**STAT235 MIDTERM EXAM VERSION – Long Answer Questions**

**Instructions**

1. In the second part, there are **<u>three</u>** long-answer problems. Show all your work because partial credit may be given in this part. Final numerical answers should have **<u>THREE</u>** significant decimal places (such as 0.00235) unless related to the *z*-table (probabilities are 4 decimal places, z-scores are 2).

2. Follow instructions carefully.

## LONG ANSWER QUESTIONS

**1. (4 marks)** Suppose that the total number of arrivals into a queue follow a Poisson distribution with an average of 2 per hour. What is the probability that the total number of arrivals in the next hour is no more than 5, given that the total number of arrivals over the next hour is at least 2?

If $X$ is the number of arrivals per hour, then $X$ follows a Poisson distribution with $\lambda = 2$.

$$P(X \le 5 \mid X \ge 2) = \frac{P(X \le 5 \cap X \ge 2)}{P(X \ge 2)} = \frac{P(2 \le X \le 5)}{1 - P(X < 2)}$$

$$= \frac{P(X=2)+P(X=3)+P(X=4)+P(X=5)}{1-[P(X=0)+P(X=1)]} = \frac{0.271+0.180+0.0902+0.0361}{1-[0.135+0.271]}$$

$$= \frac{0.577}{1-0.406} = \frac{0.577}{0.594} = 0.972$$

**2. (6 total marks)** A new battery, supposedly with a charge of 1.5 volts, actually has a voltage ($X$) following a normal distribution with a mean of 1.51 volts and a standard deviation of 0.08 volts. The battery is considered unsafe for use in children's toys if its voltage is greater than 1.57 volts.

**(a) (3 marks)** What proportion of batteries is unsafe for use in children's toys?

$$P(X > 1.57) \rightarrow P\left(\frac{X-\mu}{\sigma} > \frac{1.57-1.51}{0.08}\right) = P(Z > 0.75) = P(Z < -0.75) = 0.2266$$

**(b) (3 marks)** What values of $X$ represent the first and third quartiles for this distribution, respectively?

Look for 0.25 and 0.75 in table; $z$ = -0.67 and $z$ = 0.67, respectively.

$x_1 = \mu + z\sigma = 1.51 + (-0.67)(0.08) = 1.456$
$x_2 = \mu + z\sigma = 1.51 + (0.67)(0.08) = 1.564$

The first and third quartiles are represented by the respective values of 1.456 and 1.564 volts.

**3.** **(10 total marks)** The lifetime T of a component has the following probability density function.

$$f(t) = \begin{cases} k(t-1)(2-t) & , \quad 1 \le t \le 2 \\ 0 & , \quad otherwise \end{cases}$$

**(a) (2 marks)** Determine the value of the constant *k*. Do not use a calculator here. All steps must be shown.

$$\int_1^2 k(t-1)(2-t)dt = k\int_1^2 (-t^2+3t-2)dt = k\left[-\frac{1}{3}t^3+\frac{3}{2}t^2-2t\right]_1^2$$

$$= k\left[(-\frac{1}{3}2^3+\frac{3}{2}2^2-2(2))-(-\frac{1}{3}1^3+\frac{3}{2}1^2-2(1))\right] = k\left[(-\frac{8}{3}+6-4)-(-\frac{1}{3}+\frac{3}{2}-2)\right]$$

$$= k\left(\frac{-8+18-12+1-4.5+6}{3}\right) = k\left(\frac{0.5}{3}\right) = k\left(\frac{1}{6}\right)$$

Since $\int_1^2 k(t-1)(2-t)dt = 1$, then $k = 6$.

**(b) (3 marks)** Find the mean of *T*. Do not use a calculator here. All steps must be shown.

$$E(T) = \int_1^2 6t(-t^2+3t-2)dt = \int_1^2 6(-t^3+3t^2-2t)dt = 6\left[-\frac{1}{4}t^4+\frac{3}{3}t^3-\frac{2}{2}t^2\right]_1^2$$

$$= 6\left[(-\frac{1}{4}2^4+2^3-2^2)-(-\frac{1}{4}1^4+1^3-1^2)\right] = 6\left[(-4+8-4)-(-\frac{1}{4}+1-1)\right] = 6\left(\frac{1}{4}\right) = \frac{3}{2}$$

**(c) (5 marks)** Find the standard deviation of *T*. Do not use a calculator here. All steps must be shown.

$$E(T^2) = \int_1^2 6t^2(-t^2+3t-2)dt = \int_1^2 6(-t^4+3t^3-2t^2)dt = 6\left[-\frac{1}{5}t^5+\frac{3}{4}t^4-\frac{2}{3}t^3\right]_1^2$$

$$= 6\left[(-\frac{1}{5}2^5+\frac{3}{4}2^4-\frac{2}{3}2^3)-(-\frac{1}{5}1^5+\frac{3}{4}1^4-\frac{2}{3}1^3)\right] = 6\left[(-\frac{32}{5}+12-\frac{16}{3})-(-\frac{1}{5}+\frac{3}{4}-\frac{2}{3})\right] = 6\left(\frac{23}{60}\right) = \frac{23}{10}$$

$$V(T) = E(T^2)-[E(T)]^2 = \frac{23}{10}-\left(\frac{3}{2}\right)^2 = \frac{23}{10}-\frac{9}{4} = \frac{46-45}{20} = \frac{1}{20}$$

$$\sigma = \sqrt{V(T)} = \sqrt{1/20} = 0.224$$