The Analytic Valuation of American Options

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No analytic solution exists for the valuation of American options written on futures contracts and foreign currencies for which early exercise may be optimal. This article formulates the American option valuation problem in economically and mathematically meaningful ways. This enables us to derive valuation formulas for American options. The properties associated with the optimal exercise boundary are examined, and a numerical technique to implement the valuation formulas is presented.

In their seminal paper, Black and Scholes (1973) derived a closed-form solution for European call options. Since options traded on organized exchanges are of the American type, it is natural that the valuation problem for American options has drawn much attention. Merton (1973) has shown that, if the underlying asset does not pay dividends, American calls will not be exercised early and will have the same value as equivalent European calls. Even if the underlying asset does pay dividends, analytic solutions can be derived as long as dividends are discrete (Roll, 1977; Geske, 1979; Whaley, 1981): the analysis of premature exercise is relatively simple in this case, because optimal premature exercise will occur only an instant prior to each dividend payment. However, this type of sim-

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plification is not available for American puts and for calls on futures contracts and foreign currencies,[1] or if the underlying asset pays continuous dividends (Merton, 1965), since there is always the possibility of early exercise. As a result, European option formulas cannot be used for the valuation of American options written on futures contracts and foreign currencies, and no analytic solution exists for those options.

The mathematical difficulty associated with the valuation of American options stems from the fact that the optimal exercise boundary must be determined as a part of the solution. For this reason, theoretical work on the valuation of American options has been directed at either numerical methods or approximation methods. For example, Ramaswamy and Sundaresan (1985) and Brenner, Courtadon, and Subrahmanyam (1985) used the implicit finite-difference method in Brennan and Schwartz (1977, 1978) to account for the early exercise possibility of American options on futures contracts, while Barone-Adesi and Whaley (1987) and Whaley (1986) employed the quadratic approximation of MacMillan (1986) to value early-exercise premiums. American foreign currency options have been analyzed by Bodurtha and Courtadon (1984) using the implicit finite-difference method and a numerical method developed by Parkinson (1977). Geske and Johnson (1984) obtained a valuation formula for American puts expressed in terms of a series of compound-option functions.

The purpose of this article is to present an analytic valuation formula for American options written on assets that pay continuous dividends. Merton (1973) and McKean (1965) recognized the valuation of American calls on a stock with proportional dividends as a free-boundary problem. We modify the specification of Merton (1973) to reflect the properties of the optimal exercise boundary for American calls. Once the precise valuation problem is specified, it is straightforward to obtain an analytic solution by utilizing the results in Kolodner (1956) and McKean (1965).[2] However, in order to gain economic insight, we derive the solution as the continuous limit of the valuation formula for American calls that allow early exercise at a finite number of points in time.[3] This derivation allows us to give an economic interpretation to the mathematical results in Kolodner (1956) and McKean (1965).

[1] Ramaswamy and Sundaresan (1985), Brenner, Courtadon, and Subrahmanyam (1985), and Ball and Torous (1986) have established that American options on futures contracts are subject to early exercise even in the absence of dividend payments from the underlying asset as long as the interest rate is positive. If the interest rate is zero, American call options on futures contracts would not be exercised prematurely.

[2] There is an error in McKean's formula, which was recently verified to me by McKean.

[3] I am grateful to Michael Brennan for suggesting this approach.

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The article is organized as follows. In the next section, we formulate the precise specification of the valuation problem for American calls. We apply the risk-neutral valuation approach of Cox and Ross (1976) to American calls that allow exercise at discrete points in time. The solution to the valuation problem in continuous time is obtained by taking the limit as time intervals between exercise points tend to zero. The value of an American call is shown to consist of two components: the value of a comparable European call and an early-exercise premium. In Section 3, the critical asset price that defines the optimal exercise boundary is examined in detail. The properties of the optimal exercise boundary have important implications for the valuation of American options. The Black and Scholes formula is shown to be a special case of the American call valuation formula. In Section 3, a valuation formula for American puts is presented. Implementation of the valuation formulas is discussed in Section 4. We conclude the paper in Section 5.

1. The Valuation of American Calls

Consider an American call with exercise price of K that expires at time T. Let us denote the value of the American call at $t = T - \tau$ by $C(S, \tau)$, defined on domain $\{(S, \tau); 0 < S \leq \infty, 0 < \tau \leq T\}$, where S is the asset price and τ is the time to expiration. We employ the usual assumptions: markets are perfect and trading occurs continuously, the asset price follows a lognormal diffusion process with volatility parameter σ, and the interest rate r is constant. We consider American calls written on assets that pay continuous proportional dividends at a rate of δ. This will allow us to apply the results of this paper to the valuation of American calls on futures contracts and foreign currencies.[4]

As Ramaswamy and Sundaresan (1985) and Brenner, Courtadon, and Subrahmanyam (1985) have demonstrated in the context of American futures options, the determination of the optimal exercise boundary is an important element in the valuation of American options. The optimal exercise boundary is defined by the critical asset price, above which it is optimal to exercise the American calls. Ramaswamy and Sundaresan (1985) and Brenner, Courtadon, and Subrahmanyam (1985) employed the implicit finite-difference method to determine

[4] Black (1976), Ramaswamy and Sundaresan (1985), and Brenner, Courtadon, and Subrahmanyam (1985) have examined the valuation of options on futures contracts in this setting, in which futures prices are identical to forward prices (see Cox, Ingersoll, and Ross, 1981). Foreign currency options have been examined in the same setting by Bodurtha and Courtadon (1984).

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simultaneously the optimal exercise boundary and the value of American options on futures contracts.[5]

It would not be fruitful to tackle the complete American call function $C(S, \tau)$ directly. Instead, it is derived in two steps. First, we examine the value of a *live* American call. Let us denote the value of the *live* American call given the optimal exercise boundary $G(\cdot)$ by $V(S, \tau; G(\cdot))$, which is differentiable with respect to τ and twice-differentiable with respect to S, defined on domain $\{(S, \tau); 0 < S < G(\cdot), 0 < \tau \leq T\}$. The optimal exercise boundary $G(\cdot)$ represents the asset price *above* which American calls are exercised optimally. The function $V(S, \tau; G(\cdot))$ represents the value of the American call conditional on the asset price being below $G(\cdot)$. Merton (1973) has shown that the value of American calls is governed by the following partial differential equation:

$$\tfrac{1}{2}\sigma^2 S^2 V_{ss} + (r - \delta) S V_s - r V = V_\tau. \tag{1}$$

This equation reflects the fact that the risk-neutral asset-price process is governed by a lognormal diffusion process with the drift equal to $(r - \delta)$. The equivalence between options on futures contracts and options on assets with continuous dividend yield equal to the risk-free interest rate is apparent by comparing Equation (1) and the partial differential equation associated with the European futures option in Black (1976).[6] The partial differential equation (1) applies to any claim whose payoffs depend on the asset price. To determine the value of the *live* American call, it is necessary to specify a terminal condition and boundary conditions associated with the *live* American call:[7]

$$\lim_{\tau \to 0} V(S, \tau; G(\cdot)) = \max[0, (S - K)\phi(G(0) - S)], \tag{1a}$$

$$\lim_{S \to G(\tau)} V(S, \tau; G(\cdot)) = G(\tau) - K, \tag{1b}$$

$$\lim_{S \to 0} V(S, \tau; G(\cdot)) = 0, \tag{1c}$$

$$\lim_{S \to G(\tau)} V_S = 1, \tag{1d}$$

[5] It is conceptually useful to recognize the problem of optimal early exercise as an optimal-stopping-time problem. Van Morebeke (1976) has established the equivalence between a free-boundary problem and an optimal-stopping-time problem, and has proved the existence of solutions to free-boundary problems, which implies the existence of the optimal stopping time.

[6] Equation (1), with $\delta = r$, holds for futures options even if the commodity or the instrument underlying the futures contract pays dividends or a service stream, as long as the futures price follows a lognormal diffusion process with the volatility parameter σ. Note that sample paths of the futures price are continuous even if the underlying commodity or instrument pays discrete dividends. If the underlying spot pays certain discrete dividends during the life of the futures contract, as Roll (1977) and Geske (1979) pointed out, the spot price cannot follow a lognormal diffusion process. However, it is possible for the futures price to follow a lognormal diffusion process.

[7] It is mathematically important to note that $V(S, \tau; G(\cdot))$ need not be defined at $\tau = 0$ and $S = G(\tau)$.

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where $\phi(y)$ represents the step function, i.e., $\phi(y) = 0$ if $y < 0$ and $\phi(y) = 1$ if $y > 0$. Equation (1a) specifies the payoff of calls at expiration given that the calls have not been exercised early. It is important to note that the maximum value of the *live* American call at expiration is $(G(0) - K)$, where $G(0)$ represents the limit of $G(\tau)$ as τ tends to zero. This is because it is impossible for the asset price at expiration to be above $G(0)$ without crossing the optimal exercise boundary earlier. The American call being alive, together with the asset price being above $G(0)$ at expiration, implies violation of the optimal exercise policy. The boundary condition (1b) specifies the payoff of the call at the time of exercise. The boundary condition (1c) reflects the fact that zero is an absorbing barrier for a lognormal diffusion process. The condition (1d) is known as the "high-contact condition" (Merton, 1973) or the "smooth-pasting condition" and ensures the optimality of the exercise boundary. This specification of the boundary conditions (1b) and (1c) and the optimality condition (1d) is identical to that of Merton (1973, pp. 170–171). The expiration condition (1a) differs from that of Merton (1973) to reflect the optimal exercise policy for American calls. The partial differential equation (1), subject to (1a)–(1c) as well as (1d), is known as a free-boundary problem in mathematics. Once $V(S, \tau)$ is determined, the unconditional American call function $C(S, \tau)$ would be defined by

$$C(S, \tau) = V(S, \tau; G(\cdot)), \qquad \text{on } \{0 < S < G(\cdot), 0 < \tau \leq T\}, \tag{2a}$$
$$C(S, \tau) = S - K, \qquad \text{on } \{G(\cdot) \leq S \leq \infty, 0 < \tau \leq T\}. \tag{2b}$$

Note that, on part of the domain, $C(S, \tau)$ is defined to be $S - K$, which does not satisfy the partial differential equation (1). Hence $C(S, \tau)$ is not a solution to (1) regardless of boundary conditions. This is the reason why our focus is on *live* American calls.

In order to solve this problem, we employ the risk-neutral valuation approach of Cox and Ross (1976).[8] Under risk neutrality, the price dynamics are given by

$$dS = (r - \delta)S\, dt + \sigma S\, dz,$$

where dz is a standard Wiener process. Assume that the American call can be exercised at a finite number of points in time denoted by t_k, $k = n, n - 1, \ldots, 1$, and 0, where $t_k - t_{k+1} = \Delta t$ for all k. The call expires at t_0. Define $\psi(S_j, (i - j)\, \Delta t; S_i)$ as the transition density function of the asset price at t_j, given that the asset price at t_i is S_i under the risk-neutral price process. With one period left to expira-

[8] See the Appendix for the solution to the partial differential equation (1) subject to the boundary conditions (1a)–(1d).

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tion, the value of the *live* American call $V(S_1, \Delta t; G_0)$ would be identical to an equivalent European call value:

$$V(S_1, \Delta t; G_0) = S_1 e^{-\delta \Delta t} \aleph(d_1(S_1, \Delta t; G_0)) - Ke^{-r\Delta t} \aleph(d_2(S_1, \Delta t; G_0)),$$

where $\aleph(\cdot)$ is the unit normal distribution function and

$$V(S_1, \Delta t; G_0) = S_1 e^{-\delta \Delta t} \aleph(d_1(S_1, \Delta t; G_0)) - K e^{-r\Delta t} \aleph(d_2(S_1, \Delta t; G_0)),$$

where $\aleph(\cdot)$ is the unit normal distribution function and

$$d_1(S_1, \Delta t; G_0) = \frac{\ln(S_1/G_0) + (r - \delta + \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}},$$

$$d_2(S_1, \Delta t; G_0) = d_1(S_1, \Delta t; G_0) - \sigma\sqrt{\Delta t}.$$

Trivially, G_0, the critical asset price at expiration, is equal to the exercise price. If the value of the *live* American call is less than the immediate exercise value $(S_1 - K)$, it is optimal to exercise the call. This implies that G_1, the critical asset price at t_1, is defined implicitly by

$$G_1 - K = G_1 e^{-\delta \Delta t} \aleph(d_1(G_1, \Delta t; G_0)) - Ke^{-r\Delta t} \aleph(d_2(G_1, \Delta t; G_0)).$$

If the asset price at t_1 is above G_1, it is optimal to exercise the call. Note that G_1 is greater than K. Let us move one period back and consider the *live* American call at t_2. Its value $V(S_2, 2\Delta t)$ consists of two components. The first is the discounted expectation of the *live* American call value at t_1, conditional on the call being alive at $t_1 (S_1 \leq G_1)$, expectations being taken with respect to the risk-neutral transition density function. The second part of the American call value is the discounted expectation of the immediate exercise value at t_1, conditional on the call being exercised $(S_1 \geq G_1)$:[9]

$$V(S_2, 2\Delta t) = \int_0^{G_1} e^{-r\Delta t} V(S_1, \Delta t)\, \psi(S_1, \Delta t; S_2)\, dS_1 + \int_{G_1}^{\infty} e^{-r\Delta t}(S_1 - K)\, \psi(S_1, \Delta t; S_2)\, dS_1.$$

This equation defines the recursive relationship between the value of the *live* American call at t_2 and the value of the *live* American call at t_1. By rearranging and evaluating terms, we get

[9] The value of the *live* American call at t_k depends on the critical stock prices at t_i, for $i = 1, 2, \ldots, k - 1$. The critical asset prices are suppressed in the notation for the American call function.

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$$V(S_2, 2\Delta t) = c(S_2, 2\Delta t) + \int_{G_1}^{\infty} e^{-r\Delta t}[S_1 - K - c(S_1, \Delta t)]\, \psi(S_1, \Delta t;\, S_2)\, dS_1.$$

The first term, $c(S_2, 2\Delta t)$, is the value of a European call, whereas the second represents an early-exercise premium. This equation indicates that an American call expiring in two periods is equivalent to a portfolio of an equivalent European call and a European option expiring in one period to exchange the two-period European call (expiring at t_0) for a one-period European call (expiring at t_1). The early-exercise premium can be divided into two components:

$$\int_{G_1}^{\infty} e^{-r\Delta t}[(1 - e^{-\delta\Delta t})S_1 - (1 - e^{-r\Delta t})K]\, \psi(S_1, \Delta t;\, S_2)\, dS_1 + \int_{G_1}^{\infty} e^{-r\Delta t}\, \psi(S_1;\, S_2, \Delta t)\, dS_1 \int_{0}^{K} e^{-r\Delta t}(S_0 - K)\psi(S_0, \Delta t;\, S_1)\, dS_0.$$

The second term is of order higher than Δt, which we denote $O(\Delta t)$.[10] Intuitively, if the asset price at t_1 is above G_1, which is higher than K, the probability of the asset price falling below K within Δt time is very small. Hence, the American call function can be expressed as

$$V(S_2, 2\Delta t) = c(S_2, 2\Delta t) + O(\Delta t) + \int_{G_1}^{\infty} e^{-r\Delta t}[(1 - e^{-\delta\Delta t})S_1 - (1 - e^{-r\Delta t})K]\psi(S_1, \Delta t;\, S_2)\, dS_1. \tag{3}$$

The integrand is the discounted value of the difference between the present value at t_1 of the dividends receivable on the stock and the interest payable on the exercise price. This is the gain from early exercise, and this integration is performed over those states of which early exercise is optimal. The critical asset price at t_2 is defined by

$$G_2 - K = c(G_2, 2\Delta t) + O(\Delta t) + \int_{G_1}^{\infty} e^{-r\Delta t}[(1 - e^{-\delta\Delta t})S_1 - (1 - e^{-r\Delta t})K]\psi(S_1, \Delta t;\, G_2)\, dS_1.$$

This means that if the asset price at t_2 is above G_2, the American call

[10] See the Appendix for a proof.

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no longer exists. Otherwise, it survives one more period. By working backward recursively, we derive the American call function at t_n:[11]

$$V(S_n, n\Delta t) = c(S_n, n\Delta t) + O(n\Delta t) + \sum_{k=1}^{n-1} e^{-(n-k)r\Delta t} \int_{G_k}^{\infty} [(1 - e^{-\delta\Delta t}) S_k - (1 - e^{-r\Delta t})K] \times \psi(S_k, (n - k)\,\Delta t; S_n)\, dS_k, \tag{4}$$

where $O(n\Delta t)$ represents terms that vanish as Δt tends to zero for a fixed $n\Delta t$, and G_k, for $k = 1, 2, \ldots, n - 1$, is defined by

$$G_k - K = c(G_k, k\Delta t) + O(k\Delta t) + \sum_{i=1}^{k-1} e^{-(k-i)r\Delta t} \int_{G_i}^{\infty} [(1 - e^{-\delta\Delta t}) S_i - (1 - e^{-r\Delta t})K] \times \psi(S_i, (k - i)\Delta t; G_k)\, dS_i.$$

The American call function given in (4) can be rewritten as

$$V(S_n, n\Delta t) = c(S_n, n\Delta t) + O(n\Delta t) + \sum_{k=1}^{n-1} e^{-(n-k)r\Delta t}\Delta t \int_{G_k}^{\infty} (\delta S_k - rK)\, \psi(S_k, (n - k)\Delta t; S_n)\, dS_k.$$

By taking the limit as Δt tends to zero, setting $n\Delta t = \tau$ and defining $S = S_n$, we obtain the value of American calls that allow early exercise at any point in continuous time:

$$V(S, \tau; G(\cdot)) = c(S, \tau) + \int_0^{\tau} e^{-r(\tau-s)}\, ds \int_{G(s)}^{\infty} (\delta S_s - rK)\, \psi(S_s; S, (\tau - s))\, dS_s. \tag{5}$$

By evaluating the inner integral in Equation (5), we obtain

$$V(S, \tau; G(\cdot)) = c(S, \tau) + \int_0^{\tau} [\delta S\, e^{-\delta(\tau-s)}\, \aleph(d_1(S, \tau - s; G(s))) - rKe^{-r(\tau-s)}\, \aleph(d_2(S, \tau - s; G(s)))]\, ds. \tag{6}$$

Equation (6) expresses the value of an American call as the sum of the value of a European call value and the early-exercise premium. The early-exercise premium can be viewed as the value of a contingent claim that allows interest earned on the exercise price to be

[11] See the Appendix for the derivation.

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exchanged for dividends paid by the asset whenever the asset price is above the optimal exercise boundary. This is consistent with our intuition that early exercise gives a holder of the call the right to receive dividends at the expense of foregone interest on the exercise price. The contingent claim is valuable because dividends from the asset are always greater than interest on the exercise price, conditional on the asset price being above the optimal exercise boundary. The optimal exercise boundary $G(\cdot)$ is implicitly defined by the following integral equation:

$$\begin{aligned} G(s) - K = {} & c(G(s), s) \\ & + \int_0^s [\delta G(s) e^{-\delta(s-\xi)} \aleph(d_1(G(s), s-\xi; G(\xi)) \\ & \qquad - rKe^{-r(s-\xi)} \aleph(d_2(G(s), s-\xi; G(\xi)))]\, d\xi. \quad (7) \end{aligned}$$

This equation reflects the fact that the value of an American call at the time of optimal exercise is equal to immediate exercise value. The value of American calls is computed in two steps. In the first step, Equation (7) is solved for the function $G(\cdot)$. Given the optimal exercise boundary, Equation (6) is solved next.

Equations (6) and (7) can be solved explicitly for two special cases. First, Equation (6) can be evaluated without solving Equation (7) when both the interest rate and the dividend yield are zero: in this case, the early-exercise premium is zero. Since the value of a European call on a non-dividend paying asset never equals immediate exercise value for a finite asset price, the solution to Equation (7) is $G(\cdot) = \infty$, which implies that no early exercise would occur. This is a well-known result. In the second special case, an explicit solution to Equations (6) and (7) can be found for perpetual American calls by noting that the critical asset price for perpetual American calls is a time-invariant constant.[12] We solve Equation (7) to derive the critical asset price for perpetual American calls $G(\infty)$:[13]

$$G(\infty) = \frac{\beta K}{\beta - 1}, \quad (8)$$

where

$$\beta = \frac{-(r - \delta - \frac{1}{2}\sigma^2) + \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}.$$

[12] Although this problem was solved by Merton (1973), we present the result to demonstrate the generality of Equations (6) and (7) as valuation formulas for American calls.

[13] See the Appendix for the derivation.

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By substituting Equation (8) in Equation (6), the early-exercise premimum of perpetual American calls is reduced to[14]

$$V(S;\ G(\infty)) = \frac{K}{1-\beta}\left(\frac{(\beta-1)S}{\beta K}\right)^{\beta}. \tag{9}$$

Since $G(s)$ is nondecreasing in s, as we will show in the next section, $G(0)$ represents the lower bound and $G(\infty)$ represents the upper bound for the optimal exercise boundary $G(\cdot)$ for finitely lived American calls with otherwise similar terms.

2. The Optimal Exercise Boundary

Intuitively, early exercise occurs if the benefits outweigh the costs associated with early exercise. By exercising the call and taking the long position in the underlying asset now, one can receive dividends from the asset. The cost associated with early exercise is that one has to forgo interest to be earned on the exercise price and may regret early exercise if the asset price decreases substantially later on. This trade-off is reflected in Figure 1, which plots the critical asset price for American calls with an exercise price of \$100 as a function of time to expiration.[15] Figure 1 shows that the critical asset price is lower for calls written on an asset with a lower volatility. As shown in Figure 1, an American call will be optimally exercised at a lower critical asset price when the interest rate is lower. In order to improve our understanding of the American call valuation, let us examine in detail the behavior of the optimal exercise boundary. The optimal exercise boundary $G(\cdot)$ for an American call has the following properties.

Proposition 1. *If the asset price follows a stochastic process with continuous sample paths and the interest rate is a positive constant, then $G(\tau)$ is continuous and nondecreasing in τ.*

Proof. Continuity of the optimal exercise boundary has been proved in Van Morebeke (1976). The intuition behind continuity of $G(\cdot)$ is straightforward. Continuity of the asset price's sample paths and the definition of the optimal exercise boundary imply continuity of $G(\cdot)$. If $G(\cdot)$ were discontinuous, it would be possible for the asset

[14] If the dividend yield is positive, the value of European calls approaches zero as time to expiration increases without limit.

[15] This figure is constructed using the implicit finite-difference method. See Brennan and Schwartz (1977) for the procedure to account for the optimal early exercise in the implicit finite-difference method.

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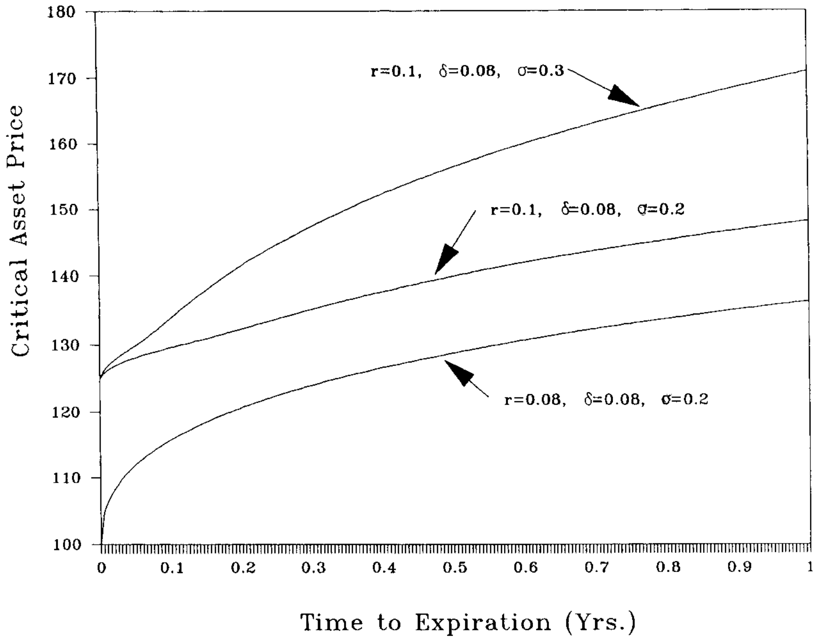

Figure 1
Critical asset price as a function of time to expiration
The critical asset price for American calls with an exercise price of \$100 as a function of time to expiration obtained from the implicit finite-difference method. r, interest rate; δ, dividend yield; σ, volatility.

price to be strictly above $G(\cdot)$ without crossing it, contradicting the definition of the optimal exercise boundary. To prove the second property, consider two American calls written on the same asset with the same exercise price. Let τ_1 and τ_2 be the times to expiration of the two calls, where $\tau_1 < \tau_2$. Cox and Rubinstein (1985, p. 140) have shown that if it is optimal to exercise the long-term call, it is never optimal to leave unexercised the short-term call. This implies that the critical asset price for the long-term call is higher than or equal to that for the short-term call, i.e., $G(\tau_1) \leq G(\tau_2)$ at a given point in time. The optimal exercise boundary is a function of time to expiration only, given the parameters associated with the underlying asset, the interest rate, and the exercise price. This means that the critical asset price for the long-term call should change from $G(\tau_2)$ to $G(\tau_1)$ over time. Since this must hold for any τ_1 and τ_2, the critical asset price should be nonincreasing as time to expiration decreases. ■

We now examine the behavior of the optimal exercise boundary near expiration.

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Proposition 2. *If the asset price follows a lognormal diffusion process and the interest rate is a positive constant,* then

$$\lim_{s\to 0} G(s) = G(0) = K, \qquad \text{if } \delta \geq r,$$

$$\lim_{s\to 0} G(s) = G(0) = (r/\delta)K, \qquad \text{if } \delta < r.$$

Proof. In order to investigate the behavior of $G(\cdot)$ near expiration, we rearrange Equation (7) into

$$\begin{aligned}\frac{G(s)}{K} = &\Bigg(e^{-rs}\,\aleph(d_2(G(s), s; K)) - 1 \\ &+ \int_0^s re^{-r(s-\xi)}\,\aleph(d_2(G(s), s-\xi; G(\xi)))\,d\xi\Bigg) \\ &\times \Bigg(e^{-\delta s}\,\aleph(d_1(G(s), s; K)) - 1 \\ &+ \int_0^s \delta e^{-\delta(s-\xi)}\,\aleph(d_1(G(s), s-\xi; G(\xi)))\,d\xi\Bigg)^{-1}. \end{aligned} \tag{10}$$

If $\delta < r$, the limit of the right-hand side of Equation (10) as s tends to zero can be evaluated using l'Hôpital's rule:[16]

$$\lim_{s\to 0}\frac{G(s)}{K} = \frac{r}{\delta}.$$

If $\delta \geq r$, the right-hand side of Equation (10) can be evaluated directly:

$$\lim_{s\to 0}\frac{G(s)}{K} = 1. \blacksquare$$

Propositions 1 and 2 imply the following restriction on the value of *live* American calls at expiration.

Proposition 3. *If the interest rate is less than or equal to the dividend yield, an American call which is optimally held to expiration will have a value of zero at T.*

Proof. Given the continuity of sample paths of both the asset price and the optimal exercise boundary, it is not possible for the asset price to end up above K at T without crossing $G(\cdot)$ at an earlier time. In other words, the fact that investors have not exercised the American

[16] Van Morebeke (1976) proved that $G(s)$ behaves like $G(0) + c\sqrt{s}$ for a small s, where c is a constant.

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calls implies that the price of the underlying asset has stayed below $G(\cdot)$. This means that $S \leq G(0) = K$ at T. Therefore, the *live* American calls will have a value of zero at T. ■

Since the implicit dividend yield is equal to the interest rate for American calls on the futures contracts, it is suboptimal to wait till expiration to exercise American calls on futures contracts. They must be exercised before expiration if they are to be exercised at all.

It is interesting to note that Equation (6) is reduced to the Black and Scholes European call formula if the dividend yield is zero. As the dividend yield approaches zero, $G(0)$ increases without a limit. Since $G(s)$ is nondecreasing in s, this means that $G(\cdot) = \infty$. In this case, it is easy to see that the early-exercise privilege is worthless. This is consistent with the well-known fact that American calls would not be exercised early if the underlying asset does not pay any dividends during the life of the calls. Note that if the dividend yield is positive, the early-exercise premium would be positive even when the interest rate is zero.

3. A Valuation Formula for American Puts

The valuation of American put options proceeds in the same way as the valuation of American calls. Consider an American put that has an exercise price of K and that expires at time T. The underlying asset is assumed to pay continuous proportional dividends at a rate of δ. As with American calls, we formulate the American put valuation problem as a free-boundary problem. Let us denote the value of a *live* American put for the optimal exercise boundary $B(\cdot)$ by $W(S, \tau; B(\cdot))$ that is differentiable with respect to τ and twice-differentiable with respect to S, defined on domain $\{(S, \tau); B(\cdot) < S \leq \infty, 0 < \tau \leq T\}$. The optimal exercise boundary $B(\cdot)$ is defined as the asset price *below* which it is optimal to exercise the American puts. The value of the *live* American put satisfies the partial differential equation (1), subject to the expiration condition and the boundary conditions:

$$\lim_{\tau \to 0} W(S, \tau; B(\cdot)) = \max[0, (K - S)\phi(S - B(0))], \tag{11a}$$

$$\lim_{S \to B(\tau)} W(S, \tau; B(\cdot)) = K - B(\tau), \tag{11b}$$

$$\lim_{S \to 0} W(S, \tau; B(\cdot)) = 0, \tag{11c}$$

$$\lim_{S \to B(\tau)} W_S = -1. \tag{11d}$$

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Equation (11a) specifies the payoff of puts at expiration, given that the puts have not been exercised early. The boundary condition (11b) specifies the payoff of the put at the time of exercise. The boundary condition (11c) reflects the fact that if the asset price increases without limit, it will never fall back within a finite time period. The boundary condition (11d) ensures the optimality of the exercise boundary.

The free-boundary problem given by (1) and (11a)–(11d) can be solved in the same way as the American call valuation problem. By applying the results of the previous section, we can express the value of a *live* American put as the sum of an early-exercise premium and the value of an equivalent European put:

$$\begin{aligned} W(S, \tau; B(\cdot)) = p(S, \tau) & \\ + \int_0^\tau & [rKe^{-r(\tau-s)} \aleph(-d_2(S, \tau - s; B(s))) \\ & - \delta S e^{-\delta(\tau-s)} \aleph(-d_1(S, \tau - s; B(s)))]\, ds, \end{aligned} \tag{12}$$

where $p(S, \tau)$ represents the Black and Scholes/Merton European put pricing formula given by

$$p(S, \tau) = Ke^{-r\tau} \aleph(-d_2(S, \tau; K)) - Se^{-\delta\tau} \aleph(-d_1(S, \tau; K)).$$

The optimal exercise boundary $B(\cdot)$ is implicitly defined by

$$\begin{aligned} K - B(s) = p(B(s), s) & \\ + \int_0^s & [rKe^{-r(s-\xi)} \aleph(-d_2(B(s), s - \xi; B(\xi))) \\ & - \delta B(s) e^{-\delta(s-\xi)} \aleph(-d_1(B(s), s - \xi; B(\xi)))]\, d\xi. \end{aligned} \tag{13}$$

The optimal exercise boundary at expiration is given by

$$\lim_{s\to 0} B(s) = B(0) = K, \qquad \text{if } \delta \le r,$$

$$\lim_{s\to 0} B(s) = B(0) = (r/\delta)K, \qquad \text{if } \delta > r.$$

As with American calls, we present explicit solutions to Equations (12) and (13) for two special cases. Equation (12) can be evaluated without solving Equation (13) when the interest is zero; in that case, $B(0) = 0$. Since $B(s)$ is nonincreasing in s, the optimal exercise boundary is zero for any s, which implies that no early exercise would occur. The American put value given by (12) is reduced to the Euro-

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pean put value as the interest rate tends to zero. This is a well-known result. The second special case involves perpetual American puts. The critical asset price for perpetual American puts $B(\infty)$ is obtained by solving Equation (13):

$$B(\infty) = \frac{\theta K}{\theta - 1}, \tag{14}$$

where

$$\theta = \frac{-(r - \delta - \frac{1}{2}\sigma^2) - \sqrt{(r - \delta - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}.$$

With this definition of the optimal exercise boundary, Equation (12) is reduced to

$$W(S; B(\infty)) = \frac{K}{1 - \theta}\left(\frac{(\theta - 1)S}{\theta K}\right)^{\theta}. \tag{15}$$

The critical asset price for perpetual American puts $B(\infty)$ represents the lower bound for the optimal exercise boundary $B(\cdot)$ for finitely lived American puts with otherwise similar terms.

4. Implementation of the Valuation Formulas

Once the integral equations (7) and (13), defining the optimal exercise boundary, are solved, computation of the American option value is a simple task that involves straightforward numerical integration. Unfortunately, it is not possible to solve the integral equations explicitly except for special cases. However, as Kolodner (1956) pointed out, the integral equations are of Volterra type and of the second kind, so that they are amenable to numerical techniques. The valuation formulas derived in this paper offer an alternative numerical method to compute the American option value.

Since s appears in the integrand as well as the upper integral limit, for each s, Equation (7) represents a separate integral equation. Let us divide τ into n subintervals s_i, for $i = 1, 2, \ldots, n$, where $s_n = \tau$. Then, to determine the optimal exercise boundary, we need to solve numerically n integral equations:

$$G(s_i) - K = c(G(s_i), s_i) + \int_0^{s_i} \Phi(s_i, \xi)\, d\xi,$$

$$\text{for } i = 1, 2, \ldots, n, \tag{16}$$

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where

$$\Phi(s_i, \xi) = \delta G(s_i) e^{-\delta(s_i-\xi)} \aleph(d_1(G(s_i), s_i - \xi; G(\xi))) \\ - rKe^{-r(s_i-\xi)} \aleph(d_2(G(s_i), s_i - \xi; G(\xi))).$$

These integral equations are solved recursively. Note that it is necessary to evaluate the integral equation over the entire range up to s_{i+1} in order to determine the optimal exercise boundary between s_i and s_{i+1}. Since the numerical method requires solving as many integral equations as the number of time intervals, it does not necessarily help us improve computational efficiency over the implicit finite-difference method.

We solve Equation (7) numerically to obtain a solution for the optimal exercise boundary $G(s)$. Then, using that numerical solution, Equation (6) is numerically integrated to compute the theoretical value of American calls. As expected, the values computed from Equations (6) and (7) are almost identical to those from the implicit finite-difference method for short-term as well as for long-term American calls. For the finite-difference methods, discretization errors occur in both the asset price and time to expiration. While discretization errors are introduced in time to expiration only for the numerical integration of Equation (6), they also occur in determining the optimal exercise boundary. We leave the task of determining the relative accuracy of the two numerical methods to future work.

5. Summary

This paper presents an analytic solution to the valuation problem of American options on assets that pay continuous dividends. To account for the possibility of early exercise in the valuation of American options, it is important to understand the implication of the optimal exercise policy on the specification of the valuation problem. Derivation of the valuation formula is possible by focusing on the value of *live* American options. We consider American options that allow exercise at discrete points in time, and take the limit as the time interval shrinks to zero, to obtain the valuation formulas. This approach helps us to improve our economic intuition for the valuation formulas. When we solve the valuation formulas numerically, as expected, they generate option values that are as accurate as those obtained using the finite-difference methods.

The focus of this paper has been on American options. However, by presenting an economically and mathematically meaningful way to analyze premature exercise of American options, this study sheds additional light on the valuation of American-type contingent claims

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in general. For example, corporate securities with American option features can be analyzed using the framework developed in this paper. This is left for future research.

Appendix

Solution to the partial differential equation (1) subject to the boundary conditions (1a)–(1d)

Instead of solving this problem directly, we transform it to a problem that can be solved using the results of Kolodner (1956) and McKean (1965). We divide the American call function into two components:

$$V(S, \tau) = R(S, \tau) + F(S, \tau).$$

The function $R(S, \tau)$ satisfies the partial differential equation (1), subject to the following boundary conditions:

$$\lim_{\tau \to 0} R(S, \tau) = 0, \tag{A1a}$$

$$\lim_{S \to G(\tau)} R(S, \tau) = G(\tau) - K, \tag{A1b}$$

$$\lim_{S \to 0} R(S, \tau) = 0, \tag{A1c}$$

$$\lim_{S \to G(\tau)} R_S = 1. \tag{A1d}$$

The function $F(S, \tau)$ is also the solution to the partial differential equation (1). The boundary conditions for $F(S, \tau)$ are given by

$$\lim_{\tau \to 0} F(S, \tau) = \max[0, (S - K)\, \phi(G(0) - S)], \tag{A2a}$$

$$\lim_{S \to 0} F(S, \tau) = 0. \tag{A2b}$$

The free-boundary problem given by the partial differential equation (1) subject to (A1a)–(A1d) has been analyzed by Kolodner (1956) and McKean (1965). By applying their results to our problem, we obtain the following solution:[17]

$$R(S, \tau) = \int_0^{\tau} e^{-r(\tau - s)} \frac{e^{-[x - H(s)]^2/2(\tau - s)}}{\sqrt{2\pi(\tau - s)}} \times \left[\frac{G(s)\sigma}{2} + \left(H'(s) - \frac{x - H(s)}{2(\tau - s)}\right)(G(s) - K)\right] ds,$$

[17] The function $R(S, \tau)$ is analytic in S and τ. See Kolodner (1956).

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where

$$H(s) = [\ln G(s) + (r - \delta - \tfrac{1}{2}\sigma^2)s]/\sigma,$$
$$x = [\ln S + (r - \delta - \tfrac{1}{2}\sigma^2)\,\tau]/\sigma.$$

The prime denotes the derivative of a function with respect to its argument. By rearranging terms, we get

$$\begin{aligned} R(S, \tau) = {} & \int_0^\tau e^{-r(\tau-s)}\, G(s)\, \frac{e^{-[x-H(s)]^2/2(\tau-s)}}{\sqrt{2\pi(\tau-s)}} \\ & \times \left[\frac{\sigma}{2} + H'(s) - \frac{x - H(s)}{2(\tau - s)}\right] ds \\ & - K \int_0^\tau e^{-r(\tau-s)}\, \frac{e^{-[x-H(s)]^2/2(\tau-s)}}{\sqrt{2\pi(\tau-s)}} \\ & \times \left[H'(s) - \frac{x - H(s)}{2(\tau - s)}\right] ds. \end{aligned} \tag{A3}$$

Note that

$$\begin{aligned} & e^{-r(\tau-s)}\, G(s) \frac{e^{-[x-H(s)]^2/2(\tau-s)}}{\sqrt{2\pi(\tau-s)}} \left[\frac{\sigma}{2} + H'(s) - \frac{x - H(s)}{2(\tau - s)}\right] \\ & \quad = -e^{-\delta(\tau-s)}\, S \frac{\partial}{\partial s} \aleph\left(\frac{x - H(s) + \sigma(\tau - s)}{\sqrt{\tau - s}}\right), \\ & e^{-r(\tau-s)}\, \frac{e^{-[x-H(s)]^2/2(\tau-s)}}{\sqrt{2\pi(\tau-s)}} \left[H'(s) - \frac{x - H(s)}{2(\tau - s)}\right] \\ & \quad = e^{-r(\tau-s)}\, \frac{\partial}{\partial s} \aleph\left(\frac{x - H(s)}{\sqrt{\tau - s}}\right). \end{aligned}$$

We integrate by parts to evaluate Equation (A3):

$$\begin{aligned} R(S, \tau) = {} & \delta S \int_0^\tau e^{-\delta(\tau-s)}\, \aleph\left(\frac{x - H(s) + \sigma(\tau - s)}{\sqrt{\tau - s}}\right) ds \\ & - rK \int_0^\tau e^{-r(\tau-s)}\, \aleph\left(\frac{x - H(s)}{\sqrt{\tau - s}}\right) ds \\ & + Se^{-\delta\tau}\, \aleph\left(\frac{\ln(S/G(0)) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\ & - Ke^{-r\tau}\, \aleph\left(\frac{\ln(S/G(0)) + (r - \delta - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

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It is straightforward to solve for the second part of the American call function $F(S, \tau)$, which satisfies the partial differential equation (1) subject to (A2a) and (A2b):

$$
\begin{aligned}
F(S, \tau) = {} & Se^{-\delta\tau}\, \aleph\left(\frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
& - Ke^{-r\tau}\, \aleph\left(\frac{\ln(S/K) + (r - \delta - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
& - Se^{-\delta\tau}\, \aleph\left(\frac{\ln(S/G(0)) + (r - \delta + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\
& + Ke^{-r\tau}\, \aleph\left(\frac{\ln(S/G(0)) + (r - \delta - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right).
\end{aligned}
$$

By adding $R(S, \tau)$ and $F(S, \tau)$, we obtain Equation (6).

Derivation of Equation (3)

Let us denote the second term of the early-exercise premium by L_1:

$$
\begin{aligned}
L_1 = {} & \int_{G_1}^{\infty} e^{-r\Delta t}\, \psi(S_1, \Delta t;\, S_2)\, dS_1 \int_0^K e^{-r\Delta t}\,(K - S_0) \\
& \times \psi(S_0, \Delta t;\, S_1)\, dS_0.
\end{aligned} \tag{A4}
$$

This term represents the discounted expectations at t_2 of a European put value at t_1, conditional on the asset price at t_1 being above G_1. Note that the European put value at t_1 is a decreasing function of S_1. Hence, we can write an upper bound of L_1 as

$$L_1 < e^{-r\Delta t}\, \aleph(d_2(S_2, \Delta t;\, G_1))\, p(G_1, \Delta t),$$

where $p(G_1, \Delta t)$ represents the value of the European put evaluated at G_1:

$$p(G_1, \Delta t) = Ke^{-\delta\Delta t}\, \aleph(-d_2(G_1, \Delta t;\, K)) - G_1 e^{-r\Delta t}\, \aleph(-d_1(G_1, \Delta t;\, K)).$$

We first show that $\aleph(d_2(S_2, \Delta t;\, G_1))$ is of order Δt or higher. Note that as the time interval between t_2 and t_1 tends to zero, G_2 approaches G_1. Since S_2 is strictly less than G_2, we have

$$\lim_{\Delta t \to 0} \aleph(d_2(S_2, \Delta t;\, G_1)) = 0.$$

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Taylor series expansion of $\aleph(d_2(S_2, \Delta t; G_1))$ in terms of Δt produces terms that are of order Δt or higher. Next, we show that $p(G_1, \Delta t)$ is of order Δt or higher. Since $G_1 \geq K$, there are two possibilities with respect to the limiting behavior of G_1:

$$\lim_{\Delta t \to 0} \frac{\ln(K/G_1)}{\sqrt{\Delta t}} = -\infty \qquad \text{or} \qquad \lim_{\Delta t \to 0} \frac{\ln(K/G_1)}{\sqrt{\Delta t}} = 0.$$

In both cases, $p(G_1, \Delta t)$ approaches zero as Δt tends to zero. Hence, Taylor series expansion of $p(G_1, \Delta t)$ in terms of Δt produces terms that are of order Δt or higher. In fact, we can evaluate the limit of G_1 as Δt tends to zero. The implicit equation defining G_1 can be rewritten as

$$\frac{G_1}{K} = \frac{e^{-r\Delta t}\aleph(d_2(G_1, \Delta t; K)) - 1}{e^{-\delta\Delta t}\aleph(d_1(G_1, \Delta t; K)) - 1}. \tag{A5}$$

Suppose $\ln(K/G_1)/\sqrt{\Delta t}$ decreases without limit as Δt tends to zero. This means that either the limit of G_1 as Δt tends to zero is strictly higher than K or the limit is equal to K, but the speed of its convergence to K is slower than $\sqrt{\Delta t}$. Then, the limit of the right-hand side of (A5) can be evaluated using l'Hôpital's rule:

$$\lim_{\Delta t \to 0} \frac{G_1}{K} = \frac{r}{\delta}.$$

This holds when $\delta < r$. We rely on the following property in evaluating the limit:

$$\lim_{x \to 0} \frac{1}{x^k} e^{-1/x} = 0, \qquad \text{for } k > 0.$$

Suppose $\ln(K/G_1)/\sqrt{\Delta t}$ approaches zero as Δt tends to zero. In this case, G_1 approaches K as Δt tends to zero and the speed of its convergence to K is faster than or equal to $\sqrt{\Delta t}$. This holds when $\delta \geq r$.

Derivation of Equation (4)

We have shown that Equation (4) holds for $m = 1$ and 2. Suppose the American call function at t_m is given by

$$\begin{aligned} V(S_m, m\Delta t) &= \sum_{k=1}^{m-1} e^{-(m-k)r\Delta t} \int_{G_k}^{\infty} [(1 - e^{-\delta\Delta t})S_k - (1 - e^{-r\Delta t})K] \\ &\quad \times \psi(S_k, (m-k)\Delta t; S_m)\, dS_k + c(S_m, m\Delta t; G_0) + O(m\Delta t). \end{aligned}$$

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Then, the critical stock price G_m is implicitly defined by

$$
\begin{aligned}
G_m &- K \\
&= \sum_{k=1}^{m-1} e^{-(m-k)r\Delta t} \int_{G_k}^{\infty} [(1 - e^{-\delta\Delta t})\ S_k - (1 - e^{-r\Delta t})K] \\
&\quad \times \psi(S_k, (m - k)\ \Delta t;\ G_m)\ dS_k + c(G_m,\ m\Delta t;\ G_0) + O(m\Delta t).
\end{aligned}
$$

This equation can be rewritten as

$$
\begin{aligned}
G_m &- K \\
&= \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta t} \int_{G_{k-1}}^{\infty} (S_{k-1} - K) \\
&\quad \times \psi(S_{k-1}, (m - k + 1)\ \Delta t;\ G_m)\ dS_{k-1} \\
&\quad - \sum_{k=1}^{m-1} e^{-(m-k)r\Delta t} \int_{G_k}^{\infty} (e^{-\delta\Delta t} S_k - e^{-r\Delta t} K) \\
&\quad \times \psi(S_k, (m - k)\ \Delta t;\ G_m)\ dS_k + e^{-r\Delta t} \int_{G_{m-1}}^{\infty} (S_{m-1} - K) \\
&\quad \times \psi(S_{m-1}, \Delta t;\ G_m)\ dS_{m-1} + O(m\Delta t).
\end{aligned}
$$

We now move back one period and consider the American call function at t_{m+1}:

$$
\begin{aligned}
V(S_{m+1}, &(m + 1)\ \Delta t) \\
&= \int_0^{G_m} e^{-r\Delta t} V(S_m,\ m\Delta t)\psi(S_m, \Delta t;\ S_{m+1})\ dS_m \\
&\quad + \int_{G_m}^{\infty} e^{-r\Delta t}(S_m - K)\psi(S_m;\ S_{m+1}, \Delta t)\ dS_m.
\end{aligned}
$$

Evaluating terms and rearranging them, we obtain

$$
\begin{aligned}
V(S_{m+1}, &(m + 1)\ \Delta t) \\
&= \sum_{k=1}^{m} e^{-(m-k+1)r\Delta t} \int_{G_k}^{\infty} [(1 - e^{-\delta\Delta t}) S_k - (1 - e^{-r\Delta t})K] \\
&\quad \times \psi(S_k, (m - k + 1)\ \Delta t;\ S_{m+1})\ dS_k \\
&\quad + c(S_{m+1}, (m + 1)\ \Delta t;\ G_0) + O(m\Delta t) \\
&\quad + \int_{G_m}^{\infty} e^{-r\Delta t}\psi(S_m, \Delta t;\ S_{m+1})\ dS_m \int_0^{G_{m-1}} e^{-r\Delta t}
\end{aligned}
$$

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$$\times\ (S_{m-1} - K)\psi(S_{m-1}, \Delta t; S_m)\ dS_{m-1}$$

$$- \sum_{k=1}^{m-1} e^{-(m-k+2)r\Delta t} \int_{G_m}^{\infty} \psi(S_m, \Delta t; S_{m+1})\ dS_m$$

$$\times \int_{G_{k-1}}^{\infty} (S_{k-1} - K)\psi(S_{k-1}; S_m, (m - k + 1)\ \Delta t)\ dS_{k-1}$$

$$+ \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta t} \int_{G_m}^{\infty} \psi(S_m; S_{m+1}, \Delta t)\ dS_m \int_{G_k}^{\infty} (e^{-\delta\Delta t} S_k - e^{-r\Delta t} K)$$

$$\times\ \psi(S_k, (m - k)\ \Delta t; S_m)\ dS_k.$$

We now show that the absolute value of the last three terms, which is denoted by L_m, is of order higher than Δt. The definition of G_m implies that, if S_m is greater than G_m, the following inequality holds:

$$S_m - K$$

$$> \sum_{k=1}^{m-1} e^{-(m-k+1)r\Delta t} \int_{G_{k-1}}^{\infty} (S_{k-1} - K)$$

$$\times\ \psi(S_{k-1}, (m - k + 1)\ \Delta t; S_m)\ dS_{k-1}$$

$$- \sum_{k=1}^{m-1} e^{-(m-k)r\Delta t} \int_{G_k}^{\infty} (e^{-\delta\Delta t}\ S_k - e^{-r\Delta t}\ K)$$

$$\times\ \psi(S_k, (m - k)\ \Delta t; S_m)\ dS_k$$

$$- e^{-r\Delta t} \int_0^{G_{m-1}} (S_{m-1} - K)\psi(S_{m-1}, \Delta t; S_m)\ dS_{m-1}$$

$$+ e^{-r\Delta t} \int_0^{\infty} (S_{m-1} - K)\psi(S_{m-1}, \Delta t; S_m)\ dS_{m-1} + O(m\Delta t).$$

This inequality allows us to obtain an upper bound for L_m:

$$L_m < \int_{G_m}^{\infty} e^{-r\Delta t}[(1 - e^{-\delta\Delta t})\,S_m - (1 - e^{-r\Delta t})\,K]\psi(S_m, \Delta t; S_{m+1})\ dS_m.$$

The right-hand side can be rewritten as

$$\begin{aligned} L_m < {} & (1 - e^{-\delta\Delta t})\,S_{m+1}e^{-\delta\Delta t}\,\aleph(d_1(S_{m+1}, \Delta t; G_m)) \\ & - (1 - e^{-r\Delta t})Ke^{-r\Delta t}\,\aleph(d_2(S_{m+1}, \Delta t; G_m)). \end{aligned}$$

It is straightforward to see that $(1 - e^{-\delta\Delta t})$ and $(1 - e^{-r\Delta t})$ are of order Δt. In order to prove that L_m is of order higher than Δt, we need to show that $\aleph(d_1(S_{m+1}, \Delta t; G_m))$ and $\aleph(d_2(S_{m+1}, \Delta t; G_m))$ are of order Δt, which requires the following conditions:

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$$\lim_{\Delta t \to 0} \aleph(d_1(S_{m+1}, \Delta t;\ G_m)) = 0,$$

$$\lim_{\Delta t \to 0} \aleph(d_2(S_{m+1}, \Delta t;\ G_m)) = 0.$$

These conditions are ensured by G_{m+1} converging to G_m as the time interval between t_{m+1} and t_m shrinks to zero, and by S_{m+1} being strictly less than G_{m+1}.

Derivation of Equations (8) and (9)

For perpetual American calls, $G(\cdot) = G(\infty)$. Let $u = s - \xi$. Then the right-hand side of Equation (7) can be rewritten as

$$\int_0^\infty \delta G(\infty)\, e^{-\delta u}\, \aleph\left(\frac{r - \delta + \frac{1}{2}\sigma^2}{\sigma}\sqrt{u}\right) du$$

$$- \int_0^\infty rKe^{-ru}\, \aleph\left(\frac{r - \delta - \frac{1}{2}\sigma^2}{\sigma}\sqrt{u}\right) du.$$

Let $\rho = r - \delta - \frac{1}{2}\sigma^2$. Integrating by parts, we get

$$\frac{G(\infty) - K}{2} + \int_0^\infty G(\infty)\, e^{-\delta u} \frac{\rho + \sigma^2}{2\sigma\sqrt{2\pi u}}\, e^{-[(\rho+\sigma^2)^2/2\sigma^2]u}\, du$$

$$- \int_0^\infty Ke^{-ru} \frac{\rho}{2\sigma\sqrt{2\pi u}}\, e^{-(\rho/2\sigma^2)u}\, du.$$

We use the following identity to evaluate the integrals:

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-au}}{\sqrt{u}}\, du = \frac{\sqrt{2}}{\sqrt{a}} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\xi^2/2}\, d\xi.$$

Then, the optimal exercise boundary is defined by

$$\frac{G(\infty) - K}{2} = \frac{1}{\sqrt{\rho^2 + 2\sigma^2 r}} \frac{(\rho + \sigma^2)\, G(\infty) - \rho K}{2}.$$

Rearranging terms and simplifying, we obtain Equation (8):

$$G(\infty) = \frac{\beta K}{\beta - 1},$$

where

$$\beta = \frac{-\rho + \sqrt{\rho^2 + 2\sigma^2 r}}{\sigma^2}.$$

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Expressing Equation (6) in terms of $u = \tau - s$ and taking the limit as $\tau \to \infty$, we get

$$V(S) = \int_0^\infty [\delta S e^{-\delta u} \aleph(d_1(S, u; G(\infty))) \\ - rKe^{-ru} \aleph(d_2(S, u; G(\infty)))]\, du.$$

Integrating by parts and rearranging terms, we get

$$V(S) = \frac{G(\infty) - K}{2} \int_0^\infty \ln(S/G(\infty))\, e^{-ru} \frac{e^{-[\ln(S/G(\infty)) + \rho u]^2/2\sigma^2 u}}{\sigma\sqrt{2\pi u^3}}\, du \\ + \frac{(\rho + \sigma^2) G(\infty) - \rho K}{2} \int_0^\infty e^{-ru} \frac{e^{-[\ln(S/G(\infty)) + \rho u]^2/2\sigma^2 u}}{\sigma\sqrt{2\pi u}}\, du.$$

The first integral is the Laplace transform of the first-passage-time density function for the risk-neutral price process to reach $G(\infty)$:

$$\frac{G(\infty) - K}{2} \left(\frac{S}{G(\infty)}\right)^\beta.$$

This is half of the perpetual American call value. The second integral is evaluated using the following identity:

$$\left(\frac{S}{G(\infty)}\right)^\beta \frac{\partial}{\partial u} \aleph\left(\frac{\ln(S/G(\infty)) - \alpha u}{\sigma\sqrt{u}}\right) \\ - \left(\frac{S}{G(\infty)}\right)^\theta \frac{\partial}{\partial u} \aleph\left(\frac{\ln(S/G(\infty)) + \alpha u}{\sigma\sqrt{u}}\right) \\ = \frac{\alpha}{\sigma\sqrt{2\pi u}} e^{-[\ln(S/G(\infty)) + \rho u]^2/2\sigma^2 u}\, e^{-ru},$$

where

$$\alpha = \sqrt{\rho^2 + 2\sigma^2 r}, \qquad \theta = \frac{-\rho - \sqrt{\rho^2 + 2\sigma^2 r}}{\sigma^2}.$$

Then, the second integral becomes

$$\frac{(\rho + \sigma^2) G(\infty) - \rho K}{2} \frac{1}{2\alpha} \left(\frac{S}{G(\infty)}\right)^\beta.$$

Using the definition of $G(\infty)$, we obtain the other half of the perpetual American call value.

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