

# Ellipsoid Method and the Amazing Oracles (I)

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# Introduction



*When you have eliminated the impossible, whatever remains, however improbable, must be the truth.*

*Sir Arthur Conan Doyle, stated by Sherlock Holmes*



# Common Perspective of Ellipsoid Method

- ▶ It is commonly believed that it is inefficient in practice for large-scale problems.
  - ▶ The convergent rate is slow, even with the use of deep cuts.
  - ▶ Cannot exploit sparsity.
- ▶ Since then, it was supplanted by interior-point methods.
- ▶ Only treated as a theoretical tool for proving the polynomial-time solvability of combinatorial optimization problems.



But...

- ▶ The ellipsoid method works very differently compared with the interior point method.
- ▶ Only require a cutting-plane oracle. Can play nicely with other techniques.
- ▶ The oracle can exploit sparsity.

## Consider Ellipsoid Method When...

- ▶ The number of optimization variables is moderate, e.g. ECO flow, analog circuit sizing, parametric problems
- ▶ The number of constraints is large, or even infinite
- ▶ Oracle can be implemented efficiently.

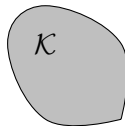


# Cutting-plane Method Revisited



# Basic Idea

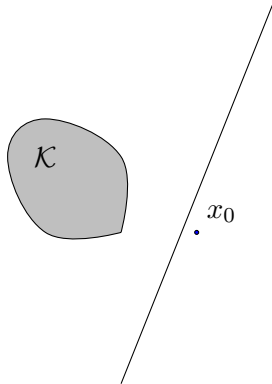
- ▶ Let  $\mathcal{K} \subseteq \mathbb{R}^n$  be a convex set.
- ▶ Consider the feasibility problem:
  - ▶ Find a point  $x^* \in \mathbb{R}^n$  in  $\mathcal{K}$ ,
  - ▶ or determine that  $\mathcal{K}$  is empty (i.e., no feasible solution)



# Separation Oracle

- ▶ When a separation oracle  $\Omega$  is *queried* at  $x_0$ , it either
  - ▶ asserts that  $x_0 \in \mathcal{K}$ , or
  - ▶ returns a separating hyperplane between  $x_0$  and  $\mathcal{K}$ :

$$g^T(x - x_0) + \beta \leq 0, \beta \geq 0, g \neq 0, \forall x \in \mathcal{K}$$



## Separation oracle (cont'd)

- ▶  $(g, \beta)$  called a *cutting-plane*, or cut, since it eliminates the halfspace  $\{x \mid g^T(x - x_0) + \beta > 0\}$  from our search.
- ▶ If  $\beta = 0$  ( $x_0$  is on the boundary of halfspace that is cut), cutting-plane is called *neutral cut*.
- ▶ If  $\beta > 0$  ( $x_0$  lies in the interior of halfspace that is cut), cutting-plane is called *deep cut*.



# Subgradient

- ▶  $\mathcal{K}$  is usually given by a set of inequalities  $f_j(x) \leq 0$  or  $f_j(x) < 0$  for  $j = 1 \cdots m$ , where  $f_j(x)$  is a convex function.
- ▶ A vector  $g \equiv \partial f(x_0)$  is called a subgradient of a convex function  $f$  at  $x_0$  if  $f(z) \geq f(x_0) + g^T(z - x_0)$ .
- ▶ Hence, the cut  $(g, \beta)$  is given by  $(\partial f(x_0), f(x_0))$

Remarks:

- ▶ If  $f(x)$  is differentiable, we can simply take  $\partial f(x_0) = \nabla f(x_0)$

# Key components of Cutting-plane method

- ▶ Cutting plane oracle  $\Omega$
- ▶ A search space  $\mathcal{S}$  initially big enough to cover  $\mathcal{K}$ , e.g.
  - ▶ Polyhedron  $\mathcal{P} = \{z \mid Cz \preceq d\}$
  - ▶ Interval  $\mathcal{I} = [l, u]$  (for one-dimensional problem)
  - ▶ Ellipsoid  $\mathcal{E} = \{z \mid (z - x_c)P^{-1}(z - x_c) \leq 1\}$

# Generic Cutting-plane method

- ▶ **Given** initial  $\mathcal{S}$  known to contain  $\mathcal{K}$ .
- ▶ **Repeat**
  1. Choose a point  $x_0$  in  $\mathcal{S}$
  2. Query the cutting-plane oracle at  $x_0$
  3. **If**  $x_0 \in \mathcal{K}$ , quit
  4. **Else**, update  $\mathcal{S}$  to a smaller set that covers:

$$\mathcal{S}^+ = \mathcal{S} \cap \{z \mid g^\top(z - x_0) + \beta \leq 0\}$$

5. **If**  $\mathcal{S}^+ = \emptyset$  or it is small enough, quit.



## Corresponding Python code

```
def cutting_plane_feas(evaluate, S, options=Options()):
    feasible = False
    status = 0
    for niter in range(options.max_it):
        cut, feasible = evaluate(S.xc)
        if feasible: # feasible sol'n obtained
            break
        status, tsq = S.update(cut)
        if status != 0:
            break
        if tsq < options.tol:
            status = 2
            break
    return S.xc, niter+1, feasible, status
```



# Convex Optimization Problem (I)

$$\begin{array}{ll}\text{minimize} & f_0(\mathbf{x}), \\ \text{subject to} & \mathbf{x} \in \mathcal{K}\end{array}$$

- ▶ The optimization problem is treated as a feasibility problem with an additional constraint  $f_0(\mathbf{x}) < t$
- ▶  $f_0(\mathbf{x})$  could be a convex function or a quasiconvex function.
- ▶  $t$  is the best-so-far value of  $f_0(\mathbf{x})$ .





## Convex Optimization Problem (II)

- Problem can be reformulated as:

$$\begin{array}{ll}\text{minimize} & t, \\ \text{subject to} & \Phi(x, t) < 0 \\ & x \in \mathcal{K}\end{array}$$

where  $\Phi(x, t) < 0$  is the  $t$ -sublevel set of  $f_0(x)$ .

- Note:  $\mathcal{K}_t \subseteq \mathcal{K}_u$  if and only if  $t \leq u$  (monotonicity)
- One easy way to solve the optimization problem is to apply the binary search on  $t$ .

```
def bsearch(evaluate, I, options=Options()):  
    feasible = False  
    l, u = I  
    t = l + (u - l)/2  
    for niter in range(options.max_it):  
        if evaluate(t): # feasible sol'n obtained  
            feasible = True  
            u = t  
        else:  
            l = t  
            tau = (u - l)/2  
            t = l + tau  
            if tau < options.tol:  
                break  
    return u, niter+1, feasible
```



```
class bsearch_adaptor:
    def __init__(self, P, E, options=Options()):
        self.P = P
        self.E = E
        self.options = options

    @property
    def x_best(self):
        return self.E.xc

    def __call__(self, t):
        E = self.E.copy()
        self.P.update(t)
        x, _, feasible, _ = cutting_plane_feas(
            self.P, E, self.options)
        if feasible:
            self.E._xc = x.copy()
            return True
        return False
```



# Shrinking

- ▶ Another possible way is, to update the best-so-far  $t$  whenever a feasible solution  $x_0$  is found such that  $\Phi(x_0, t) = 0$ .
- ▶ We assume that the oracle takes the responsibility for that.

# Generic Cutting-plane method (Optim)

- ▶ **Given** initial  $\mathcal{S}$  known to contain  $\mathcal{K}_t$ .
- ▶ **Repeat**
  1. Choose a point  $x_0$  in  $\mathcal{S}$
  2. Query the separation oracle at  $x_0$
  3. **If**  $x_0 \in \mathcal{K}_t$ , update  $t$  such that  $\Phi(x_0, t) = 0$ .
  4. Update  $\mathcal{S}$  to a smaller set that covers:

$$\mathcal{S}^+ = \mathcal{S} \cap \{z \mid g^\top(z - x_0) + \beta \leq 0\}$$

5. **If**  $\mathcal{S}^+ = \emptyset$  or it is small enough, quit.

```
def cutting_plane_dc(evaluate, S, t, options=Options()):
    feasible = False # no sol'n
    x_best = S.xc
    for niter in range(options.max_it):
        cut, t1 = evaluate(S.xc, t)
        if t != t1: # best t obtained
            feasible = True
            t = t1
            x_best = S.xc
        status, tau = S.update(cut)
        if status == 1:
            break
        if tau < options.tol:
            status = 2
            break
    return x_best, t, niter+1, feasible, status
```



## Example: Profit Maximization Problem [Aliabadi and Salahi, 2013]

$$\begin{array}{ll}\text{maximize} & p(Ax_1^\alpha x_2^\beta) - v_1x_1 - v_2x_2 \\ \text{subject to} & x_1 \leq k.\end{array}$$

- ▶  $p(Ax_1^\alpha x_2^\beta)$  : Cobb-Douglas production function
- ▶  $p$ : the market price per unit
- ▶  $A$ : the scale of production
- ▶  $\alpha, \beta$ : the output elasticities
- ▶  $x$ : input quantity
- ▶  $v$ : output price
- ▶  $k$ : a given constant that restricts the quantity of  $x_1$

## Example: Profit maximization (cont'd)

- ▶ The formulation is not in the convex form.
- ▶ Rewrite the problem in the following form:

$$\begin{array}{ll}\text{maximize} & t \\ \text{subject to} & t + v_1x_1 + v_2x_2 < pAx_1^\alpha x_2^\beta \\ & x_1 \leq k.\end{array}$$



# Profit maximization in Convex Form

- ▶ By taking the logarithm of each variable:
  - ▶  $y_1 = \log x_1, y_2 = \log x_2$ .
- ▶ We have the problem in a convex form:

$$\begin{array}{ll}\max & t \\ \text{s.t.} & \log(t + v_1 e^{y_1} + v_2 e^{y_2}) - (\alpha y_1 + \beta y_2) < \log(pA) \\ & y_1 \leq \log k.\end{array}$$

# Python code (Profit oracle) I

```
class profit_oracle:
    def __init__(self, params, a, v):
        p, A, k = params
        self.log_pA = np.log(p * A)
        self.log_k = np.log(k)
        self.v = v
        self.a = a

    def __call__(self, y, t):
        fj = y[0] - self.log_k # constraint
        if fj > 0.:
            g = np.array([1., 0.])
            return (g, fj), t
        log_Cobb = self.log_pA + np.dot(self.a, y)
        x = np.exp(y)
        vx = np.dot(self.v, x)
        te = t + vx
        fj = np.log(te) - log_Cobb
        if fj < 0.:
            te = np.exp(log_Cobb)
            t = te - vx
            fj = 0.
```

## Python code (Profit oracle) II

```
g = (self.v * x) / te - self.a  
return (g, fj), t
```

## Python code (Main program) I

```
import numpy as np
from profit_oracle import *
from cutting_plane import *
from ell import *

p, A, k = 20.0, 40.0, 30.5
params = p, A, k
a = np.array([0.1, 0.4])
v = np.array([10.0, 35.0])
y0 = np.array([0., 0.]) # initial x0
E = ell(200, y0)
P = profit_oracle(params, a, v)
yb1, fb, niter, feasible, status = \
    cutting_plane_dc(P, E, 0.0)
print(fb, niter, feasible, status)
```



# Area of Applications

- ▶ Robust convex optimization
  - ▶ oracle technique: affine arithmetic
- ▶ Parametric network potential problem
  - ▶ oracle technique: negative cycle detection
- ▶ Semidefinite programming
  - ▶ oracle technique: Cholesky factorization



# Robust Convex Optimization



# Robust Optimization Formulation

- Consider:

$$\begin{array}{ll}\text{minimize} & \sup_{q \in \mathbb{Q}} f_0(\mathbf{x}, q) \\ \text{subject to} & f_j(\mathbf{x}, q) \leq 0, \quad \forall q \in \mathbb{Q}, \quad j = 1, 2, \dots, m,\end{array}$$

where  $q$  represents a set of varying parameters.

- The problem can be reformulated as:

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & f_0(\mathbf{x}, q) < t \\ & f_j(\mathbf{x}, q) \leq 0, \quad \forall q \in \mathbb{Q}, \quad j = 1, 2, \dots, m,\end{array}$$

# Oracle in Robust Optimization Formulation

- ▶ The oracle only needs to determine:
  - ▶ If  $f_j(\mathbf{x}_0, q) > 0$  for some  $j$  and  $q = q_0$ , then
    - ▶ the cut  $(g, \beta) = (\partial f_j(\mathbf{x}_0, q_0), f_j(\mathbf{x}_0, q_0))$
  - ▶ If  $f_0(\mathbf{x}_0, q) \geq t$  for some  $q = q_0$ , then
    - ▶ the cut  $(g, \beta) = (\partial f_0(\mathbf{x}_0, q_0), f_0(\mathbf{x}_0, q_0) - t)$
  - ▶ Otherwise,  $\mathbf{x}_0$  is feasible, then
    - ▶ Let  $q_{\max} = \operatorname{argmax}_{q \in \mathbb{Q}} f_0(\mathbf{x}_0, q)$ .
    - ▶  $t := f_0(\mathbf{x}_0, q_{\max})$ .
    - ▶ The cut  $(g, \beta) = (\partial f_0(\mathbf{x}_0, q_{\max}), 0)$
- ▶ Random sampling trick



## Example: Profit Maximization Problem (convex)

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & \log(t + \hat{v}_1 e^{y_1} + \hat{v}_2 e^{y_2}) - (\hat{\alpha} y_1 + \hat{\beta} y_2) \leq \log(\hat{p} A) \\ & y_1 \leq \log \hat{k}, \end{aligned}$$

- ▶ Now assume that:
  - ▶  $\hat{\alpha}$  and  $\hat{\beta}$  vary  $\bar{\alpha} \pm e_1$  and  $\bar{\beta} \pm e_2$  respectively.
  - ▶  $\hat{p}$ ,  $\hat{k}$ ,  $\hat{v}_1$ , and  $\hat{v}_2$  all vary  $\pm e_3$ .

## Example: Profit Maximization Problem (oracle)

By detail analysis, the worst case happens when:

- ▶  $p = \bar{p} + e_3, k = \bar{k} + e_3$
- ▶  $v_1 = \bar{v}_1 - e_3, v_2 = \bar{v}_2 - e_3,$
- ▶ if  $y_1 > 0$ ,  $\alpha = \bar{\alpha} - e_1$ , else  $\alpha = \bar{\alpha} + e_1$
- ▶ if  $y_2 > 0$ ,  $\beta = \bar{\beta} - e_2$ , else  $\beta = \bar{\beta} + e_2$

**Remark:** for more complicated problems, affine arithmetic could be used  
[Liu et al., 2007].

## profit\_rb\_oracle

```
class profit_rb_oracle:
    def __init__(self, params, a, v, vparams):
        ui, e1, e2, e3 = vparams
        self.uie = [ui * e1, ui * e2]
        self.a = a
        p, A, k = params
        p -= ui * e3
        k -= ui * e3
        v_rb = v.copy()
        v_rb += ui * e3
        self.P = profit_oracle((p, A, k), a, v_rb)

    def __call__(self, y, t):
        a_rb = self.a.copy()
        for i in [0, 1]:
            a_rb[i] += self.uie[i] if y[i] <= 0. \
                               else -self.uie[i]
```



# Multi-parameter Network Problem



# Parametric Network Problem

Given a network represented by a directed graph  $G = (V, E)$ .

Consider:

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & u_i - u_j \leq h_{ij}(x, t), \quad \forall (i, j) \in E, \\ \text{variables} & x, u,\end{array}$$

- ▶  $h_{ij}(x, t)$  is the weight function of edge  $(i, j)$ ,
- ▶ Assume: network is large but the number of parameters is small.

## Network Potential Problem (cont'd)

Given  $x$  and  $t$ , the problem has a feasible solution if and only if  $G$  contains no negative cycle. Let  $\mathcal{C}$  be a set of all cycles of  $G$ .

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & W_k(x, t) \geq 0, \forall C_k \in \mathcal{C}, \\ \text{variables} & x\end{array}$$

- ▶  $C_k$  is a cycle of  $G$
- ▶  $W_k(x, t) = \sum_{(i,j) \in C_k} h_{ij}(x, t).$

# Negative Cycle Finding

There are lots of methods to detect negative cycles in a weighted graph [Cherkassky and Goldberg, 1999], in which Tarjan's algorithm [Tarjan, 1981] is one of the fastest algorithms in practice [Dasdan, 2004, Cherkassky and Goldberg, 1999].



# Oracle in Network Potential Problem

- ▶ The oracle only needs to determine:
  - ▶ If there exists a negative cycle  $C_k$  under  $x_0$ , then
    - ▶ the cut  $(g, \beta) = (-\partial W_k(x_0), -W_k(x_0))$
  - ▶ If  $f_0(x_0) \geq t$ , then
    - ▶ the cut  $(g, \beta) = (\partial f_0(x_0), f_0(x_0) - t)$
  - ▶ Otherwise,  $x_0$  is feasible, then
    - ▶  $t := f_0(x_0)$ .
    - ▶ The cut  $(g, \beta) = (\partial f_0(x_0), 0)$



## Python Code

```
class network_oracle:
    def __init__(self, G, f, p):
        self.G = G
        self.f = f
        self.p = p # partial derivative of f w.r.t x
        self.S = negCycleFinder(G)

    def __call__(self, x):
        def get_weight(G, e):
            return self.f(G, e, x)

        self.S.get_weight = get_weight
        C = self.S.find_neg_cycle()
        if C is None:
            return None, 1

        f = -sum(self.f(self.G, e, x) for e in C)
        g = -sum(self.p(self.G, e, x) for e in C)
```



## Example: Optimal Matrix Scaling [Orlin and Rothblum, 1985]

- ▶ Given a sparse matrix  $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ .
- ▶ Find another matrix  $B = UAU^{-1}$  where  $U$  is a nonnegative diagonal matrix, such that the ratio of any two elements of  $B$  in absolute value is as close to 1 as possible.
- ▶ Let  $U = \text{diag}([u_1, u_2, \dots, u_N])$ . Under the min-max-ratio criterion, the problem can be formulated as:

$$\begin{array}{ll} \text{minimize} & \pi/\psi \\ \text{subject to} & \psi \leq u_i |a_{ij}| u_j^{-1} \leq \pi, \quad \forall a_{ij} \neq 0, \\ & \pi, \psi, u, \text{ positive} \\ \text{variables} & \pi, \psi, u. \end{array}$$

## Optimal Matrix Scaling (cont'd)

By taking the logarithms of variables, the above problem can be transformed into:

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & \pi' - \psi' \leq t \\ & u'_i - u'_j \leq \pi' - a'_{ij}, \quad \forall a_{ij} \neq 0, \\ & u'_j - u'_i \leq a'_{ij} - \psi', \quad \forall a_{ij} \neq 0, \\ \text{variables} & \pi', \psi', u'.\end{array}$$

where  $k'$  denotes  $\log(|k|)$  and  $x = (\pi', \psi')^\top$ .

## Corresponding Python Code

```
def constr(G, e, x):
    u, v = e
    i_u = G.node_idx[u]
    i_v = G.node_idx[v]
    cost = G[u][v]['cost']
    return x[0] - cost if i_u <= i_v else cost - x[1]

def pconstr(G, e, x):
    u, v = e
    i_u = G.node_idx[u]
    i_v = G.node_idx[v]
    return np.array([1., 0.] if i_u <= i_v else [0., -1.])

class optscaling_oracle:
    def __init__(self, G):
        self.network = network_oracle(G, constr, pconstr)

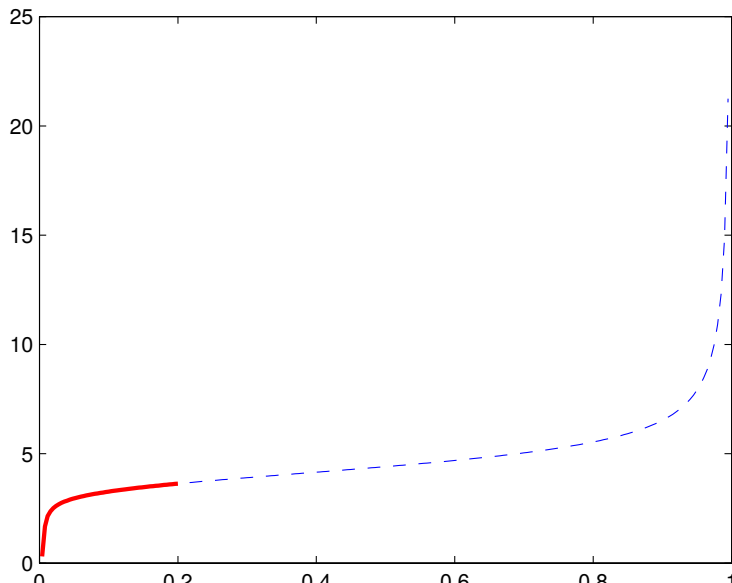
    def __call__(self, x, t):
        cut, feasible = self.network(x)
        if not feasible: return cut, t
        s = x[0] - x[1]
        fj = s - t
        if fj < 0.:
            t = s
```

## Example: clock period & yield-driven co-optimization

$$\begin{array}{ll}\text{minimize} & T_{CP} - w_{\beta}\beta \\ \text{subject to} & u_i - u_j \leq T_{CP} + F_{ij}^{-1}(1 - \beta), \quad \forall (i, j) \in E_s, \\ & u_j - u_i \leq F_{ij}^{-1}(1 - \beta), \quad \forall (j, i) \in E_h, \\ & T_{CP} \geq 0, 0 \leq \beta \leq 1, \\ \text{variables} & T_{CP}, \beta, u.\end{array}$$

- ▶ Note that  $F_{ij}^{-1}(x)$  is not concave in general in  $[0, 1]$ .
- ▶ Fortunately, we are most likely interested in optimizing circuits for high yield rather than the low one in practice.
- ▶ Therefore, by imposing an additional constraint to  $\beta$ , say  $\beta \geq 0.8$ , the problem becomes convex.

# Inverse CDF



# Matrix Inequalities

# Problems With Matrix Inequalities

Consider the following problem:

$$\begin{array}{ll}\text{minimize} & t, \\ \text{subject to} & F(x, t) \succeq 0,\end{array}$$

- ▶  $F(x, t)$ : a matrix-valued function
- ▶  $A \succeq 0$  denotes  $A$  is positive semidefinite.





# Problems With Matrix Inequalities

- ▶ Recall that a matrix  $A$  is positive semidefinite if and only if  $v^T A v \geq 0$  for all  $v \in \mathbb{R}^N$ .
- ▶ The problem can be transformed into:

$$\begin{array}{ll} \text{minimize} & t, \\ \text{subject to} & v^T F(x, t) v \geq 0, \forall v \in \mathbb{R}^N \end{array}$$

- ▶ Consider  $v^T F(x, t) v$  is concave for all  $v \in \mathbb{R}^N$  w. r. t.  $x$ , then the above problem is a convex programming.
- ▶ Reduce to *semidefinite programming* if  $F(x, t)$  is linear w.r.t.  $x$ , i.e.,  
 $F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n$

# Oracle in Matrix Inequalities

The oracle only needs to:

- ▶ Perform a *row-based* Cholesky factorization such that  $F(x_0, t) = R^\top R$ .
- ▶ Let  $A_{:p,:p}$  denotes a submatrix  $A(1:p, 1:p) \in \mathbb{R}^{p \times p}$ .
- ▶ If Cholesky factorization fails at row  $p$ ,
  - ▶ there exists a vector  $e_p = (0, 0, \dots, 0, 1)^\top \in \mathbb{R}^p$ , such that
    - ▶  $v = R_{:p,:p}^{-1} e_p$ , and
    - ▶  $v^\top F_{:p,:p}(x_0) v < 0$ .
  - ▶ The cut  $(g, \beta) = (-v^\top \partial F_{:p,:p}(x_0) v, -v^\top F_{:p,:p}(x_0) v)$

```

class lmi_oracle:
    ''' Oracle for LMI constraint  $F*x \leq B$  '''

    def __init__(self, F, B):
        self.F = F
        self.F0 = B
        self.Q = chol_ext(len(self.F0))

    def __call__(self, x):
        n = len(x)

        def getA(i, j):
            return self.F0[i, j] - sum(
                self.F[k][i, j] * x[k] for k in range(n))

        self.Q.factor(getA)
        if self.Q.is_spd():
            return None, True
        v, ep = self.Q.witness()
        g = np.array([self.Q.sym_quad(v, self.F[i])
                       for i in range(n)])
        return (g, ep), False

```



## Example: Matrix Norm Minimization

- ▶ Let  $A(\mathbf{x}) = A_0 + x_1 A_1 + \cdots + x_n A_n$
- ▶ Problem  $\min_{\mathbf{x}} \|A(\mathbf{x})\|$  can be reformulated as

$$\begin{aligned} & \text{minimize } t, \\ & \text{subject to } \begin{pmatrix} tI & A(\mathbf{x}) \\ A^\top(\mathbf{x}) & tI \end{pmatrix} \succeq 0, \end{aligned}$$

- ▶ Binary search on  $t$  can be used for this problem.

```

class qmi_oracle:
    t = None
    count = 0

    def __init__(self, F, F0):
        self.F = F
        self.F0 = F0
        self.Fx = np.zeros(F0.shape)
        self.Q = chol_ext(len(F0))

    def update(self, t): self.t = t

    def __call__(self, x):
        self.count = 0; nx = len(x)

        def getA(i, j):
            if self.count < i + 1:
                self.count = i + 1
                self.Fx[i] = self.F0[i]
                self.Fx[i] -= sum(self.F[k][i] * x[k]
                                for k in range(nx))
            a = -self.Fx[i].dot(self.Fx[j])
            if i == j: a += self.t
            return a

        self.Q.factor(getA)
        if self.Q.is_spd(): return None, True

```



## Example: Estimation of Correlation Function

$$\begin{array}{ll} \min_{\kappa, p} & \|\Omega(p) + \kappa I - Y\| \\ \text{s. t.} & \Omega(p) \succcurlyeq 0, \kappa \geq 0. \end{array}$$

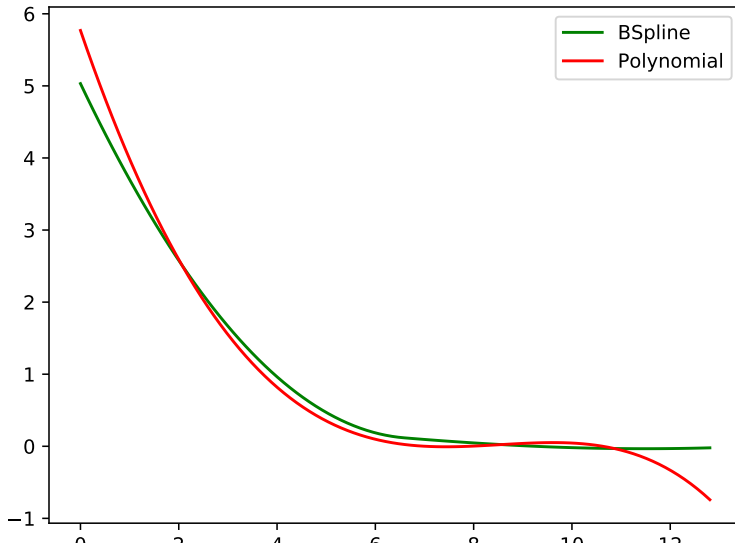
- ▶ Let  $\rho(h) = \sum_i^n p_i \Psi_i(h)$ , where
  - ▶  $p_i$ 's are the unknown coefficients to be fitted
  - ▶  $\Psi_i$ 's are a family of basis functions.
- ▶ The covariance matrix  $\Omega(p)$  can be recast as:

$$\Omega(p) = p_1 F_1 + \cdots + p_n F_n$$

where  $\{F_k\}_{i,j} = \Psi_k(\|s_j - s_i\|_2)$



# Experimental Result



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