Arbitrage pricing through risk measures

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Objective

This dissertation aims to present an extension of the framework presented by **Madan and Cherny (2010)**, through the introduction of a spread function $R^{\gamma}_{\psi_1,\psi_2}$ and its derivatives.

Context

- The set $\mathcal{A}$ is a convex cone formed by financial positions, $X \in L^\infty$, which are traded in the market.
- The acceptability index will determine whether the position X is traded or not.
- The bid and ask prices of the position X can be represented as a coherent risk measure.

Risk Measures

Definition

A measure ρ is called a coherent risk measure if it satisfies the following conditions, $\forall\ X, Y \in L^\infty$:

- (Monotonicity) If $X \leq Y$, then $\rho(X) \geq \rho(Y)$,
- (Cash Invariance) If $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) - m$,
- (Convexity) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, for $\lambda \in [0,1]$,
- (Positive homogeneity) If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda\rho(X)$.

Theorem

*(**Föllmer and Schied (2016)**, Proposition 4.6.) Let ρ be a monetary risk measure with acceptance set $\mathcal{A} := \mathcal{A}_\rho$ then:*

1. *$\mathcal{A}$ is non-empty.*
2. $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty$.
3. *$X \in \mathcal{A}, Y \in L^\infty, Y \geq X$, then $Y \in \mathcal{A}$.*
4. *$\{\lambda \in [0,1] : \lambda X + (1-\lambda) Y \in \mathcal{A}\}$ is closed in $[0,1]$, for $X \in \mathcal{A}$ and $Y \in L^\infty$.*
5. *ρ can be recovered from $\mathcal{A}$*

$$\rho(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}.$$

6. *ρ is a convex risk measure if and only if $\mathcal{A}$ is a convex set.*
7. *ρ is positively homogeneous if and only if $\mathcal{A}$ is a cone. In particular, ρ is coherent if and only if $\mathcal{A}$ is a convex cone.*

A very important relation that was explored by **Madan and Cherny (2010)** is the following equivalence relation presented in the seminal work of **Artzner et al. (1999)**. The authors show that any convex cone of acceptable financial positions $\mathcal{A}$ is defined by a non-empty closed convex set of probability measures,

$$X \in \mathcal{A} \iff \mathbb{E}_{\mathbb{Q}}[X] \geq 0, \ \forall \, \mathbb{Q} \in \mathcal{D}, \tag{1}$$

where $\mathcal{D}$, called supporting set or set of test measures, is a set of probability measures that are absolutely continuous with respect to $\mathbb{P}$. The Equation (1) tells us that the financial position X will be in the cone of acceptable positions $\mathcal{A}$ if, and only if, all probability measures that are in the set of test measures, $\mathcal{D}$, approve the acceptability of the random variable X.

Definition

Let ψ^γ be a concave distortion function and X be a financial position. The Choquet integral is defined as

$$\int_\Omega Xd(\psi^\gamma \circ \mathbb{P}) = \int_{-\infty}^{0} [(\psi^\gamma(\mathbb{P}(X > x)) - 1]dx + \int_0^\infty (\psi^\gamma(\mathbb{P}(X > x))dx, \quad (2)$$

and the function $\mathbb{E}^\psi[.] : L^\infty \to \mathbb{R}$ given by

$$\mathbb{E}^\psi[X] := -\int_{-\infty}^{0} \psi^\gamma(F_X(x))dx + \int_0^\infty [1 - \psi^\gamma(F_X(x))]dx \quad (3)$$

is called a distorted expectation or Choquet expectation.

Conic Finance

The first function to be introduced is the acceptability index. This function is chosen by the agent and is responsible for telling us which financial position will be traded in the market for a given level of liquidity.

Definition

The function $\alpha : L^\infty \to [0, \infty]$ is an acceptability index. We say that a financial position X is acceptable at $\gamma \geq 0$ if

$$\alpha(X) \geq \gamma. \tag{4}$$

The coefficient γ can be interpreted as the market liquidity level, **Leippold and Schärer (2017)**. In this context, we will have acceptable positions for each liquidity level, where the higher γ the more illiquid the market is (in a complete market $\gamma = 0$).

Theorem

The function $\alpha : L^\infty \to [0,\infty]$ is an index of acceptability if and only if

$$\alpha(X) = \sup \left\{ \gamma \in \mathbb{R}_+ : \inf_{\mathbb{Q} \in \mathcal{D}_\gamma} \mathbb{E}_\mathbb{Q}[X] \geq 0 \right\}, \tag{5}$$

where $\inf \emptyset = \infty$ *and* $\sup \emptyset = 0$*, and there exists a family of subsets* $\{\mathcal{D}_\gamma\}_{\gamma \in \mathbb{R}_+}$ *of* $\mathcal{P}$ *such that* $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma'}$ *for* $\gamma \leq \gamma'$*.*

As suggested in **Cherny and Madan (2009)** and **Madan and Cherny (2010)**, an acceptability index can be constructed from a family of concave distortions. Therefore, the acceptability index, Theorem (5), can be represented as

$$\alpha(X) = \left\{ \gamma \geq 0 : \int_{-\infty}^{\infty} x d\psi^{\gamma}(F_X(x)) \geq 0 \right\}. \tag{6}$$

Therefore, the bid and ask price for the framework presented by **Madan and Cherny (2010)** can be written as

$$\begin{aligned} a_{\psi}^{\gamma}(X) &= -\int_{-\infty}^{\infty} x d\psi^{\gamma}(F_{-X}(x)) \\ &= -\mathbb{E}_{\mathbb{Q}}^{\gamma}[-X], \end{aligned}$$

$$\begin{aligned} b_{\psi}^{\gamma}(X) &= \int_{-\infty}^{\infty} x d\psi^{\gamma}(F_X(x)) \\ &= \mathbb{E}_{\mathbb{Q}}^{\gamma}[X]. \end{aligned}$$

Proposed Approach

To present the spread function, we need to introduce a new configuration that will take into account two acceptability indices, α_1 and α_2. One index is related to one direction of the trade and the other index will be related to the other direction, both indices will be represented by a family of concave distortions. In this case, the following equivalence relations hold for $X \in L^\infty$,

$$\alpha_1(X) \geq \gamma \iff \int_{-\infty}^{\infty} x d\psi_1^\gamma(F_X(x)) \geq 0, \quad (7)$$

$$\alpha_2(X) \geq \gamma \iff \int_{-\infty}^{\infty} x d\psi_2^\gamma(F_X(x)) \geq 0. \quad (8)$$

In this new framework we will have three sets of financial positions, $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}^*$.

- $\mathcal{A}_1$ is the convex cone of traded positions in the bid direction, i. e. $\alpha_1(X) \geq \gamma, X \in L^\infty$.
- $\mathcal{A}_2$ is the convex cone of positions traded in the ask direction, i.e. $\alpha_2(X) \geq \gamma, X \in L^\infty$.
- $\mathcal{A}^*$ is the set of financial positions that are traded in both directions, i.e. $\mathcal{A}^* = \mathcal{A}_1 \cap \mathcal{A}_2$. We know that the finite intersection of convex sets is again a convex set and the intersection of cones is again a cone. Therefore, $\mathcal{A}^*$ is a convex cone.

Assuming that the market is incomplete and will trade only financial positions acceptable by the α_1 and α_2 indices, the ask and bid prices will be as follows.

Definition

Let ψ_1 and ψ_2 be two concave distortion functions and γ the liquidity level for two fixed acceptability indexes, α_1 and α_2. Then the bid and ask prices of a financial position are given by:

$$b^{\gamma}_{\psi_1}(X) := \mathbb{E}^{\psi_1,\gamma}_{\mathbb{Q}}[X] \tag{9}$$

$$= -\int_{-\infty}^{0} \psi_1^{\gamma}(F_X(x))dx + \int_0^{\infty} [1 - \psi_1^{\gamma}(F_X(x))]dx \tag{10}$$

$$a^{\gamma}_{\psi_2}(X) := -\mathbb{E}^{\psi_2,\gamma}_{\mathbb{Q}}[-X].$$

$$= \int_{-\infty}^{0} [\psi_2^{\gamma}(F_{-X}(x))]dx + \int_0^{\infty} [\psi_2^{\gamma}(F_{-X}(x)) - 1]dx \tag{11}$$

The spread function represents the difference between the ask and bid prices for a given financial position and is commonly presented in the literature.

Definition

Let $a^{\gamma}_{\psi_2}$ and $b^{\gamma}_{\psi_1}$ be the ask and bid prices under the concave distortions ψ_2 and ψ_1, respectively. The spread function, $R^{\gamma}_{\psi_1,\psi_2} : L^{\infty} \to \mathbb{R}$, for a financial position X is given by

$$R^{\gamma}_{\psi_1,\psi_2}(X) := a^{\gamma}_{\psi_2}(X) - b^{\gamma}_{\psi_1}(X). \tag{12}$$

Our spread function encompasses all cases presented in the literature and allows us to add more cases.

The papers that compute the spread using a single distortion such as **Bannor and Scherer (2014)**, **Chen et al. (2019)** and **Luo and Chen (2021)**, can be considered a particular case of $R^{\gamma}_{\psi_1,\psi_2}$ considering $\psi_1(u) = \psi_2(u), \forall\, u \in [0,1]$.

We can consider more cases, where ψ_1 and ψ_2 functions are different. These cases can be interpreted as a scenario where an agent is under a certain restriction to buy or sell the payoff, for example.

Theorem

Let ψ_1 and ψ_2 be two concave distortions and a given $\gamma \geq 0$. Then, the bid and ask prices will satisfy the following properties:

1. $b^{\gamma}_{\psi_1}$ *and* $a^{\gamma}_{\psi_2}$ *are continuous functions,*
2. *If* $\psi_1(u) = \psi_2(u)$, $\forall u \in [0, 1]$, *then* $b^{\gamma}_{\psi_1}(X) \leq a^{\gamma}_{\psi_2}(X)$ *for all* $X \in L^{\infty}$,
3. *The ask price can be represented as*

$$a^{\gamma}_{\psi_2}(X) = \int_0^1 F_X^{-1}(1-p)d\psi_2^{\gamma}(p),$$

4. *The bid price can be represented as*

$$b^{\gamma}_{\psi_1}(X) = \int_0^1 F_X^{-1}(p)d\psi_1^{\gamma}(p).$$

Theorem

Let ψ_1 and ψ_2 be concave distortions and a given γ. The spread function, $R^{\gamma}_{\psi_1,\psi_2}$, satisfies the following properties, for the financial positions $X, Y \in L^{\infty}$:

1. $R^{\gamma}_{\psi_1,\psi_2}$ *is a continuous function,*
2. *If $\psi_1(u) = \psi_2(u), \forall\, u \in [0,1]$, then the spread function is non-negative, $R^{\gamma}_{\psi_1,\psi_2}(X) \geq 0$,*
3. *If $\psi_1(u) = \psi_2(u), \forall\, u \in [0,1]$, then the map $\gamma \to R^{\gamma}_{\psi_1,\psi_2}$ is increasing,*
4. *If $\gamma = 0$, $R^{\gamma}_{\psi_1,\psi_2}(X) = 0$, for all $X \in L^{\infty}$,*
5. *If $\gamma \to \infty$, then $R^{\gamma}_{\psi_1,\psi_2}(X) = \mathrm{range}(X)$, i.e.* $\lim_{\gamma\to\infty} R^{\gamma}_{\psi_1,\psi_2}(X) = \mathrm{range}(X)$.

Remark

The numerical examples presented by ***Chen et al. (2019)*** *and* ***Luo and Chen (2021)*** *corroborate the results obtained in items 2, 3 and 4. The authors performed such examples for exotic derivatives and found that: when $\gamma = 0$, we have that the bid and ask prices are equivalent; γ has an increasing relation with the spread function; and the spread function is always non-negative, as the authors consider a special case where Wang distortion distorts the accumulations of both prices.*

Example

Madan and Schoutens (2016) present a closed expression for the price of an option, under the Black-Scholes hypothesis, using the distortion ψ_γ^{WANG}. In the Black-Scholes model, we have that $\ln S_t$ is normally distributed with mean $\ln S_0 + (r - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$, this is equivalent to

$$S_t \sim LN\big(\ln S_0 + (r - \frac{1}{2}\sigma^2)t, \sigma^2 t\big).$$

The Wang distortion of the cumulative price is given by

$$\psi_\gamma^{WANG}(F_{S_t}(x)) = N\left(\frac{\ln x - (\ln S_0 - (r - \frac{1}{2}\sigma^2)t + \gamma\sigma\sqrt{t})}{\sigma\sqrt{t}}\right).$$

Example (Continuation)

The bid price presented by the authors is given by

$$b^{\gamma}_{\psi^{WANG}}(c_t) = S_t e^{-\gamma\sigma\sqrt{T-t}} N(d_1) - K e^{-r(T-t)} N(d_2), \tag{13}$$

where

$$d_1 = \frac{\ln\left(S_t/K\right) + \left(r + \sigma^2/2\right)(T-t) - \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

The closed formula for the ask price using Wang distortion is similar to the expression found for the bid,

$$a^{\gamma}_{\psi^{WANG}}(c_t) = S_t e^{\gamma\sigma\sqrt{T-t}} N(d_1) - K e^{-r(T-t)} N(d_2), \tag{14}$$

Example (Continuation)

where

$$d_1 = \frac{\ln\left(S_t/K\right) + \left(r + \sigma^2/2\right)(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

However, it is possible to go a little further and find the Greeks for this two-price economy with the Wang distortion. Below are the main Greeks for the bid price.

1. (Delta)

$$\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial S_t} = e^{-\gamma\sigma\sqrt{T-t}} N(d_1),$$

2. (Gamma)

$$\frac{\partial}{\partial S}\left(\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial S_t}\right) = \frac{1}{\sigma S\sqrt{T-t}} e^{-\gamma\sigma\sqrt{T-t}} N'(d_1),$$

Example (Continuation)

3. (Vega)

$$\frac{\partial b^{\gamma}_{\psi WANG}(c_t)}{\partial \sigma} = -\gamma\sqrt{T-t}S_t e^{-\gamma\sigma\sqrt{T-t}}N(d_1)$$

$$+S_t e^{-\gamma\sigma\sqrt{T-t}}N'(d_1)\sqrt{T-t},$$

4. (Theta)

$$\frac{\partial b^{\gamma}_{\psi WANG}(c_t)}{\partial t} = \frac{\gamma\sigma}{\sqrt{T-t}}S_t e^{-\gamma\sigma\sqrt{T-t}}N(d_1) - rKe^{-r(T-t)}N(d_2)$$

$$-\frac{1}{2}\frac{\sigma}{\sqrt{T-t}}S_t e^{-\gamma\sigma\sqrt{T-t}}N'(d_1),$$

5. (Rho)

$$\frac{\partial b^{\gamma}_{\psi WANG}(c_t)}{\partial r} = (T-t)Ke^{-r(T-t)}N(d_2).$$

Example (Continuation)

In addition to the traditional Greeks of the Black-Scholes model, we have a new Greek, the partial derivative with respect to the liquidity parameter γ. This new partial derivative for the bid price is given by,

6.
$$\frac{\partial b^{\gamma}_{\psi WANG}(c_t)}{\partial \gamma} = -\gamma S_t e^{-\gamma\sigma\sqrt{T-t}} N(d_1).$$

The partial derivative of the ask price with respect to γ is given by

7.
$$\frac{\partial a^{\gamma}_{\psi WANG}(c_t)}{\partial \gamma} = \gamma S_t e^{\gamma\sigma\sqrt{T-t}} N(d_1).$$

Remark

In Example 10, for the special case where $\gamma = 0$, we have that the bid and ask prices are equal, which was already expected by Theorem 9, and the Greeks are the same as in the Black-Scholes model.

Final remarks

Important contributions have been made and the remaining research has been identified and described.

The steps necessary to achieve our goal are as follows:

1. To derive the Greeks for each of the convex distortions,
2. To derive the Greeks for different dynamics for the assets,
3. To perform a comprehensive numerical example to identify the behavior of the Greeks.