Intelligent Systems Engineering

FS-01 Introduction to Fuzzy Sets

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Outline

These notes are based on [Pedrycz and Gomide 1998] chapter 1

Motivation for Fuzzy Sets: Uncertainty

Uncertainty is universal! As an example, consider a computer system to recognize trees in a visual image; sources of uncertainty in this task include (but are not limited to)

- noise in the sensed imagery,
- distortion due to pose and lens conditions,
- variability of the class of interest (what is a "tree"?),
- faithfulness of the features used to described a tree,
- missing features,
- spatial context (a tree in a forest versus a tree in Edmonton),
- temporal context (a tree in summer versus a tree in winter),
- the choice of recognition algorithm, and so on.

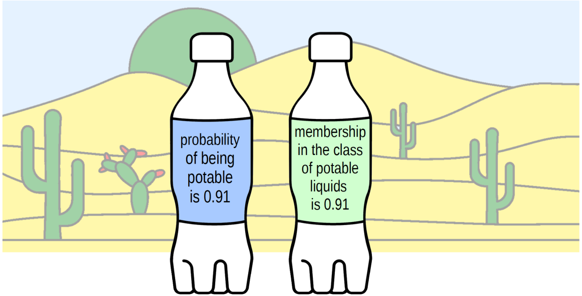
Motivation for Fuzzy Sets: Uncertainty

- Uncertainty has been traditionally addressed using **probability theory.**
- But, sometimes, uncertainty takes other forms than **randomness**.
- Often, instead of asking whether something is true or not, we ask
  - how much it is true, i.e., how much is a certain property exhibited in a particular instance
  - e.g. how much a particular object matches an ideal prototype

Example [Jim Bezdek, inaugural editorial of IEEE Trans. on Fuzzy Systems, 1993]:

You are dying of thirst in a desert when you come across two bottles.

Which one would you drink?



Motivation for Fuzzy Sets: Other examples of uncertainty

- Data processing: Making sense of data, linguistic summaries
  - Textual description of numerical weather forecasts
  - Interpretation of polls or election results
- Decision-making and control problems
  - Decide on car purchase given brand, price, gas consumption, customer rating, etc.
  - Design a controller that maintains a comfortable room temperature
  - Design a highway traffic control system that assures safe driving environment
  - Design a system that can park a car

Fuzzy Sets: A Motivation

- Image processing and computer vision: selection of image processing algorithm, interpretation of a scene (image understanding)

- Domain-oriented, common sense knowledge:
  - if a region has color of skin, it is round, and located in the upper part of the filed of view, then confidence of face is high
  - if an object in the filed of view is moving, then use a short exposition time

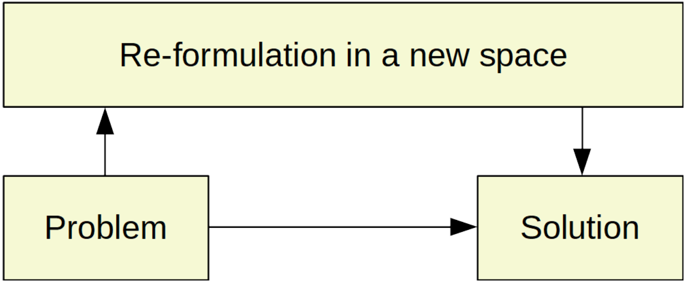
Rule-based systems

- Expressing domain knowledge in a form of rules

  IF condition THEN conclusion

- Easy to understand and acquire
- Modular system
- Rules are generalizations of existing patterns of decision-making, classification, control, ...

Problem Solving



Examples of alternative spaces: Laplace, Fourier, fuzzy sets

Sets

Used to embrace elements to form some general concepts (granules)

- even numbers
- capital cities of Europe
- sport cars
- ...

but there are also situations like the following

Sets, really?

. . . but there are also situations like the following

- large cities in Canada
- low temperature
- high inflation rate

and even terms like these

- small approximation error
- medium size software system
- fast response of a dynamic system
- ill-defined system of linear equations

Dichotomy

Another way of looking at problems associated with sets

One seed does not constitute a pile nor two ... from the other side everybody will agree that 100 million seeds constitute a pile.

What therefore is the appropriate limit? Can we say that 325 647 seeds don't constitute a pile but 325 648 do?

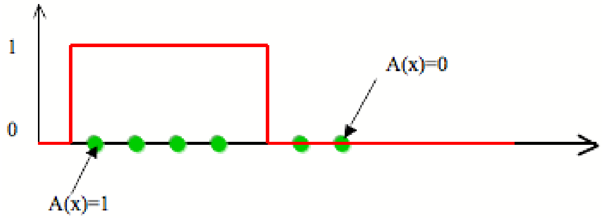
[Borel, 1950]

Description of Sets

- Based on the concept of belongingness
  - inclusion $\in$, and
  - exclusion $\notin$
- Described by
  - inclusion - enumeration (characterization) of elements belonging to set A
  - characteristic function

Characteristic Functions

$$A:\ X \rightarrow \{0,1\}$$



Sets subscribe to the concept of dichotomy:

$$x \in A \Leftrightarrow A(x) = 1$$

$$x \notin A \Leftrightarrow A(x) = 0$$

History of Fuzzy Sets

1920: J. Lukasiewicz, E. Post (three-valued logic and many valued logic)

1965: L. A. Zadeh (fuzzy sets)

1968: L. A. Zadeh (fuzzy algorithm)

1975: E.H. Mamdani (fuzzy control by linguistic rules)

1987: Fuzzy boom - Industrial applications of fuzzy sets in Japan & Korea

- Home electronics
- Vehicle control, process control
- Pattern recognition, image processing
- Expert systems
- Military systems, space research

1990s: Applications to very complex control problems (e.g. helicopter autopilot, Japan 1991)

Fuzzy Sets

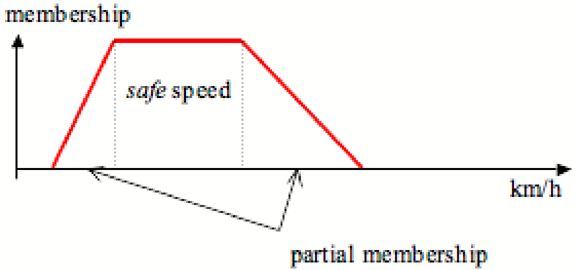
Definition:

Fuzzy set A is characterized by a membership function

Membership Functions:

- admit a notion of partial membership of element to the concept
- the higher the membership value $A(x)$, the more typical x is in A

Membership Function $A: X \to [0,1]$



Membership function describing concept "safe speed" on highway

$A(x) = 1$: complete membership (belongingness)
$A(x) = 0$: complete exclusion
but also partial membership $0 < A(x) < 1$

Example: Belongingness (the universe of discourse is a group of people X.)

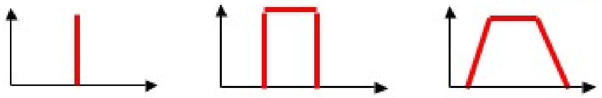
Q1: Who has a driver's license? – a crisp subset $A: \ X \to \{0, 1\}$ (characteristic function)

Q2: Who can drive very well? – a fuzzy subset $A: \ X \to [0, 1]$ (membership function)

Example: A control problem

Defining control objective

- single numeric setpoint
- interval (set-based) setpoint
- fuzzy set setpoint



Vector Representation of Sets and Fuzzy Sets

For Sets and Fuzzy Sets in finite spaces

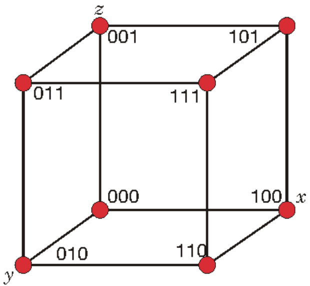
Sets: vectors with entries 0, 1

$$A = [0\ 0\ 1\ 1\ 0\ 0\ 0\ 1]$$

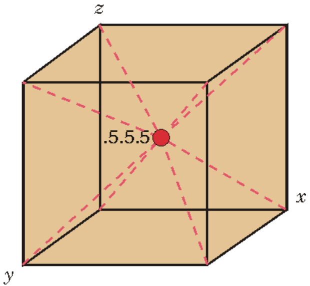
Fuzzy sets: vectors with entries in [0, 1]

$$B = [0.2\ \ 0.5\ \ 0.9\ \ 1\ \ 1\ \ 0.8\ \ 0.6\ \ 0.4\ \ 0.3]$$

Geometry of Sets and Fuzzy Sets



Sets



Fuzzy Sets

Operations on Sets

(Standard) Union $(A \cup B) = \max\big(A(x), B(x)\big)$

(Standard) Intersection $(A \cap B) = \min\big(A(x), B(x)\big)$

(Standard) Complement $(\bar{A}) = 1 - A(x)$

Sets and Two-valued Logic

Sets	Propositions
inclusion	truth - assignment
$A(x) = 1$ (inclusion) $A(y) = 0$ (exclusion)	$t(p) = 1$ (true) $t(p) = 0$ (false)
operations	operations
$(A \cap B) = \min\big(A(x), B(x)\big)$ $(A \cup B) = \max\big(A(x), B(x)\big)$ $(\bar{A}) = 1 - A(x)$	$(A \,\&\, B) = A(x) \wedge B(x)$ $(A \mathrm{or} B) = A(x) \vee B(x)$ $t(\neg p) = 1 - t(p)$

Operations on Fuzzy Sets

Union	$(A \cup B) = \max\big(A(x), B(x)\big)$
Intersection	$(A \cap B) = \min\big(A(x), B(x)\big)$
Complement	$(\bar{A}) = 1 - A(x)$

Note:

This formulation is the same as for conventional sets, but:

- $A(x)$, $B(x)$ can attain values from $[0, 1]$ not just $\{0, 1\}$
- for std. operations, sets and fuzzy sets evaluate the same for (boundary) values of 0 and 1
- there are other possibilities besides the standard union, intersection, and complement

Properties of Sets and Fuzzy Sets (1/2)

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cap B = B \cap A$ $A \cup B = A \cup B$
Associativity	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cap A = A;\ A \cup A = A$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Identity	$A \cup \emptyset = A;\ A \cap X = A$

Properties of Sets and Fuzzy Sets (2/2)

DeMorgan's Law	$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
Law of contradiction	$A \cap \overline{A} = \emptyset$
Law of excluded middle	$A \cup \overline{A} = X$

Overlap and underlap of Fuzzy Sets

Law of contradiction does not hold: overlap property

$$A \cap \overline{A} \supseteq \emptyset$$

Law of excluded middle does not hold: underlap property

$$A \cup \overline{A} \subseteq X$$

Dichotomy Problem

"…the law of excluded middle is true when precise symbols are employed, but it is not true when symbols are vague, as, in fact, all symbols are." Russel, 1923