ORIENTABILITY OF A REGULAR LEVEL SET

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Theorem. *Let M and N be oriented smooth manifolds and $F : M \longrightarrow N$ a smooth map. Suppose $m \doteq \dim M > \dim N \doteq n$. If $a \in N$ is a regular value of F, the smooth manifold $S \doteq F^{-1}(a)$ is orientable.*

Proof. Denote by $i : S \hookrightarrow M$ the inclusion map. We shall prove the existence of a smooth nowhere-vanishing $(m-n)$-form on S.

Since M and N are orientable, there are nowhere-vanishing smooth forms $\omega \in \Omega^m(M)$ and $\alpha \in \Omega^n(N)$. We claim that, in order to conclude the proof, it suffices to show that, for some $\tau \in \Omega^{m-n}(M)$, we have

$$(\tau \wedge F^*\alpha)_q = \omega_q \quad \forall\, q \in S. \tag{1}$$

Indeed, let τ be such a differential form. Given $q \in S$, consider a basis $\{v_1, \dots, v_m\}$ for T_qM such that $\{v_1, \dots, v_{m-n}\}$ is a basis for $T_qS \cong \ker dF_q$. Then

$$\begin{aligned} 0 &\neq \omega_q(v_1, \dots, v_m) \\ &= (\tau \wedge F^*\alpha)_q(v_1, \dots, v_m) \\ &= \tau_q(v_1, \dots, v_{m-n})\, \alpha_a(dF_q\, v_{m-n+1}, \dots, dF_q\, v_m), \end{aligned} \tag{2}$$

where the last equality holds since, for $\sigma \in S_m$,

$$\alpha_a(dF_q\, v_{\sigma(m-n+1)}, \dots, dF_q\, v_{\sigma(m)})$$

is non-zero if and only if $\sigma(\{m-n+1, \dots, m\}) \subseteq \{m-n+1, \dots, m\}$. It then follows from (2) that $\tau_q(v_1, \dots, v_{m-n}) \neq 0$ and thus $(i^*\tau)_q \neq 0$. Therefore we conclude that $i^*\tau$ is a nowhere-vanishing smooth $(m-n)$-form on S.

Having established this, our goal is to prove that (1) holds for some $\tau \in \Omega^{m-n}(M)$. We achieve this using a patching argument with a partition of unity.

First notice that $(F^*\alpha)_p \neq 0$ for all $p \in S$ by the surjectivity of dF_p. Now, consider $p \in S$ and a chart $(U_p, x^1, \dots, x^m)$ on M about p. On U_p, write

$$\omega = g\, dx^1 \wedge \cdots \wedge dx^m, \;\; g \in C^\infty(U_p),$$

and

$$F^*\alpha = \sum_I a_I\, dx^I, \;\; a_I \in C^\infty(U_p).$$

Since $(F^*\alpha)_p \neq 0$, there is an increasing multi-index $I_p = (i_1, \dots, i_n)$ such that $a_{I_p}(p) \neq 0$ and we may assume by continuity that a_{I_p} does not vanish on U_p. Denote by $j_1, \dots, j_{m-n}$ the elements of $\{1, \dots, m\} \setminus I_p$ and define, on U_p,

$$\tau^p \doteq \frac{g}{(\operatorname{sgn}\sigma)\, a_{I_p}}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{m-n}} \in \Omega^{m-n}(U_p),$$

where σ is the unique permutation on S_m that maps

$$(j_1, \dots, j_{m-n}, i_1, \dots, i_n) \longmapsto (1, \dots, m).$$

Then, on U_p, we have

$$\begin{aligned}
\tau^p \wedge F^*\alpha &= \left(\frac{g}{(\operatorname{sgn}\sigma)\, a_I}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{m-n}} \right) \wedge \left(\sum_I a_I\, dx^I \right) \\
&= \frac{g}{(\operatorname{sgn}\sigma)}\, dx^{j_1} \wedge \cdots \wedge dx^{j_{m-n}} \wedge dx^I = \omega.
\end{aligned} \tag{3}$$

Finally, we glue the local forms $\{\tau_p\}_{p\in S}$ together to obtain the desired $(m-n)$-form τ. Let $\{\psi_p\}_{p\in S} \cup \{\psi_{M\setminus S}\}$ be a partition of unity on M subordinate to the open cover $\{U_p\}_{p\in S} \cup \{M \setminus S\}$. Define

$$\tau \doteq \sum_{p\in S} \psi_p \tau^p \in \Omega^{m-n}(M).$$

Note that:

- For each $p \in S$, the form $\psi_p \tau^p$ is defined on U_p. Nonetheless, it may be trivially extended to a smooth form on M by defining it to be identically zero outside U_p since $\operatorname{supp}\psi_p$ is closed in M and contained in U_p.
- Since $\{\operatorname{supp}\psi_p\}_{p\in S}$ is a locally finite family, τ is indeed well defined and smooth on M.

We claim that τ is the desired $(m-n)$-form. In fact, if $q \in S$, then

$$\begin{aligned}
(\tau \wedge F^*\alpha)_q &= \tau_q \wedge (F^*\alpha)_q \\
&= \left(\sum_{p\in S} \psi_p(q)\, \tau_q^p \right) \wedge (F^*\alpha)_q \\
&= \sum_{p\in S} \psi_p(q)\, \tau_q^p \wedge (F^*\alpha)_q \\
&= \sum_{p\in S} \psi_p(q)\, [\tau^p \wedge (F^*\alpha)]_q \\
&\overset{(1)}{=} \sum_{p\in S} \psi_p(q)\, \omega_q \\
&\overset{(2)}{=} \left(\sum_{p\in S} \psi_p(q) + \psi_{M\setminus S}(q) \right) \omega_q = \omega_q,
\end{aligned}$$

where (1) holds by (3) since the sum is taken over $p \in S$ with $q \in \operatorname{supp}\psi_p \subseteq U_p$; and (2) holds since $q \in S$. This concludes the proof of (1). □