STAT235 MIDTERM EXAM VERSION 111 SOLUTION

Complete the following (please **print**):

Section Number: ___________ Surname, First name: ______________________________

I.D. Number: ___________

Instructions

1. Enter your lecture section number, your last name (the same one that is illustrated on your One Card), first name (the same one that is illustrated on your One Card), and student ID # into the space provided above.

2. Make sure you use ONLY PENCIL to put and mark your name, student ID #, and your exam version number on the scantron sheet. Your name (surname and first name) should be entered into NAME block, your student ID # into IDENTIFICATION NUMBER block and finally the one-digit exam version into SPECIAL CODES block. The exam version is specified at the top of the exam. Make sure to shade in the circle that corresponds to the letter or digit in the box at the top of each column.

3. This is a multiple-choice closed book exam. There are **33** questions in the exam. For each question, carry out the appropriate analysis and put your answer on the scantron sheet by shading the letter A, B, C, D or E that corresponds to your chosen answer. Make sure your answers are clearly marked with ONLY PENCIL. Otherwise, no marks will be given. For each question exactly one of the five answers is correct. If you fill in more than one answer to a question, the question will be scored incorrect. Each question is worth 1 mark. All answers are rounded.

4. Please note that only the answers in the scantron sheet will be considered. If you initially mark your answers in the exam sheet, make sure that you copy them correctly into the scantron sheet. If you change an answer on the scantron sheet, be sure that you erase your first mark completely; then blacken the circle of the answer choice you prefer. Your score will be based on the number of questions you answer correctly. No credit will be given for omitted answers and no credit will be lost for wrong answers. You must mark all your answers on the scantron sheet during the allotted time. No additional time will be allowed at the end of the session for this purpose.

5. You are permitted to use a **non-programmable** calculator approved by the Faculty of Engineering. During the lab exam you are forbidden to use any devices with communication capabilities including cell phones and pagers. You are also forbidden to use any photographically capable devices in the exam room. Copying questions or answers on paper to take from the exam room is prohibited.

6. Note that the formula sheet and the table of the cumulative distribution function of the standard normal distribution are attached to the exam. There are **7** pages in the exam. The exam is graded out of **33** points.

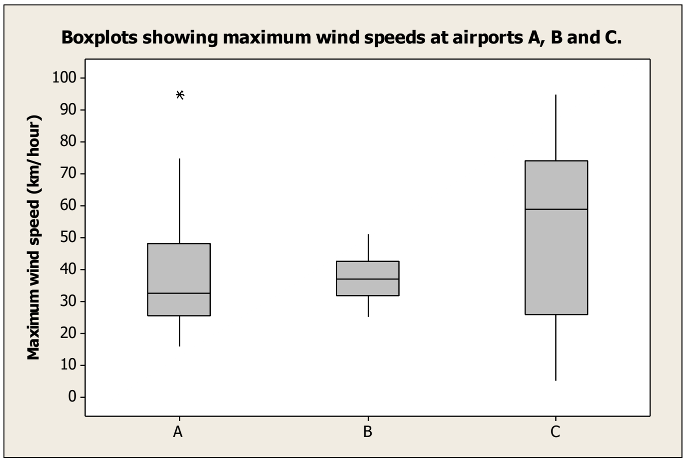
7. You must return your scantron and exam booklet when you finish the exam. You have 80 minutes to complete the exam.

8. Sign the exam booklet in the space provided below.

SIGNATURE: ________________________________

PART 1 (MULTIPLE-CHOICE QUESTIONS, 1 POINT EACH)

1. The side-by-side boxplots below show the maximum wind speeds reached on 200 randomly sampled days at three different airports (A, B, and C).



Given the above boxplots, which of the following statements about each distribution is most likely TRUE?

(a) A is left skewed, B is symmetric, C is right skewed, and the maximum wind speed in C exceeds 50 km/hour on more than 100 out of 200 days sampled,
(b) A is right skewed, B is symmetric, C is left skewed, and the maximum wind speed in C exceeds 50 km/hour on more than 100 out of 200 days sampled,
(c) A is right skewed, B is symmetric, C is left skewed, and the maximum wind speed in C does not exceed 50 km/hour on more than 100 out of 200 days sampled,
(d) A and C are both left skewed, B is symmetric, and the maximum wind speed in C exceeds 50 km/hour on more than 100 out of 200 days sampled,
(e) A is right skewed, B is symmetric, C is left skewed, and the maximum wind speed in C exceeds 70 km/hour on more than 150 out of 200 days sampled.

Solution: For A, median is closer to Q_1 than Q_3, upper whisker is longer than lower whisker, and outlier is present, so data are right-skewed. For B, the median is in the middle, whiskers are the same length, and there are no outliers, so data are symmetric. For C, median is closer to Q_3 than Q_1, so data are left-skewed. Median to upper whisker is 50% of the data, so more than half of data are above 50.

2. The maximum wind speeds (in km/hour) recorded on 16 random selected days at a certain airport are shown below (already arranged in ascending order):

7	30	33	35	37	39	39	40	41	43	44	52	55	60	82	106

The three quartiles (Q_1, Q_2, and Q_3, respectively) of this distribution are:

(a) 36, 40.5, 53.5
(b) 35, 40.5, 55
(c) 37, 40.5, 52
(d) 36, 41, 53.5
(e) 37, 41, 55

Solution: Even number of observations, so median is average of middle two numbers (40, 41). Each half also even, so Q_1 is average of middle two numbers of first half (35, 37) and Q_3 is average of middle two numbers of second half (52, 55).

3. Refer to the data in the previous question. Identify the outliers (if any) based on 1.5*IQR Rule.

(a) Only 106 **(b) 7, 82, and 106** (c) 7 and 106 (d) Only 7 (e) None

Solution: The outliers are any observations smaller than Q_1 – 1.5*IQR or larger than Q_3 + 1.5*IQR. Q_1 – 1.5*IQR = 36 – 1.5*(53.5 – 36) = 36 – 26.25 = 9.75 and Q_3 + 1.5*IQR = 53.5 + 26.25 = 79.75. Thus 7, 82, and 106 are the only outliers based on 1.5*IQR Rule.

4. If a probability distribution is left-skewed, which of the following is/are TRUE?

I. $Q_3 - Q_2 > Q_2 - Q_1$
II. $Q_3 - Q_2 < Q_2 - Q_1$
III. The median is less than the mean
IV. The median is greater than the mean
V. 50% of the observations are less than the mean
VI. The median is a more suitable measure of center than the mean and the interquartile range is a more suitable measure of spread than the standard deviation

(a) I and III **(b) II, IV, and VI** (c) I, IV, and VI (d) I, III, and VI (e) II, IV, V, and VI

Solution: In a left-skewed distribution, Q_3 is closer to the median than Q_1, the median is higher in magnitude than the mean, and resistant measures of center/spread are more suitable.

5. Hydraulic landing assemblies coming from an aircraft rework facility are each inspected for defects. Historical records indicate that 8% have defects in shafts, 6% have defects in bushings, and 2% have defects in both shafts and bushings. One of the hydraulic assemblies is selected randomly. What is the probability that the assembly has neither type of defect?

(a) 0.84 (b) 0.86 (c) 0.87 **(d) 0.88** (e) 0.90

Solution: Let *SH* denote the event that an assembly has defects in shafts and *B* the event that an assembly has defects in bushings.
$P(SH' \cap B') = 1 - P(SH \cup B)$ = 1 – (0.08 + 0.06 – 0.02) = 0.88

6. The investigation into the explosion of the space shuttle Challenger showed that the cause of the explosion was a gas leak through a joint that should have been sealed by large rubber O-rings. The shuttle's two solid rocket motors have a total of six such joints. If the probability of failure for a single joint in the shuttle is 0.023 and failures of those joints are independent, what is the probability that at least one joint fails?

(a) 0.003 **(b) 0.130** (c) 0.152 (d) 0.163 (e) 0.212

Solution: *P*(at least one joint fails) = 1 – *P*(all joints work) = $1 - (1 - 0.023)^6 = 0.130$

7. An urn contains 6 blue marbles and 4 white marbles. Two of the marbles are chosen at random. What is the probability that exactly one blue and one white are selected if drawing without replacement?

(a) 0.267 (b) 0.298 (c) 0.481 **(d) 0.533** (e) 0.621

Solution: Let B denote that a blue marble is selected and W that a white marble is selected.
$P((B \cap W) \cup (W \cap B)) = (6/10)*(4/9) + (4/10)*(6/9) = 8/15 \approx 0.533$.

8. The number of possible four-letter code words, selected from the 26 letters in the alphabet, in which all four letters are different is

(a) 234,772 (b) 261,567 (c) 279,841 **(d) 358,800** (e) 456,976

Solution: The first letter can be selected in 26 ways, the second letter in 25 ways (as it must be different from the first one), the third letter in 24 ways (as it must be different from the first two letters selected), and the last one in 23 ways. Thus, the number of four-letter code words in which all four letters are different is 26*25*24*23 = 358,800.

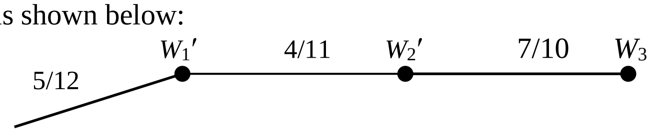
OR The number of four-letter codes is the number of four-letter permutations from the set of 26:

$$P_4^{26} = \frac{26!}{(26-4)!} = \frac{26\times 25\times 24\times 23\times 22!}{22!} = 26\times 25\times 24\times 23 = 358,800$$

9. A junk box in your room contains a dozen old batteries, five of which are totally dead. You start picking batteries one at a time (without replacement) and testing them. The probability that you have to pick 3 batteries in order to find one that works is

(a) 0.025 (b) 0.058 (c) 0.084 **(d) 0.106** (e) 0.127

Solution: Denote by W_i the event that an ith randomly selected battery works (i = 1, 2, 3). The tree graph is shown below:



$$P(W_1' \cap W_2' \cap W_3) = \frac{5}{12}\cdot\frac{4}{11}\cdot\frac{7}{10} = \frac{7}{66} \approx 0.106$$

10. The probability is 0.01 that an electrical connector, if kept dry, fails during the warranty period. If the connector is ever wet, the probability of a failure during the warranty period is 0.05. If 90% of the connectors are kept dry and 10% are wet, the proportion of connectors that will fail during the warranty period is

(a) 0.014 (b) 0.022 (c) 0.045 (d) 0.052 (e) 0.084

Solution: Let F denote that a connector fails and D that a connector is dry.
$P(F) = P(F \cap D) + P(F \cap D') = 0.01*0.90 + 0.10*0.05 = 0.014$.

11. In automotive repair, experience shows that engine problems can be attributed to bad ignition wires, bad spark plugs, or both. The probabilities of the occurrences are provided in the table given below.

		IGNITION WIRES	
		Bad	Good
Spark Plugs	Bad	0.20	0.60
	Good	0.15	0.05

What is the probability that the problem is either due to bad ignition wires alone or due to bad spark plugs alone?

(a) 0.42 (b) 0.48 (c) 0.60 (d) 0.65 **(e) 0.75**

Solution: P(bad ignition wires alone) **+** P(bad spark plugs alone) = 0.60 + 0.15 = 0.75

12. Refer to the previous question. A mechanic diagnosed the problem and found that the spark plugs are bad. What is the probability that the ignition wires are also bad?

(a) 0.2 (b) 0.8 (c) 0.78 **(d) 0.25** (e) 0.21

Solution: $P(\text{bad wires} \,|\, \text{bad spark plugs}) = \dfrac{0.2}{0.8} = 0.25$

13. Refer to the previous two questions. Let A denote that a selected engine problem is due to bad ignition wires and B that the selected engine problem is due to bad spark plugs. The events A and B are

(a) independent and mutually exclusive,
(b) independent but not mutually exclusive,
(c) not independent but mutually exclusive,
(d) not independent and not mutually exclusive,
(e) none of the above.

Solution: Not independent because $P(A \cap B) = 0.20 \neq P(A) \times P(B) = 0.35*0.80$.
Not mutually exclusive because $A \cap B$ is non-empty or, equivalently, $P(A \cap B) = 0.20 \neq 0$.

14. An urn contains three balls numbered 2, 3, and 4. If two balls are drawn from the urn at random without replacement and X is the sum of numbers on the two balls drawn, what is the probability distribution of X?

(a)

x	5	6	7
$P(X = x)$	0.20	0.16	0.64

(b)

x	5	6	7
$P(X = x)$	0.04	0.16	0.80

(c)

x	5	6	7
$P(X = x)$	0.20	0.32	0.48

(d)

x	5	6	7
$P(X = x)$	1/3	1/3	1/3

(e) The probability distribution cannot be obtained based on the information provided.

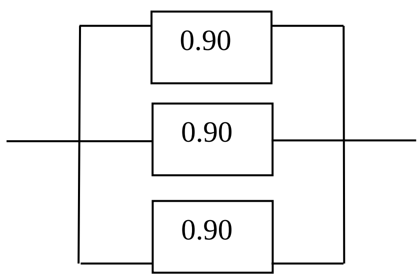
Solution: The possible sum of two numbers selected from the set {2, 3, 4} is either 5, 6, or 7. Let X_1 be the first number and X_2 the second number. Then, $X = X_1 + X_2$ and
$P(X = 5) = P((X_1 = 2 \cap X_2 = 3) \cup (X_1 = 3 \cap X_2 = 2)) = P(X_1 = 2 \cap X_2 = 3) + P(X_1 = 3 \cap X_2 = 2)$
$$= \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} = \frac{2}{6} = \frac{1}{3}$$
Similarly, $P(X = 6) = 1/3$ and $P(X = 7) = 1/3$.

15. Suppose that someone offers you the bet of tossing a fair coin and rolling a fair die. If a head and a 6 are obtained, you win $10. Otherwise, you must pay $2. What are your expected winnings?

(a) -$4 (b) -$3 (c) -$2 **(d) -$1** (e) $1

Solution: Let X be the expected winnings. The possible values of X are either $10 with the probability (1/2)*(1/6) = 1/12 or –$2 with the probability (1 – 1/12). Thus, the expected value of X is
$$E(X) = 10\left(\frac{1}{12}\right) + (-2)\left(1 - \frac{1}{12}\right) = -1$$

16. The following circuit consisting of three devices operates if and only if there is a path of functional devices from left to right. The probability that each device functions is 0.90 and the devices operate independently of one another.



The EXACT probability that the circuit operates is

(a) 0.7290 (b) 0.9801 (c) 0.9900 **(d) 0.9990** (e) 0.9999

Solution: The probability that the circuit fails (all devices fail) is $(1 - 0.90)^3 = 0.001$. Thus, the probability that the circuit operates is $1 - 0.001 = 0.999$.

17. The density function of a continuous random variable X is below. Find $P(X < 4)$.

$$f(x) = \begin{cases} x/4, & 0 \le x < 1, \\ \dfrac{8-x}{28}, & 1 \le x < 8. \end{cases}$$

(a) 4/14 (b) 35/28 **(c) 10/14** (d) 12/16 (e) 28/32

Solution: $P(X < 4) = 1 - P(X \ge 4) = 1 - \int_4^8 \left(\frac{8-x}{28}\right)dx = 1 - \frac{4}{14} = \frac{10}{14}$

18. The density function of a continuous random variable X is

$$f(x) = \begin{cases} (x+1)/2, & -1 \le x < 1, \\ 0, & \textit{elsewhere.} \end{cases}$$

What is the mean (expected value) of X?

(a) 1/54 **(b) 1/3** (c) 1/18 (d) 1/9 (e) 2/9

Solution: $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{1} x\frac{(x+1)}{2}dx = \frac{1}{2}\int_{-1}^{1}(x^2+x)dx = \frac{1}{2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{3}$

19. Refer to the previous question. What is the 75th percentile of X?

(a) 0.669 (b) 0.682 (c) 0.703 **(d) 0.732** (e) 0.785

Solution: The area under the density function to the left of the 75th percentile is the area of a triangle (half the area of a rectangle):

$\frac{1}{2}(x+1)(\frac{x+1}{2}) = 0.75$ *or equivalently* $(x+1)^2 = 3.$

The only feasible solution is $x+1 = \sqrt{3},$ *so* $x = \sqrt{3}-1 \approx 0.732.$

Another solution:

$F(x) = \int_{-1}^{x}\frac{t+1}{2}dt = \frac{1}{2}\left[\frac{t^2}{2}+t\right]_{-1}^{x} = \frac{1}{2}\left[\frac{x^2}{2}+x-\frac{1}{2}+1\right] = 0.75$, so $x^2+2x+1-3 = (x+1)^2-3 = 0$.

20. The temperature gauge of a cooling system has its temperature uniformly distributed between 64.5 and 89.5 degrees Kelvin. What percentage of time does the gauge represent temperatures below 70 or above 80 degrees Kelvin?

(a) 10% (b) 25% (c) 40% **(d) 60%** (e) 77%

Solution: Let X be the gauge temperature. Then X follows a uniform distribution between 64.5 and 89.5 with a density function being a flat line between 64.5 and 89.5. As the area under the density function between 64.5 and 89.5 is 1, the height of the density function is 1/25.
$P(X < 70 \cup X > 80) = 1 - P(70 < X < 80) = 1 - (10)(1/25) = 1 - 0.4 = 0.6$

21. A missile protection system consists of 5 radar sets operating independently, each with a probability of 0.9 of detecting a missile entering a zone that is covered by all of the units. If a missile enters the zone, what is the probability that exactly four radar sets detect the missile?

(a) 0.656 (b) 1.000 (c) 0.066 (d) 0.672 **(e) 0.328**

Solution: Let X be the number of radar sets that detect the missile. Then, $X \sim B(n = 5, p = 0.90)$.

$P(X = 4) = \binom{5}{4}(0.9)^4(1-0.9)^{5-4} = 5(0.9)^4(0.1)^1 = 0.328$

22. The half-life of the chemical isotope, Bohrium-260, has an exponential distribution with mean 35 milliseconds. If a particular isotope has not yet met its half-life after 25 milliseconds, what is the probability that it will meet its half-life in less than 5 additional milliseconds?

(a) $e^{-\frac{1}{5}}$ **(b)** $1 - e^{-\frac{1}{7}}$ (c) $e^{-\frac{5}{7}}$ (d) $e^{-\frac{1}{7}}$ (e) $1 - e^{-\frac{1}{5}}$

Solution: Let X be the half-life of the isotope, so X follows an exponential distribution with $E(X) = 35$, so $\lambda = 1/35$. Based on the lack of memory property of the exponential distribution, the probability that the isotope will meet its half-life in less than 5 milliseconds, given it has not met its half-life in the previous 25 milliseconds is the same as the probability of less than 5 milliseconds for a new isotope. In other words, the past history has no effect on the isotope's half-life.

$$P(X < 25+5 \mid X > 25) = P(X < 5) = P(X \le 5) = F(5) = 1 - e^{-(1/35)(5)} = 1 - e^{-1/7}$$

23. In comparing probability distributions, which of the following statements is TRUE?

(a) The Poisson distribution has the lack of memory property.
(b) The uniform and normal distributions are both symmetric.
(c) The exponential and Poisson distributions are interchangeable because they have the exact same random variable.
(d) The binomial distribution with $n = 10$ and $p = 0.10$ is symmetric.
(e) The negative binomial distribution is a special case of a geometric distribution.

Solution: Review all the mentioned distributions.

24. Suppose the number of times a student checks the clock during an exam follows a Poisson distribution with an average of 3 times per hour. What is the probability that the student will check the clock within the next 30 minutes?

(a) 0.664 (b) 0.681 (c) 0.732 **(d) 0.777** (e) 0.950

Solution: Let Y be the number of times the student checks the clock in the next 30 minutes (1/2 hour). As the number of times that the student checks the clock follows a Poisson distribution with the mean of 3 per hour, then Y follows a Poisson distribution with a mean of 1.5 per ½ hour.

$$P(Y \ge 1) = 1 - P(Y = 0) = 1 - \frac{e^{-1.5} 1.5^0}{0!} = 1 - e^{-1.5} = 0.777$$

OR Denote W = time until the next checking of the clock. Then, W follows an exponential distribution with $\lambda = 3$.

$$P(W < 1/2) = F(1/2) = 1 - e^{-3(1/2)} = 1 - e^{-1.5} = 0.777$$

25. A weather detection system consists of 7 weather stations operating independently, each with a probability of 0.57 of detecting violent weather such as hurricanes, typhoons, earthquakes, and smog. If violent weather passes through the region where the system is in place, what is the approximate probability that the fifth station is the first to detect the violent weather?

(a) 0.020 (b) 0.008 (c) 0.234 (d) 0.004 (e) 0.045

Solution: Let X be the number of stations that can detect violent weather. Then, X follows a geometric distribution with $p = 0.57$.
$P(X = 5) = (1 - 0.57)^4(0.57) = 0.020$

26. Suppose a particular technician at a ski production company in Canada inspects and decides which skis are great enough to be used at the 2014 Sochi Olympics. Based off the manufacturing process, each ski is produced independently and the probability of each one being great enough is 0.8. What is the approximate probability the technician needs to inspect 10 skis to find the eighth great ski?

    (a) 0.196 **(b) 0.242** (c) 0.331 (d) 0.383 (e) 0.461

    Solution: Let X be the number of skis to be inspected. Then, X follows a negative binomial distribution with the parameters $r = 8$ and $p = 0.8$.
    $$P(X = 10) = \binom{10-1}{8-1}(1-0.8)^{10-8}(0.8)^8 = 0.242$$

27. The resistance in milliohms of 1 meter of copper cable at a fixed temperature is normally distributed with a mean of 26.4 and a standard deviation of 1.32. What is the probability that a 1-meter segment of copper cable has a resistance between 23.4 and 27.6?

    (a) 0.9100 (b) 0.1930 **(c) 0.8070** (d) 0.7122 (e) 0.8302

    Solution: $P(23.4 \le X \le 27.6) = P\left(\frac{23.4-26.4}{1.32} \le Z = \frac{X-\mu}{\sigma} \le \frac{27.6-26.4}{1.32}\right) = P(-2.27 \le Z \le 0.91)$
    $= P(Z \le 0.91) - P(Z \le -2.27) = 0.8186 - 0.0116 = 0.8070$

28. Refer to the previous question. What is the resistance of a 1-meter segment of copper cable at the 99th percentile?

    (a) 29.5 (b) 29.8 (c) 23.3 (d) 27.5 (e) 30.5

    Solution: Let x_0 be the resistance at the 99th percentile. Thus, $P(X \le x_0) = 0.99$.
    $$P(X \le x_0) = P(Z = \frac{X-\mu}{\sigma} \le \frac{x_0 - 26.4}{1.32} = z_0) = 0.99$$
    From the table of Z, $z_0 = 2.33$. Thus, solving the equation gives
    $\frac{x_0 - 26.4}{1.32} = z_0 = 2.33$ for x_0 and we obtain $x_0 = 26.4 + (2.33)(1.32) = 29.5$.

29. The thicknesses of steel washers produced by a certain machine are normally distributed with a mean of 2.2 mm and a standard deviation of 0.03 mm. What are the mean and variance of the total thickness of 4 such steel washers?

    (a) 2.2, 0.0009 (b) 2.2, 0.04 (c) 8.8, 0.04 **(d) 8.8, 0.0036** (e) 8.8, 0.12

    Solution: $E(X_1 + X_2 + X_3 + X_4) = E(X_1) + E(X_2) + E(X_3) + E(X_4) = 4*2.2 = 8.8$
    $Var(X_1 + X_2 + X_3 + X_4) = Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) = 4*0.03^2 = 0.0036$

30. Refer to the previous question. What is the probability that the total thickness of four randomly selected washers exceeds 8.92 mm?

(a) 0.0228 (b) 0.0418 (c) 0.0516 (d) 0.0822 (e) 0.1075

Solution: The sum of four variables, each having a normal distribution in the previous question, also follows a normal distribution with the mean 8.8 and standard deviation $\sqrt{0.0036} = 0.06$. Thus,

$$P(X_1 + X_2 + X_3 + X_4 > 8.92) = P\left(Z = \frac{X_1 + X_2 + X_3 + X_4 - \mu_{X_1+X_2+X_3+X_4}}{\sigma_{X_1+X_2+X_3+X_4}} > \frac{8.92 - 8.8}{0.06} = 2 \right)$$

$$= P(Z > 2) = P(Z < -2) = 0.0228$$

31. Suppose that a certain model of laptops have a 75% chance of operating four years without defect. If a company purchases 800 such laptops (selecting them randomly and independently), what are the mean and standard deviation of the number of laptops that would be expected to operate four years without defect?

(a) 600, 12.25 (b) 600, 150 (c) 750, 150 (d) 200, 12.25 (e) 200, 150

Solution:

$E(X) = np = (800)(0.75) = 600$ Standard deviation $= \sqrt{np(1-p)} = \sqrt{(800)(0.75)(0.25)} = 12.25$

32. Refer to the previous question. What is the probability that at least 630 of these laptops will operate four years without defect?

(a) 0.5793 (b) 0.4207 **(c) 0.0080** (d) 0.9920 (e) 2.41

Solution:

$$P(X \geq 630) \approx P\left(\frac{Y - np}{\sqrt{np(1-p)}} \geq \frac{629.5 - 600}{12.25} \right) = P(Z \geq 2.41) = P(Z \leq -2.41) = 0.0080$$

33. Suppose the marks (denoted *X*) of students in a final exam follow a normal distribution, having a mean of 55 and a variance of 28. The instructor decides to apply the following linear transformation: *Y* = 1.1*X* + 5. What are the mean and variance of the transformed marks?

(a) 65.5, 30.8 (b) 60.5, 38.88 (c) 60.5, 33.88 **(d) 65.5, 33.88** (e) 65.5, 38.88

Solution: $E(Y) = E(1.1X + 5) = 1.1E(X) + 5 = 1.1(55) + 5 = 65.5$

$$Var(Y) = Var(1.1X + 5) = (1.1)^2 Var(X) + 0 = (1.1)^2 (28) = 33.88$$