3. $$\boxed{\mathbb{E}\left[\left|\hat{\phi}_n(t) - \phi_X(t)\right|^2\right] = \frac{1}{n}\left(1 - \phi_X(t)^2\right)}$$

$$= \underline{n^{-1}}\left(1 - \phi_X(t)^2\right)$$

↳ the order of magnitude is $O(n^{-1})$

Since $\hat{\phi}_n$ is unbiased, its quadratic risk is equal to its variance.

2.) Fix $t \in \mathbb{R}$, $\tilde{\Phi}(t)$ is an unbiased estimator of $\Phi_X(t)$

$$\mathbb{E}[\Phi_n(t)] = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[e^{itX_k}] = \Phi_X(t)$$

Lemma 1.2

$$\mathbb{E}[\Phi_n(t)] = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[e^{itX_k}] = \frac{1}{n}\cdot n \cdot \Phi_X(t)$$
$$= \Phi_X(t)$$

$$\mathbb{E}[\phi_n(t)^2]$$
$$= \mathbb{E}[\phi_n(t)\phi_n(-t)] = \mathbb{E}\left[\sum_{k=1}^{n}\sum_{j=1}^{n} e^{i(X_k-X_j)t}\right]\frac{1}{n^2}$$

when $k=j$, $e^{i(X_k-X_j)t} = 1$

$$\mathbb{E}\left[\sum_{k,j:k\neq j} e^{i(X_k-X_j)t} + \sum_{k=j} 1\right]\frac{1}{n^2} = \mathbb{E}\left[\sum_{k,j:k\neq j} e^{i(X_k-X_j)t} + n\right]\frac{1}{n^2}$$

$$= \frac{1}{n^2}\sum_{k\neq j}\mathbb{E}\left[e^{i(X_k-X_j)t}\right] + \frac{1}{n} =$$

for k in range (n)
for j in range (n) $\Rightarrow$ $\boxed{n(n-1)}$
if j = k {
}

$$= \boxed{\frac{(n-1)}{n} |\phi(t)|^2 + \frac{1}{n}}$$

$$\boxed{\operatorname{var}(\hat{\phi}_n(t))} = \mathbb{E}\left[\left(\hat{\phi}_n(t) - \mathbb{E}[\hat{\phi}_n(t)]\right)^2\right]$$

$$= \mathbb{E}\left[\left(\hat{\phi}_n(t) - \phi(t)\right)^2\right] = \mathbb{E}\left[\hat{\phi}_n(t)^2\right] - \mathbb{E}\left[\hat{\phi}_n(t)\right]^2$$

$$= \frac{n-1}{n} |\phi(t)|^2 + \frac{1}{n} - |\phi(t)|^2 = \left(1 - \frac{1}{n}\right)|\phi|^2 + \frac{1}{n} - |\phi|^2$$

$$= -\frac{|\phi(t)|^2}{n} + \frac{1}{n} = \boxed{\frac{1}{n}\left(1 - |\phi(t)|^2\right)}$$

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx} \, dF_X(x) = \underbrace{\int_{\mathbb{R}} e^{itx} f_X(x) \, dx}_{\text{def. Expectation real valued random variables}}$$

characteristic function

X is a real valued random variable defined on a prob. space (Ω, Σ, P)

Lebesgue Integral

1.)

$$\Phi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx} p(t) \, dt$$

def of expectation arbitrary real-valued random var.

$p(t)$: prob dens. function

$$= \int_{\mathbb{R}} e^{itx} \, dF(t)$$

$$\Phi_n(w) = \int_{-\infty}^{\infty} e^{itw} \, dF_n(t) = \frac{1}{n} \sum_{j=1}^{n} e^{iX_t}$$

The estimator is obtained by replacing the probability function $F(t)$ by the empirical distribution function

(Ω, F, P)

$$\hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(X_i \leq t) \; *$$