ECE 420 Parallel and Distributed Programming
Assignment 3
Solutions

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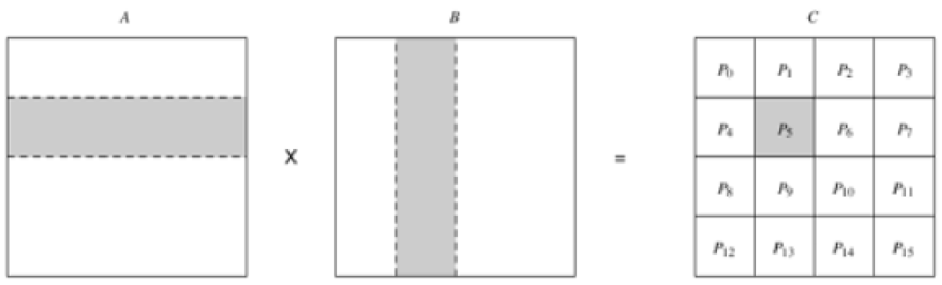
Note: this assignment provides some sample questions for the final exam.

1. Consider the problem of multiplying two $n \times n$ dense, square matrices A and B to yield the product matrix $C = A \times B$. The simple 2-D partitioning to this problem is described as follows. Partition the two $n \times n$ matrices A and B, respectively, into p blocks (submatrices) $A_{i,j}$ and $B_{i,j}$ ($0 \le i, j < p$), each of size $(n/p^{0.5}) \times (n/p^{0.5})$. Process $P_{i,j}$ initially stores $A_{i,j}$ and $B_{i,j}$ and computes block $C_{i,j}$ of the result matrix. Computing submatrix $C_{i,j}$ requires all submatrices $A_{i,k}$ and $B_{k,j}$ for $0 \le k < p^{0.5}$, as shown in the figure below.



Answer the following questions:

1) What's the serial run time of this algorithm, i.e., the time to run the algorithm above on a single processor?

2) Assume that t_s is the latency or the startup time for each data transfer, and t_w the per-word transfer time. Derive an expression for the parallel run time of the procedure above on p processes.

3) Under what values of p is this parallel algorithm cost-optimal?

Answer: 1) Serial runtime is $O(n^3)$.

2) $\sqrt{p}$ rows of all-to-all broadcasts, each is among a group of $\sqrt{p}$ processes. A message size is $\frac{n^2}{p}$, communication time: $t_s log\sqrt{p} + t_w \frac{n^2}{p}(\sqrt{p}-1)$

$\sqrt{p}$ columns of all-to-all broadcasts, communication time: $t_s log\sqrt{p} + t_w \frac{n^2}{p}(\sqrt{p}-1)$

Computation time: $\sqrt{p} \times (n/\sqrt{p})^3 = n^3/p$

Parallel time: $T_p = \frac{n^3}{p} + 2\left(t_s log\sqrt{p} + t_w \frac{n^2}{p}(\sqrt{p}-1)\right)$

3) $\text{cost} = n^3 + t_s p \log p + 2t_w n^2(\sqrt{p}-1)$

The parallel algorithm is cost-optimal for $p = O(n^2)$.

2. Assume that t_s is the latency or the startup time for each data transfer, and t_w the per-word transfer time. Assume that p processes participate in the operation and the data to be broadcast or reduced contains m words. Analyze the time cost of one-to-all broadcast and all-to-one reduction.

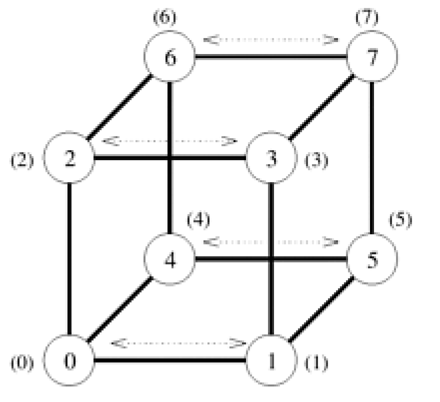
Answer: The broadcast or reduction procedure involves $\log p$ point-to-point simple message transfers, each at a time cost of $t_s + t_w m$. Therefore, the total time taken by the procedure is

$$T = (t_s + t_w m) \log p.$$

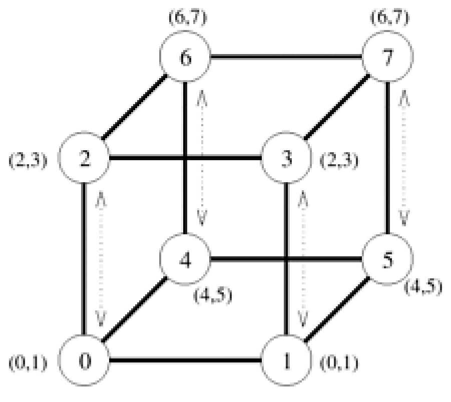
3. The following algorithm performs an all-to-all broadcast on a d-dimensional hypercube.

```
procedure ALL_TO_ALL_BC_HCUBE(my_id, my_msg, d, result)
begin
   result := my_msg;
   for i := 0 to d - 1 do
       partner := my id XOR $2^i$;
       send result to partner;
       receive msg from partner;
       result := result $\cup$ msg;
   endfor;
end ALL_TO_ALL_BC_HCUBE
```

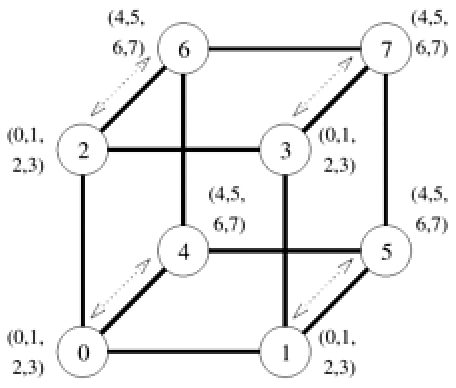
And the following figure illustrates the algorithm procedure on an 8-node (3D) hypercube. On a p-node hypercube, show that the completion time of this algorithm is $t_s \log p + t_w m (p\text{-}1)$.



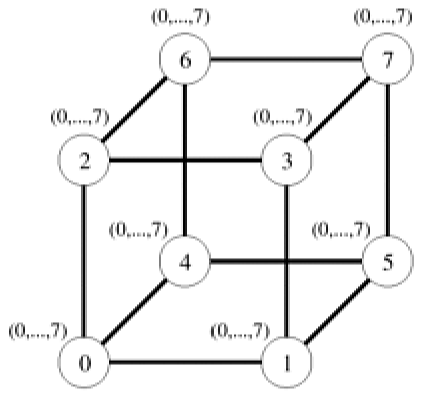
(a) Initial distribution of messages



(b) Distribution before the second step



(c) Distribution before the third step



(d) Final distribution of messages

Answer: on a p-node hypercube, the size of each message exchanged in the i th of the $\log p$ steps is $2^{i\text{-}1}m$. It takes a pair of nodes time $t_s + 2^{i\text{-}1}t_w m$ to send and receive messages from each other during the i th step. Hence, the time to complete the entire procedure is

$$T = \sum_{i=1}^{\log p} (t_s + 2^{i-1} t_w m)$$

$$= t_s \log p + t_w m(p-1).$$

4. Suppose we use MPI_Send and MPI_Recv to do the send and receive operations in two processes performed in the following order:

Process 0	Process 1
Send Data to Process 1	Send Data to Process 0
Receive from Process 1	Receive from Process 0

Is the program safe? If not, suggest at least three ways to make the program safe.

Answer: No, it's unsafe. It's implementation dependent. (It may work if the system implements buffered send. It will fail if the send is non-buffered.)

1) Re-order the operations

Process 0	Process 1
Send Data to Process 1	Receive from Process 0
Receive from Process 1	Send Data to Process 0

2) Supply receive buffer at the same time as send, with `MPI_sendrecv`.

Process 0	Process 1
Sendrecv to/from Process 1	Sendrecv to/from Process 0

3) Use nonblocking operations for either the send, or receive or both.

Process 0	Process 1
Isend to Process 1	Isend to Process 0
Irecv from Process 1	Irecv from Process 0
Waitall	Waitall

5. Derive the exact cost of the following parallel programs for matrix-vector multiplication $x = A \times b$ using p processes. A is n by n. Assume n is both a multiple of p and a multiple of $p^{0.5}$. Assume that t_s is the latency or the startup time for each data transfer, t_w the per-word transfer time, and t_c is the time to perform each multiplication-addition.

1) 1-D row-wise partitioning. Assume the pointer a points to the locally stored n/p rows of A, and the pointer b points to the corresponding locally stored n/p elements of b.

```
RowMatrixVectorMultiply(int n, double *a, double *b, double *x,
                        MPI_Comm comm)
{
  int i, j;
  int nlocal;        /* Number of locally stored rows of A */
  double *fb;        /* Will point to a buffer that stores the entire vector 
  int npes, myrank;
  MPI_Status status;

  /* Get information about the communicator */
  MPI_Comm_size(comm, &npes);
  MPI_Comm_rank(comm, &myrank);

  /* Allocate the memory that will store the entire vector b */
  fb = (double *)malloc(n*sizeof(double));

  nlocal = n/npes;

  /* Gather the entire vector b on each processor using MPI's ALLGATHER operat
  MPI_Allgather(b, nlocal, MPI_DOUBLE, fb, nlocal, MPI_DOUBLE,
      comm);

  /* Perform the matrix-vector multiplication involving the locally stored su
  for (i=0; i<nlocal; i++) {
    x[i] = 0.0;
    for (j=0; j<n; j++)
      x[i] += a[i*n+j]*fb[j];
  }

  free(fb);
}
```

Answer:

$$T_P = t_c \cdot \frac{n^2}{p} + t_s \log p + t_w \cdot \frac{n}{p} \cdot (p-1)$$

$$\text{cost} = t_c n^2 + t_s p \log p + t_w n(p-1)$$

2) 1-D column-wise partitioning. Assume the pointer *a* points to the locally stored *n*/*p* columns of A, and the pointer *b* points to the corresponding locally stored *n*/*p* elements of b.

```
ColMatrixVectorMultiply(int n, double *a, double *b, double *x,
                         MPI_Comm comm)
{
  int i, j;
  int nlocal;
  double *px;
  double *fx;
  int npes, myrank;
  MPI_Status status;

  /* Get identity and size information from the communicator */
  MPI_Comm_size(comm, &npes);
  MPI_Comm_rank(comm, &myrank);

  nlocal = n/npes;

  /* Allocate memory for arrays storing intermediate results. */
  px = (double *)malloc(n*sizeof(double));
  fx = (double *)malloc(n*sizeof(double));

  /* Compute the partial-dot products that correspond to the local columns of
  for (i=0; i<n; i++) {
    px[i] = 0.0;
    for (j=0; j<nlocal; j++)
      px[i] += a[i*nlocal+j]*b[j];
  }

  /* Sum-up the results by performing an element-wise reduction operation */
  MPI_Reduce(px, fx, n, MPI_DOUBLE, MPI_SUM, 0, comm);

  /* Redistribute fx in a fashion similar to that of vector b */
  MPI_Scatter(fx, nlocal, MPI_DOUBLE, x, nlocal, MPI_DOUBLE, 0,
      comm);

  free(px); free(fx);
}
```

Answer:

$$T_P = t_c \cdot \frac{n^2}{p} + (t_s + nt_w)\log p + (t_s + \frac{n}{p} \cdot t_w)(p-1)$$

$$\text{cost} = t_c n^2 + (t_s + nt_w)p\log p + (t_s p + nt_w)(p-1)$$

In the solution provided above, Scatter takes time (t_s + m t_w)(p-1) where message size m = n/p. This is assuming the messages of size n/p are sent one by one (from the source to individual destinations). And (p-1) such single message transfers are needed. Furthermore, using a similar algorithm, Gather will take the same time as Scatter, since it's the reverse process of Scatter.

If a hypercube algorithm is used, the Scatter or Gather time could be as low as t_s log p + m t_w (p-1), where m = n/p. (Explanation: http://parallelcomp.uw.hu/ch04lev1sec4.html)

3) 2-D Partitioning. Assume the pointer *a* points to the locally stored sub-matrix of *A*. Initially, each process in the first column of the process grid holds a part of *b*. And for these processes, the pointer *b* points to the locally stored portions of the vector *b*. For other processes, the pointer *b* points to empty memory cells initially.

```
void MatrixVectorMultiply_2D(int n, double *a, double *b, double *x,MPI_Comm
comm)
{
    int ROW=0, COL=1; /* Improve readability */
    int i, j, nlocal;
    double *px; /* Will store partial dot products */

    int npes, dims[2], periods[2], keep_dims[2];
    int myrank, my2drank, mycoords[2];
    int other_rank, coords[2];

    MPI_Status status;
    MPI_Comm comm_2d, comm_row, comm_col;

    MPI_Comm_size(comm, &npes);
    MPI_Comm_rank(comm, &myrank);

    dims[ROW]= dims[COL] = sqrt(npes);

    nlocal = n/dims[ROW];

    px = malloc(nlocal*sizeof(double));
    periods[ROW] = periods[COL] = 1;

    //Create a 2D Cartesian topology and get the rank and coordinates of the
process
    MPI_Cart_create(comm, 2, dims, periods, 1, &comm_2d);
    MPI_Comm_rank(comm_2d, &my2drank); /* Get my rank in the new topology */
    MPI_Cart_coords(comm_2d, my2drank, 2, mycoords); /* Get my coordinates */

    /* Create the row-based sub-topology*/
    keep_dims[ROW] = 0;
    keep_dims[COL] = 1; /* Column is still connected*/
    MPI_Cart_sub(comm_2d, keep_dims, &comm_row);
    /* Create the column-based sub-topology*/
    keep_dims[ROW] = 1;
    keep_dims[COL] = 0;
    MPI_Cart_sub(comm_2d, keep_dims, &comm_col);

    /* Redistribute the b vector. The vector b is in the first column*/
    /* Step 1. The processes along the 0th column send their data to the
diagonal processes.
    if (mycoords[COL] == 0 && mycoords[ROW] != 0) {
        coords[ROW] = mycoords[ROW];
        coords[COL] = mycoords[ROW];
        MPI_Cart_rank(comm_2d, coords, &other_rank);
        MPI_Send(b, nlocal, MPI_DOUBLE, other_rank, 1, comm_2d);
    }
    if (mycoords[ROW] == mycoords[COL] && mycoords[ROW] != 0) {
        coords[ROW] = mycoords[ROW];
        coords[COL] = 0;
        MPI_Cart_rank(comm_2d, coords, &other_rank);
        MPI_Recv(b, nlocal, MPI_DOUBLE, other_rank, 1, comm_2d, &status);
    }

    /* Step 2. The diagonal processes perform a column-wise broadcast*/
    coords[0] = mycoords[COL];
    MPI_Cart_rank(comm_col, coords, &other_rank);
    MPI_Bcast(b, nlocal, MPI_DOUBLE, other_rank, comm_col);

    for (i=0; i<nlocal; i++) {
        px[i] = 0.0;
        for (j=0; j<nlocal; j++){
            px[i] += a[i*nlocal+j]*b[j];
        }
    }

    /* Perform the sum-reduction along the rows to add up the partial dot
products */
    coords[0] = mycoords[ROW];
    MPI_Cart_rank(comm_row, coords, &other_rank);
    MPI_Reduce(px, x, nlocal, MPI_DOUBLE, MPI_SUM, other_rank, comm_row);

    MPI_Comm_free(&comm_2d); /* Free up communicator */
    MPI_Comm_free(&comm_row); /* Free up communicator */
    MPI_Comm_free(&comm_col); /* Free up communicator */

    free(px);
}
```

Answer:

$$T_P = t_c \cdot \frac{n^2}{p} + (t_s + t_w n/\sqrt{p}) + (t_s + t_w n/\sqrt{p})\log(\sqrt{p}) + (t_s + t_w n/\sqrt{p})\log(\sqrt{p})$$

$$\text{cost} = t_c n^2 + (t_s p + t_w n\sqrt{p}) + (t_s + t_w n/\sqrt{p}) p \log p$$

4) An alternative implementation to 1-D column-wise partitioning in 2) will be to use `MPI_Allreduce` to perform the required reduction operation and then have each process copy the locally stored elements of vector *x* from the vector `fx`. What will be the cost of this implementation?

$$T_p = t_c \cdot \frac{n^2}{p} + \log p(t_s + nt_w)$$

Answer:

$$\text{cost} = t_c n^2 + p \log p (t_s + n t_w)$$

In this solution, AllReduce takes time t_s log p + m t_w (p-1), where message size m = n. This is assuming the following algorithm: we do an all-to-all broadcast of messages of size n and then do local reduction at each node.

A second solution is to do an all-to-one reduction followed by a one-to-all broadcast of the reduced result, which takes time 2(t_s + t_w m) log p, where the message size m = n.

Alternatively, if a hypercube algorithm is used, AllReduce time could be as low as (t_s + t_w m) log p, where message size m = n here. (Explanation: http://parallelcomp.uw.hu/ch04lev1sec3.html)

6. A popular serial algorithm for sorting an array of *n* elements whose values are uniformly distributed over an interval [a, b] is the bucket sort algorithm. In this algorithm, the interval [a, b] is divided into *p* equal-sized subintervals referred to as buckets. The algorithm first places each element in an appropriate bucket. Since the *n* elements are uniformly distributed over the interval [a, b], the number of elements in each bucket is roughly *n*/*p*. The algorithm then sorts the elements in each bucket, yielding a sorted sequence. The following is an MPI program implementing a parallel bucket sort algorithm:

```
/* parallel_bucket_sort.c -- sort a list of evenly distributed numbers.
 * The numbers are randomly generated and are evenly distributed in
 * the range between 0 and 2*n
 *
 * Input: n, the number of numbers
 *
 * Output: the sorted list of numbers
 *
 * Note:  Arrays are allocated using malloc.  A single large array
 *        is used.
 *
 * Uses the bucket sort and selection sort algorithms.
 */
#include <stdio.h>
#include <stdlib.h>    /* for random function */
#include "mpi.h"
```

```
void Make_numbers(long int [], int, int, int);
void Sequential_sort(long int [], int);
int  Get_minpos(long int [], int);
void Put_numbers_in_bucket(long int [], long int [], int, int, int, int);

main(int argc, char* argv[]) {
    long int  * big_array;
    long int  * local_array;
    int       n=80;   /* default is 80 elements to sort */
    int       n_bar;  /* = n/p */
    long int  number;
    int       p;
    int       my_rank;
    int       i;
    double    start, stop;   /* for timing */

    MPI_Init(&argc, &argv);
    MPI_Comm_size(MPI_COMM_WORLD, &p);
    MPI_Comm_rank(MPI_COMM_WORLD, &my_rank);

    n = atoi(argv[1]);  /* first parameter is the number of numbers */

    if (my_rank == 0) {

      /* check if parameters are valid */
      if (n%p != 0) {
            fprintf(stderr,"The number of processes must evenly divide total
                  elements.\n");
            MPI_Abort( MPI_COMM_WORLD, 2 );
            exit(1);
      }

      /* make a big input array */
      big_array = malloc(n*sizeof(long int));
      printf("\nTotal elements = %d; Each process sorts %d elements.\n",
           n, n/p);
      Make_numbers(big_array, n, n/p, p);
      start = MPI_Wtime();  /* Time measurement */
    }

    n_bar = n/p;

    local_array = malloc(n_bar*sizeof(long int));

    Put_numbers_in_bucket(big_array, local_array, n, n_bar, p,
        my_rank);

    Sequential_sort(local_array, n_bar);

    MPI_Gather(local_array, n_bar, MPI_LONG,
             big_array,   n_bar, MPI_LONG, 0, MPI_COMM_WORLD);

    stop = MPI_Wtime();
    if (my_rank==0) {
      printf("\nAfter sorting:\n");
      for(i=0; i<n; i++) printf("%7ld %c", big_array[i],
            i%8==7 ? '\n'  : ' ');
      printf("\n\nTime to sort using %d processes = %lf msecs\n",
           p, (stop - start)/0.001);
    }

    free(local_array);
    if (my_rank==0) free(big_array);
    MPI_Finalize();

    }  /* main */


/*****************************************************************/
void Make_numbers(long int  big_array[]  /* out */,
              int       n            /* in  */,
              int       n_bar        /* in  */,
              int       p            /* in  */)
{
  /* Puts numbers in "buckets" but we can treat it otherwise  */
    int       i, q;
    MPI_Status status;

    printf("Before sorting:\n");
    for (q = 0; q < p; q++) {
      printf("\nP%d: ", q);
      for (i = 0; i < n_bar; i++) {
        big_array[q*n_bar+i] =  random() % (2*n/p) + (q*2*n/p); /* Assuming
            the range of elements in big_array is [0, 2n]*/

        printf("%7ld %s", big_array[q*n_bar+i], i%8==7 ? "\n    " :
            " ");
      }
      printf("\n");
    }
    printf("\n");
}  /* Make_numbers */



/*****************************************************************/
void Sequential_sort(long int array[]   /* in/out */,
                 int  size          /* in     */)
{
  /* Use selection sort to sort a list from smallest to largest */
  int      eff_size, minpos;
  long int temp;

  for(eff_size = size; eff_size > 1; eff_size--) {
    minpos = Get_minpos(array, eff_size);
    temp = array[minpos];
    array[minpos] = array[eff_size-1];
    array[eff_size-1] = temp;
  }
}

/* Return the index of the smallest element left */
int Get_minpos(long int array[], int eff_size)
{
  int i, minpos = 0;

  for (i=0; i<eff_size; i++)
    minpos = array[i] > array[minpos] ? i: minpos;
  return minpos;
}


/*****************************************************************/
void Put_numbers_in_bucket(long int  big_array[]   /* in */,
                     long int  local_array[] /* out */,
                     int       n             /* in  */,
                     int       n_bar         /* in  */,
                     int       p             /* in  */,
                     int       my_rank       /* in  */)
{
  /* Assume that numbers in big_array are evenly distributed at root,
     but are unsorted.  Send numbers to the process (bucket that should have
      them. This version uses unsafe messaging and may fail in some cases!!
      */

  int       i, q, bucket;
  MPI_Status status;

  if (my_rank == 0) {
    for (i=0; i<n; i++) {
      bucket = big_array[i]/(2*n_bar);  /* Assuming the range of the
            elements in big_array is [0, 2n] */
      MPI_Send(&big_array[i], 1, MPI_LONG, bucket,
             0, MPI_COMM_WORLD);
      /*      printf("P%d:%ld ", bucket, big_array[i]);     */
    }
    /*    printf("\n"); */
  }
  for (i=0; i<n_bar; i++) {
    MPI_Recv(&local_array[i], 1, MPI_LONG, MPI_ANY_SOURCE,
           0, MPI_COMM_WORLD, &status);
  }
}
```

1) Assume the number of processes can evenly divide the total number of elements in `big_array`. Given n elements and p processes, each handling a different bucket, analyze the parallel running time complexity and cost of the above MPI program, between the time `start` and `stop`, ignoring the time of generating `big_array`. Note that selection sort has $O(n^2)$ time complexity when sorting n numbers.

2) Under what value of *p* will the above program be cost-optimal? Assume that the best sequential algorithm you can use is the serial bucket sort with selection sort applied in each bucket.

Answer:

$$T_p = \Theta\left(\frac{n^2}{p^2}\right) + \Theta(n) + \Theta(n)$$

$$\text{cost} = \Theta\left(\frac{n^2}{p}\right) + \Theta(np)$$

$$T_s = \Theta\left(\frac{n^2}{p^2}\right) \cdot p + \Theta(n) = \Theta\left(\frac{n^2}{p}\right) + \Theta(n)$$

It's cost optimal when $p = O(\sqrt{n})$.

Here Gather takes time Θ(n). But it's also right if your answer for the Gather part was t_s log p + m t_w (p-1), where message size m = n/p, when a hypercube algorithm is used. However, t_s log p + n/p * t_w (p-1) is still equal to O(n), in the big O notation, which is required by this question. Therefore, the answer will be the same in the big O notation.

Notes:

1. One-to-All Broadcast or All-to-One Reduction will always take time (t_s + mt_w) log p on a tree-based algorithm given in the slides. That's the only default algorithm we assume we're using. And in the exam, no other algorithms will be used for One-to-All Broadcast or All-to-One Reduction.

2. All-to-All Broadcast (AllGather) will always take time t_s log p + m t_w (p-1), assuming a hypercube algorithm is used as given in the slides. No other algorithms will be used for this operation.

3. For all other communication operations, in the exam, either the question will describe the algorithm it is using, or you will be asked to explain (briefly) what algorithms you used to justify your answer.