1 Notation

- n: number of teams;
- N: number of matches;
- T_k^q: last minute of q-th half of match;
- m_k^q: total goals in q-th half of match k;
- $t_{k,l}^q$: minute of the l-th goal in the q-th half of match k;
- $t_{k,0}^1 = 0$, $t_{k,m_k^1+1}^1 = T_k^1$, $t_{k,0}^2 = 45$ and $t_{k,m_k^2+1}^2 = T_k^2$;
- $H_{k,l}^q = \begin{cases} 1, & \text{if the home team scored a goal in interval } [t_{k,l}, t_{k,l+1}); \\ 0, & \text{otherwise;} \end{cases}$
- $A_{k,l}^q = \begin{cases} 1, & \text{if the away team scored a goal in interval } [t_{k,l}, t_{k,l+1}); \\ 0, & \text{otherwise;} \end{cases}$
- $\lambda_k(t)$: rate of goals for the home team in match k;
- $\mu_k(t)$: rate of goals for the away team in match k;
- $\rho = [t_k^1, t_k^2, H_k^1, H_k^2, A_k^1, A_k^2]$;
- $\mathbf{\Lambda}(t_1, t_2) = \int_{t_1}^{t_2} \lambda_k(t)\ dt$;
- $\mathbf{Y}(t_1, t_2) = \int_{t_1}^{t_2} \mu_k(t)\ dt$.

2 Likelihood for match k

We define $T_k^1 = 45 + U_k^1$ and $T_k^2 = 90 + U_k^2$ and model U_k^1 and U_k^2 as

$$U_k^1 \sim \text{Poisson}(\pi_1)$$

$$U_k^2 \sim \text{Poisson}(\pi_2)$$

therefore,

$$L(\rho \mid T_k^1, T_k^2) = \prod_{l=0}^{m_k^1} \Bigg[\exp\Bigg\{ -\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1) \Bigg\} \exp\Bigg\{ -\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1) \Bigg\} \Bigg(\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1) \Bigg)^{H_{k,l}^1} \Bigg(\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1) \Bigg)^{A_{k,l}^1} \Bigg]$$
$$\prod_{l=0}^{m_k^2} \Bigg[\exp\Bigg\{ -\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2) \Bigg\} \exp\Bigg\{ -\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2) \Bigg\} \Bigg(\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2) \Bigg)^{H_{k,l}^2} \Bigg(\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2) \Bigg)^{A_{k,l}^2} \Bigg]$$

and

$$L(\rho, T_k^1, T_k^2) = \prod_{l=0}^{m_k^1} \Bigg[\exp\Bigg\{ -\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1) \Bigg\} \exp\Bigg\{ -\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1) \Bigg\} \Bigg(\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1) \Bigg)^{H_{k,l}^1} \Bigg(\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1) \Bigg)^{A_{k,l}^1} \Bigg]$$
$$\prod_{l=0}^{m_k^2} \Bigg[\exp\Bigg\{ -\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2) \Bigg\} \exp\Bigg\{ -\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2) \Bigg\} \Bigg(\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2) \Bigg)^{H_{k,l}^2} \Bigg(\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2) \Bigg)^{A_{k,l}^2} \Bigg]$$
$$\frac{\pi_1^{U_k^1} \exp\{-\pi_1\}}{U_k^1!} \frac{\pi_2^{U_k^2} \exp\{-\pi_2\}}{U_k^2!}.$$

The log-likelihood is

$$l(\rho, T_k^1, T_k^2) = \sum_{l=0}^{m_k^1} \Bigg[-\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1) - \mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1) + H_{k,l}^1 \ \log(\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1)) + A_{k,l}^1 \ \log(\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1)) \Bigg]$$
$$+ \sum_{l=0}^{m_k^2} \Bigg[-\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2) - \mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2) + H_{k,l}^2 \ \log(\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2)) + A_{k,l}^2 \ \log(\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2)) \Bigg]$$
$$+ U_k^1 \log(\pi_1) - \pi_1 - \log(U_k^1!) + U_k^2 \log(\pi_2) - \pi_2 - \log(U_k^2!).$$

3 DCP

$$\begin{aligned}\lambda_k(t) &= \alpha_i\ \beta_j\ \gamma_h\ \lambda_{xy}\ e^{f(t)\xi_1} \\ &= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^* + f(t)\xi_1\}\end{aligned}$$

where $\alpha_i^* = \log(\alpha_i)$, $\beta_j^* = \log(\beta_j)$, $\gamma_h^* = \log(\gamma_h)$ and $\lambda_{xy}^* = \log(\lambda_{xy})$.

If $f(t) = \log(t+1)$ then

$$\begin{aligned}\mathbf{\Lambda}(t_1, t_2) &= \int_{t_1}^{t_2} \lambda_k(t)\ dt = \int_{t_1}^{t_2} \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^* + \log(t+1)\xi_1\}\ dt \\ &= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \int_{t_1}^{t_2} \exp\{\log(t+1)\xi_1\}\ dt \\ &= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \int_{t_1}^{t_2} (t+1)^{\xi_1}\ dt \\ &= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \left(\frac{(t_2+1)^{\xi_1+1} - (t_1+1)^{\xi_1+1}}{\xi_1+1} \right)\end{aligned}$$

If $f(t) = t$ then

$$\begin{aligned}\mathbf{\Lambda}(t_1, t_2) &= \int_{t_1}^{t_2} \lambda_k(t)\ dt = \int_{t_1}^{t_2} \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^* + t\xi_1\}\ dt \\ &= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \int_{t_1}^{t_2} \exp\{t\xi_1\}\ dt \\ &= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \left(\frac{\exp\{t_2\xi_1\} - \exp\{t_1\xi_1\}}{\xi_1} \right) \\ &= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \left(\frac{\exp\{t_2\xi_1\}(1 - \exp\{\xi_1(t_1 - t_2)\})}{\xi_1} \right)\end{aligned}$$