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P1: $\frac{1}{m}\sum_{i=1}^{m}(x_i)^2 = \frac{1}{m}(x\cdot x) = \frac{x\cdot x}{m} = \frac{|x|^2}{m}$

$\frac{1}{m}\sum_{i=1}^{M}(x_i-\mu)^2 = \frac{1}{m}\sum_{i=1}^{m}(x_i^2-2\mu x_i+\mu^2)$

$= \frac{x\cdot x}{m} - \frac{2\mu}{m}\sum_{i=1}^{m}x_i + \frac{m\mu^2}{m}$ $\quad\mu = \frac{1}{m}\sum_{i=1}^{M}x_i$

$= \frac{x\cdot x}{M} - 2\mu^2 + \mu^2$

$= \frac{x\cdot x}{m} - \mu^2 = \frac{|x|^2}{m} - \mu^2.$

P2: ①. $x_1^2+x_2^2=1$ is an empty circle. area

as graph shows, line l_1 is not on the circle.

So set $\{(x_1,x_2): x_1^2+x_2^2=1\}$ is not convex.

②. $|x_1|+|x_2|\le 1$ is a solid. square. area.

lines joining any two points from the set belong to the set.

So set $\{(x_1,x_2): |x_1|+|x_2|\le 1\}$ is convex.

P3. (a). $f(x_1) = x_1^2+x_2^2-4x_2x_1$

$f'(x_1) = 2x_1-4x_2$

$f''(x_1) = 2 \quad f''>0 \quad \therefore f$ is convex in x_1.

(b). $f'(x_2) = 2x_2-4x_1$

$f''(x_2) = 2>0. \quad \therefore f$ is convex in x_2.

(c). $Hf(x_1,x_2) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} \\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2}\end{bmatrix}.$

$\frac{\partial^2 f}{\partial x_1\partial x_2} = \frac{\partial}{\partial x_1}(2x_2-4x_1) = -4. \quad \frac{\partial^2 f}{\partial x_2\partial x_1} = \frac{\partial}{\partial x_2}(2x_1-4x_2) = -4.$

$Hf(x_1,x_2) = \begin{bmatrix} 2 & -4 \\ -4 & 2\end{bmatrix}.$

By code, eigenval $= [6,-2]$. and $-2<0$.

$\therefore f$ is not convex in (x_1,x_2).

P4: Assume f is convex, $x, y \in \mathrm{dom}\, f$, $\forall \lambda \in [0,1]$

$$x + \lambda (y-x) \in \mathrm{dom}\, f$$

$$\Rightarrow f(x+\lambda(y-x)) \leq (1-\lambda) f(x) + \lambda f(y).$$

$$\Rightarrow f(y) \geq f(x) + \frac{f(x+\lambda(y-x)) - f(x)}{t}.$$

$$\lambda \downarrow 0 \Rightarrow f(y) - f(x) \geq \nabla f^T(x)(y-x)$$