STAT235 MIDTERM EXAM VERSION 111 SOLUTION

Complete the following (please **print**):

Section Number: ___________ Last name, First name: ______________________________

Instructions

1. Enter your lecture section number, your last name (the same one that is illustrated on your One Card), and first name (the same one that is illustrated on your One Card) into the space provided above. Enter your student ID # in the upper right hand corner on all the *other* pages of the exam.

2. Make sure you use ONLY PENCIL to put and mark your name, student ID #, and your exam version number on the Scantron sheet. Your name (last name and first name) should be entered into the NAME block, your student ID # into the IDENTIFICATION NUMBER block, and finally, the three-digit exam version into the SPECIAL CODES block. The exam version is specified at the top of the exam. Make sure to shade in the circle that corresponds to the letter, digit, or empty space in the box at the top of each column.

3. This is a multiple-choice closed book exam. There are **30** questions in the exam. For each question, carry out the appropriate analysis and put your answer on the Scantron sheet by shading the letter A, B, C, D, or E that corresponds to your chosen answer. Make sure your answers are clearly marked with ONLY PENCIL. Otherwise, no marks will be given. For each question exactly one of the five answers is correct. If you fill in more than one answer to a question, the question will be scored incorrect. Each question is worth 1 mark.

4. Please note that only the answers in the Scantron sheet will be considered. If you initially mark your answers in the exam sheet, make sure that you copy them correctly into the Scantron sheet. If you change an answer on the Scantron sheet, be sure that you erase your first mark completely and then blacken the circle of the answer choice you prefer. Your score will be based on the number of questions you answer correctly. No credit will be given for omitted answers and no credit will be lost for wrong answers. You must mark all your answers on the Scantron sheet during the allotted time. No additional time will be allowed at the end of the session for this purpose.

5. You are permitted to use a **non-programmable** calculator approved by the Faculty of Engineering. During the lab exam you are forbidden to use any devices with communication capabilities including cell phones and pagers. You are also forbidden to use any photographically capable devices in the exam room. Copying questions or answers on paper to take from the exam room is prohibited.

6. Note that the formula sheet and the table of the cumulative distribution function of the standard normal distribution are attached to the exam. There are **6** pages in the exam. The exam is graded out of **30** points.

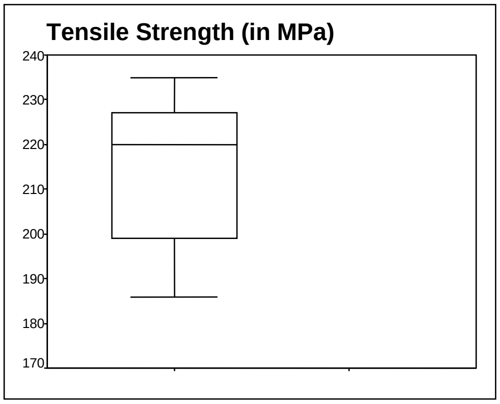
7. You must return your Scantron and exam booklet when you finish the exam. You have 180 minutes to complete the exam.

8. Sign the exam booklet in the space provided below.

SIGNATURE: ________________________________

PART 1 (MULTIPLE-CHOICE QUESTIONS, 1 POINT EACH)

1. The following boxplot of tensile strength (in MPa) of 900 iron specimens was obtained:



Based on the boxplot, what is the most likely shape of the distribution?

(a) Symmetric
(b) Skewed to the right
(c) Skewed to the left
(d) Neither skewed to the right nor to the left
(e) Uniform

Solution: Since the median is much closer to Q_3 than Q_1, the data are left-skewed.

2. Refer to the boxplot in the previous question. Which sentence about the tensile strength of iron specimens is TRUE?

(a) Half of the specimens have tensile strength below 210,
(b) Half of the specimens have tensile strength above 210,
(c) More than half of the specimens have tensile strength below 210,
(d) **Less than half of the specimens have tensile strength below 210,**
(e) Three quarters of the specimens have tensile strength above 210.

Solution: With no outliers, 25% of the data is between the minimum (lower whisker) and Q_1, 25% is between Q_1 and the median, 25% is between the median and Q_3, and 25% is between Q_3 and the maximum (upper whisker). Using these percentages, determine what option is correct knowing that the value of 210 is between Q1 and the median.

3. A device for checking welds in pipes is designed to signal if the weld is defective. The probability that the device signals if the weld is defective is 5/6 and the probability that the device does not signal if the weld is not defective is 92/94. It is known (from previous experience) that 6% of the pipes are defective. What is the probability that the device will not signal?

(a) 0.81 (b) 80/96 (c) 86/94 **(d) 0.93** (e) 92/96

Solution: Let D be the event that the device is defective and G be the event that the device signals. As $P(G \mid D) = 5/6$, $P(G' \mid D') = 92/94$, and $P(D) = 0.06$, then

$$P(G \mid D) = \frac{P(G \cap D)}{P(D)} = \frac{P(G \cap D)}{0.06} = \frac{5}{6} \rightarrow P(G \cap D) = 0.05.$$

$$P(G' \mid D') = \frac{P(G' \cap D')}{P(D')} = \frac{P(G' \cap D')}{1-0.06} = \frac{92}{94} \rightarrow P(G' \cap D') = 0.92$$

	G	*G′*	
D	0.05	0.01	0.06
D′	0.02	0.92	0.94
	0.07	**0.93**	1.00

4. Refer to the previous question. If the device signals, what is the probability that the weld is defective?

   (a) 0.4286 (b) 0.523 (c) 0.622 (d) 0.681 **(e) 0.714**

   Solution: From the above table, $P(D \mid G)$ = 0.05/0.07 = 0.714.

5. A quality control plan calls for rejecting a large lot of crankshaft bearings if a sample of seven is drawn and at least one is defective. What is the probability of rejecting the lot if the proportion of defectives in the lot is 1/10?

   (a) 0.331 (b) 0.478 (c) 0.491 (d) 0.512 **(e) 0.522**

   Solution: $P(X \geq 1) = 1 - P(X = 0) = 1 - 0.90^7 = 0.522$

6. There are 17 broken lightbulbs in a box of 100 lightbulbs. A random sample of 3 lightbulbs is chosen without replacement. What is the probability that the sample contains no broken lightbulbs?

   (a) 0.462 (b) 0.493 (c) 0.518 **(d) 0.568** (e) 0.613

   Solution: $(83/100) * (82/99) * (81/98) = 0.568$

7. How many four-digit even numbers can be formed if digits may be repeated within any one number and the first digit must be different from zero?

   (a) 2520 **(b) 4500** (c) 5000 (d) 5400 (e) 9000

   Solution: The number of all 4 digit numbers with the first digit different from zero and the last digit even is 9*10*10*5 = 4500.

8. Suppose that someone offers you the bet of rolling two fair dice. If the sum of dots is 7 or 11, you win \$6. Otherwise, you must pay \$2. What are your expected winnings?

   (a) -\$0.624 (b) -\$0.444 **(c) -\$0.222** (d) -\$0.144 (e) \$0.232

   Solution: The sample space (*S*) consists of 6*6 = 36 equally likely outcomes. The sum of dots being 7 or 11 consists of 8 outcomes: {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (5,6), (6,5)}.
   $E(X) = (8/36)*6 + (-2)*(28/36) = -8/36 = -0.222$

9. Let X and Y be independent random variables. The mean of X is 100, the mean of Y is 50, the standard deviation of X is 10 and the standard deviation of Y is 5. The mean and standard deviation of $2X - 3Y$ are respectively

(a) 25, 3.873 (b) 50, 9.220 (c) 50, 15 **(d) 50, 25** (e) 50, 625

Solution: $E(2X-3Y) = 2E(X) - 3E(Y) = 2*100 - 3*50 = 50$

$Var(2X-3Y) = 2^2 * Var(X) + 3^2 * Var(Y) = 4*100 + 9*25 = 625$

$S.D.(2X-3Y) = \sqrt{625} = 25$

10. A coin is twice as likely to turn up tails as heads. The coin is tossed twice. Let X be the number of heads obtained. Which of the following is the probability distribution of X?

(a)

x	0	1	2
$P(X = x)$	1/9	4/9	4/9

(b)

x	0	1	2
$P(X = x)$	1/9	5/9	3/9

(c)

x	0	1	2
$P(X = x)$	4/9	4/9	1/9

(d)

x	0	1	2
$P(X = x)$	4/9	2/9	3/9

(e) $P(X = 0) = P(X = 1) = P(X = 2) = 1/3$

Solution: If tails is twice as likely, then $P(H) = 1/3$ and $P(T) = 2/3$. The coin tosses are independent so $P(X = 0) = (2/3)^2$, $P(X = 1) = (1/3)(2/3) + (2/3)(1/3)$, and $P(X = 2) = (1/3)^2$.

11. Suppose that a light bulb manufacturing company claims that their 100 W bulbs have a 75% chance of lasting 4000 hours without burning out. A home owner takes this claim cautiously and buys one of these light bulbs at a time. What is the probability that the owner has to buy at most two bulbs to obtain a bulb that lasts 4000 hours without burning out?

(a) 0.188 **(b) 0.938** (c) 0.518 (d) 0.750 (e) 0.062

Solution: X = number of bulbs needed to obtain the first bulb that lasts 4000 hours
X follows a geometric distribution with $p = 0.75$.

$P(X \le 2) = P(X = 1) + P(X = 2) = 0.75 + 0.25*0.75 = 0.938$

12. Refer to the previous question. Once the home owner is satisfied that the claim of the manufacturer is probably true, he decides to purchase such light bulbs until he gets 5 light bulbs that last 4000 hours. What is the probability that he has to purchase exactly 7 light bulbs to obtain 5 that last 4000 hours?

(a) 0.250 (b) 0.238 (c) 0.756 **(d) 0.223** (e) 0.312

Solution: X = number of bulbs needed to obtain 5 that last 4000 hours
X follows a negative binomial distribution with $p = 0.75$, $r = 5$.

$$P(X = 7) = \binom{7-1}{5-1} 0.75^5.(1-0.75)^{7-5} = 0.223$$

13. Refer to the previous two questions. The home owner decides to purchase 10 such light bulbs. What is the probability that between 8 and 10 of these light bulbs, inclusive, last 4000 hours?

(a) 0.188 **(b) 0.526** (c) 0.224 (d) 0.750 (e) 0.282

Solution: X = number of bulbs that last 4000 hours among the 10 bulbs purchased
X follows a binomial distribution with n = 10 and p = 0.75.

$$P(8 \le X \le 10) = P(X = 8) + P(X = 9) + P(X = 10) =$$

$$\binom{10}{8} 0.75^8 \cdot 0.25^{10-8} + \binom{10}{9} 0.75^9 \cdot 0.25^{10-9} + \binom{10}{10} 0.75^{10} \cdot 0.25^{10-10} =$$

$$45 * 0.75^8 \cdot 0.25^2 + 10 * 0.75^9 \cdot 0.25 + 0.75^{10} \cdot 0.25^0 = 0.526$$

14. The number of power surges in an electric grid has a Poisson distribution with a mean of 1 power surge every 12 hours. What is the probability that there will be no more than 1 power surge in a 24-hour period?

(a) $2e^{-2}$ **(b) $3e^{-2}$** (c) $e^{-1/2}$ (d) $(3/2)\, e^{-2}$ (e) $3e^{-1}$

Solution: $P(X \le 1) = P(X = 0) + P(X = 1) = \frac{2^0}{0!} e^{-2} + \frac{2^1}{1!} e^{-2} = 3e^{-2}$

15. A system has two components placed in series so that the system fails if either of the two components fails. The first component fails with the probability 0.10 and the second component fails with the probability 0.20. If the two components operate independently, what is the probability that the entire system will fail?

(a) 0.02 **(b) 0.28** (c) 0.34 (d) 0.38 (e) 0.72

Solution: The probability that the first component works is 1 – 0.10 = 0.90 and the probability that the second component works is 1 – 0.20 = 0.80. Thus, the probability that both components work is 0.9*0.8 = 0.72 by independence. Thus, the probability the system fails is 1 – 0.72 = 0.28.

16. Let X be a random variable following the exponential distribution with the parameter λ:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

If the median of the distribution is 1/3, then λ is equal to

(a) 0.231 (b) 0.811 (c) 1.355 **(d) 2.079** (e) 3

Solution: $\int_0^{1/3} \lambda \cdot e^{-\lambda x} dx = 0.50 \rightarrow \left[-e^{-\lambda x}\right]_0^{1/3} = -e^{-\lambda/3} - (-1) = 1 - e^{-\lambda/3} = 0.5$

$\rightarrow e^{-\lambda/3} = 0.5 \rightarrow -\lambda/3 = \ln(0.5) \rightarrow \lambda = -3\ln(0.5) = 2.079$.

17. The cumulative distribution function $F(x)$ of a random variable X has the following form

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ x^2 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

What is the expected value of X?

(a) 1/6 (b) 1/3 **(c) 2/3** (d) 3/4 (e) 4/5

Solution: The density function $f(x)$ in [0,1] is the derivative of $F(x)$, or $2x$. Thus,

$$E(X) = \int_0^1 f(x) \cdot x\, dx = \int_0^1 2x \cdot x\, dx = 2 \cdot \left[\frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

18. The thickness of a flange on an aircraft component is uniformly distributed between 0.95 and 1.05 millimeters. The fraction of flanges that exceed 1.02 is

(a) 0.15 **(b) 0.30** (c) 0.35 (d) 0.40 (e) 0.45

Solution: The height of the density function (flat line) is 10, as the area under the density function has to be equal to 1: (1.05 – 0.95)*10 = 1. Thus, the probability = area of the rectangle with the width (1.02 – 1.05) and height 10 or 0.03*10 = 0.30.

19. The scores of students in an exam follow a normal distribution with a mean of 60 and an unknown standard deviation σ. It is known that 2.5% of the students had scores exceeding 70. Thus the standard deviation σ is approximately

(a) 2.5 **(b) 5** (c) 6 (d) 7 (e) 8

Solution: From the 68-95-99.7 Rule, 95% of observations lie within 2 standard deviations of the mean. Thus, 2.5% of observations are more than two standard deviations above the mean. Hence, the standard deviation (σ) must be 5.

20. Suppose the variable Z follows a standard normal distribution. Find $P(-1 < Z < 2)$.

(a) 0.6328 (b) 0.6823 (c) 0.7574 **(d) 0.8185** (e) 0.8842

Solution: $P(-1 < Z < 2) = P(Z < 2) - P(Z < -1) = 0.9772 - 0.1587 = 0.8185$

21. The volume (given in litres) of container 1 follows a normal distribution N(10.5, 1.5) and the volume of the second container follows a normal distribution N(10, 0.5). The first container is filled with gas as full as possible. What is the probability that you will be able to get all of the gas from container 1 into container 2; in other words, the volume of container 2 is at least as large as that of container 1?

(a) 0.3085 (b) 0.6368 **(c) 0.3632** (d) 0.4013 (e) 0.3536

Solution: Let V_1 be the volume of the first container and V_2 be the volume of the second container. As each of V_1 and V_2 follows a normal distribution, the difference $V_2 - V_1$ follows a normal distribution with a mean of 10 – 10.5 = –0.5 and a variance of
$Var(V_2 - V_1) = Var(V_2) + Var(V_1) = 0.5 + 1.5 = 2$.

$$P(V_2 - V_1 \geq 0) = P\left(Z = \frac{V_2 - V_1 - \mu_{V_2 - V_1}}{\sigma_{V_2 - V_1}} \geq \frac{0-(-0.5)}{\sqrt{2}} \right) \simeq P(Z \geq 0.35) = 0.3632$$

22. Consider a random variable *X* following a normal distribution with a mean of 1 and a variance of 1. The 95th percentile of *X*, or the value *x* such that $P(X \leq x) = 0.95$, is about

    (a) 1.280 (b) 1.645 (c) 1.746 (d) 2.228 **(e) 2.645**

    Solution:
    $P(X \leq x) = P\left(Z = \frac{X-\mu}{\sigma} \leq \frac{x-1}{1} \right) = 0.95$ or $P(Z \leq z_0) = 0.95,$ where $z_0 = x - 1$.
    From the *z*-table, $z_0 = 1.645.$ Thus, *x* = 2.645.

23. Suppose the distance *X* between a point target and a shot aimed at the point in a coin-operated target game is a continuous random variable with the probability density function

$$f(x) = \begin{cases} x/18 & \text{for } 0 \leq x \leq 6, \\ 0, & \text{otherwise.} \end{cases}$$

    What is the mean of *X*?

    (a) 2.5 (b) 3.0 (c) 3.5 **(d) 4.0** (e) 4.5
    Solution: $\mu = \int_0^6 xf(x)dx = \int_0^6 x(\tfrac{x}{18})dx = \int_0^6 \tfrac{x^2}{18}dx = \tfrac{1}{18}\left[\tfrac{x^3}{3}\right]_0^6 = \tfrac{1}{18}[72-0] = 4$

24. Refer to the previous question. What is the variance of *X*?

    (a) 1.5 **(b) 2.0** (c) 2.5 (d) 3.0 (e) 4.0

    Solution: $E[X^2] = \int_0^6 x^2 f(x)dx = \int_0^6 x^2(\tfrac{x}{18})dx = \int_0^6 \tfrac{x^3}{18}dx = \tfrac{1}{18}\left[\tfrac{x^4}{4}\right]_0^6 = \tfrac{1}{18}[324-0] = 18$
    $\sigma^2 = V(X) = E(X^2) - [E(X)]^2 = 18 - [4]^2 = 2$

25. Refer to the previous two questions. Find $P(0 < X < 1)$.

    (a) 1/36 (b) 1/18 (c) 1/12 (d) 1/8 (e) ¼

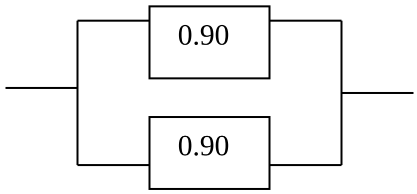
    Solution: $P(0 < X < 1) = \int_0^1 f(x)dx = \int_0^1 (\tfrac{x}{18})dx = \tfrac{1}{18}\left[\tfrac{x^2}{2}\right]_0^1 = \tfrac{1}{18}[\tfrac{1}{2} - 0] = \tfrac{1}{36}$

26. The lifetime of automobile tires of a certain brand are found to follow an exponential distribution with mean 60 (in thousands of km). The probability that one of these tires bought today will last over 70,000 km is

    (a) 0.135 (b) 0.223 **(c) 0.311** (d) 0.368 (e) 0.607

    Solution: $E(X) = 1/\lambda = 60$ so $\lambda = 1/60$.
    $P(X > 70) = 1 - P(X \leq 70) = 1 - F(70) = 1 - (1 - e^{(-1/60)\,70}) = e^{-7/6}$

27. The following circuit consisting of two devices operates if and only if there is a path of functional devices from left to right. The probability that each device functions is 0.90 and the devices operate independently of one another.



The EXACT probability that the circuit operates is

(a) 0.8100 (b) 0.9801 **(c) 0.9900** (d) 0.9979 (e) 0.9999

Solution: The probability that the circuit fails is (1 – 0.90)*(1 – 0.90) = 0.01. Thus, the probability that the circuit operates is 1 – 0.01 = 0.99.

Use the following information for questions 28 – 30:

The number of surface flaws in plastic panels used in the interior of automobiles has a Poisson distribution with a mean of 0.05 flaws per square foot of plastic panel. Assume an automobile interior contains 10 square feet of plastic panel.

28. What is the probability that there are no surface flaws in an auto's interior?

(a) 0.995 (b) 0.049 **(c) 0.607** (d) 0.951 (e) 0.393

Solution: Let X denote the number of flaws in 10 square feet of plastic panel.
Then, X follows a Poisson distribution with $\lambda = 0.05*10 = 0.5$. Thus,

$$P(X = 0) = \frac{(0.5)^0}{0!} e^{-0.5} = 0.607$$

OR

Let Y be number of consecutive square feet of the plastic panel. Then Y follows an exponential distribution with $\lambda = 0.05$. Thus, the probability that there are no surface flaws in an auto's interior (each car contains 10 square feet) is

$$P(Y > 10) = \int_{10}^{\infty} 0.05 e^{-0.05y} dy = \left[-e^{-0.05y}\right]_{10}^{\infty} = e^{-0.5} = 0.607$$

29. Refer to the previous question. How many automobiles should be examined on the average to obtain the first one with surface flaws in the auto's interior?

(a) 2.541 (b) 6.459 (c) 1.649 (d) 1.051 (e) 20.504

Solution: Let W be the number of automobiles to be examined until the first automobile with at least one flaw in its interior is obtained. The probability that an automobile has at least one flaw is 1 – 0.607 = 0.393. Thus,

$$E(W) = \frac{1}{p} = \frac{1}{0.393} = 2.541$$

30. Refer to the previous two questions. If 100 cars are sold to a rental company, what is the probability that at least 45 of them have any surface flaws?

(a) 0.9995 **(b) 0.1469** (c) 0.4129 (d) 0.8531 (e) 0.5871

Solution: Let U denote the number of cars with surface flaws among the 100 selected. Hence, U is binomial with $n = 100$ and $p = 0.393$. As $np = 39.347 > 5$ and $n(1 – p) = 60.653 > 5$, a normal approximation to binomial is justified in this case.

$$P(U \geq 45) \approx P\left(\frac{Y - np}{\sqrt{np(1-p)}} \geq \frac{45 - 0.5 - 39.347}{\sqrt{100(0.393)(0.607)}}\right) = P(Z \geq 1.05)$$

$= 1 – P(Z \leq 1.05) = 1 – 0.8531 = 0.1469$