

Ellipsoid Method and the Amazing Oracles (I)

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Introduction

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Introduction



When you have eliminated the impossible, whatever remains, however improbable, must be the truth.

Sir Arthur Conan Doyle, stated by Sherlock Holmes



Common Perspective of Ellipsoid Method

- ▶ It is commonly believed that it is inefficient in practice for large-scale problems.
 - ▶ The convergent rate is slow, even with the use of deep cuts.
 - ▶ Cannot exploit sparsity.
- ▶ Since then, it was supplanted by interior-point methods.
- ▶ Only treated as a theoretical tool for proving the polynomial-time solvability of combinatorial optimization problems.



But...

- ▶ The ellipsoid method works very differently compared with the interior point method.
- ▶ Only require a *separation oracle*. Can play nicely with other techniques.
- ▶ The ellipsoid method itself cannot exploit sparsity, but the oracle can.

Consider Ellipsoid Method When...

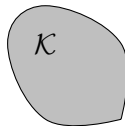
- ▶ The number of optimization variables is moderate, e.g. ECO flow, analog circuit sizing, parametric problems
- ▶ The number of constraints is large, or even infinite
- ▶ Oracle can be implemented efficiently.



Cutting-plane Method Revisited

Basic Idea

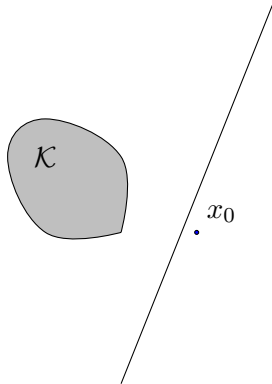
- ▶ Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a convex set.
- ▶ Consider the feasibility problem:
 - ▶ Find a point $x^* \in \mathbb{R}^n$ in \mathcal{K} ,
 - ▶ or determine that \mathcal{K} is empty (i.e., no feasible solution)



Separation Oracle

- ▶ When a separation oracle Ω is *queried* at x_0 , it either
 - ▶ asserts that $x_0 \in \mathcal{K}$, or
 - ▶ returns a separating hyperplane between x_0 and \mathcal{K} :

$$g^\top(x - x_0) + \beta \leq 0, \beta \geq 0, g \neq 0, \forall x \in \mathcal{K}$$



Separation oracle (cont'd)

- ▶ (g, β) is called a *cutting-plane*, or cut, since it eliminates the halfspace $\{x \mid g^\top(x - x_0) + \beta > 0\}$ from our search.
- ▶ If $\beta = 0$ (x_0 is on the boundary of halfspace that is cut), cutting-plane is called *neutral cut*.
- ▶ If $\beta > 0$ (x_0 lies in the interior of halfspace that is cut), cutting-plane is called *deep cut*.
- ▶ If $\beta < 0$ (x_0 lies in the exterior of halfspace that is cut), cutting-plane is called *shallow cut*.

Subgradient

- ▶ \mathcal{K} is usually given by a set of inequalities $f_j(x) \leq 0$ or $f_j(x) < 0$ for $j = 1 \cdots m$, where $f_j(x)$ is a convex function.
- ▶ A vector $g \equiv \partial f(x_0)$ is called a subgradient of a convex function f at x_0 if $f(z) \geq f(x_0) + g^T(z - x_0)$.
- ▶ Hence, the cut (g, β) is given by $(\partial f(x_0), f(x_0))$

Remarks:

- ▶ If $f(x)$ is differentiable, we can simply take $\partial f(x_0) = \nabla f(x_0)$

Key components of Cutting-plane method

- ▶ Cutting plane oracle Ω
- ▶ A search space \mathcal{S} initially big enough to cover \mathcal{K} , e.g.
 - ▶ Polyhedron $\mathcal{P} = \{z \mid Cz \preceq d\}$
 - ▶ Interval $\mathcal{I} = [l, u]$ (for one-dimensional problem)
 - ▶ Ellipsoid $\mathcal{E} = \{z \mid (z - x_c)P^{-1}(z - x_c) \leq 1\}$

Generic Cutting-plane method

- ▶ **Given** initial \mathcal{S} known to contain \mathcal{K} .
- ▶ **Repeat**
 1. Choose a point x_0 in \mathcal{S}
 2. Query the cutting-plane oracle at x_0
 3. **If** $x_0 \in \mathcal{K}$, quit
 4. **Else**, update \mathcal{S} to a smaller set that covers:

$$\mathcal{S}^+ = \mathcal{S} \cap \{z \mid g^\top(z - x_0) + \beta \leq 0\}$$

5. **If** $\mathcal{S}^+ = \emptyset$ or it is small enough, quit.

Corresponding Python code

```
def cutting_plane_feas(evaluate, S, options=Options()):
    feasible = False
    status = 0
    for niter in range(options.max_it):
        cut, feasible = evaluate(S.xc)
        if feasible: # feasible sol'n obtained
            break
        status, tsq = S.update(cut)
        if status != 0: # empty cut
            break
        if tsq < options.tol:
            status = 2
            break
    return S.xc, niter+1, feasible, status
```

From Feasibility to Optimization

$$\begin{array}{ll}\text{minimize} & f_0(\mathbf{x}), \\ \text{subject to} & \mathbf{x} \in \mathcal{K}\end{array}$$

- ▶ The optimization problem is treated as a feasibility problem with an additional constraint $f_0(\mathbf{x}) \leq t$
- ▶ $f_0(\mathbf{x})$ could be a convex function or a *quasiconvex function*.
- ▶ t is also called the *best-so-far* value of $f_0(\mathbf{x})$.



Convex Optimization Problem

- Consider the following general form:

$$\begin{array}{ll}\text{minimize} & t, \\ \text{subject to} & \Phi(x, t) \leq 0 \\ & x \in \mathcal{K}\end{array}$$

where $\mathcal{K}'_t = \{x \mid \Phi(x, t) \leq 0\}$ is the t -sublevel set of $\{x \mid f_0(x) \leq t\}$.

- Note: $\mathcal{K}'_t \subseteq \mathcal{K}'_u$ if and only if $t \leq u$ (monotonicity)
- One easy way to solve the optimization problem is to apply the binary search on t .



```
def bsearch(evaluate, I, options=Options()):
    feasible = False
    l, u = I
    t = l + (u - l)/2
    for niter in range(options.max_it):
        if evaluate(t): # feasible sol'n obtained
            feasible = True
            u = t
        else:
            l = t
            tau = (u - l)/2
            t = l + tau
        if tau < options.tol:
            break
    return u, niter+1, feasible
```



```
class bsearch_adaptor:
    def __init__(self, P, E, options=Options()):
        self.P = P
        self.E = E
        self.options = options

    @property
    def x_best(self):
        return self.E.xc

    def __call__(self, t):
        E = self.E.copy()
        self.P.update(t)
        x, _, feasible, _ = cutting_plane_feas(
            self.P, E, self.options)
        if feasible:
            self.E._xc = x.copy()
            return True
        return False
```



Shrinking

- ▶ Another possible way is, to update the best-so-far t whenever a feasible solution x_0 is found by solving the equation $\Phi(x_0, t_{\text{new}}) = 0$.

Generic Cutting-plane method (Optim)

- ▶ **Given** initial \mathcal{S} known to contain \mathcal{K}_t .
- ▶ **Repeat**
 1. Choose a point x_0 in \mathcal{S}
 2. Query the separation oracle at x_0
 3. **If** $x_0 \in \mathcal{K}_t$, update t such that $\Phi(x_0, t) = 0$.
 4. Update \mathcal{S} to a smaller set that covers:

$$\mathcal{S}^+ = \mathcal{S} \cap \{z \mid g^\top(z - x_0) + \beta \leq 0\}$$

5. **If** $\mathcal{S}^+ = \emptyset$ or it is small enough, quit.

```

def cutting_plane_dc(evaluate, S, t, options=Options()):
    feasible = False # no sol'n
    x_best = S.xc
    for niter in range(options.max_it):
        cut, t1 = evaluate(S.xc, t)
        if t != t1: # best t obtained
            feasible = True
            t = t1
            x_best = S.xc
        status, tau = S.update(cut)
        if status != 0: # empty cut
            break
        if tau < options.tol:
            status = 2
            break
    return x_best, t, niter+1, feasible, status

```

Example: Profit Maximization Problem

This example is taken from [?].

$$\begin{array}{ll}\text{maximize} & p(Ax_1^\alpha x_2^\beta) - v_1x_1 - v_2x_2 \\ \text{subject to} & x_1 \leq k.\end{array}$$

- ▶ $p(Ax_1^\alpha x_2^\beta)$: Cobb-Douglas production function
- ▶ p : the market price per unit
- ▶ A : the scale of production
- ▶ α, β : the output elasticities
- ▶ x : input quantity
- ▶ v : output price
- ▶ k : a given constant that restricts the quantity of x_1

Example: Profit maximization (cont'd)

- ▶ The formulation is not in the convex form.
- ▶ Rewrite the problem in the following form:

$$\begin{array}{ll}\text{maximize} & t \\ \text{subject to} & t + v_1x_1 + v_2x_2 < pAx_1^\alpha x_2^\beta \\ & x_1 \leq k.\end{array}$$

Profit maximization in Convex Form

- ▶ By taking the logarithm of each variable:
 - ▶ $y_1 = \log x_1, y_2 = \log x_2$.
- ▶ We have the problem in a convex form:

$$\begin{array}{ll}\max & t \\ \text{s.t.} & \log(t + v_1 e^{y_1} + v_2 e^{y_2}) - (\alpha y_1 + \beta y_2) < \log(pA) \\ & y_1 \leq \log k.\end{array}$$

Python code (Profit oracle) I

```
class profit_oracle:
    def __init__(self, params, a, v):
        p, A, k = params
        self.log_pA = np.log(p * A)
        self.log_k = np.log(k)
        self.v = v
        self.a = a

    def __call__(self, y, t):
        fj = y[0] - self.log_k # constraint
        if fj > 0.:
            g = np.array([1., 0.])
            return (g, fj), t
        log_Cobb = self.log_pA + np.dot(self.a, y)
        x = np.exp(y)
        vx = np.dot(self.v, x)
        te = t + vx
        fj = np.log(te) - log_Cobb
        if fj < 0.:
            te = np.exp(log_Cobb)
            t = te - vx
            fj = 0.
```

Python code (Profit oracle) II

```
g = (self.v * x) / te - self.a  
return (g, fj), t
```

Python code (Main program) I

```
import numpy as np
from profit_oracle import *
from cutting_plane import *
from ell import *

p, A, k = 20.0, 40.0, 30.5
params = p, A, k
a = np.array([0.1, 0.4])
v = np.array([10.0, 35.0])
y0 = np.array([0., 0.]) # initial x0
E = ell(200, y0)
P = profit_oracle(params, a, v)
yb1, fb, niter, feasible, status = \
    cutting_plane_dc(P, E, 0.0)
print(fb, niter, feasible, status)
```

Area of Applications

- ▶ Robust convex optimization
 - ▶ oracle technique: affine arithmetic
- ▶ Parametric network potential problem
 - ▶ oracle technique: negative cycle detection
- ▶ Semidefinite programming
 - ▶ oracle technique: Cholesky factorization



Robust Convex Optimization



Robust Optimization Formulation

- Consider:

$$\begin{array}{ll}\text{minimize} & \sup_{q \in \mathbb{Q}} f_0(\mathbf{x}, q) \\ \text{subject to} & f_j(\mathbf{x}, q) \leq 0, \quad \forall q \in \mathbb{Q}, \quad j = 1, 2, \dots, m,\end{array}$$

where q represents a set of varying parameters.

- The problem can be reformulated as:

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & f_0(\mathbf{x}, q) \leq t \\ & f_j(\mathbf{x}, q) \leq 0, \quad \forall q \in \mathbb{Q}, \quad j = 1, 2, \dots, m,\end{array}$$

Oracle in Robust Optimization Formulation

- ▶ The oracle only needs to determine:
 - ▶ If $f_j(\mathbf{x}_0, q) > 0$ for some j and $q = q_0$, then
 - ▶ the cut $(g, \beta) = (\partial f_j(\mathbf{x}_0, q_0), f_j(\mathbf{x}_0, q_0))$
 - ▶ If $f_0(\mathbf{x}_0, q) \geq t$ for some $q = q_0$, then
 - ▶ the cut $(g, \beta) = (\partial f_0(\mathbf{x}_0, q_0), f_0(\mathbf{x}_0, q_0) - t)$
 - ▶ Otherwise, \mathbf{x}_0 is feasible, then
 - ▶ Let $q_{\max} = \operatorname{argmax}_{q \in \mathbb{Q}} f_0(\mathbf{x}_0, q)$.
 - ▶ $t := f_0(\mathbf{x}_0, q_{\max})$.
 - ▶ The cut $(g, \beta) = (\partial f_0(\mathbf{x}_0, q_{\max}), 0)$
- ▶ Random sampling trick

Example: Profit Maximization Problem (convex)

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & \log(t + \hat{v}_1 e^{y_1} + \hat{v}_2 e^{y_2}) - (\hat{\alpha} y_1 + \hat{\beta} y_2) \leq \log(\hat{p} A) \\ & y_1 \leq \log \hat{k}, \end{aligned}$$

- ▶ Now assume that:
 - ▶ $\hat{\alpha}$ and $\hat{\beta}$ vary $\bar{\alpha} \pm e_1$ and $\bar{\beta} \pm e_2$ respectively.
 - ▶ \hat{p} , \hat{k} , \hat{v}_1 , and \hat{v}_2 all vary $\pm e_3$.

Example: Profit Maximization Problem (oracle)

By detail analysis, the worst case happens when:

- ▶ $p = \bar{p} + e_3, k = \bar{k} + e_3$
- ▶ $v_1 = \bar{v}_1 - e_3, v_2 = \bar{v}_2 - e_3,$
- ▶ if $y_1 > 0, \alpha = \bar{\alpha} - e_1$, else $\alpha = \bar{\alpha} + e_1$
- ▶ if $y_2 > 0, \beta = \bar{\beta} - e_2$, else $\beta = \bar{\beta} + e_2$

Remark:

- ▶ for more complicated problems, affine arithmetic could be used [?].

profit_rb_oracle

```
class profit_rb_oracle:
    def __init__(self, params, a, v, vparams):
        ui, e1, e2, e3 = vparams
        self.uie = [ui * e1, ui * e2]
        self.a = a
        p, A, k = params
        p -= ui * e3
        k -= ui * e3
        v_rb = v.copy()
        v_rb += ui * e3
        self.P = profit_oracle((p, A, k), a, v_rb)

    def __call__(self, y, t):
        a_rb = self.a.copy()
        for i in [0, 1]:
            a_rb[i] += self.uie[i] if y[i] <= 0. \
                               else -self.uie[i]

        self.P.a = a_rb
        return self.P(y, t)
```

Multi-parameter Network Problem



Parametric Network Problem

Given a network represented by a directed graph $G = (V, E)$.

Consider:

$$\begin{array}{ll} \text{find} & \mathbf{x}, \mathbf{u} \\ \text{subject to} & \mathbf{u}_j - \mathbf{u}_i \leq h_{ij}(\mathbf{x}), \forall (i, j) \in E, \end{array}$$

- ▶ $h_{ij}(\mathbf{x})$ is the concave function of edge (i, j) ,
- ▶ Assume: network is large but the number of parameters is small.

Network Potential Problem (cont'd)

Given x , the problem has a feasible solution if and only if G contains no negative cycle. Let \mathcal{C} be a set of all cycles of G .

$$\begin{array}{ll}\text{find} & x \\ \text{subject to} & w_k(x) \geq 0, \forall C_k \in \mathcal{C},\end{array}$$

- ▶ C_k is a cycle of G
- ▶ $w_k(x) = \sum_{(i,j) \in C_k} h_{ij}(x)$.

Negative Cycle Finding

There are lots of methods to detect negative cycles in a weighted graph [?], in which Tarjan's algorithm [?] is one of the fastest algorithms in practice [??].



Oracle in Network Potential Problem

- ▶ The oracle only needs to determine:
 - ▶ If there exists a negative cycle C_k under x_0 , then
 - ▶ the cut $(g, \beta) = (-\partial w_k(x_0), -w_k(x_0))$
 - ▶ Otherwise, the shortest path solution is u

Python Code

```
class network_oracle:
    def __init__(self, G, f, p):
        self.G = G
        self.f = f
        self.p = p # partial derivative of f w.r.t x
        self.S = negCycleFinder(G)

    def __call__(self, x):
        def get_weight(G, e):
            return self.f(G, e, x)

        self.S.get_weight = get_weight
        C = self.S.find_neg_cycle()
        if C is None:
            return None, 1
        f = -sum(self.f(self.G, e, x) for e in C)
        g = -sum(self.p(self.G, e, x) for e in C)
        return (g, f), 0
```



Example: Optimal Matrix Scaling [?]

- ▶ Given a sparse matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$.
- ▶ Find another matrix $B = UAU^{-1}$ where U is a nonnegative diagonal matrix, such that the ratio of any two elements of B in absolute value is as close to 1 as possible.
- ▶ Let $U = \text{diag}([u_1, u_2, \dots, u_N])$. Under the min-max-ratio criterion, the problem can be formulated as:

$$\begin{array}{ll}\text{minimize} & \pi/\psi \\ \text{subject to} & \psi \leq u_i |a_{ij}| u_j^{-1} \leq \pi, \quad \forall a_{ij} \neq 0, \\ & \pi, \psi, u, \text{ positive} \\ \text{variables} & \pi, \psi, u.\end{array}$$

Optimal Matrix Scaling (cont'd)

By taking the logarithms of variables, the above problem can be transformed into:

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & \pi' - \psi' \leq t \\ & u'_i - u'_j \leq \pi' - a'_{ij}, \quad \forall a_{ij} \neq 0, \\ & u'_j - u'_i \leq a'_{ij} - \psi', \quad \forall a_{ij} \neq 0, \\ \text{variables} & \pi', \psi', u'.\end{array}$$

where k' denotes $\log(|k|)$ and $x = (\pi', \psi')^\top$.



Corresponding Python Code

```
def constr(G, e, x):
    u, v = e
    i_u = G.node_idx[u]
    i_v = G.node_idx[v]
    cost = G[u][v]['cost']
    return x[0] - cost if i_u <= i_v else cost - x[1]

def pconstr(G, e, x):
    u, v = e
    i_u = G.node_idx[u]
    i_v = G.node_idx[v]
    return np.array([1., 0.] if i_u <= i_v else [0., -1.])

class optscaling_oracle:
    def __init__(self, G):
        self.network = network_oracle(G, constr, pconstr)

    def __call__(self, x, t):
        cut, feasible = self.network(x)
        if not feasible: return cut, t
        s = x[0] - x[1]
        fj = s - t
        if fj < 0.:
            t = s
```

Example: clock period & yield-driven co-optimization

$$\begin{array}{ll}\text{minimize} & T_{CP} - w_{\beta}\beta \\ \text{subject to} & u_i - u_j \leq T_{CP} - F_{ij}^{-1}(\beta), \quad \forall (i, j) \in E_s, \\ & u_j - u_i \leq F_{ij}^{-1}(1 - \beta), \quad \forall (j, i) \in E_h, \\ & T_{CP} \geq 0, \quad 0 \leq \beta \leq 1, \\ \text{variables} & T_{CP}, \beta, u.\end{array}$$

- ▶ Note that $F_{ij}^{-1}(x)$ is not concave in general in $[0, 1]$.
- ▶ Fortunately, we are most likely interested in optimizing circuits for high yield rather than the low one in practice.
- ▶ Therefore, by imposing an additional constraint to β , say $\beta \geq 0.8$, the problem becomes convex.

Matrix Inequalities

Problems With Matrix Inequalities

Consider the following problem:

$$\begin{array}{ll} \text{find} & x, \\ \text{subject to} & F(x) \succeq 0, \end{array}$$

- ▶ $F(x)$: a matrix-valued function
- ▶ $A \succeq 0$ denotes A is positive semidefinite.

Problems With Matrix Inequalities

- ▶ Recall that a matrix A is positive semidefinite if and only if $v^T A v \geq 0$ for all $v \in \mathbb{R}^N$.
- ▶ The problem can be transformed into:

$$\begin{array}{ll} \text{find} & \mathbf{x}, \\ \text{subject to} & v^T F(\mathbf{x}) v \geq 0, \forall v \in \mathbb{R}^N \end{array}$$

- ▶ Consider $v^T F(\mathbf{x}) v$ is concave for all $v \in \mathbb{R}^N$ w. r. t. \mathbf{x} , then the above problem is a convex programming.
- ▶ Reduce to *semidefinite programming* if $F(\mathbf{x})$ is linear w.r.t. \mathbf{x} , i.e.,
 $F(\mathbf{x}) = F_0 + x_1 F_1 + \cdots + x_n F_n$

Oracle in Matrix Inequalities

The oracle only needs to:

- ▶ Perform a *row-based* LDLT factorization such that $F(x_0) = LDL^\top$.
- ▶ Let $A_{:p,:p}$ denotes a submatrix $A(1:p, 1:p) \in \mathbb{R}^{p \times p}$.
- ▶ If the process fails at row p ,
 - ▶ there exists a vector $e_p = (0, 0, \dots, 0, 1)^\top \in \mathbb{R}^p$, such that
 - ▶ $v = R_{:p,:p}^{-1} e_p$, and
 - ▶ $v^\top F_{:p,:p}(x_0) v < 0$.
 - ▶ The cut $(g, \beta) = (-v^\top \partial F_{:p,:p}(x_0) v, -v^\top F_{:p,:p}(x_0) v)$

```

class lmi_oracle:
    ''' Oracle for LMI constraint  $F*x \leq B$  '''

    def __init__(self, F, B):
        self.F = F
        self.F0 = B
        self.Q = chol_ext(len(self.F0))

    def __call__(self, x):
        n = len(x)

        def getA(i, j):
            return self.F0[i, j] - sum(
                self.F[k][i, j] * x[k] for k in range(n))

        self.Q.factor(getA)
        if self.Q.is_spd():
            return None, True
        v, ep = self.Q.witness()
        g = np.array([self.Q.sym_quad(v, self.F[i])
                       for i in range(n)])
        return (g, ep), False

```



Example: Matrix Norm Minimization

- ▶ Let $A(\mathbf{x}) = A_0 + x_1 A_1 + \cdots + x_n A_n$
- ▶ Problem $\min_{\mathbf{x}} \|A(\mathbf{x})\|$ can be reformulated as

$$\begin{aligned} & \text{minimize } t, \\ & \text{subject to } \begin{pmatrix} tI & A(\mathbf{x}) \\ A^\top(\mathbf{x}) & tI \end{pmatrix} \succeq 0, \end{aligned}$$

- ▶ Binary search on t can be used for this problem.

```

class qmi_oracle:
    t = None
    count = 0

    def __init__(self, F, F0):
        self.F = F
        self.F0 = F0
        self.Fx = np.zeros(F0.shape)
        self.Q = chol_ext(len(F0))

    def update(self, t): self.t = t

    def __call__(self, x):
        self.count = 0; nx = len(x)

        def getA(i, j):
            if self.count < i + 1:
                self.count = i + 1
                self.Fx[i] = self.F0[i]
                self.Fx[i] -= sum(self.F[k][i] * x[k]
                                for k in range(nx))
            a = -self.Fx[i].dot(self.Fx[j])
            if i == j: a += self.t
            return a

        self.Q.factor(getA)
        if self.Q.is_spd(): return None, True

```



Example: Estimation of Correlation Function

$$\begin{array}{ll} \min_{\kappa, p} & \|\Omega(p) + \kappa I - Y\| \\ \text{s. t.} & \Omega(p) \succcurlyeq 0, \kappa \geq 0. \end{array}$$

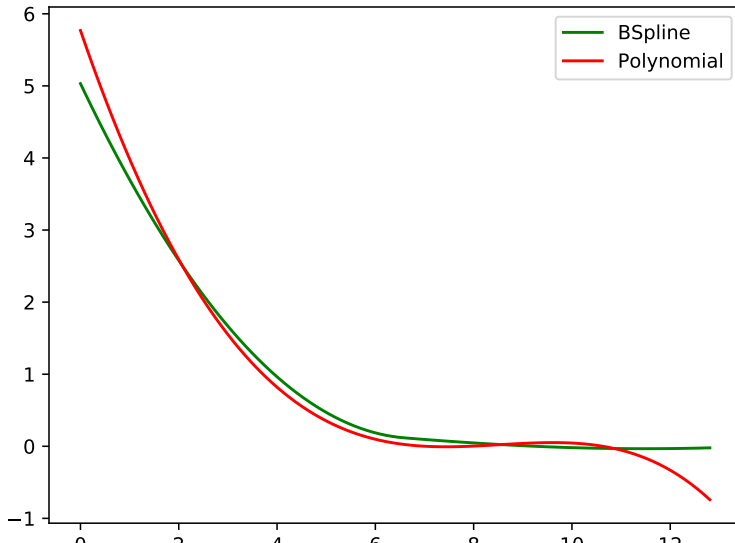
- ▶ Let $\rho(h) = \sum_i^n p_i \Psi_i(h)$, where
 - ▶ p_i 's are the unknown coefficients to be fitted
 - ▶ Ψ_i 's are a family of basis functions.
- ▶ The covariance matrix $\Omega(p)$ can be recast as:

$$\Omega(p) = p_1 F_1 + \cdots + p_n F_n$$

where $\{F_k\}_{i,j} = \Psi_k(\|s_j - s_i\|_2)$



Experimental Result



Reference I

