S_t)\, dS_T$ denotes the price of an otherwise identical European call. The second term represents the early exercise premium of the American call. In addition, each term of (13) can be given a similar dynamic trading strategy interpretation as given to the American put option. Moreover, it can be easily established that for the American call option under consideration, its ex-expiration-date early exercise boundary B_T is

$$B_T = \max\left(\frac{rK}{\alpha}, K\right).$$

Therefore, the less the dividend payouts, the less likely that the call will be exercised on or before the expiration. In particular, when the underlying security does not pay dividends or $\alpha \to 0$, $B_T \to \infty$, consistent with the well-known result that the American call reduces to an otherwise identical European call (Merton, 1973).

Note that equations (10) and (13) hold for general diffusion processes having positive and continuous local volatility function, $\sigma^2(S, t)$, as represented by (1), and with well-defined transition density function. Therefore, once the risk-neutral transition density function of the diffusion process is explicitly specified, more explicit valuation formulas for American options could be obtained. In many cases, the American option values can be numerically computed using equations (10) and (13) by first estimating the early exercise boudnary $\mathcal{B}$ from solving the value matching condition or the supercontact condition recursively, such as (8) and (6) for an American put, whichever is more convenient for implementation. Moreover, for a diffusion process that admits explicit valuation formulas, various option hedge parameters can also be conveniently derived and computed. This is the advantage of the analytic valuation approach over other approximation methods for computing American option prices. For the implementation procedures of these valuation formulas, see Huang, Subrahmanyam, and Yu (1996).

3. Option valuation for alternative diffusion processes

Based on the fact that the previously derived analytic formulas hold for very general diffusion processes, in this section we derive closed-form valuation formulas of American options for some typical diffusion processes that have been studied in finance literature. These processes offer alternative asset price dynamics other than the widely used lognormal diffusion. For ease of presentation, we focus on American put price and assume the underlying security does not pay any dividends throughout the option's expiration date. All the results presented below can be easily generalized to the case where the underlying security pays proportional dividends and can be extended to American call options. Furthermore, for conciseness, we also present only the valuation formulas for the put prices and leave out the analytic formulas of various option hedging ratios and the implicit equations that define the early exercise boundaries. The formulas for option hedge parameters are straightforward to obtain. The precise representations of the early exercise boundary equation should be clear from equation (8). In addition, the typical diffusion processes are chosen to simplify the presentation, the analysis on more general diffusion processes can be easily carried out by employing the time and scale transformations used in Goldenberg (1991) in pricing European options.

3.1. Absorbing guassian diffusion

Consider the diffusion process S_t that follows a Gaussian process with constant volatility σ^2. Under risk neutrality, S_t has the representation

$$dS_t = rS_t dt + \sigma dz_t. \tag{14}$$

Our first example considers the case when the security price, also denoted as S_t, follows the Gaussian process (14) with an absorption state at zero.[6] Such process is usually called the Ornstein-Uhlenbeck process with absorption at zero and diffusion coefficient σ and is also called by Cox and Ross (1976) as the absolute process.

This diffusion process gives a simple process of security price movements. Using this process to describe the security price movements essentially assumes that the security price follows a random walk but with the possibility of bankruptcy. As a direct application, it can be used to model the prices of some corporate securities for which the firms become financially distressed but fail to reorganize.

The absorbing Gaussian process is a continuous Markov process whose distribution consists of two parts: a discrete part and a continuous part. The discrete part gives the probability of absorption at any given time; the continuous part of the distribution describes the likelihood of transition between two non-absorbing states. For put options, both parts of the distrubution accrue values to the options. The transition density of the continuous part is (see, for example, Karlin and Taylor, 1975; Goldenberg 1991)

$$\Psi(S_t; S_0) = q(S_t; S_0) - q(S_t; -s_0), \quad S_0 \geq 0, S_t \geq 0, \tag{15}$$

where $q(S_t; S_0)$ is the transition density function of process (14) without absorption—namely,

$$q(S_t; S_0) = \frac{1}{\sqrt{2\pi\Phi_t}} \exp\left\{-\frac{(S_t - S_0 e^{rt})^2}{2\Phi_t}\right\}, \quad \text{with} \quad \Phi_t = \frac{\sigma^2}{2r}\left(e^{2rt} - 1\right). \tag{16}$$

The discrete part of the distribution (i.e., the probability of absorption at time t) is

$$\begin{aligned} Pr(S_t = 0 \mid S_0) &= 1 - \int_0^\infty \Psi(S_t; S_0)\, dS_t \\ &= 2N\left(-\frac{S_0 e^{rt}}{\sqrt{\Phi_t}}\right), \end{aligned} \tag{17}$$

where $N(x) \equiv \int_{-\infty}^{x} \frac{\exp(-y^2/2)}{\sqrt{2\pi}}\, dy$ is the standard normal distribution function.

Table 1. Put prices under absorbing Gaussian process: $T = 0.5$, $r = 8\%$, $S = 100$ and $K = 100$

σ	5	20	35	50	65
European put	0.2235	3.7901	7.8447	11.9552	16.0829
American put	0.5502	4.1789	8.1930	12.2749	16.3822
Premium	0.3267	0.3888	0.3483	0.3197	0.2993

Denote the American put price by P_0, then from (10), (15), and (17), the American put price can be obtained as

$$P_0 = p_0 + rK \int_0^T e^{-rt} \left[N\left(\frac{B_t - S_0 e^{rt}}{\sqrt{\Phi_t}}\right) + N\left(-\frac{B_t + S_0 e^{rt}}{\sqrt{\Phi_t}}\right)\right] dt, \tag{18}$$

where p_0 is the corresponding European put price

$$\begin{aligned} p_0 &= e^{-rT} E\{\max[K - S_T, 0] \mid S_0\} \\ &= (e^{-rT} - S_0) N\left(\frac{K - S_0 e^{rT}}{\sqrt{\Phi_T}}\right) + (Ke^{-rT} + S_0) N\left(-\frac{K + S_0 e^{rT}}{\sqrt{\Phi_T}}\right) \\ &\quad + e^{-rT}\sqrt{\Phi_T}\left[N'\left(\frac{K - S_0 e^{rT}}{\sqrt{\Phi_T}}\right) - N'\left(\frac{K + S_0 e^{rT}}{\sqrt{\Phi_T}}\right)\right], \end{aligned}$$

where $E(\cdot)$ represents the expectation taken under the risk neutral transition density function $\Psi(S_T; S_0)$. The pricing formulas for European calls under this absorbing Gaussian diffusion has been recently obtained by Goldenberg (1991).

From (18), we can easily see the effect of absorption on the early exercise premium of the American put. Notice that from (15), the possibility of absorption at state 0 makes the transition of stock price trajectories from S_0 to S_t less likely. Therefore, given the likelihood of early exercise at one point of time, the investor will be less likely to regret the early exercise decision in the future. In return, there is higher probability that the investor will exercise the option early. This would result in larger early exercise premium to the put than the case when the security price does not absorb at state 0. This is consistent with the intuition that when a firm faces the possibility of bankruptcy, a put option written on its stock becomes more valuable since it does not have any downside risk but retains all the upside potentials.

We compare the early exercise boundaries of American puts under absorbing Gaussian diffusion for different volatility parameters. The curves obtained from using the recursive implementation method are depicted in Figure 2, and the corresponding European and American put prices are given in Table 1.

There are two points that can be observed from the above Table 1 and Figure 2. First, when the volatility of the absorbing Gaussian increases, both the European and American put prices become larger. This is consistent with our intuition that when the underlying stock price becomes more volatile, put option prices increase due to the increased likelihood of the stock price to reach the lower states.

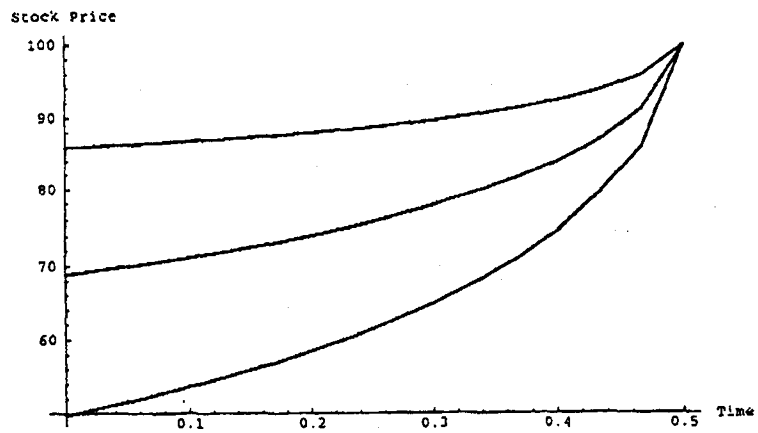

Figure 1. Early exercise boundaries under absorbing Gaussian diffusion. The early exercise boundary is approximated using the recursive implementation method detailed in Huang, Subrahmanyam and Yu (1996) with $n = 15$. The three early exercise boundaries in the graph correspond to, from top down, the absorbing Gaussian diffusion with volatility level $\sigma = 20$, $\sigma = 35$, and $\sigma = 50$. Other parameters are $T = 6$ months, $r = 8\%$, and $K = 100$.

On the other hand, we observe that when the volatility level increases, the early exercise premium to the American put first increases and then gradually decreases. This can be understood by the fact that when the volatility level is low, increases in the volatility level will increase the likelihood of early exercise and with larger exercise payoff. However, when the volatility level is high, because of the possibility of absorption, further increases in the volatility level reduce the likelihood of early exercise. The reason for this is that with the possibility of absorption and with higher volatility, the investor will tend to delay exercising the option unless the stock price is extremely low. Thus, essentially the incentive of early exercise is somehow reduced when there is a very high level of volatility. This is also reflected in the position of early exercise boundary when the volatility level is very high (like in the case of $\sigma = 50$, the boundary is much lower compared with the other two cases).

3.2. *Lognormal diffusions*

We now return to the widely studied lognormal diffusion processes. We first summarize the valuation results under the regular lognormal diffusion process and then provide an important generalization to this literature.

If the security price S_t follows a lognormal diffusion with constant return volatility, $\sigma^2 > 0$, then S_t has the risk-neutral representation

$$\frac{dS_t}{S_t} = r\,dt + \sigma dz_t. \tag{19}$$

Under (19), the price of alive American put, $P(S_t, t)$, is the solution to the following Black-Scholes (1973) fundamental partial differential equation subject to the boundary conditions (3) through (6):

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 P_t}{\partial S^2} + rS_t\frac{\partial P_t}{\partial S} - rP_t + \frac{\partial P_t}{\partial t} = 0, \tag{20}$$

As a result, the American put price (10) reduces to the analytic valuation formula obtained by Kim (1990), Jacka (1991), and Carr, Jarrow, and Myneni (1992):

$$P_0 = p_0 + rK\int_0^T e^{-rt} N\left(\frac{\ln(B_t/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right) dt, \tag{21}$$

where as before $N(x) \equiv \int_{-\infty}^{x} \frac{\exp(-y^2/2)}{\sqrt{2\pi}} dy$ is the standard normal distribution function and p_0 is the corresponding Black-Scholes European put price:

$$p_0 = Ke^{-rT} N\left(\frac{\ln(K/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) - S_0 N\left(\frac{\ln(K/S_0) - (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right).$$

Moreover, from (13), the price of an American call on the proportional dividend paying security is

$$C_0 = c_0 + \int_0^T \left[\alpha S_0 e^{-\alpha t} N(d_1) - rKe^{-rt} N(d_1 - \sigma\sqrt{t})\right] dt, \tag{22}$$

where $d_1 = \frac{\ln(S_0/K) + (r - \alpha + \sigma^2/2)t}{\sigma\sqrt{t}}$ and c_0 is the corresponding Black-Scholes European call price. Equation (22) is also given in Kim (1990).

Finally, in the lognormal diffusion case, we can also explicitly obtain the valuation formula and the early exercise boundary of a perpetual American option. For a perpetual American put, the fundamental partial differential equation (20) reduces to an Euler-type ordinary differential equation—namely, the last term of (20) vanishes. Consequently, the fundamental partial differential equation and the supercontact condition (6) can be readily solved to obtain the American put price and the perpetual early exercise boundary. From Merton (1973), the early exercise boundary and the option price are, respectively,

$$\begin{aligned} B_t &= \frac{K\theta}{1+\theta}, \quad \text{for any } t \geq 0, \\ P_0 &= \frac{K}{1+\theta}\left[\frac{(1+\theta)S_0}{\theta K}\right]^{-\theta}, \end{aligned}$$

where $\theta = 2r/\sigma^2$. They respectively serve as the natural lower boundaries of the option price and the early exercise boundary of a short-term American put option.

Due to the properties of price positivity and the resulting analytical tractability, the lognormal diffusion has been widely used in the asset-pricing and option-pricing literature to describe security price movements. However, the lognormal diffusion does not permit the possibility of bankruptcy and does not allow changing volatility. This is the reason that researchers have sometimes used the absorbing Gaussian diffusion to introduce the possibility of bankruptcy and the CEV diffusion to model bankruptcy and changing return volatility. In the rest of this section, we extend the literature on option valuation under lognormal diffusion to include barriers. In this context, we consider one case of lognormal diffusion with an absorbing barrier at a lower state. This *absorbing lognormal diffusion* provides us with another way of introducing bankruptcy but still retains the tractability that the lognormal diffusion offers.

3.2.1. Lognormal diffusion with an absorbing barrier at $A < K$

We consider the case in which the security price follows a lognormal diffusion but with an absorbing barrier at a constant A, $0 < A < K$. This price behavior of this process is that it follows (19) as long as $S_t > A$. Once it hits the barrier A from above, the process is absorbed at state A permanently.

This type of security price dynamics arises when the security is backed by some collaterals or insurance commitments. When the firm underlying the traded security becomes financially distressed but is not able to reorganize either through formal Chapter 11 proceedings or private restructurings, in both cases the firm is liquidated and the security holders either receive a pro rata share of the liquidation value of the firm (depending on the seniority of the security) or receive the guaranteed payoffs from the collaterals or insurance commitments, which are greater than zero under many circumstances. Thus, by modeling the security price behavior this way, we introduce the possibility of bankruptcy into the lognormal diffusion process.

To value American options under this diffusion process, we apply the same technique used for absorbing Gaussian diffusion. Since a lognormal process can never reach the state zero, the price dynamics for this diffusion is that the logarithmic stock price process

$$d(\ln S_t) = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dz_t \tag{23}$$

is absorbed at $\ln A$. This implies that the transition density for the continuous part of the distribution of this diffusion process is, for $S_0 \geq A$, $S_t \geq A$,

$$\Psi(S_t; S_0) = \frac{1}{\sqrt{2\pi\sigma^2 t S_t^2}} \left(e^{-\frac{[\ln(S_t/S_0)-(r-\sigma^2/2)t]^2}{2\sigma^2 t}} - e^{-\frac{[\ln(S_t S_0/A^2)-(r-\sigma^2/2)t]^2}{2\sigma^2 t}} \right). \tag{24}$$

The discrete part of the distribution representing the probability of absorption at time t is

$$Pr(S_t = A \mid S_0) = 1 - \int_A^\infty \Psi(S_t; S_0)\, dS_t$$

$$= N\left(\frac{\ln(A/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right)$$
$$+ N\left(\frac{\ln(A/S_0) + (r - \sigma^2/)t}{\sigma\sqrt{t}}\right). \tag{25}$$

Consequently, the American put price can be obtained from (10), (24), and (25):

$$P_0 = p_0 + rK\int_0^T e^{-rt}\left[N\left(\frac{\ln(B_t/S_0) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right)\right.$$
$$\left. + N\left(-\frac{\ln(B_tS_0/A^2) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right)\right] dt, \tag{26}$$

where p_0 is the corresponding European put price

$$p_0 = Ke^{-rT}\left[N\left(\frac{\ln(K/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) + N\left(-\frac{\ln(KS_0/A^2) - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right)\right]$$
$$-S_0\left[N\left(\frac{\ln(K/S_0) - (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right) + N\left(-\frac{\ln(KS_0/A^2) - (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right)\right].$$

The possibility of absorption at state $A > 0$ has two offsetting effects on the option price. On the one hand, the valuation formula (26) indicates that the possibility of absorption at $A > 0$, like in the case of absorbing Gaussian diffusion, results in higher values for both European and American puts. This is because the possibility of absorption makes it more likely that the security price will stay in the lower states and less likely to bounce back. Therefore, the investor is less likely to regret the early exercise decision in the future and the option values increase because of the increased likelihood of higher payoff from exercising the option.

On the other hand, the absorbing state A also restricts the state space of the stock price and decreases the option price. This is because when the absorbing state is relatively high, the less the values the stock price can take. Eventually, even with the possibility of absorption at A, the payoff from the option exercise may be quite small (compared with the case without the absorbing state). In the end, the option prices (both European and American) depend on joint effect of these two factors.

To examine the final effect on option prices and on the early exercise boundary, we compute option prices under absorbing lognormal diffusion with different absorbing states. The results are listed in Tables 2 and 3 and their corresponding early exercise boundaries are given in Figures 2 and 3.

These results support our earlier arguments. As can be seen from Table 2 and Figure 3, when the absorbing state is low, we do not see much effect of absorption on option prices. The changes on the option prices are quite small, and the early exercise boundaries almost cluster with that under the regular lognormal diffusion process. However, when the absorbing state is high, the actual option prices decrease because of the prevailing effect of the second factor (see Table 3). This is also reflected in Figure 4 in which the early exercise boundaries are pushed upward because the high absorbing state restricts the possible paths of the stock price and therefore results in reduced payoffs from option exercise.

Table 2. Put prices under absorbing lognormal diffusion: $T = 1$ year, $\sigma = 0.3$, $r = 8\%$, $S = 90$ and $K = 100$.

Absorbing state A	—	35	50	65
European put price	12.0633	12.0633	12.0596	11.2198
American put price	13.7562	13.7562	13.7539	13.1803
Premium	1.6929	1.6929	1.6943	1.9605

Table 3. Put prices under absorbing lognormal diffusion: $T = 1$ year, $\sigma = 0.4$, $r = 8\%$, $S = 90$ and $K = 100$

Absorbing state A	—	45	60	75
European put price	15.6344	15.6041	14.0092	6.9864
American put price	17.0889	17.0674	15.8477	10.8153
Premium	1.4545	1.4633	1.8385	3.8289

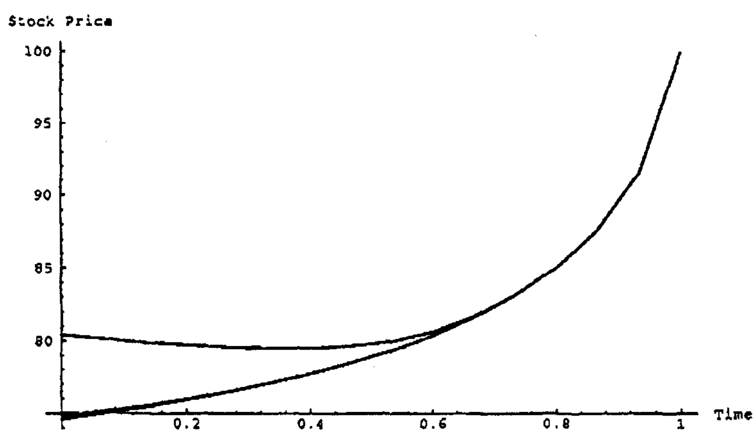

Figure 2. Early exercise boundaries under absorbing lognormal diffusion and under lognormal diffusion. The early exercise boundary is approximated using the recursive implementation method detailed in Huang, Subrahmanyam and Yu (1996) with $n = 15$. The four early exercise boundaries in the graph correspond to, from top down, absorbing lognormal diffusions with $A = 65$, $A = 50$, $A = 35$, and regular lognormal diffusion, respectively. Other parameters are $T = 1$ year, $\sigma = 0.3$, $r = 8\%$, and $K = 100$. The last three boundaries almost clustered together.

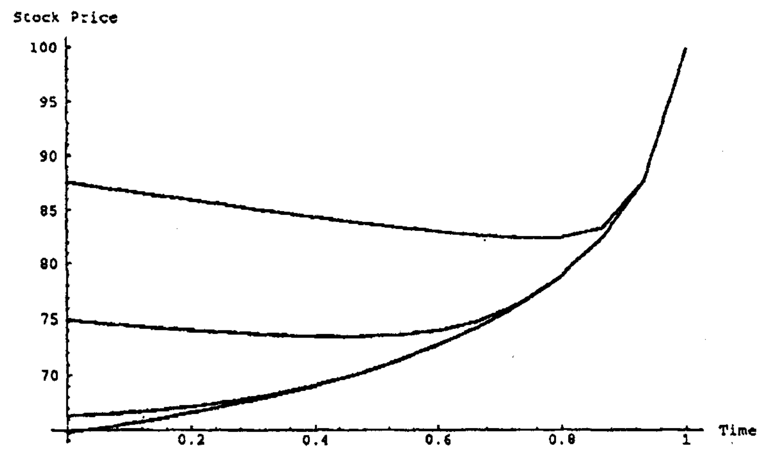

Figure 3. Early exercise boundaries under absorbing lognormal diffusion and under regular lognormal diffusion. The early exercise boundary is approximated using the recursive implementation method detailed in Huang, Subrahmanyam and Yu (1996) with $n = 15$. The four early exercise boundaries in the graph correspond to, from top down, absorbing lognormal diffusions with $A = 75$, $A = 60$, $A = 45$, and regular lognormal diffusion, respectively. Other parameters are $T = 1$ year, $\sigma = 0.4$, $r = 8\%$, and $K = 100$.

Another fact observed from the above tables is that even though the final option prices decrease as a result of the restricted stock price sample paths, the early exercise premium of the American option increases with the increases of the absorbing state. This is because when the absorbing state is higher, the stock price is more likely to be absorbed and therefore the option is more likely to be exercised earlier.

We also observe that as the absorbing state becomes higher, the early exercise boundary becomes slightly nonmonotonic: it moves downward slightly and then shoots up. This could be explained by the time effect of absorption. As the absorbing state becomes higher and as time goes on, in order to maximize the payoffs from early exercise so as to offset the decrease of the option's time value, the investor would tend to slightly reduce the critical value for option exercise and grab the possibility of absorption to the maximum extent. However, when the time to option expiration is very short, the chance of stock price being absorbed becomes slim, this leads the critical exercise prices to be pushed up.

Finally, we also examine the effect of volatility on the American option prices and early exercise premiums for a fixed absorbing state. The numerical results are given in Table 4 and the early exercise boundaries are graphed in Figure 5.

As we can see from the table, the increases in the volatility level lead to a higher American option value. However, the increase in option value does not come from increases in early exercise premium but rather from the increases in the corresponding European put value, resulted from the increased likelihood of absorption. Because of the possibility of absorption, the early exercise premium does not fluctuate much with the changes in volatility, since the increase in volatility somehow decreases the incentive of early exercise.

Table 4. Put prices under absorbing lognormal diffusion: $T = 6$ months, $\sigma = 0.3, r = 8\%, S = K = 100$ and $A = 60$

σ	0.1	0.2	0.3	0.4	0.5
European put price	1.2353	3.7854	6.4669	9.1457	11.5676
American put price	1.6549	4.2090	6.8808	9.5600	12.0189
Premium	0.4196	0.4237	0.4139	0.4144	0.4514

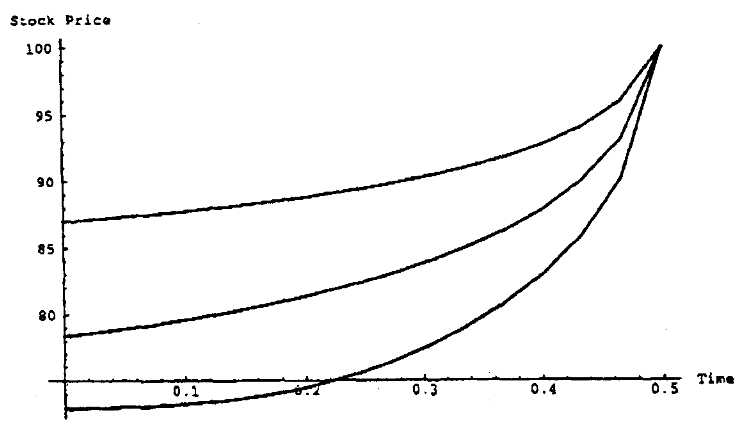

Figure 4. Early exercise boundaries under absorbing lognormal diffusion. The early exercise boundary is approximated using the recursive implementation method detailed in Huang, Subrahmanyam and Yu (1996) with $n = 15$. The three early exercise boundaries in the graph correspond to, from top down, absorbing lognormal diffusions with $\sigma = 0.2$, $\sigma = 0.3$, and $\sigma = 0.4$, respectively. Other paremeters are $T = 6$ months, $A = 60$, $r = 8\%$, and $K = 100$.

This intuition is clearly reflected in Figure 4 in the way of the shifts of the early exercise boundary with the changes in the volatility level.

The above numerical results also demonstrate that since it behaves quite like the widely studied lognormal diffusion, the aborbing lognormal diffusion can be a good choice for modeling the stock price movements, in which both price positivity and possibility of bankruptcy are preserved in a simple manner. By properly choosing the absorbing state, we can utilize it to model different security price processes, such as corporate securities under given capital structures.

3.3. *Residual volatility diffusion*

Our third typical example considers the case when the risk-neutral security price S_t follows the residual volatility process as represented by

$$dS_t = rS_t dt + \sqrt{2rS_t^2 + D}\, dz_t, \tag{27}$$

for some constant $D \geq 0$. When $D = 0$, (27) reduces to a lognormal diffusion (19) with return volatility $\sigma^2 = 2r$.

Intuitively, the constant D can be interpreted as the residual volatility of the security price due to other factors unrelated to the security itself, such as trading noise or market imperfections. In this sense, the lognormal diffusion (19) assumes zero residual volatility whereas the absorbing Gaussian diffusion considered before assume that all security price volatility is residual—that is, is generated by random factors other than the fundamentals of the traded security. Thus, this process offers a useful alternative of introducing changing volatility into security prices other than the CEV diffusion discussed below. For the same reason, it could be adapted to applications of market micro-structure studies of options tradings.

We assume that state 0 is the absorbing state for this process. Then the residual volatility diffusion process (27) is reducible to a standard Brownian motion process absorbed at zero by the following transformation:

$$F(S_t) = \frac{1}{\sqrt{2r}} \ln(S_t\sqrt{2r} + \sqrt{2rS_t^2 + D}) - \frac{\ln\sqrt{D}}{\sqrt{2r}}. \tag{28}$$

Therefore, for the continuous distribution of residual volatility diffusion, the risk-neutral transition density function of S_t given S_0 is

$$\begin{aligned} \Psi(S_t; S_0) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{[\ln(S_t\sqrt{2r}+\sqrt{2rS_t^2+D}) - \ln(S_0\sqrt{2r}+\sqrt{2rS_0^2+D})]^2}{4rt}} \cdot \frac{1}{\sqrt{2rS_t^2 + D}} \\ &\quad - \frac{1}{\sqrt{2\pi t}} e^{-\frac{[\ln(S_t\sqrt{2r}+\sqrt{2rS_t^2+D}) + \ln(S_0\sqrt{2r}+\sqrt{2rS_0^2+D})]^2}{4rt}} \cdot \frac{1}{\sqrt{2rS_t^2 + D}}. \end{aligned} \tag{29}$$

The discrete part of the distribution is

$$\begin{aligned} Pr(S_t = 0 \mid S_0) &= 1 - \int_0^\infty \Psi(S_t; S_0)\, dS_t, \\ &= 2N\left(-\frac{\ln(S_0)\sqrt{2r} + \sqrt{2rS_0^2 + D}}{\sqrt{2rt}} \right). \end{aligned} \tag{30}$$

From (10), (29) and (30), we can obtain the American put price as follows:

$$P_0 = p_0 + rK \int_0^T e^{-rt} [N(d_1) + N(-d_2)]\, dt, \tag{31}$$

where

$$d_1 = \frac{1}{\sqrt{2rt}} \ln \left[\frac{B_t + \sqrt{B_t^2 + \frac{D}{2r}}}{S_0 + \sqrt{S_0^2 + \frac{D}{2r}}} \right],$$

$$d_2 = \frac{1}{\sqrt{2rt}} \ln \left[\left(B_t + \sqrt{B_t^2 + \frac{D}{2r}} \right) \left(S_0 + \sqrt{S_0^2 + \frac{D}{2r}} \right) \right],$$

and p_0 is the corresponding European put price:

$$\begin{aligned} p_0 = {} & \frac{1}{2}\sqrt{S_0^2 + \frac{D}{2r}} \left[N\left(\frac{\ln \Delta_{S_0,K} + 2rT}{\sqrt{2rT}} \right) + N(-\beta - \sqrt{2rT}) \right] \\ & - \frac{S_0}{2} \left[N\left(-\frac{\ln \Delta_{S_0,K} + 2rT}{\sqrt{2rt}} \right) + N(\beta + \sqrt{2rT}) \right] \\ & - \frac{D}{2\left(2rS_0 + \sqrt{4rS_0^2 + D}\right)} \left[N\left(\frac{\ln \Delta_{S_0,K} - 2rT}{\sqrt{2rt}} \right) + N(-\beta + \sqrt{2rT}) \right] \\ & + Ke^{-rT} \left[N\left(-\frac{\ln \Delta_{S_0} K}{\sqrt{2rT}} \right) + N(-\beta) \right], \end{aligned}$$

with

$$\Delta_{S_0,K} = \frac{S_0 + \sqrt{S_0^2 + \frac{D}{2r}}}{K + \sqrt{K^2 + \frac{D}{2r}}},$$

$$\beta = \frac{1}{\sqrt{2\pi T}} \ln \left[\frac{(S_0\sqrt{2r} + \sqrt{2rS_0^2 + D})(K\sqrt{2r} + \sqrt{2rK^2 + D})}{D} \right].$$

Goldenberg (1991) has recently derived the European option pricing forumlas under this residual volatility diffusion process. As can be easily seen, when $D \to 0$, the American put price (31) reduces to the American put price under the lognormal diffusion process (21) with $\sigma^2 = 2r$. This reconciles with the fact that when $D = 0$ the process (27) becomes a lognormal diffusion with return volatility $\sigma^2 = 2r$.

Similar to the case of absorbing residual volatility diffusion, when $D \to 0$, the American put price (31) reduces to the American put price under the lognormal diffusion process (21) with $\sigma^2 = 2r$. Thus, equation (31) provides us with another alternative American put pricing model.

Furthermore, like under the absorbing Gaussian diffusion (18), the form of American put price (31) also reflects the effect of absorption behavior of the diffusion on the early exercise premium of the American put: the put price increases because of the increased likelihood

Table 5. Put prices under residual volatility diffusion: $T = 0.5$ year, $r = 8\%$, $\sigma^2 = 2r$ and $K = 100$

Residual volatility D	—	300	500	700
European put price	9.1746	33.3102	35.0195	36.6492
American put price	9.5835	34.0926	35.8018	37.4315
Premium	0.4089	0.7824	0.7823	0.7823

Table 6. Put prices under residual volatility diffusion: $T = 1$ year, $r = 8\%$, $\sigma^2 = 2r$ and $K = 10$

Residual volatility D	—	2	5	8
European put price	1.1698	3.4759	3.7544	4.0136
American put price	1.2633	3.6448	3.9235	4.1828
Premium	0.0935	0.1689	0.1691	0.1692

of the stock price to stay in lower states. Consequently, the early exercise boundary of the American put is pushed upward.

Formula (31) also reflects the effect of residual volatility on the early exercise premiums. The residual volatility tends to increases the early exercise premium in another way. While the possibility of absorption at the origin makes the investor less likely to regret the early exercise decision, the residual volatility makes the security price more volatile and increases the likelihood of transition to a lower state in the future—making the investor *more* likely to regret the early exercise decision in the future. This tends to push the early exercise boundary downward.

These two factors jointly determine the put prices and the early exercise boundary of an American put. To clearly see the final effect of these factors on put prices, we compute and compare few option prices and graph their corresponding early exercise boundaries. The put prices are given in Tables 5 and 6, and the corresponding early exercise boundaries are depicted in Figure 5 and 6.

These results indicate that the put prices increase as a result of the possibility of absorbing at the origin and residual volatility. The combined effect of absorption and residual volatility is also reflected in both the European and American put prices, as well as in the positions of the early exercise boundaries. Compared with the early exercise boundary under the lognormal diffusion, both the European and American put prices increase, and the early exercise boundary is pushed up because of the absorption possibility. On the other hand, other things being equal, the larger the residual volatility, the higher the European put prices and early exercise premiums (and hence the higher the American put prices), the lower the early exercise boundary. This is also consistent with our intuition on the effect of volatility changes on the early exercise boundary of an American put. In short, while the possibility of absorption makes it less likely for the stock price to bounce back, the residual volatility accentuates such possibility.

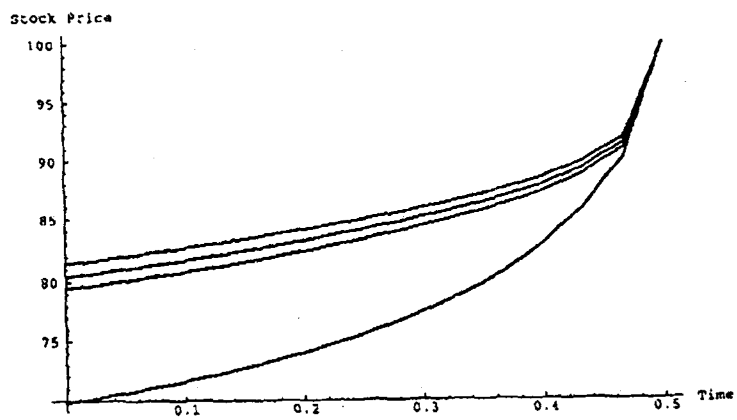

Figure 5. Early exercise boundaries under residual volatility diffusion and under regular lognormal diffusion. The early exercise boundary is approximated using the recursive implementation method detailed in Huang, Subrahmanyam and Yu (1996) with $n = 15$. The four early exercise boundaries in the graph correspond to, from top down, residual volatility diffusions with $D = 300$, $D = 500$, $D = 700$, and under lognormal diffusion, respectively. Other parameters are $K = 100$, $r = 8\%$, $\sigma^2 = 2r$, and $T = 0.5$ year. The early exercise boundaries of the residual volatility diffusion are quite close.

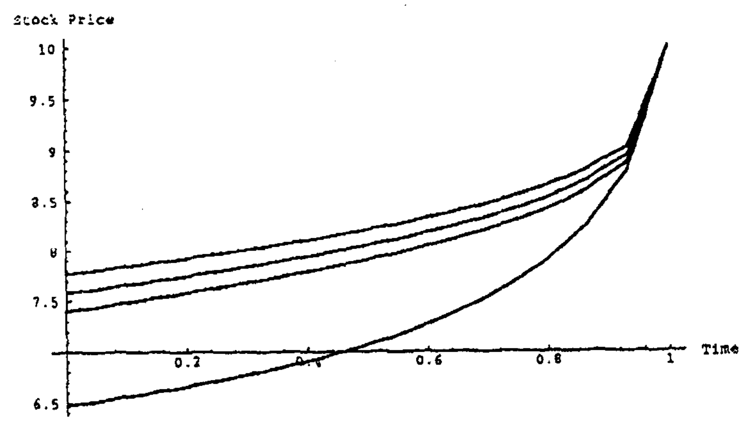

Figure 6. Early exercise boundaries under residual volatility diffusion and under regular lognormal diffusion. The early exercise boundary is approximated using the recursive implementation method detailed in Huang, Subrahmanyam and Yu (1996) with $n = 15$. The four early exercise boundaries in the graph correspond to, from top down, residual volatility diffusions with $D = 8$, $D = 5$, $D = 2$, and under lognormal diffusion, respectively. Other parameters are $K = 10$, $r = 8\%$, $\sigma^2 = 2r$, and $T = 1$ year. The three early exercise boundaries of the residual volatility diffusion are quite close.

3.4. CEV Diffusion

Finally, we consider the American option pricing formulas when the risk-neutral security price S_t follows a constant elasticity of variance (CEV) process:

$$dS_t = rS_t dt + \sigma S_t^{\frac{\beta}{2}} dz_t, \quad 0 \leq \beta \leq 2. \tag{32}$$

Using the CEV diffusion process (32) as the description of security price movements allows one to introduce decreasing return volatility in a simple but nontrivial way. In addition, since the CEV diffusion postulates the state 0 as an absorbing state, it permits the possibility of bankruptcy. These two features of the CEV diffusion and the empirical evidence of Black (1976), MacBeth and Merville (1980), Christie (1982), and Swidler and Diltz (1992) on the behaviors of stock prices make CEV diffusion (32) a superior alternative to model corporate security prices than the lognormal diffusion process (19).

Moreover, the CEV diffusion (32) is also rich in the sense that it contains three widely discussed diffusion processes as special cases. When $\beta = 2$, it becomes the lognormal process (19). When $\beta = 1$, the process (32) reduces to the widely discussed square root process:

$$dS_t = rS_t dt + \sigma S_t^{\frac{1}{2}} dz_t. \tag{33}$$

When $\beta = 0$ and state 0 is an absorbing state, (32) reduces to the absorbing Gaussian diffusion process or the absolute process (14):

$$dS_t = rS_t dt + \sigma dz_t.$$

In addition, Cox and Ross (1976) show that both the square root and the absorbing Gaussian processes can be asymptotically approximated by pure Markov jump processes.

Previous studies on the pricing and computing European options for the CEV diffusion had been carried out by Cox (1975), Beckers (1980) and Schroder (1989). The procedures of estimating the CEV diffusion parameters are given in Jacklin and Gibbons (1992), which can be adapted for the valuation of American options in practice. From Cox (1975), we have the risk-neutral transition density function of the CEV diffusion process (32):

$$\Psi(S_t; S_0) = (2-\beta)\theta^{\frac{1}{2-\beta}}(xw^{1-2\beta})^{\frac{1}{4-2\beta}} e^{-x-w} I_{\frac{1}{2-\beta}}(2\sqrt{xw}), \tag{34}$$

where

$$\begin{aligned} \theta_t &= \frac{2r}{\sigma^2(2-\beta)[e^{r(2-\beta)t}-1]}, \\ x_t &= \theta_t S_0^{2-\beta} e^{r(2-\beta)t}, \\ w &= \theta_t S_t^{2-\beta}, \end{aligned}$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q (see, for example, Karlin and Taylor, 1981). The discrete part of distribution (the probability of bankruptcy) is

$$\begin{aligned} Pr(S_t = 0 \mid S_0) &= 1 - \int_0^\infty \Psi(S_t; S_0)\, dS_t, \\ &= G\left(\frac{1}{2-\beta}, x_t\right), \end{aligned} \tag{35}$$

where

$$G(n_1, h_1) = \int_{h_1}^\infty g(n_1, \xi)\, d\xi$$

is the complementary Gamma distribution function and $g(n, h_2)$ is the gamma density function

$$g(n_1, h_2) = \frac{e^{-h_2} h_2^{n_1 - 1}}{\Gamma(n_1)}.$$

From (10), (34), and (35), a nontrivial simplification yields the American put price:

$$P_0 = p_0 + rK \int_0^T e^{-rt} \left\{ Q[2x_t, 2/(2-\beta), 2y_1] + G\left(\frac{1}{2-\beta}, x_t\right) \right\} dt, \tag{36}$$

where $y_1 = \theta_t B_t^{2-\beta}$ and

$$Q[2x_t, 2m, 2y_1] = \int_{y_1}^\infty p(2x_t, 2m, 2w)\, dw, \tag{37}$$

and

$$p(2x_t, 2m, 2w) = e^{-r-w} (x_t/w)^{\frac{m-1}{2}} I_{m-1}(2\sqrt{x_t w}),$$

is the noncentral chi-square density function with $2m$ degrees of freedom and noncentral parameter $2w$ (see Johnson and Kotz, 1970, p. 133) and p_0 is the corresponding European put price:

$$\begin{aligned} p_0 &= Ke^{-rT} \left\{ Q[2x_T, 2/(2-\beta), 2y] + G\left(\frac{1}{2-\beta}, x_T\right) \right\} \\ &\quad - S_0\{1 - Q[2y, 2 + 2/(2-\beta), 2x_T]\}, \end{aligned} \tag{38}$$

with $y = \theta_T K^{2-\beta}$ and

$$Q[2x_T, 2m, 2y] = \int_y^\infty p(2x_T, 2m, 2w)\, dw, \tag{39}$$

Notice that the functional $Q[2h_1, 2m, 2h_2]$ in (36) and (38) and as defined by (37) and (39) is called the complementary noncentral chi-square distribution function. When $2/(2-\beta)$

is an integer, $Q[2h_1, 2m, 2h_2]$ satisfies (see Schroder, 1989, app.)

$$Q[2h_1, 2m, 2h_2] + Q[2h_2, 2 - 2m, 2h_1] = 1, \tag{40}$$

which can be used to obtain a slightly simpler form for the European put price:

$$p_0 = Ke^{-rT}\left\{Q[2x_T, 2/(2-\beta), 2y] + G\left(\frac{1}{2-\beta}, x_T\right)\right\} - S_0 Q[2x_T, -2/(2-\beta), 2y]. \tag{41}$$

Furthermore, equations (37) and (39) imply that $Q[2h_1, 2m, 2h_2]$ can be represented by

$$Q[2h_1, 2m, 2h_2] = \sum_{n=1}^{\infty} g(n, h_2)G(n + m - 1, h_1). \tag{42}$$

A computing algorithm for the complementary noncentral chi-square distribution functions $Q[2h_1, 2m, 2h_2]$ is explicitly given in Schroder (1989), which can be adopted in the numerical computation of American put price (36).

Finally, from (40) and (42), it is straightforward to verify that for the case $\beta = 2$, the American put price formula (36) reduces to (21), the American put pricing formula under the lognormal diffusion (19). For the square root diffusion (33), the American put price is readily obtained from (36), (41), and (42):

$$\begin{aligned} P_0 &= Ke^{-rT}\left[\sum_{n=1}^{\infty} g(n, y)G(n, x_T) + e^{-x_T}\right] - S_0 \sum_{n=1}^{\infty} g(n, y)G(n-2, x_T) \\ &\quad + rK\left[\sum_{n=1}^{\infty}\int_0^T e^{-rt} g(n, y_1)G(n, x_t)\, dt + \int_0^T e^{-rt-x_t}\, dt\right], \end{aligned}$$

with

$$\begin{aligned} \theta_t &= \tfrac{2r}{\sigma^2(e^{rt}-1)}, & x_t &= \theta_t S_0 e^{rt}, \\ y &= \theta_T K, & y_1 &= \theta_t B_t. \end{aligned}$$

4. Concluding remarks

This paper generalizes the current literature on American option pricing in many ways. We offer an alternative but intuitive method to derive the analytic valuation formulas for American options. This approach takes into account of the two sources of American option values and the balance of costs and benefits associated with premature exercise decisions. In addition, this approach circumvents the difficulty of relying on the partial differential equations to determine the option prices, thereby permitting us to apply it to much more general diffusion processes for the underlying security price than the existing literature.

The feature of our approach enables us to generate analytic valuation formulas for American option under various diffusion processes. In particular, we are able to obtain analytic valuation formulas under a host of diffusion processes, capturing some interesting and important aspects of actual security price behaviors, such as changing volatility, bankruptcy possibility. In this regard, we contribute to the option-pricing literature by introducing the recent American option valuation technology to many interesting underlying security price processes, some of them can have potential applications in many finance problems. For example, by offering wider choices of American option pricing models, this work is beneficial to financial theorists examining the behaviors of options trading prices and the efficiency of options markets. In addition, since the analysis can be easily extended to the cases of currency exchange options, index options, and futures options, and the resultant analytic valuation formulas are easily amenable to obtain various option hedge parameters, this work is also beneficial to financial practitioners in carrying out options arbitrage activities in commodity, foreign currency and futures markets.

While the analysis of this paper has been on typical diffusion processes with simple forms, one can easily extend the methodolgy to diffusion processes with more complex volatility structure, such as those examined in Goldenberg (1991) in pricing European options. In addition, the analysis of this paper can also be generalized to other options valuation context, such as American option pricing under stochastic volatility and under exotic exercise payoff policies.

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Notes

1. Jacka (1991) also obtains these valuation formulas by examining the equivalence of formulating the valuation problem of an American option as an optimal stopping problem and as a free-boundary problem. For a rigorous survey on the theory of American option pricing, see Myneni (1992).
2. Other derivative securities having this property include options involving discontinuous payoffs, such as various barrier options, collars and floors.
3. Kim (1990) and Carr, Jarrow, and Myneni (1992) also formulate the problem this way.
4. See Karlin and Taylor (1981) and Cox and Miller (1977) for some expositions of the first passage time and the associated transition density function.
5. Kim (1990) obtained the same characterizations under the lognormal diffusion assumption on the underlying security price process.
6. For ease of presentation, we assume in this case the absorbing barrier is the state zero. The analysis can be easily extended to other types of deterministic barriers, such as the flat barriers of the form $A > 0$. See Section 3.2 for a related example.

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