Folho 6

6.1

a) $\int (3x^2 - 2x^5)\,dx = x^3 - \frac{1}{3}x^6 + C, \ C \in \mathbb{R}$

2. $\int (\sqrt{x} + 2)^2\,dx \underset{x>0}{=} \int x + 4\sqrt{x} + 4\,dx = \frac{x^2}{2} + \frac{8}{3}x^{3/2} + 4x + C,$
$C \in \mathbb{R}$

3) $\int (2x+10)^{20}\,dx = \frac{1}{2}\int \underbrace{2}_{u'}\,\underbrace{(2x+10)^{20}}_{u^n}\,dx = \frac{1}{2}\,\frac{(2x+10)^{21}}{21} + C,$
$C \in \mathbb{R}$

4) $\int x^2 e^{x^3}\,dx = \frac{1}{3}\int 3x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C, \ C \in \mathbb{R}$

5) $\int x^4 (x^5 + 10)^9\,dx = \frac{1}{5}\int 5x^4 (x^5+10)^9\,dx$
$= \frac{1}{5}\,\frac{(x^5+10)^{10}}{10} + C, \ C \in \mathbb{R}$

6) $\int \frac{2x+1}{x^2+x+3}\,dx \underset{\frac{u'}{u}}{=} \ln|x^2+x+3| + C, \ C \in \mathbb{R}$

7) $\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \underbrace{2}_{u'}\,\underbrace{(2x+1)^{1/2}}_{u^n}\,dx = \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}(2x+1)^{3/2} + C, \ C \in \mathbb{R}$

8) $\int \frac{x}{3-x^2}\,dx = \frac{-1}{2}\int \frac{-2x}{3-x^2}\,dx \underset{\frac{u'}{u}}{=} -\frac{1}{2}\ln|3-x^2| + C, \ C \in \mathbb{R}$

9) $\int \frac{1}{4-3x}\,dx = -\frac{1}{3}\int \frac{-3}{4-3x}\,dx \underset{\frac{u'}{u}}{=} -\frac{1}{3}\ln|4-3x| + C,$
$C \in \mathbb{R}$

10) $\displaystyle\int \frac{1}{e^{3x}}\,dx = \int e^{-3x}\,dx = -\frac{1}{3}\int -3\,e^{-3x}\,dx$

$= -\frac{1}{3}\,e^{-3x} + C,\ C\in\mathbb{R}$

11) $\displaystyle\int \frac{-7}{\sqrt{1-5x}}\,dx = \frac{7}{5}\int \underbrace{-5}_{u'}\,\underbrace{(1-5x)^{-1/2}}_{u^n}\,dx = \frac{7}{5}\,\frac{(1-5x)^{1/2}}{1/2} + C;\ C\in\mathbb{R}$

$= \frac{14}{5}\sqrt{1-5x} + C,\ C\in\mathbb{R}$

12) $\displaystyle\int \frac{\sqrt{1+3\ln x}}{x}\,dx = \frac{1}{3}\int \underbrace{3\cdot\frac{1}{x}}_{u'}\,\underbrace{(1+3\ln x)^{1/2}}_{u^n}\,dx$

$= \frac{1}{3}\,\frac{(1+3\ln x)^{3/2}}{3/2} + C,\ C\in\mathbb{R} = \frac{2}{9}\,(1+3\ln x)^{3/2} + C,\ C\in\mathbb{R}$

13) $\displaystyle\int x\,\sin x^2\,dx = -\frac{1}{2}\int \underbrace{-2x}_{u'}\,\underbrace{\sin(x^2)}_{\sin(u)}\,dx = -\frac{1}{2}\int (\cos x^2)'\,dx$

$= -\frac{1}{2}\cos x^2 + C,\ C\in\mathbb{R}$

14) $\displaystyle\int \frac{1}{x[\ln^2 x + 1]}\,dx = \int \underbrace{\frac{\frac{1}{x}}{(\ln x)^2+1}\,dx}_{\frac{u'}{1+u^2}} = \operatorname{arctg}(\ln x) + C,\ C\in\mathbb{R}$

15) $\displaystyle\int \left(\frac{2}{x}-3\right)^2\frac{1}{x^2}\,dx = -\frac{1}{2}\int \underbrace{-\frac{2}{x^2}}_{u'}\,\underbrace{\left(\frac{2}{x}-3\right)^2}_{u^n} = -\frac{1}{6}\left[\frac{2}{x}-3\right]^3$

16) $\displaystyle\int \sin(\pi-2x)\,dx = \frac{1}{2}\int \underbrace{+2}_{u'}\,\sin\underbrace{(\pi-2x)}_{u}\,dx = \frac{1}{2}\cos(\pi-2x) + C,$
$C\in\mathbb{R}$

$= -\frac{1}{2}\cos(2x) + C$

18) $\int \underbrace{\sin x}_{u} \underbrace{\cos x}_{u'} \, dx = \frac{1}{2} \sin^2 x + C, \quad C \in \mathbb{R}$

17) $\int \operatorname{th} x \, dx = \int \frac{\operatorname{sh} x}{\operatorname{ch} x} \, dx \underset{\frac{u'}{u}}{=} \ln |\operatorname{ch} x| + C, \quad C \in \mathbb{R}$

$= \ln(\operatorname{ch} x) + C, \quad C \in \mathbb{R}$

19) $\int \sin(2x) \cos x \, dx = \int 2 \sin x \cos x \cos x \, dx = \int 2 \sin x \cos^2 x \, dx$

$= -\frac{2}{3} \int 3(-\sin x) \cos^2 x \, dx$

$= -\frac{2}{3} \cos^3 x + C, \quad C \in \mathbb{R}$

20) $\int \sin^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \, dx$

$= \frac{1}{2} \left[x - \frac{1}{2} \sin(2x) \right] + C, \quad C \in \mathbb{R}$

21) $\int \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} \, dx = \frac{1}{4} \int 4 \sin^2 \frac{x}{2} \cos^2 \frac{x}{2} \, dx$

$= \frac{1}{4} \int \left(2 \sin \frac{x}{2} \cos \frac{x}{2} \right)^2 dx$

$= \frac{1}{4} \int \sin^2 x \, dx$

$= \frac{1}{8} \left[x - \frac{1}{2} \sin(2x) \right] + C, \quad C \in \mathbb{R}$

22) $\int \cos^3 x \, dx = \int \cos x \cos^2 x \, dx$

$= \int \cos x \, (1 - \sin^2 x) \, dx$

$= \int [\cos x - \cos x \sin^2 x] \, dx$

$= \sin x - \frac{1}{3} \sin^3 x + C, \quad C \in \mathbb{R}$

23) $$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx$$
$$= \frac{1}{2}\ln|x^2-1| + C, \quad C \in \mathbb{R}$$

24) $$\int \frac{x}{\sqrt{x^2-1}}\,dx = \frac{1}{2}\int \underbrace{2x}_{u'}\underbrace{(x^2-1)^{-1/2}}_{u^n} = \frac{1}{2}\,\frac{1}{1/2}\cdot(x^2-1)^{1/2} + C$$
$$= \sqrt{x^2-1} + C, \quad C \in \mathbb{R}$$

25) $$\int \underbrace{\frac{1}{x}}_{u'}\underbrace{\operatorname{sen}(\ln x)}_{\operatorname{sen}(u)}\,dx = \int \left(-\cos \ln(x)\right)'\,dx$$
$$= -\cos(\ln x) + C, \quad C \in \mathbb{R}$$

26) $$\int \frac{-3}{x(\ln x)^3}\,dx = -3\int \underbrace{\frac{1}{x}}_{u'}\underbrace{(\ln x)^{-3}}_{u^n}\,dx = -3\,\frac{1}{-2}\cdot(\ln x)^{-2} + C$$
$$= \frac{3}{2}\,\frac{1}{(\ln x)^2} + C, \quad C \in \mathbb{R}$$

27) $$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{(e^x)^2+1}\,dx \underset{\frac{u'}{u^2+1}}{=} \operatorname{arctg} e^x + C, \quad C \in \mathbb{R}$$

28) $$\int \frac{e^x}{1-2e^x}\,dx = -\frac{1}{2}\int \frac{-2e^x}{1-2e^x}\,dx$$
$$\underset{\frac{u'}{u}}{=} -\frac{1}{2}\ln|1-2e^x| + C, \quad C \in \mathbb{R}$$

29) $$\int \frac{1}{\cos^2(7x)}\,dx = \frac{1}{7}\int \frac{7}{\cos^2(7x)}\,dx$$
$$= \frac{1}{7}\operatorname{tg}(7x) + C, \quad C \in \mathbb{R}$$

29) $\displaystyle\int \sqrt{2x-1} - \sqrt{1+3x}\, dx = \frac{1}{2}\int 2\,(2x-1)^{1/2}\, dx - \frac{1}{3}\int 3\,(1+3x)^{1/2}\, dx$

$$= \frac{1}{2}\,\frac{(2x-1)^{3/2}}{3/2} - \frac{1}{3}\,\frac{(1+3x)^{3/2}}{3/2} + C$$

$$= \frac{1}{3}\sqrt{(2x-1)^3} - \frac{2}{9}\sqrt{(1+3x)^3} + C, \quad C \in \mathbb{R}$$

31) $\displaystyle\int \frac{1}{x}\left(1 + \ln^2 x\right) dx = \int \frac{1}{x} + \frac{1}{x}\ln^2 x\, dx$

$$= \ln|x| + \frac{1}{3}\ln^3|x| + C, \quad C \in \mathbb{R}$$

32) $\displaystyle\int \frac{2 + \sqrt{\operatorname{arctg}(2x)}}{1+4x^2}\, dx = \int \frac{2}{1+(2x)^2} + \frac{1}{1+(2x)^2}\operatorname{arctg}^{1/2}(2x)\, dx$

$$= \operatorname{arctg}(2x) + \frac{1}{2}\times\frac{2}{3}\operatorname{arctg}^{3/2}(2x) + C, \quad C \in \mathbb{R}$$

$$= \frac{1}{3}\operatorname{arctg}(2x)\left[3 + \sqrt{\operatorname{arctg}(2x)}\right] + C, \quad C \in \mathbb{R}$$

33) $\displaystyle\int \frac{e^{\operatorname{arctg} x}}{1+x^2}\, dx \underset{u'e^u}{=} e^{\operatorname{arctg} x} + C, \quad C \in \mathbb{R}$.

34) $\displaystyle\int \frac{\sin x}{\sqrt{1+\cos x}}\, dx = -\int -\sin x\,(1+\cos x)^{-1/2}\, dx$

$$\underset{u'u^n}{=} -2\,(1+\cos x)^{\frac{1}{2}} + C, \quad C \in \mathbb{R}$$

6.2

a) $\int \ln x \, dx = \int 1 \cdot \ln x \, dx$

$= x \ln x - \int x \cdot \frac{1}{x} \, dx$

$= x \ln x - \int dx = x \ln x - x + C, \quad C \in \mathbb{R}$

b) $\int \underset{v}{x} \, \underset{u'}{\operatorname{sen}(2x)} \, dx = -\frac{1}{2} \cos(2x) \cdot x + \frac{1}{2} \int \cos(2x) \cdot 1 \, dx$

$= -\frac{x}{2} \cos(2x) + \frac{1}{4} \operatorname{sen}(2x) + C, \quad C \in \mathbb{R}$

c) $\int 1 \cdot \operatorname{arctg} x \, dx = x \cdot \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx$

$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$

$= x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C, \quad C \in \mathbb{R}$

d) $\int \underset{v}{x} \, \underset{u'}{\cos x} \, dx = x \operatorname{sen} x - \int \operatorname{sen} x \, dx$

$= x \operatorname{sen} x + \cos x + C, \quad C \in \mathbb{R}$

e) $\int 1 \cdot \ln(1-x) \, dx = x \ln(1-x) - \int x \left(-\frac{1}{1-x}\right) dx$

$= x \ln(1-x) + \int \frac{x}{1-x} \, dx$

$= x \ln(1-x) - \int 1 - \frac{1}{1-x} \, dx$

$= x \ln(1-x) - x - \ln|1-x| + C, \quad C \in \mathbb{R}$

$= \ln(1-x)(x-1) - x + C, \quad C \in \mathbb{R}$

Obs

$$\begin{array}{r|l} x & -x+1 \\ \hline -x+1 & -1 \\ 1 & \end{array}$$

$\therefore \frac{x}{1-x} = -1 + \frac{1}{1-x}$

f) $\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

$= \frac{x^2}{4} \left[2 \ln x - 1 \right] + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

g) $\int \underset{\sim}{x^2} \underset{u'}{\operatorname{sen} x} \, dx = -\cos x \, x^2 + \int 2x \cos x \, dx$

$= -x^2 \cos x + 2 \left[x \operatorname{sen} x - \int \operatorname{sen} x \, dx \right]$

$= -x^2 \cos x + 2x \operatorname{sen} x + 2 \cos x + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

$= (2 - x^2) \cos x + 2x \operatorname{sen} x + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

h) $\int x \operatorname{sen} x \cos x \, dx = \frac{1}{2} \int x \operatorname{sen}(2x) \, dx$

$= \frac{1}{2} \left[\frac{-x}{2} \cos(2x) + \int \frac{1}{2} \cos(2x) \, dx \right]$

$= \frac{1}{4} \left[-x \cos 2x + \frac{1}{2} \operatorname{sen}(2x) \right] + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

$= \frac{1}{8} \left[-2x \cos 2x + \operatorname{sen}(2x) \right] + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

i) $\int \ln^2 x \, dx = x \cdot \ln^2 x - \int x \cdot 2 \ln x \cdot \frac{1}{x} \, dx$

$= x \cdot \ln^2 x - 2 \int \ln x \, dx$

$= x \ln^2 x - 2 \left[x \ln x - x \right] + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

$= x \ln x \, (\ln x - 2) + 2x + \mathcal{C}, \ \mathcal{C} \in \mathbb{R}$

j) $\int \underset{u'}{e^x} \underset{v}{\cos x}\, dx = e^x \cos x + \int \underset{u'}{e^x} \underset{v}{\sin x}\, dx$

$\Rightarrow \int e^x \cos x\, dx = e^x \cos x + e^x \cdot \sin x - \underbrace{\int e^x \cdot \cos x\, dx}$

$\Rightarrow 2 \int e^x \cos x\, dx = e^x (\cos x + \sin x) + C, \quad C \in \mathbb{R}$

$\Rightarrow \int e^x \cos x\, dx = e^x \dfrac{\cos x + \sin x}{2} + C, \quad C \in \mathbb{R}$

k) $\int \arcsin x\, dx = x \arcsin x - \int \dfrac{x}{\sqrt{1-x^2}}\, dx$

$= x \arcsin x + \frac{1}{2} \int -2x \, (1-x^2)^{-1/2}\, dx$

$= x \arcsin x + (1-x^2)^{\frac{1}{2}} + C, \quad C \in \mathbb{R}$

$= x \arcsin x + \sqrt{1-x^2} + C$

l) $\int e^{\sin x} \sin x \cos x\, dx = \int \cos x\, e^{\sin x} \cdot \sin x\, dx$

$= e^{\sin x} \sin x - \int e^{\sin x} \cos x\, dx$

$= e^{\sin x} \sin x - e^{\sin x} + C, \quad C \in \mathbb{R}$

$= e^{\sin x} (\sin x - 1) + C, \quad C \in \mathbb{R}$

m) $\int \dfrac{\arcsin \sqrt{x}}{\sqrt{x}}\, dx = \int \underset{u'}{\dfrac{1}{\sqrt{x}}} \underset{v}{\arcsin(\sqrt{x})}\, dx$

$= 2 \sqrt{x} \arcsin \sqrt{x} - 2 \int \cancel{\sqrt{x}} \cdot \dfrac{1/2\sqrt{x}}{\sqrt{1+(\sqrt{x})^2}}\, dx$

$= 2\sqrt{x} \arcsin \sqrt{x} - \int \dfrac{1}{\sqrt{1+x}}\, dx$

$= 2\sqrt{x} \arcsin \sqrt{x} - \dfrac{(1+x)^{1/2}}{1/2} + C$

$= 2\sqrt{x} \arcsin \sqrt{x} + 2 \sqrt{1+x} + C, \quad C \in \mathbb{R}$

η) $\int x \operatorname{arctg} x \, dx = \frac{x^2}{2} \cdot \operatorname{arctg} x - \frac{1}{2} \int x^2 \cdot \frac{1}{1+x^2} \, dx$

$= \frac{x^2}{2} \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \, dx$

$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \left[1 - \frac{1}{1+x^2}\right] dx$

$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \left[x - \operatorname{arctg} x\right] + C, \; C \in \mathbb{R}$

$= \frac{1}{2} (x^2+1) \operatorname{arctg} x - x/2 + C, \; C \in \mathbb{R}$

θ) $\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx$

$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx$

$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C, \; C \in \mathbb{R}$

ι) $\int \sin(\ln x) \, dx = x \sin(\ln x) - \int x \frac{1}{x} \cos(\ln x) \, dx$

$= x \sin(\ln x) - \int \cos(\ln x) \, dx$

$= x \sin(\ln x) - \left[x \cdot \cos \ln x + \int x \cdot \frac{1}{x} \sin(\ln x) \, dx\right]$

$= x \sin(\ln x) - x \cos(\ln x) - \underbrace{\int \sin(\ln x) \, dx}$

$\Rightarrow 2 \int \sin(\ln(x)) \, dx = x \left(\sin(\ln x) - \cos(\ln x)\right) + C, \; C \in \mathbb{R}$

$\Rightarrow \int \sin(\ln(x)) \, dx = \frac{x}{2} \left[\sin(\ln x) - \cos(\ln x)\right] + C, \; C \in \mathbb{R}$

q) $\int \operatorname{ch} x \sin(3x)\,dx = \operatorname{sh} x \cdot \sin(3x) - 3\int \operatorname{sh} x \cos(3x)\,dx$

$= \operatorname{sh} x \sin(3x) - 3\left[\operatorname{ch} x \cdot \cos(3x) + 3\int \operatorname{ch} x \sin(3x)\,dx\right]$

$= \operatorname{sh} x \sin(3x) - 3\operatorname{ch} x \cos(3x) - 9\int \operatorname{ch} x \sin 3(x)\,dx$

$\Rightarrow \int \operatorname{ch} x \sin(3x)\,dx = \frac{1}{10}\operatorname{sh} x \sin(3x) - \frac{3}{10}\operatorname{ch} x \cos(3x) + C.$

$C \in \mathbb{R}$

r) $\int x^3 e^{x^2}\,dx = \int x^2 \cdot x\, e^{x^2}\,dx$

$= \frac{1}{2} e^{x^2} \cdot x^2 - \frac{2}{2}\int e^{x^2} \cdot x\,dx$

$= \frac{x^2}{2} e^{x^2} - \frac{1}{2} e^{x^2} + C, \ C \in \mathbb{R}$

$= \frac{1}{2} e^{x^2}(x^2 - 1) + C, \ C \in \mathbb{R}$

6.3

a) $\int x\,(x+3)^{1/3}\,dx$

- Substituição

$$x+3 = t^3 \Rightarrow x = t^3 - 3$$
$$g(t) = t^3 - 3 \Rightarrow g'(t) = 3t^2 \quad , \; t>0$$

Para desfazer a substituição, há que inverter

$$x+3 = t^3 \Rightarrow t = \sqrt[3]{x+3} \qquad x \in \mathbb{R}$$

- Como

$$f(x) = x\,(x+3)^{1/3} \quad , \quad f(g(t)) = (t^3-3) \cdot t$$

A nova primitiva é

$$\int f(g(t))\,g'(t)\,dt = \int \left[(t^3-3)\,t\right] 3t^2\,dt$$

$$= 3 \int (t^3-3)\,t^3\,dt$$

$$= 3 \int t^6 - 3t^3\,dt = 3\left[\frac{t^7}{7} - \frac{3}{4}t^4\right] + C$$

- Regressando à variável inicial como $t = \sqrt[3]{x+3}$

vem

$$\int f(x)\,dx = 3\left[\frac{(x+3)^{7/3}}{7} - \frac{3}{4}(x+3)^{4/3}\right] + C, \; C \in \mathbb{R}$$

$$= 3(x+3)^{4/3}\left[\frac{x+3}{7} - \frac{3}{4}\right] + C = \frac{3}{28}(x+3)^{4/3}(4x-9) + C$$

b) $\int \frac{1}{\operatorname{sen} x}\,dx$

Note-se que

$$D_{1/\operatorname{sen}} = \mathbb{R} \setminus \{k\pi, \; k \in \mathbb{Z}\}$$

• substituir 8

$$x = 2\,\mathrm{arctg}\, t \quad , \quad t>0 \quad (\text{ou } t<0) \qquad x \in]-\pi/,\pi[$$

$$g(t) = 2\,\mathrm{arctg}\, t \Rightarrow g'(t) = \frac{2}{1+t^2}$$

$$\mathrm{sen}\, x = \mathrm{sen}\,(2\,\mathrm{arctg}\, t)$$
$$= 2\,\mathrm{sen}(\mathrm{arctg}\, t)\cos(\mathrm{arctg}\, t)$$

Mas

$$\mathrm{sen}^2\alpha + \cos^2\alpha = 1 \Rightarrow \mathrm{tg}^2\alpha + 1 = 1/\cos^2\alpha$$
$$\Rightarrow \cos^2\alpha = \frac{1}{\mathrm{tg}^2\alpha + 1}$$

donde

$$\boxed{\cos(\mathrm{arctg}\, t) = \frac{1}{\sqrt{t^2+1}}}$$

Além disso

$$\mathrm{sen}^2\alpha = 1 - \cos^2\alpha$$

donde

$$\mathrm{sen}(\mathrm{arctg}\, t) = \sqrt{1 - \frac{1}{t^2+1}} = \frac{t}{\sqrt{t^2+1}}$$

• Como $f(x) = \frac{1}{\mathrm{sen}\, x}$ vem

$$f(g(t)) = \frac{1}{\mathrm{sen}(2\,\mathrm{arctg}\, t)} = \frac{1}{2\frac{t}{\sqrt{t^2+1}}\cdot\frac{1}{\sqrt{t^2+1}}} = \frac{t^2+1}{2t}$$

E a nova primitiva é

$$\int f(g(t))\, g'(t)\, dt = \int \frac{t^2+1}{2t}\cdot\frac{2}{t^2+1}\, dt = \int \frac{1}{t}\, dt$$
$$= \ln t + C, \; C \in \mathbb{R}$$

• Regressando à variável inicial x vem

$$\int \frac{1}{\mathrm{sen}\, x}\, dx = \ln\left(\mathrm{tg}\,\frac{x}{2}\right) + C, \quad C \in \mathbb{R}$$

c) $\int \frac{x}{\sqrt{2-3x}}\, dx$

. Substituição

$$t^2 = 2-3x \Rightarrow x = \frac{2-t^2}{3} \quad , t>0$$

$$g(t) = \frac{2-t^2}{3} \quad , \quad g'(t) = -\frac{2t}{3}$$

Para desfazer a substituição há que inverter

$$t = \sqrt{2-3x} \qquad x < 2/3$$

. Como

$$f(x) = \frac{x}{\sqrt{2-3x}} \quad , \quad f(g(t)) = \frac{2-t^2}{3} \cdot \frac{1}{t} = \frac{1}{3}\left[\frac{2}{t} - t\right]$$

a nova primitiva é

$$\int f(g(t))\, g'(t)\, dt = \int \frac{1}{3}\left[\frac{2}{t} - t\right] \cdot \left(-\frac{2t}{3}\right) dt$$

$$= -\frac{2}{9} \int 2 - t^2\, dt = -\frac{2}{9}\left(2t - \frac{t^3}{3}\right) + C$$

$C \in \mathbb{R}$

. Regressando à variável inicial, como

$$t = \sqrt{2-3x}$$

vem

$$\int \frac{x}{\sqrt{2-3x}}\, dx = -\frac{2}{9}\left[2\sqrt{2-3x} - \frac{(2-3x)^{3/2}}{3}\right] + C, \quad C \in \mathbb{R}$$

$$= -\frac{2}{9}\sqrt{2-3x}\left[2 - \frac{2-3x}{3}\right] + C = -\frac{2}{27}\sqrt{2-3x}\,(4+3x) + C,$$

$C \in \mathbb{R}$

d) $\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\, dx$

. substituição

$$\sqrt{x} = t \Rightarrow x = t^2 \quad , t>0$$

$$g(t) = t^2 \qquad g'(t) = 2t$$

. Como

$$f(x) = \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}} \quad \text{vem} \quad f(g(t)) = \frac{\operatorname{sen} t}{t}$$

e a nova primitiva é

$$\int f(g(t)) \cdot g'(t)\,dt = \int \frac{\operatorname{sen} t}{t} \cdot 2t\,dt$$

$$= -\cos t + C, \quad C \in \mathbb{R}$$

. Regressando à variável inicial, como $t = \sqrt{x}$, $x > 0$

vem

$$\int f(x)\,dx = \int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx = -\cos(\sqrt{x}) + C, \quad C \in \mathbb{R}$$

$\boxed{\text{obs}}$

$$\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx \underset{u' \operatorname{sen} u}{=} -\cos\sqrt{x} + C, \quad C \in \mathbb{R} \;!$$

e) $\displaystyle\int \frac{e^{2x}}{3+e^{x}}\,dx$

. substituição

$e^{x} = t \Rightarrow x = \ln t$, $t > 0$

$g(t) = \ln t$, $g'(t) = \frac{1}{t}$

. Como $f(x) = \dfrac{e^{2x}}{3+e^{x}}$, $f(g(t)) = \dfrac{t^2}{3+t}$

a nova primitiva é

$$\int f(g(t)) \cdot g'(t)\,dt = \int \frac{t^2}{3+t} \cdot \frac{1}{t}\,dt$$

$$= \int \frac{t}{3+t}\,dt = \int 1 - \frac{3}{3+t}\,dt$$

$$\begin{array}{r|l} \cancel{t} & t+3 \\ \hline -\cancel{t} - 3 & 1 \\ \hline -3 & \end{array}$$

$$= t - 3\ln|3+t| + C, \quad C \in \mathbb{R}$$

. Regressando à variável x, como $t = e^{x}$

vem

$$\int f(x)\,dx = \int \frac{e^{2x}}{3+e^{x}}\,dx = e^{x} - 3\ln(3+e^{x}) + C, \quad C \in \mathbb{R}$$

f) $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

- Substituição

$x = \cos t$

$g(t) = \cos t \qquad g'(t) = -\operatorname{sen} t \qquad t \in \left]0, \pi\right[$ (1)

- Tem-se

$$f(x) = \frac{x^2}{\sqrt{1-x^2}}, \quad f(g(t)) = \frac{\cos^2 t}{\sqrt{1-\cos^2 t}} = \frac{\cos^2 t}{\sqrt{\operatorname{sen}^2 t}} = \frac{\cos^2 t}{\operatorname{sen} t},$$

pelo que a nova primitiva vem

$$\int f(g(t))\cdot g'(t)\,dt = -\int \frac{\cos^2 t}{\operatorname{sen} t}\cdot \operatorname{sen} t\,dt$$

$$= -\int \cos^2 t\,dt$$

$$= -\int \frac{1+\cos(2t)}{2}\,dt$$

$$= -\frac{1}{2}\left[t + \frac{1}{2}\operatorname{sen}(2t)\right] + C, \quad C \in \mathbb{R}$$

- Regressando à variável inicial $t = \arccos x$

vem

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\left[\arccos x + \frac{1}{2}\operatorname{sen}(2\arccos x)\right]$$

$$= -\frac{1}{2}\left[\arccos x + x\sqrt{1-x^2}\right] + C, \quad C \in \mathbb{R}$$

já que $\operatorname{sen}(2\arccos x) = 2\operatorname{sen}(\arccos x)\cdot \underbrace{\cos(\arccos x)}_{x}$

e

$\operatorname{sen}\alpha = \sqrt{1-\cos^2\alpha}, \quad \operatorname{sen}\alpha > 0$

(1) Obs $t \in [0, \pi]$ por ser a restrição usada para definir o arccos e $t \neq 0, \pi$ para que $g'(t) \neq 0$.

g) $\int \frac{\sqrt{x}}{x - \sqrt[3]{x}}\, dx$

. substituição

$x = t^6 \Rightarrow g(t) = t^6, \quad g'(t) = 6t^5, \quad t > 0$

. Como

$f(x) = \frac{\sqrt{x}}{x - \sqrt[3]{x}}, \quad f(g(t)) = \frac{t^3}{t^6 - t^2} = \frac{t}{t^4 - 1}$

a nova primitiva é

$$\int f(g(t)) \cdot g'(t)\, dt = \int \frac{t^3}{t^6 - t^2} \cdot 6t^5\, dt$$

$$= 6 \int \frac{t^8}{t^2(t^4-1)}\, dt = 6 \int \frac{t^6}{t^4 - 1}\, dt$$

$$\begin{array}{r|l} t^6 & t^4 - 1 \\ \hline -t^6 \quad +t^2 & t^2 \\ t^2 & \end{array}$$

$$= 6 \int t^2 + \frac{t^2}{t^4 - 1}\, dt$$

$$\overset{(*)}{=} 6 \int t^2 + \frac{1/4}{t-1} - \frac{1/4}{t+1} + \frac{1/2}{t^2+1}\, dt$$

$$= 6 \left[\frac{t^3}{3} + \tfrac{1}{4} \ln|t-1| - \tfrac{1}{4} \ln|t+1| + \tfrac{1}{2} \operatorname{arctg} t \right] + C, \; C \in \mathbb{R}$$

$$= 2t^3 + \tfrac{3}{2} \ln \left| \frac{t-1}{t+1} \right| + 3 \operatorname{arctg} t + C, \; C \in \mathbb{R}$$

. Regressando à variável inicial $t = x^{1/6}$

vem $$\int \frac{\sqrt{x}}{x - \sqrt[3]{x}}\, dx = 2\sqrt{x} + \tfrac{3}{2} \ln \left| \frac{x^{1/6} - 1}{x^{1/6} + 1} \right| + 3 \operatorname{arctg} x^{1/6} + C$$

$C \in \mathbb{R}$

(*) Método das frações parciais

$N(x) = t^2$

$D(x) = t^4 - 1 \Rightarrow D(x) = 0 \Leftrightarrow (t^2 - 1)(t^2 + 1) = 0$

$\Leftrightarrow (t-1)(t+1)(t^2+1) = 0$

$\frac{t^2}{t^4 - 1} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Cx + D}{t^2 + 1} \Rightarrow A = 1/4 \; ; \; B = -1/4 \, , \; C = 0 \, , \; D = 1/2$

h) $\int \sqrt{1+x^2}\, dx$ $\qquad ch^2 t - sh^2 t = 1$

. Substituição

$x = sh\, t$

$g(t) = sh\, t \qquad g'(t) = ch\, t, \quad t \in \mathbb{R}$

. Como

$f(x) = \sqrt{1+x^2}, \quad f(g(t)) = \sqrt{1+sh^2 t} = \sqrt{ch^2 t}$
$= ch\, t$

a nova primitiva é

$$\int f(g(t)) \cdot g'(t)\, dt = \int ch\, t \cdot ch\, t\, dt = \int ch^2 t\, dt$$

$$= \int \frac{ch(2t)+1}{2}\, dt = \frac{1}{2} \int ch(2t) + 1\, dt$$

$$= \frac{1}{2}\left[\frac{1}{2} sh(2t) + t\right] + C, \; C \in \mathbb{R}$$

Regressando à variável inicial $t = arg\, sh\, x$

vem

$$\int \sqrt{1+x^2}\, dx = \frac{1}{4}\left[sh(2\, arg\, sh\, x) + 2\, arg\, sh\, x \right] + C, \; C \in \mathbb{R}$$

$$= \frac{1}{4}\left[2\, sh(arg\, sh\, x)\, ch(arg\, sh\, x) + 2\, arg\, sh\, x \right] + C$$

$$= \frac{1}{2}\left[x\, ch(arg\, sh\, x) + arg\, sh\, x \right] + C$$

$$= \frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2}\right) \right] + C$$

$C \in \mathbb{R}$

<u>Obs</u>

$ch^2 x - sh^2 x = 1 \Rightarrow ch\, x = \sqrt{1 + sh^2 x}$

6.4

a) $\int \frac{2x^2+x+1}{(x-1)(x+1)^2}\,dx$ $\quad 2 = $ grau $N <$ grau $D = 3$

$N(x) = 2x^2+x+1$

$D(x) = (x-1)(x+1)^2$ $\qquad$ grau $N <$ grau D

. Factorizar D

$D(x) = (x-1)(x+1)(x+1)$

$x = 1 \Rightarrow$ zero real simples $\to x-1$; $\frac{A}{x-1}$

$x = -1 \Rightarrow$ zero real duplo $\to x+1$: $\frac{B}{x-1} + \frac{C}{(x-1)^2}$

. Decomposição em fracções simples

$$\frac{N(x)}{D(x)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow 2x^2+x+1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$x = -1$: $2 = -2C \Rightarrow C = -1$

$x = 1$: $4 = 2A \Rightarrow A = 1/2$

$x = 0$: $1 = \frac{1}{2} - B + 1 \Rightarrow B = 1/2$

$$\frac{N(x)}{D(x)} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1} - \frac{1}{(x+1)^2}$$

. Cálculo das primitivas

$$\int \frac{\frac{1}{2}}{x-1}\,dx = \frac{1}{2}\ln|x-1|$$

$$\frac{1}{2}\int \frac{1}{x+1}\,dx = \frac{1}{2}\ln|x+1|$$

$$\int -\frac{1}{(x+1)^2}\,dx = \frac{1}{x+1}$$

Assim

$$\int \frac{N(x)}{D(x)}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + \frac{1}{x+1} + C\,,\ C\in\mathbb{R}$$

$$= \ln\sqrt{x^2-1} + \frac{1}{x+1} + C\,,\ C\in\mathbb{R}$$

b) $\displaystyle\int \frac{3x^2-4x-1}{(x^2-1)(x-2)}\,dx$

- $N(x) = 3x^2-4x-1$
- $D(x) = (x^2-1)(x-2)$ — grau N < grau D

- Factorizar D

$D(x)=0 \iff x=1 \lor x=-1 \lor x=2$

$x=1$ zero real simples $\to x-1$: $\frac{A}{x-1}$

$x=-1$ " $\to x+1$: $\frac{B}{x+1}$

$x=2$ " $\to x-2$: $C/x-2$

- Decompor N/D

$$\frac{N(x)}{D(x)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$\Rightarrow 3x^2-4x-1 = A(x+1)(x-2) + B(x-1)(x-2) + C(x-1)(x+1)$$

$x=-1$: $6 = 6B \Rightarrow B=1$

$x=1$: $-2=-2A \Rightarrow A=1$

$x=2$: $3 = 3C \Rightarrow C=-1$

$$\therefore \frac{N(x)}{D(x)} = \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2}$$

- Cálculo das primitivas

$\int \frac{1}{x-1}\,dx = \ln|x-1|$

$\int \frac{1}{x+1}\,dx = \ln|x+1|$

$\int \frac{1}{x-2} = \ln|x-2|$

Assim

$$\int \frac{N(x)}{D(x)} = \ln|x-1| + \ln|x+1| + \ln|x-2| + C$$
$$= \ln|(x^2-1)(x-2)| + C, \quad C\in\mathbb{R}$$

c) $\int \frac{2x^2 - x - 2}{x^2(x-2)}\,dx$

· Sejam

$N(x) = 2x^2 - x - 2$ grau N < grau D

$D(x) = x^2(x-2)$

· Factorizar D:

$D(x) = 0 \Leftrightarrow x = 0 \lor x = 2$

$x=0$: zero real duplo

$x = 0 \to$ zero real duplo $\to x$: $\frac{A}{x} + \frac{B}{x^2}$

$x = 2 \to$ zero real simples $\to x-2$: $\frac{C}{x-2}$

· Decompor N/D

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$$\Rightarrow 2x^2 - x - 2 = Ax(x-2) + B(x-2) + Cx^2$$

· $x = 0$ $-2 = -2B \Rightarrow B = 1$

· $x = 2$ $4 = 4C \Rightarrow C = 1$

· $x = 1$ $-1 = -A - 1 + 1 \Rightarrow A = 1$

$$\therefore \frac{N(x)}{D(x)} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x-2}$$

· Cálculo das primitivas

$$\int \frac{1}{x}\,dx = \ln x + \mathcal{C}$$

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + \mathcal{C}$$

$$\int \frac{1}{x-2}\,dx = \ln|x-2| - \mathcal{C} \quad , \mathcal{C} \in \mathbb{R}$$

Assim

$$\int \frac{2x^2 - x - 2}{x^2(x-2)}\,dx = \ln|x| + \ln|x-2| - \frac{1}{x} + \mathcal{C}$$

$$= \ln|x(x-2)| - \frac{1}{x} + \mathcal{C}, \quad \mathcal{C} \in \mathbb{R}$$

d) $\displaystyle\int \frac{2x^3+5x^2+6x+2}{x(x+1)^3}\,dx$

. $N(x) = 2x^3 - 5x^2 + 6x + 2$

$D(x) = x(x+1)^3$

. Zeros de D

$D(x) = 0 \Leftrightarrow x = 0 \quad \vee \quad x = -1$

↳ zero real triplo

. Decompor N/D

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

$\Rightarrow 2x^3+5x^2+6x+2 = A(x+1)^3 + B(x+1)^2x + Cx(x+1) + Dx$

. $x = 0$: $2 = A$

. $x = -1$: $-1 = D$

. $x = 1$: $15 = \cancel{2\times}8 + 4B + 2C - 1$

. $x = -2$: $-6 = -2 + (-2)B + 2C + 2$

$\Rightarrow \begin{cases} 2B + C = 0 \\ B - C = 6 \end{cases}$; $3B = 6$ $\Rightarrow \begin{cases} C = -4 \\ B = 2 \end{cases}$

$$\therefore \frac{N(x)}{D(x)} = \frac{2}{x} + \frac{2}{x+1} - \frac{4}{(x+1)^2} - \frac{1}{(x+1)^3}$$

. Cálculo dos primitivos

$\displaystyle\int \frac{2}{x}\,dx = \ln x^2 + C$

$\displaystyle\int \frac{2}{x+1}\,dx = \ln (x+1)^2 + C$

$\displaystyle -\int \frac{4}{(x+1)^2}\,dx = 4\,\frac{1}{x+1} + C,$

$\displaystyle\int -\frac{1}{(x+1)^3}\,dx = \frac{1}{2(x+2)^2}$

Assim

$$\int \frac{N(x)}{D(x)}\,dx = \ln x^2 + \ln (x+1)^2 + \frac{4}{x+1} + \frac{1}{2(x+2)^2} + C, C \in \mathbb{R}$$

$$= \ln [x(x+1)]^2 + \frac{4}{x+1} + \frac{1}{2(x+1)^2} + C, C \in \mathbb{R}$$

e) $\int \frac{x^2 - x + 2}{x(x^2-1)} dx$

. $N(x) = x^2 - x + 2$

$D(x) = x(x^2-1)$

. Zeros de D

$D(x) = 0$

$\Leftrightarrow x = 0 \lor x - 1 = 0 \lor x + 1 = 0$

$x = 0 \rightarrow$ zero real simples $\rightarrow x : \frac{A}{x}$

$x = 1 \rightarrow$ " $\rightarrow x-1 : \frac{B}{x-1}$

$x = -1 \rightarrow$ " $\rightarrow x+1 : C/_{x+1}$

. Decompor N/D

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \quad , \quad A, B, C \in \mathbb{R}$$

$$\Rightarrow x^2 - x + 2 = A(x^2-1) + B(x+1)x + C\,x(x-1)$$

. $x = 0 \qquad -2 = A$

. $x = 1 \qquad 1 = B$

. $x = -1 \qquad 2 = C$

$$\frac{N(x)}{D(x)} = \frac{-2}{x} + \frac{1}{x-1} + \frac{2}{x+1}$$

. Cálculo das primitivas

$\int -\frac{2}{x} dx = -\ln x^2 + C$

$\int \frac{1}{x-1} dx = \ln|x-1| + C$

$\int \frac{2}{x+1} dx = \ln(x+1)^2 + C \qquad , \ C \in \mathbb{R}$

Assim

$$\int \frac{N(x)}{D(x)} dx = \ln\left(\frac{x+1}{x}\right)^2 + \ln|x-1| + C \ , \ C \in \mathbb{R}$$

4) $\displaystyle\int \frac{4x^2+x+1}{x^3-x}\,dx$

Seja $N(x) = 4x^2+x+1$
$D(x) = x^3 - x$ $\quad$ grau N < grau D

Zeros de D

$D(x)=0 \Leftrightarrow x(x^2-1)=0 \Leftrightarrow x=0 \vee x=\pm 1 \quad \to$ 3 zeros reais

Decomposição de N/D

- $x=0 \quad \frac{A}{x}$
- $x=1 \quad B/_{x-1}$
- $x=-1 \quad C/_{x+1}$

$$\frac{N}{D}(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\Rightarrow 4x^2+x+1 = A(x^2-1) + B(x+1)x + Cx(x-1)$$

$x=0 \quad 1 = -A \Rightarrow A=-1$

$x=1 \quad 6 = 2B \Rightarrow B=3$

$x=-1 \quad 4 = 2C \Rightarrow C=2$

Logo

$$\frac{N(x)}{D(x)} = \frac{-1}{x} + \frac{3}{x-1} + \frac{2}{x+1}$$

Cálculo das primitivas

$$\int \frac{N(x)}{D(x)}\,dx = \int -\frac{1}{x} + \frac{3}{x-1} + 2/_{x+1}\,dx$$

$$= -\ln|x| + 3\ln|x-1| + 2\ln|x+1| + C,\ C\in\mathbb{R}$$

$$= \ln\frac{|x-1|^3}{|x|} + \ln(x+1)^2 + C,\ C\in\mathbb{R}$$

g) $\int \frac{27}{x^4-3x^3}\,dx$

$N(x) = 27$

$D(x) = x^4 - 3x^3$

Zeros de D

$D(x)=0 \Leftrightarrow x^3\,(x-3)=0$

$\Rightarrow \quad x = 0 \;\vee\; x = 3 \quad \Rightarrow$ 4 zeros reais

$x=0 \Rightarrow$ zero real triplo $\rightarrow x : \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3}$

$x = 3 \Rightarrow$ zero real simples $\rightarrow x-3 : \frac{D}{x-3}$

Decomposição de N/D

$27 = Ax^2(x-3) + Bx(x-3) + C(x-3) + Dx^3$

$x=0 \Rightarrow 27 = -3C \Rightarrow C = -9$

$x=3 \Rightarrow 27 = 27D \Rightarrow D = 1$

$x=1 \Rightarrow 27 = -2A - 2B + 18 + 1 \Rightarrow A + B = -4$

$x=-1 \Rightarrow 27 = -4A + 4B + 36 - 1 \Rightarrow A - B = 2$

$\Rightarrow \begin{cases} B = -3 \\ A = -1 \end{cases}$

$$\frac{27}{x^4-3x^3} = -\frac{1}{x} - \frac{3}{x^2} - \frac{9}{x^3} + \frac{1}{x-3}$$

Cálculo da Primitiva

$$\int \frac{27}{x^4-3x^3}\,dx = -\int \frac{1}{x}\,dx - 3\int \frac{1}{x^2}\,dx - 9\int \frac{1}{x^3}\,dx + \int \frac{1}{x-3}\,dx$$

$$= -\ln|x| + 3\,\frac{1}{x} + \frac{9}{2}\,\frac{1}{x^2} + \ln|x-3| + \mathcal{C}$$

$$= \ln\left|\frac{x-3}{x}\right| + \frac{3}{x} + \frac{9}{2x^2} + \mathcal{C},\ \mathcal{C} \in \mathbb{R}$$

h) $\displaystyle\int \frac{x^4 - 8}{x^3 - 2x^2}\, dx$

Sejam

$N(x) = x^4 - 8$

$D(x) = x^3 - 2x^2$ $\Big\rangle$ grau $N >$ grau D

Divisão euclidiana

$$\begin{array}{l|l} x^4 \qquad\qquad -8 & x^3 - 2x^2 \\ \hline -x^4 - 2x^3 & x + 2 \\ \quad -2x^3 & \\ \quad +2x^3 + 4x^2 & \\ \hline \qquad 4x^2 - 8 & \end{array}$$

$$N(x) = (x^3 - 2x^2)(x + 2) + \underbrace{(4x^2 - 8)}_{R(x)}$$

Logo

$$\frac{N(x)}{D(x)} = x + 2 + 4\, \frac{x^2 - 2}{x^2 (x-2)}$$

Zeros de D

$D(x) = 0 \Leftrightarrow x = 0$ ou $x = 2$ 2 zeros reais

Decomposição R/D

$$\frac{R(x)}{D(x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$\Rightarrow 4x^2 - 8 = A\, x\, (x-2) + B\, (x-2) + C\, x^2$

$\Rightarrow 4x^2 - 8 = (A + C)\, x^2 + (B - 2A)\, x - 2B$

$$\Rightarrow \begin{cases} A + C = 4 \\ B - 2A = 0 \\ -2B = -8 \end{cases} \Rightarrow \begin{cases} C = 2 \\ A = 2 \\ B = 4 \end{cases} \qquad \therefore \frac{R(x)}{D(x)} = \frac{2}{x} + \frac{4}{x^2} + \frac{2}{x-2}$$

Cálculo das primitivas

$$\int \frac{4x^4 - 8}{x^3 - 2x^2}\, dx = \int (x + 2)\, dx + 2 \int \frac{1}{x}\, dx + 4 \int \frac{1}{x^2}\, dx + 2 \int \frac{1}{x-2}\, dx$$

$$= \frac{x^2}{2} + 2x + 2 \ln|x| - \frac{4}{x} + 2 \ln|x-2| + C, \; C \in \mathbb{R}$$

$$= \frac{x^2}{2} + 2x - \frac{4}{x} + \ln (x (x-2))^2 + C, \; C \in \mathbb{R}$$

i) $\displaystyle\int \frac{x+3}{(x-2)(x^2-2x+5)}\,dx$

Sejam

$N(x) = x+3$

$D(x) = (x-2)(x^2-2x+5)$ grau N < grau D

$= (x-2)\left[(x-1)^2+4\right]$

Zeros de D

$D(x)=0 \Leftrightarrow x=2 \ \vee\ (x-1)^2=-4$

$\Leftrightarrow x=2 \ \vee\ x=1\pm 2i$

$x=2 \to$ zero real simples $\to x-2$: $\dfrac{A}{x-2}$

$x=1\pm 2i \to$ zeros complexos simples $\to (x-1)^2+4$: $\dfrac{Bx+C}{(x-1)^2+4}$

Decomposição de N/D

$$\frac{N(x)}{D(x)} = \frac{A}{x-2} + \frac{Bx+C}{(x-1)^2+4}$$

$$\Rightarrow x+3 = A\left[(x-1)^2+4\right] + (Bx+C)(x-2)$$

- $x=2$ $\quad 5 = 5A \Rightarrow A=1$
- $x=0$ $\quad 3 = 5-2C \Rightarrow C=1$
- $x=1$ $\quad 4 = 4-(B+1) \Rightarrow B=-1$

$$\frac{x+3}{(x-2)\left[(x-1)^2+4\right]} = \frac{1}{x-2} + \frac{1-x}{(x-1)^2+4}$$

Cálculo da Primitiva

$$\int \frac{x+3}{(x-2)\left[(x-1)^2+4\right]}\,dx = \int \frac{1}{x-2} - \frac{x-1}{(x-1)^2+4}\,dx$$

$$= \ln|x-2| - \frac{1}{2}\ln\left[(x-1)^2+4\right] + C, \ C\in\mathbb{R}$$

$$= \ln|x-2| - \ln\sqrt{(x-1)^2+4} + C, \ C\in\mathbb{R}$$

j) $\displaystyle\int \frac{x+1}{x(x^2+1)^2}\,dx$

Sejam

$N(x) = x+1$

$D(x) = x(x^2+1)^2$ grau N < grau D

O polinomio D já está factorizado

$x=0$ zero real simples : $\dfrac{A}{x}$

$x=\pm i$ zeros complexos duplos : $\dfrac{Bx+C}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

Decomposição de N/D

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

$$x+1 = A(x^2+1)^2 + (Bx+C)\,x\,(x^2+1) + (Ex+F)\,x$$

. $x=0$ $\quad 1 = A$

. $x=i$, $x=-i$ $\quad \begin{cases} 1+i = (Ei+F)\,i \\ 1-i = (-Ei+F)(-i) \end{cases} \Rightarrow \begin{cases} 1+i = -E+Fi \\ 1-i = -E-Fi \end{cases} \Rightarrow \begin{cases} E=-1 \\ F=1 \end{cases}$

. $x=1$ $\quad 2 = 4+2(B+C) \Rightarrow B+C=-1$

. $x=-1$ $\quad 0 = 4-2(C-B)-2 \Rightarrow C-B=1$

$C = 0$, $B=-1$

$$\therefore \frac{N(x)}{D(x)} = \frac{1}{x} - \frac{x}{x^2+1} + \frac{1-x}{(x^2+1)^2}$$

Cálculo das primitivas (ver no fim da folha)

$$\int \frac{x+1}{x(x^2+1)^2}\,dx = \int \frac{1}{x} - \frac{x}{x^2+1} + \frac{1-x}{(x^2+1)^2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\operatorname{arctg} x + \frac{1}{2}\,\frac{x+1}{x^2+1} + C,$$

$C \in \mathbb{R}$

k) $\int \frac{x+2}{2x(x-1)^2(x^2+1)}\,dx$

Sejam

$N(x) = x+2$

$D(x) = 2x(x-1)^2(x^2+1)$) grau N < grau D

Decomposição de N/D

x 1 zero real simples $\frac{A}{x}$

$(x-1)^2$ 1 zero real duplo $\frac{B}{x-1} + \frac{C}{(x-1)^2}$

x^2+1 2 zeros complexos conjugados $\frac{Ex+F}{x^2+1}$

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Ex+F}{x^2+1}$$

$$= \frac{1}{x} + \frac{-5/4}{x-1} + \frac{3/4}{(x-1)^2} + \frac{1}{4}\,\frac{x+2}{x^2+1}$$

Cálculo das primitivas

$$\int \frac{x+2}{2x(x-1)^2(x^2+1)}\,dx = \int\left[\frac{1}{x} + \frac{5/4}{x-1} + \frac{3/4}{(x-1)^2} + \frac{1}{4}\,\frac{x+2}{x^2+1}\right]dx$$

$$= \ln|x| + \tfrac{5}{4}\ln|x-1| - \tfrac{3}{4}\,\frac{1}{x-1} + \frac{1}{4}\int \frac{x}{x^2+1} + \frac{2}{x^2+1}\,dx$$

$$= \ln\left|x(x-1)^{5/4}\right| - \frac{3}{4(x-1)} + \frac{1}{8}\ln(x^2+1) + \frac{1}{2}\operatorname{arctg} x + C,$$

$C \in \mathbb{R}$

l) $\displaystyle\int \frac{3x^3+x^2-1-x}{x^2(x^2-1)}\,dx$

Sejam

$N(x) = 3x^3 + x^2 - 1 - x$

$D(x) = x^2(x^2-1) = x^2(x-1)(x+1)$

Decomposição de N/D

$x^2 \Rightarrow x=0$ é zero real duplo $\qquad \frac{A}{x} + \frac{B}{x^2}$

$x-1 \Rightarrow x=1$ " " simples $\qquad \frac{C}{x-1}$

$x+1 \Rightarrow x=-1$ " " " $\qquad \frac{E}{x+1}$

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{E}{x+1}$$

$$\Rightarrow \frac{3x^3+x^2-x-1}{x^2(x^2-1)} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x-1} + \frac{1}{x+1}$$

Cálculo da primitiva

$$\int \frac{N(x)}{D(x)}\,dx = \int \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x-1} + \frac{1}{x+1}\,dx$$

$$= \ln|x| - 1/x + \ln|x-1| + \ln|x+1| + C, \; C \in \mathbb{R}$$

$$= \ln|x(x-1)(x+1)| - \frac{1}{x} + C$$

$$= \ln|x(x^2-1)| - \frac{1}{x} + C, \; C \in \mathbb{R}$$

6.5 Calcule

a) $\int \frac{1}{(2+\sqrt{x})^7\sqrt{x}}\,dx = 2\int \frac{1}{2}\frac{1}{\sqrt{x}}\,(2+\sqrt{x})^{-7}\,dx$

$= -\frac{1}{3}\,(2+\sqrt{x})^{-6} + C,\; C\in\mathbb{R}$

b) $\int tg^2 x\,dx = \int \frac{sen^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx$

$= \int \frac{1}{\cos^2 x} - 1\,dx = tg\,x - x + C,\; C\in\mathbb{R}$

c) $\int \frac{x + [\arcsen(3x)]^2}{\sqrt{1-9x^2}}\,dx = \int \underbrace{\frac{x}{\sqrt{1-9x^2}}}_{u'u^n} + \underbrace{\frac{1}{\sqrt{1-(3x)^2}}\,[\arcsen(3x)]^2}_{u'u^n}\,dx$

$= -\frac{1}{9}\,(1-9x^2)^{1/2} + \frac{1}{6}\,[\arcsen(3x)]^3 + C,\; C\in\mathbb{R}$

d) $\int \frac{x\,e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}\,dx = -\int \underbrace{\frac{2x}{\sqrt{1-x^2}}}_{u'}\,\underbrace{e^{\sqrt{1-x^2}}}_{e^u}\,dx$

$= -e^{\sqrt{1-x^2}} + C,\; C\in\mathbb{R}$

e) $\int \frac{1}{\cos^2 x\,sen^2 x}\,dx = \int \frac{sen^2 x + \cos^2 x}{\cos^2 x\,sen^2 x}\,dx$

$= \int \frac{1}{\cos^2 x} + \frac{1}{sen^2 x}\,dx$

$= tg^2 x - cotg^2 x + C,\; C\in\mathbb{R}$

ou

$\int \frac{1}{\cos^2 x\,sen^2 x}\,dx = \int \frac{1}{(\cos x\,sen\,x)^2}\,dx = 4\int \frac{1}{(2\,sen\,x\cos x)^2}\,dx = 4\int \frac{1}{sen^2(2x)}\,dx$

$= -2\,cotg(2x) + C,\; C\in\mathbb{R}$

f) $\int \cos^2 x \cdot \operatorname{sen}^2 x \, dx$

Note-se que

$$\cos^2 x \operatorname{sen}^2 x = (\operatorname{sen} x \cos x)^2 = \Big[\tfrac{1}{2} \underbrace{2 \operatorname{sen} x \cos x}\Big]^2$$

$$= \frac{1}{4} \operatorname{sen}^2(2x)$$

Mas

$$\cos(2\alpha) = \cos^2\alpha - \operatorname{sen}^2\alpha$$

$$= 1 - 2\operatorname{sen}^2\alpha \Rightarrow \operatorname{sen}^2\alpha = \frac{1-\cos(2\alpha)}{2}$$

Assim

$$\cos^2 x \cdot \operatorname{sen}^2 x = \frac{1}{4} \frac{1-\cos(4x)}{2}$$

E

$$\int \cos^2 x \cdot \operatorname{sen}^2 x \, dx = \frac{1}{8} \int 1 - \cos(4x) \, dx$$

$$= \frac{1}{8}\left[x - \frac{1}{4}\operatorname{sen}(4x)\right] + C, \quad C \in \mathbb{R}$$

g) $\int \frac{1}{1+e^x} dx = \int \frac{1+e^x - e^x}{1+e^x} dx = \int \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x} dx$

$$= \int 1 - \frac{e^x}{1+e^x} dx$$

$$= x - \ln(1+e^x) + C, \quad C \in \mathbb{R}$$

h) $\int \frac{1}{x^2\sqrt{4-x^2}} dx$ Seja $\varphi:]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[\longrightarrow]-2,2[$ $\varphi(t) = 2\operatorname{sen} t$

φ é bijectiva, $\varphi'(t) = 2\cos t \neq 0$, pelo que se pode fazer $x = \varphi(t)$.

A nova primitiva é

$$\int \frac{1}{4\operatorname{sen}^2 t \sqrt{4 - 4\operatorname{sen}^2 t}} \cdot 2\cos t \, dt$$

$$= \frac{1}{4}\int \frac{1}{\operatorname{sen}^2 t \cdot \cancel{\sqrt{1-\operatorname{sen}^2 t}}} \cancel{\cos t}\, dt = \frac{1}{4}\int \frac{1}{\operatorname{sen}^2 t} dt = -\tfrac{1}{4}\operatorname{cotg} t + C.$$

$$= -\tfrac{1}{4} \frac{\cos t}{\operatorname{sen} t} + C = -\frac{1}{4} \frac{\cos(\operatorname{arcsen} x/2)}{x/2} + C \qquad x = 2\operatorname{sen} t \Rightarrow \boxed{\frac{x}{2} = \operatorname{sen} t}$$

$$= -\tfrac{1}{2} \frac{\sqrt{1-\left(\frac{x}{2}\right)^2}}{x} + C = -\frac{\sqrt{4-x^2}}{4x} + C, \quad C \in \mathbb{R}$$

6.6 a $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \operatorname{sen} x$ $\quad (\pi/2, \pi)$

Do exercício 6.2 g) sabemos que

$$\int x^2 \operatorname{sen} x \, dx = (2-x^2)\cos x + 2x \operatorname{sen} x + C, \quad C \in \mathbb{R}$$

Isto é

$$F_C(x) = (2-x^2)\cos x + 2x \operatorname{sen} x + C, \quad C \in \mathbb{R}$$

é uma família de primitivas de f. Destas funções queremos a que verifica

$$F_C(\pi/2) = \pi.$$

Ora

$$F_C(\pi/2) = \pi \Leftrightarrow \left(2 - \frac{\pi^2}{4}\right) \times 0 + \pi + C = \pi$$

$$\Leftrightarrow C = 0$$

A primitiva de f cujo gráfico passa em $(\pi/2, \pi)$ é a função

$$F_0(x) = (2-x^2)\cos x + 2x \operatorname{sen} x$$

6.7 $f: \mathbb{R} \to \mathbb{R}$

a) $f''(x) = 4x - 1$, $x \in \mathbb{R}$ $\quad f(1) = 3$, $f'(2) = -2$

Se

$$f''(x) = 4x - 1 \Rightarrow f'(x) = 2x^2 - x + C$$

$$\Rightarrow f(x) = \frac{2x^3}{3} - \frac{x^2}{2} + Cx + A, \quad A, C \in \mathbb{R}.$$

Mas

$$f'(2) = 2 \Rightarrow 2 \cdot 4 - 2 + C = -2 \Rightarrow C = -8$$

isto é

$$f'(x) = 2x^2 - x - 8$$

Por outro lado

$$f(1) = 3 \Rightarrow \frac{2}{3} - \frac{1}{2} - 8 + A = 3 \Rightarrow A = \frac{65}{6}$$

Assim

$$f(x) = \frac{2x^3}{3} - \frac{x^2}{2} - 4x + 65/6$$

b) $f''(x) = \operatorname{sen} x \cos x \qquad f(0)=0 \qquad f'(0)=1$

$= \frac{1}{2} \operatorname{sen}(2x)$

Ora

$$f''(x) = \frac{1}{2} \operatorname{sen}(2x) \Rightarrow f'(x) = -\frac{1}{4} \cos(2x) + B$$

$$\Rightarrow f(x) = -\frac{1}{8} \operatorname{sen}(2x) + Bx + A, \quad A, B \in \mathbb{R}$$

Como $f(0)=0$ e $f'(0)=1$ tem

$$\begin{cases} A = 0 \\ -\frac{1}{4} + B = 1 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 5/4 \end{cases}$$

$$\therefore f(x) = 5/4 - \frac{1}{8} \operatorname{sen}(2x)$$

6.4 j) Cálculo das primitivas

$$\int \frac{1}{x}\,dx = \ln|x| + C_1$$

$$\int \underbrace{\frac{x}{x^2+1}}_{\frac{u'}{u}}\,dx = \frac{1}{2}\ln(x^2+1) + C_2$$

$$\int \frac{1-x}{(x^2+1)^2}\,dx = \int \frac{1}{(x^2+1)^2} - \frac{x}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{(x^2+1)^2}\,dx - \frac{1}{2}\int \underbrace{2x}_{u'}\,\underbrace{(x^2+1)^{-2}}_{u^n}\,dx$$

$$= \int \frac{(1+x^2)-x^2}{(x^2+1)^2}\,dx + \frac{1}{2}\,\frac{1}{x^2+1} + C$$

$$= \int \frac{\cancel{1+x^2}}{(x^2+1)^{\cancel{2}}} - \frac{x^2}{(x^2+1)^2}\,dx + \frac{1}{2(x^2+1)} + C$$

$$= \int \frac{1}{1+x^2}\,dx - \int \frac{x^2}{(x^2+1)^2}\,dx + \frac{1}{2(x^2+1)} + C$$

$$= \operatorname{arctg} x - \int \underbrace{x}_{v}\cdot\underbrace{\frac{x}{(x^2+1)^2}}_{u'}\,dx + \frac{1}{2(x^2+1)} + C$$

$$= \operatorname{arctg} x - \left[-\frac{1}{2}\,\frac{1}{(x^2+1)}\cdot x + \int \frac{1}{2}\,\frac{1}{x^2+1}\cdot 1\,dx\right] + \frac{1}{2(x^2+1)} + C$$

$$= \operatorname{arctg} x + \frac{x}{2(x^2+1)} - \frac{1}{2}\int \frac{1}{1+x^2}\,dx + \frac{1}{2(x^2+1)} + C$$

$$= \operatorname{arctg} x + \frac{x+1}{2(x^2+1)} - \frac{1}{2}\operatorname{arctg} x + C$$

$$= \frac{1}{2}\operatorname{arctg} x + \frac{1}{2}\,\frac{x+1}{x^2+1} + C,\quad C\in\mathbb{R}$$