OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

# Fuzzy Systems

## Fuzzy Sets and Fuzzy Logic

**Prof. Dr. Rudolf Kruse** **Christoph Doell**

{kruse,doell}@ovgu.de

Otto-von-Guericke University of Magdeburg

Faculty of Computer Science

Institute of Intelligent Cooperating Systems

# Motivation

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Every day humans use imprecise linguistic terms
*e.g. big*, *fast*, *about 12 o'clock*, *old*, etc.

All complex human actions are decisions based on such concepts:

- driving and parking a car,
- financial/business decisions,
- law and justice,
- giving a lecture,
- listening to the professor/tutor.

So, these terms and the way they are processed play a crucial role.

Computers need a mathematical model to express and process such complex semantics.

Concepts in classical mathematics are inadequate for such models.

# Lotfi Asker Zadeh (1965)

Classes of objects in the real world do not have precisely defined criteria of membership.

Such imprecisely defined "classes" play an important role in human thinking,

Particularly in domains of pattern recognition, communication of information, and abstraction.

Zadeh in 2004 (born 1921)

# Imprecision

**Any notion is said to be imprecise when its *meaning* is not fixed by sharp boundaries.**

Can be applied fully/to certain degree/not at all.

Gradualness ("membership gradience") also called *fuzziness*.

A Proposition is imprecise if it contains gradual predicates.

Such propositions may be neither true nor false, but in-between.

They are true *to a certain degree*, *i.e.* partial truth.

Forms of such degrees can be found in natural language, *e.g. very*, *rather*, *almost not*, etc.

# Example – The Sorites Paradox

If a sand dune is small, adding one grain of sand to it leaves it small.
A sand dune with a single grain is small.

---

Hence all sand dunes are small.

Paradox comes from all-or-nothing treatment of *small*.

Degree of truth of “heap of sand is small” decreases by adding one grain after another.

Certain number of words refer to continuous numerical scales.

# Example – The Sorites Paradox

How many grains of sand has a sand dune at least?

Statement $A(n)$: "$n$ grains of sand are a sand dune."

Let $d_n = T(A(n))$ denote "degree of acceptance" for $A(n)$.

Then

$$0 = d_0 \leq d_1 \leq \ldots \leq d_n \leq \ldots \leq 1$$

can be seen as truth values of a **many valued logic**.

# Applications of Fuzzy Systems

**Control Engineering**

**Approximate Reasoning**

**Data Analysis**

**Image Analysis**

*Advantages:*

Use of imprecise or uncertain information

Use of expert knowledge

Robust nonlinear control

Time to market

Marketing aspects

# Washing Machines Use Fuzzy Logic


Source: `http://www.siemens-home.com/`

# Fuzzy Sets

# Membership Functions

**Lotfi A. Zadeh (1965)**
**“A fuzzy set is a class with a continuum of membership grades.”**

An imprecisely defined set $M$ can often be characterized by a *membership function* $\mu_M$.

$\mu_M$ associates real number in $[0, 1]$ with each element $x \in X$.

Value of $\mu_M$ at $x$ represents *grade of membership* of $x$ in $M$.

A Fuzzy set is defined as mapping

$$\mu : X \mapsto [0, 1].$$

Fuzzy sets $\mu_M$ generalize the notion of a characteristic function

$$\chi_M : X \mapsto \{0, 1\}.$$

# Membership Functions

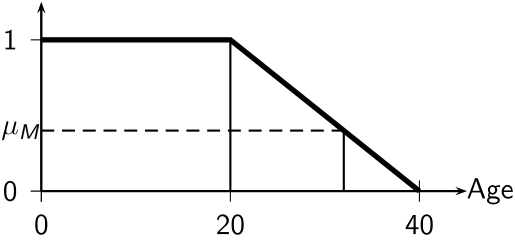
$\mu_M(u) = 1$ reflects full membership in $M$.

$\mu_M(u) = 0$ expresses absolute non-membership in $M$.

Sets can be viewed as special case of fuzzy sets where only full membership and absolute non-membership are allowed.

Such sets are called *crisp sets* or Boolean sets.

Membership degrees $0 < \mu_M < 1$ represent *partial membership*.




Representing *young* in "a young person"

# Membership Functions

A Membership function attached to a given linguistic description (such as *young*) depends on context:
A young retired person is certainly older than young student.
Even idea of young student depends on the user.

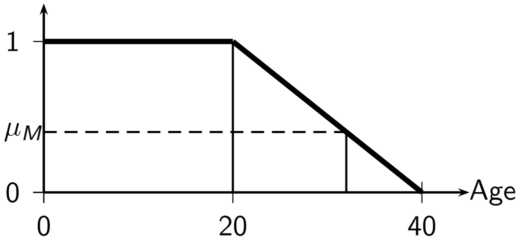
Membership degrees are fixed only *by convention*:
Unit interval as range of membership grades is arbitrary.
Natural for modeling membership grades of fuzzy sets of real numbers.

# Membership Functions

Consider again representation for predicate *young*

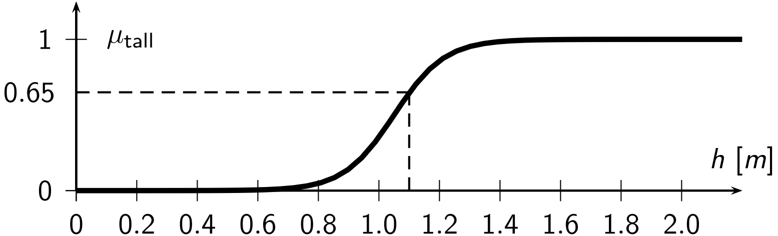
There is no precise threshold between
prototypes of *young* and
prototypes of *not young*.

Fuzzy sets offer natural interface between linguistic and numerical representations.




Representing *young* in “a young person”

# Example – Body Height of 4 Year Old Boys



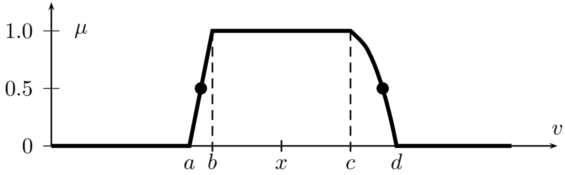

Membership function for $\mu_{\text{tall}}$

1.5 m is for sure tall, 0.7 m is for sure small, *but in-between?!*

Imprecise predicate *tall* modeled as sigmoid function,
*e.g.* height of 1.1 m has membership degree of 0.65.

So, height of 1.1 m satisfies predicate *tall* with 0.65.

# Example – Velocity of Rotating Hard Disk




Fuzzy set $\mu$ characterizing velocity of rotating hard disk.

Let $x$ be velocity $v$ of rotating hard disk in revolutions per minute.

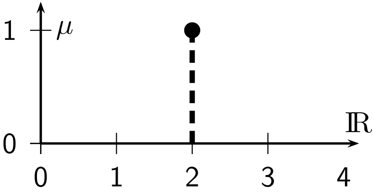
If no observations about $x$ available, use expert's knowledge:
"It's *impossible* that $v$ drops under $a$ or exceeds $d$.
"It's highly certain that any value between $[b, c]$ can occur."

Additionally, values of $v$ with membership degree of 0.5 are provided.

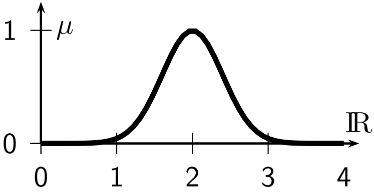
Interval $[a, d]$ is called *support* of the fuzzy set.

Interval $[b, c]$ is denoted as *core* of the fuzzy set.

# Examples for Fuzzy Numbers




exactly two




around two

Exact numerical value has membership degree of 1.

Left: monotonically increasing, right: monotonically decreasing, *i.e.* unimodal function.

Terms like *around* modeled using triangular or Gaussian function.

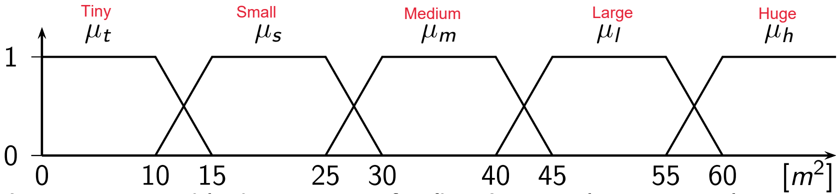
# Linguistic Variables and Linguistic Values

**Linguistic variables** represent attributes in fuzzy systems.

They are **partitioned** into **linguistic values** (not numerical!).

Partition is usually chosen subjectively (based on human intuition).

All linguistic values have a meaning, not a precise numerical value.




Linguistic variable *living area of a flat* $A$ stores linguistic values: *e.g. tiny*, *small*, *medium*, *large*, *huge*

Every $x \in A$ has $\mu(x) \in [0,1]$ to each value, *e.g.* let $a = 42.5m^2$.
So, $\mu_t(a) = \mu_s(a) = \mu_h(a) = 0, \;\; \mu_m(a) = \mu_l(a) = 0.5$.

# Support and Core of a Fuzzy Set

**Definition**
The *support* $S(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of $X$ that have nonzero membership. Formally

$$S(\mu) = [\mu]_{\underline{0}} = \{x \in X \mid \mu(x) > 0\}.$$

**Definition**
The *core* $C(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of $X$ that have membership of one. Formally,

$$C(\mu) = [\mu]_1 = \{x \in X \mid \mu(x) = 1\}.$$

# Height of a Fuzzy Set

**Definition**
The *height* $h(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the largest membership grade obtained by any element in that set. Formally,

$$h(\mu) = \sup_{x \in X} \{\mu(x)\}.$$

$h(\mu)$ may also be viewed as supremum of $\alpha$ for which $[\mu]_\alpha \neq \emptyset$.

**Definition**
A fuzzy set $\mu$ is called *normal, iff* $h(\mu) = 1$.
It is called *subnormal, iff* $h(\mu) < 1$.

# Semantics of Fuzzy Sets

## What membership grades may mean?

Fuzzy sets are relevant in three types of information-driven tasks:

classification and data analysis,

decision-making problems,

approximate reasoning.

These three tasks exploit three semantics of membership grades:

similarity

preference

possibility

# Degree of Similarity

Oldest interpretation of membership grades.

$\mu(u)$ is degree of proximity of $u$ from prototype elements of $\mu$.

Goes back to interests of fuzzy set concept in pattern classification.

Still used today for cluster analysis, regression, etc.
Here, proximity between pieces of information is modelled.

Also, in fuzzy control: similarity degrees are measured between current situation and prototypical ones.

# Degree of Preference

$\mu$ represents both:

- set of more or less preferred objects and
- values of a decision variable $X$.

$\mu(u)$ represents both:

- intensity of preference in favor of object $u$ and
- feasibility of selecting $u$ as value of $X$.

Fuzzy sets then represent criteria or flexible constraints.

This has been used in

- fuzzy optimization (especially fuzzy linear programming) and
- decision analysis.

Typical applications: engineering design and scheduling problems.

# Degree of Possibility

This interpretation was implicitly proposed by Zadeh when he introduced possibility theory and developed his theory of approximate reasoning.

$\mu(u)$ can be viewed as:

- degree of possibility that parameter $X$ has value $u$
- given the only information "$X$ is $\mu$".

Then support values are mutually exclusive and membership degrees rank these values by their possibility.

This view has been used in expert systems and artificial intelligence.

# Fuzzy Numbers

**Definition**

$\mu$ is a fuzzy number if and only if $\mu$ is normal and $[\mu]_\alpha$ is bounded, closed, and convex $\forall \alpha \in (0, 1]$.

**Example:**

The term *approximately* $x_0$ is often described by a parametrized class of membership functions, *e.g.*

$$\mu_1(x) = \max\{0,\ 1 - c_1|x - x_0|\}, \qquad c_1 > 0,$$
$$\mu_2(x) = \exp(-c_2\|x - x_0\|_p), \qquad c_2 > 0, \quad p \geq 1.$$

# Multi-valued Logics

# Set Operators. . .

**. . . are defined by using traditional logics operator**

Let $X$ be universe of discourse (universal set):

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$$
$$A \cup B = \{x \in X \mid x \in A \vee x \in B\}$$
$$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg(x \in A)\}$$

$$A \subseteq B \text{ if and only if } (x \in A) \to (x \in B) \text{ for all } x \in X$$

One idea to define fuzzy set operators: use fuzzy logics.

# The Traditional or Aristotlelian Logic

## What is logic about? Different schools speak different languages!

There are raditional, linguistic, psychological, epistemological and mathematical schools.

Traditional logic has been founded by Aristotle (384-322 B.C.).

Aristotlelian logic can be seen as formal approach to human reasoning.

It's still used today in Artificial Intelligence for knowledge representation and reasoning about knowledge.

Detail of "The School of Athens" by R. Sanzio (1509) showing Plato (left) and his student Aristotle (right).

# Classical Logic: An Overview

Logic studies methods/principles of **reasoning**.

Classical logic deals with **propositions** (either *true* or *false*).

The *propositional logic* handles combination of *logical variables*.

Key idea: how to express $n$-ary logic functions with **logic primitives**, *e.g.* $\neg, \wedge, \vee, \rightarrow$.

A set of logic primitives is **complete** if any logic function can be composed by a finite number of these primitives,
*e.g.* $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$, $\{\downarrow\}$ (NOR), $\{|\}$ (NAND)
(this was also discussed during the 1st exercise).

# Inference Rules

When a variable represented by logical formula is:
*true* for all possible truth values, *i.e.* it is called **tautology**,
*false* for all possible truth values, *i.e.* it is called **contradiction**.

Various forms of tautologies exist to perform **deductive inference**

They are called **inference rules**:

$$(a \land (a \to b)) \to b \qquad \textit{(modus ponens)}$$
$$(\neg b \land (a \to b)) \to \neg a \qquad \textit{(modus tollens)}$$
$$((a \to b) \land (b \to c)) \to (a \to c) \qquad \textit{(hypothetical syllogism)}$$

*e.g.* modus ponens: given two true propositions $a$ and $a \to b$ (*premises*), truth of proposition $b$ (*conclusion*) can be inferred.

Every tautology remains a tautology when any of its variables is replaced with an arbitrary logic formula.

# The Basic Principle of Classical Logic

*The Principle of Bivalence:*

"Every proposition is either true or false."

It has been formally developed by Tarski.

Alfred Tarski (1902-1983)

Łukasiewicz suggested to replace it by
*The Principle of Valence:*

"Every proposition has a truth value."

Propositions can have intermediate truth value, expressed by a number from the unit interval $[0, 1]$.

Jan Łukasiewicz (1878-1956)

# Three-valued Logics

A 2-valued logic can be extended to a 3-valued logic *in several ways*, *i.e.* different three-valued logics have been well established:

truth, falsity, indeterminacy are denoted by 1, 0, and $1/2$, resp.

The negation $\neg a$ is defined as $1 - a$, *i.e.* $\neg 1 = 0, \neg 0 = 1$ and $\neg 1/2 = 1/2$.

Other primitives, *e.g.* $\wedge, \vee, \rightarrow, \leftrightarrow$, differ from logic to logic.

Five well-known three-valued logics (named after their originators) are defined in the following.

# Primitives of Some Three-valued Logics

| | | Łukasiewicz | | | | Bochvar | | | | Kleene | | | | Heyting | | | | Reichenbach | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | b | $\wedge$ | $\vee$ | $\rightarrow$ | $\leftrightarrow$ | $\wedge$ | $\vee$ | $\rightarrow$ | $\leftrightarrow$ | $\wedge$ | $\vee$ | $\rightarrow$ | $\leftrightarrow$ | $\wedge$ | $\vee$ | $\rightarrow$ | $\leftrightarrow$ | $\wedge$ | $\vee$ | $\rightarrow$ | $\leftrightarrow$ |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| 0 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | 0 | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ |
| 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

All of them fully conform the usual definitions for $a, b \in \{0, 1\}$.

They differ from each other only in their treatment of $1/2$.

**Question:** Do they satisfy the law of contradiction $(a \wedge \neg a = 0)$ and the law of excluded middle $(a \vee \neg a = 1)$?

# $n$-valued Logics

After the three-valued logics: generalizations to $n$-valued logics for arbitrary number of truth values $n \geq 2$.

In the 1930s, various $n$-valued logics were developed.

Usually truth values are assigned by rational number in $[0, 1]$.

Key idea: uniformly divide $[0, 1]$ into $n$ truth values.

**Definition**
The set $T_n$ of truth values of an $n$-valued logic is defined as

$$T_n = \left\{0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\right\}.$$

These values can be interpreted as degree of truth.

# Primitives in $n$-valued Logics

Łukasiewicz proposed first series of $n$-valued logics for $n \geq 2$.

In the early 1930s, he simply generalized his three-valued logic.

It uses truth values in $T_n$ and defines primitives as follows:

$$\begin{aligned} \neg a &= 1 - a \\ a \wedge b &= \min(a, b) \qquad\qquad & a \rightarrow b &= \min(1,\ 1 + b - a) \\ a \vee b &= \max(a, b) & a \leftrightarrow b &= 1 - |a - b| \end{aligned}$$

The $n$-valued logic of Łukasiewicz is denoted by $L_n$.

The sequence $(L_2, L_3, \ldots, L_\infty)$ contains the classical two-valued logic $L_2$ and an infinite-valued logic $L_\infty$ (rational **countable** values $T_\infty$).

The infinite-valued logic $L_1$ (**standard Łukasiewicz logic**) is the logic with all real numbers in $[0, 1]$ ($1 =$ cardinality of continuum).

# From Logic to Fuzzy Logic

# Zadeh's fuzzy logic proposal was much simpler

In 1965, he proposed a logic with values in $[0, 1]$:

$$\begin{aligned} \neg a &= 1 - a, \\ a \wedge b &= \min(a, b), \\ a \vee b &= \max(a, b). \end{aligned}$$

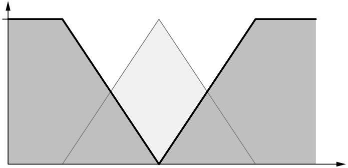
The set operators are defined pointwise as follows for $\mu, \mu'$:

$$\begin{aligned} \neg\mu : X \to X, \neg\mu(x) &= 1 - \mu(x), \\ \mu \wedge \mu' : X \to X (\mu \wedge \mu')(x) &= \min\{\mu(x), \mu'(x)\}, \\ \mu \vee \mu' : X \to X (\mu \vee \mu')(x) &= \max\{\mu(x), \mu'(x)\}. \end{aligned}$$

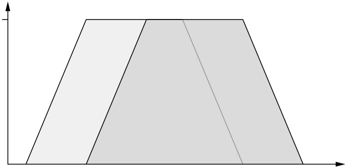
Zadeh in 2004

(born 1921)

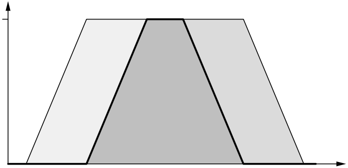
# Standard Fuzzy Set Operators – Example



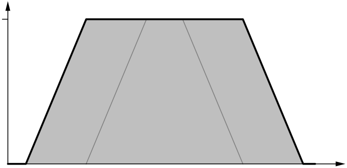
fuzzy complement



two fuzzy sets



fuzzy intersection



fuzzy union

# Fuzzy Set Theory

STOP

**Definition**
Let $X \neq \emptyset$ be a set.

$2^X \stackrel{\text{def}}{=} \{A \mid A \subseteq X\}$ power set of $X$,

$A \in 2^X$, $\quad \chi_A : X \to \{0, 1\}$ characteristic function,

$\mathcal{X}(X) \stackrel{\text{def}}{=} \{\chi_A \mid A \in 2^X\}$ set of characteristic functions.

**Theorem**

$(2^X, \cap, \cup, ^c)$ *is Boolean algebra,*

$\phi : 2^X \to \mathcal{X}(X), \quad \phi(A) \stackrel{\text{def}}{=} \chi_A$ *is bijection.*

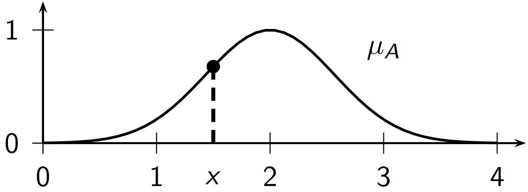
**Theorem**
$(\mathcal{X}(X), \wedge, \vee, \neg)$ *is Boolean algebra where*

$$\chi_{A \wedge B} \stackrel{\text{def}}{=} \min\{\chi_A, \chi_B\}, \quad \chi_{A \vee B} \stackrel{\text{def}}{=} \max\{\chi_A, \chi_B\}, \quad \chi_{\neg A} \stackrel{\text{def}}{=} 1 - \chi_A.$$

# What does a fuzzy set represent?

Consider fuzzy proposition $A$ (“approximately two”) on $\mathbb{R}$

fuzzy logic offers means to construct such imprecise sentences




$A$ defined by membership function $\mu_A$, *i.e.* truth values $\forall x \in \mathbb{R}$

let $x \in \mathbb{R}$ be a subject/observation

$\mu_A(x)$ is the degree of truth that $x$ is $A$

# Standard Fuzzy Set Operators

**Definition**

We define the following algebraic operators on $\mathcal{F}(X)$:

$$(\mu \wedge \mu')(x) \stackrel{\text{def}}{=} \min\{\mu(x), \mu'(x)\} \qquad \text{intersection ("AND")},$$

$$(\mu \vee \mu')(x) \stackrel{\text{def}}{=} \max\{\mu(x), \mu'(x)\} \qquad \text{union ("OR")},$$

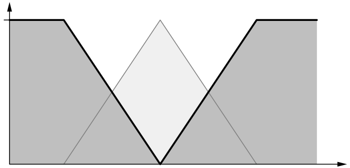
$$\neg\mu(x) \stackrel{\text{def}}{=} 1 - \mu(x) \qquad \text{complement ("NOT")}.$$

$\mu$ is subset of $\mu'$ if and only if $\mu \leq \mu'$.

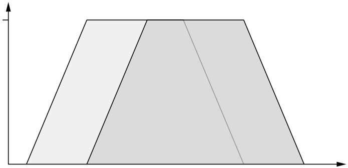
**Theorem**

$(\mathcal{F}(X), \wedge, \vee, \neg)$ *is a complete distributive lattice but no boolean algebra.*

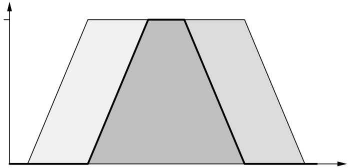
# Standard Fuzzy Set Operators – Example



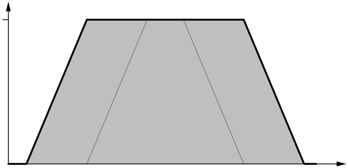
fuzzy complement



two fuzzy sets



fuzzy intersection



fuzzy union

# Fuzzy Set Complement

# Fuzzy Complement/Fuzzy Negation

## Definition

Let $X$ be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) :=\sim (\mu(x))$ where $\sim : [0,1] \to [0,1]$ satisfies the conditions

$$\sim (0) = 1, \quad \sim (1) = 0$$

and

$$\text{for } x, y \in [0,1],\ x \leq y \implies \sim x \geq \sim y \quad (\sim \text{ is non-increasing}).$$

Abbreviation: $\sim x :=\sim (x)$

# Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1], \; x < y \Longrightarrow \sim x > \sim y$ ($\sim$ is strictly decreasing)
- $\sim$ is continuous
- $\sim \sim x = x$ for all $x \in [0, 1]$ ($\sim$ is involutive)

According to conditions, two subclasses of negations are defined:

**Definition**
A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive, too.

$\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

# Families of Negations

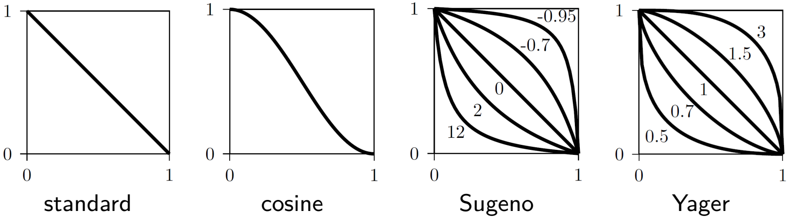
standard negation: $\sim x = 1 - x$

threshold negation: $\sim_\theta (x) = \begin{cases} 1 & \text{if } x \le \theta \\ 0 & \text{otherwise} \end{cases}$

Cosine negation: $\sim x = \frac{1}{2}(1 + \cos(\pi x))$

Sugeno negation: $\sim_\lambda (x) = \frac{1-x}{1+\lambda x}, \quad \lambda > -1$

Yager negation: $\sim_\lambda (x) = (1 - x^\lambda)^{\frac{1}{\lambda}}$



# Two Extreme Negations

$$\textit{intuitionistic}\text{ negation } \sim_i(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\textit{dual intuitionistic}\text{ negation } \sim_{di}(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Both negations are not strictly increasing, not continuous, not involutive

Thus they are neither strict nor strong

They are “optimal” since their notions are nearest to crisp negation

$\sim_i$ and $\sim_{di}$ are two extreme cases of negations

For any negation $\sim$ the following holds

$$\sim_i \leq \sim \leq \sim_{di}$$

# Inverse of a Strict Negation

Any strict negation $\sim$ is strictly decreasing and continuous.

Hence one can define its inverse $\sim^{-1}$.

$\sim^{-1}$ is also strict but in general differs from $\sim$.

$\sim^{-1} = \sim$ if and only if $\sim$ is involutive.

Every strict negation $\sim$ has a unique value $0 < s_\sim < 1$ such that $\sim s_\sim = s_\sim$.

$s_\sim$ is called *membership crossover point*.

$A(a) > s_\sim$ if and only if $A^c(a) < s_\sim$ where $A^c$ is defined via $\sim$.

$\sim^{-1}(s_\sim) = s_\sim$ always holds as well.

# Representation of Negations

**Any strong negation can be obtained from standard negation.**

Let $a, b \in \mathbb{R},\ a \leq b$.

Let $\varphi : [a, b] \to [a, b]$ be continuous and strictly increasing.

$\varphi$ is called *automorphism* of the interval $[a, b] \subset \mathbb{R}$.

**Theorem**

*A function* $\sim : [0, 1] \to [0, 1]$ *is a strong negation if and only if there exists an automorphism* $\varphi$ *of the unit interval such that for all* $x \in [0, 1]$ *the following holds*

$$\sim_\varphi (x) = \varphi^{-1}(1 - \varphi(x)).$$

$\sim_\varphi (x) = \varphi^{-1}(1 - \varphi(x))$ is called $\varphi$-transform of the standard negation.

# Fuzzy Set Intersection and Union

# Classical Intersection and Union

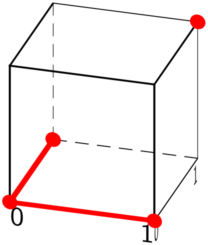
Classical set intersection represents logical conjunction.

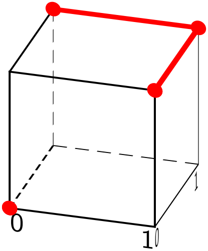
Classical set union represents logical disjunction.

Generalization from $\{0, 1\}$ to $[0, 1]$ as follows:

| $x \wedge y$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $x \vee y$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |



# Fuzzy Set Intersection and Union

Let $A, B$ be fuzzy subsets of $X$, *i.e.* $A, B \in \mathcal{F}(X)$.

Their **intersection** and **union** can be defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \quad \text{where} \quad \top : [0,1]^2 \to [0,1]$$
$$(A \cup B)(x) = \bot(A(x), B(x)) \quad \text{where} \quad \bot : [0,1]^2 \to [0,1].$$

# Triangular Norms and Conorms I

$\top$ is a *triangular norm (t-norm)* $\iff$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\iff$ $\bot$ satisfies C1-C4

for all $x, y \in [0, 1]$, the following laws hold

**Identity Law**

**T1:** $\top(x, 1) = x \quad (A \cap X = A)$
**C1:** $\bot(x, 0) = x \quad (A \cup \emptyset = A)$.

**Commutativity**

**T2:** $\top(x, y) = \top(y, x) \quad (A \cap B = B \cap A)$,
**C2:** $\bot(x, y) = \bot(y, x) \quad (A \cup B = B \cup A)$.

# Triangular Norms and Conorms II

for all $x, y, z \in [0, 1]$, the following laws hold

## Associativity

**T3:** $\top(x, \top(y, z)) = \top(\top(x, y), z) \quad (A \cap (B \cap C)) = ((A \cap B) \cap C)$,
**C3:** $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z) \quad (A \cup (B \cup C)) = ((A \cup B) \cup C)$.

## Monotonicity

$y \leq z$ implies
**T4:** $\top(x, y) \leq \top(x, z)$
**C4:** $\bot(x, y) \leq \bot(x, z)$.

# Triangular Norms and Conorms III

$\top$ is a *triangular norm (t-norm)* $\Longleftrightarrow$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\Longleftrightarrow$ $\bot$ satisfies C1-C4

Both identity law and monotonicity respectively imply

$$\begin{aligned} \forall x \in [0,1] &: \top(0,x) = 0, \\ \forall x \in [0,1] &: \bot(1,x) = 1, \\ \text{for any } t\text{-norm } \top &: \top(x,y) \leq \min(x,y), \\ \text{for any } t\text{-conorm } \bot &: \bot(x,y) \geq \max(x,y). \end{aligned}$$

note: $x = 1 \Rightarrow T(0,1) = 0$ and
$x \leq 1 \Rightarrow T(x,0) \leq T(1,0) = T(0,1) = 0$

# De Morgan Triplet I

For every $\top$ and strong negation $\sim$, one can define *t*-conorm $\perp$ by

$$\perp(x, y) = \sim \top(\sim x, \sim y), \quad x, y \in [0, 1].$$

Additionally, in this case $\top(x, y) = \sim \perp(\sim x, \sim y), \; x, y \in [0, 1]$.

$\perp, \top$ are called *N-dual t-conorm* and *N-dual t-norm* to $\top, \perp$, resp.

In case of the standard negation $\sim x = 1 - x$ for $x \in [0, 1]$,
N-dual $\perp$ and $\top$ are called *dual t-conorm* and *dual t-norm*, resp.

$\perp(x, y) = \sim \top(\sim x, \sim y)$ expresses "fuzzy" De Morgan's law.

note: De Morgan's laws $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

# De Morgan Triplet II

**Definition**

The triplet $(\top, \perp, \sim)$ is called *De Morgan triplet* if and only if

$\top$ is $t$-norm, $\perp$ is $t$-conorm, $\sim$ is strong negation,

$\top, \perp$ and $\sim$ satisfy $\perp(x, y) = \sim \top(\sim x, \sim y)$.

In the following, some important De Morgan triplets will be shown,

only the most frequently used and important ones.

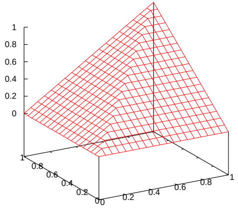
In all cases, the standard negation $\sim x = 1 - x$ is considered.

# The Minimum and Maximum I

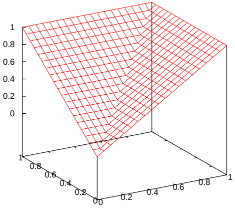
$\top_{\min}(x, y) = \min(x, y), \quad \perp_{\max}(x, y) = \max(x, y)$

Minimum is the greatest $t$-norm and max is the weakest $t$-conorm.

$\top(x, y) \leq \min(x, y)$ and $\perp(x, y) \geq \max(x, y)$ for any $\top$ and $\perp$




$\top_{\min}$

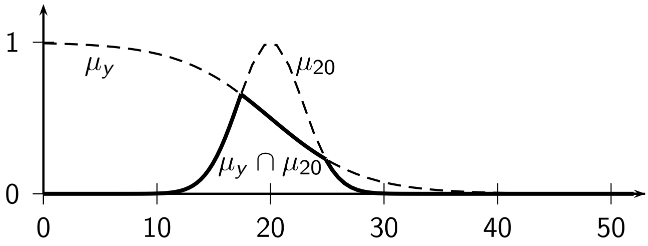



$\perp_{\max}$

# The Minimum and Maximum II

$\top_{\min}$ and $\bot_{\max}$ can be easily processed numerically and visually, *e.g.* linguistic values *young* and *approx. 20* described by $\mu_y$, $\mu_{20}$.

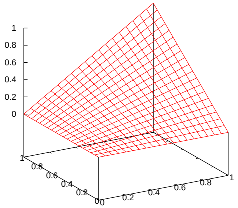
$\top_{\min}(\mu_y, \mu_{20})$ is shown below.



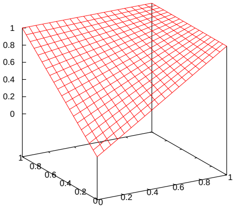
# The Product and Probabilistic Sum

$$\top_{\text{prod}}(x, y) = x \cdot y, \quad \bot_{\text{sum}}(x, y) = x + y - x \cdot y$$

Note that use of product and its dual has nothing to do with probability theory.




$\top_{\text{prod}}$

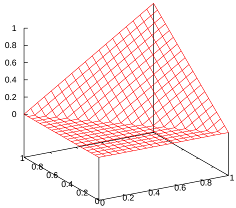



$\bot_{\text{sum}}$

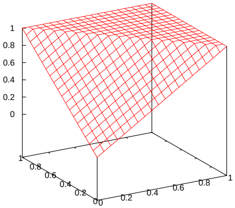
# The Łukasiewicz $t$-norm and $t$-conorm

$$\top_{\text{Łuka}}(x, y) = \max\{0,\ x + y - 1\}, \quad \bot_{\text{Łuka}}(x, y) = \min\{1,\ x + y\}$$

$\top_{\text{Łuka}}$, $\bot_{\text{Łuka}}$ are also called *bold intersection* and *bounded sum*.




$\top_{\text{Łuka}}$



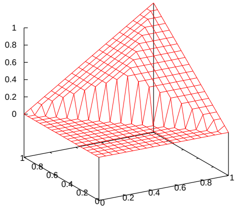

$\bot_{\text{Łuka}}$

# The Nilpotent Minimum and Maximum

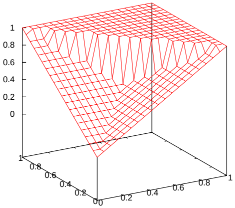
$$\top_{\min_0}(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\bot_{\max_1}(x, y) = \begin{cases} \max(x, y) & \text{if } x + y < 1 \\ 1 & \text{otherwise} \end{cases}$$

Found in 1992 and independently rediscovered in 1995.




$\top_{\min_0}$




$\bot_{\max_1}$

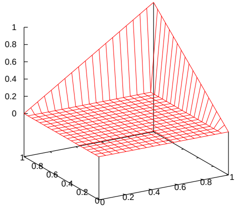
# The Drastic Product and Sum

$$\top_{-1}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

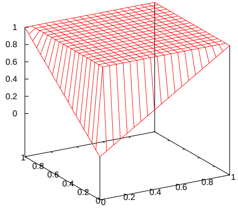
$$\perp_{-1}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$\top_{-1}$ is the weakest $t$-norm, $\perp_{-1}$ is the strongest $t$-conorm.

$\top_{-1} \leq \top \leq \top_{\min}, \quad \perp_{\max} \leq \perp \leq \perp_{-1}$ for any $\top$ and $\perp$

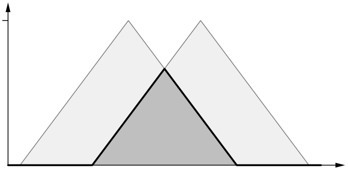



$\top_{-1}$

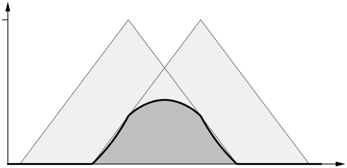



$\perp_{-1}$

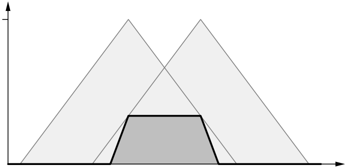
# Examples of Fuzzy Intersections



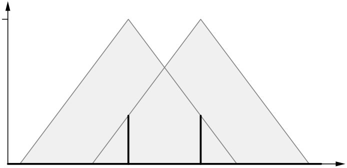
$t$-norm $\top_{\min}$



$t$-norm $\top_{\mathrm{prod}}$



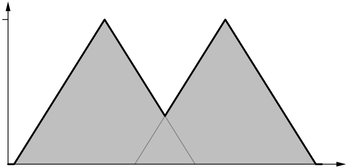
$t$-norm $\top_{\mathrm{Łuka}}$



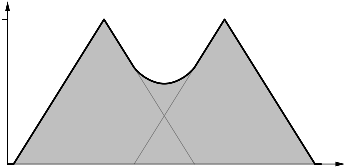
$t$-norm $\top_{-1}$

Note that all fuzzy intersections are contained within upper left graph and lower right one.

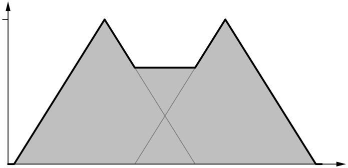
# Examples of Fuzzy Unions



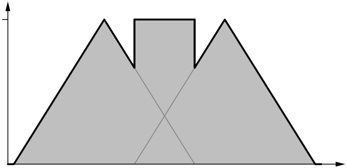
$t$-conorm $\perp_{\max}$



$t$-conorm $\perp_{\text{sum}}$



$t$-conorm $\perp_{\text{Łuka}}$



$t$-conorm $\perp_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.

# The Special Role of Minimum and Maximum I

$\top_{\min}$ and $\bot_{\max}$ play key role for intersection and union, resp.

In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0, 1]$:

**Distributivity**

$$\bot_{\max}(x, \top_{\min}(y, z)) = \top_{\min}(\bot_{\max}(x, y), \bot_{\max}(x, z)),$$
$$\top_{\min}(x, \bot_{\max}(y, z)) = \bot_{\max}(\top_{\min}(x, y), \top_{\min}(x, z))$$

**Continuity**

$\top_{\min}$ and $\bot_{\max}$ are continuous.

# The Special Role of Minimum and Maximum II

## Strict monotonicity on the diagonal

$x < y$ implies $\top_{\min}(x, x) < \top_{\min}(y, y)$ and $\bot_{\max}(x, x) < \bot_{\max}(y, y)$.

## Idempotency

$$\top_{\min}(x, x) = x, \quad \bot_{\max}(x, x) = x$$

## Absorption

$$\top_{\min}(x, \bot_{\max}(x, y)) = x, \quad \bot_{\max}(x, \top_{\min}(x, y)) = x$$

## Non-compensation

$x < y < z$ imply $\top_{\min}(x, z) \neq \top_{\min}(y, y)$ and $\bot_{\max}(x, z) \neq \bot_{\max}(y, y)$.

# The Special Role of Minimum and Maximum III

**Is $(\mathcal{F}(X), \top_{\text{min}}, \bot_{\text{max}}, \sim)$ a boolean algebra?**

Consider the properties (B1)-(B9) of any Boolean algebra.

For $(\mathcal{F}(X), \top_{\text{min}}, \bot_{\text{max}}, \sim)$ with strong negation $\sim$
only complementary (B7) does not hold.

Hence $(\mathcal{F}(X), \top_{\text{min}}, \bot_{\text{max}}, \sim)$ is a *completely distributive lattice* with identity element $\mu_X$ and zero element $\mu_\emptyset$.

**No lattice** $(\mathcal{F}(X), \top, \bot, \sim)$ forms a Boolean algebra

due to the fact that complementary (B7) does not hold:

- There is no complement/negation $\sim$ with $\top(A, \sim A) = \mu_\emptyset$.
- There is no complement/negation $\sim$ with $\bot(A, \sim A) = \mu_X$.

# Complementary Property of Fuzzy Sets

Using fuzzy sets, it's **impossible** to keep up a Boolean algebra.

Verify, *e.g.* that law of contradiction is violated, *i.e.*

$$(\exists x \in X)(A \cap A^c)(x) \neq \emptyset.$$

We use $\min, \max$ and strong negation $\sim$ as fuzzy set operators.

So we need to show that

$$\min\{A(x), 1 - A(x)\} = 0$$

is violated for at least one $x \in X$.

easy: This Equation is violated for all $A(x) \in (0, 1)$.

It is satisfied only for $A(x) \in \{0, 1\}$.

# The concept of a pseudoinverse

## Definition

Let $f : [a, b] \to [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to $f$ is the function $f^{(-1)} : [c, d] \to [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$

# Continuous Archimedean $t$-norms and $t$-conorms

broad class of problems relates to representation of multi-place functions by composition of a "simpler" function, *e.g.*

$$K(x, y) = f^{(-1)}(f(x) + f(y))$$

So, one should consider suitable subclass of all $t$-norms.

**Definition**

A $t$-norm $\top$ is

(a) *continuous* if $\top$ as function is continuous on unit interval,

(b) *Archimedean* if $\top$ is continuous and $\top(x, x) < x$ for all $x \in ]0, 1[$.

**Definition**

A $t$-conorm $\perp$ is

(a) *continuous* if $\perp$ as function is continuous on unit interval,

(b) *Archimedean* if $\perp$ is continuous and $\perp(x, x) > x$ for all $x \in ]0, 1[$.

# Continuous Archimedean $t$-norms

**Theorem**

*A t-norm $\top$ is continuous and Archimedean if and only if there exists a strictly decreasing and continuous function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ such that*

$$\top(x,y) = f^{(-1)}(f(x) + f(y)) \tag{1}$$

*where*

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \textit{if } x \leq f(0) \\ 0 & \textit{otherwise} \end{cases}$$

*is the pseudoinverse of $f$. Moreover, this representation is unique up to a positive multiplicative constant.*

$\top$ *is generated by $f$* if $\top$ has representation (1).

$f$ is called *additive generator* of $\top$.

# Additive Generators of $t$-norms – Examples

**Find an additive generator $f$ of $\top_{\text{Łuka}}(x, y) = \max\{x + y - 1,\ 0\}$.**

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f_{\text{Łuka}}^{(-1)}(x) = \max\{1 - x,\ 0\}$

thus $\top_{\text{Łuka}}(x, y) = f_{\text{Łuka}}^{(-1)}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

**Find an additive generator $f$ of $\top_{\text{prod}}(x, y) = x \cdot y$.**

to be discussed in the exercise

hint: use of logarithmic and exponential function

# Continuous Archimedean $t$-conorms

**Theorem**

*A t-conorm $\bot$ is continuous and Archimedean if and only if there exists a strictly increasing and continuous function $g : [0,1] \to [0,\infty]$ with $g(0) = 0$ such that*

$$\bot(x,y) = g^{(-1)}(g(x) + g(y)) \tag{2}$$

*where*

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

*is the pseudoinverse of $g$. Moreover, this representation is unique up to a positive multiplicative constant.*

$\bot$ *is generated by $g$* if $\bot$ has representation (2).

$g$ is called *additive generator* of $\bot$.

# Additive Generators of $t$-conorms – Two Examples

**Find an additive generator $g$ of $\perp_{\text{Łuka}}(x, y) = \min\{x + y,\ 1\}$.**

for instance $g_{\text{Łuka}}(x) = x$

then, $g_{\text{Łuka}}^{(-1)}(x) = \min\{x,\ 1\}$

thus $\perp_{\text{Łuka}}(x, y) = g_{\text{Łuka}}^{(-1)}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y))$

**Find an additive generator $g$ of $\perp_{\text{sum}}(x, y) = x + y - x \cdot y$.**

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

# Hamacher Family I

$$\top_\alpha(x,y) = \frac{x \cdot y}{\alpha + (1-\alpha)(x+y+x\cdot y)}, \quad \alpha \geq 0,$$
$$\bot_\beta(x,y) = \frac{x+y+(\beta-1)\cdot x \cdot y}{1+\beta \cdot x \cdot y}, \quad \beta \geq -1,$$
$$\sim_\gamma(x) = \frac{1-x}{1+\gamma x}, \quad \gamma > -1$$

**Theorem**
$(\top, \bot, \sim)$ *is a De Morgan triplet such that*

$$\top(x,y) = \top(x,z) \Longrightarrow y = z,$$
$$\bot(x,y) = \bot(x,z) \Longrightarrow y = z,$$
$$\forall z \leq x\ \exists y, y' \text{ such that } \top(x,y) = z,\ \bot(z,y') = x$$

*and* $\top$ *and* $\bot$ *are rational functions if and only if there are numbers* $\alpha \geq 0,\ \beta \geq -1$ *and* $\gamma > -1$ *such that* $\alpha = \frac{1+\beta}{1+\gamma}$ *and* $\top = \top_\alpha,\ \bot = \bot_\beta$ *and* $\sim = \sim_\gamma$.

# Hamacher Family II

Additive generators $f_\alpha$ of $\top_\alpha$ are

$$f_\alpha = \begin{cases} \frac{1-x}{x} & \text{if } \alpha = 0 \\ \log \frac{\alpha+(1-\alpha)x}{x} & \text{if } \alpha > 0. \end{cases}$$

Each member of these families is strict $t$-norm and strict $t$-conorm, respectively.

Members of this family of $t$-norms are decreasing functions of parameter $\alpha$.

# Sugeno-Weber Family I

For $\lambda > 1$ and $x, y \in [0, 1]$, define

$$\top_\lambda(x, y) = \max\left\{\frac{x + y - 1 + \lambda xy}{1 + \lambda},\ 0\right\},$$
$$\bot_\lambda(x, y) = \min\left\{x + y + \lambda xy,\ 1\right\}.$$

$\lambda = 0$ leads to $\top_{\text{Łuka}}$ and $\bot_{\text{Łuka}}$, resp.

$\lambda \to \infty$ results in $\top_{\text{prod}}$ and $\bot_{\text{sum}}$, resp.

$\lambda \to -1$ creates $\top_{-1}$ and $\bot_{-1}$, resp.

# Sugeno-Weber Family II

Additive generators $f_\lambda$ of $\top_\lambda$ are

$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter $\lambda$.

Additive generators of $\perp_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$.

# Yager Family

For $0 < p < \infty$ and $x, y \in [0, 1]$, define

$$\top_p(x, y) = \max\left\{1 - ((1 - x)^p + (1 - y)^p)^{1/p},\ 0\right\},$$
$$\bot_p(x, y) = \min\left\{(x^p + y^p)^{1/p},\ 1\right\}.$$

Additive generators of $\top_p$ are

$$f_p(x) = (1 - x)^p,$$

and of $\bot_p$ are

$$g_p(x) = x^p.$$

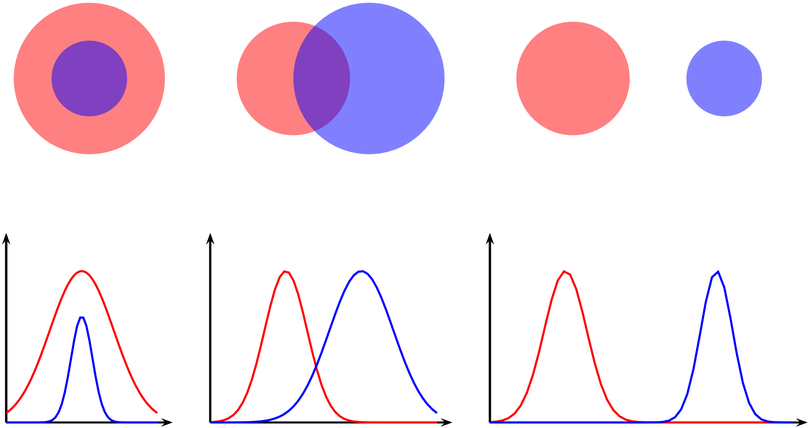
$\{\top_p\}_{0<p<\infty}$ are strictly increasing in $p$.

Note that $\lim_{p \to +0} \top_p = \top_{Łuka}$.

# Fuzzy Sets Inclusion

# Fuzzy Implications

**crisp:** $x \in A \Rightarrow x \in B$, **fuzzy:** $x \in \mu \Rightarrow x \in \mu'$



# Definitions of Fuzzy Implications

One way of defining $I$ is to use $\forall a, b \in \{0, 1\}$

$$I(a, b) = \neg a \vee b.$$

In fuzzy logic, disjunction and negation are $t$-conorm and fuzzy complement, resp., thus $\forall a, b \in [0, 1]$

$$I(a, b) = \bot(\sim a, b).$$

Another way in classical logic is $\forall a, b \in \{0, 1\}$

$$I(a, b) = \max\{x \in \{0, 1\} \mid a \wedge x \leq b\}.$$

In fuzzy logic, conjunction represents $t$-norm, thus $\forall a, b \in [0, 1]$

$$I(a, b) = \sup\{x \in [0, 1] \mid \top(a, x) \leq b\}.$$

So, classical definitions are equal, fuzzy extensions are not.

# Definitions of Fuzzy Implications

$I(a,b) = \bot(\sim a, b)$ may also be written as either

$$I(a,b) = \neg a \vee (a \wedge b) \qquad \text{or}$$
$$I(a,b) = (\neg a \wedge \neg b) \vee b.$$

Fuzzy logical extensions are thus, respectively,

$$I(a,b) = \bot(\sim a, \top(a,b)),$$
$$I(a,b) = \bot(\top(\sim a, \sim b), b)$$

where $(\top, \bot, n)$ must be a *De Morgan triplet*.

So again, classical definitions are equal, fuzzy extensions are not.

reason: Law of absorption of negation does not hold in fuzzy logic.

# $S$-Implications

Implications based on $I(a, b) = \perp(\sim a, b)$ are called $S$-**implications**.

Symbol $S$ is often used to denote $t$-conorms.

Four well-known $S$-implications are based on $\sim a = 1 - a$:

| Name | $I(a, b)$ | $\perp(a, b)$ |
|---|---|---|
| Kleene-Dienes | $I_{\max}(a, b) = \max(1 - a, b)$ | $\max(a, b)$ |
| Reichenbach | $I_{\text{sum}}(a, b) = 1 - a + ab$ | $a + b - ab$ |
| Łukasiewicz | $I_{Ł}(a, b) = \min(1,\ 1 - a + b)$ | $\min(1,\ a + b)$ |
| largest | $I_{-1}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$ | $\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$ |

# $S$-Implications

The drastic sum $\perp_{-1}$ leads to the largest $S$-implication $I_{-1}$ due to the following theorem:

**Theorem**
*Let $\perp_1, \perp_2$ be t-conorms such that $\perp_1(a, b) \leq \perp_2(a, b)$ for all $a, b \in [0, 1]$. Let $I_1, I_2$ be $S$-implications based on same fuzzy complement $\sim$ and $\perp_1, \perp_2$, respectively. Then $I_1(a, b) \leq I_2(a, b)$ for all $a, b \in [0, 1]$.*

Since $\perp_{-1}$ leads to the largest $S$-implication, similarly, $\perp_{\max}$ leads to the smallest $S$-implication $I_{\max}$.

Furthermore,

$$I_{\max} \leq I_{\text{sum}} \leq I_{Ł} \leq I_{-1}.$$

# $R$-Implications

$I(a, b) = \sup\{x \in [0, 1] \mid \top(a, x) \leq b\}$ leads to $R$-**implications**.

Symbol $R$ represents close connection to residuated semigroup.

Three well-known $R$-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a, b) = \sup\{x \mid \min(a, x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a, b) = \sup\{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz $t$-norm leads to **Łukasiewicz implication**

$$I_{Ł}(a, b) = \sup\{x \mid \max(0, a + x - 1) \leq b\} = \min(1,\ 1 - a + b).$$

# $R$-Implications

| Name | Formula | $\top(a,b) =$ |
|---|---|---|
| Gödel | $I_{\min}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$ | $\min(a,b)$ |
| Goguen | $I_{\text{prod}}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$ | $ab$ |
| Łukasiewicz | $I_{Ł}(a,b) = \min(1,\ 1 - a + b)$ | $\max(0,\ a + b - 1)$ |
| largest | $I_{\text{L}}(a,b) = \begin{cases} b, & \text{if } a = 1 \\ 1, & \text{otherwise} \end{cases}$ | not defined |

$I_{\text{L}}$ is actually the limit of all $R$-implications.

It serves as least upper bound.

It cannot be defined by $I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}$.

# $R$-Implications

**Theorem**

*Let $\top_1, \top_2$ be t-norms such that $\top_1(a, b) \leq \top_2(a, b)$ for all $a, b \in [0, 1]$. Let $I_1, I_2$ be $R$-implications based on $\top_1, \top_2$, respectively. Then $I_1(a, b) \geq I_2(a, b)$ for all $a, b \in [0, 1]$.*

It follows that Gödel $I_{\min}$ is the smallest $R$-implication.

Furthermore,

$$I_{\min} \leq I_{\mathrm{prod}} \leq I_{Ł} \leq I_{\mathrm{L}}.$$

# $QL$-Implications

Implications based on $I(a, b) = \bot(\sim a, \top(a, b))$ are called $QL$-**implications** ($QL$ from quantum logic).

Four well-known $QL$-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**

$$I_Z(a, b) = \max[1 - a, \min(a, b)].$$

- The algebraic product and sum lead to

$$I_{\mathrm{p}}(a, b) = 1 - a + a^2 b.$$

- Using $\top_{Ł}$ and $\bot_{Ł}$ leads to **Kleene-Dienes implication** again.
- Using $\top_{-1}$ and $\bot_{-1}$ leads to

$$I_{\mathrm{q}}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

# Axioms

All $I$ come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following axioms:

1) $a \leq b$ implies $I(a, x) \geq I(b, x)$ *(monotonicity in 1st argument)*
2) $a \leq b$ implies $I(x, a) \leq I(x, b)$ *(monotonicity in 2nd argument)*
3) $I(0, a) = 1$ *(dominance of falsity)*
4) $I(1, b) = b$ *(neutrality of truth)*
5) $I(a, a) = 1$ *(identity)*
6) $I(a, I(b, c)) = I(b, I(a, c))$ *(exchange property)*
7) $I(a, b) = 1$ if and only if $a \leq b$ *(boundary condition)*
8) $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$ *(contraposition)*
9) $I$ is a continuous function *(continuity)*

# Generator Function

$I$ that satisfy all listed axioms are characterized by this theorem:

**Theorem**
*A function* $I : [0,1]^2 \to [0,1]$ *satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement* $\sim$ *if and only if there exists a strict increasing continuous function* $f : [0,1] \to [0,\infty)$ *such that* $f(0) = 0$,

$$I(a,b) = f^{(-1)}(f(1) - f(a) + f(b))$$

*for all* $a, b \in [0,1]$, *and*

$$\sim a = f^{-1}(f(1) - f(a))$$

*for all* $a \in [0,1]$.

# Example

Consider $f_\lambda(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$.

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1+\lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy complement generated by $f$ for all $a \in [0, 1]$ is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_\lambda(a, b) = \min\left(1, \ \frac{1 - a + b + \lambda b}{1 + \lambda a}\right).$$

If $\lambda \in (-1, 0)$, then $I_\lambda$ is called **pseudo-Łukasiewicz implication**.

# List of Implications in Many Valued Logics

| Name | Class | Form $I(a, b) =$ | Axioms | Complement |
|---|---|---|---|---|
| Gaines-Rescher | | $\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$ | 1–8 | $1 - a$ |
| Gödel | R | $\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$ | 1–7 | |
| Goguen | R | $\begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases}$ | 1–7, 9 | |
| Kleene-Dienes | S,QL | $\max(1 - a, b)$ | 1–4, 6, 8, 9 | $1 - a$ |
| Łukasiewicz | R, S | $\min(1, 1 - a + b)$ | 1–9 | $1 - a$ |
| Pseudo-Łukasiewicz 1 | R, S | $\min\left[1, \frac{1-a+(1+\lambda)b}{1+\lambda a}\right]$ | 1–9 | $\frac{1-a}{1+\lambda a}$, $(\lambda > -1)$ |
| Pseudo-Łukasiewicz 2 | R, S | $\min\left[1, 1 - a^w + b^w\right]$ | 1–9 | $(1 - a^w)^{\frac{1}{w}}$, $(w > 0)$ |
| Reichenbach | S | $1 - a + ab$ | 1–4, 6, 8, 9 | $1 - a$ |
| Wu | | $\begin{cases} 1 & \text{if } a \leq b \\ \min(1 - a, b) & \text{otherwise} \end{cases}$ | 1–3,5,7,8 | $1 - a$ |
| Zadeh | QL | $\max[1 - a, \min(a, b)]$ | 1–4, 9 | $1 - a$ |

# Which Fuzzy Implication?

Since the meaning of $I$ is not unique, we must resolve the following question:

Which $I$ should be used for calculating the fuzzy relation $R$?

Hence meaningful criteria are needed.

They emerge from various fuzzy inference rules, *i.e.* modus ponens, modus tollens, hypothetical syllogism.

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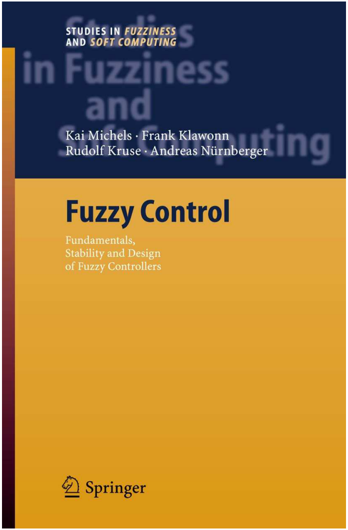

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