1 Notation

- n: number of teams;
- N: number of matches;
- T_k^q: last minute of q-th half of match;
- m_k^q: total goals in q-th half of match k;
- $t_{k,l}^q$: minute of the l-th goal in the q-th half of match k;
- $t_{k,0}^1 = 0$, $t_{k,m_k^1+1}^1 = T_k^1$, $t_{k,0}^2 = 45$ and $t_{k,m_k^2+1}^2 = T_k^2$;
- $H_{k,l}^q = \begin{cases} 1, & \text{if the home team scored a goal in interval } [t_{k,l}, t_{k,l+1}); \\ 0, & \text{otherwise;} \end{cases}$
- $A_{k,l}^q = \begin{cases} 1, & \text{if the away team scored a goal in interval } [t_{k,l}, t_{k,l+1}); \\ 0, & \text{otherwise;} \end{cases}$
- $\lambda_k(t)$: rate of goals for the home team in match k;
- $\mu_k(t)$: rate of goals for the away team in match k;
- $\rho = [t_k^1, t_k^2, H_k^1, H_k^2, A_k^1, A_k^2]$;
- $\mathbf{\Lambda}(t_1, t_2) = \int_{t_1}^{t_2} \lambda_k(t)\ dt$;
- $\mathbf{Y}(t_1, t_2) = \int_{t_1}^{t_2} \mu_k(t)\ dt$.

2 Likelihood for match k

In a match k and half q, conditional on the score being (x, y), the distribution of the time to the next home goal is exponential with rate $\lambda_k(t)$ and the distribution of time to the next away goal is exponential with rate $\mu_k(t)$.

So in an interval $(t_1, t_2]$ of match k, half q and score (x, y), three scenarios can happen:

1. Home team scores one goal

$$\begin{aligned} P(t, H = 1) &= P(\text{home team scores in } (t_1, t_2] \text{ and away team does not}) \\ &= \exp\{-\mathbf{\Lambda}(t_1, t_2)\}\mathbf{\Lambda}(t_1, t_2) \exp\{-\mathbf{Y}(t_1, t_2)\} \end{aligned}$$

2. Away team scores one goal

$$\begin{aligned} P(t, A=1) &= P(\text{away team scores in } (t_1, t_2] \text{ and home team does not}) \\ &= \exp\{-\mathbf{Y}(t_1,t_2)\}\mathbf{Y}(t_1,t_2)\exp\{-\mathbf{\Lambda}(t_1,t_2)\} \end{aligned}$$

3. Neither team scores

$$P(t, H=0, A=0) = \exp\{-\mathbf{\Lambda}(t_1,t_2)\}\exp\{-\mathbf{Y}(t_1,t_2)\}.$$

We define $T_k^1 = 45 + U_k^1$ and $T_k^2 = 90 + U_k^2$ and model U_k^1 and U_k^2 as

$$U_k^1 \sim \text{Poisson}(\pi_1)$$

$$U_k^2 \sim \text{Poisson}(\pi_2)$$

then taking the product over all intervals of both halves of match k, we have

$$\begin{aligned} L(\rho \mid T_k^1, T_k^2) = &\prod_{l=0}^{m_k^1}\left[\exp\left\{-\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1)\right\}\exp\left\{-\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1)\right\}\left(\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1)\right)^{H_{k,l}^1}\left(\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1)\right)^{A_{k,l}^1}\right] \\ &\prod_{l=0}^{m_k^2}\left[\exp\left\{-\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2)\right\}\exp\left\{-\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2)\right\}\left(\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2)\right)^{H_{k,l}^2}\left(\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2)\right)^{A_{k,l}^2}\right] \end{aligned}$$

and

$$\begin{aligned} L(\rho, T_k^1, T_k^2) = &\prod_{l=0}^{m_k^1}\left[\exp\left\{-\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1)\right\}\exp\left\{-\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1)\right\}\left(\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1)\right)^{H_{k,l}^1}\left(\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1)\right)^{A_{k,l}^1}\right] \\ &\prod_{l=0}^{m_k^2}\left[\exp\left\{-\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2)\right\}\exp\left\{-\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2)\right\}\left(\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2)\right)^{H_{k,l}^2}\left(\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2)\right)^{A_{k,l}^2}\right] \\ &\frac{\pi_1^{U_k^1}\exp\{-\pi_1\}}{U_k^1!}\frac{\pi_2^{U_k^2}\exp\{-\pi_2\}}{U_k^2!}. \end{aligned}$$

Finally, the log-likelihood is

$$\begin{aligned} l(\rho, T_k^1, T_k^2) = &\sum_{l=0}^{m_k^1}\left[-\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1) - \mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1) + H_{k,l}^1\ \log(\mathbf{\Lambda}(t_{k,l}^1, t_{k,l+1}^1)) + A_{k,l}^1\ \log(\mathbf{Y}(t_{k,l}^1, t_{k,l+1}^1))\right] \\ &+\sum_{l=0}^{m_k^2}\left[-\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2) - \mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2) + H_{k,l}^2\ \log(\mathbf{\Lambda}(t_{k,l}^2, t_{k,l+1}^2)) + A_{k,l}^2\ \log(\mathbf{Y}(t_{k,l}^2, t_{k,l+1}^2))\right] \\ &+ U_k^1\log(\pi_1) - \pi_1 - \log(U_k^1!) + U_k^2\log(\pi_2) - \pi_2 - \log(U_k^2!). \end{aligned}$$

3 DCP

$$
\begin{aligned}
\lambda_k(t) &= \alpha_i \ \beta_j \ \gamma_h \ \lambda_{xy} \ e^{f(t)\xi_1} \\
&= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^* + f(t)\xi_1\}
\end{aligned}
$$

where $\alpha_i^* = \log(\alpha_i)$, $\beta_j^* = \log(\beta_j)$, $\gamma_h^* = \log(\gamma_h)$ and $\lambda_{xy}^* = \log(\lambda_{xy})$.

If $f(t) = \log(t+1)$ then

$$
\begin{aligned}
\mathbf{\Lambda}(t_1, t_2) = \int_{t_1}^{t_2} \lambda_k(t) \ dt &= \int_{t_1}^{t_2} \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^* + \log(t+1)\xi_1\} \ dt \\
&= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \int_{t_1}^{t_2} \exp\{\log(t+1)\xi_1\} \ dt \\
&= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \int_{t_1}^{t_2} (t+1)^{\xi_1} \ dt \\
&= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \left(\frac{(t_2+1)^{\xi_1+1} - (t_1+1)^{\xi_1+1}}{\xi_1 + 1} \right)
\end{aligned}
$$

If $f(t) = t$ then

$$
\begin{aligned}
\mathbf{\Lambda}(t_1, t_2) = \int_{t_1}^{t_2} \lambda_k(t) \ dt &= \int_{t_1}^{t_2} \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^* + t\xi_1\} \ dt \\
&= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \int_{t_1}^{t_2} \exp\{t\xi_1\} \ dt \\
&= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \left(\frac{\exp\{t_2\xi_1\} - \exp\{t_1\xi_1\}}{\xi_1} \right) \\
&= \exp\{\alpha_i^* + \beta_j^* + \gamma_h^* + \lambda_{xy}^*\} \left(\frac{\exp\{t_2\xi_1\}(1 - \exp\{\xi_1(t_1 - t_2)\})}{\xi_1} \right)
\end{aligned}
$$