

Intelligent Systems Engineering

10. Fuzzy Arithmetic

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Outline

- 1 Fuzzy numbers
- 2 Interval analysis
- 3 Fuzzy arithmetic
 - Triangular fuzzy numbers
 - Addition of TFNs
 - Multiplication of TFNs
- 4 Measures of fuzziness

These notes are based on [Pedrycz, 2007] and [Keller et al. 2016]

Fuzzy Numbers

Fuzzy sets defined over the domain of real numbers \mathbb{R} . They allow working with uncertain/fuzzy quantities, such as

- about 5
- smaller than 200
- close to 0.1

Example: Resistors are produced with certain tolerance around their nominal value of resistance, e.g. a 220k resistor has, in fact, resistance “about 220k Ω ”

In turn, we may need to incorporate this uncertainty into our calculations

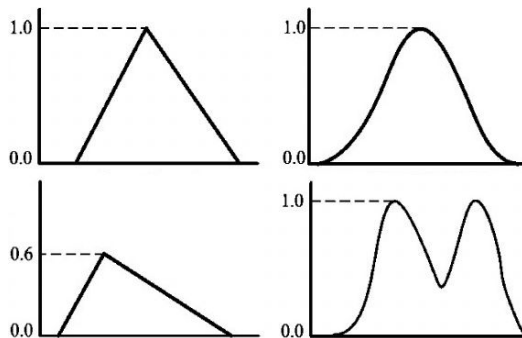
Fuzzy numbers

Fuzzy number is a fuzzy set

$$A : \mathbb{R} \rightarrow [0, 1]$$

that is

- normal,
- unimodal,
- continuous, and has
- bounded support



Interval analysis

Roots of computing with fuzzy numbers originate from interval analysis (a.k.a. calculus of tolerances). The basic objects of interval analysis are intervals I , such as $[1, 2]$, $[a, b]$, etc.

Basic arithmetic operations on intervals are:

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] * [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$[a, b] \div [c, d] = [\min(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}), \max(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d})]$$

Fuzzy arithmetic

Computing with fuzzy numbers leads to **fuzzy arithmetic**, which is based on **extension principle**.

Consider a function F taking as arguments two fuzzy numbers A and B and producing a third fuzzy number $C = F(A, B)$. The extension principle determines membership function of C as

$$C(z) = \sup_{x,y \in \mathbb{R}: z=f(x,y)} [A(x) \wedge B(y)]$$

where $z \in \mathbb{R}$, and $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a real function that is point-wise consistent with F , i.e.

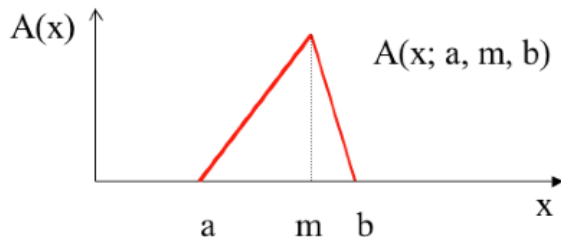
$$F(\{x\}, \{y\}) = f(x, y)$$

where $\{x\}$ and $\{y\}$ are fuzzy singletons corresponding to values x and y , respectively. The expression for extension principle is also called *sup-min convolution*.

Triangular fuzzy numbers

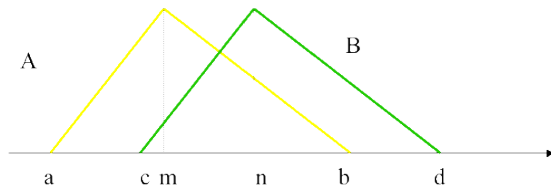
A triangular fuzzy number (TFN) is defined in terms of variable $x \in \mathbb{R}$ with parameters a, m , and b , describing lower boundary, modal value, and upper boundary, respectively

$$A(x; a, m, b) = \begin{cases} \frac{x-a}{m-a} & \text{if } x \in [a, m], \\ \frac{b-x}{b-m} & \text{if } x \in [m, b]. \end{cases}$$



Addition of TFNs (in-class exercise, slide 1/5)

Consider two TFNs $A(x; a, m, b)$ and $B(x; c, n, d)$



To derive the formula for fuzzy addition, the general form of the extension principle has to be parameterized by $f(x, y) = x + y$, i.e

$$C(z) = \sup_{x, y \in \mathbb{R}: z=x+y} [A(x) \wedge B(y)]$$

Addition of TFNs (in-class exercise, slide 2/5)

To start derivation of the formula for fuzzy addition, the general form of the extension principle has to be parameterized by $f(x, y) = x + y$, i.e

$$C(z) = \sup_{x, y \in \mathbb{R}: z=x+y} [A(x) \wedge B(y)]$$

Now, let 's consider two cases: (i) $z < m + n$, (ii) $z > m + n$

(i) $z < m + n$: Consider values of x and y for which memberships $A(x)$ and $B(y)$ are equal to a constant ω , i.e.

$$A(x) = B(y) = \omega$$

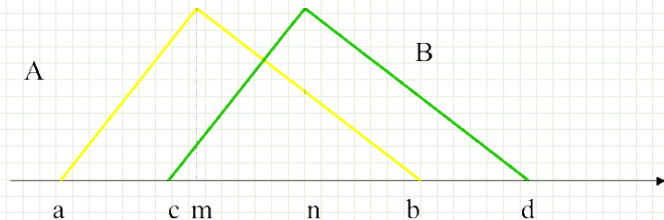
ω can be expressed in terms of parameters/variables of fuzzy numbers A and B as follows

Addition of TFNs (in-class exercise, slide 3/5)

The two expressions

$$\frac{x - a}{m - a} = \omega; \quad \frac{y - c}{n - c} = \omega$$

can be rewritten, respectively, as



Finally, since $z = x + y$, we can express z as follows

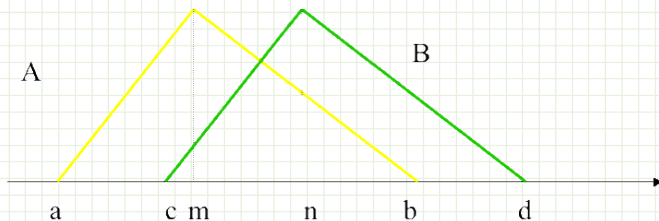
Addition of TFNs (in-class exercise, slide 4/5)

(ii) $z > m + n$: In this case, ω can be expressed as

$$1 - \frac{x - m}{b - m} = \omega; 1 - \frac{y - n}{d - n} = \omega$$

which can be rewritten as $x = m + (1 - \omega)(b - m)$, and $y = n + (1 - \omega)(d - n)$, respectively. Using the same reasoning as before, z can be expressed as

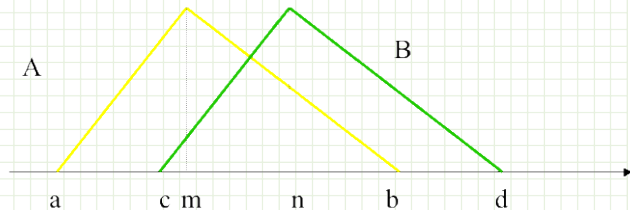
$$z = m + n + (1 - \omega)(b + d - m - n)$$



Addition of TFNs (in-class exercise, slide 5/5)

By expressing $C(z) = \omega$ (in terms of z and the parameters a, b, c, d, m, n) and putting the two partial results together, we obtain

$$C(z) = \begin{cases} \frac{z-(a+c)}{(m+n)-(a+c)} & \text{if } z < m+n \\ 1 & \text{if } z = m+n \\ \frac{(b+d)-z}{(b+d)-(m+n)} & \text{if } z > m+n \end{cases}$$



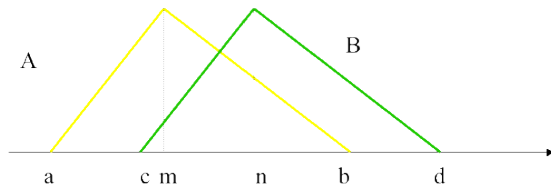
By substituting $e = a + c$, $f = b + d$, $o = m + n$, the resulting fuzzy number C is

$$C(z; e, o, f) = C(z; a + c, m + n, b + d)$$

i.e. the resulting fuzzy number C is also of triangular shape, and its parameters can be determined from the parameters of the variables A and B .

Multiplication of TFNs

Again, consider two TFNs $A(x; a, m, b)$ and $B(x; c, n, d)$



The general form of the extension principle has to be parameterized by $f(x, y) = xy$, i.e

$$C(z) = \sup_{x, y \in \mathbb{R}: z=xy} [A(x) \wedge B(y)]$$

Multiplication of TFNs (continued)

As in case of addition (done in class), consider two cases:

a) $z < mn$, b) $z > mn$

a) Similarly to addition, we would consider such values of x and y for which memberships $A(x)$ and $B(y)$ are equal and to a constant ω , i.e.

$$A(x) = B(y) = \omega$$

ω can be expressed in terms of parameters/variables of fuzzy numbers A and B as follows

$$\frac{x - a}{m - a} = \omega, \quad \frac{y - c}{n - c} = \omega$$

Multiplication of TFNs (continued)

The above expressions can be rewritten as $x = a + (m - a)\omega$ and $y = c + (n - c)\omega$, respectively. So far, this results is the same as for addition. However, since z is now xy , it can be expressed as follows

$$z = ac + \omega c(m - a) + \omega a(n - c) + \omega^2(m - a)(n - c),$$

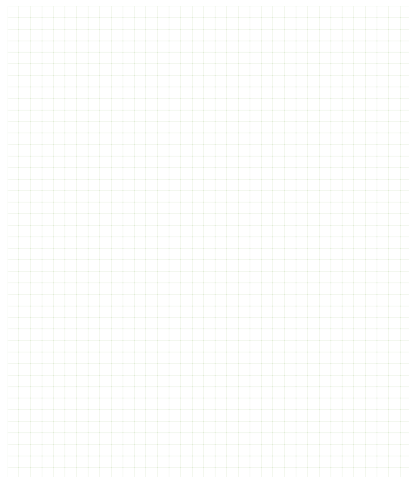
i.e. z is a quadratic function of ω , and thus the resulting FN is no longer triangular.

[Note: Derivation of expression for case b) is similar]

Measures of fuzziness

Which set is fuzzier? Degree of fuzziness is a measure of difficulty of ascertaining membership of elements of a fuzzy set:

- if the membership of an element is less than 0.5, the possibility of the element being outside of the fuzzy set is greater and hence it is easier to exclude it from the set
- if the membership of an element is greater than 0.5, the possibility of the element being inside the fuzzy set is greater and hence it is easier to include it in the set



Fuzziness: Simple energy index

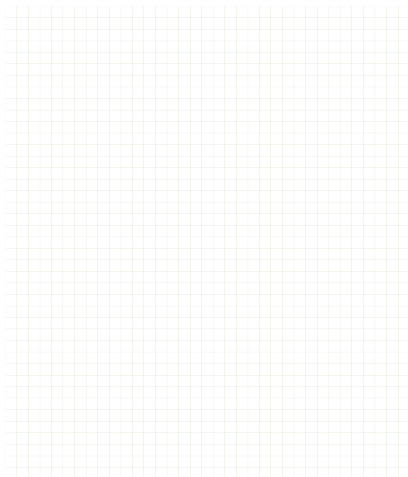
Area under the curve

$$F_0 = \int_{x \in \text{Supp}(A)} A(x) dx$$

Any problems here? Compare fuzziness of fuzzy set $A(x) = \text{trimf}(x, [0, 2, 4])$ and that of $B(x) = 1; x \in [1, 3]$?

$B(x)$ is a crisp interval nearest to $A(x)$, but this measure does not distinguish between them

$$F_0(A) = F_0(B)$$



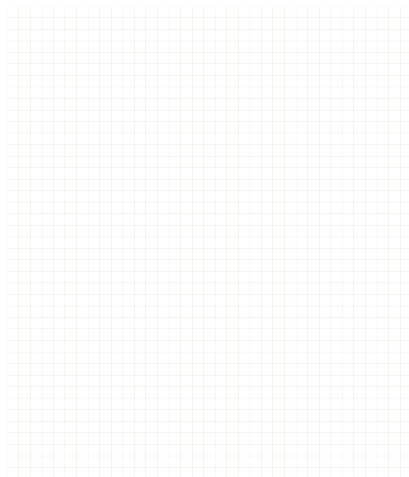
Fuzziness: Closeness to grade 0.5

The closeness to the most fuzzy grade

$$F_1 = \int_{x \in \text{Supp}(A)} f(x),$$

where the function $f(x)$ is defined in the support of the set A , such that $f(x) : \text{Supp}(A) \leftarrow [0, 0.5]$, i.e.

$$f(x) = \begin{cases} A(x) & \text{for } A(x) \leq 0.5, \\ 1 - A(x) & \text{otherwise.} \end{cases}$$



Fuzziness: Distance from 0.5-cut

Distance from 0.5-cut

$$F_2 = \int_{x \in \text{Supp}(A)} |A(x) - A_{0.5}(x)| dx,$$

The value of $A_{0.5}$ is 0 for x where $A(x) < 0.5$, and $A_{0.5}$ is 1 for x where $A(x) \geq 0.5$. In effect, $F_1 = F_2$.

Other measures have been proposed, e.g. quadratic energy index or inverse of distance from the complement.

