Intelligent Systems Engineering

FS-02 Membership Functions

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Outline

Description of (traditional, crisp) subsets

Given a "universe" set X, a subset A of X can be defined in several ways.

- **Enumeration**; e.g. X is the set of integers; the subset of prime numbers less than 10 can be specified by listing its members: $A = \{2, 3, 5, 7\}$
- **Description** (providing defining properties); the same subset can be described as $A = \{x \in X | x$ is a positive integer < 10 that has only two distinct divisors: 1 and x
- **Characteristic function** (which is also denoted by the set name) $A : X \to \{0, 1\}$; from X into the binary set $\{0, 1\}$ given by

$$A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Description of fuzzy subsets

Zadeh [1965] defined a fuzzy subset of X as a function

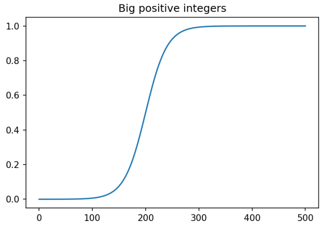
$$A : X \to [0,1],$$

that is, a function from X into the interval $[0,1]$, called **membership function**.
The value $A(x)$ is called

- the membership of the point x in the fuzzy set A, or
- the degree to which the point x belongs to the set A.

E.g., the fuzzy subset with **linguistic description** “big positive integers” could be defined as

$$A(x) = \frac{1}{1 + e^{0.05(200-x)}},$$



Where do membership functions come from?

All fuzzy set theory is based on the concept of a **membership function**. These functions

- are based on some common sense definitions that convey some linguistic expression (as in the examples above)
- in a more general sense, they
  - come from expert knowledge directly, or
  - they can be derived from questionnaires, heuristics, and so on.

This is a human-centric view and is certainly open to debate.

In many cases, the membership functions take on specific functional forms, for convenience in representation and computation: triangular, trapezoidal, S-functions, pi-functions, sigmoids, and even Gaussians.

Membership Functions

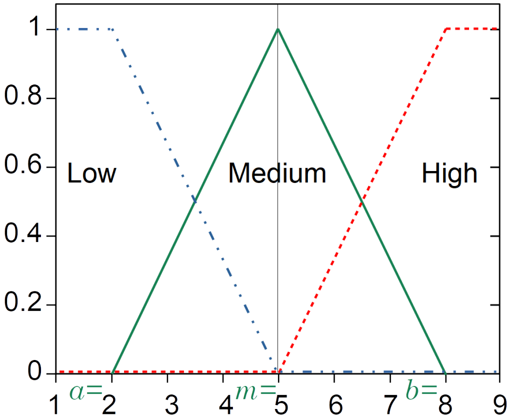
Membership functions are used to represent (capture) notions expressed in problem description:

- elements with complete membership
- elements excluded from fuzzy set
- the form of transition between complete membership and complete exclusion

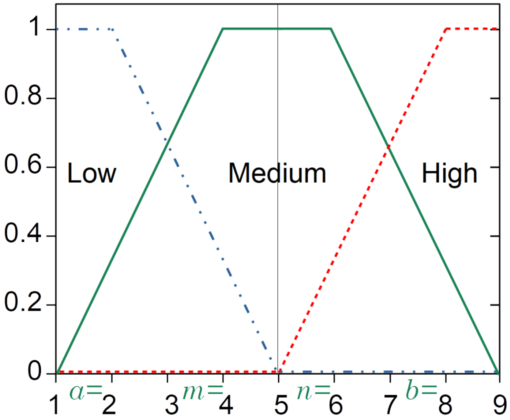
Triangular membership functions

Triangular fuzzy sets:

- modal value m
- lower and upper bounds a and b



Trapezoidal membership functions



Trapezoidal fuzzy sets:

- modal values m and n
- lower and upper bounds a and b

Other types of membership functions

Gaussian m.f. $A(x) = e^{-\frac{(x-m)^2}{\sigma^2}}$

Exponential m.f. $A(x) = \frac{k(x-m)^2}{1+k(x-m)^2}$

Sigmoidal m.f. $A(x) = \frac{1}{1+e^{x-m}}$

Examples of fuzzy sets

Fuzzy sets represent concepts in real world (example linguistic description)

- Size (large container)
- Age (young man)
- Complexity (very complex problem)
- Error (small approximation error)
- Reliability (high reliability)
- Power consumption (ultra-low power consumption)
- Microprocessor speed (fast microprocessor)

Membership Function Determination

How to determine shape/functional description of membership from empirical data?

- Horizontal approach
- Vertical approach
- Pairwise comparison
- Clustering (grouping) method

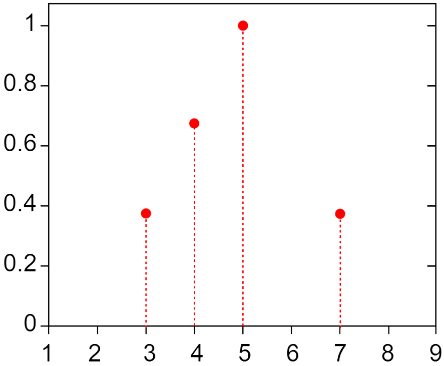
Horizontal approach

Gather information about membership values at selected elements of the universe of discourse (space) X - polling mechanism (likelihood measure):

Experiment: A group of experts are asked to answer the question: "Can x_i be accepted as compatible with concept A (yes/no)?" The estimated value of membership function at x_i corresponds to the number of positive replies to the total number of replies:

$$A(x_i) = \frac{n_{\text{yes}}}{N}$$

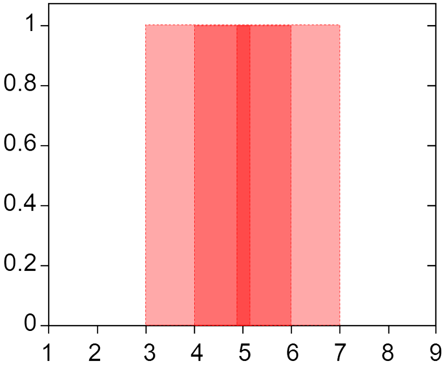
Standard deviation can be attached to the experimental results.



Vertical approach

Identify α-cuts and "reconstruct" a fuzzy set using these sets:

Experiment: Identifying α-cuts. After several α levels are selected, experts are asked to identify the corresponding subset of X whose elements belong to A to a degree not less than α. FS is then build by stacking up the successive α–cuts.



Horizontal and Vertical Approaches

- Pro: easy to use
- Con: “local” character of experiments (isolated experiments dealing with single elements of the universe of discourse)

Pairwise Comparison Method

Rationale:

- assume that membership values $A_{x1}, A_{x2}, \ldots, A_{xn}$ are given
- arrange them as a reciprocal matrix $\boldsymbol{A}$
  - reciprocity $a_{ij} = 1/a_{ji}$
  - transitivity $a_{ik} = a_{ij}a_{jk}$
- multiply $\boldsymbol{A}$ by the vector of membership values $\boldsymbol{a}$

$$\begin{aligned} \boldsymbol{A}\boldsymbol{a} &= n\boldsymbol{a} \\ (\boldsymbol{A} - n\,\boldsymbol{I})\,\boldsymbol{a} &= 0 \end{aligned}$$

where I is a unit matrix, and $\boldsymbol{a}$ and n are eigenvector and eigenvalue of the matrix $\boldsymbol{A}$, respectively.

Pairwise Comparison Method

Realization:

- compare objects pair–wise in the context of $\boldsymbol{A}$ (e.g. ratio scale 1, 2, . . . , 7) and construct the matrix based on these ratios
- assess consistency of the matrix (and, in turn, consistency of the gathered data) by looking at the value of n (should be comparable to the dimension of the matrix)
- normalize the eigenvector $\boldsymbol{a}$ to get an estimate of the membership function

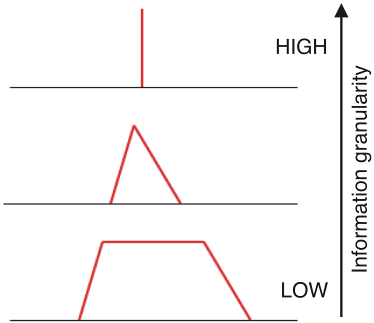
Characteristics of Fuzzy Sets

Height	$\mathrm{hgt}(A) = \sup_x A(x)$
Normality	$A(x)$: $\sup_x A(x) = 1$ (otherwise $A(x)$ is "subnormal")
Support	$\mathrm{Supp}(A) = \{x \in X \mid A(x) > 0\}$
Core	$\mathrm{Core}(A) = \{x \in X \mid A(x) = 1\}$
Cardinality	$\mathrm{Card}(A) = \sum_{x \in X} A(x)$ (or $\int$ for continuous fuzzy sets)

$\sup$ Supremum (pl. suprema) of a subset A of a partially ordered set P is the least element in P that is greater than or equal to all elements of A, if such an element exists. [Wikipedia]

Hierarchy of fuzzy sets

Fuzzy sets representing the same concept but with different resolution lead to different levels of information granularity



Higher granularity: need more constructs (fuzzy sets) to describe the universe of discourse

Unary operations on fuzzy sets

Operations with single argument – a fuzzy set. They serve for manipulation of fuzzy sets to change their meaning or strength of the concept they represent (i.e. change membership function)

Normalization – converting subnormal fuzzy set into its normal counterpart

$$\mathrm{Norm}(A) = \frac{A(x)}{\mathrm{hgt}(A)}$$

Unary operations on fuzzy sets

Concentration – concentrating the membership function around points with higher membership values

$$\mathrm{Con}(A) = A^2(x)$$

Dilation – effect opposite to concentration

$$\mathrm{Dil}(A) = A^{1/2}(x)$$

Of course, concentration and dilatation can be generally expressed in form

$$A^p(x)$$

with concentration corresponding to values $p > 1$ and dilation to values $p < 1$.

Equality and Inclusion Relations

Two fuzzy sets A and B can be related with the following two relations

Equality:

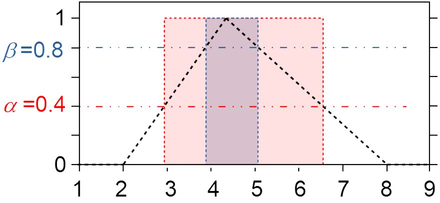
$$A = B \text{ iff } \forall x: \; A(x) = B(x)$$

Inclusion:

$$A \subset B \text{ iff } \forall x: \; A(x) \leq B(x)$$

Representing Fuzzy Sets using α-cuts

α-cut of fuzzy set A is a set, A_α defined as $A_\alpha = \{x | A(x) \geq \alpha\}$



if $\alpha < \beta$ then $A_\alpha \supseteq A_\beta$

Representation Theorem

Any fuzzy set can be represented as a family of sets

$$A = \bigcup_{\alpha} \alpha A_\alpha$$