Intelligent Systems Engineering

FS-03 Operations on Fuzzy Sets

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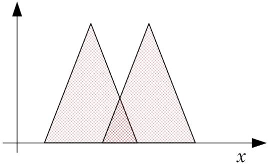
Outline

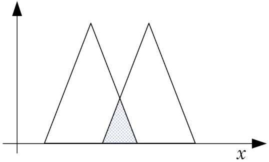
These notes are based on [Pedrycz, 2007] and [Keller et al. 2016]

Standard (Fuzzy) Set Operations

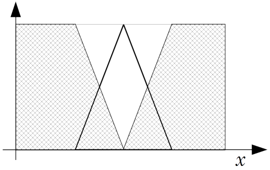
Union	$A \cup B$	$\max[A(x), B(x)]$



Intersection	$A \cap B$	$\min[A(x), B(x)]$



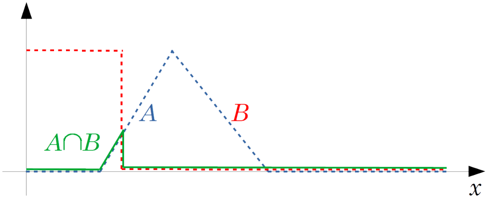
Complement	$\overline{A}$	$1 - A(x)$



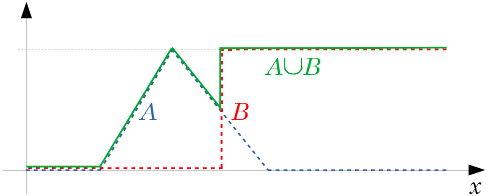
Problems of standard models

Although the entire range [0,1] is available for the values of membership, these operations provide no interaction among the variables, i.e.

Intersection: no matter how large the other variable is, the result of min operation is given by the smaller value alone



Union: no matter how small the other variable, the result of max operation is given by the larger value alone



Alternative definitions of FS operations

Triangular norms and co-norms

- Triangular norms (originally introduced in probabilistic metric spaces) are operations that satisfy reasonable axioms for the definition of intersection and union
- models of fuzzy set operations
  - Intersection: **t**-norms (triangular norm)
  - Union: **s**-norms (triangular co-norm)

Nomenclature:

In the following definitions, we use symbols w, x, y, z for short of membership of a specific point of a universe of dicourse X in a fuzzy set. For example, x may stand for $A(x)$, y for $B(x)$, etc.

Triangular norms and co-norms are represented by symbols **t** an **s**, respectively.

Axioms (intersection)

Commutativity:	$x \,\mathbf{t}\, y = y \,\mathbf{t}\, x$
Associativity:	$x \,\mathbf{t}\, (y \,\mathbf{t}\, z) = (x \,\mathbf{t}\, y) \,\mathbf{t}\, z$
Monotonicity:	if $x \leq y$ and $w \leq z$ then $x \,\mathbf{t}\, w \leq y \,\mathbf{t}\, z$
Boundary Conditions:	$0 \,\mathbf{t}\, x = 0$, $1 \,\mathbf{t}\, x = x$

Triangular Norms - models of intersection

Minimum t–norm

$$x \,\mathbf{t}\, y = \min(x, y)$$

$\cap$

Product t–norm

$$x \,\mathbf{t}\, y = xy$$

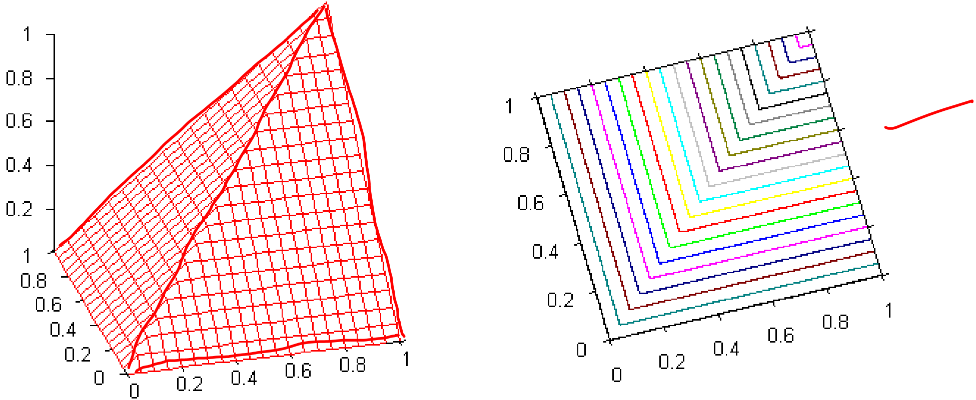
Lukasiewicz t–norm

$$x \,\mathbf{t}\, y = \max(x + y - 1, 0)$$

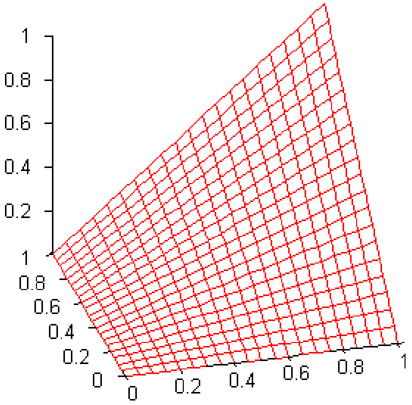
Drastic product t–norm

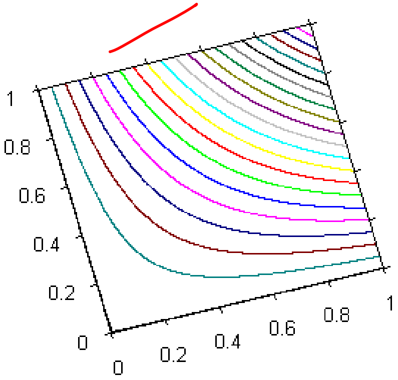
$$x \,\mathbf{t}\, y = \begin{cases} 0 & \text{if} \max(x, y) < 1, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Standard $\mathbf{t}$–norm $x\,\mathbf{t}\,y = \min(x,y)$

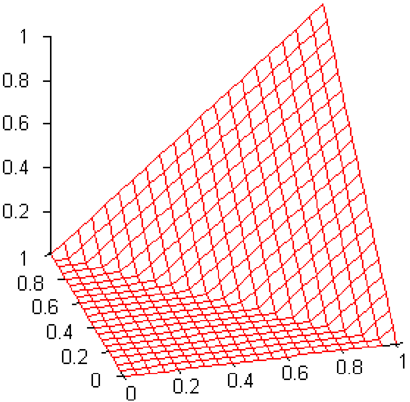


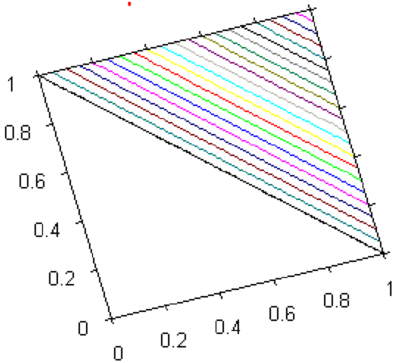
Product $\mathbf{t}$–norm $x \,\mathbf{t}\, y = xy$



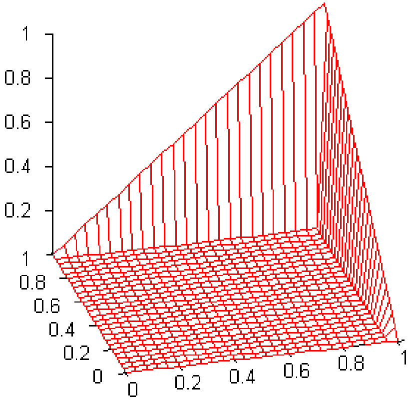


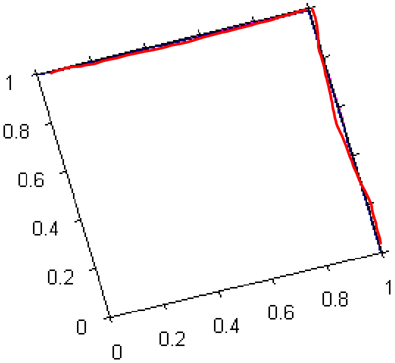
Lukasiewicz t−norm $x \, \mathbf{t} \, y = \max(x + y - 1, 0)$





Drastic t−norm





Axioms (union)

Commutativity:	$x\,\mathsf{s}\,y = y\,\mathsf{s}\,x$
Associativity:	$x\,\mathsf{s}\,(y\,\mathsf{s}\,z) = (x\,\mathsf{s}\,y)\,\mathsf{s}\,z$
Monotonicity:	if $x \leq y$ and $w \leq z$ then $x\,\mathsf{s}\,w \leq y\,\mathsf{s}\,z$
Boundary Conditions:	$0\,\mathsf{s}\,x = x$, $1\,\mathsf{s}\,x = 1$

Triangular Co-norms - models of union

Maximum s–norm

$$x \,\mathsf{s}\, y = \max(x, y)$$ $\cup$

Probabilistic sum s–norm

$$x \,\mathsf{s}\, y = x + y - xy$$ $[0,1]$

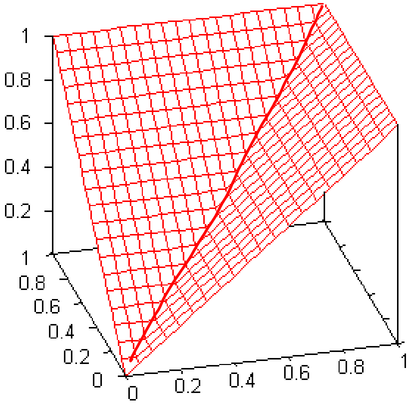
Bounded sum s–norm

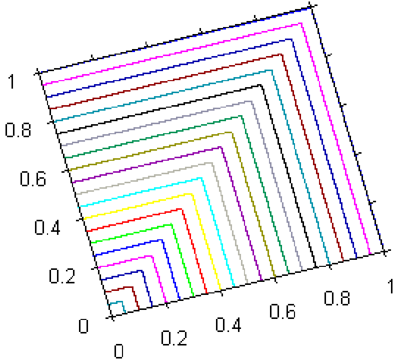
$$x \,\mathsf{s}\, y = \min(x + y, 1)$$

Drastic sum s–norm

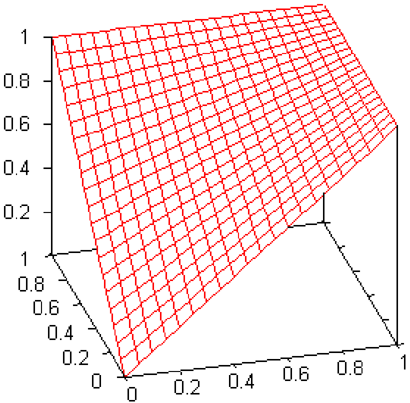
$$x \,\mathsf{s}\, y = \begin{cases} 1 & \text{if } \min(x, y) > 0, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

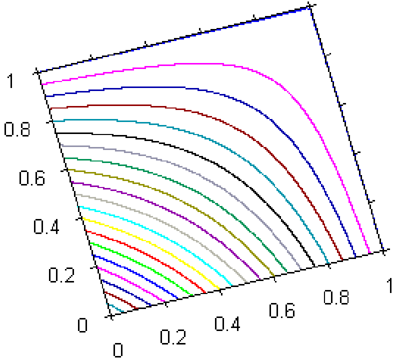
Standard **s**–norm $x \, \mathbf{s} \, y = \max(x, y)$



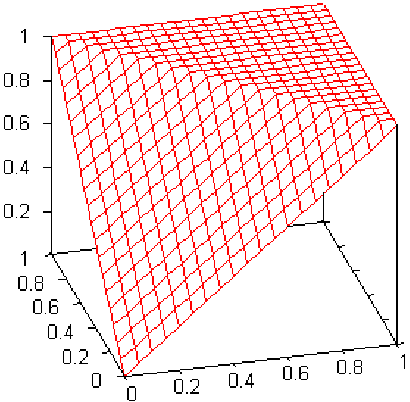


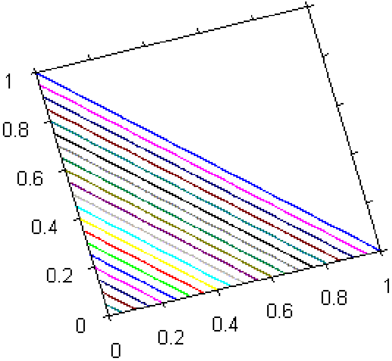
Probabilistic sum s–norm $x \, \mathbf{s} \, y = x + y - xy$



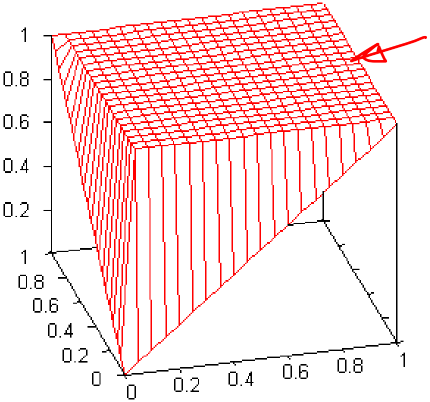


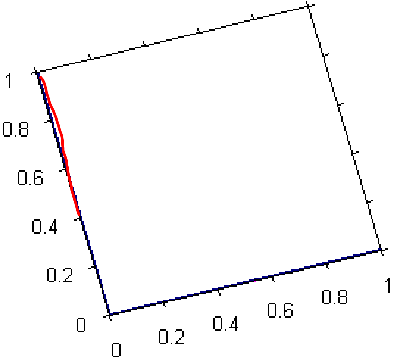
Bounded sum $\mathbf{s}$–norm $x \,\mathbf{s}\, y = \min(x+y, 1)$





Drastic s–norm





Triangular Norms and Co-norms

Cannot be linearly ordered.

However, there are bounds on their values:

$$\text{drastic product} \leq \mathsf{t} \leq \min$$

and

$$\max \leq \mathsf{s} \leq \text{drastic sum}$$

Triangular Norms and Co-norms: Duality

For each t−norm, there exists a dual s−norm.

Corresponding to DeMorgan's laws:

$$x \,\mathsf{s}\, y = 1 - (1 - x) \,\mathsf{t}\, (1 - y)$$

and

$$x \,\mathsf{t}\, y = 1 - (1 - x) \,\mathsf{s}\, (1 - y)$$

Axioms (Fuzzy Complement)

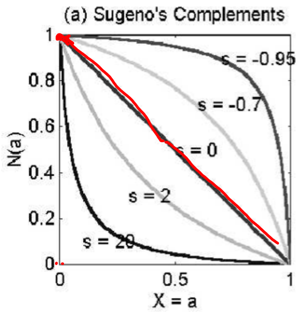
$N(a) = \bar{a} = 1 - a$

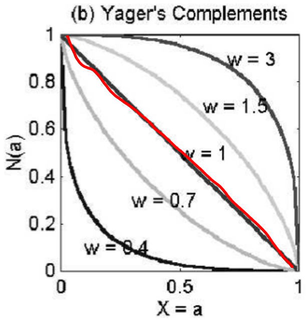
Monotonicity:	if $a < b$ then $\mathsf{N}(a) > \mathsf{N}(b)$
Involution:	$\mathsf{N}(\mathsf{N}(a)) = a$
Boundary Conditions:	$\mathsf{N}(0) = 1$ and $\mathsf{N}(1) = 0$

$[0, 1]$

Fuzzy Complements

Standard fuzzy complement	$N(a) = 1 - a$
Sugeno's fuzzy complement	$N(a) = (1-a)/(1+sa)$
Yager's fuzzy complement	$N(a) = (1-a^w)^{1/w}$





Comparison Operations on Fuzzy Sets

So far: set operations on fuzzy sets; i.e. how to combine two fuzzy sets and how to find a fuzzy set which complements another fuzzy set.
Now: comparison operations, to find how similar two fuzzy sets are; there are several possibilities to measure this:

- distance measures
- possibility measure
- necessity measure

$v_1 \in \mathbb{R}^n$

v_2

Distance Measures

A = {0.2 0.3 0.4}
B = {0.7 0.8 0}

In general, distance between two fuzzy sets can be measured using their membership functions

$$d(A, B) = \sqrt[p]{\int_X |A(x) - B(x)|^p dx}; \; p \geq 1$$

For different values of p, we obtain different measures, such as Hamming distance ($p = 1$), Euclidean distance ($p = 2$), Tchebyschev distance ($p = \infty$), etc.

Set based comparison operations

Calculation of distance involves two functions (membership functions A and B) – this measure therefore emphasizes functional aspect of fuzzy sets and not their set–based characteristic.

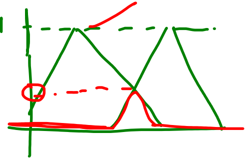
Set based comparison operations

- possibility
- necessity

alleviate this problem by taking into account set–based operations instead of functions.

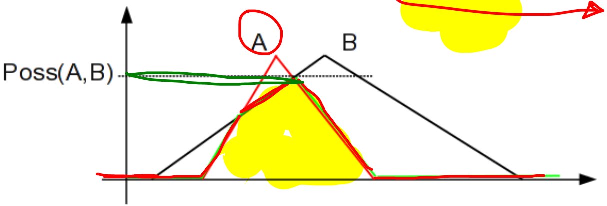
Possibility Measure

Possibility measure of fuzzy set A with respect to fuzzy set B is defined as



$$\mathrm{Poss}(A,B) = \sup_{x \in X} \left[\min \left(A(x), B(x) \right) \right]$$

$\cap$; intersection



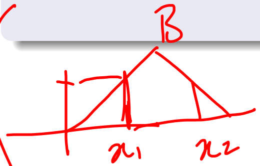
This measure quantifies the extent to which fuzzy sets A and B overlap.

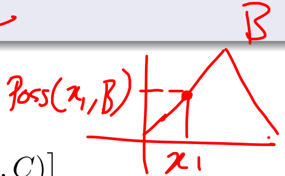
Possibility measure: main properties

Possibility

B(x1) ; B(x2

$$\text{Poss}(A,B) = \sup_{x \in X} \left[\min\left(A(x), B(x) \right) \right]$$





$$\text{Poss}(A,B) = \text{Poss}(B,A)$$

$$\text{Poss}(A \cup B, C) = \max\left[\text{Poss}(A,C), \text{Poss}(B,C) \right]$$

Fuzzy singleton → $\text{Poss}(\{x\},B) = B(x)$

POSS{x1,B)=B(x1)

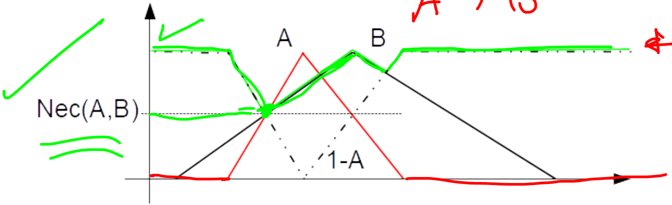
Necessity Measure

Necessity measure of fuzzy set A with respect to fuzzy set B is defined as

$$\mathrm{Nec}(A, B) = \inf_{x \in X} \left[\max \left(1 - A(x), B(x) \right) \right]$$

$\cup$

$\bar{A}$, B



$\bar{A}$

This measure quantifies the extent to which fuzzy sets A is included in fuzzy set B.

Necessity measure: main properties

Necessity

min

$$\mathrm{Nec}(A, B) = \inf_{x \in X} \left[\max \left(A(x), 1 - B(x) \right) \right]$$

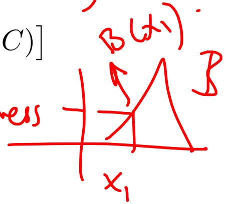
$\cup$

$\overline{A}$; $\overline{B}$

$$\mathrm{Nec}(A, B) \neq \mathrm{Nec}(B, A)$$

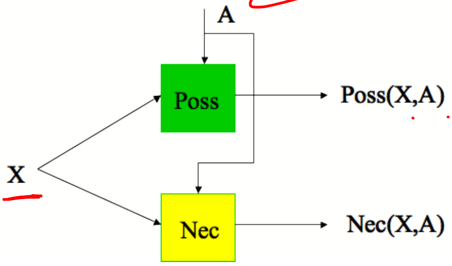
$$\mathrm{Nec}(A \cap B, C) = \min \left[\mathrm{Nec}(A, C), \mathrm{Nec}(B, C) \right]$$

singletons: $$\mathrm{Nec}(\{x\}, B) = B(x)$$

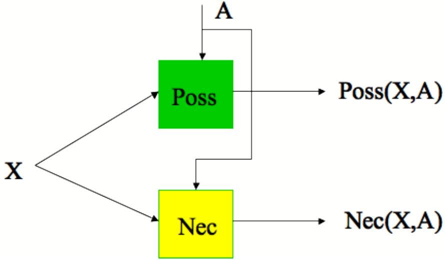


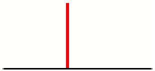
Possibility and Necessity: a matching interface

Fuzzy set.



Possibility and Necessity: a matching interface

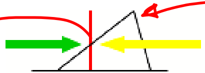




$\{x_0\}$

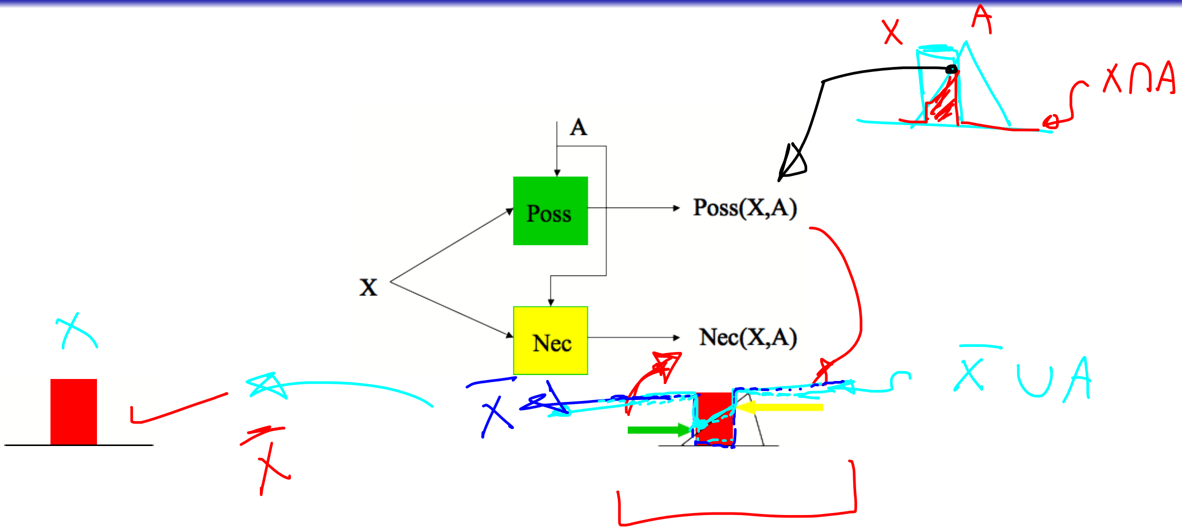
$A(x_0)$ ·

A



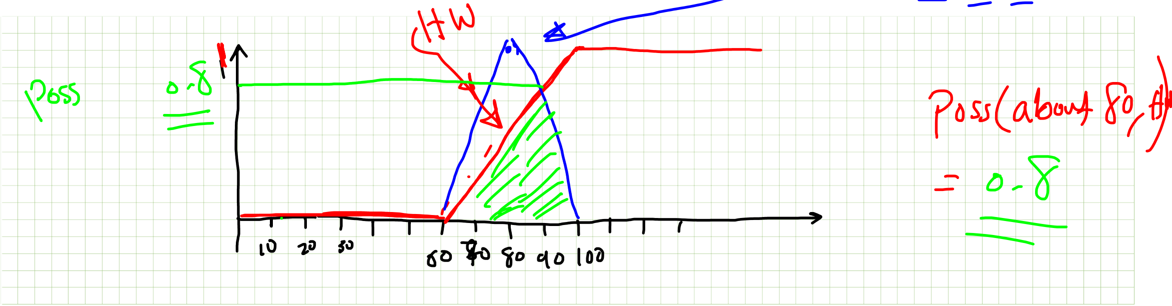
$\{x_0\}$

Possibility and Necessity: a matching interface



Application of Possibility Measure

Example: Fuzzy set B represents a concept of *high* speed on a HW with speed limit 100km/h; membership in this set is 0 for speeds below 60km/h, then it linearly increases and attains value 1 for speed 100km/h; speeds above the speed limit have constant membership equal to 1.
A car is moving at speed *about* 80km/h, defined by triangular MF $A\{\text{speed}, 60, 80, 100\}$.



To which degree is about 80km/h *high* speed?

Equality index

Consider two fuzzy sets A and B defined in the same finite space $X = \{x_1, x_2, \ldots, x_n\}$. Their equality ($\equiv$) can be assessed as follows:

$$A \equiv B : \ (A \subset B) \wedge (B \subset A)$$

min.

Inclusion ($\subset$) can be modelled with implication ($\rightarrow$)

$$A \equiv B = \frac{1}{n} \sum_{i=1}^{n} \min \left[\left(A(x_i) \rightarrow B(x_i)\right), \left(B(x_i) \rightarrow A(x_i)\right) \right]$$

Operation of Implication

In case of fuzzy sets $a, b, \in [0, 1]$:

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases}$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b/a & \text{if } a > b. \end{cases}$$