Intelligent Systems Engineering

FS04 Fuzzy Relations

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Outline

These notes are based on [Pedrycz, 2007] and [Keller et al. 2016]

Relations

Definition

Let X and Y be two universes of discourse. A relation R defined as $X \times Y$ is any subset of the Cartesian product of these two universes:

$$R : X \times Y \to \{0, 1\}$$

If $R(x, y) = 1$, we say x and y are *related* (in sense of R). Otherwise, if $R(x, y) = 0$, x and y are *unrelated* (in sense of R).

Relations: Examples

Example 1. X is the domain of countries and Y is the domain of currencies. Let's define a relation R describing usage of currencies in countries. Then, for example

$$\begin{aligned} R(\text{Canada,CAD}) &= 1 \text{ while} \\ R(\text{Russia, CAD}) &= 0 \end{aligned}$$

Example 2. Equality relation $\text{Equal}(x, y)$ can be defined as $\{(x, y)|x = y\}$ for $(x, y) \in X$. Then for universe $X = \{1, 2, 3\}$, one can write

$$\begin{aligned} &\text{Equal}(1,1) = 1, \text{Equal}(2,2) = 1, \text{Equal}(3,3) = 1; \text{ and} \\ &\text{Equal}(1,2) = 0, \text{Equal}(1,3) = 0, \text{Equal}(2,1) = 0, \\ &\text{Equal}(2,3) = 0, \text{Equal}(3,1) = 0, \text{Equal}(3,2) = 0 \end{aligned}$$

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

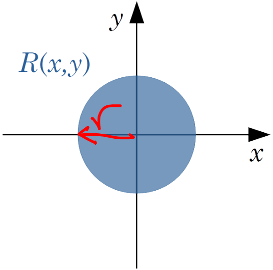
Relations: Representation

Relation can be also represented as a matrix

$$\text{Equal} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Id.

Relation can be also defined using an expression, e.g. $R(x, y) = \{(x, y)|x^2 + y^2 \leq r^2\}$ which defines a disc of radius r centered at (0,0).

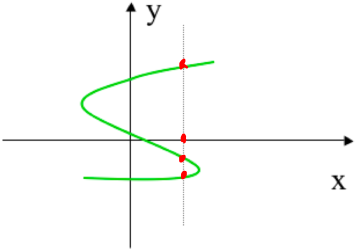


Relations vs. functions

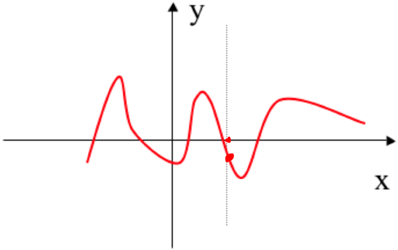
Relations are more general than functions:

- Function – a unique y for each x
- Relation – (possibly) more y for each x

As a consequence, all functions are relations but not all relations are functions.



Relation



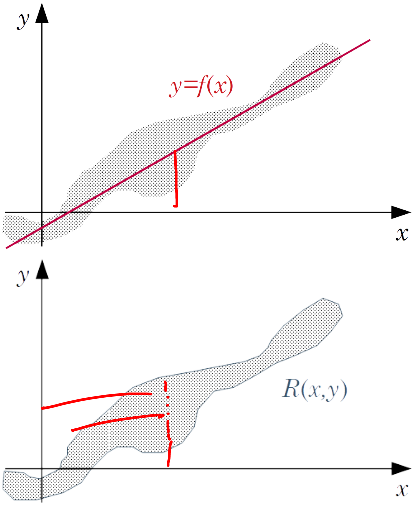
Function

Noisy Function or Relation

From experiments, we usually get "clouds" of data.

- sometimes, this data can be modelled with a function
- other times, this is not possible because
  - condition of unique mapping is not satisfied
  - causal relationship among variables is not clear

In such cases, relation would be a more appropriate model of the data: it can express that data are related but does not attempt to identify independent or dependent variable.



Fuzzy Relations

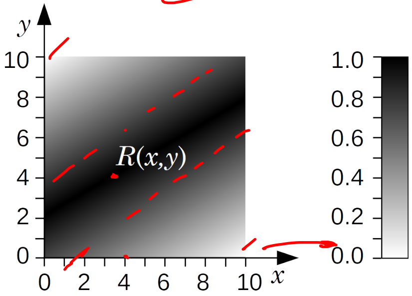
A fuzzy relation R can be defined in a similar fashion as (crisp) relation; however the values of R are taken from the entire interval $[0,1]$ instead of the binary values $\{0,1\}$, i.e.

$$R : X \times Y \to [0,1]$$

In addition to related/unrelated, fuzzy relation can also express a degree of relationship in sense of relation R.

Which of the following examples is depicted on right?

- x is much smaller than y
- x and y are approximately equal
- x is salary, y is age
- x is speed on highway, y is intensity of accidents



Similarity among the members of Duck family

R=

	Dewey	Louie
Huey	0.8	0.9

Fuzzy Relations - Basic Notions

Domain of R

$$\text{dom}(R)(x) = \sup_{y \in Y} R(x, y)$$

Co-domain of R (term co-domain is used instead of range to follow the concept of *direction-free* relation)

$$\text{co}(R)(y) = \sup_{x \in X} R(x, y)$$

α–cut of R (representation theorem; pyramids)

$$R = \cup_{\alpha \in (0,1]} (\alpha R_\alpha)$$

Basic Operations on Fuzzy Relations

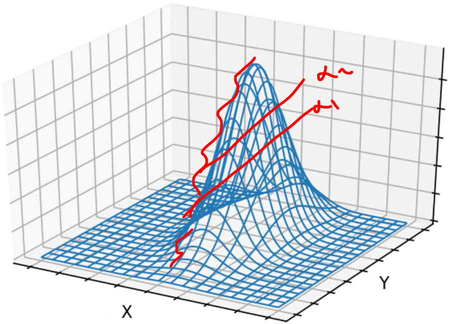
Union $(R \cup S)(x,y) = R(x,y)\,\mathbf{s}\,S(x,y)$

Intersection $(R \cap S)(x,y) = R(x,y)\,\mathbf{t}\,S(x,y)$

Complement $\overline{R}(x,y) = 1 - R(x,y)$

Transpose $R^{\mathsf{T}}(x,y) = R(y,x)$

with properties: $(R^{\mathsf{T}})^{\mathsf{T}} = R \quad (\overline{R})^{\mathsf{T}} = \overline{R^{\mathsf{T}}}$



Composition of Fuzzy Relations

Composition is operation executed on two compatible (fuzzy) relations composition

- $R : X \times Y \to [0,1]$ that relates (or maps) elements from set X to set Y
- $S : Y \times Z \to [0,1]$ that relates (or maps) elements from set Y to set Z

to form a single new relation $T : X \times Z \to [0,1]$ that relates the elements in set X to the elements in set Z

Sup-t composition (e.g. max-min)

$$T = R \circ S$$
$$T(x,z) = \sup_{y \in Y} [R(x,y) \ \mathbf{t} \ S(y,z)]$$

Inf-s composition (e.g. min-max)

$$T = R \bullet S$$
$$T(x,z) = \inf_{y \in Y} [R(x,y) \ \mathbf{s} \ S(y,z)]$$

Example

$* \rightarrow \min$

$+ \rightarrow \max$

Consider two relations R and S defined as follows

$$R = \begin{bmatrix} 1 & 0.6 & 0.5 \\ 0.7 & 0.1 & 1 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ 0.5 & 1 \\ 0.9 & 0 \end{bmatrix}$$

$2 \times 3 \quad 3 \times 2$

considering $\min$ operation in place of triangular norm **t**, the max-min composition of these two relations is $R \circ S =$

$$\begin{bmatrix} \max[\min(1,0), \min(0.6,0.5), \min(0.5,0.9)] & \max(\min(1,1), \min(0.6,1), \min(0.5,0)) \\ \max(\min(0.7,0), \min(0.1,0.5), \min(1,0.9)) & \max(\min(0.7,1), \min(0.1,1), \min(1,0)) \end{bmatrix}$$

$(0.5, 1, 0)$

$$= \begin{bmatrix} \max(0, 0.5, 0.5) & \max(1, 0.6, 0) \\ \max(0, 0.1, 0.9) & \max(0.7, 0.1, 0) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \\ 0.9 & 0.7 \end{bmatrix}$$

2×2

Compositions of Fuzzy Relations: Interpretation

max min

$\sup$–**t** composition involves *matching* and *inference*

- Data in composed relations is matched using **t** operation $\cap$
- Values in the non–overlapping regions $(X \times \cancel{Y} Z)$ are then selected using $\sup$ operation (providing the inferred result)

$\inf$–**s** composition involves *blending* and *compression* $\cup$

- Data in composed relations is blended using **s** operation, reinforcing membership in overlapping region $(\cancel{Z} Y)$ and projecting to the non–overlapping regions $(X \times \cancel{Y} Z)$
- The membership of the common region $(\cancel{Z} Y)$ is compressed using the $\inf$ operation, ignoring the the non–overlapping regions $(X \times \cancel{Y} Z)$

Compositions of Fuzzy Relations: Properties

$T = R \circ S$	$T = R \bullet S$
$T(x,z) = \sup_{y \in Y}[R(x,y)\ \mathbf{t}\ S(y,z)]$	$T(x,z) = \inf_{y \in Y}[R(x,y)\ \mathbf{s}\ S(y,z)]$
max min	min max

Associativity

$P \circ (R \circ S) = (P \circ R) \circ S$	$P \bullet (R \bullet S) = (P \bullet R) \bullet S$

Distributivity

$P \circ (R \cup S) = (P \circ R) \cup (P \circ S)$	$P \bullet (R \cap S) = (P \bullet R) \cap (P \bullet S)$

Weak Distributivity

$P \circ (R \cap S) \subseteq (P \circ R) \cap (P \circ S)$	$P \bullet (R \cup S) \supseteq (P \bullet R) \cup (P \bullet S)$

Monotonicity

$R \subset S$ then $(P \circ R) \subseteq (P \circ S)$	$R \subset S$ then $(P \bullet R) \supseteq (P \bullet S)$

Composition of Fuzzy Relations: Application

Y

X

R=

	Dewey	Louie
Huey	0.8	0.9

Z

S=

	Donald
Dewey	0.9
Louie	0.7

Y

How similar is Huey to Donald?

$$T(\text{Huey}, \text{Donald}) = R \circ S = [\max[\min(0.8, 0.9), \min(0.9, 0.7)]]$$

$$= \max(0.8, 0.7) = 0.8$$

Fuzzy relation

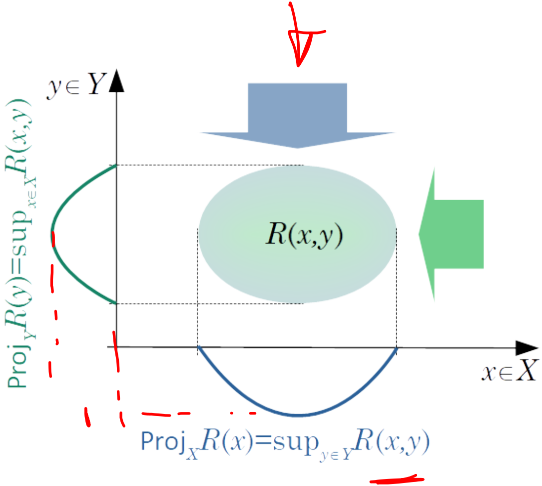
Projections of Fuzzy Relations

Let R be defined in $X \times Y$. The projection of R on X is defined as

$$R_{|X}(x) = \text{Proj}_X R(x) = \sup_{y \in Y} R(x, y).$$

This reduces the dimension of the relation (e.g. for a 2D relation, we arrive at a fuzzy set). Analogously, projection of R on Y is defined as

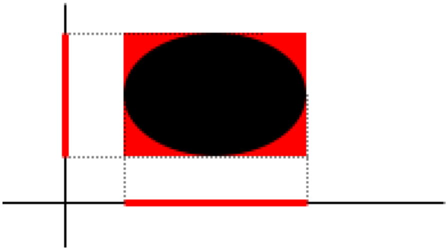
$$R_{|Y}(y) = \text{Proj}_Y R(y) = \sup_{x \in X} R(x, y).$$



Projections of Fuzzy Relations

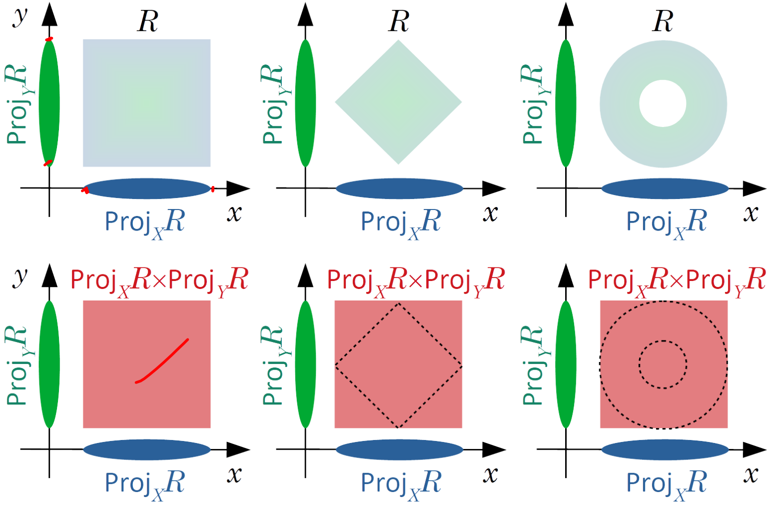
In general, the original relation R cannot be reconstructed from its projections, i.e.

$$\text{Proj}_X R(x) \times \text{Proj}_Y R(y) \neq R \text{ but } \text{Proj}_X R(x) \times \text{Proj}_Y R(y) \supseteq R$$



In fact, only the 'envelope' of the original relation can be reconstructed. Reason: by projection, the data has been compressed (dimensionality has been reduced).

Fuzzy relations, Projections and Reconstructions

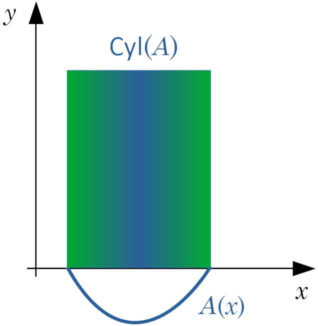


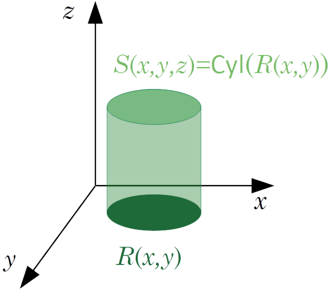
Cylindric Extension

The opposite of projection: increasing dimensionality of a fuzzy set/relation so that it can be combined it with other construct of the same (higher) dimension.

$$\mathrm{Cyl}(A)(x, y) = A(x) \text{ for all } y \in Y$$

2,2





Variants of Composition

Composition of two fuzzy relations is the most general case. However, there are other possibilities as well.

fuzzy relation	fuzzy relation	fuzzy relation
fuzzy set	fuzzy relation	fuzzy set
value	fuzzy relation	fuzzy set
value	(crsip) function	value
fuzzy set	(crisp) function	fuzzy set

Case of crisp function applied to fuzzy set is described by so called *extension principle* (leading to fuzzy numbers and fuzzy arithmetic).

Examples of Composition

fuzz. R = F. set.

singleton v.s

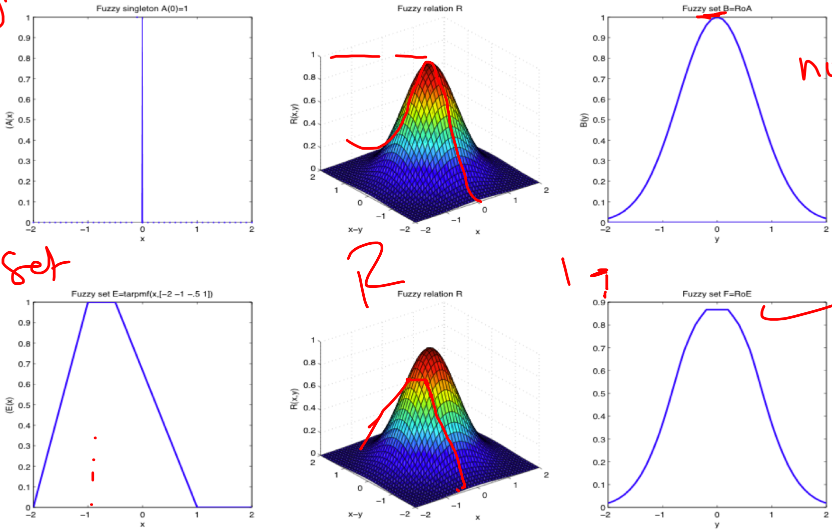
nor. fuzz. set.

fuzz. set

R

1 ?

F. set.



Fuzzy rule–based systems

In knowledge based systems, knowledge can be expressed in form of rules

IF	*condition 1*	AND *condition 2*	THEN	*action*,	or
IF	*premise 1*	AND *premise 2*	THEN	*conclusion*,	or
IF	*antecedent 1*	AND *antecedent 2*	THEN	*consequent*	

In fuzzy systems, such rules are linguistic statements of expert knowledge in which *conditions* and *actions* are fuzzy sets (e.g. positive small, around zero, fast, etc.). These rules are *fuzzy relations* based on fuzzy implication (IF–THEN).

Inference as composition

R

- A set of fuzzy rules forms a knowledge base of a fuzzy system. Let's denote K this collection of rules (i.e. K is the fuzzy relation describing knowledge about the system).
- In fuzzy decision–making (e.g. fuzzy control), the rule base K is first matched with available data describing context in form of a fuzzy set D.
- Then, inference is made on another fuzzy variable represented in K. This matching–inference process is done using the sup–min composition shown previously.

Compositional rule of inference (CRI)

Application of composition to make inferences:

max – min

$$\boxed{I = D \circ K}$$

Making inference

Membership function of the inference (conclusion, consequence, decision, action) can be determined using the CRI

$$I(y) = \sup_{x \in X} \min\left[D(x), K(x,y)\right]$$

X denotes the space in which data (inputs) D are defined, and it is a subspace on which the knowledge (rule) base K is defined.

- K consists of rules containing AND connectives and fuzzy implication (IF–THEN), which can be both represented using $\min$ operation.
- Individual rules are connected by ELSE (corresponding to OR) connectives, which can be represented by $\max$ operation applied to membership of individual rules.

Example of making inference

Consider a control system

- Data (context) is described in terms of outputs, Y, of the controlled process [e.g. room temperature]
- The control action that drives the process is C (normally y and c are crisp, but let's consider them as fuzzy sets for now) [e.g. fan speed]
- Control rule–base is denoted R [e.g. how room temperature and fan speed are related]

Applying CRI $I = D \circ K$ we get the fuzzy control action C as

$$C(c) = \max_{Y} \min \left(Y(y), R(y, c) \right)$$

Composition vs. extension principle

- While, in general, *composition* applies to manipulating fuzzy data (fuzzy relations or sets) with fuzzy knowledge (another fuzzy relation),
- *extension principle* describes a special case of composition which applies to manipulation of fuzzy data (fuzzy sets) with crisp knowledge (crisp relation or function).

Consider a crisp relation $y = f(x)$. This may be considered a fuzzy relation R with the following membership function

$$R(x, y) = \begin{cases} 1 & \text{if } y = f(x), \\ 0 & \text{otherwise.} \end{cases}$$

Extension Principle

Now, consider a set of fuzzy data A with membership function $A(x)$. Using compositional rule of inference, one can obtain

$$B(y) = A(x) \circ R(x, y) = \sup_{x \in X} \min \left(A(x), R(x, y) \right)$$

Region	sup (general)	$R(x, y)$	sup evaluates to
$x = f^{-1}(y)$	$\sup_x \min \left(A(x), R(x, y) \right)$	1	$\sup_x \min \left(A(x), 1 \right)$
$x \neq f^{-1}(y)$	$\sup_x \min \left(A(x), R(x, y) \right)$	0	$\sup_x \min \left(A(x), 0 \right)$

Clearly, the second term (for $x \neq f^{-1}(y)$) gives 0 and can be dropped entirely. The first term takes $\min$ and thus the value 1 can be dropped here. Finally, the following formulation of extension principle is obtained

$$B(y) = A(x) \circ R(x, y) = \sup_{x = f^{-1}(y)} \min \left(A(x) \right)$$

Extension Principle: graphical method