Arbitrage pricing through risk measures

Master’s Thesis

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Abstract

This work provides an extension of the conic finance framework, through a new spread function that allows each side of the trade to be calculated using a distinct distortion. In this way, we return to the original conic finance framework when the distortions are equal. Based on this new framework, we prove useful results when the liquidity parameter takes on extreme values. We prove that in a special case the spread function will be a deviation measure. Additionally, closed forms are provided for the Greeks of the Black-Scholes model and an additional partial derivative with respect to the parameter gamma is presented, for the case where the bid and ask prices are computed using the Wang distortion.

Keywords: Conic Finance, Risk Measures, Option Pricing, Distorted probabilities.

Contents

1 Introduction

Since the first pricing model presented by Bachelier in his doctoral thesis in 1900, interest in option pricing models has been growing. Over time, the models have become more and more sophisticated, seeking to incorporate the latest stylized facts.

Most pricing models, e.g., Black e Scholes (1973) and Merton (1973), assume that the financial market is complete. From this assumption, we have what is known as the law of one price, where the price of a given option is independent of whether the agent wants to buy or sell.

Although the classical approach says that the market is complete, there will be the possibility of perfect hedging, this is not what is observed in actual markets. As long as the market presents the bid and ask spread, perfect replication is not possible and the law of one price is not consistent and is replaced by the law of two prices. Several authors have tried to explain this spread presented in two-price economies. Davis et al. (1993), Soner et al. (1995) and Barles e Soner (1998) have tried to explain this spread by including transaction or inventory costs. However, the authors did not fully explain the magnitude of the spread.

A new approach was presented by D. B. Madan e Cherny (2010) seeking to address this empirical evidence. The authors present an approach, referred to as the *conic finance theory*, where agents are still modeled as a passive counterparty. However, there is a difference in the price according to the direction of the trade. That is, there is a price where the market is interested in selling, the ask price, and the price where the market is interested in buying, the bid price.

In this new approach, the model for the market is defined as a convex cone, $\mathcal{A}$, containing the financial positions that agents are willing to trade in the market. Building on the results of the seminal work of Artzner et al. (1999), the authors describe the relationship between the financial positions in the convex cone and their respective expected values. However, this convex cone of financial positions is not constant over time. To capture the dynamics of this set of financial positions traded in the market, the authors use an acceptability index α, which are introduced in Cherny e Madan (2009), which is directly related to the parameter γ, the liquidity level. The authors show that in this framework, the two prices in this economy are represented as the supremum and infimum of the expectations with respect to a set of probability measures, $\mathcal{D}_\gamma$.

A new hedging methodology was presented by D. B. Madan et al. (2016) within this framework, this new methodology allowing systematic hedging choices with wide applications. Van Bakel et al. (2020) studied the impact of this framework on CVA and DVA of option positions. A study to measure the liquidity of exotic options, also within this framework, was conducted by Guillaume e Schoutens (2015). Chen et al. (2019) and Luo e Chen (2021) present explicit formulas for exotic options using the Wang distortion

Our objective in this work is to extend the conic finance framework, enabling the use of two acceptability indexes, α_1 and α_2, in such a way that the agent is free to represent the different directions of trade with certain distortions, ψ_1 and ψ_2. In our main context, α_1 and α_2 are acceptability indexes in the sense of Cherny e Madan (2009), whereas ψ_1 and ψ_2 are distortion functions related to these acceptability indexes. In this way, ψ_1 is used to distort the bid price and, likewise, ψ_2 distorts the ask price. In this new framework, when ψ_1 and ψ_2 are equal we get the configuration proposed by D. B. Madan e Cherny (2010).

Within this framework, we present the main object of this work, the *spread function*, $R^{\gamma}_{\psi_1,\psi_2}$, which is now defined as the difference between the bid and ask prices with, possibly, distinct distortions. We prove that, for the extreme cases $\gamma = 0$ and $\gamma \to \infty$, it holds that $R^{0}_{\psi_1,\psi_2}(X) = 0$, the bid and ask prices are equivalent, and $R^{\gamma\to\infty}_{\psi_1,\psi_2}(X) = \text{range}(X) = \operatorname{ess\,sup}(X) - \operatorname{ess\,inf}(X)$. We also demonstrate a handful of mathematical and financial properties enjoyed by the spread function, and in particular we highlight that it is a deviation measure, whenever $\gamma > 0$ and the distortions are equal. Additionally, within the conic finance framework, we derive the main Greeks for a European call option using the Wang distortion and we present the partial derivative with respect to the parameter γ

To the best of our knowledge, there is no literature research that expands the conic finance approach in this way or that demonstrates the properties of the spread function for a financial position $X \in L^{\infty}$. Chen et al. (2019) and Luo e Chen (2021), have presented, through numerical examples, results that are in line with ours, but these studies are centered on a specific derivative, rather than a random financial position X.

The remainder of this paper is structured in this format: Section 2 presents the notation, definitions, and preliminaries from the literature; Section 3 presents an extension of the conic finance framework and exposes our main results related to the spread function and presents the Greeks for a European call option using the Wang distortion; Section 4 presents the concluding remarks; and finally, we present an appendix containing a definition of concepts that will be used throughout the text and some important results related to option pricing.

2 Background

Hereafter we adopt the following notation and conventions: random variables are denoted by X and Y and represent financial positions. Equality in distribution is denoted by $\stackrel{d}{=}$; aside from that, equalities and inequalities involving random variables are to be understood in an $\mathbb{P}$-almost sure sense. Let $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the space of essentially bounded random variables in $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with L^∞-norm $||\cdot||_\infty$, where $||X||_\infty = \operatorname{ess\,sup} X = \inf\{k \in \mathbb{R} : \mathbb{P}(X > k) = 0\}$ and the essential infimum is defined as $\operatorname{ess\,inf} X = \sup\{k \in \mathbb{R} : \mathbb{P}(X < k) = 0\}$. $E[X]$ is the expected value of X under $\mathbb{P}$. We define $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$. F_X is the probability function of X and its inverse is F_X^{-1}, defined as $F_X^{-1}(p) = \inf\{x : F_X(x) \geq p\}$, where the following equivalence holds

$$p \leq F_X(x) \iff F_X^{-1}(p) \leq x, \ x \in \mathbb{R}, \ p \in [0, 1]. \tag{1}$$

2.1 Arbitrage

We consider a model with $d{+}1$ assets, whose price processes $S = \{(S_t^0, S_t^1 \ldots, S_t^d)\}_{t \in [0,T]}$ is assumed to be an adapted, continuous and strictly positive semi-martingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. The price process S^0 is interpreted as a risk-free bond, i.e. $S_t^0 = e^{rt}$, and $S_t^i \geq 0, i \in \{0, \ldots, d\}, t \in [0, T]$.

Definition 2.1. *1. A **trading strategy** is a stochastic process*

$$\phi = (\phi^0, \phi^1, \ldots, \phi^d) : \Omega \times [0, T] \to \mathbb{R}^{d+1}, \tag{2}$$

where ϕ_t^i corresponds to the amount of each asset held by the investor, such that $\int_0^T \mathbb{E}[\phi_s^0]ds < \infty$ and $\sum_{i=0}^d \int_0^T \mathbb{E}[(\phi_s^i)^2]ds < \infty$.

*2. The **value of the portfolio** ϕ at time t is given by*

$$V_t^\phi := \phi_t \cdot S_t = \sum_{i=0}^d \phi_t^i S_t^i, \quad t \in [0, T]. \tag{3}$$

*3. The **gain process** from a trading strategy ϕ is given by*

$$G_t^\phi := \int_0^t \phi_s dS_s = \sum_{i=1}^n \int_0^t \phi_s^i dS_s^i. \tag{4}$$

*4. A portfolio strategy ϕ is called **self financing**, if*

$$V_t^\phi = V_0^\phi + G_t^\phi, \quad t \in [0, T], \tag{5}$$

i.e. $dV_t^\phi = \phi_t dS_t$.

Remark 1. A trading strategy is self-financing precisely when the value of the portfolio is preserved over time. This type of portfolio is extremely important for the replication of a claim, as will be shown later.

Before we continue, we are often interested in the relative price of an asset. If we normalize the asset prices in our model by the risk-free asset, S^0, we obtain what is known as a discounted pricing process. This particular pricing process is most useful when we want to find the present value of a future payoff.

Definition 2.2. *A* ***numeraire*** *is a price process X on $[0,T]$, if*

$$\mathbb{P}\big(\{\omega : X_t(\omega) > 0,\ \forall t \leq T\}\big) = 1. \tag{6}$$

Our numeraire will be S^0, this way we will be able to obtain a discounted pricing process. Our goal now is to extend the items presented in the definition (2.1) to a normalized model.

Theorem 2.1. *(Bingham e Kiesel (2013), Proposition 6.1.1) A self financing portfolio remains self financing after a numeraire change.*

Therefore, we can present another version of the definition (2.1) as follows.

Definition 2.3.

1. *The* ***discounted price process*** *$\hat{S}$ is defined as*

$$\hat{S} = (\hat{S}^0, \hat{S}^1, \dots, \hat{S}^d) = \left(1, \frac{S^1}{S^0}, \dots, \frac{S^d}{S^0}\right) \tag{7}$$

where S^0 is a numeraire.

2. *The* ***discounted value process*** *of the portfolio ϕ is given by*

$$\hat{V}_t^\phi := \frac{V_t^\phi}{S_t^0} = \phi_t^0 + \sum_{i=1}^d \phi_t^i \hat{S}_t^i, \quad t \in [0,T]. \tag{8}$$

3. *The* ***discounted gains process*** *of the portfolio ϕ is given by*

$$\hat{G}_t^\phi := \sum_{i=1}^d \int_0^t \phi_s^i d\hat{S}_s^i, \tag{9}$$

4. *A portfolio strategy ϕ is* ***self financing*** *if and only if*

$$\hat{V}_t^\phi = \hat{V}_0^\phi + \hat{G}_t^\phi \tag{10}$$

i.e. $d\hat{V}_t^\phi = \phi_t d\hat{S}_t$.

A fundamental requirement for pricing a claim is that there are no arbitrage possibilities. If arbitrage exists in a market model, we are saying that there are opportunities to make gains with certainty at zero cost.

Definition 2.4. *A self financing portfolio ϕ is an* ***arbitrage opportunity*** *if the value process V^ϕ satisfies, $\mathbb{P}$-almost surely,*

$$V_0^\phi = 0, \quad V_T^\phi \geq 0, \quad V_T^\phi > 0.$$

Now that we know when a certain portfolio is an arbitrage opportunity, we can consider the following questions: 1) when will we have an arbitrage-free market model? and 2) what is the arbitrage-free price for a given derivative? Before answering the question, we must introduce the concept of Risk Neutral measures.

Definition 2.5. *A probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ is called a* ***equivalent martingale measure (EMM)*** *with respect to S^0, if*

1. *$\mathbb{Q}$ and $\mathbb{P}$ are equivalent,*
2. *The discounted price processes $\hat{S}^i$, $\forall\ i \in \{0, \dots, d\}$, are martingales under the measure $\mathbb{Q}$.*

The set of all equivalent martingales measures is denoted by $\mathcal{P}$.

To sum up, an EMM is a probability measure that makes the discounted price process of all $d + 1$ market assets to be martingale processes. EMM measures are also known as risk-neutral measures. We can now answer the first question with the next theorem.

Theorem 2.2. *(Bingham e Kiesel (2013), Theorem 6.1.1) The market model is arbitrage-free if and only if the set $\mathcal{P}$ is non-empty.*

That is, the model is arbitrage-free precisely when the price process of all assets is a martingale under a certain measure equivalent to $\mathbb{P}$.

Returning to the second question, we want to know what is the arbitrage-free price of a derivative with a maturity T. In the literature, a derivative security is also called a contingent claim, or just claim. A claim is an asset whose payoff depends exactly on the behavior of other assets, $S^0, S^1, \dots, S^d$. For a given claim C we have the following result.

Definition 2.6. *Let ϕ be a portfolio strategy and a contingent claim C, then*

1. *ϕ is called **admissible**, for a finite u, if*

$$V_t^\phi \geq -u. \tag{11}$$

2. *C is called **replicable** if there is at least one admissible portfolio strategy such that*

$$V_T^\phi = C. \tag{12}$$

If the claim is replicable, holding the claim or a portfolio strategy whose process value is V_T^ϕ generates the same financial result. Therefore, the price of a claim at a given period t is given by $\Pi_t(C) = V_t^\phi$. Thus, we can answer our second question.

Theorem 2.3. *(Bingham e Kiesel (2013), Theorem 6.1.4) The arbitrage free price for a replicable claim is given by*

$$\Pi_t(C) = S_t^0 \mathbb{E}_{\mathbb{Q}}\left[\frac{C}{S_T^0}\middle|\mathcal{F}_t\right]. \tag{13}$$

When all claims are replicable, we will have a market model known as the complete model.

Theorem 2.4. *(Bingham e Kiesel (2013),Theorem 6.1.5) In an arbitrage-free model, if all claims are replicable, then the model is called complete. Furthermore, this arbitrage-free model is complete if and only if there is exactly one risk-neutral measure.*

2.2 Option pricing models

In this section, we will present the main pricing models considering $d = 1$, i.e. a model with 2 assets, where one of the assets is a risk free bond. For the reader interested in a more comprehensive reference of models, see Smith Jr (1976), Haug (2007a) and Haug (2007b).

The following terminology is used: $S_t \geq 0$, a random variable, is the stock price in period t; K is the strike price of the option; r is the risk-free interest rate; T is the time to expiration in years; $\sigma \geq 0$ is the volatility of the underlying asset return; μ is the expected rate of return on the underlying asset; and $c_t = c(S_t, t)$ is the price of the European call option at time t.

2.2.1 The Bachelier model

In his doctoral thesis of 1900, see Bachelier, Cootner et al. (1964), Bachelier assumes that the price process of the asset, S, is described by an Arithmetic Brownian Motion, which is given by

$$dS_t = \mu dt + \sigma dW_t,$$

where μ and σ are constants. Under a risk-neutral measure, $\mathbb{Q}$, the asset price dynamics are described as follows:

$$dS_t = \sigma dW_t^{\mathbb{Q}}, \tag{14}$$

since it is implicitly assumed that $r = 0$. In Equation (14), we can see one of the biggest objections raised by Smith Jr (1976), the asset price can assume negative values.

The call option price for the Bachelier model is given by

$$\begin{aligned} c_t &= \mathbb{E}[(S_t - K)^+] \\ &= (S_t - K)N(d_1) + \sigma\sqrt{T-t}N'(d_1), \end{aligned} \tag{15}$$

where N is the cumulative standard normal, N' is the standard normal density function and $d_1 = \frac{S_t - K}{\sigma\sqrt{T-t}}$. Note in Equation (15) the positive relationship between $\sqrt{T-t}$ and c_t, this is another objection to using Bachelier's model, because the maximum value which the call price can assume is not equal to the stock price, Smith Jr (1976)

2.2.2 The Sprenkle model

Sprenkle (1964) assumes that the price dynamics is given by a Geometric Brownian Motion, GBM,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{16}$$

where μ is the expected value of the asset return and σ is the volatility of the return. Under Equation (16) the stock price has a lognormal distribution, thus prohibiting negative prices, and the return has a normal distribution.

The formula for the call price for the Sprenkle model is given by:

$$c_t = S_t e^{\zeta(T-t)} N(d_1) - (1-k)KN(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (\zeta + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

k is the degree of risk aversion of the market and ζ is the average rate of growth of the asset price, $e^{\zeta(T-t)} = \mathbb{E}[S_T/S_t]$. Note that, in the Sprenkle model, the strike value, K, is not discounted.

2.2.3 The Boness model

Boness (1964) assumes that the asset price is log normally distributed, as is Sprenkle, and the price dynamics is given by (16). The call option formula derived by Boness is

$$c_t = S_t N(d_1) - K e^{-\eta(T-t)} N(d_2), \tag{17}$$

where

$$d_1 = \frac{\ln(S_t/K) + (\eta + \sigma^2/2)(T-t)}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

The Equation (17) is the same as the formula obtained by Black e Scholes (1973), but Boness considers that each asset will have a growth rate η. This suggests that Boness considers η as a proxy for the call option price growth, $\mathbb{E}(c_T/c_t)$, Smith Jr (1976).

2.2.4 The Black-Scholes model

Black e Scholes (1973) present a closed formula for pricing European options. The authors greatest insight is the portfolio replication argument, (Haug, 2007a).

The market consists of a risky asset, a stock S, and a riskless asset, a government bond B, whose dynamics are given by B:

$$dB_t = rB_t dt, \tag{18}$$

$$dS_t = S_t \mu_t dt + S_t \sigma_t dW_t, \tag{19}$$

where the risk-free rate r is constant.

Suppose the European call c_t price is a twice differentiable function. Using Itô's formula, we have that dc_t is given by

$$dc_t = \left\{ S_t \mu_t \frac{\partial c_t}{\partial S_t} + \frac{\partial c_t}{\partial t} + \sigma_t^2 S_t^2 \frac{1}{2} \frac{\partial^2 c_t}{\partial S_t^2} \right\} dt + \frac{\partial c_t}{\partial S_t} \sigma_t S_t dW_t. \tag{20}$$

Note that the dynamics of c_t, like the dynamics of S_t, can be described as an Itô process.

Consider a portfolio consisting of a short position in a call option and a long position in the stock. We will denote the value of this portfolio at time t by Π_t, such that

$$\Pi_t = \Delta S_t - c_t. \tag{21}$$

The portfolio dynamics is given by

$$d\Pi_t = \Delta dS_t - dc_t. \tag{22}$$

Substituting (18) and (19) into (22) we have

$$\begin{aligned} d\Pi_t =& \Delta\Big\{S_t\mu_t dt + S_t\sigma_t dW_t\Big\} - \Big\{S_t\mu_t\frac{\partial c_t}{\partial S_t} + \frac{\partial c_t}{\partial t} + \sigma_t^2 S_t^2\frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^2}\Big\}dt \\ & - \frac{\partial c_t}{\partial S_t}\sigma_t S_t dW_t. \end{aligned}$$

If we define the long position in stocks as $\Delta = \frac{\partial c_t}{\partial S_t}$, we will have the following dynamics

$$d\Pi_t = \Big\{ - \frac{\partial c_t}{\partial t} - \frac{1}{2}\frac{\partial c_t^2}{\partial S_t^2}\sigma_t^2 S_t^2\Big\}dt. \tag{23}$$

Note that the portfolio dynamics given in (23) is risk-free during the period dt. We still have that

$$d\Pi_t = r\Pi_t dt, \tag{24}$$

if the growth of Π_t is different from r we will have an arbitrage opportunity, which is not possible. Substituting (21) and (23) into (24) we obtain the famous Black-Scholes PDE

$$\frac{\partial c_t}{\partial t} + \frac{1}{2}\frac{\partial c_t^2}{\partial S_t^2}\sigma_t^2 S_t^2 + rS_t\frac{\partial c_t}{\partial S_t} = rc_t. \tag{25}$$

Note that to obtain the Black-Scholes PDE we need to be in an arbitrage-free model. Consequently, there will be an exact replication of the payoff through a portfolio strategy. Using Feynman-Kac to solve this PDE and Girsanov theorem to replace the physical measure $\mathbb{P}$ by the risk neutral measure $\mathbb{Q}$, we have that

$$c_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+]. \tag{26}$$

Solving for (26), we get the following formula for the call option

$$c_t = c_t^{BS}(S_t, K, T, \sigma, r) = S_t N(d_1) - e^{-r(T-t)}KN(d_2), \tag{27}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Where $N(d_2)$ can be interpreted as the probability that the underlying asset ends above the strike price, $\mathbb{Q}(S_T > K)$. Or simply, the probability of the underlying asset ending up in the money.

2.2.5 The Merton model

The model proposed by Merton (1973) allows pricing options where the underlying asset pays dividends. This model is most recommended for options on a stock index where it is assumed that the index pays out a continuous dividend yield. For the case of options on a single stock, it is more appropriate to treat the dividends as discrete, see Haug (2007b) Chapter 9.

According to Merton, the price of an asset that pays dividends continuously at an annualized rate q follows the following dynamics

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t. \tag{28}$$

The PDE obtained from the SDE presented in (28) is given by

$$\left[\frac{\partial c_t}{\partial t} + \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^2}\sigma^2 S_t^2 + (r-q)\frac{\partial c_t}{\partial S_t}S_t\right]dt = rc_t, \tag{29}$$

for a more detailed demonstration see Lyuu (2002), Chapter 15. The solution to the PDE presented above is given by

$$c_t = e^{-q(T-t)}S_t N(d_1) - e^{-r(T-t)}KN(d_2), \tag{30}$$

where

$$d_1 = \frac{\ln(S_t/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{31}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{32}$$

2.2.6 The Jump-Diffusion model

In the model proposed by Merton (1976) the price changes are formed by a random component, Wiener process, with drift and a jump component that is modeled by a compound Poisson process. The jumps in prices occur independently and identically distributed. This jump term that is added in the GBM tends to cause incompleteness, due to the greater difficulty of exact replication of a payoff, since stock prices are affected by random size jumps, (Staum, 2007).

The price for the stock S at the period t is given by

$$S_t = S_0 e^{L_t}, \tag{33}$$

where the Lévy process, L, can be described as

$$L_t = (\mu - \frac{\sigma^2}{2} - \lambda k)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i. \tag{34}$$

The first two terms on the right represent a Brownian motion with drift process and the last term is a compound Poisson jump process.

The compound Poisson jump process has two sources of randomness. The first source of randomness comes from the moment in time when the jump will occur. Merton uses a Poisson process dN_t with intensity λ to model this first phenomenon. And the second source of randomness is the size of the jump, given that a jump has occurred. Merton assumes that the size of the jump in the log of the asset price is distributed as a normal with mean μ_J and variance σ_J^2.

The probability that the asset price jumps during a small time interval dt is obtained using the Poisson process dN_t,

$$\mathbb{P}(\{\text{an asset price jumps once in } dt\}) = \mathbb{P}(\{dN_t = 1\}) \approx \lambda dt, \tag{35}$$

$$\mathbb{P}(\{\text{an asset price jumps more than once in } dt\}) = \mathbb{P}(\{dN_t \geq 2\}) \approx 0, \tag{36}$$

$$\mathbb{P}(\{\text{an asset price does not jump in } dt\}) = \mathbb{P}(\{dN_t = 0\}) \approx 1 - \lambda dt, \tag{37}$$

where $\lambda \in \mathbb{R}^+$ is the mean number of jumps per unit of time which is independent of time t.

The SDE proposed by the author that incorporates the above mentioned properties is represented as follows,

$$\frac{dS_t}{S_t} = (\mu - \lambda k)S_t dt + \sigma S_t dW_t + (y_t - 1)dN_t, \tag{38}$$

where σ is the instantaneous volatility of the asset return conditional on that jump does not occur, N_t is an Poisson process with intensity λ, $y_t - 1$ is the relative price jump size of S_t, such that

$$\frac{dS_t}{S_t} = \frac{y_t S_t - S_t}{S_t} = y_t - 1 \sim LN(k = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1, e^{2\mu_J + \sigma_J^2}(e^{\sigma_J^2} - 1)). \tag{39}$$

Furthermore, the processes W_t, N_t and y_t given in equation (38) are independent. The price of a call option for the Merton model is given by

$$c_t = \sum_{j \geq 0} \frac{e^{-\lambda(T-t)}(\lambda(T-t))^j}{j!} c_t^{BS}(S_j \equiv S_t e^{j\mu_J + \frac{j\sigma_J^2}{2} - \lambda(e^{\mu_J + \frac{\sigma_J^2}{2}} - 1)(T-t)}, K, T, \sqrt{\sigma^2 + \frac{j\sigma_J^2}{T-t}}, r). \tag{40}$$

where j represents the number of jumps that occur during the $T - t$ time period.

2.2.7 The Garman and Kohlhagen model

The Model proposed by Garman e Kohlhagen (1983) is a model that aims to price foreign exchange options (FX options). In the case of an FX option, we are buying the right to trade a currency pair at a specific price at a specific date. This means that we are buying the base currency and selling the quote currency. Therefore, S_t will represent the price of a given currency (domestic currency) per unit of quote currency at time t, and K is the option strike price, also in local currency per unit of foreign currency. The standard Black-Scholes model does not apply well since in this context, we will have two interest rates, which differs from the assumptions of Black-Sholes.

The authors assume that the S dynamics is governed by a GBM, as in the Black-Scholes model, and thus the following PDE is obtained,

$$\left[\frac{\partial c_t}{\partial t} + \frac{1}{2}\frac{\partial^2 c_t}{\partial S_t^2}\sigma^2 S_t^2 + (r - r_f)\frac{\partial c_t}{\partial S_t}S_t\right]dt = rc_t, \tag{41}$$

where r_f is the foreign interest rate. Note that the PDE above is very similar to the PDE presented by Merton (1973), equation (29), the main difference is that in the model for dividend paying stocks we will replace r_f by q. The solution to the PDE in the equation (41) is given by

$$c_t = e^{-r_f(T-t)}S_tN(d_1) - e^{-r(T-t)}KN(d_2), \tag{42}$$

where

$$d_1 = \frac{\ln(S_t/K) + (r - r_F + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{43}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{44}$$

2.2.8 The Hull-White model

The Hull-White model, (Hull & White, 1987), offers a closed-form solution to the European option price problem when we have another equation that describes the behavior of volatility over time. The dynamics of the asset price and the volatility obey the following system of equations

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t^1 \tag{45}$$

$$dv_t = \beta v_t dt + \varepsilon v_t dW_t^2 \tag{46}$$

where $v_t = \sigma_t^2$, and $\varepsilon > 0$. In the equations above, W_t^1 and W_t^2 denote independent Wiener process. Note that, in this model we have two sources of risk, W^1 and W^2. In the case where W_1 and W_2 are two independent Brownian motions, the authors presented a closed-form solution for a specific risk-neutral measure.

The so-called average future variance is defined as the random variable

$$\tilde{\sigma}_t^2 = \frac{1}{T-t}\int_t^T \sigma_s^2 ds. \tag{47}$$

The option price for the Hull-White model is obtained taking the expected value, under a risk neutral measure $\mathbb{Q}$, of the Black-Scholes formula and replacing the constant volatility by the average future variance, equation (47), i.e.

$$c_t = c_t^{BS}(S_t, K, T, \sqrt{\tilde{\sigma}_t^2}, r) = S_t N(d_1) - e^{-r(T-t)} K N(d_2) \tag{48}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \tilde{\sigma}_t^2/2)(T-t)}{\sqrt{\tilde{\sigma}_t^2(T-t)}}, \\ d_2 &= d_1 - \sqrt{\tilde{\sigma}_t^2(T-t)}. \end{aligned}$$

For $\varrho \neq 0$, there is correlation between W^1 and W^2, is possible to obtain the option prices using Monte Carlo simulation.

2.2.9 The Heston model

The model presented by Heston (1993) stands out from other stochastic volatility models because it allows a correlation, ϱ, between the shocks that drive the asset price and its volatility and also presents an analytical solution for European options. Heston assumes that the dynamics of the asset is given by a Geometric Brownian Motion, while volatility evolves according to the process proposed by Cox et al. (1985). The underlying processes of asset price and its volatility are presented below:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^1 \tag{49}$$

$$dv_t = \kappa[\theta - v_t]dt + \xi\sqrt{v_t} dW_t^2 \tag{50}$$

where $\kappa > 0$ is the mean reversion speed, i.e. the rate at which the variance converges to its unconditional mean, $\theta > 0$ is the long-term mean of the variance, $\xi \geq 0$ represents the volatility of the variance and v_t is the level of variance at time t.

Note that while the Black-Scholes model presents only one source of risk, the Heston model admits the change in volatility as another source of risk, where W_1 and W_2 are Wiener process with covariance $dW_1 dW_2 = \varrho dt$, $\varrho \in (-1, 1)$. As with the Hull-White model, Heston model will also be incomplete.

The price c_t of an European call option is defined as

$$c_t = S_t P_1(\ln(S_t), v_t, T) - e^{-r(T-t)} K P_2(\ln(S_t), v_t, T). \tag{51}$$

The right side of the equation above is similar to the Black-Scholes formula, where P_1 and P_2 are two distinct conditional probabilities that can be interpreted as probabilities that a call option expires in the money, under different measures.

2.3 Risk measures and Conic finance.

Let L^∞ be the space of essentially bounded financial positions X, where $X \geq 0$ is a gain and $X < 0$ is a loss. We call any functional $\rho : L^\infty \to \mathbb{R}$ a risk measure. Our goal is to study risk measures that can be interpreted as bid or ask prices of an asset, as presented in D. B. Madan e Cherny (2010).

2.3.1 Risk Measures

Definition 2.7. *A measure ρ is called a* ***monetary measure*** *of risk if it satisfies the following conditions,* $\forall\, X, Y \in L^\infty$

- *(Monotonicity) If $X \leq Y$, then $\rho(X) \geq \rho(Y)$,*
- *(Cash Invariance) If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.*

In general, by monotonicity we can say that if the payoff of X is less than the payoff of Y, in all states of nature, then of course more capital is required to make the financial position X acceptable. By cash invariance, we can say that the required capital of the position $X + m$ will be $\rho(X) - m$, since m is invested in a risk-free manner.

Definition 2.8. *A* ***convex risk measure*** *ρ is a monetary risk measure that satisfies:*

- *(Convexity) $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for $\lambda \in [0, 1]$.*

The convexity property captures the idea of diversification. The required amount of capital of a financial position formed by the convex combination of two other positions, $\lambda X + (1 - \lambda)Y$, must be less than or equal to the convex combination of the required capital of positions X and Y.

Definition 2.9. *A **coherent risk measure** ρ is a convex measure that satisfies:*

- *(Positive homogeneity) If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda\rho(X)$.*

  A coherent risk measure can also be defined using another property

- *(Sub Additivity) $\rho(X + Y) \leq \rho(X) + \rho(Y)$.*

The convexity property is equivalent to the sub additivity property, under the assumption of positive homogeneity.

A very important object in conic finance theory is the acceptance set, $\mathcal{A}$. The set $\mathcal{A}$ will represent the financial positions traded in the market, i.e. if $X \in \mathcal{A}$, then the market is willing to pay a certain amount to buy X and a certain amount to sell X. The results below show the relationship between a risk measure and an acceptance set.

Definition 2.10. *A monetary risk measure ρ induces the following acceptance set*

$$\mathcal{A}_\rho \coloneqq \{X \in L^\infty : \rho(X) \leq 0\}.$$

If X is acceptable, $X \in \mathcal{A}_\rho$, then X do not require surplus capital.

Theorem 2.5. *(Föllmer e Schied (2016), Proposition 4.6.) Let ρ be a monetary risk measure with acceptance set $\mathcal{A} \coloneqq \mathcal{A}_\rho$ then:*

1. *$\mathcal{A}$ is non-empty.*
2. $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty$.
3. *$X \in \mathcal{A}, Y \in L^\infty, Y \geq X$, then $Y \in \mathcal{A}$.*
4. *$\{\lambda \in [0, 1] : \lambda X + (1 - \lambda)Y \in \mathcal{A}\}$ is closed in $[0, 1]$, for $X \in \mathcal{A}$ and $Y \in L^\infty$.*
5. *ρ can be recovered from $\mathcal{A}$*

$$\rho(X) = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}.$$

6. *ρ is a convex risk measure if and only if $\mathcal{A}$ is a convex set.*

7. *ρ is positively homogeneous if and only if $\mathcal{A}$ is a cone. In particular, ρ is coherent if and only if $\mathcal{A}$ is a convex cone.*

A very important relation that was explored by D. B. Madan e Cherny (2010) is the following equivalence relation presented in the seminal work of Artzner et al. (1999). The authors show that any convex cone of acceptable financial positions $\mathcal{A}$ is defined by a non-empty closed convex set of probability measures,

$$X \in \mathcal{A} \iff \mathbb{E}_{\mathbb{Q}}[X] \geq 0, \ \ \forall\, \mathbb{Q} \in \mathcal{D}, \tag{52}$$

where $\mathcal{D}$, called supporting set or set of test measures, is a set of probability measures that are absolutely continuous with respect to $\mathbb{P}$. The Equation (52) tells us that the financial position X will be in the cone of acceptable positions $\mathcal{A}$ if, and only if, all probability measures that are in the set of test measures, $\mathcal{D}$, approve the acceptability of the random variable X.

The supporting set in Equation (52) formed by probability measures absolutely continuous with respect to $\mathbb{P}$ is not unique. We can present the largest $\mathcal{D}$ set as

$$\mathcal{D} = \{\mathbb{Q} \in \mathcal{P} : \mathbb{E}_{\mathbb{Q}}[X] \geq -\rho(X), \forall X \in L^{\infty}\}, \tag{53}$$

where $\mathcal{P}$ is the set of probability measures absolutely continuous with respect to $\mathbb{P}$.

In the conic finance framework, the set $\mathcal{A}$ is modeled as a convex cone. Thus, the equation (52) tells us that: if a financial position is traded in the market, i.e., $X \in \mathcal{A}$, then the infimum of the financial position expectations will be non-negative for a given set of probability measures, i.e., $\inf_{\mathbb{Q}\in\mathcal{D}} \mathbb{E}_{\mathbb{Q}}[X] \geq 0$. The equation (53) tells us what is the largest set $\mathcal{D}$ in which the expectation, under each measure $\mathbb{Q} \in \mathcal{D}$, is non-negative.

Another object that is of fundamental importance and will be used to represent each of the prices in this economy is the expected value. The expected value as detailed by Lo (2018), can be written as

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} X dF_X(x) = -\int_{-\infty}^{0} F_X(x)dx + \int_{0}^{\infty} (1 - F_X(x))dx. \tag{54}$$

where all probabilities are treated uniformly. However, in some cases we are not interested in uniform treatment of the probabilities. In our case, a non-uniform treatment of probabilities can help us interpret the bid and ask price.

One area that commonly does not treat probabilities uniformly is the area of insurance. In this area, the insurance company will assume an individual's risk and in return the insured will pay an amount to the insurance company, this amount is called the premium. To ensure that the insurance company does not go bankrupt, the premium has to be larger than the expected loss. In this case the expected loss is defined as in equation (54), but for a non-negative random variable X. The premium is obtained through a distortion applied to the survival function, $\psi(1 - F_X(x))$, such that the following property holds

$$\psi(q) \geq q, \quad q \in [0, 1]. \tag{55}$$

Consequently, the difference between the premium charged by the insurance company, $\mathbb{E}^{\psi}[X]$, and the expected loss is

$$\int_0^\infty x d\psi(F_X(x)) - \int_0^\infty x dF_X(x) \geq 0, \tag{56}$$

$$\mathbb{E}^{\psi}[X] - \mathbb{E}[X] \geq 0, \tag{57}$$

and its called risk premium. Note that the condition established in (55) guarantees that the risk premium is non-negative.

The concepts presented from the insurance area are useful for describing the behavior of prices, in a two-price economy. Since it is possible to interpret the ask price as the result of an underweighting of losses and an overweighting of gains, while the bid price is the result of an overweighting of losses and an underweighting of gains, (Leippold & Schärer, 2017). A formal definition for the distortion and for the distorted expectation will be given below.

Definition 2.11. *A function $\psi^\gamma : [0, 1] \to [0, 1]$ is called a **concave distortion function** if and only if ψ^γ is monotone, $\psi^\gamma(0) = 0$ and $\psi^\gamma(1) = 1$. The set $\{\psi^\gamma\}_{\gamma \geq 0}$ is called a **family of concave distortions** if the following conditions hold*

1. *$\psi^{\gamma_1}(u) \leq \psi^{\gamma_2}(u)$ for $\gamma_1 \leq \gamma_2$, $u \in [0, 1]$,*

2. *the map $\gamma \mapsto \psi^\gamma(u)$ is continuous $\forall\ u \in [0, 1]$.*

Additionally, let us assume the following conditions,

3. *the map $u \mapsto \psi^\gamma(u)$ is continuous on $(0, 1]$,*

4. *For $\gamma = 0$, $\psi^0(u) = u$, for $u \in [0, 1]$,*

5. $\lim_{\gamma\to\infty}\psi^\gamma(u) = 1$, *for* $u \in (0, 1]$.

If ψ^γ is a distortion function and $\mathbb{P}$ is a probability measure on $\mathcal{F}$, then the mapping $U \mapsto \psi^\gamma(P(U))$ is called a distorted probability measure, for $U \in \mathcal{F}$. Clearly, the distorted probability measure $\psi^\gamma \circ \mathbb{P}$ is not a probability measure in general, since the additivity property is not satisfied. However, since ψ^γ is a concave function, $\psi^\gamma \circ \mathbb{P}$ will be 2-alternating or submodular.

The premium that an insurance company charges, presented in the equation (56), is called a Choquet expectation. This type of expectation will be very important to describe the prices of a financial position. The concept of a Choquet expectation is directly related to the Choquet integral and both concepts are defined below.

Definition 2.12. *Let ψ^γ be a concave distortion function and X be a financial position. The* ***Choquet integral of X*** *is defined as*

$$\int_\Omega X d(\psi^\gamma \circ \mathbb{P}) = \int_{-\infty}^{0} [(\psi^\gamma(\mathbb{P}(X > x)) - 1]dx + \int_{0}^{\infty} (\psi^\gamma(\mathbb{P}(X > x))dx, \tag{58}$$

and the function $\mathbb{E}^\psi[.] : L^\infty \to \mathbb{R}$ given by

$$\mathbb{E}^\psi[X] := -\int_{-\infty}^{0} \psi^\gamma(F_X(x))dx + \int_{0}^{\infty} [1 - \psi^\gamma(F_X(x))]dx \tag{59}$$

is called a distorted expectation or ***Choquet expectation****.*

Note that, the Choquet expectation can be written as $\mathbb{E}^\psi[X] = -\int_\Omega(-X)d(\psi^\gamma \circ \mathbb{P})$. Representing the Choquet expectation by it is integral, it is possible to verify that the expectation without distortion and the Choquet expectation share the properties of monotonicity, translation invariance and positivite homogeneity. For a list of properties satisfied by Choquet integral, see Ridaoui e Grabisch (2016).

Table 1 presents the seven most commonly used distortions in the literature. Among the functions presented, only the function ψ^γ_{VaR} is not a concave distortion, all the others are concave and, consequently, are adequate to distort the expectation and represent the premium charged by an insurance company or the bid and ask prices of a given financial position.

The most common distortions in the literature are ψ^γ_{VaR} and ψ^γ_{CVaR}. The Wang distortion, ψ^γ_{WANG}, is also used frequently and was presented in S. S. Wang (2000). The other distortions will be discussed briefly below.

Table 1 – Distortion Functions

Risk Measures	Function
VaR	$\psi^{\gamma}_{VaR}(u) = 1_{\{u \geq \frac{1}{1+\gamma}\}}, \gamma \geq 0$
CVaR	$\psi^{\gamma}_{CVaR}(u) = \min\{u(1+\gamma), 1\}, \gamma \geq 0$
MINVAR	$\psi^{\gamma}_{MINVAR}(u) = 1 - (1-u)^{1+\gamma}, \gamma \geq 0$
MAXVAR	$\psi^{\gamma}_{MAXVAR}(u) = u^{\frac{1}{1+\gamma}}, \gamma \geq 0$
MAXMINVAR	$\psi^{\gamma}_{MAXMINVAR}(u) = (1-(1-u)^{\gamma+1})^{\frac{1}{1+\gamma}}, \gamma \geq 0$
MINMAXVAR	$\psi^{\gamma}_{MINMAXVAR}(u) = 1-(1-u^{\frac{1}{1+\gamma}})^{\gamma+1}, \gamma \geq 0$
WANG	$\psi^{\gamma}_{WANG}(u) = N(N^{-1}(u)+\gamma), \gamma \geq 0$

Source: Elaborated by the author.

The third distortion presented in Table 1,

$$\psi^{\gamma}_{MINVAR}(u) = 1 - (1-u)^{\gamma+1}, \quad \gamma \geq 0, \quad u \in [0,1]. \tag{60}$$

is associated with a risk measure called MINVAR. The MINVAR risk measure is defined as $\rho^{MINVAR}_{\gamma}(X) = -\mathbb{E}[Y]$, where

$$Y \overset{d}{=} \min\{X_1, \ldots, X_{\gamma+1}\}$$

and $X_1, \ldots, X_{\gamma+1}$ are independent draws of X. Then, the risk measure obtained by computing the distorted expectation using the concave distortion (60) is equal to the negative of the expected value of the minimum of the $\gamma + 1$ draws of X.

The concave distorted function below

$$\psi^{\gamma}_{MAXVAR}(u) = u^{\frac{1}{\gamma+1}}, \quad \gamma \geq 0, \quad u \in [0,1]. \tag{61}$$

is associated with a risk measure $\rho^{MAXVAR}_{\gamma}(X) = \mathbb{E}[Y]$, called MAXVAR, where Y is a random variable,

$$X \overset{d}{=} \max\{Y_1, \ldots, Y_{\gamma+1}\}$$

and $Y_1, \ldots, Y_{\gamma+1}$ are independent draws of Y. Then, the risk measure obtained by computing the distorted expectation using the distortion (61) is equal to the expected value of Y.

The risk measure $\rho^{MINMAXVAR}_{\gamma}(X) = -\mathbb{E}[Y]$, is called MINMAXVAR risk measure and is associate with a combination of two concave distortions, MINVAR and MAXVAR,

$$\psi^{\gamma}_{MINMAXVAR}(u) = (1-(1-u^{\frac{1}{1+\gamma}}))^{1+\gamma}, \tag{62}$$

where Y is a random variable that satisfies

$$Y \stackrel{d}{=} \min\{Z_1, \dots, Z_{\gamma+1}\} \text{ and } \max\{Z_1, \dots Z_{\gamma+1}\} \stackrel{d}{=} X, \tag{63}$$

and $Z_1, \dots, Z_{\gamma+1}$ are independent draws of Z.

The distortion below is the combination of MINVAR and MAXVAR distortions,

$$\psi^{\gamma}_{MAXMINVAR}(u) = (1 - (1-u)^{1+\gamma})^{\frac{1}{1+\gamma}} \tag{64}$$

The risk measure associate to this distortion, $\rho_{\gamma}^{MAXMINVAR}(X) = -\mathbb{E}[Y]$, is called MAXMINVAR, where Y is a random variable that satisfy the following property,

$$\max\{Y_1, \dots, Y_{\gamma+1}\} \stackrel{d}{=} \min\{X_1, \dots, X_{\gamma+1}\}, \tag{65}$$

i.e., the minimum of $\gamma + 1$ independent draws of X has the same distribution as the maximum $\gamma + 1$ independent draws of Y.

The example below makes clear the relationship between the Choquet integral and the risk measures.

Example 2.1. *Let X be a non-negative random variable, $X : \Omega \to \mathbb{R}_+$, and ψ_{γ}^{VaR} a distortion function as presented in Table 1. The distorted function is given by*

$$\psi^{\gamma}_{VaR}(F_X(x)) = \begin{cases} 0, & 0 \leq F_X(x) \leq 1-\alpha, \\ 1, & 1-\alpha \leq F_X(x) \leq 1. \end{cases}$$

Using the Choquet integral,

$$\begin{aligned} \int_0^{+\infty} [1 - \psi^{\gamma}_{VaR}(F_X(x))]dx &= \int_0^{F_X^{-1}(1-\alpha)} dx + \int_{F_X^{-1}(1-\alpha)}^{\infty} [1-1]dx \\ &= \int_0^{F_X^{-1}(1-\alpha)} dx = F_X^{-1}(1-\alpha) \end{aligned}$$

Then, it is clear that it is possible to represent the VaR risk measure using a Choquet integral with a distortion function, ψ_{γ}^{VaR}.

2.3.2 Conic finance

This subsection contains the main definitions and results for modeling the bid and ask price according to conic finance theory.

The first function to be introduced is the acceptability index. This function is chosen by the agent and is responsible for telling us which financial position will be traded in the market for a given level of liquidity.

Definition 2.13. *The function* $\alpha : L^\infty \to [0, \infty]$ *is an acceptability index. We say that a financial position* X *is* ***acceptable*** *at* $\gamma \geq 0$ *if*

$$\alpha(X) \geq \gamma. \tag{66}$$

The coefficient γ can be interpreted as the market liquidity level, Leippold e Schärer (2017). In this context, we will have acceptable positions for each liquidity level, where the higher γ the more illiquid the market is (in a complete market $\gamma = 0$). The acceptability index, α, must satisfy the following properties

- (Quase-concavity): $\alpha(X) \geq \gamma$ and $\alpha(Y) \geq \gamma$, then $\alpha(\lambda X + (1-\lambda)Y) \geq \gamma$ for $\lambda \in [0, 1]$.
- (Monotonicty): If $X \leq Y$. then $\alpha(X) \geq \alpha(Y)$.
- (Scale Invariance): If $\lambda \geq 0$, then $\alpha(\lambda X) = \alpha(X)$.
- (Fatou property): If a sequence of financial positions, $\{X_n\}$, such that $|X_n| \leq 1$, with X_n converging to X in probability and the position X_n is acceptable at the γ level, $\alpha(X_n) \geq \gamma$, then $\alpha(X) \geq x$.

Theorem 2.6. *The function* $\alpha : L^\infty \to [0, \infty]$ *is an* ***acceptability index*** *if and only if there exists a family of subsets* $\{\mathcal{D}_\gamma\}_{\gamma \in \mathbb{R}_+}$ *of* $\mathcal{P}$ *such that*

$$\alpha(X) = \sup \left\{ \gamma \in \mathbb{R}_+ : \inf_{\mathbb{Q} \in \mathcal{D}_\gamma} \mathbb{E}_\mathbb{Q}[X] \geq 0 \right\}, \tag{67}$$

where $\inf \emptyset = \infty$ *and* $\sup \emptyset = 0$, *and* $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma'}$ *for* $\gamma \leq \gamma'$.

There is a inherent relationship between acceptability indeces and risk measures. According to Cherny e Madan (2009), if α is an acceptability index, then it can be written as

$$\alpha(X) = \sup\{\gamma \in \mathbb{R}_+ : \rho_\gamma(X) \leq 0\}, \tag{68}$$

where $\rho_\gamma(X) = -\inf_{\mathbb{Q} \in \mathcal{D}_\gamma} \mathbb{E}_\mathbb{Q}[X]$ is a coherent risk measure and $\{\rho_\gamma\}_{\gamma \in \mathbb{R}_+}$ is a family of coherent risk measures that are increasing in γ with the property that $\alpha(X)$ is the largest level γ such that the financial position X is acceptable to the level γ,

$$\alpha(X) \geq \gamma \iff \mathbb{E}_\mathbb{Q}[X] \geq 0, \text{ for any } \mathbb{Q} \in \mathcal{D}_\gamma \tag{69}$$

and $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma'}$ for any $\gamma' > \gamma$.

Assuming that the market will trade only financial positions acceptable at a certain level of liquidity, γ, for a fixed acceptability index α and assuming further that the market is not complete, we will have a price range where trades will occur. The two most relevant prices in this range are the bid and ask prices.

The market will accept to sell the financial position X at the minimum price a, ask price. However, this residual cash flow $a - X$ must be α-acceptable at the γ level. For $a - X$ to be acceptable at the γ level the price a must exceed $\mathbb{E}_{\mathbb{Q}}[X], \ \forall\, \mathbb{Q} \in \mathcal{D}_\gamma$. Consequently, the minimum price will be given by

$$\begin{aligned}
a^\gamma(X) &= \inf\{a : \alpha(a - X) \geq \gamma\} \\
&= \inf\left\{a : \left(\inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[a - X]\right) \geq 0\right\} \\
&= \inf\left\{a : \left(a + \inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[-X]\right) \geq 0\right\} \\
&= \inf\left\{a : a \in \left[-\inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[-X], \infty\right)\right\} \\
&= -\inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[-X] = \rho^\gamma(-X)
\end{aligned}$$

where $\rho^\gamma(-X)$ is increasing in γ. On the other hand, the market will be willing to pay a maximum price b, bid price, for the financial position X. Where the cash flow $X - b$ must be α-acceptable at the γ level. This maximum price is given by

$$\begin{aligned}
b^\gamma(X) &= \sup\{b : \alpha(X - b) \geq \gamma\} \\
&= \sup\left\{b : \left(\inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[X - b]\right) \geq 0\right\} \\
&= \sup\left\{b : \left(\inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[X] - b\right) \geq 0\right\} \\
&= \sup\left\{b : b \in \left(0, \inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[X]\right]\right\} \\
&= \inf_{\mathbb{Q}\in\mathcal{D}_\gamma} \mathbb{E}_{\mathbb{Q}}[X] = -\rho^\gamma(X)
\end{aligned}$$

Therefore, if the price b at the γ level does not exceed $\mathbb{E}_{\mathbb{Q}}[X], \ \forall\, \mathbb{Q} \in \mathcal{D}_\gamma$, then $X - b$ will be acceptable. Note that, both prices can be represented by a family of coherent increasing measures in γ.

As suggested in Cherny e Madan (2009) and D. B. Madan e Cherny (2010), an acceptability index can be constructed from a family of concave distortions. Therefore, the acceptability index, Theorem (2.6), can be represented as

$$\alpha(X) = \left\{ \gamma \geq 0 : \int_{-\infty}^{\infty} x d\psi^{\gamma}(F_X(x)) \geq 0 \right\}. \tag{70}$$

For this new representation of the acceptability index, the following equivalence relation will hold

$$\alpha(X) \geq \gamma \iff \int_{-\infty}^{\infty} x d\psi^{\gamma}(F_X(x)) \geq 0.$$

Therefore, the bid and ask price for the framework presented by D. B. Madan e Cherny (2010) can be written as

$$\begin{aligned} a_{\psi}^{\gamma}(X) &= \inf\{a : \alpha(a - X) \geq \gamma\} \\ &= \inf\left\{ a : \int_{-\infty}^{\infty} x d\psi^{\gamma}(F_{a-X}(x)) \geq 0 \right\} \\ &= -\int_{-\infty}^{\infty} x d\psi^{\gamma}(F_{-X}(x)) \\ &= -\mathbb{E}_{\mathbb{Q}}^{\gamma}[-X], \end{aligned}$$

$$\begin{aligned} b_{\psi}^{\gamma}(X) &= \sup\{b : \alpha(X - b) \geq \gamma\} \\ &= \sup\left\{ b : \int_{-\infty}^{\infty} x d\psi^{\gamma}(F_{X-b}(x)) \geq 0 \right\} \\ &= \int_{-\infty}^{\infty} x d\psi^{\gamma}(F_X(x)) \\ &= \mathbb{E}_{\mathbb{Q}}^{\gamma}[X]. \end{aligned}$$

Note that, the bid and ask prices can be obtained through a Choquet expectation, just as is done in the insurance field.

A remark about the integral given in equation (70). If the density function of X is given by $f_X(x) = F'_X(x)$ and if ψ^{γ} is differentiable, then

$$\int_{-\infty}^{\infty} x d\psi^{\gamma}(F_X(x)) = \int_{-\infty}^{\infty} x \psi^{\gamma\prime}(F_X(x)) f_X(x) dx.$$

Note that in this case, the distorted expectaion can be interpreted as the expectation under a probability measure $\mathbb{Q}^{\gamma}$, whose density with respect to $\mathbb{P}$, obtained by the Radon-Nikodym Theorem, is given by $\psi^{\gamma\prime}(F_X(x))$. As such, the density of the financial position X distorts more and more to the left as γ increases.

3 Proposed Approach

The main objective of this dissertation is the introduction of a spread function that allows modeling bid and ask quotes. We introduce a new configuration that will take into account two acceptability indices, α_1 and α_2. Each index is represented by a family of concave distortions. If $\alpha_1 \neq \alpha_2$, the two families of concave distortions will be different. In this case, the following equivalence relations hold for $X \in L^\infty$,

$$\alpha_j(X) \geq \gamma \iff \int_{-\infty}^{\infty} x d\psi_j^\gamma(F_X(x)) \geq 0, \tag{71}$$

for a family $\psi_j^\gamma, \gamma \geq 0, j \in \{1, 2\}$.

In this new framework we have three sets of financial positions, $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}^*$. $\mathcal{A}_1$ is the convex cone of traded positions in the bid direction, i. e. $X \in \mathcal{A}_1$ if and only if $\alpha_1(X) \geq \gamma, X \in L^\infty$. $\mathcal{A}_2$ is the convex cone of positions traded in the ask direction, i.e. $X \in \mathcal{A}_2$ if and only if $\alpha_2(X) \geq \gamma, X \in L^\infty$. We will denote by $\mathcal{A}^*$ the set of financial positions that are traded in both directions, i.e. $\mathcal{A}^* = \mathcal{A}_1 \cap \mathcal{A}_2$. We know that the finite intersection of convex sets is again a convex set and the intersection of cones is again a cone. Therefore, $\mathcal{A}^*$ is a convex cone. Note that our convex cone contains the same financial positions that are in the convex cone of the framework presented by D. B. Madan e Cherny (2010), i.e. $\mathcal{A}^* = \mathcal{A}$.

Assuming that the market is incomplete and will trade only financial positions acceptable by the α_1 and α_2 indices at the γ level, the ask and bid prices are defined as follows.

Definition 3.1. *Let ψ_1 and ψ_2 be two concave distortion functions and γ the liquidity level for two fixed acceptability indexes, α_1 and α_2. Then the* ***bid and ask prices*** *of a financial position are given by:*

$$\begin{aligned}
b_{\psi_1}^\gamma(X) &:= \sup\{b \in \mathbb{R}_+ : \alpha_1(X - b) \geq \gamma\} \\
&= \sup\left\{b \in \mathbb{R}_+ : \int_{-\infty}^{\infty} x d\psi_1^\gamma(F_{X-b}(x)) \geq 0\right\} \\
&= \int_{-\infty}^{\infty} x d\psi_1^\gamma(F_X(x)) \\
&= \mathbb{E}_{\mathbb{Q}}^{\psi_1,\gamma}[X] \qquad (72) \\
&= -\int_{-\infty}^{0} \psi_1^\gamma(F_X(x))dx + \int_{0}^{\infty} [1 - \psi_1^\gamma(F_X(x))]dx \qquad (73)
\end{aligned}$$

$$\begin{aligned}
a^{\gamma}_{\psi_2}(X) &\coloneqq \inf\{a \in \mathbb{R}_+ : \alpha_2(a - X) \geq \gamma\} \\
&= \inf\left\{a \in \mathbb{R}_+ : \int_{-\infty}^{\infty} x d\psi_2^{\gamma}(F_{a-X}(x)) \geq 0\right\} && (74) \\
&= -\mathbb{E}_{\mathbb{Q}}^{\psi_2,\gamma}[-X]. \\
&= \int_{-\infty}^{0} [\psi_2^{\gamma}(F_{-X}(x))]dx + \int_{0}^{\infty} [\psi_2^{\gamma}(F_{-X}(x)) - 1]dx && (75)
\end{aligned}$$

Before we go on and introduce the main object of study of this dissertation, it is necessary to introduce the properties that bid and ask will satisfy.

Theorem 3.1. *Let $b^{\gamma}_{\psi_1}$ and $a^{\gamma}_{\psi_2}$ be defined in 3.1, a given γ, $m \in \mathbb{R}$ and $\lambda \geq 0$. Then the following properties will hold,*

1. *(Monotonicity) If $X \geq Y$, then $b^{\gamma}_{\psi_1}(X) \geq b^{\gamma}_{\psi_1}(Y)$ and $a^{\gamma}_{\psi_2}(X) \geq a^{\gamma}_{\psi_2}(Y)$,*
2. *(Translation Invariance) If $m \in \mathbb{R}$, then $b^{\gamma}_{\psi_1}(X + m) = b^{\gamma}_{\psi_1}(X) + m$ and $a^{\gamma}_{\psi_2}(X + m) = a^{\gamma}_{\psi_2}(X) + m$,*
3. *(Positive homogeneity) If $\lambda \geq 0$, then $b^{\gamma}_{\psi_1}(\lambda X) = \lambda b^{\gamma}_{\psi_1}(X)$ and $a^{\gamma}_{\psi_2}(\lambda X) = \lambda a^{\gamma}_{\psi_2}(X)$,*
4. *(Comonotonic additivity) If X and Y are comonotonic, i.e. $\big(X(\omega_0) - X(\omega_1)\big)\big(Y(\omega_0) - Y(\omega_1)\big) \geq 0, \forall\ \omega_0, \omega_1 \in \Omega$, then $b^{\gamma}_{\psi_1}(X + Y) = b^{\gamma}_{\psi_1}(X) + b^{\gamma}_{\psi_1}(Y)$ and $a^{\gamma}_{\psi_2}(X + Y) = a^{\gamma}_{\psi_2}(X) + a^{\gamma}_{\psi_2}(Y)$,*

In addition, the following property will hold only for the bid price

5. *(Superadditivity) $b^{\gamma}_{\psi_1}(X + Y) \geq b^{\gamma}_{\psi_1}(X) + b^{\gamma}_{\psi_1}(Y)$,*

and this next property will hild only to the ask price,

6. *(Subadditivity) $a^{\gamma}_{\psi_2}(X + Y) \leq a^{\gamma}_{\psi_2}(X) + a^{\gamma}_{\psi_2}(Y)$.*

Proof. The properties are easily checked if we rewrite the prices as Choquet integrals, i.e.

$$b^{\gamma}_{\psi_1}(X) = -\int_{\Omega} (-X) d\psi_1 \circ \mathbb{P} \quad \text{and} \quad a^{\gamma}_{\psi_2}(X) = \int_{\Omega} X d\psi_2 \circ \mathbb{P}$$

□

Once this new configuration has been presented, we can now introduce the main object of this work, the spread function. This function represents the difference between the ask and bid prices for a given financial position and is commonly presented in the literature. However, the spread function that will be presented next encompasses all the cases presented in the literature and allows us to add more cases.

Definition 3.2. *Let $a^{\gamma}_{\psi_2}$ and $b^{\gamma}_{\psi_1}$ be the ask and bid prices under the concave distortions ψ_2 and ψ_1, respectively. The* ***spread function****, $R^{\gamma}_{\psi_1,\psi_2} : L^{\infty} \to \mathbb{R}$, for a financial position X is given by*

$$R^{\gamma}_{\psi_1,\psi_2}(X) \coloneqq a^{\gamma}_{\psi_2}(X) - b^{\gamma}_{\psi_1}(X). \tag{76}$$

A simpler definition of the spread function, with $\psi_1 = \psi_2$, has been previously considered by Bannör e Scherer (2014), Chen et al. (2019) and Luo e Chen (2021). However, we can consider more cases, where ψ_1 and ψ_2 functions are different. These cases can be interpreted as a scenario where an agent is under a certain restriction to buy or sell the payoff, for example.

The Theorem below presents the main properties satisfied by the spread function.

Theorem 3.2. *Let ψ_1 and ψ_2 be concave distortions, a given γ, $m \in \mathbb{R}$ and $\lambda \geq 0$. The spread function, $R^{\gamma}_{\psi_1,\psi_2}$, satisfies the following properties, for the financial positions $X, Y \in L^{\infty}$:*

1. *(Positive homogeneity) $R^{\gamma}_{\psi_1,\psi_2}(\lambda X) = \lambda R^{\gamma}_{\psi_1,\psi_2}(X)$;*
2. *(Law invariance) If X and Y have the same distribution function, i.e. $X \overset{d}{=} Y$, then $R^{\gamma}_{\psi_1,\psi_2}(X) = R^{\gamma}_{\psi_1,\psi_2}(Y)$;*
3. *(Comonotonic additivity) If X and Y are comonotonic, then $R^{\gamma}_{\psi_1,\psi_2}(X+Y) = R^{\gamma}_{\psi_1,\psi_2}(X) + R^{\gamma}_{\psi_1,\psi_2}(Y)$;*
4. *(Location invariance) $R^{\gamma}_{\psi_1,\psi_2}(X + m) = R^{\gamma}_{\psi_1,\psi_2}(X)$;*
5. *(Subadditivity) $R^{\gamma}_{\psi_1,\psi_2}[X + Y] \leq R^{\gamma}_{\psi_1,\psi_2}[X] + R^{\gamma}_{\psi_1,\psi_2}[Y]$;*
6. *(Standardization) $R^{\gamma}_{\psi_1,\psi_2}(m) = 0$.*

Proof. 1. As the bid and ask prices satisfy the positive homogeneity property, we have that

$$\begin{aligned} R^{\gamma}_{\psi_1,\psi_2}(\lambda X) &= a^{\gamma}_{\psi_2}(\lambda X) - b^{\gamma}_{\psi_1}(\lambda X) \\ &= \lambda a^{\gamma}_{\psi_2}(X) - \lambda b^{\gamma}_{\psi_1}(X) \\ &= \lambda\Big(a^{\gamma}_{\psi_2}(X) - b^{\gamma}_{\psi_1}(X)\Big) \\ &= \lambda R^{\gamma}_{\psi_1,\psi_2}(X) \end{aligned}$$

2. The Location invariance property is obtained directly from the Cash invariance property of prices,

$$\begin{aligned} R^{\gamma}_{\psi_1,\psi_2}(X+m) &= a^{\gamma}_{\psi_2}(X+m) - b^{\gamma}_{\psi_1}(X+m) \\ &= a^{\gamma}_{\psi_2}(x) + m - (b^{\gamma}_{\psi_1}(X) + m) \\ &= a^{\gamma}_{\psi_2}(X) - b^{\gamma}_{\psi_1}(X) \\ &= R^{\gamma}_{\psi_1,\psi_2}(X) \end{aligned}$$

3. From the Law invariance of prices we obtain the Law invariance of the spread,

$$\begin{aligned} a^{\gamma}_{\psi_2}(X) &= a^{\gamma}_{\psi_2}(Y), \\ b^{\gamma}_{\psi_1}(X) &= b^{\gamma}_{\psi_1}(Y), \end{aligned}$$

if $X \overset{d}{=} Y$. Therefore,

$$R^{\gamma}_{\psi_1,\psi_2}(X) = a^{\gamma}_{\psi_2}(X) - b^{\gamma}_{\psi_1}(X) = a^{\gamma}_{\psi_2}(Y) - b^{\gamma}_{\psi_1}(Y) = R^{\gamma}_{\psi_1,\psi_2}(Y).$$

4. By additive comonoticity of the prices, we have the following equalities

$$\begin{aligned} a^{\gamma}_{\psi_2}(X+Y) &= a^{\gamma}_{\psi_2}(X) + a^{\gamma}_{\psi_2}(Y), \\ b^{\gamma}_{\psi_1}(X+Y) &= b^{\gamma}_{\psi_1}(X) + b^{\gamma}_{\psi_1}(Y). \end{aligned}$$

Then,

$$\begin{aligned} R^{\gamma}_{\psi_1,\psi_2}(X+Y) &= a^{\gamma}_{\psi_2}(X+Y) - b^{\gamma}_{\psi_1}(X+Y) \\ &= a^{\gamma}_{\psi_2}(X) + a^{\gamma}_{\psi_2}(Y) - b^{\gamma}_{\psi_1}(X) - b^{\gamma}_{\psi_1}(Y) \\ &= R^{\gamma}_{\psi_1,\psi_2}(X) + R^{\gamma}_{\psi_1,\psi_2}(Y). \end{aligned}$$

5. We have seen,

$$a^{\gamma}_{\psi_2}(X+Y) \leq a^{\gamma}_{\psi_2}(X) + a^{\gamma}_{\psi_2}(Y)$$
$$-b^{\gamma}_{\psi_1}(X+Y) \leq -b^{\gamma}_{\psi_1}(X) - b^{\gamma}_{\psi_1}(Y)$$

Then,

$$\begin{aligned} a^{\gamma}_{\psi_2}(X+Y) - b^{\gamma}_{\psi_1}(X+Y) &\leq a^{\gamma}_{\psi_2}(X) + a^{\gamma}_{\psi_2}(Y) - b^{\gamma}_{\psi_1}(X+Y) \\ &\leq a^{\gamma}_{\psi_2}(X) + a^{\gamma}_{\psi_2}(Y) - b^{\gamma}_{\psi_1}(X) - b^{\gamma}_{\psi_1}(Y) \end{aligned}$$

Therefore,

$$R^{\gamma}_{\psi_1,\psi_2}(X+Y) \leq R^{\gamma}_{\psi_1,\psi_2}(X) + R^{\gamma}_{\psi_1,\psi_2}(Y),$$

the spread function is subadditivity.

6. Note that for the Choquet integral, we have that

$$a^{\gamma}_{\psi_2}(m) = -\mathbb{E}_{\mathbb{Q}}[-m] = \int m d\psi_2 \circ \mathbb{P} = m$$
$$b^{\gamma}_{\psi_1}(m) = \mathbb{E}_{\mathbb{Q}}[m] = -\int (-m) d\psi_1 \circ \mathbb{P} = m.$$

Therefore,

$$R^{\gamma}_{\psi_1,\psi_2}(m) = a^{\gamma}_{\psi_2}(m) - b^{\gamma}_{\psi_1}(m) = 0.$$

□

Remark 2. The forward implication of items 1. and 5. of Theorem 3.2 is that the spread function satisfies the convexity property. By induction, we have that the spread of a portfolio of financial positions will be less than or equal to the sum of the spreads of all the financial positions in that portfolio. Equality between the spread of a portfolio and the sum of all spreads will occur when all financial positions have the same distribution function.

Theorem 3.3. *Let ψ_1 and ψ_2 be two concave distortions and a given $\gamma \geq 0$. Then, the bid and ask prices will satisfy the following properties:*

1. *$b^{\gamma}_{\psi_1}$ and $a^{\gamma}_{\psi_2}$ are Lipschitz continuous functionals on L^{∞},*

2. *If $\psi_1(u) = \psi_2(u),\ \forall u \in [0,1]$, then $b^{\gamma}_{\psi_1}(X) \leq a^{\gamma}_{\psi_2}(X)$ for all $X \in L^{\infty}$,*

3. *The ask price can be represented as*

$$a^{\gamma}_{\psi_2}(X) = \int_0^1 F_X^{-1}(1-p)d\psi_2^{\gamma}(p),$$

4. *The bid price can be represented as*

$$b^{\gamma}_{\psi_1}(X) = \int_0^1 F_X^{-1}(p)d\psi_1^{\gamma}(p).$$

Proof. 1. Note that, since the bid price is increasing function, we have

$$\begin{aligned} b^{\gamma}_{\psi_1}(Y) &\leq b^{\gamma}_{\psi_1}(X + \|X-Y\|_{\infty}) = b^{\gamma}_{\psi_1}(X) + \|X-Y\|_{\infty} \\ b^{\gamma}_{\psi_1}(Y) - b^{\gamma}_{\psi_1}(X) &\leq \|X-Y\|_{\infty} \\ |b^{\gamma}_{\psi_1}(Y) - b^{\gamma}_{\psi_1}(X)| &\leq \|X-Y\|_{\infty} \end{aligned}$$

Therefore, $b^{\gamma}_{\psi_1}$ is Lipschitz continuous with respect L^{∞}-norm. In a similiar way what was done for the bid price, since

$$a^{\gamma}_{\psi_2}(Y) \leq a^{\gamma}_{\psi_2}(X + \|X-Y\|_{\infty}),$$

we are able to prove that

$$|a^{\gamma}_{\psi_2}(X) - a^{\gamma}_{\psi_2}(Y)| \leq \|X-Y\|_{\infty}.$$

2. For this prove, rewrite the bid and ask functions as

$$a^{\psi_2,\gamma}(X) = \int_0^{\infty} \psi_2^{\gamma}(F_{-X}(-x))dx + \int_{-\infty}^0 [\psi_2^{\gamma}(F_{-X}(-x)) - 1]dx, \tag{77}$$

$$b^{\psi_1,\gamma}(X) = \int_{-\infty}^0 [-\psi_1^{\gamma}(F_X(x))]dx + \int_0^{\infty} [1 - \psi_1^{\gamma}(F_X(x))]dx. \tag{78}$$

From the concavity of the distortion, we have that $\psi_1^{\gamma}(u) + \psi_1^{\gamma}(1-u) \geq 1$. Then,

$$-\psi_1^{\gamma}(1-u) \leq \psi_1^{\gamma}(u) - 1, \tag{79}$$

$$1 - \psi_1^{\gamma}(1-u) \leq \psi_1^{\gamma}(u). \tag{80}$$

Using the equations (79) and (80), by Choquet monotonicity, we have that

$$\begin{aligned} \int_{-\infty}^0 [-\psi_1^{\gamma}(1-u)]dv &\leq \int_{-\infty}^0 [\psi_1^{\gamma}(u) - 1]dv, \\ \int_0^{\infty} [1 - \psi_1^{\gamma}(1-u)]dv &\leq \int_0^{\infty} \psi_1^{\gamma}(u)dv. \end{aligned}$$

Let $u = F_{-X}(-v)$, we obtain

$$\int_{-\infty}^{0}[-\psi_1^\gamma(1-F_{-X}(-v))]dv \leq \int_{-\infty}^{0}[\psi_1^\gamma(F_{-X}(-v))-1]dv,$$
$$\int_{0}^{\infty}[1-\psi_1^\gamma(1-F_{-X}(-v))]dv \leq \int_{0}^{\infty}\psi_1^\gamma(F_{-X}(-v))dv.$$

Therefore

$$\begin{aligned}\int_{-\infty}^{0}[-\psi_1^\gamma(1-F_{-X}(-v))]dv+\int_{0}^{\infty}[1-\psi_1^\gamma(1-F_{-X}(-v))]dv&\\ \leq \int_{-\infty}^{0}[\psi_1^\gamma(F_{-X}(-v))-1]dv+\int_{0}^{\infty}\psi_1^\gamma(F_{-X}(-v))dv&,\\ b_{\psi_1}^\gamma(X)\leq a_{\psi_2}^\gamma(X).&\end{aligned}$$

We conclude that, when distortions are equal, the ask function is always greater than or equal to the bid function, for a given γ and $X \in L^\infty$.

3. We can rewrite the ask price equation as

$$a_{\psi_2}^\gamma(X) = \int_{0}^{\infty}\psi_2^\gamma(\mathbb{P}(X\geq x))dx - \int_{-\infty}^{0}[1-\psi_2^\gamma(\mathbb{P}(X\geq x))]dx. \tag{81}$$

Note that, $\psi_2^\gamma(\mathbb{P}(X \geq x) = \int_0^{\mathbb{P}(X\geq x)} d\psi_2^\gamma(p)$ and $1-\psi_2^\gamma(\mathbb{P}(X\geq x) = \int_{\mathbb{P}(X\geq x)}^1 d\psi_2^\gamma(p)$. Applying Fubini's theorem and using the equivalence relation of the equation (1), the first integral of the ask price is transformed into

$$\begin{aligned}\int_0^\infty\int_0^{\mathbb{P}(X\geq x)} d\psi_2^\gamma(p)dx &= \int_0^{\mathbb{P}(X\geq 0)}\int_0^{F_X^{-1}(1-p)} dx d\psi_2^\gamma(p)\\ &= \int_0^{\mathbb{P}(X\geq 0)} F_X^{-1}(1-p)d\psi_2^\gamma(p).\end{aligned} \tag{82}$$

Similarly, we can rewrite the second integral of (81) as

$$\begin{aligned}\int_{-\infty}^0\int_{\mathbb{P}(X\geq x)}^1 d\psi_2^\gamma(p)dx &= \int_{\mathbb{P}(X\geq 0)}^1\int_{F_X^{-1}(1-p)}^0 dx d\psi_2^\gamma(p)\\ &= -\int_{\mathbb{P}(X\geq 0)}^1 F_X^{-1}(1-p)d\psi_2^\gamma(p).\end{aligned} \tag{83}$$

Replacing (82) and (83) in (81), we obtain

$$a_{\psi_2}^\gamma(X) = \int_0^1 F_X^{-1}(1-p)d\psi_2^\gamma(p).$$

4. We can express $\psi_1^\gamma(\mathbb{P}(X \leq x)$ and $1-\psi_1^\gamma(\mathbb{P}(X \leq x)$ as $\int_0^{\mathbb{P}(X\leq x)} d\psi_1^\gamma(p)$ and $\int_{\mathbb{P}(X\leq x)}^1 d\psi_1^\gamma(p)$, respectively. So, the first integral of the right-hand side of the equation (73), the bid price equation, can be rewritten using Fubini's theorem and the equivalence relation (1) as

$$\begin{aligned} -\int_{-\infty}^{0}\int_{0}^{\mathbb{P}(X\leq x)} d\psi_1^\gamma(p)dx &= -\int_{0}^{\mathbb{P}(X\leq 0)}\int_{F_X^{-1}(p)}^{0} dxd\psi_1^\gamma(p) \\ &= \int_{0}^{\mathbb{P}(X\leq 0)} F_X^{-1}(p)d\psi_1^\gamma(p). \end{aligned} \tag{84}$$

Similarly, for the second equation on the right-hand side of (73),

$$\begin{aligned} \int_{0}^{\infty}\int_{\mathbb{P}(X\leq x)}^{1} d\psi_1^\gamma(p)dx &= \int_{\mathbb{P}(X\leq 0)}^{1}\int_{0}^{F_X^{-1}(p)} dxd\psi_1^\gamma(p) \\ &= \int_{\mathbb{P}(X\leq 0)}^{1} F_X^{-1}(p)d\psi_1^\gamma(p). \end{aligned} \tag{85}$$

Inserting (84) and (85) in (73), we conclude our demonstration.

□

For the special case where both distortions are equal, we will have that the bid price of a financial position will always be greater than or equal to the ask price, which is empirically adequate. In items 3. and 4. a form of representation for bid and ask prices using quantiles is exposed. The above representations can be understood as a particular case of the representations presented in R. Wang e Wei (2020), Lemma 3. In that work, the authors represented using quantiles a larger class of Choquet Integrals, as signed Choquet Integrals.

Theorem 3.4. *Let ψ_1 and ψ_2 be concave distortions and a given γ. The spread function, $R^\gamma_{\psi_1,\psi_2}$, satisfies the following properties, for the financial positions $X, Y \in L^\infty$:*

1. *$R^\gamma_{\psi_1,\psi_2}$ is a Lipschitz continuous functionals on L^∞,*
2. *If $\psi_1(u) = \psi_2(u), \forall\ u \in [0,1]$, then the spread function is non-negative, $R^\gamma_{\psi_1,\psi_2}(X) \geq 0$,*
3. *If $\psi_1(u) = \psi_2(u), \forall\ u \in [0,1]$, then the map $\gamma \to R^\gamma_{\psi_1,\psi_2}$ is increasing,*
4. *If $\gamma = 0$, $R^\gamma_{\psi_1,\psi_2}(X) = 0$, for all $X \in L^\infty$,*
5. *If $\gamma \to \infty$, then $R^\gamma_{\psi_1,\psi_2}(X) = \mathrm{range}(X)$, i.e. $\lim_{\gamma\to\infty} R^\gamma_{\psi_1,\psi_2}(X) = \mathrm{range}(X)$.*

Proof. 1. To prove that the spread function will be continuous we will use the inequalities obtained above. First, note that the difference of the spread function for two financial positions X and Y in the co-domain metric is given by

$$|R^{\gamma}_{\psi_1,\psi_2}(X) - R^{\gamma}_{\psi_1,\psi_2}(Y)| = |a^{\gamma}_{\psi_2}(X) - b^{\gamma}_{\psi_1}(X) - a^{\gamma}_{\psi_2}(Y) + b^{\gamma}_{\psi_1}(Y)|.$$

By the triangular inequality we have that

$$|R^{\gamma}_{\psi_1,\psi_2}(X) - R^{\gamma}_{\psi_1,\psi_2}(Y)| \leq |a^{\gamma}_{\psi_2}(X) - a^{\gamma}_{\psi_2}(Y)| + |b^{\gamma}_{\psi_1}(X) - b^{\gamma}_{\psi_1}(Y)|.$$

Therefore, from the inequalities above

$$|R^{\gamma}_{\psi_1,\psi_2}(X) - R^{\gamma}_{\psi_1,\psi_2}(Y)| \leq 2||X - Y||_{\infty},$$

where the Lipschitz constant is 2.

2. This property follows directly from item 2 of Theorem 3.3.

3. By definition, we have that

$$\psi_1^{\gamma_1}(u) \leq \psi_1^{\gamma_2}(u) \leq \psi_1^{\gamma_3}(u) \leq \dots, \tag{86}$$

for a given u and $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \dots$. Writing the negative of the equation (86), in terms of $1-u$, we will get

$$-\psi_1^{\gamma_1}(1-u) \geq -\psi_1^{\gamma_2}(1-u) \geq -\psi_1^{\gamma_3}(1-u) \geq \dots. \tag{87}$$

Using the inequalities presented in (79) and (80), we can easily verify that

$$\begin{aligned}\dots \leq -\psi_1^{\gamma_3}(1-u) \leq -\psi_1^{\gamma_2}(1-u) &\leq -\psi_1^{\gamma_1}(1-u) \qquad (88)\\ &\leq \psi_1^{\gamma_1}(u) - 1 \leq \psi_1^{\gamma_2}(u) - 1 \leq \psi_1^{\gamma_3}(u) - 1 \leq \dots,\\ \dots \leq 1-\psi_1^{\gamma_3}(1-u) \leq 1-\psi_1^{\gamma_2}(1-u) &\leq 1-\psi_1^{\gamma_1}(1-u) \qquad (89)\\ &\leq \psi_1^{\gamma_1}(u) \leq \psi_1^{\gamma_2}(u) \leq \psi_1^{\gamma_3}(u) \leq \dots.\end{aligned}$$

We have seen that

$$-\psi_1^{\gamma_1}(1-u) + 1 - \psi_1^{\gamma_1}(1-u) \leq \psi_1^{\gamma_1}(u) - 1 + \psi_1^{\gamma_1}(u), \tag{90}$$

where if we integrate both sides of the equation (90) and replace u by $F_X(-v)$, we obtain the result of item 2. of Theorem 3.3, i.e. $b^{\gamma}_{\psi_1}(X) \leq a^{\gamma}_{\psi_2}(X)$. Then, repeating the procedure presented in equation (90) for all terms of equations (88) and (89), we have that

$$\cdots \leq -\psi_1^{\gamma_2}(1-u) + 1 - \psi_1^{\gamma_2}(1-u) \leq -\psi_1^{\gamma_1}(1-u) + 1 - \psi_1^{\gamma_1}(1-u)$$
$$\psi_1^{\gamma_1}(u) - 1 + \psi_1^{\gamma_1}(u) \leq \psi_1^{\gamma_2}(u) - 1 + \psi_1^{\gamma_2}(u) \leq \psi_1^{\gamma_3}(u) - 1 + \psi_1^{\gamma_3}(u) \leq \dots$$

Integrating and using $u = F_{-X}(-x)$, we have that

$$-b^{\gamma_1}_{\psi_1} \leq -b^{\gamma_2}_{\psi_1} \leq -b^{\gamma_3}_{\psi_1} \leq \dots$$
$$a^{\gamma_1}_{\psi_1} \leq a^{\gamma_2}_{\psi_1} \leq a^{\gamma_3}_{\psi_1} \leq \dots .$$

We know that the sum of two increasing functions is another increasing function. Therefore, we obtain that $R^{\gamma}_{\psi_1,\psi_2}$ is an increasing function

4. In the case where $\gamma = 0$, we know by Definition 2.11, that the function ψ_1 will not distort the probabilities, i.e. $\psi_1^0(u) = u$ for $u \in [0,1]$. So, the bid and ask prices can be writen as

$$a^0_{\gamma_2}(X) = -\Big\{ -\int_{-\infty}^{0} F_{-X}(x)dx + \int_0^{\infty} [1 - F_{-X}(x)]dx \Big\},$$
$$b^0_{\gamma_1}(X) = -\int_{-\infty}^{0} F_X(x)dx + \int_0^{\infty} [1 - F_X(x)]dx.$$

Therefore, rewriting the ask as in equation (77), is easy to verify that

$$R^0_{\psi_1,\psi_2} = a^0_{\gamma_2}(X) - b^0_{\gamma_1}(X) = 0,$$

i.e. for the market with perfect liquidity, $\gamma = \infty$, we will have that the bid is equal to the ask.

5. Note that, we have $\{\psi^{\gamma}\}_{\gamma \geq 0}$, where

$$\psi^{\gamma}(u) = \begin{cases} 1, \quad u = 1 \\ \psi^{\gamma}(u) \in (0,1), \quad u \in (0,1) \\ 0, \quad u = 0 \end{cases}$$

If $\gamma \to \infty$, by Definition 2.11,

$$\lim_{\gamma \to \infty} \psi^{\gamma}(u) = f(u) = \begin{cases} 1, \quad u \in (0,1] \\ 0, \quad u = 0 \end{cases}$$

i.e., $\forall\, \varepsilon > 0, \forall\, u \in [0,1], \exists\, \gamma \in \mathbb{R}$, such that,

$$\gamma > \gamma^* \implies |\psi^\gamma(u) - f(u)| < \varepsilon.$$

Basically, from the definition, if $\gamma > \gamma^*$, $\psi^\gamma(u)$ is inside a open ball with radius ε and center $f(u)$,

$$\psi^\gamma(u) \in (f(u) - \varepsilon, f(u) + \varepsilon),$$

So, ψ^γ converges pointwise to f. It is clear that the sequence $\{\psi^\gamma\}$ is dominated by $k \geq 1$, in the sense that

$$|\psi^\gamma(u)| \leq k, \quad u \in [0,1].$$

Since ψ^γ converges pointwise to f and ψ^γ is bounded, then the dominated convergence theorem holds. For the case where $\psi^\gamma(u) = \psi^\gamma(F_X(x)), x \in \mathbb{R}$, and $f(u) = \mathbf{1}_{\{F_X(x) \in (0,1]\}}(x)$, by dominated converge theorem, we have

$$\lim_{\gamma \to \infty} \int_{-\infty}^{\infty} \psi^\gamma(F_X(x))dx = \int_{-\infty}^{\infty} \lim_{\gamma \to \infty} \psi^\gamma(F_X(x))dx = \int_{-\infty}^{\infty} \mathbf{1}_{\{F_X(x) \in (0,1]\}}(x)dx.$$

For the bid price, defined in equation (73), the limit of $b^\gamma_{\psi_1}$ as γ goes to ∞ is given by

$$\begin{aligned}
\lim_{\gamma \to \infty} b^\gamma_{\psi_1}(X) &= \lim_{\gamma \to \infty} \left\{ - \int_{-\infty}^{0} \psi_1^\gamma(F_X(x))dx + \int_0^\infty [1 - \psi_1^\gamma(F_X(x))]dx \right\} \\
&= \lim_{\gamma \to \infty} \left\{ - \int_{-\infty}^{0} \psi_1^\gamma(F_X(x))dx \right\} + \lim_{\gamma \to \infty} \left\{ \int_0^\infty [1 - \psi_1^\gamma(F_X(x))]dx \right\} \\
&= - \int_{-\infty}^{0} \lim_{\gamma \to \infty} \psi_1^\gamma(F_X(x))dx + \int_0^\infty \lim_{\gamma \to \infty} [1 - \psi_1^\gamma(F_X(x))]dx \\
&= - \int_{-\infty}^{0} \mathbf{1}_{\{F_X(x) \in (0,1]\}}(x)dx + \int_0^\infty [1 - \mathbf{1}_{\{F_X(x) \in (0,1]\}}(x)]dx
\end{aligned}$$

If $0 < F_X(x) = \mathbb{P}(X \leq x) \leq 1$, then $\mathbf{1}_{\{F_X(x) \in (0,1]\}}(x) = 1$. On the other hand, if $0 = F_X(x) = \mathbb{P}(X \leq x)$, then $\mathbf{1}_{\{F_X(x) \in (0,1]\}}(x) = 0$. Therefore, using the relation presented in (1), the bid price when γ goes to infinity is

$$- \int_{F_X^{-1}(0)}^{0} dx + \int_0^\infty [1 - \mathbf{1}_{\{F_X(x) \in (0,1]\}}(x)]dx = F_X^{-1}(0), \tag{91}$$

where $F_X^{-1}(0) = \operatorname{ess\,inf}(X)$.

The ask price when $\gamma \to \infty$ is obtained in an analogous way. From the dominated convergence theorem, we obtain,

$$\begin{aligned}
\lim_{\gamma\to\infty} a^{\gamma}_{\psi_2}(X) &= \lim_{\gamma\to\infty}\left\{\int_{-\infty}^{0}\psi_2^{\gamma}(F_{-X}(x))dx + \int_{0}^{\infty}[\psi_2^{\gamma}(F_{-X}(x)) - 1]dx\right\} \\
&= \lim_{\gamma\to\infty}\left\{\int_{-\infty}^{0}\psi_2^{\gamma}(F_{-X}(x))dx\right\} + \lim_{\gamma\to\infty}\left\{\int_{0}^{\infty}[\psi_2^{\gamma}(F_{-X}(x)) - 1]dx\right\} \\
&= \int_{-\infty}^{0}\lim_{\gamma\to\infty}\psi_2^{\gamma}(F_{-X}(x))dx + \int_{0}^{\infty}\lim_{\gamma\to\infty}[\psi_2^{\gamma}(F_{-X}(x)) - 1]dx \\
&= \int_{-\infty}^{0}\mathbf{1}_{\{F_{-X}(x)\in(0,1]\}}(x)dx + \int_{0}^{\infty}[\mathbf{1}_{\{F_{-X}(x)\in(0,1]\}}(x) - 1]dx \\
&= \int_{F_{-X}^{-1}(0)}^{0}dx + \int_{0}^{\infty}[\mathbf{1}_{\{F_{-X}(x)\in(0,1]\}}(x) - 1]dx = -F_{-X}^{-1}(0)
\end{aligned}$$

where $-F_{-X}^{-1}(0) = -\operatorname{ess\,inf}(-X) = \operatorname{ess\,sup}(X)$. Consequently, in this scenario of extreme illiquidity, the spread function can be written as

$$\begin{aligned}
\lim_{\gamma\to\infty} R^{\gamma}_{\psi_1,\psi_2}(X) &= \lim_{\gamma\to\infty} a^{\gamma}_{\psi_2}(X) - \lim_{\gamma\to\infty} b^{\gamma}_{\psi_1}(X) \\
&= \operatorname{ess\,sup}(X) - \operatorname{ess\,inf}(X) \\
&= \operatorname{range}(X)
\end{aligned}$$

□

In items 2 and 3, we are in a special case, the most common in the literature, and these items tell us that the spread function will be non-negative for any financial position and that the higher the γ, the greater will be the difference between the bid and ask prices, i.e. the more liquid the market, the smaller the spread will be. Items 4. and 5. present the two extreme cases, when $\gamma = 0$ and $\gamma \to \infty$. For the case where $\gamma = 0$, perfectly liquid market, the bid and ask prices are equal, i.e. the law of one price holds. For $\gamma \to \infty$, we have that the spread function for the financial position X will be equal to $\operatorname{range}(X)$, regardless of the distortion. Basically, in the limit, we have that the bid price is such that the probability of the financial position receiving values less than $F_X^{-1}(0)$ is zero and the ask price is such that the probability of the financial position receiving values greater than $-F_{-X}^{-1}(0)$ is zero. Thus, the $\operatorname{range}(X)$ can be interpreted as the greatest possible difference between these prices that are tied with a non-zero probability.

Remark 3. Items 2, 3 and 4 in Theorem 3.4 mirror the properties found in the numerical examples of Chen et al. (2019) and Luo e Chen (2021). The authors performed such examples for exotic derivatives and found that: when $\gamma = 0$, the bid and ask prices are equivalent; γ has an increasing relation with their spread function, wich is always non-negative, as the authors consider a special case where Wang distortion distorts the accumulations of both prices.

Remark 4. The direct implication of item 4. is that the spread function $R^{\gamma}_{\psi_1,\psi_2} : L^{\infty} \to [0, \infty]$ will be a deviation measure, provided $\gamma > 0$ and as properties of positive homogeneity, location invariance, and subadditivity are satisfied.

Example 3.1. *D. Madan e Schoutens (2016) present a closed expression for the price of an option, under the Black-Scholes hypothesis, using the distortion ψ^{γ}_{WANG} of Table 1. In the Black-Scholes model, we have that $\ln S_t$ is normally distributed with mean $\ln S_0 + (r - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$, this is equivalent to*

$$S_t \sim LN\big(\ln S_0 + (r - \frac{1}{2}\sigma^2)t, \sigma^2 t\big).$$

The Wang distortion of the cumulative price is given by

$$\psi^{WANG}_{\gamma}(F_{S_t}(x)) = N\bigg(\frac{\ln x - (\ln S_0 - (r - \frac{1}{2}\sigma^2)t + \gamma\sigma\sqrt{t})}{\sigma\sqrt{t}}\bigg).$$

The bid price presented by the authors is given by

$$b^{\gamma}_{\psi_{WANG}}(c_t) = S_t e^{-\gamma\sigma\sqrt{T-t}} N(d_1) - K e^{-r(T-t)} N(d_2), \tag{92}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\big(S_t/K\big) + \big(r + \sigma^2/2\big)(T-t) - \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

The closed formula for the ask price using Wang distortion is similar to the expression found for the bid,

$$a^{\gamma}_{\psi_{WANG}}(c_t) = S_t e^{\gamma\sigma\sqrt{T-t}} N(d_1) - K e^{-r(T-t)} N(d_2), \tag{93}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\big(S_t/K\big) + \big(r + \sigma^2/2\big)(T-t) + \gamma\sigma\sqrt{T-t}}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

However, it is possible to go a little further and find the Greeks for this two-price economy with the Wang distortion. Below are the main Greeks for the bid price.

1. (Delta)

$$\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial S_t} = e^{-\gamma\sigma\sqrt{T-t}} N(d_1),$$

2. (Gamma)

$$\frac{\partial}{\partial S}\left(\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial S_t}\right) = \frac{1}{\sigma S\sqrt{T-t}} e^{-\gamma\sigma\sqrt{T-t}} N'(d_1),$$

3. (Vega)

$$\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial \sigma} = -\gamma\sqrt{T-t} S_t e^{-\gamma\sigma\sqrt{T-t}} N(d_1) + S_t e^{-\gamma\sigma\sqrt{T-t}} N'(d_1)\sqrt{T-t},$$

4. (Theta)

$$\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial t} = \frac{\gamma\sigma}{\sqrt{T-t}} S_t e^{-\gamma\sigma\sqrt{T-t}} N(d_1) - rKe^{-r(T-t)}N(d_2) - \frac{1}{2}\frac{\sigma}{\sqrt{T-t}} S_t e^{-\gamma\sigma\sqrt{T-t}} N'(d_1),$$

5. (Rho)

$$\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial r} = (T-t)Ke^{-r(T-t)}N(d_2).$$

The partial derivatives for the ask price are given by

6. (Delta)

$$\frac{\partial a^{\gamma}_{\psi^{WANG}}(c_t)}{\partial S_t} = e^{\gamma\sigma\sqrt{T-t}} N(d_1),$$

7. (Gamma)

$$\frac{\partial}{\partial S}\left(\frac{\partial a^{\gamma}_{\psi^{WANG}}(c_t)}{\partial S_t}\right) = \frac{1}{\sigma S\sqrt{T-t}} e^{\gamma\sigma\sqrt{T-t}} N'(d_1),$$

8. (Vega)

$$\frac{\partial a^{\gamma}_{\psi^{WANG}}(c_t)}{\partial \sigma} = \gamma\sqrt{T-t} S_t e^{\gamma\sigma\sqrt{T-t}} N(d_1) + S_t e^{\gamma\sigma\sqrt{T-t}} N'(d_1)\sqrt{T-t},$$

9. (Theta)

$$\frac{\partial a^{\gamma}_{\psi^{WANG}}(c_t)}{\partial t} = -\frac{\gamma\sigma}{\sqrt{T-t}} S_t e^{\gamma\sigma\sqrt{T-t}} N(d_1) - rKe^{-r(T-t)}N(d_2) - \frac{1}{2}\frac{\sigma}{\sqrt{T-t}} S_t e^{\gamma\sigma\sqrt{T-t}} N'(d_1),$$

10. (Rho)

$$\frac{\partial a^{\gamma}_{\psi^{WANG}}(c_t)}{\partial r} = (T-t)Ke^{-r(T-t)}N(d_2).$$

In addition to the traditional Greeks of the Black-Scholes model, we have a new Greek, the partial derivative with respect to the liquidity parameter γ. This new partial derivative for the bid price is given by,

11.
$$\frac{\partial b^{\gamma}_{\psi^{WANG}}(c_t)}{\partial \gamma} = -\gamma S_t e^{-\gamma\sigma\sqrt{T-t}} N(d_1).$$

The partial derivative of the ask price with respect to $gamma$ is given by

12.
$$\frac{\partial a^{\gamma}_{\psi^{WANG}}(c_t)}{\partial \gamma} = \gamma S_t e^{\gamma\sigma\sqrt{T-t}} N(d_1).$$

Remark 5. In Example 3.1, for the special case where $\gamma = 0$, we have that the bid and ask prices are equal, which was already expected by Theorem 3.4, and the Greeks are the same as in the Black-Scholes model.

Remark 6. In the Black-Scholes model, the price behavior is described by a GBM, equation (16). However, a change of measure is made, where the price dynamics under the new meaure, the risk neutral measure, is the following

$$d\ln S_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t^{\mathbb{Q}}, \tag{94}$$

From this dynamic we obtain the cumulative price distribution function and find the option prices for the Black-Scholes model, equation (27). In the case where the cumulative is distorted by ψ^{WANG}, as in the example above, we can describe the price dynamics as

$$d\ln S_t = \left(r - \frac{\sigma^2}{2} - \frac{\gamma\sigma}{2t^{1/2}}\right)dt + \sigma dW_t^{\mathbb{Q}^\gamma} \tag{95}$$

where $\mathbb{Q}^\gamma$ is interpreted as a probability measure whose dynamics of the Brownian Motion under this new measure is given by

$$dW_t^{\mathbb{Q}^\gamma} = \left(\frac{\mu - r}{\sigma} + \frac{\gamma}{2t^{1/2}}\right)dt + dW_t. \tag{96}$$

Basically, we are saying that we can interpret the bid and ask prices, presented in (92) and (93) as a change of measure from $\mathbb{P}$ to $\mathbb{Q}^\gamma$. Therefore, if it is possible to find a closed form for the cumulative price distortion, we can find a relation, similar to equation (96), between a supposed measure $\mathbb{Q}^\gamma$ and the physical measure $\mathbb{P}$.

4 Final remarks

This work aims to contribute to the evolution of the conic finance framework. Our goal is too ambitious and has not been fully achieved. However, important contributions have been made and the remaining research has been identified and described.

The steps necessary to achieve our goal are as follows:

1. To derive a new framework that enables the implementation of the spread function,
2. To derive the properties for bid and ask prices,
3. To derive the properties for spread function,
4. To derive the properties of map $\gamma \to R^{\gamma}_{\psi_1,\psi_2}(X)$,
5. To derive the properties of map $\psi_1, \psi_2 \to R^{\gamma}_{\psi_1,\psi_2}(X)$,
6. To characterize the acceptability indices,
7. To derive the Greeks for each of the convex distortions shown in Table 1,
8. To derive the Greeks for different dynamics for the assets,
9. To perform a comprehensive numerical example to identify the behavior of the Greeks.

Through the theorems presented in Section 3, we were able to achieve the first 4 objectives almost completely. By Example 3.1, we were able to demonstrate the Greeks for one of the distortions presented in Table 1.

A Appendix

In this appendix, we will introduce the definition of concepts that will be used throughout the text and some important results related to option pricing.

A.1 Wiener Process

Definition A.1. *A **stochastic process** $\{X_t\}_{t\in\mathcal{T}}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a set of random variables, $X_t : \Omega \to \mathbb{R}$.*

If the set of index $\mathcal{T}$ is countable, we say that the stochastic process is discrete and if the set $\mathcal{T}$ is continuous, the process is continuous. In this appendix, $\mathcal{T} = [0, 1]$ and we will use $\mathcal{T}$ or $[0, 1]$ interchangeably.

Definition A.2. *Let $\{X_t\}_{t\in\mathcal{T}}$ and $\mathcal{F}$ be a stochastic process and a sigma-algebra, respectively. We have the following definitions*

1. *A function $f : \Omega \to \mathbb{R}$ is said to be **$\mathcal{F}$-measurable** if, $\forall\ I \subseteq \mathbb{R}$ we have that $f^{-1}(I) \in \mathcal{F}$.*
2. *A **sigma-algebra generated by X** over the interval $[0, t]$ is defined as*

$$\mathcal{F}_t^X = \sigma\{X(s) : s \in [0, t]\}.$$

3. *A **filtration** $\{\mathcal{F}_t\}_{t\geq 0}$, is an indexed family of sigma-algebras in Ω such that*

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T,\ \forall\ s, t \text{ with } s < t < T.$$

4. *$\{X_t\}_{t\in T}$ is said to be an **adapted process** with respect to filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if*

$$X_t \in \mathcal{F}_t,\ \forall t \geq 0.$$

We interpret 1. and 2. in Definition A.2 as: if a function f is $\mathcal{F}$-measurable then, we can measure the probability that f belongs to some subset I and we can interpret $\mathcal{F}_t^X$ as the information generated by observing the process X over the interval $[0, t]$. The interpretation of 3 and 4 is as follows: The amount of available information increases, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_T$ with each period of time, $s < t < T$ and if the process is adapted, then we can say that the value of the variable X_t is determined by the information we have access to at t, $\mathcal{F}_t$.

Definition A.3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* X *be a random variable in* $L^1(\Omega, \mathcal{F}, \mathbb{P})$*, and let* $\mathcal{G}$ *be a sigma-algebra such that* $\mathcal{G} \subset \mathcal{F}$*. The* ***conditional expectation of*** $\boldsymbol{X}$ ***given the sigma algebra*** $\boldsymbol{\mathcal{G}}$*,* $\mathbb{E}[X|\mathcal{G}]$*, is a random variable that satisfies*

1. $\mathbb{E}[X|\mathcal{G}]$ *is* $\mathcal{G}$*-measurable;*
2. *For every* $G \in \mathcal{G}$ *it holds that*

$$\int_G \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_G X d\mathbb{P}.$$

A very important type of process when we talk about asset pricing is the martingale process.

Definition A.4. *A stochastic process* X *in a filtered probability space is a* ***martingale process*** *if satisfies*

1. $\mathbb{E}[|X_t|] < \infty, \forall t \in [0, T]$*,*
2. X *is an adapted process,*
3. $X_s = \mathbb{E}[X_t|\mathcal{F}_s]$*, for* $0 \leq s \leq t \leq T$*.*

This process is directly related to the Risk Neutral measures presented in the text. The most classic example of a martingale process is the Wiener process defined below.

Definition A.5. *A stochastic process* W *is called a* ***Wiener process*** *if it satisfies the following conditions:*

1. W_0 *= 0;*
2. W_t *has independent increments, i.e.*

$$W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_k} - W_{t_{k-1}} \tag{97}$$

   are independent for all $0 \leq t_1 < t_2 < \cdots < t_k$*. From this it can be deduced that* $W_u - W_t$ *is independent of* $\mathcal{F}_t$ *if* $u > t$*.*

3. $W_t - W_s$ *is normally distributed for* $s < t$ *and given by* $W_t - W_s = \varepsilon\sqrt{t-s}$*, where* $\varepsilon \sim N(0, 1)$*;*

4. W_t *has continuous trajectories.*

Proposition A.1. *A Wiener process W is a martingale process.*

Proof.

$$\begin{aligned}
\mathbb{E}[W_t|\mathcal{F}_s] &= \mathbb{E}[W_t - W_s + W_s|\mathcal{F}_s] \\
&= \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] \\
&= \mathbb{E}[\varepsilon\sqrt{t-s}|\mathcal{F}_s] + W_s \\
&= \sqrt{t-s}\mathbb{E}[\varepsilon] + W_s \\
&= W_s
\end{aligned} \tag{98}$$

□

A.2 Itô integral and process

Definition A.6. *A process* f*,*

$$f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$$

belongs to the class $\mathcal{V} = \mathcal{V}(S,T)$ *if the following conditions are satisfied:*

1. *The process* f *is adapted to the* $\mathcal{F}_t^W$*-filtration.*

2. $\mathbb{E}[\int_S^T f_t^2 dt] < \infty$.

We will show how to define the following integral

$$\int_S^T f_t dW_t,$$

known as the Itô Integral, where $f \in \mathcal{V}$ and W_t is a Wiener process.

The steps presented here to define (99) are the same steps presented in Øksendal (2003) and Focardi, Fabozzi et al. (2004). First we define (99) for a simple process. Then we will show that if $f \in \mathcal{V}$, then it can be approximated by elementary functions.

Definition A.7. *A stochastic process* $\phi \in \mathcal{V}$ *is called an* ***elementary process*** *if it has the following form*

$$\phi_t = \sum_j e_j.1_{[t_j,t_{j+1})}(t)$$

where

$$t_j = t_j^{(n)} = \begin{cases} j.2^{-n} & \textit{if } S \leq j.2^{-n} \leq T \\ S & \textit{if } j.2^{-n} < S \\ T & \textit{if } j.2^{-n} > T \end{cases}$$

and $\{t_j\}_{j\geq 0}$ *is a strictly monotone sequence in* $[0,\infty)$ *and* $\{e_j\}_{j\geq 0}$ *is a* $\mathcal{F}_{t_j}$*-measurable sequence of random variables, since* $\phi \in \mathcal{V}$.

For an elementary ϕ process, the stochastic integral is defined as follows,

$$\int_S^T \phi_t dW_t = \sum_{j\geq 0} e_j.[W_{t_{j+1}} - W_{t_j}]. \tag{99}$$

Note that (99) is a random variable, the case where Itô's integral is interpreted as a stochastic process will be presented later in this section. However, before we continue, we will present a result that will be very useful for the further construction of Itô's integral.

Proposition A.2. *If* f *is a process that satisfies conditions 1. and 2. given in Definition A.6, then*

$$\mathbb{E}\bigg[\Big(\int_S^T \phi_t dW_t\Big)^2\bigg] = \mathbb{E}\bigg[\int_S^T \phi_t^2 dt\bigg].$$

Proof.

$$\begin{aligned} \mathbb{E}\bigg[\Big(\int_S^T \phi_t dW_t\Big)^2\bigg] &= \mathbb{E}\bigg[\Big(\sum_{j\geq 0} e_j.[W_{t_{j+1}} - W_{t_j}]\Big)^2\bigg] \\ &= \mathbb{E}\bigg[\Big(\sum_{j\geq 0} e_j.[W_{t_{j+1}} - W_{t_j}]\Big)\Big(\sum_{i\geq 0} e_i.[W_{t_{i+1}} - W_{t_i}]\Big)\bigg] \end{aligned} \tag{100}$$

Note that the right-hand side of the equation (100) will be represented by a summation of expressions of type $\mathbb{E}\bigg[e_ie_j[W_{t_{i+1}} - W_{t_i}][W_{t_{j+1}} - W_{t_j}]\bigg]$. We need to compute the result of this expression for when $i \neq j$ and $i = j$.

If $i < j$, we will have that $e_i, e_j \in \mathcal{F}_{t_j}$, $[W_{t_{i+1}} - W_{t_i}] \in \mathcal{F}_{t_i} \subseteq \mathcal{F}_{t_j}$ and by item 2. of Definition A.5 we have that $[W_{t_{i+1}} - W_{t_i}]$ will be independent of $\mathcal{F}_{t_j}$. Therefore,

$$\begin{aligned} \mathbb{E}\bigg[e_ie_j[W_{t_{i+1}} - W_{t_i}][W_{t_{j+1}} - W_{t_j}]\bigg] &= \mathbb{E}[\mathbb{E}(e_ie_j[W_{t_{i+1}} - W_{t_i}][W_{t_{j+1}} - W_{t_j}]|\mathcal{F}_{t_j})] \\ &= E[e_ie_j[W_{t_{i+1}} - W_{t_i}]\mathbb{E}(W_{t_{j+1}} - W_{t_j}|\mathcal{F}_{t_j})]. \end{aligned} \tag{101}$$

Since $W_{t_{j+1}} - W_{t_j}$ is independent of $\mathcal{F}_{t_j}$, we have that

$$\begin{aligned}
\mathbb{E}(W_{t_{j+1}} - W_{t_j}|\mathcal{F}_{t_j}) &= \mathbb{E}(W_{t_{j+1}} - W_{t_j}) \\
&= \mathbb{E}(\varepsilon\sqrt{t_{j+1} - t_j}) \\
&= \sqrt{t_{j+1} - t_j}\mathbb{E}(\varepsilon) \\
&= 0,
\end{aligned} \tag{102}$$

where $\varepsilon \sim N(0,1)$, Definition A.5. Substituting (102) into (101), we have that

$$\mathbb{E}\Big[e_i e_j [W_{t_{i+1}} - W_{t_i}][W_{t_{j+1}} - W_{t_j}]\Big] = 0,$$

if $i < j$. The case where $j < i$ is obtained analogously.

If $i = j$, we have that

$$\begin{aligned}
\mathbb{E}\Big[e_i e_j [W_{t_{i+1}} - W_{t_i}][W_{t_{j+1}} - W_{t_j}]\Big] &= \mathbb{E}\Big[e_j^2\big(\varepsilon\sqrt{t_{j+1} - t_j}\big)^2\Big] \\
&= (t_{j+1} - t_j)\mathbb{E}\big[e_j^2\varepsilon^2\big] \\
&= (t_{j+1} - t_j)\mathbb{E}\big[e_j^2\mathbb{E}(\varepsilon^2|\mathcal{F}_{t_j})\big] \\
&= (t_{j+1} - t_j)\mathbb{E}\big[e_j^2\mathbb{E}(\varepsilon^2)\big] \\
&= (t_{j+1} - t_j)\mathbb{E}\big[e_j^2\big].
\end{aligned}$$

So, for both cases we will have the following expressions,

$$\mathbb{E}\Big[e_i e_j [W_{t_{i+1}} - W_{t_i}][W_{t_{j+1}} - W_{t_j}]\Big] = \begin{cases} 0 & \text{if } i \neq j \\ (t_{j+1} - t_j)\mathbb{E}\big[e_j^2\big] & \text{if } i = j \end{cases} \tag{103}$$

Therefore,

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int_S^T \phi_t dW_t\Big)^2\Big] &= \sum_{i,j}\mathbb{E}\Big[e_i e_j [W_{t_{i+1}} - W_{t_i}][W_{t_{j+1}} - W_{t_j}]\Big] \\
&= \sum_j (t_{j+1} - t_j)\mathbb{E}\big[e_j^2\big] \\
&= \mathbb{E}\Big[\int_S^T \phi_t^2 dt\Big]
\end{aligned}$$

□

The goal now is to denote the stochastic integral for any $f \in \mathcal{V}$, to do so we will use Itô's isometry and approximation procedures. In the next 3 steps we will present the procedures to approximate any f using elementary functions. The proof of each of these steps is given in Øksendal (2003), page 28.

Step 1: Let be a function $g \in \mathcal{V}$ bounded and $g(.,\omega)$ is continuous for each state of nature, ω. Then g can be approximated by

$$\phi_t^n = \sum_j g_{t_j} \mathbf{1}_{[t_j, t_{j+1})}(t)$$

such that

$$\mathbb{E}\bigg[\int_S^T (g - \phi_t^n)^2 dt\bigg] \to 0,\ n \to \infty, \forall\, \omega \in \Omega.$$

Step 2: Let $h \in \mathcal{V}$ be a bounded function. Then h can be approximated by functions $g^n \in \mathcal{V}$ that are bounded and $g^n(.,\omega)$ is continuous for all $g^n \omega$ and n such that

$$\mathbb{E}\bigg[\int_S^T (h_t - g_t^n)^2 dt\bigg] \to 0,\ n \to \infty.$$

Step 3: Let f be a function $f \in \mathcal{V}$, not necessarily continuous or bounded. Then, f can be approximated by a sequence of bounded functions $\{h^n\} \in \mathcal{V}$ such that

$$\mathbb{E}\bigg[\int_S^T (f - h^n)^2 dt\bigg] \to 0,\ n \to \infty.$$

We are now able to define Itô's integral for any function within the class $\mathcal{V}$. If we take a $f \in \mathcal{V}$, we can choose a sequence of elementary functions $\phi^n \in \mathcal{V}$ such that

$$\mathbb{E}\bigg[\int_S^T (f - \phi^n)^2 dt\bigg] \to 0.$$

Therefore we can define Itô's integral as

$$\int_S^T f_t dW_t := \lim_{n\to\infty} \int_S^T \phi_t^n dW_t.$$

The limit exists as an element of L^2. A formal definition of the Itô's integral is given below.

Definition A.8. *The* ***Itô's Integral*** *for a function $f \in \mathcal{V}$ is given by*

$$\int_S^T f_t dW_t := \lim_{n\to\infty} \int_S^T \phi_t^n dW_t$$

where the sequence of elementary functions, ϕ^n, satisfies

$$\mathbb{E}\bigg[\int_S^T (f_t - \phi_t^n)^2 dt\bigg] \to 0,\ n \to \infty.$$

The Itô's integral defined above has the following properties definied below.

Theorem A.1. *(Øksendal (2003), Theorem 3.2.1.) Suppose that $f, g \in \mathcal{V}(0,T)$, let $0 < S < U < T$ and $c, d \in R$. Then the following properties hold:*

1. $\int_S^T f_t dW_t = \int_S^U f_t dW_t + \int_U^T f_t dW_t$ *for a.e.* ω,

2. $\int_S^T (cf_t + dg_t) dW_t = c \int_S^T f_t dW_t +_S^T g_t dW_t$ *for a.e.* ω,

3. $\int_S^T (cf_t + g_t) dW_t$ *is* $\mathcal{F}_T$ *measurable.*

4. $\mathbb{E}\bigg[\int_S^T f_t dt\bigg] = 0$

So far we have represented the Itô integral only as a random variable, we were looking at a fixed interval (S, T). If we let the interval vary $(0, t)$ we have that Itô's integral becomes a stochastic process

$$\int_0^t f_s dWs := \int_0^T f_s \mathbf{1}_{[0,t]}(s) dWs.$$

The properties 1., 2. and 4. of Theorem A.1 still hold for this integral.

One type of stochastic process that has a direct relationship with the Itô integral and is widely used in asset modeling is the so-called Itô process (or stochastic integral). The Itô process is a stochastic process, which is obtained by adding an ordinary integral to an Itô integral.

Definition A.9. *Let W be a Wiener process defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The stochastic process defined by*

$$X_t = X_0 + \int_0^t u_s ds + \int_0^t v_s dW_s, \tag{104}$$

*$t \in [0, T]$ is called an **Itô process**, where v is a stochastic process that belongs to $\mathcal{V}(0,T)$ with*

$$\mathbb{P}\bigg(\int_0^t v_s^2 ds < \infty, \forall\, t \geq 0\bigg) = 1,$$

is u a $\mathcal{F}_t$-adpated process with

$$\mathbb{P}\bigg(\int_0^t |u_s| ds < \infty, \forall\, t \geq 0\bigg) = 1.$$

If X_t is an Itô process the equation (104) can be written in the shorter differential form as

$$dX_t = u_t dt + v_t dW_t. \tag{105}$$

The equation (105) is called a stochastic differential equation (SDE), but this SDE will have some meaning only in its integral form. For it is not possible to rewrite the differential equation as

$$\frac{dX_t}{dt} = u_t + v_t \frac{dW_t}{dt}$$

since the Wiener process is not differentiable, for more details see Focardi, Fabozzi et al. (2004), Chapter 10.

The next theorem is the main result in the theory of stochastic calculus. Com esse teorema seguinte teremos uma ideia do comportamento de uma função de um Itô process.

Theorem A.2. *(Øksendal (2003),Theorem 4.1.2.) Seja X um Itô process dado por*

$$dX_t = u_t dt + v_t dW_t. \tag{106}$$

Let $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ a function of class $C^2([0, \infty) \times \mathbb{R})$ (i.e. s twice continuously differentiable). Then, the process Y defined by $Y_t := g(t, X_t)$ is an Itô process with

$$\begin{aligned} dY(t, X_t) = &\left\{ \frac{\partial g}{\partial t}(t, X_t) + u_t \frac{\partial g}{\partial x}(t, X_t) + v_t^2 \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \right\} dt \\ &+ \sigma \frac{\partial g}{\partial x}(t, X_t) dW_t. \end{aligned}$$

Example A.1. *Let X be an Itô process given by*

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t, \tag{107}$$

and let $g(t, X_t) = \ln X_t$. Compute dY_t and the solution of the SDE in (107).

First, note that $\frac{\partial g}{\partial t} = 0$, $\frac{\partial g}{\partial x} = \frac{1}{x}$ e $\frac{\partial^2 g}{\partial x^2} = \frac{-1}{x^2}$. Applying Itô's formula and replacing u_t and v_t by μX_t and σX_t in (A.2), we obtain

$$dY_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t. \tag{108}$$

The dynamics of Y_t is shown in equation (108). Integrating,

$$\begin{aligned} \int_0^t dY_s &= \int_0^t \left(\mu - \frac{\sigma^2}{2} \right) ds + \int_0^t \sigma dW_s, \\ Y_t &= Y_0 + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t, \\ \ln X_t &= \ln X_0 + \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t, \\ X_t &= X_0 \exp \left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}. \end{aligned} \tag{109}$$

The solution of the differential equation (107) is the stochastic process in (109).

A.3 Change of measure

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ the non-negative random variable, Z, with $\mathbb{E}[Z] = 1$, is defined as follows

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}, \tag{110}$$

where Z is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$. By Equation (110), we can define the probability measure $\mathbb{Q}$ by the following formula

$$\mathbb{Q}(A) = \int_A Z d\mathbb{Q}, \ \forall\ A \in \mathcal{F}.$$

Note that we will now have an expectation under the original probability measure, $\mathbb{E}$, and an expectation under the new probability measure, $\mathbb{E}_{\mathbb{Q}}$.

Definition A.10. *In a probability space,* $(\Omega, \mathcal{F}, \mathbb{P})$*, we can define the* ***Radon-Nikodym derivative process*** Z*, on* $0 \leq t \leq T$*, as*

$$Z_t = \mathbb{E}[Z_T | \mathcal{F}_t]$$

Proposition A.3. *If* Y *is* $\mathcal{F}_t$*-mensurable, then*

$$\mathbb{E}_{\mathbb{Q}}[Y] = \mathbb{E}[Y Z_t].$$

Proof.

$$\begin{aligned} \mathbb{E}[Y Z_t] &= \int_\Omega Y Z_t d\mathbb{P} \\ &= \int_\Omega Y d\mathbb{Q} \end{aligned}$$

□

Proposition A.4. *If the random variable* Y *is* $\mathcal{F}_t$*-mensurable, then*

$$\mathbb{E}_{\mathbb{Q}}[Y | \mathcal{F}_t] = \frac{1}{Z_s} \mathbb{E}[Y Z_t | \mathcal{F}_s],$$

where $0 \leq s < t \leq T$.

Proof. By Defintion A.3, we have that

$$\int_G \mathbb{E}_{\mathbb{Q}}[Y | \mathcal{F}_s] d\mathbb{Q} = \int_G Y d\mathbb{Q}$$

for every $G \in \mathcal{G}$. Therefore, we can rewrite Equation (111) as

$$\int_G \frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]d\mathbb{Q} = \int_G Y d\mathbb{Q}$$

for every $G \in \mathcal{G}$. Then, we need only prove the Equation (111). Note that $\frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]$ is $\mathcal{F}_s$-mensurable and

$$\int_G \frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]d\mathbb{Q} = \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{\omega\in G\}}\frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]].$$

By Proposition A.3 and by iterated expectations, we have that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{\omega\in G\}}\frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]] =& \mathbb{E}[\mathbf{1}_{\{\omega\in G\}}\frac{1}{Z_s}\mathbb{E}[YZ_t|\mathcal{F}_s]Z_s] \\
=& \mathbb{E}[\mathbf{1}_{\{\omega\in G\}}\mathbb{E}[YZ_t|\mathcal{F}_s]] \\
=& \mathbb{E}[\mathbf{1}_{\{\omega\in G\}}YZ_t] \\
=& \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{\omega\in G\}}Y] = \int_G Y d\mathbb{Q}.
\end{aligned}$$

□

To price an option in a risk neutral scenario, we will need Girsanov's theorem, which is presented below.

Theorem A.3. *(Shreve (2004), Theorem 5.2.3.) Let W be a Wiener process on* $(\Omega, \mathcal{F}, \mathbb{P})$*, and let* φ *be an adapted process. Define the process* Z *on* $[0, T]$ *by*

$$Z_t = \exp\left\{ -\int_0^t \varphi_u dW_u - \frac{1}{2}\int_0^t \varphi_u^2 dW_u \right\}.$$

Assume that

$$\mathbb{E}[\int_0^T \varphi_u^2 Z_u^2 du < \infty],$$

and a probability measure $\mathbb{Q}$ *on* $\mathcal{F}_T$ *is defined by*

$$Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

Then, the dynamics of the $\mathbb{Q}$*-Wiener process,* $W^{\mathbb{Q}}$*, is given by*

$$dW_t^{\mathbb{Q}} = \varphi_t d_t - dW_t.$$

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